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Efficient multipowers*

Aleksey Kolokolov and Roberto Renò

SAFE, Goethe University Frankfurt
ALEXEIUO@GMAIL.COM

Dipartimento di Scienze Economiche, Università di Verona.
Via Cantarane 24, 37129, Verona
ROBERTO.RENO@UNIVR.IT

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Abstract

Multipower estimators, widespread for their robustness to the presence of jumps, are also useful for reducing the estimation error of integrated volatility powers even in the absence of jumps. Optimizing linear combinations of multipowers can indeed drastically reduce the variance with respect to traditional estimators. In the case of quarticity, we also prove that the optimal combination is a nearly efficient estimator, being arbitrarily close to the nonparametric efficiency bound as the number of consecutive returns employed diverges. We provide guidance on how to select the optimal number of consecutive returns to minimize mean square error. The implementation on US stock prices corroborates our theoretical findings and further shows that our proposed quarticity estimator noticeably reduces the number of detected jumps, and improves the quality of volatility forecasts.

Keywords: Efficiency, Volatility, Quarticity, Multipower, Jumps, Threshold

JEL Classification Codes: C58, G1.

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1 Motivation

A very active literature in financial econometrics focuses on the estimation of integrated volatility powers. The most popular example is realized volatility, which is an estimator of the integrated variance and attracted enormous recent interest. The integral of squared variance, dubbed *quarticity*, is also quite important since it is a necessary ingredient in relevant applications, such as estimating the confidence bands for integrated variance estimates (see, e.g., Andersen et al., 2014), applying jump tests (see, e.g., Dumitru and Urga, 2012 for a review), forecasting volatility (Bollerslev et al., 2016), computing the optimal sampling frequency in the presence of market microstructure noise (see, e.g., Bandi and Russell, 2006). The integrated third volatility power is needed, for example, to build confidence bands in the estimator developed by Reiß (2011) to obtain efficiency in the presence of microstructure noise. Integrated volatility powers are useful, among other things, to estimate the activity of processes modeled via a jump-diffusion (Todorov and Tauchen, 2011).

Disentangling price volatility due to continuous movements to that due to discontinuous movements is also of fundamental importance in a number of equally relevant applications. In this field, a primary advancement has been represented by the introduction of multipower estimators, a tool introduced by Barndorff-Nielsen and Shephard (2004) to reduce the bias, due to jumps, in estimating integrated volatility powers. Successful modifications of multipower variation estimators have been proposed in Andersen et al. (2012), and Corsi et al. (2010), among others.

Grounding on the success of this literature in estimating integrated volatility powers in the presence of price discontinuities, this paper concentrates on *efficiency*, and specifically on how to minimize the variance of multipower estimators. The theory is first developed in the case without jumps. This allows to study the variance minimization problem in the simplified case of continuous semimartingales. Our first contribution is to provide a general criterion to find the optimal (in the sense of achieving minimum variance) linear combination of N multipower estimators given the number m of consecutive returns employed (for example, $m = 2$ for bipower variation, $m = 3$ for tripower variation, and so on). We present explicit solutions in relevant cases. This shows that multipower estimators are useful even in the absence of jumps, since their usage reduces the variance of the estimator. The (empirically compelling) case with jumps is then recovered by applying the same logic to *truncated* returns, in the spirit of Mancini (2009). Multipower estimators are indeed robust to jumps only for powers

less than 2, and their asymptotic variance depends on jumps as well. Applying the truncation method we get rid of both problems.

When estimating integrated variance, standard realized variance (i.e., the sum of squared returns) is an efficient estimator, achieving the nonparametric efficiency bound (Renault et al., 2016). For other integrated volatility powers the variance minimization problem is not so straightforward. As discussed in Renault et al. (2016), Mykland and Zhang (2009) provide a nearly efficient estimator, meaning that its variance can be made arbitrarily close to the nonparametric efficiency bound, while Jacod and Rosenbaum (2013) provide a fully efficient estimator. In the case of quarticity, both these estimators can be viewed as linear combinations of multipower variations, with the number of consecutive returns m growing to infinity. Our methodology allows to optimize the variance of the estimator when the number of consecutive returns m is fixed. In this respect, our second contribution is to propose a novel quarticity estimator which is nearly efficient, as that proposed by Mykland and Zhang (2009), but with a smaller variance.

Minimization of the variance can be readily exploited to improve the mean square error of any multipower estimator, as well as of the “nearest neighbor” estimators of Andersen et al. (2012), which do not require truncation. Indeed, the gain in variance as m gets larger comes at the cost of a deterioration of the bias. However, these two effects can be traded off to select the value of m which minimizes the mean square error, so that the loss in terms of bias is more than compensated by the gain in terms of variance. Thus, by construction, our nearly efficient estimator can only lower the mean square error with respect to the original estimator. Our theoretical contributions have then the immediate practical application of delivering superior estimates of integrated volatility powers.

Our theoretical results are corroborated by simulated experiments and empirical applications, both focused on estimating quarticity. Using simulations, we show that the quarticity estimator we propose performs better than competing alternatives, including standard multipowers, nearest neighbor estimators and the Jacod and Rosenbaum (2013) estimator. This result is robust to the presence of frictions (market microstructure noise and flat pricing) which are not dealt with by our theory but are introduced in the data generating process for simulated returns.

When we apply the proposed quarticity estimator to a set of intraday prices of liquid US stocks, we confirm that estimates obtained with our estimator are less variable than competing estimators, and suffer less distortion. An immediate sanity check is

that estimated quarticity should be larger than squared estimated integrated variance, the relation being exact asymptotically due to Jensen's inequality. In our sample, with our estimator this happens in 87% of the cases, a figure much larger than what obtained with standard multipowers (58% with tripower, 49% with quadpower) and with standard nearest neighbor estimators (64% with min, 67% with med). This result is consistent with much smaller estimation error due to efficiency gains.

Having at disposal superior quarticity estimates has relevant empirical implications. We provide two examples. The first is that the adoption of our quarticity estimator for standard jump tests delivers drastically less jumps than what obtained with standard multipowers. This finding supports the recent empirical literature (see, e.g., Christensen et al., 2014 and Bajgrowicz et al., 2016) claiming that the number of jumps typically detected in high frequency data is spuriously excessive, suggesting that a significant part of spuriously detected jumps could be due to inefficient quarticity estimation. In the second example, we adopt the specification proposed by Bollerslev et al. (2016) to forecast volatility. In this setting, quarticity is used to correct the coefficients of the HAR model of Corsi (2009). Also in this case, we show that our quarticity estimator improves the quality of realized volatility forecasts with respect to standard quarticity measures. Both results clearly emphasize the empirical potential of our contribution.

The paper is structured as follows. In Section 2, we describe the theoretical framework, define the class of multipower estimators and provide an expression for minimum variance estimators. Section 3 studies our proposed (nearly) efficient quarticity estimator. Section 4 reports the Monte Carlo study, while Section 5 reports the empirical application. Section 6 concludes.

2 Efficient multipowers

In a filtered probability space satisfying the usual conditions (see e.g. Protter, 2004), denote by X_t (e.g., the log-price of a financial stock or a stock index) an Itô semimartingale of the form

$$dX_t = \mu_t dt + \sigma_t dW_t + dJ_t, \quad (2.1)$$

where μ_t is predictable, σ_t is non-negative and càdlàg, and J_t is a jump process. The estimation target is the integrated volatility power over a finite interval,

$$V_T(R) = \int_0^T (\sigma_s)^R ds,$$

for a given $R > 0$. The most important cases in practice are $V_T(2)$ (integrated variance) and $V_T(4)$ (quarticity).

2.1 The case $J = 0$

We start with the case without jumps, in which the methodology can be outlined clearly. The empirically compelling case with jumps is treated in the following Section.

Define $\Delta_i X = X_{iT/n} - X_{(i-1)T/n}$, for $i = 1, \dots, n$, and write $\Delta_n = T/n$. Multipower estimators (Barndorff-Nielsen and Shephard, 2004, 2006) are defined as follows. Consider an m -valued real vector $\mathbf{r} = [r_1, \dots, r_m]$ with positive coefficients, and let

$$R = \sum_{j=1}^m r_j.$$

We define the *multipower variation* estimator $\text{MPV}(\mathbf{r})$ with powers $\mathbf{r}$ as

$$\text{MPV}(\mathbf{r}) = c_{\mathbf{r}} \cdot \sum_{i=1}^{n-m+1} \left(\prod_{j=1}^m |\Delta_{i+j-1} X|^{r_j} \right), \quad (2.2)$$

where the constant $c_{\mathbf{r}}$, meant to make the estimator unbiased in small samples under the assumption of constant volatility, is defined by

$$c_{\mathbf{r}} = \left(\frac{n}{T} \right)^{\frac{R}{2}-1} \frac{n}{n - (m-1)} \left(\prod_{j=1}^m (\mu_{r_j})^{-1} \right),$$

where $\mu_r = E(|u|^r)$ with u being a standard normal. Important special cases are *bipower variation*, used to estimate integrated variance:

$$\text{MPV}([1, 1]) = \frac{\pi}{2} \frac{n}{n-1} \sum_{i=1}^{n-1} |\Delta_i X| |\Delta_{i+1} X|$$

and tripower and quadpower variation, both used to estimate quarticity:

$$\begin{aligned}\text{MPV}([4/3, 4/3, 4/3]) &= \frac{n}{T} \frac{1}{\mu_{\frac{4}{3}}^3} \frac{n}{n-2} \sum_{i=1}^{n-2} |\Delta_i X|^{\frac{4}{3}} |\Delta_{i+1} X|^{\frac{4}{3}} |\Delta_{i+2} X|^{\frac{4}{3}}, \\ \text{MPV}([1, 1, 1, 1]) &= \frac{n}{T} \left(\frac{\pi}{2}\right)^2 \frac{n}{n-3} \sum_{i=1}^{n-3} |\Delta_i X| |\Delta_{i+1} X| |\Delta_{i+2} X| |\Delta_{i+3} X|.\end{aligned}$$

When $J_t = 0$, it has been proven, see e.g. Barndorff-Nielsen et al. (2006a), that $\text{MPV}(\mathbf{r})$ is a consistent and asymptotically normally distributed estimator, as $n \rightarrow \infty$, of $\int_0^T (\sigma_s)^R ds$. Under mild assumptions on the model (2.1), the following stable central limit theorem holds:

$$\sqrt{\frac{1}{\Delta_n}} \left(\text{MPV}(\mathbf{r}) - \int_0^T (\sigma_s)^R ds \right) \xrightarrow[n \rightarrow \infty]{} \mathcal{MN} \left(0, V_{\mathbf{r}} \int_0^T (\sigma_s)^{2R} ds \right) \quad (2.3)$$

where $\mathcal{MN}$ denotes a standard mixed normal distribution, and

$$V_{\mathbf{r}} = \frac{\prod_{j=1}^m \mu_{2r_j} - (2m-1) \prod_{j=1}^m \mu_{r_j}^2 + 2 \sum_{i=1}^{m-1} \prod_{j=1}^i \mu_{r_j} \prod_{j=m-i+1}^m \mu_{r_j} \prod_{j=1}^{m-i} \mu_{r_j+r_{j+i}}}{\prod_{j=1}^m \mu_{r_j}^2}, \quad (2.4)$$

see Theorem 3 in Barndorff-Nielsen et al. (2006a) for details.

In the large empirical literature that uses multipower estimators, these are invariably implemented with equal powers: bipower variation with $\mathbf{r} = [1, 1]$, tripower variation with $\mathbf{r} = [4/3, 4/3, 4/3]$, quadpower variation with $\mathbf{r} = [1, 1, 1, 1]$ and so forth. However, minimizing $V_{\mathbf{r}}$ in formula (2.4) reveals that the variance of the estimator can be reduced considerably by changing powers. For example, while with the standard tripower variation we have $V_{[4/3, 4/3, 4/3]} = 13.65$, the optimal choice with three powers would be $V_{[3.5455, 0.2182, 0.2362]} = 9.70$, that is a gain of roughly 30% on the asymptotic variance. This fact has also been remarked in the paper of Mancini and Calvori (2012). As we show next, combining estimators with different powers is even more efficient.¹ This also implies that it makes perfect sense to use multipower estimators even in the absence of jumps.

¹For example, a more efficient estimator defined as $c_n \sum_{i=1}^{n-2} \left(|\Delta_i X|^2 + |\Delta_{i+1} X|^2 + |\Delta_{i+2} X|^2 \right)^2$ for an appropriate constant c_n can be viewed as a linear combination of multipower variations with different powers.

Central limit theorems of the form (2.3) hold for virtually all estimators of integrated volatility powers used in the financial econometrics. Consequently, it is natural to study efficiency in the class of such estimators in the first place. Denote by $\mathcal{E}_R$ the class of estimators in the model defined by (2.1), satisfying: if $\mathcal{E}_{R,n} \in \mathcal{E}_R$,

(i)

$$\mathcal{E}_{R,n} - \int_0^T (\sigma_s)^R ds \xrightarrow{p} 0$$

(ii)

$$\sqrt{\frac{1}{\Delta_n}} \left(\mathcal{E}_{R,n} - \int_0^T (\sigma_s)^R ds \right) \xrightarrow[n \rightarrow \infty]{\Rightarrow} \mathcal{MN} \left(0, V_{\mathcal{E}_{R,n}} \int_0^T (\sigma_s)^{2R} ds \right),$$

where $\mathcal{MN}$ denotes a standard mixed normal distribution, and $V_{\mathcal{E}_{R,n}}$ is a constant.

We say, that an estimator $\mathcal{E}_{R,n} \in \mathcal{E}_R$ is efficient with respect to another estimator $\mathcal{E}'_{R,n} \in \mathcal{E}_R$, if

$$V_{\mathcal{E}_{R,n}} \leq V_{\mathcal{E}'_{R,n}}.$$

Clearly, this implies that for every fixed T and every volatility path, the variance of random variable $\mathcal{E}_{R,n}$ is not larger than the variance of $\mathcal{E}'_{R,n}$. The lower bound V^* for the constants $V_{\mathcal{E}_{R,n}}$ among all estimators $\mathcal{E}_{R,n}$ from $\mathcal{E}_R$ is equal to the Cramer-Rao lower bound in the model with constant volatility ($\sigma_t = 1$), that is $R^2/2$. Indeed, any estimator $\mathcal{E}_{R,n} \in \mathcal{E}_R$ is in particular a consistent and asymptotically normal with variance $V_{\mathcal{E}_{R,n}}$ in the model with $\sigma_t = 1$. Hence, $V_{\mathcal{E}_{R,n}}$ must not exceed the Cramer-Rao lower bound. If for an estimator $\mathcal{E}_{R,n} \in \mathcal{E}_R$, $V_{\mathcal{E}_{R,n}}$ is equal to the Cramer-Rao lower bound, the inequality $V_{\mathcal{E}_{R,n}} \leq V_{\mathcal{E}'_{R,n}}$ holds for any other estimator $\mathcal{E}'_{R,n} \in \mathcal{E}_R$. In that case, $\mathcal{E}_{R,n}$ is efficient in the class $\mathcal{E}_R$, since the constants $V_{\mathcal{E}_{R,n}}$ and $V_{\mathcal{E}'_{R,n}}$ are independent from the realization of the volatility process. Renault et al. (2016) have recently established a similar but stronger result (see Example 1 in their paper): the nonparametric lower bound for all “regular” estimators of integrated volatility powers $\int_0^T (\sigma_s)^R ds$ in the model (2) is given by

$$\frac{R^2}{2} \int_0^T (\sigma_s)^{2R} ds. \quad (2.5)$$

This implies, that an efficient estimator in the class $\mathcal{E}_R$ is also efficient among all regular estimators in the sense of Renault et al. (2016).

We start by investigating efficiency among multipowers only. Clearly, this does not include all possible estimators in the class $\mathcal{E}_R$, so that the minimum variance estimator

among multipowers will not necessarily be efficient in the above sense. In the next section, we specialize on quarticity ($R = 4$) and show that the corresponding estimator is nearly efficient, again in the sense of Renault et al. (2016), meaning that its variance can get arbitrary close to the nonparametric lower bound.

We now define the class of estimators inside which we look for the one with minimum variance. We call this estimator *MPV-efficient*, to highlight this is efficient only in the restricted class in which we fix the number N of multipower estimators employed in the linear combination, and the (maximum) number m of consecutive returns employed. Minimization is achieved with respect to the weights of the linear combination and to the powers to which returns are raised.

We write a linear combination of N different $\text{MPV}(\mathbf{r})$ estimators which have the maximum number of adjacent returns equal to m as:

$$\text{GMPV}(N, m; \sigma^R) = \sum_{j=1}^N w_j \text{MPV}(\mathbf{r}^{(j)}), \quad (2.6)$$

where $\mathbf{G}$ stands for “generalized”, and the weights $w_j, j = 1, \dots, N$ are such that

$$\sum_{j=1}^N w_j = 1$$

and for each $j = 1, \dots, N$, $\mathbf{r}^{(j)}$ is a vector in $\mathbb{R}^m$ with non-negative components such that $\sum_{i=1}^m r_i^{(j)} = R$. By construction, $\text{GMPV}(N, m; \sigma^R)$ is a consistent estimator of $V_T(p)$, that is

$$\text{GMPV}(N, m; \sigma^R) \xrightarrow{p} \int_0^T (\sigma_s)^R ds$$

as $n \rightarrow \infty$ with fixed N and m . Our problem is to find the efficient estimator in this class, that is the problem of finding the optimal weights w_j and power vectors $\mathbf{r}^{(j)}$ delivering the minimum asymptotic variance of $\text{GMPV}(N, m; \sigma^R)$. We denote by $\text{GMPV}^*(N, m; \sigma^R)$ the MPV-efficient estimator in this class for fixed N, m, R . The following proposition provides the desired result.

Proposition 2.1. *The MPV-efficient estimator in the class of linear combinations of multipower estimators, for given N, m and R , is given by:*

$$\text{GMPV}^*(N, m; \sigma^R) = \sum_{j=1}^N w_j^* \text{MPV}(\mathbf{r}^{*(j)}), \quad (2.7)$$

where the efficient $\mathbb{R}^m$ -valued power vectors $\mathbf{r}^{*(1)}, \dots, \mathbf{r}^{*(N)}$ minimize the quantity

$$\mathcal{V}(N, m, R, \mathbf{r}^{(1)}, \dots, \mathbf{r}^{(N)}) = \frac{1}{\sum_{i=1}^N \sum_{j=1}^N C_{ij}^{-1}(\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(N)})},$$

where C^{-1} is the inverse of the $N \times N$ symmetric matrix C defined as:

$$\begin{aligned} C_{ij}(\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(N)}) = & \left(\prod_{k=1}^m \mu_{r_k^{(i)}} \mu_{r_k^{(j)}} \right)^{-1} \left(\prod_{k=1}^m \mu_{r_k^{(i)} + r_k^{(j)}} + \sum_{k=1}^{m-1} \prod_{\ell=1}^k \mu_{r_\ell^{(i)}} \prod_{\ell=1}^{m-k} \mu_{r_{k+\ell}^{(i)} + r_\ell^{(j)}} \prod_{\ell=m-k+1}^m \mu_{r_\ell^{(i)}} \right. \\ & \left. + \sum_{k=1}^{m-1} \prod_{\ell=1}^k \mu_{r_\ell^{(j)}} \prod_{\ell=1}^{m-k} \mu_{r_{k+\ell}^{(j)} + r_\ell^{(i)}} \prod_{\ell=m-k+1}^m \mu_{r_\ell^{(j)}} - (2m-1) \prod_{k=1}^m \mu_{r_k^{(i)}} \mu_{r_k^{(j)}} \right), \end{aligned} \quad (2.8)$$

for $i, j = 1, \dots, N$; and the efficient weights coefficient w_j^* , $j = 1, \dots, N$, are given by:

$$w_j^* = \tilde{V}(N, m; \sigma^R) \sum_{i=1}^N C_{ij}^{-1}(\mathbf{r}^{*(1)}, \dots, \mathbf{r}^{*(N)}), \quad (2.9)$$

where $\tilde{V}(N, m; \sigma^R) = \mathcal{V}(N, m, R; \mathbf{r}^{*(1)}, \dots, \mathbf{r}^{*(N)})$. For the MPV-efficient estimator it holds:

$$\sqrt{\frac{1}{\Delta_n}} \left(\text{GMPV}^*(N, m; \sigma^R) - \int_0^T (\sigma_s)^R ds \right) \xrightarrow{n \rightarrow \infty} \mathcal{MN} \left(0, \tilde{V}(N, m; \sigma^R) \int_0^T (\sigma_s)^{2R} ds \right), \quad (2.10)$$

where the above convergence is stable in law.

Proof. See Appendix A. □

The above proposition provides a procedure that allows to find, by numerical optimization, the minimum variance combination with fixed N , m and R . For example, when $R = 3$ the MPV-efficient estimator with $N = 1$, $m = 2$ is found to be $\text{GMPV}^*(1, 2; \sigma^3) = \text{MPV}([0.1358, 2.8642])$, with $\tilde{V}(1, 2; \sigma^3) = 4.7947$, the lower bound imposed by maximum likelihood being 4.5 in this case. Adding more estimators in a linear combination would further minimize variance. So, when $N = 2$ the MPV-efficient estimator is found to be

$$\text{GMPV}^*(2, 2; \sigma^3) = 0.2634 \text{MPV}([1.5, 1.5]) + 0.7366 \text{MPV}([3, 0]), \quad (2.11)$$

Table 1: Reports the values of $\tilde{V}(m, N; \sigma^4)$ for several values of m and N . The globally efficient value is 8.

<hr/>					
N					
m	1	2	3	4	5
1	10.666				
2	10.050	9.600			
3	9.701	9.314	9.143		
4	9.478	9.119	8.989	8.965	
5	9.323	9.019	8.856	8.850	8.801
<hr/>					

with $\tilde{V}(2, 2; \sigma^3) = 4.7324$; while when $N = 3$ it is

$$\begin{aligned} \text{GMPV}^*(3, 2; \sigma^3) = & -0.1345 \text{ MPV}([0.6868, 2.3132]) + 0.3639 \text{ MPV}([1.3020, 1.6980]) \\ & + 0.7706 \text{ MPV}([0.0016, 2.9984]), \end{aligned} \quad (2.12)$$

with $\tilde{V}(3, 2; \sigma^3) = 4.7320$. It is interesting to look, always in the case $R = 3$, at the MPV-efficient estimator with $N = 3, m = 3$, which is found to be

$$\text{GMPV}^*(3, 3; \sigma^3) = 0.1865 \text{ MPV}([0, 1.5, 1.5]) + 0.1865 \text{ MPV}([1.5, 0, 1.5]) \quad (2.13)$$

$$+ 0.6270 \text{ MPV}([0, 3, 0]), \quad (2.14)$$

with $\tilde{V}(3, 3; 3) = 4.6666$. We can conjecture the emergence a clear structure for the case $N = m$. We will exploit this structure in the definition of our efficient quarticity estimator.

Tables 1 and 2 report the values of $\tilde{V}(m, N; \sigma^R)$ when fixing m and N in the cases $R = 3$ and $R = 4$ respectively. We can see that, as m and N increase, these values become smaller and closer to the globally efficient value $R^2/2$. In Section 3, we will show that, when $R = 4$, the MPV-efficient estimator with $m = N$ can achieve the efficient value as $m \rightarrow \infty$.

2.2 The case $J \neq 0$

The case with $J \neq 0$ makes the analysis of the efficiency of multipower estimators more complicated. Consistency of multipower estimators is indeed lost, in the presence of

Table 2: Reports the values of $\tilde{V}(m, N; \sigma^3)$ for several values of m and N . The globally efficient value is 4.5.

	N				
m	1	2	3	4	5
1	4.891				
2	4.795	4.732			
3	4.745	4.693	4.667		
4	4.714	4.667	4.666	4.641	
5	4.693	4.653	4.643	4.629	4.620

jumps, if $\max\{r_1, \dots, r_m\} \geq 2$, and the asymptotic variance of the multipower estimator also contains a part influenced by jumps if $\max\{r_1, \dots, r_m\} \geq 1$, see Barndorff-Nielsen et al. (2006a,b); Veraart (2010); Vetter (2010); Woerner (2006). Thus, the central limit Theorem in Eq. (2.3) only holds with these two constraints.

However, a simple solution at hand is provided by applying multipower estimators to truncated returns. Consider a positive stochastic process ϑ_t , which we call a *threshold* (Mancini, 2009). Write $\vartheta_t = \xi_t \Theta(\Delta)$, with $\Theta(\Delta)$ being a real function satisfying

$$\lim_{\Delta \rightarrow 0} \Theta(\Delta) = 0, \quad \lim_{\Delta \rightarrow 0} \frac{\Delta \log \frac{1}{\Delta}}{\Theta(\Delta)} = 0, \quad (2.15)$$

and ξ_t being a stochastic process on $[0, T]$ which is a.s. bounded and with a strictly positive lower bound. Using the threshold, we define *truncated* equally-spaced returns, observed over the interval $[0, T]$ as

$$\overline{\Delta}_i X = (\Delta_i X) I_{\{|\Delta_i X| \leq \vartheta_{(i-1)T/n}\}}, \quad i = 1, \dots, n \quad (2.16)$$

where $I_{\{A\}}$ is the indicator function of the set A . The truncation is meant to annihilate returns larger than a given threshold, while leaving all the remaining returns unchanged. We define the *threshold multipower variation* estimator $\text{TMPV}(\mathbf{r})$ (Corsi et al., 2010) as

$$\text{TMPV}(\mathbf{r}) = c'_{\mathbf{r}} \cdot \sum_{i=1}^{n-m+1} \left(\prod_{j=1}^m |\overline{\Delta}_{i+j-1} X|^{r_j} \right). \quad (2.17)$$

where the constant $c'_{\mathbf{r}}$, meant to make the estimator unbiased in small sample under

the assumption of constant volatility, is now defined by

$$c'_{\mathbf{r}} = \left(\frac{n}{T}\right)^{\frac{R}{2}-1} \frac{n}{n - (m-1) - n_J} \left(\prod_{j=1}^m (\mu_{r_j})^{-1} \right), \quad (2.18)$$

where n_J is the number of terms vanishing in the sum in Eq. (2.17) because of the indicator function. Then, under mild conditions for the process (2.1), the following stable central limit continues to hold as $n \rightarrow \infty$:

$$\sqrt{\frac{1}{\Delta_n}} \left(TMPV(\mathbf{r}) - \int_0^T (\sigma_s)^R ds \right) \Rightarrow \mathcal{MN} \left(0, V_{\mathbf{r}} \int_0^T (\sigma_s)^{2R} ds \right), \quad (2.19)$$

see Theorem 2.3 in Corsi et al. (2010) for the case with finite activity jumps, and Theorem 13.2.1 in Jacod and Protter (2012), for the general case allowing for infinite activity jumps.

Using truncated returns, the results in Section 2.1 can be readily recovered. The disadvantage of the truncation is that it entails an additional parameter, that is the threshold ϑ . Moreover, truncating implied a finite sample bias due to the elimination of large, but continuous, returns. However, using a reasonably high threshold (in this paper, we always use 5 “local” standard deviations) will leave out very big jumps only, whose impact would be the largest, while remaining small jumps are dealt with the multipower technique. This double-sword feature smoothes the hurdle of having to select an additional threshold and makes the proposed estimators virtually immune from the bias due to the presence of the jumps, so that we can concentrate on variance reduction.²

An alternative way to avoid threshold selection has been proposed by Andersen et al. (2012) and robustified in Andersen et al. (2014), consisting in a “comparison” method based on the nearest neighbors. We also consider this kind of estimators in the Monte Carlo study and empirical application, namely the **minRQ** estimator, defined as

$$\text{minRQ} = \frac{\pi}{3\pi - 8} \frac{n^2}{n-1} \sum_{i=1}^{n-1} \min(|\Delta_i X|, |\Delta_{i+1} X|)^4, \quad (2.20)$$

²An alternative to truncation could be minimizing using powers with the constraint that all of them are less than 2, so that multipowers are sufficient to eliminate jumps asymptotically. However, this would not only, from a theoretical point of view, sacrifice efficiency; but also, from an empirical point of view, compromise the performance of the estimator given the well known small sample bias of multipower estimators (Corsi et al., 2010), which is particularly severe when small powers are next to big powers (e.g. in the $[0.5, 3.5]$ case).

and the **medRQ** estimators, defined as

$$\text{medRQ} = \frac{3\pi}{9\pi + 72 - 52\sqrt{3}} \frac{n^2}{n-2} \sum_{i=1}^{n-2} \text{med}(|\Delta_i X|, |\Delta_{i+1} X|, |\Delta_{i+2} X|)^4. \quad (2.21)$$

Both estimators are consistent for quarticity in the presence of jumps, allow for a central limit theorem in the same form of Eq. (2.19) with $V_{\mathbf{r}}$ replaced by 18.54 for the **minRQ** estimator, and by 14.16 for the **medRQ** estimator. Thus, the nonlinear structure of these estimators implies a quite large asymptotic variance, which means that the combined use of neighbor estimators and efficient multipowers, as suggested in Section 3.1, also leads to improvement of the estimator in the mean square error sense. Moreover, also nearest neighbor estimators might be truncated. Finally, changing nearest neighbor powers could also be of help with the variance, even if this theoretical problem appears more challenging.

In what follows, we cast the theory in the more realistic case in which $J_t \neq 0$ using truncated returns, and we restrict our attention to threshold multipower estimators.

3 A nearly efficient quarticity estimator

In practice, two cases appear to be more relevant: the case $R = 2$ (integrated volatility), and the case $R = 4$ (integrated quarticity). In the case $R = 2$, the efficient multipower estimator, for every value of m and N , is simply **TMPV**([2]), that is the threshold realized variance proposed by Mancini (2009). The case with $R = 2$ is the unique value of R for which an efficient estimator can be constructed so easily.

The case $R = 4$ (quarticity) is more intriguing and particularly important in financial applications: it is used, for example, in determining the optimal sampling frequency in the presence of market microstructure noise (Bandi and Russell, 2006); in computing jump tests; in determining the confidence intervals of realized variance; in forecasting volatility; see, for example, the discussion in Balter (2015), who provides an estimator based on the observation of the whole price path.

Based on the analysis in Section 2.1, we propose to use the **MPV**-efficient quarticity

estimator when $N = m$, that is:

$$\text{GTMPV}^{**}(m; \sigma^4) = \frac{3}{2m+1} \text{TMPV}([4]) + \frac{2}{2m+1} \sum_{j=0}^{m-2} \text{TMPV}([2, \underbrace{0, \dots, 0}_j, 2]), \quad (3.1)$$

The weights in Eq. (3.1) sum up to 1, so that $\text{GTMPV}^{**}(m; \sigma^4)$ is a consistent estimator, as $n \rightarrow \infty$, of $V_T(4)$ for every fixed value of m . The next proposition provides the asymptotic distribution of $\text{GTMPV}^{**}(m; \sigma^4)$ for a fixed m .

Proposition 3.1. *As $n \rightarrow \infty$, if m is fixed,*

$$\sqrt{\frac{1}{\Delta_n}} \left(\text{GTMPV}^{**}(m; \sigma^4) - \int_0^T \sigma_s^4 ds \right) \Rightarrow \mathcal{MN} \left(0, V^{**}(m, \sigma^4) \int_0^T \sigma_s^8 ds \right), \quad (3.2)$$

where

$$V^{**}(m, \sigma^4) = 8 + \frac{8}{2m+1}, \quad (3.3)$$

and the above convergence is stable in law.

Proof. See Appendix A. □

Proposition 3.1 shows that the estimator (3.1), when defined for a fixed m , is nearly efficient in the sense of Renault et al. (2016), since its variance converges to $8 \int_0^T \sigma_s^8 ds$ (the nonparametric efficiency bound) as $m \rightarrow \infty$ (see their discussion in Section 4).

Our estimator is also closely related to that proposed by Jacod and Rosenbaum (2013). Proposition 3.2, stated below, implies that, when $m, n \rightarrow \infty$ jointly with $m^2/n \rightarrow 0$, the estimator (3.1) is fully efficient. Indeed the Jacod and Rosenbaum quarticity estimator can also be written, ignoring negligible end-effects, as a linear combination of multipower estimators (see the proof of Proposition 3.2), so that the two estimators share the same asymptotic properties, including full efficiency in the case of regular sampling times as demonstrated by Renault et al. (2016).

When m is fixed, the difference between the Jacod and Rosenbaum quarticity estimator and $\text{GTMPV}^{**}(m; \sigma^4)$ consists in the weights of the linear combination. For the estimator (3.1), the weights are purposely designed to deliver the minimum variance with a fixed m . Thus, the estimator (3.1) has smaller variance than the Jacod and Rosenbaum (2013) estimator implemented with the same window (while sharing its asymptotic properties as $m \rightarrow \infty$ at a given rate). As we discuss below, this fact is very important in practice. Indeed, larger m also implies larger bias. When the

objective is the minimization of a loss function which depends on both variance and bias, as it is customary, an intermediate value of m is optimal. Thus, minimizing the variance in the fixed m case is beneficial even if full efficiency is lost.

The problem of improving efficiency when m is fixed has also been studied by Mykland and Zhang (2009), who also propose a nearly efficient block estimator for integrated volatility powers which is UMVU (Uniformly Minimum Variance Unbiased) in each block.³ This clearly improves the asymptotic variance, but does not explore (as we do here) the possibility of interaction among blocks. For this reason, our estimator has smaller variance. The asymptotic variance of the Mykland and Zhang (2009) estimator when $R = 4$ is indeed given, from Eq. (58) in their paper, by:

$$V^{MZ}(m, \sigma^4) = 8 + 8 \frac{m^2 + 2m}{m^2 - 1} \quad (3.4)$$

so that the relative efficiency compares favorably for $\text{GTMPV}^{**}(m; \sigma^4)$ since

$$\frac{V^{MZ}(m, \sigma^4)}{V^{**}(m, \sigma^4)} = \frac{1}{2} \frac{m(2m+1)(m+2)}{(m+1)^2(m-1)} > 1. \quad (3.5)$$

Figure 1 shows the Asymptotic Relative Efficiency (ARE) of both estimators as a function of m , showing that the advantage in using the efficient multipowers estimator can be quite large for small values of m . The figure also shows the ARE of the standard multipower estimator $\text{TMPV}(\underbrace{[4/m, \dots, 4/m]}_{m \text{ terms}})$ with all equal powers (for $m = 1$ it trivially coincides with the efficient estimator; for $m = 2$ this is bipower variation, for $m = 3$ tripower variation, for $m = 4$ quadpower variation, and so on), which shows that using equal powers for large m is definitively not the best option; and the ARE of the **minRQ** (compared with the case $m = 2$) and the **medRQ** estimator (compared with the case $m = 3$), which shows that the min-med estimators have lower relative efficiency than traditional multipowers. For both $\text{GTMPV}^{**}(m; \sigma^4)$ and the block estimator, the asymptotic relative efficiency increases with m , converging to 1 when $m \rightarrow \infty$. The convergence is however faster for $\text{GTMPV}^{**}(m; \sigma^4)$.

³The setting of Mykland and Zhang (2009) is without jumps. It can however be reconciled with our framework by applying their estimator to truncated returns, as we do here, instead of the original one.

Figure 1: Asymptotic Relative Efficiency (with respect to the case $m \rightarrow \infty$) for quarticity estimation. Several estimators are compared: the GTMPV** estimator in Eq. (3.1), the Mykland and Zhang (2009) block estimator, the standard multipower estimator with all equal powers, and the minRQ and medRQ estimator.

3.1 The choice of the optimal m : bias considerations

Achieving (near) efficiency is clearly not the end of the story. Typically, the objective is the minimization of a loss function which depends also on the bias, such as the mean square error. While the variance tends to decrease with m , the bias tends to increase with m , thus originating the usual bias-variance tradeoff.

A convenient expression of the asymptotic bias is provided by the next proposition, in which m is allowed to diverge at a suitable rate. The result borrows from the work of Jacod and Rosenbaum (2013).

Proposition 3.2. *Assume that volatility is driven by the process*

$$d\sigma_t^2 = \mu_t^\sigma dt + \Lambda_t dW_t + dJ_t^\sigma,$$

where μ_t is a predictable process, Λ_t is càdlàg and J_t^σ is a jump process. If $n, m \rightarrow \infty$ in such a way that $m^2/n \rightarrow \theta$ with $\theta > 0$, we have

$$\sqrt{\frac{1}{\Delta_n}} \left(\text{GTMPV}^{**}(m; \sigma^4) - \int_0^T \sigma_s^4 ds \right) \Rightarrow B_1 + B_2 + B_3 + \mathcal{MN} \left(0, 8 \int_0^T \sigma_s^8 ds \right), \quad (3.6)$$

where

$$B_1 = -\frac{\theta}{2} (\sigma_0^4 + \sigma_T^4), \quad (3.7)$$

$$B_2 = -\frac{\theta}{6} \int_0^T \Lambda_s^2 ds, \quad (3.8)$$

$$B_3 = -\frac{\theta}{6} \sum (\Delta \sigma_s^2)^2, \quad (3.9)$$

and $\Delta \sigma_s^2 = \sigma_s^2 - \sigma_{s-}^2$ are the jumps in the variance process.

The (asymptotic) bias thus consists of three negative terms, the first due to the border-effect, and the second and the third due to the variability of volatility.

We can take advantage of the small sample approximation of the bias provided by Proposition 3.2 with the small sample approximation of the variance provided by Proposition 3.1 to get the following approximation of the mean square error:

$$MSE \approx \Delta_n \left(8 + \frac{8}{2m+1} \right) Q^2 + \Delta_n^3 m^4 B^2 \quad (3.10)$$

where $Q^2 = \int_0^T \sigma_s^8 ds$ and $B = \frac{1}{2} (\sigma_0^4 + \sigma_T^4) + \frac{1}{6} \left(\int_0^T \Lambda_s^2 ds + \sum (\Delta \sigma_s^2)^2 \right)$.

In principle, this MSE could be estimated from the data as a function of m , which could then be optimized. The main problem would be to estimate the volatility of volatility term $\int_0^T \Lambda_s^2 ds + \sum (\Delta \sigma_s^2)^2$ with sufficiently low error. To gain feeling of what we can get, we set $\int_0^T \sigma_t dt = 1$, we ignore the end-effect term $\frac{1}{2} (\sigma_0^4 + \sigma_T^4)$ since we use the constant c_r in Eq. (2.18) to compensate for it, and optimize the MSE for various values of $\bar{\Lambda} = \left(\int_0^T \Lambda_s^2 ds + \sum (\Delta \sigma_s^2)^2 \right)^{1/2}$. Table 3 reports the optimal m we found for different choices of n , and shows that the optimal m would strongly depend on the volatility-of-volatility estimate. Given the notorious difficulty in estimating this parameter, we propose the following alternative approach.

We assume to start from a quarticity estimator $\widehat{Q}$, which might not be efficient, but which is assumed to be unbiased (for example, any multipower or nearest neighbor estimator). Under this assumption, we can always improve (in the mean square error sense, or in the sense of any alternative loss function combining bias and variance) with respect to the estimator $\widehat{Q}$ by choosing m^* that minimizes the MSE:

$$\widetilde{MSE} = \left[\text{GTMPV}^{**}(m; \sigma^4) - \widehat{Q} \right]^2 + \text{Var}(\text{GTMPV}^{**}(m; \sigma^4)) \quad (3.11)$$

Table 3: Reports the values of m^{opt} that optimize the mean square error (3.10) for different values of n and $\bar{\Lambda} = \left(\int_0^T \Lambda_s^2 ds + \sum (\Delta \sigma_s^2)^2 \right)^{1/2}$, in the case $\int_0^T \sigma_t^8 dt = \sigma_0 = \sigma_T = 1$.

	$\bar{\Lambda} = 0.1$	$\bar{\Lambda} = 0.5$	$\bar{\Lambda} = 1$	$\bar{\Lambda} = 2$	$\bar{\Lambda} = 3$	$\bar{\Lambda} = 5$
	optimal m					
$n = 40$	56	15	9	5	4	2
$n = 80$	73	20	12	7	5	3
$n = 400$	109	38	22	13	9	6

where the variance of $\text{GTMPV}^{**}(m; \sigma^4)$ can be estimated, using the asymptotic expression (3.2), by

$$\text{Var}(\text{GTMPV}^{**}(m; \sigma^4)) \approx \left(8 + \frac{8}{2m+1} \right) \text{TMPV}([8/3, 8/3, 8/3]),$$

or replacing $\text{TMPV}([8/3, 8/3, 8/3])$ by another consistent estimator of $\int_0^T \sigma_s^8 ds$.⁴ By construction, the estimator $\text{GTMPV}^{**}(m^*; \sigma^4)$ will have a smaller mean square error than the original estimator $\widehat{Q}$. This is the technique we use in the empirical applications on jump testing and volatility forecasting, using $\widehat{Q} = \text{TMPV}([4])$.

Note that in finite samples there is another source of negative bias, due to the truncation of the largest observations. However, it is of smaller order with respect to the other two biases, hence it is not considered here. Moreover, with observed prices, another sources of bias is market microstructure noise. The bias due to the microstructure noise can be eliminated by using pre-averaging approach (see, e.g., Hautsch and Podolskij, 2013). The variance of multi-power variations based on pre-averaged returns can be minimized by changing powers as in the present paper. However, since the variance of pre-averaged estimators depends (among other terms) on the moments of the noise, the lower bound for the variance of such estimators would be different from the one

⁴Alternatively, the variance of $\text{GTMPV}^{**}(m; \sigma^4)$ could be estimated, in small samples, using wild bootstrap. Our numerical experiments indicate that the two ways of estimating the variance are equivalent, so that we suggest to use the handy asymptotic approximation.

considered in the present paper. The determination of this bound, possibly by methods similar to the ones used by Renault et al. (2016), is a self-contained problem outside the scope of this paper. This said, in the Monte Carlo section below we quantify the impact of market microstructure noise and flat trading explicitly.

4 Monte Carlo simulations

In this Section, we perform a series of Monte Carlo experiments focusing on quarticity estimation at different frequencies. In particular, on a simulated typical trading day in the US market of 6.5 hours, we consider $n = 40, 80, 400$ which roughly corresponds to 10, 5 and 1-minutes returns respectively.

In order to generate a realistic price dynamics, we simulate the jump-diffusion model:

$$\begin{aligned} d \log p_t &= \mu dt + \gamma_t \sigma_t dW_{p,t} + dJ_t \\ d \log \sigma_t^2 &= (\alpha - \beta \log \sigma_t^2) dt + \eta dW_{\sigma,t}, \end{aligned} \tag{4.1}$$

where W_p and W_σ are standard Brownian motions with $\text{corr}(dW_p, dW_\sigma) = \rho$, σ_t is a stochastic volatility factor and γ_t is an *intraday effect*. We use the model parameters estimated by Andersen et al. (2002) on S&P500 prices: $\mu = 0.0304$, $\alpha = -0.012$, $\beta = 0.0145$, $\eta = 0.1153$, $\rho = -0.6127$, where the parameters are expressed in daily units and returns are in percentage, and we use:

$$\gamma_{t,\tau} = \frac{1}{0.1033} (0.1271\tau^2 - 0.1260\tau + 0.1239),$$

as estimated by us on S&P500 intraday returns. We discretize model (4.1) in the interval $[0, 1]$ with the Euler scheme, using a discretization step of $\sqrt{1/n}$.

Instead of specifying the jump process as a compound Poisson process with random jump sizes we restrict its realizations to a fixed number of jumps of a known size. In particular, we consider the case of absence of jumps and the case of a single jump with deterministic size equal to $3\sqrt{1/n}$ (small jump, notice that in the simulations $\sigma_t \simeq 1$) and $10\sqrt{1/n}$ (big jump).

In order to make the Monte Carlo simulations more realistic, we further simulate additional frictions which are not considered in the theoretical framework of this paper, but are known to be present in the data. The frictions we consider are microstructure

noise, in the form of distortions to the price process, and flat prices, that is prices that do not change due, for example, to liquidity reasons.⁵ We thus consider three possible scenarios:

1. The price process is observed without frictions.
2. The price process is contaminated by microstructure noise.
3. The price process is contaminated by flat prices.

In order to save space, since microstructure noise and flat pricing have the largest impact at high frequency, we study these frictions only at $n = 400$.

To estimate quarticity we implement standard (non truncated) quadpower variations $\text{MPV}([1, 1, 1, 1])$, threshold tripower and quadpower variations, that is $\text{TMPV}([\frac{4}{3}, \frac{4}{3}, \frac{4}{3}])$ and $\text{TMPV}([1, 1, 1, 1])$ respectively; the single power $\text{TMPV}([4])$ estimator; and multi-power estimators $\text{GTMPV}^{**}(m; \sigma^4)$ for different values of m (for simplicity, we omit the σ^4 in the notation). We also implement the nearest neighbor **minRQ** and **medRQ** estimators defined in Eq. (2.20) and Eq. (2.21) Finally, we consider the efficient estimator $\text{QVeff}(k_n)$ of Jacod and Rosenbaum (2013), see Eq. (A.4), implemented with the bias correction proposed in Jacod and Rosenbaum (2015) for different values of k_n .

In order to set up the threshold for truncated multipowers, we use

$$\vartheta_t = c_\vartheta \cdot \widehat{\sigma}_t^n \quad (4.2)$$

with $c_\vartheta = 5$ and $\widehat{\sigma}_t^n$ is an estimator of local standard deviation (that is over the interval $\sqrt{1/n}$) obtained as in Corsi et al. (2010). The choice of $c_\vartheta = 5$ is meant to truncate only returns that are extremely large with respect to the estimated local standard deviation.

On each replication we compute the generated quarticity value IQ_k and the estimated quarticity value $\widehat{IQ}_k$ according to different estimators ($k = 1, \dots, M$). We report the relative bias,

$$\text{Bias} = \frac{1}{M} \sum_{k=1}^M \frac{\widehat{IQ}_k - IQ_k}{IQ_k},$$

⁵On flat pricing, a theoretical analysis is offered by Phillips and Yu (2009) and Bandi et al. (2014).

the relative standard deviation,

$$\text{Std} = \sqrt{\frac{1}{M} \sum_{k=1}^M \left(\frac{\widehat{IQ}_k - IQ_k}{IQ_k} - \text{bias} \right)^2},$$

and the relative Root Mean Square Error:

$$\text{RMSE} = \sqrt{\text{Std}^2 + \text{Bias}^2}.$$

The figures are computed with $M = 10,000$ replications.

4.1 Estimation without frictions

Tables 4, 5, 6 report the results in the case $n = 40$ (10 minutes), $n = 80$ (5 minutes), $n = 400$ (1 minute) respectively for the competing estimators, and in three cases: absence of jumps, presence of a single small jump and presence of a single large jump.

Generally speaking, the performance of truncated estimators is largely better than non-truncated ones. In the case with a big jump, MPV is literally shattered: the RMSE decreases with increasing frequency, but it is still roughly +400% at the one-minute frequency. Basically, non-truncated estimators are severely biased. Standard threshold multipower estimators reduce the bias considerably. For example, in the 5-minute case in the presence of a small jump, the bias goes from the +151% of the standard MPV([1, 1, 1, 1]) estimator to the -6% of the truncated TMPV([1, 1, 1, 1]) estimator. However, as we discuss below, the RMSE of TMPV([1, 1, 1, 1]) (and TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)) is higher than that of the GTMPV** estimator because of the efficiency loss.

The estimator based on multipowers also perform better than the neighbor truncation min-med estimators. At the frequency of 5 minutes (the most typical in applications) the RMSE of min-med estimators is more than double than that of GTMPV** estimators, both in the case in which there are small or large jumps, and still roughly 50% higher on paths without jumps, which again reflects the efficiency loss of these estimators.

Regarding the GTMPV**($m; \sigma^4$) estimators, we can see that they generally display a small negative bias, which is consistent with the theory. Using GTMPV**($m; \sigma^4$), we obtain a gain in terms of relative RMSE, with respect to the standard multipower estimators TMPV([1, 1, 1, 1]) and TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$), of roughly 20% at 10 minutes (from 60% to 40%), of 13% at 5 minutes (from 43% to 30%) and of 5% at 1 minute (from

20% to 15%). Thus, the gain is substantial and is entirely due to the smaller variance of $\text{GTMPV}^{**}(m; \sigma^4)$, which more than offsets the loss in bias due to the higher value of m employed.

The performance of the efficient estimator of Jacod and Rosenbaum (2013) reflects its asymptotic nature: it improves with large k_n . In order to compare it with the $\text{GTMPV}^{**}(m; \sigma^4)$, we fix $k_n = m + 1$ and compute the two estimators for various m . The relative RMSE for the three different frequencies used in the simulation study in the case without jumps is shown in Figure 2. It is clear that using the estimator $\text{GTMPV}^{**}(m; \sigma^4)$, which has a lower variance than the Jacod and Rosenbaum estimator for fixed m , can obtain a better result with respect to an estimator which is designed to have the lowest possible variance when m (that is, $k_n - 1$) diverges to $+\infty$. Again, this is important in practice, since the number of multipowers (or the window used for preliminary estimates of spot variance) is actually fixed. The performance of the two estimators tend to be similar when m is large, since the two estimators coincide (excluding finite sample bias corrections) for large m . The Jacod and Rosenbaum estimator is also better when n is large (at the 1-minute frequency) for intermediate values of m .

Summarizing, the Monte Carlo experiments in this section show that i) truncating returns is essential to get reasonable quarticity estimates in the presence of jumps; ii) standard multipower estimators, including the min-med estimator, suffer substantial efficiency loss with respect to the (nearly) efficient multipower estimators, which results in a deteriorated estimate in terms of the mean square error; iii) a minimum variance estimator designed for fixed m can be beneficial, in terms of mean square error, with respect to an efficient estimator designed for diverging m .

4.2 Estimation in the presence of microstructure noise

The first type of friction we introduce in simulated experiments is microstructure noise in the form of autocorrelated price distortion. The observed prices, X_j , are generated as follows:

$$X_j = X_j^* + \epsilon_j, \quad (4.3)$$

where X_j^* is simulated as in the previous section, and

$$\epsilon_j = \rho_\epsilon \epsilon_{j-1} + \epsilon_j^*, \quad \epsilon_j^* \sim \mathcal{N}(0, \sigma_\epsilon^2). \quad (4.4)$$

Figure 2: Relative RMSE obtained on simulations for the $\text{GTMPV}^{**}(m; \sigma^4)$ and QVeff , the Jacod and Rosenbaum estimator with $k_n = m + 1$ for different values of m , at three different sampling frequencies.

Table 4: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $\mathbf{n} = \mathbf{40}$ intraday returns generated by model (4.1). The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$.

	no jumps			small jump			big jump		
	RMSE	Std	Bias	RMSE	Std	Bias	RMSE	Std	Bias
MPV([1, 1, 1, 1])	0.6036	0.5998	-0.0674	3.6144	3.0225	1.9820	11.9077	9.6054	7.0378
TMPV([4])	0.5353	0.5353	-0.0007	0.5286	0.5277	-0.0306	0.5388	0.5379	-0.0320
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	0.5914	0.5898	-0.0441	0.5652	0.5520	-0.1218	0.5864	0.5747	-0.1167
TMPV([1, 1, 1, 1])	0.6041	0.6002	-0.0686	0.5758	0.5519	-0.1641	0.5946	0.5738	-0.1558
minRQ	0.7094	0.7092	-0.0166	1.1396	1.1123	0.2478	1.0556	1.0307	0.2278
medRQ	0.6053	0.6042	-0.0359	0.9878	0.9566	0.2461	0.9228	0.8931	0.2326
GTMPV**(4)	0.4751	0.4743	-0.0272	0.4719	0.4695	-0.0478	0.4878	0.4855	-0.0480
GTMPV**(8)	0.4432	0.4392	-0.0594	0.4405	0.4330	-0.0810	0.4546	0.4475	-0.0801
GTMPV**(12)	0.4278	0.4202	-0.0803	0.4299	0.4178	-0.1010	0.4385	0.4265	-0.1021
GTMPV**(16)	0.4220	0.4121	-0.0912	0.4253	0.4106	-0.1109	0.4326	0.4180	-0.1115
GTMPV**(20)	0.4198	0.4095	-0.0924	0.4249	0.4105	-0.1094	0.4328	0.4187	-0.1099
GTMPV**(24)	0.4213	0.4132	-0.0820	0.4259	0.4149	-0.0961	0.4344	0.4236	-0.0963
QVeff(4)	0.4785	0.4708	-0.0857	0.4727	0.4556	-0.1261	0.4894	0.4732	-0.1250
QVeff(8)	0.4777	0.4773	-0.0192	0.4636	0.4592	-0.0636	0.4799	0.4756	-0.0635
QVeff(12)	0.4557	0.4546	-0.0325	0.4460	0.4389	-0.0790	0.4566	0.4495	-0.0802
QVeff(16)	0.4376	0.4339	-0.0567	0.4310	0.4180	-0.1052	0.4387	0.4258	-0.1056
QVeff(20)	0.4268	0.4188	-0.0821	0.4256	0.4061	-0.1275	0.4319	0.4126	-0.1278
QVeff(24)	0.4266	0.4157	-0.0957	0.4267	0.4024	-0.1418	0.4325	0.4087	-0.1412

We consider a persistent noise process ($\rho_\epsilon = 0.5$). The microstructure noise is virtually not present at moderate frequencies, hence we do not consider the 5- and 10-minute frequencies in the present subsection. For $n = 400$ (1-minute), as in the simulation design of Podolskij and Vetter (2009), we set $\sigma_\epsilon^2 = 0.0005IV_t$, with IV_t denoting the daily integrated variance.

Results, shown in Table 7, show that microstructure noise induces a strong distortion in all estimates, in form of an upper bias of roughly +50% which is very similar across all considered estimators. Still, the observed variance of $\text{GTMPV}^{**}(m; \sigma^4)$ is the lowest among competitors.

Even if the magnitude of the bias due to microstructure noise could vary given a different data generating process for the price dynamics and the noise itself, these results suggest that the impact of the noise is anyway translated into an upper bias which affects the competing estimators very similarly, suggesting that it would still be beneficial to concentrate on variance reduction even in the presence of this kind of friction.

Table 5: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $\mathbf{n} = \mathbf{80}$ intraday returns generated by model (4.1). The threshold is set as in Eq. (4.2) with $c_{\vartheta} = 5$.

	no jumps			small jump			big jump		
	RMSE	Std	Bias	RMSE	Std	Bias	RMSE	Std	Bias
MPV([1, 1, 1, 1])	0.4376	0.4362	-0.0356	2.7541	2.3016	1.5124	9.1583	7.4072	5.3860
TMPV([4])	0.3761	0.3759	-0.0130	0.3806	0.3802	-0.0174	0.3805	0.3801	-0.0161
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	0.4248	0.4240	-0.0263	0.4263	0.4209	-0.0676	0.4247	0.4202	-0.0620
TMPV([1, 1, 1, 1])	0.4380	0.4364	-0.0366	0.4413	0.4319	-0.0905	0.4342	0.4257	-0.0854
minRQ	0.5051	0.5049	-0.0153	0.6991	0.6877	0.1255	0.6497	0.6376	0.1246
medRQ	0.4362	0.4354	-0.0259	0.5935	0.5799	0.1265	0.5781	0.5629	0.1317
GTMPV**(5)	0.3361	0.3350	-0.0278	0.3406	0.3389	-0.0342	0.3391	0.3377	-0.0306
GTMPV**(10)	0.3208	0.3167	-0.0510	0.3221	0.3167	-0.0591	0.3217	0.3169	-0.0551
GTMPV**(15)	0.3138	0.3060	-0.0696	0.3136	0.3037	-0.0781	0.3134	0.3043	-0.0748
GTMPV**(20)	0.3091	0.2973	-0.0845	0.3100	0.2958	-0.0930	0.3090	0.2957	-0.0898
GTMPV**(30)	0.3065	0.2890	-0.1020	0.3081	0.2879	-0.1098	0.3070	0.2878	-0.1068
GTMPV**(40)	0.3072	0.2894	-0.1030	0.3078	0.2877	-0.1094	0.3070	0.2878	-0.1069
QVeff(5)	0.3427	0.3338	-0.0776	0.3461	0.3332	-0.0935	0.3460	0.3343	-0.0889
QVeff(10)	0.3419	0.3414	-0.0192	0.3411	0.3389	-0.0389	0.3414	0.3397	-0.0346
QVeff(15)	0.3349	0.3341	-0.0231	0.3315	0.3286	-0.0438	0.3320	0.3296	-0.0399
QVeff(20)	0.3261	0.3237	-0.0388	0.3215	0.3159	-0.0599	0.3219	0.3168	-0.0569
QVeff(30)	0.3117	0.3023	-0.0762	0.3126	0.2971	-0.0973	0.3111	0.2962	-0.0951
QVeff(40)	0.3095	0.2900	-0.1081	0.3118	0.2838	-0.1291	0.3102	0.2827	-0.1276

Table 6: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $\mathbf{n} = \mathbf{400}$ intraday returns generated by model (4.1). The threshold is set as in Eq. (4.2) with $c_{\vartheta} = 5$.

	no jumps			small jump			big jump		
	RMSE	Std	Bias	RMSE	Std	Bias	RMSE	Std	Bias
MPV([1, 1, 1, 1])	0.2094	0.2092	-0.0096	1.2485	1.0169	0.7245	4.1396	3.3419	2.4431
TMPV([4])	0.1772	0.1772	-0.0019	0.1748	0.1746	-0.0068	0.1749	0.1748	-0.0062
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	0.2017	0.2016	-0.0069	0.1953	0.1948	-0.0148	0.1986	0.1981	-0.0141
TMPV([1, 1, 1, 1])	0.2094	0.2092	-0.0099	0.2034	0.2024	-0.0199	0.2069	0.2060	-0.0188
minRQ	0.2373	0.2373	-0.0015	0.2449	0.2437	0.0243	0.2501	0.2490	0.0236
medRQ	0.2046	0.2046	-0.0049	0.2158	0.2144	0.0243	0.2198	0.2185	0.0238
GTMPV**(20)	0.1516	0.1499	-0.0229	0.1505	0.1482	-0.0260	0.1506	0.1483	-0.0261
GTMPV**(30)	0.1501	0.1462	-0.0337	0.1493	0.1447	-0.0367	0.1493	0.1447	-0.0369
GTMPV**(40)	0.1497	0.1432	-0.0436	0.1491	0.1417	-0.0463	0.1493	0.1418	-0.0468
GTMPV**(50)	0.1504	0.1409	-0.0527	0.1499	0.1393	-0.0554	0.1499	0.1392	-0.0558
QVeff(20)	0.1582	0.1581	-0.0044	0.1558	0.1554	-0.0104	0.1568	0.1565	-0.0099
QVeff(30)	0.1569	0.1569	-0.0042	0.1547	0.1544	-0.0105	0.1549	0.1546	-0.0099
QVeff(40)	0.1549	0.1547	-0.0083	0.1534	0.1528	-0.0141	0.1537	0.1530	-0.0141
QVeff(50)	0.1534	0.1528	-0.0139	0.1522	0.1510	-0.0192	0.1521	0.1509	-0.0198

4.3 Estimation in the presence of flat prices

We finally consider a form of friction which we document to be present at the highest frequencies, that is flat pricing. Flat pricing consists in the observation of spurious zero returns, which might be due to lack of liquidity in the market or asymmetric information (see the discussion in Bandi et al., 2014). In our simulation setting, we assume that the generated returns are given by:

$$\Delta_j X = \Delta_j X^* \cdot \psi_j, \quad (4.5)$$

where X^* is the process generated without frictions, and

$$\psi_j = \begin{cases} 0, & \text{with probability } p_\psi \\ 1, & \text{with probability } (1 - p_\psi) \end{cases} \quad (4.6)$$

We set $p_\psi = 0.3$ and, as before, we consider only the case $n = 400$ (1-minute data). This choice is motivated by data analysis, since the phenomenon of flat prices tends to fade away, for the stocks we consider in the empirical application, at the 5-minutes frequency. Results are shown in Table 8.

Even if the probability of flat trading we use in the simulated experiments is quite high with respect to the observed values (reported in Table 9), the impact on final estimates is quite small. For $\text{GTMPV}^{**}(m; \sigma^4)$, the relative RMSE increases from about 15% to 17% only. This is basically due to increased estimator variance, in line with the theoretical predictions of Phillips and Yu (2009) for realized variance. In this sense, microstructure noise and flat trading have an impact which is completely different on estimators, the first kind of friction mostly affecting the estimator bias, whereas the second kind of friction mostly affecting the estimator variance.

Interestingly, $\text{GTMPV}^{**}(m; \sigma^4)$ is much more robust than competing estimators to this form of friction. Indeed, the relative RMSE standard TMPV estimators increases from roughly 20% to 45% for tripower and 60% for quadpower; for **minRQ** it increases from 23% to 30%; and for **medRQ** it increases from 20% to 23%. The gap between $\text{GTMPV}^{**}(m; \sigma^4)$ and competing estimator becomes then wider in the presence of flat trading.

Table 7: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $n = 400$ intraday returns generated by model (4.3), which includes **microstructure noise**. The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$. The autocorrelation of the noise process is $\rho_\epsilon = 0.5$.

	no jumps			small jump			big jump		
	RMSE	Std	Bias	RMSE	Std	Bias	RMSE	Std	Bias
MPV([1, 1, 1, 1])	0.6615	0.3249	0.5762	2.1188	1.3929	1.5966	5.9511	4.4924	3.9030
TMPV([4])	0.6352	0.2703	0.5748	0.6358	0.2646	0.5781	0.6251	0.2588	0.5690
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	0.6546	0.3116	0.5756	0.6529	0.3144	0.5722	0.6366	0.3009	0.5610
TMPV([1, 1, 1, 1])	0.6612	0.3253	0.5756	0.6555	0.3329	0.5647	0.6381	0.3204	0.5518
minRQ	0.6912	0.3640	0.5876	0.7336	0.3839	0.6251	0.7388	0.3802	0.6335
medRQ	0.6705	0.3225	0.5878	0.7060	0.3354	0.6212	0.7040	0.3257	0.6242
GTMPV**(30)	0.5889	0.2298	0.5422	0.5812	0.2252	0.5358	0.5718	0.2201	0.5277
GTMPV**(40)	0.5765	0.2262	0.5303	0.5688	0.2207	0.5243	0.5596	0.2173	0.5157
GTMPV**(50)	0.5654	0.2245	0.5190	0.5567	0.2166	0.5128	0.5493	0.2152	0.5054
QVeff(30)	0.6287	0.2430	0.5798	0.6155	0.2388	0.5673	0.6076	0.2320	0.5616
QVeff(40)	0.6233	0.2393	0.5755	0.6113	0.2349	0.5644	0.6036	0.2302	0.5579
QVeff(50)	0.6141	0.2339	0.5678	0.6056	0.2342	0.5585	0.5960	0.2282	0.5506

Table 8: Quarticity estimators. The table reports the relative RMSE, standard deviation and bias computed with $M = 10,000$ replications of $n = 400$ intraday returns generated by model (4.5), which includes **flat pricing**. The threshold is set as in Eq. (4.2) with $c_\vartheta = 5$. The probability of observing a zero return is $p_\epsilon = 0.3$.

	no jumps			small jump			big jump		
	RMSE	Std	Bias	RMSE	Std	Bias	RMSE	Std	Bias
MPV([1, 1, 1, 1])	0.5742	0.1347	-0.5582	0.7554	0.6872	-0.3135	2.1987	2.1738	0.3301
TMPV([4])	0.3895	0.2752	0.2757	0.4212	0.2868	0.3085	0.4084	0.2875	0.2900
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	0.4301	0.1555	-0.4010	0.4916	0.1465	-0.4692	0.4672	0.1470	-0.4435
TMPV([1, 1, 1, 1])	0.5760	0.1347	-0.5600	0.6283	0.1265	-0.6154	0.6465	0.1142	-0.6363
minRQ	0.3009	0.2286	-0.1957	0.3318	0.2575	-0.2093	0.3139	0.2627	-0.1719
medRQ	0.2337	0.2183	-0.0834	0.2596	0.2531	-0.0577	0.2575	0.2527	-0.0495
GTMPV**(30)	0.1737	0.1694	-0.0384	0.1793	0.1736	-0.0449	0.1780	0.1725	-0.0442
GTMPV**(40)	0.1731	0.1652	-0.0518	0.1789	0.1694	-0.0574	0.1777	0.1682	-0.0573
GTMPV**(50)	0.1736	0.1622	-0.0618	0.1795	0.1654	-0.0696	0.1781	0.1648	-0.0677
QVeff(30)	0.1833	0.1833	-0.0018	0.1880	0.1879	-0.0050	0.1862	0.1860	-0.0096
QVeff(40)	0.1801	0.1799	-0.0092	0.1866	0.1861	-0.0148	0.1834	0.1825	-0.0178
QVeff(50)	0.1790	0.1781	-0.0181	0.1835	0.1815	-0.0271	0.1817	0.1792	-0.0301

5 Empirical Application

The purpose of the empirical application is to apply our proposed quarticity estimator to real data from the financial market, and to compare its performance with respect to competing estimators in relevant applications, namely jump testing and volatility

Table 9: Reports the list of the sixteen blue chip stocks used in the empirical application, their ticker and the percentage of zero-returns at one and five minutes.

Company	Ticker	1-minute zero re- turns (%)	5-minute zero re- turns (%)
Bank of America	BAC	9.46	4.71
Citigroup Inc.	C	12.51	7.77
JPMorgan Chase & Co.	JPM	6.24	2.76
Wells Fargo & Company	WFC	6.24	2.76
Boeing	BA	5.60	2.39
Caterpillar Inc.	CAT	4.45	1.90
FedEx Corporation	FDX	4.93	2.05
Honeywell International Inc.	HON	7.96	3.41
Hewlett-Packard Company	HPQ	8.65	3.75
International Business Machines Corp.	IBM	4.02	1.83
AT&T Inc.	T	9.55	4.44
Texas Instruments Incorporated	TXN	12.94	6.09
Kraft Foods Inc.	KFT	13.33	6.32
PepsiCo, Inc.	PEP	8.19	3.61
The Procter & Gamble Company	PG	7.94	3.54
Time Warner Inc.	TWX	10.72	4.70

forecasting.

As in the Monte Carlo section, we restrict our attention to the $\text{GTMPV}^{**}(m^*)$ estimator in Eq. (3.1) (we omit the σ^4 in the notation for simplicity). By m^* we indicate the value that minimizes the MSE in Eq. (3.11), using $\hat{Q} = \text{TMPV}([4])$.

The data set we use is the collection of sixteen blue chip stocks quoted on the New York Stock Exchange. The stocks are all very liquid and they are listed, together with their corresponding ticker, in Table 9. One-minute prices were recovered from the TickData One Minute Equity Data (OMED) dataset, from 3 January 2007 to 29 June 2012, for a total of 1,385 trading days. Our sample then lies in the middle of the credit crunch crisis, characterized by very high volatility levels and a supposedly high number of jumps. The data went through a standard filtering procedure. TickData one-minute equity data are adjusted for corporate actions such as mergers and acquisitions or symbol changes. Moreover, the underlying tick data used to build 1-minute time series are first controlled for cancelled trades, or records not temporally aligned with previous/subsequent data; then filtered to identify bad ticks which are corrected using validation with third-party sources. All the measures reported here are for daily units

and percentage returns.

Table 9 also shows the percentage of zero returns at 1 minute and 5 minutes. We can see that the impact of flat trading at 1 minute is substantial in all stocks, while it is much less impactful at 5 minutes. On each day in the sample, we compute the same quarticity estimators used in the Monte Carlo section. The 1-minute frequency corresponds to $n = 390$; the 5-minutes frequency to $n = 78$, and the 10-minutes frequency to $n = 39$.

5.1 Estimating quarticity

Table 10 reports summary statistics on pooled daily quarticity estimates ($\widehat{IQ}$) and the ratio $\sqrt{\widehat{IQ}/\widehat{IV}}$, where we use $\text{TRV}([2])$ as an estimator $\widehat{IV}$ of integrated volatility. Asymptotically, we expect this ratio to be always greater than 1, by Jensen inequality. However, since we are using estimated quantities, the ratio could be less than one. Using the same normalization for all measures allows the comparison of different days and different stocks. We apply the estimators at three sampling frequencies: 1, 5 and 10 minutes. In Table 10, we exclude the Flash Crash day (May 6, 2010) since the difference between truncated and non-truncated estimators in this specific day is many orders of magnitudes away than what observed in the other days.

We observe a large difference between mean and median quarticity estimates at all the considered frequencies, indicating a very skewed distribution. The difference is less pronounced for the ratios. We start the discussion with the five-minute frequency. Here, we can see that $\text{GTMPV}^{**}(m^*)$ has the lowest in-sample standard deviation, both in terms of average estimates and ratios. The median (data-driven) m^* used in these computation is 5. To gain insight, we also report $\text{GTMPV}^{**}(m)$ at several values of m , and compare them with the Jacod-Rosenbaum estimator $\text{QVeff}(m)$. We can see that, as m increases, the in-sample standard deviation and average of both estimators decreases, with $\text{GTMPV}^{**}(m^*)$ being less variable at all fixed frequencies. The choice of m^* balances the bias and the variance. All these results are thus in line with the theory.

Figure 3 shows the estimated probability density functions of the ratio $\sqrt{\widehat{IQ}/\widehat{IV}}$ for pooled daily quarticity estimates, for different quarticity estimators: $\text{TMPV}([\frac{4}{3}, \frac{4}{3}, \frac{4}{3}])$ (labelled threshold tripower), $\text{TMPV}([1, 1, 1, 1])$ (threshold quadpower), minRV , medRV , $\text{GTMPV}^{**}(m^*)$. We use 5-minute returns. The Figure shows clearly the empirical potential of the theory. The $\text{GTMPV}^{**}(m^*)$ estimator delivers a much more concentrated

Table 10: Summary statistics of pooled quarticity estimators $\widehat{IQ}$ and the ratios $\sqrt{\widehat{IQ}/\widehat{IV}}$, where $\widehat{IV}$ is threshold realized variance ($\text{TRV}([2])$). The Flash Crash day is excluded.

	$\widehat{IQ}$			$\sqrt{\widehat{IQ}/\widehat{IV}}$			
	1-minute frequency			mean	median	std	kurtosis
mean	median	std	mean	median	std	kurtosis	
MPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	316.7	3.5	9314.0	1.578	1.366	1.005	570.7
MPV($[1, 1, 1, 1]$)	343.5	4.0	9852.4	1.721	1.463	1.227	526.5
minRQ	512.3	4.4	19736.3	1.895	1.530	3.477	12383.1
medRQ	464.7	4.4	17007.9	1.838	1.522	2.467	8600.2
TMPV($[4]$)	337.7	4.6	9290.8	1.769	1.582	0.810	54.3
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	310.3	3.6	8677.6	1.576	1.394	0.782	81.0
TMPV($[1, 1, 1, 1]$)	293.7	3.1	8601.8	1.480	1.310	0.744	99.0
GTMPV**(m^*)	331.3	4.5	9219.9	1.750	1.564	0.798	54.9
(m^*)	(5.7)	(3.0)	(7.2)				
5-minute frequency							
mean	median	std	mean	median	std	kurtosis	
MPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	207.3	1.9	7595.9	1.104	1.018	0.502	427.7
MPV($[1, 1, 1, 1]$)	238.6	2.2	7893.6	1.203	1.081	0.765	2801.0
minRQ	315.9	2.3	10433.0	1.275	1.114	1.119	2102.6
medRQ	249.0	2.3	7799.7	1.236	1.103	0.957	2771.3
TMPV($[4]$)	209.8	2.5	6418.3	1.219	1.179	0.204	4.3
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	194.2	1.9	7490.8	1.075	1.033	0.265	5.7
TMPV($[1, 1, 1, 1]$)	177.9	1.7	7377.2	1.015	0.975	0.269	5.8
GTMPV**(m^*)	203.2	2.5	6296.7	1.203	1.160	0.201	4.6
(m^*)	(8.1)	(5.0)	(8.2)				
GTMPV**(5)	189.9	2.3	6228.4	1.144	1.109	0.150	5.0
GTMPV**(10)	165.1	2.0	5530.0	1.081	1.054	0.112	4.6
GTMPV**(20)	141.1	1.7	4621.9	1.005	0.990	0.070	5.5
GTMPV**(30)	125.0	1.6	3868.2	0.962	0.957	0.054	1.7
GTMPV**(35)	120.0	1.5	3634.4	0.948	0.947	0.052	7.0
QVeff(5)	191.0	2.3	6364.7	1.149	1.108	0.179	5.2
QVeff(10)	181.4	2.2	6236.2	1.129	1.096	0.144	5.0
QVeff(20)	162.3	1.9	5759.2	1.049	1.030	0.096	4.5
QVeff(30)	137.1	1.7	4539.8	0.982	0.973	0.069	5.0
QVeff(35)	128.6	1.6	4125.7	0.955	0.948	0.063	3.1
10-minute frequency							
mean	median	std	mean	median	std	kurtosis	
MPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	134.0	1.2	5062.8	0.914	0.836	0.484	1305.1
MPV($[1, 1, 1, 1]$)	154.4	1.5	5477.1	1.009	0.906	0.601	836.2
minRQ	180.6	1.6	5316.7	1.075	0.929	0.788	756.2
medRQ	159.6	1.6	5372.0	1.043	0.936	0.706	1545.6
TMPV($[4]$)	165.4	1.7	7657.7	1.043	0.991	0.377	9.0
TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	123.7	1.2	4565.5	0.891	0.839	0.363	7.3
TMPV($[1, 1, 1, 1]$)	111.8	1.0	4473.8	0.830	0.783	0.348	6.7
GTMPV**(m^*)	158.3	1.7	6959.0	1.036	0.985	0.369	9.3
(m^*)	(5.2)	(5.0)	(3.4)				

ratio, with considerably thinner tails, as shown by the drop in the sample kurtosis in Table 10. The estimated ratio is spuriously less than one for 54.17% of estimates using threshold quadpower, 43.92% using threshold tripower, 35.55% using minRV and 33.37% using medRV; the violation is instead observed only in 12.95% of cases with

Figure 3: Estimated probability density functions of pooled daily ratios $\sqrt{\widehat{IQ}/\widehat{IV}}$, where $\widehat{IQ}$ are different quarticity estimators and $\widehat{IV}$ is threshold realized variance (TRV([2])). Estimates are obtained at the 5-minute frequency.

GTMPV^{**}(m^*).⁶

At 1 minute and 10 minutes, results are qualitatively the same. At 1 minute, where the impact of market microstructure noise and flat trading is likely to be higher, standard multipowers display a smaller variability and a smaller average than GTMPV^{**}(m^*). At 10 minutes, where the bias is relatively higher since the number of observations is smaller, we observe the same phenomenon. Again, the GTMPV^{**}(m^*) estimator is implemented here to balance bias and variance in an optimal way. Finally, notice that, at all frequencies, the difference in the kurtosis between non-truncated and truncated ratios is particularly pronounced, indicating that truncated estimator have much thinner tails.

⁶We are taking advantage here also of the fact that TMPV[4] delivers, by construction, a ratio greater than one, and is also used to compute the bias when optimizing m^* .

5.2 Impact on jump testing

A popular way for testing for jumps is to take the difference between realized variance (which is impacted by jumps) and bipower variation (which is not), and standardize this with the standard deviation of the difference. The technique has been basically laid out in Barndorff-Nielsen and Shephard (2006), see also Huang and Tauchen (2005), and is largely used in empirical work. We follow this empirical strategy using the version of the test proposed by Corsi et al. (2010), namely:

$$CTz(QV) = \frac{1}{n^{1/2}} \frac{\frac{RV - C \text{TMPV}([1,1])}{RV}}{\sqrt{\tilde{\theta} \max\left(\frac{QV}{(C \text{TMPV}([1,1])^2}, 1\right)}}, \quad (5.1)$$

where $\tilde{\theta} = (\pi^2/4 + \pi - 5)$. In the definition (5.1), QV is a consistent quarticity estimator. Consistently with their proposed estimators, Corsi et al. (2010) suggest to use standard multipowers $\text{TMPV}([4/3, 4/3, 4/3])$ or $\text{TMPV}([1, 1, 1, 1])$ to estimate quarticity. In what follows, we study the sensitivity of the test to different quarticity estimators. In total we have $2392 \times 16 \approx 40\,000$ tests in our sample.

Table 11 reports the number of detected jumps when standard (threshold tripower, and threshold quadpower) and the $\text{GTMPV}^{**}(m^*)$ multipower estimators are used. At all confidence intervals, at all sampling frequencies, the number of detected jumps is significantly larger than what predicted by the confidence interval, as exhaustively reported by the empirical literature. However, when using $\text{GTMPV}^{**}(m^*)$, the percentage of detected jumps reduces drastically, reducing the number of detection of roughly a half.

This empirical finding might help in explaining why, in the literature, it appears that “too many” jumps are detected, as documented in Christensen et al. (2014) using ultra-high frequency data, and by Bajgrowicz et al. (2016). We suggest, indeed, that most of the jumps are spurious artifact of inefficient quarticity measurements. Our results show also that the impact of such spurious detections can be substantially reduced by employing a (nearly) efficient quarticity estimator.

Table 11: Reports the percentage of detected jumps in our sample for three confidence intervals (99%,99.9%,99.99%); for three different quarticity estimators (the standard threshold tripower and quadpower variation, and $\text{GTMPV}^{**}(m^*)$); and for three different sampling frequencies (1, 5, 10 minutes). Results indicate a drastic reduction of detected jumps when $\text{GTMPV}^{**}(m^*)$ is used.

c.i.	TMPV($[\frac{4}{3}, \frac{4}{3}, \frac{4}{3}]$)	TMPV($[1, 1, 1, 1]$)	$\text{GTMPV}^{**}(m^*)$
1 minute frequency - percentage of detected jumps			
99%	45.12	48.11	35.22
99.9%	31.29	34.66	20.31
99.99%	22.77	26.22	13.68
5 minutes frequency - percentage of detected jumps			
99%	20.56	21.65	12.30
99.9%	12.84	13.56	8.23
99.99%	8.69	9.14	5.68
10 minutes frequency - percentage of detected jumps			
99%	18.31	18.96	11.92
99.9%	10.38	10.72	6.99
99.99%	6.06	6.26	3.92

5.3 Impact on volatility forecasting

As suggested by Bollerslev et al. (2016), quarticity estimation can assist volatility forecasting. Denote by $\widehat{IV}_t$ at day t an integrated variance estimator, and define $\widehat{IV}_{t-j|t-h} = \frac{1}{h} \sum_{i=j}^h \widehat{IV}_{t-i}$. One of the most popular model for forecasting daily integrated variance is the Heterogeneous Autoregression (HAR) model of Corsi (2009):

$$\widehat{IV}_t = \beta_0 + \beta_1 \widehat{IV}_1 + \beta_2 \widehat{IV}_{t-1|t-5} + \beta_3 \widehat{IV}_{t-1|t-22} + u_t, \quad (5.2)$$

where u_t is a stationary error process. It is well known that the β coefficients estimated with the HAR model are affected by measurement errors in the realized volatilities. In order to account for the presence of the measurement errors, Bollerslev et al. (2015) introduce the idea of “dynamic attenuation” with the HARQ model, whose specification

is:

$$\widehat{IV}_t = \beta_0 + \left(\beta_1 + \beta_{1,Q} \widehat{IQ}_{t-1}^{1/2} \right) \widehat{IV}_1 + \beta_2 \widehat{IV}_{t-1|t-5} + \beta_3 \widehat{IV}_{t-1|t-22} + u_t, \quad (5.3)$$

where $\widehat{IQ}_{t-1}$ is an integrated quarticity estimator at day t . We then study the sensitivity of the results obtained with model (5.3) to different quarticity estimator.

The variable to be forecasted in our exercise is $\widehat{IV} = \text{TRV}$, that is threshold realized variance. The different quarticity estimators are those examined so far. Our exercise is fully out-of-sample. We obtain the forecast of $\widehat{IV}_t$ at time t forecast using estimation of model (5.3) on the past year till day $t - 1$, so that estimates are performed on a rolling window. As the loss function, we use the traditional Root Mean Square (relative) Error. Table 12 shows the RMSE of the HARQ model corresponding to different quarticity estimates, standardized by RMSE obtained with the HAR model, a value less than one meaning that the forecasting performance of the proposed specification is better than the HAR model.

We confirm and corroborate the empirical evidence of Bollerslev et al. (2016): The HARQ model provides superior forecasts with respect to the HAR model, since the ratio is generally less than one, with rare exceptions, for all stocks and all quarticity estimators. When it comes to the choice of the quarticity estimator, the estimator we propose achieves the best value of the loss function quite often, and is also the best estimator on average, indicating a clear advantage in using our estimator. The second-best estimator is $\text{TMPV}([4])$, which is also a nearly efficient estimator (with $m = 1$), and which has been indicated by Bollerslev et al. (2016) as their best-performing quarticity estimator. Again, this is an indication that an estimator built to be more precise can yield substantial improvement also in terms of volatility forecasting.

6 Conclusions

We provided methods to find minimum variance estimators in the class of linear combinations of multipower estimators. In particular, we propose a specific quarticity estimator which is nearly efficient (in the sense of Renault et al., 2016) and with a smaller variance than traditional estimators in the literature in the case in which the numbers of multipowers employed is fixed. This result can be employed to obtain, by construction, superior estimators (in the mean square error or some other loss function sense) with respect to any unbiased estimator.

Table 12: Reports the out-of-sample forecast relative RMSE of $HARQ$ model for TRV with different quarticity estimators, standardized by the relative RMSE of the HAR model, so that a value less than 1 indicates superiority with respect to the HAR model. In bold, we indicate the estimator with the best performance for the given stock.

Ticker	Quarticity estimators					
	minRQ	medRQ	TMPV($([\frac{4}{3}, \frac{4}{3}, \frac{4}{3}])$)	TMPV($([1, 1, 1])$)	TMPV($([4])$)	GTMPV $^{**}(m^*)$
BAC	0.8584	0.8504	0.8704	0.8767	0.7439	0.7425
C	0.7600	0.7584	0.7842	0.7853	0.7958	0.7948
JPM	0.9008	0.9156	0.9125	0.9433	0.8610	0.8619
WFC	0.9565	0.9517	0.9539	0.9651	0.9332	0.9329
BA	0.9654	0.9602	0.9519	0.9570	0.9630	0.9641
CAT	0.9348	0.9405	0.9385	0.9342	0.9328	0.9318
FDX	0.9754	0.9607	0.9608	0.9696	0.9458	0.9460
HON	0.8769	0.8721	0.8707	0.8785	0.8672	0.8681
IBM	0.9267	0.9194	0.9182	0.9167	0.9131	0.9100
HPQ	0.8918	0.8932	0.9139	0.9345	0.9263	0.9273
TXN	0.9903	0.9812	0.9686	0.9574	0.9642	0.9640
T	0.9978	0.9153	0.9245	0.9311	0.9023	0.9020
KFT	1.0094	1.0078	0.9888	0.9858	0.9799	0.9798
PEP	0.9646	0.9653	0.9497	0.9549	0.9541	0.9541
PG	0.8581	0.8559	0.8610	0.8628	0.8620	0.8627
TWX	0.9663	0.9533	0.9545	0.9503	0.9614	0.9607
average	0.9271	0.9188	0.9201	0.9252	0.9066	0.9064

Based on these results, we show, on simulated data, that optimally weighted multi-powers outperform benchmark estimators in terms of mean square error. With respect to the efficient Jacod and Rosenbaum (2013) quarticity estimator, we improve in the fact that the estimator we propose has a smaller variance when m , the number of multipowers employed, is fixed.

We use our theoretical results to implement a data-driven selection of m . The empirical application shows that improving the efficiency of quarticity measures by means of our estimator helps in delivering substantially less jumps than what previously found with high-frequency data, and in improving the quality of realized volatility forecasts. We thus conclude that our quarticity estimator could replace existing alternatives for empirical work.

References

- Andersen, T., Benzoni, L., Lund, J., 2002. An empirical investigation of continuous-time equity return models. *Journal of Finance* 57, 1239–1284.
- Andersen, T., Dobrev, D., Schaumburg, E., 2012. Jump robust volatility estimation using nearest neighbor truncation. *Journal of Econometrics* 169, 75–93.
- Andersen, T., Dobrev, D., Schaumburg, E., 2014. A robust neighborhood truncation approach to estimation of integrated quarticity. *Econometric Theory* 30, 3–59.
- Bajgrowicz, P., Scaillet, O., Treccani, A., 2016. Jumps in high-frequency data: Spurious detections, dynamics, and news. *Management Science* 62, 2198–2217.
- Balter, J., 2015. Quarticity estimation on OHLC data. *Journal of Financial Econometrics* 13, 505–519.
- Bandi, F., Pirino, D., Renò, R., 2014. EXcess Idle Time. Working paper.
- Bandi, F., Russell, J., 2006. Separating microstructure noise from volatility. *Journal of Financial Economics* 79, 655–92.
- Barndorff-Nielsen, O.E., Graversen, E., Jacod, J., Shephard, N., 2006a. Limit theorems for bipower variation in financial econometrics. *Econometric Theory* 22, 677–719.
- Barndorff-Nielsen, O.E., Shephard, N., 2004. Power and bipower variation with stochastic volatility and jumps. *Journal of Financial Econometrics* 2, 1–48.
- Barndorff-Nielsen, O.E., Shephard, N., 2006. Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics* 4, 1–30.
- Barndorff-Nielsen, O.E., Shephard, N., Winkel, M., 2006b. Limit theorems for multipower variation in the presence of jumps. *Stochastic Processes and Their Applications* 116, 798–806.
- Bollerslev, T., Patton, A.J., Quaedvlieg, R., 2016. Exploiting the errors: A simple approach for improved volatility forecasting. *Journal of Econometrics* 192, 1–18.
- Christensen, K., Oomen, R.C., Podolskij, M., 2014. Fact or friction: Jumps at ultra high frequency. *Journal of Financial Economics* 114, 576–599.
- Corsi, F., 2009. A simple approximate long-memory model of realized volatility. *Journal of Financial Econometrics* 7, 174–196.
- Corsi, F., Pirino, D., Renò, R., 2010. Threshold bipower variation and the impact of jumps on volatility forecasting. *Journal of Econometrics* 159, 276–288.
- Dumitru, A.M., Urga, G., 2012. Identifying jumps in financial assets: a comparison between nonparametric jump tests. *Journal of Business & Economic Statistics* 30, 242–255.
- Hautsch, N., Podolskij, M., 2013. Preaveraging-based estimation of quadratic variation in the presence of noise and jumps: theory, implementation, and empirical evidence. *Journal of Business & Economic Statistics* 31, 165–183.
- Huang, X., Tauchen, G., 2005. The relative contribution of jumps to total price variance. *Journal of Financial Econometrics* 3, 456–499.

- Jacod, J., Protter, P., 2012. Discretization of Processes. Springer-Verlag.
- Jacod, J., Rosenbaum, M., 2013. Quarticity and other functionals of volatility: efficient estimation. *Annals of Statistics* 41, 1462–1484.
- Jacod, J., Rosenbaum, M., 2015. Estimation of volatility functionals: the case of a $\sqrt{n}$ window, in: Friz, P., Gatheral, J., Gulisashvili, A., Jacquier, A., Teichmann, J. (Eds.), *Large Deviations and Asymptotic Methods in Finance*. Springer, pp. 559–590.
- Mancini, C., 2009. Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps. *Scandinavian Journal of Statistics* 36, 270–296.
- Mancini, C., Calvori, F., 2012. Jumps, in: Bauwens, L., Hafner, C., Laurent, S. (Eds.), *Handbook of volatility models and their applications*. Wiley, pp. 427–475.
- Mykland, P., Zhang, L., 2009. Inference for continuous semimartingales observed at high frequency. *Econometrica* 77, 1403–1445.
- Phillips, P.C.B., Yu, J., 2009. Information loss in volatility measurement with flat price trading. Working paper.
- Podolskij, M., Vetter, M., 2009. Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps. *Bernoulli* 15, 634–668.
- Protter, P., 2004. *Stochastic Integration and Differential Equations*. Springer.
- Reiß, M., 2011. Asymptotic equivalence for inference on the volatility from noisy observations. *Annals of Statistics* 39, 772–802.
- Renault, E., Sarisoy, C., Werker, B., 2016. Efficient estimation of integrated volatility and related processes. *Econometric Theory* Forthcoming.
- Todorov, V., Tauchen, G., 2011. Volatility jumps. *Journal of Business and Economic Statistics* 29, 356–371.
- Veraart, A., 2010. Inference for the jump part of quadratic variation of Ito semimartingales. *Econometric Theory* 26, 331–368.
- Vetter, M., 2010. Limit theorems for bipower variation of semimartingales. *Stochastic Processes and their Applications* 120, 22–38.
- Woerner, J., 2006. Power and multipower variation: inference for high frequency data, in: Shiryaev, A.N., Grossinho, M.d.R., Oliveira, P., Esquivel, M. (Eds.), *Stochastic Finance*. Springer, pp. 343–364.

A Proofs

Proof of Proposition 2.1. By Theorem 11.2.1 in Jacod and Protter (2012), the joint asymptotic distribution of

$$\left[\text{MPV}(\mathbf{r}^{(1)}), \dots, \text{MPV}(\mathbf{r}^{(N)}) \right]'$$

is mixed normal with conditional variance

$$\Sigma_T = C \cdot \int_0^T (\sigma_s)^{2R} ds.$$

By standard mathematical arguments used in portfolio choice problems (or equivalently by straightforward calculations), we deduce that the asymptotic variance of a linear combination of multipower estimators, $\sum_{j=1}^N w_j \text{MPV}(\mathbf{r}^{(j)})$, is minimal for

$$w_j^* = \frac{\sum_{i=1}^N (\Sigma_T^{-1})_{ij}}{\sum_{i=1}^N \sum_{j=1}^N (\Sigma_T^{-1})_{ij}} = \frac{\sum_{i=1}^N (C^{-1})_{ij}}{\sum_{i=1}^N \sum_{j=1}^N (C^{-1})_{ij}}, \quad (\text{A.1})$$

where Σ_T^{-1} is the inverse of $N \times N$ covariance matrix Σ_T , and the last equality follows by reducing the common multiplicative term $\int_0^T (\sigma_s)^{2R} ds$.

Straightforward computations show that, for a fixed set of powers, the variance $\sum_{j=1}^N w_j^* \text{MPV}(\mathbf{r}^{(j)})$ is indeed given by

$$\tilde{V}(N, m; \sigma^R) \int_0^T (\sigma_s)^{2R} ds = \frac{1}{\sum_{i=1}^N \sum_{j=1}^N (C^{-1})_{ij}} \int_0^T (\sigma_s)^{2R} ds.$$

By the properties of multivariate normal distribution,

$$\text{GMPV}^*(N, m; \sigma^R) = \sum_{j=1}^N w_j^* \text{MPV}(\mathbf{r}^{*(j)}), \quad (\text{A.2})$$

it holds:

$$\sqrt{\frac{1}{\Delta_n}} \left(\text{GMPV}^*(N, m; \sigma^R) - \int_0^T (\sigma_s)^R ds \right) \xrightarrow[n \rightarrow \infty]{\Rightarrow} \mathcal{MN} \left(0, \tilde{V}(N, m; \sigma^R) \int_0^T (\sigma_s)^{2R} ds \right), \quad (\text{A.3})$$

where the above convergence is stable in law.

□

Proof of Proposition 3.1. Without loss of generality, we can restrict to the model $X_t = \sigma W_t$ with constant σ , and set $\vartheta_t = +\infty, T = 1$. The proof of Eq. (3.2) and (3.3) is then straightforward. $\square$

Proof of Proposition 3.2. The Jacod and Rosenbaum (2013) estimator takes the form:

$$\text{QVeff}(k_n) = \frac{T}{n} \left(1 - \frac{2}{k_n}\right) \sum_{i=1}^{n-k_n+1} \left(\widehat{c}_i^n\right)^2. \quad (\text{A.4})$$

Write:

$$\begin{aligned} \text{QVeff}(k_n) &= \frac{T}{n} \left(1 - \frac{2}{k_n}\right) \sum_{i=1}^{n-k_n+1} \left(\frac{n}{T} \frac{1}{k_n} \sum_{j=0}^{k_n-1} (\overline{\Delta}_{i+j} X)^2 \right)^2 \\ &= \frac{n}{T} \left(1 - \frac{2}{k_n}\right) \frac{1}{k_n^2} \sum_{i=1}^{n-k_n+1} \left(\sum_{j=0}^{k_n-1} (\overline{\Delta}_{i+j} X)^2 \right)^2 \\ &= \frac{n}{T} \left(1 - \frac{2}{k_n}\right) \frac{1}{k_n^2} \left(\sum_{j=0}^{k_n-1} \sum_{i=1}^{n-k_n+1} (\overline{\Delta}_{i+j} X)^4 + 2 \sum_{j_1=0}^{k_n-1} \sum_{j_2=j_1+1}^{k_n-1} \sum_{i=1}^{n-k_n+1} (\overline{\Delta}_{i+j_1} X)^2 (\overline{\Delta}_{i+j_2} X)^2 \right) \end{aligned}$$

Now, for all $j = 0, \dots, k_n - 1$ we have:

$$\frac{1}{3} \frac{n}{T} \sum_{i=1}^{n-k_n+1} (\overline{\Delta}_{i+j} X)^4 = \text{TMPV}([4]) + O_p(k_n/n),$$

and, for all $j_1 = 0, \dots, k_n - 1$ and $j_2 = j_1 + 1, \dots, k_n - 1$ we have

$$\frac{n}{T} \sum_{i=1}^{n-k_n+1} (\overline{\Delta}_{i+j_1} X)^2 (\overline{\Delta}_{i+j_2} X)^2 = \text{TMPV}([2, \underbrace{0, \dots, 0}_{j_2-j_1-1 \text{ terms}}, 2]) + O_p(k_n/n)$$

This proves that the QVeff estimator can be written in a generalized form of the GTMPV**($k_n - 1, \sigma^4$) estimator, since it is a linear combination of the $k_n - 1$ estimators $\text{TMPV}([4])$, $\text{TMPV}([2, 2])$, $\text{TMPV}([2, 0, 2])$, $\text{TMPV}([2, 0, 0, 2])$, $\dots$, $\text{TMPV}([2, \underbrace{0, \dots, 0}_{k_n-2 \text{ terms}}, 2])$ plus end-effects which are of order k_n^2/n (since there are $k_n - 1$ terms which are $O_p(k_n/n)$). The generalization is in the fact that the sum of the weights in the linear combination is not necessarily 1; but it has to converge to 1, as $n, k_n \rightarrow \infty$, since the two quantities converge to the same object.

Now, when $k_n^2/n \rightarrow \theta$, Theorem 2.2 and 2.5 in Jacod and Rosenbaum (2015) provides the asymptotic bias for the GTMPV** estimator. $\square$