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The distribution of the minimum of a positive sample

by

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Abstract: We give expansions for the distribution function, density function and moments of the sample minimum when sampling from a distribution on $[0, x_2]$ that is nearly analytic at zero. Similar results are given for the sample maximum when sampling from a distribution on $[x_1, x_2]$ that is nearly analytic at x_2 . When these distributions are analytic, the expansions are in inverse powers of the sample size n . If not, they require a double expansion.

Keywords: Density; Maximum; Moments

1 Introduction

Many variables in real life take values in finite intervals. The best known examples are proportions. Often interest is on extreme values of such variables, for example, lowest and highest proportions of people affected by a deadly disease.

Besides, recent years have witnessed a renewed interest in order statistics. Broadly speaking, the work divide into two categories: those who assume a parametric form of the sampled distribution (see, for example, Loperfido et al. (2007)) and those who just assume some features of the sampled distribution (see, for example, Kim et al. (2018)). See also Balakrishnan and Selvitella (2017).

The aim of this note is to provide expansions for the distributions of the minimum and maximum of a random sample of values in a finite interval. We provide expansions for the distribution function, density function and moments. These expansions can be used for improved statistical modeling of the minimum and maximum. For example, the expansions for the distribution function can be used for improved percentile estimation. The expansions for the density function can be used for improved maximum likelihood estimation. The expansions for the moments can be used for improved moments estimation.

Suppose that we have a random sample of size n with minimum m_n and maximum M_n from a distribution function $F(x)$ on $[x_0, x_1]$, where x_0 or x_1 is finite. In Section 2, we suppose that $F(x)$ is analytic at x_1 , and give expansions in powers of n^{-1} for the distribution function and moments of the sample maximum M_n . In Section 3, we suppose that for some $a > 0$, $F(x)/x^a$ is analytic at zero: we give expansions in powers of $n^{-1/a}$ for the distribution function, density function and moments of the sample minimum m_n . But the coefficient of $n^{-i/a}$ is now a polynomial of degree i in $n^{1/a-1}$. Examples include $F(x) = x^a$ on $[0, 1]$ and $F(x)$ the gamma distribution function with mean a . Section 4 gives similar results for the maximum.

Set $\delta_{i,j} = I(i = j)$, the Kronecker delta, and $[x]_r = \Gamma(x + r)/\Gamma(x) = x(x + 1) \cdots (x + r - 1)$. For a sequence $\mathbf{w} = (w_1, w_2, \dots)$ of real or complex numbers, Comtet (1974) defines their partial

ordinary Bell polynomial, $B_{r,k} = \tilde{B}_{r,k}(\mathbf{w})$, by

$$\left(\sum_{r=1}^{\infty} w_r t^r \right)^k = \sum_{r=k}^{\infty} B_{r,k} t^r$$

for $t \in C$ and $k = 0, 1, \dots$. Comtet (1974) tabled $B_{r,k}$ on page 309 for $1 \leq r \leq 10$.

In-built routines for Bell polynomials are available in most computer algebra packages. For example, see `BellY` in Mathematica and `IncompleteBellPoly` in Matlab. So, the expansions given will be accessible to most practitioners.

2 Expansions for the distribution function and moments of the sample maximum

Let M_n be the maximum of a random sample of size n from a distribution function $F(x) = P(X \leq x)$ on $[x_0, x_1]$ that is analytic at x_1 , where $-\infty \leq x_0 < x_1$. So,

$$F(x) = \sum_{i=0}^{\infty} (x_1 - x)^i f_i, \quad (1)$$

where $f_0 = 1$ and $f_1 < 0$ since $F(x)$ is non-decreasing near x_1 . We shall see that

$$Y_n = -f_1 n (M_n - x_1) \xrightarrow{\mathcal{L}} Y \quad (2)$$

as $n \rightarrow \infty$, where $P(Y \leq y) = G_3(x, 1) = e^y$ on $(-\infty, 0]$, known as the third extreme value distribution function with parameter 1. See Fisher and Tippett (1928). Note that $Y_n \leq 0$. It follows from Withers and Nadarajah (2015) that

$$E (-Y_n)^t \rightarrow E (-Y)^t = \Gamma(1+t)$$

for $Re(t) > -1$. A similar result holds for $(Y_{n,1}, \dots, Y_{n,r})$, where $Y_{n,r} = -f_1 n (M_{n,r} - x_1)$ and $M_{n,r}$ is the r -th largest sample value.

Example 2.1 *The class of distributions (1) includes the uniform distribution on $[0, 1]$, $F(x) = x = 1 - (1 - x)$. That is, $f_1 = -1$, $f_i = 0$ for $i \geq 2$. So, $Y_n = n(M_n - 1)$.*

Example 2.2 *For the truncated Pareto distribution on $[0, 1]$, $F(x) = 1 + \gamma - \gamma x^{-\theta}$ with $\gamma > 0$, $\theta > 0$ and $f_i = -\gamma [\theta]_i$ for $i \geq 1$. So, $Y_n = \gamma \theta n (M_n - 1)$.*

For $y \leq 0$, set

$$z = y/f_1, \quad C_i = \sum_{k=1}^i (-1)^{k-1} \tilde{B}_{i,k}(\mathbf{f})/k, \quad c_j = C_{j+1}, \quad (3)$$

$$t_j = t_j(z) = \sum_{k=0}^j \tilde{B}_{j,k}(\mathbf{c}) z^k / k!. \quad (4)$$

Theorem 2.1 For $y \leq 0$, Y_n of (2) has distribution and density functions

$$P_n(y) = P(Y_n \leq y) = e^y \sum_{j=0}^{\infty} (z/n)^j t_j(z) \quad (5)$$

and

$$\begin{aligned} p_n(y) &= P_n(y) + e^y \sum_{j=0}^{\infty} (nf_1)^{-j} \sum_{k=0}^j \tilde{B}_{j,k}(\mathbf{c})(j+k) y^{j+k-1} f_1^{-k}/k! \\ &= e^y \sum_{j=0}^{\infty} (nf_1)^{-j} \sum_{k=0}^j \tilde{B}_{j,k}(\mathbf{c}) \left[y^{j+k} + (j+k) y^{j+k-1} \right] f_1^{-k}/k! \end{aligned} \quad (6)$$

for z , $t_j(z)$ of (3), (4).

Proof: Set

$$S = \sum_{i=1}^{\infty} (z/n)^i f_i, \quad P_n = P_n(y) = F(x_1 - z/n)^n = (1 + S)^n, \quad T_0 = \sum_{j=1}^{\infty} (z/n)^j c_j.$$

Then

$$\begin{aligned} T_0^k &= \sum_{j=k}^{\infty} \tilde{B}_{j,k}(\mathbf{c}) (z/n)^j, \quad S^k = \sum_{i=k}^{\infty} \tilde{B}_{i,k}(\mathbf{f}) (z/n)^i, \\ \ln P_n &= n \ln(1 + S) = n \sum_{k=1}^{\infty} (-1)^{k-1} S^k/k = n \sum_{i=1}^{\infty} (z/n)^i C_i = y + zT_0, \\ P_n &= e^y \sum_{k=0}^{\infty} (zT_0)^k/k! = e^y \sum_{j=0}^{\infty} (z/n)^j t_j. \end{aligned}$$

Now differentiate. $\square$

Example 2.3 Take $F(x) = x^\theta$ on $[0, 1]$, where $\theta > 0$. Then (1), (5) hold with $f_i = (-1)^i \binom{\theta}{i}$, $Y_n = \theta n(M_n - 1)$, $z = -y/\theta$ and $P_n(y) = (1 - z/n)^{n\theta}$ satisfies (5) by the proof of Theorem 2.1 with $t_j = t_j(z)$ of (5) given by

$$t_j = \sum_{k=0}^j \tilde{B}_{j,k}(\mathbf{w}) y^k/k!$$

for $w_j = 1/(j+1)$. Also Y_n has *i-th* moment

$$E Y_n^i = (-1)^i i \sum_{j=0}^{\infty} (n\theta)^{-j} H_{i,j}$$

for $i = 1, 2, \dots$, where $H_{i,j}$ is $D_{i,j}$ of (13) below at $a = 1$. For, by (5),

$$E Y_n^i = \sum_{j=0}^{\infty} (-n\theta)^{-j} \tilde{B}_{j,k}(\mathbf{w}) A_{i,j+k}/k!,$$

where

$$A_{i,j} = \int_{-\infty}^0 y^i d(e^y y^j) = i(-1)^{i+j}(i+j-1)!.$$

Note that $\{t_j\}$ do not depend on θ . Putting $\theta = 1$ gives the distribution of the maximum from a uniform distribution.

Corollary 2.1 For c_j of (3), Y_n of (2) has *i-th* moment

$$E Y_n^i = (-1)^i i \sum_{j=0}^{\infty} (-nf_1)^{-j} H_{i,j} \quad (7)$$

for $i = 1, 2, \dots$, where

$$H_{i,j} = \sum_{k=0}^j \tilde{B}_{j,k}(\mathbf{c}) (-f_1)^{-k} (i+j+k-1)!/k!.$$

Proof: By (6),

$$E M_n^i = \int_{-\infty}^0 y^i p_n(y) dy = \sum_{j=0}^{\infty} (nf_1)^{-j} H'_{i,j}$$

for

$$H'_{i,j} = \sum_{k=0}^j \tilde{B}_{j,k}(\mathbf{c}) h_{i,j+k} f_1^{-k} / k!,$$

where

$$h_{i,j} = I_{i+j} + jI_{i+j-1} = i(-1)^{i+j}(i+j-1)!, \quad I_j = \int_{-\infty}^0 y^j e^y dy = \int_0^{\infty} (-y)^j e^{-y} dy = (-1)^j j!.$$

So, $H'_{i,j} = (-1)^{i+j} i H_{i,j}$. $\square$

Corollary 2.2 Set $F_j = f_j/f_1^j$. Then Y_n of (2) has variance

$$(f_1 n)^2 \text{var}(M_n) = \text{var}(Y_n) = \sum_{j=0}^{\infty} n^{-j} K_{2,j}, \quad (8)$$

where

$$K_{2,j} = (-f_1)^{-j} k_{2,j}, \quad k_{2,j} = 2H_{2,j} - \sum [H_{1,a} H_{1,b} : a+b=j].$$

Proof: By Corollary 2.1,

$$\text{var}(Y_n) = \sum_{j=0}^{\infty} (-nf_1)^{-j} k_{2,j}$$

and hence the result. $\square$

Example 2.4 For $F(x) = x$ of Example 2.1, $F_i = 0$ for $i \geq 2$, so that $K_{2,1} = -44$, $K_{2,2} = 11$. So, $n^2 \text{var}(M_n) = 1 - 4n^{-1} + 11n^{-2} + \dots$.

Example 2.5 For $F(x)$ of Example 2.2, $F_i = (-c)^{1-i}[\theta]_i/\theta^i$.

By (5),

$$P(Y_n \leq y) = e^y \sum_{j=0}^{J-1} (z/n)^j t_j(z) + O(n^{-J}) \quad (9)$$

for $J \geq 1$. The right hand side only needs c_j, f_{j+1} for $j < J$. So, (9) holds if we weaken (1) to

$$F(x) = \sum_{i=0}^J (x_1 - x)^i f_i + O\left((x_1 - x)^{J+1}\right),$$

where $f_0 = 1$ and $f_1 < 0$. That is, if $F(x)$ has a Taylor expansion of order J at x_1 . Similar results hold for the other expansions in this note.

3 Expansions for the distribution function and moments of the sample minimum

We give two methods. The first is to use results in Section 2. Suppose that $X'_1, \dots, X'_n$ is a random sample from $X \sim F'(x)$, some continuous distribution function on $[x'_0, x'_1]$ with minimum m'_n and maximum M'_n . Then $-X_j \equiv X'_j$ is a random sample from $[x_0, x_1]$, where $x_0 = -x'_1, x_1 = -x'_0$ with minimum $m_n = -M'_n$ and maximum $M_n = -m'_n$.

Theorem 3.1 Suppose that

$$F'(x) = \sum_{i=1}^{\infty} (x - x'_0)^i g_i, \quad Y_n = g_1 n (x'_0 - m'_n).$$

So, $F'(x_0) = 0, Y_n < 0$ and Y_n has distribution function, density function, moments and variance given by (5), (6), (7) and (8), respectively, with $f_i = -g_i$.

We now drop the dashes and take m_n as the minimum from a distribution function $F(x)$ on $[x_0, x_1]$. Clearly, for $a \neq 1$, **Theorem 3.1 does not cover the case described in the following example.**

Example 3.1 Take $F(x) = x^a$ on $[0, 1]$, where $a > 0$. Set $N = n^{1/a}, Y_n = Nm_n$. Then, for $0 \leq y \leq N$,

$$\begin{aligned} P(Y_n > y) &= (1 - y^a/n)^n \rightarrow \exp(-y^a), \\ Y_n &\xrightarrow{\mathcal{L}} Y, \\ E Y_n^t &\rightarrow E Y^t = \Gamma(1 + t/a) \end{aligned} \quad (10)$$

as $n \rightarrow \infty$, where $\text{Re}(t) > -a$, $P(Y \leq y) = 1 - \exp(-y^a)$ and $P(-Y \leq y) = \exp(-z) = G_3(y, a)$ say, where $z = (-y)^a$. For $y \leq 0$, set $w_j = 1/(j+1)$, $z = (-y)^a$ and $P_n = P(-Y_n \leq y)$. Then,

$$P_n = e^{-z} \sum_{j=0}^{\infty} (z/n)^j t_j(-z) \quad (11)$$

for $t_j(y)$ of Example 2.3. $-Y_n$ has density function

$$p_n(y) = dP_n/dy = dP_n/dz \, dz/dy,$$

where

$$dz/dy = -az^{1-1/a}, \quad dP_n/dz = -e^{-z} \sum_{j=0}^{\infty} n^{-j} \sum_{k=0}^j t_{j,k} \left[z^{j+k} - (j+k)z^{j+k-1} \right] \quad (12)$$

and $t_{j,k} = \tilde{B}_{j,k}(\mathbf{w})(-1)^k/k!$. Y_n has moments

$$E Y_n^i = (i/a) \sum_{j=0}^{\infty} D_{i,j} n^{-j}, \quad (13)$$

where

$$D_{i,j} = \sum_{k=0}^j t_{j,k} \Gamma(i/a + j + k).$$

Proof: $G_3(y, a) = \exp[-(-y)^a]$ is known as the third extreme value distribution function, see Fisher and Tippett (1928). (10) holds by Withers and Nadarajah (2015). Since $P_n = (1 - z/n)^n$,

$$\ln P_n = n \ln(1 - z/n) = - \sum_{k=1}^{\infty} z^k n^{1-k}/k = -z \sum_{j=0}^{\infty} (z/n)^j/(j+1) = -z(1 + S_n)$$

for

$$S_n = \sum_{j=1}^{\infty} (z/n)^j w_j, \quad S_n^k = \sum_{j=k}^{\infty} (z/n)^j \tilde{B}_{j,k}(\mathbf{w}), \quad P_n e^z = e^{S_n} = \sum_{k=0}^{\infty} (-z S_n)^k / k!$$

which yield (12) and (13). Set

$$J_{i,j-1} = \int_0^{\infty} z^{i/a+j-1} e^{-z} dz = \Gamma(i/a + j).$$

Then

$$E Y_n^i = \int_{-\infty}^i (-u)^i p_n(y) dy = - \int_0^{\infty} z^{i/a} dP_n(z)/dz \, dz = \sum_{j=0}^{\infty} n^{-j} \sum_{k=0}^j t_{j,k} I_{i,j+k},$$

where $I_{i,j} = J_{i,j} - j J_{i,j-1} = (i/a) J_{i,j-1}$. $\square$

It is striking that Examples 2.3 and 3.1 use the same $t_j(y)$ even though they have different types of limits. Example 3.1 is a special case of

Theorem 3.2 Suppose that $F(x)$ is a distribution function on $[0, x_1]$ with

$$F(x) = x^{a-1} \sum_{i=1}^{\infty} x^i g_i, \quad (14)$$

where $a > 0$ and $g_1 > 0$. For $y \geq 0$, set

$$\gamma = a - 1, \quad N = (ng_1)^{1/a}, \quad y_N = y/N, \quad Y_n = Nm_n, \quad (15)$$

$$G(y) = \exp(-y^a), \quad C_j(x) = C_j = \sum_{k=1}^j \tilde{B}_{j,k}(\mathbf{g}) x^{k\gamma}/k, \quad d_i(x) = d_i = xC_{i+1}(x),$$

$$E_i = N^a y^i d_i(y_N) = \sum_{j=0}^i E_{i,j} \delta^j,$$

where $\delta = N^{1-a} = (ng_1)^{1/a-1}$, $E_{i,j} = b_{i,j} y^{(j+1)a+i-j}$ and $b_{i,j} = \tilde{B}_{i+a,j+1}(\mathbf{g})/(j+1)$. Set $h_k = (-g_1)^{-k}/k!$. Then $P_n = P(Y_n > y)$ is given by

$$P_n = G(y) Q_n, \quad (16)$$

where

$$Q_n = \sum_{i=0}^{\infty} N^{-i} P_{n,i}, \quad P_{n,i} = \sum_{k=0}^i \tilde{B}_{i,k}(\mathbf{E}) h_k.$$

Also,

$$\tilde{B}_{i,k}(\mathbf{E}) = \sum_{j=0}^i B_{i,k,j}(y) \delta^j, \quad (17)$$

where $B_{i,k,j}(y) = b_{i,k,j} y^{(j+k)a+i-j}$, $b_{i,k,0} = \tilde{B}_{i,k}(\mathbf{e})$, $e_i = g_{i+1}$, $b_{i,1,0} = g_{i+1}$, $b_{i,i,0} = g_2^i$ and $b_{i,1,j}$ is equal to the coefficient of δ^j in the E_i above. So, $P_{n,0} = 1$ and, for $i \geq 1$, $P_{n,i}$ of (16) can be written as

$$P_{n,i} = \sum_{j=0}^i P_{i,j}(y) \delta^j, \quad (18)$$

where

$$P_{i,j}(y) = \sum_{k=1}^i B_{i,k,j}(y) h_k.$$

So, these $b_{i,k,j}$ with (17) are enough to give $P_{n,i}$ for $i = 1, 2, 3$.

The coefficient of N^{-i} or $n^{-i/a}$ in the expansion for the distribution function of M_n is a polynomial of degree i in δ or $n^{1/a-1}$.

Note that since $\tilde{B}_{i,1}(\mathbf{E}) = E_i$, we can read $b_{i,1,j}$ off from Theorem 3.2. For example, $b_{4,1,1} = g_1 g_4 + g_2 g_3$ and $b_{4,1,2} = g_1^2 g_3 + g_1 g_2^2$.

Proof: C_j can be read off from C_j of Section 2 by replacing f_j by g_j and $(-1)^{k-1}$ by $x^{k\gamma}$. Note that

$$F(x)^k = x^{k\gamma} \sum_{j=k}^{\infty} x^j \tilde{B}_{j,k}(\mathbf{g})$$

and $P_n = P(Y_n > y) = [1 - F(y_N)]^n$. So,

$$\ln P_n = n \ln [1 - F(y_N)] = -n \sum_{k=1}^{\infty} F(y_N)^k / k = -n \sum_{j=1}^{\infty} y_N^j C_j(y_N)$$

with leading term $-y^a$. So, $\ln P_n = -y^a - nS_n$, where

$$S_n = \sum_{j=2}^{\infty} y_N^j C_j(y_N).$$

Set

$$A_n = g_1 n S_n = N^a S_n = N^a \sum_{i=1}^{\infty} y_N^i d_i(y_N) = \sum_{i=1}^{\infty} N^{-i} E_i.$$

Then,

$$A_n^k = \sum_{i=k}^{\infty} N^{-i} \tilde{B}_{i,k}(\mathbf{E}), \quad \exp(-nS_n) = \exp(-A_n/g_1) = \sum_{k=0}^{\infty} (-A_n/g_1)^k / k! = \sum_{i=0}^{\infty} N^{-i} P_{n,i}$$

and $P_n = G(y) \exp(-nS_n)$, yielding (16). (17) implies (18). $\square$

Theorem 3.2 gives $b_{i,j,k}$ needed for $m_i(t, z)$, $i \leq 4$. To apply Theorem 3.2 to Example 3.1, set $x_1 = 1$, $g_i = \delta_{i,1}$, $W_i = y^a/(i+1)$, $z = y^a \delta$ and $E_i = W_i z^i$. Then $\tilde{B}_{i,k}(\mathbf{E}) = z^i \tilde{B}_{i,k}(\mathbf{W})$ and (11) follows.

To get the moments of $Y_n = Nm_n$, we need its density function

$$p_n(y) = -dP_n/dy = G(y) \mathcal{D} Q_n, \tag{19}$$

where $\mathcal{D} = ay^{a-1} - d/dy$ and $G(y)' = -ay^{a-1}G(y)$. Define $b_{i,k,j}$ by (17).

Theorem 3.3 $Y_n = Nm_n$ has density function

$$p_n(y) = G(y) \sum_{i=0}^{\infty} N^{-i} p_{n,i}, \tag{20}$$

where

$$p_{n,i} = \mathcal{D} P_{n,i} = \sum_{j=0}^i p_{i,j}(y) \delta^j, \quad p_{i,j}(y) = \mathcal{D} P_{i,j}(y) = \sum_{k=1}^i b_{i,k,j} p_{i,j,k}(y) h_k$$

and $p_{i,j,k}(y) = \mathcal{D} y^c = ay^{a+c-1} - cy^{c-1} = (ay^a - c) y^{c-1}$, $c = (j+k)a + i - j$.

Proof: Follows from (16), (18) and (19). $\square$

This implies

Corollary 3.1 For $t \in C$ with real part greater than -1 , the *t-th moment* of $Y_n = Nm_n$ is given by

$$E Y_n^t = \sum_{i=0}^{\infty} N^{-i} m_i(t, \delta),$$

where

$$m_i(t, \delta) = \sum_{j,k=0}^i \delta^j m_{i,j,k}(t) b_{i,k,j} h_k,$$

$$m_{i,j,k}(t) = \int_0^{\infty} y^t \exp(-y^a) p_{i,j,k}(y) dy = \mathcal{A} O_{t+(j+k)a+i-j-1}$$

for

$$\mathcal{A} O_{t+c} = a O_{t+c+a} - (c+1) O_{t+c}, \quad O_t = \int_0^{\infty} y^t \exp(-y^a) dy = \Gamma((t+1)/a) / a.$$

We have given the distribution function, density function and moments of Y_n as double expansions in powers of N^{-i} and $\delta^j = N^{(1-a)j}$, say

$$R_n = R + \sum_{i=1}^{\infty} N^{-i} \sum_{j=0}^i \delta^j R_{i,j}.$$

We now show which terms can be dropped to obtain a result which is good to $O(N^{-5})$.

The case $a > 1$. We need to include $R_{i,j}$ for which $1 \leq i \leq 4$, $0 \leq j \leq i$ and $j < (5-i)/(a-1)$. Note that $i < (5-i)/(a-1)$ if and only if $i < 5/a$. This is only true for $i = 4$ if $a < 5/4$. So, if $a \geq 5/4$, this second restriction on j applies.

The case $a = 2$, then one must include $R_{1,j}$ for $j = 0, 1$, $R_{2,j}$ for $j = 0, 1, 2$, $R_{3,j}$ for $j = 0, 1$, and $R_{4,0}$.

The case $a = 1.5$, then one must include $R_{1,j}$ for $j = 0, 1$, $R_{2,j}$ for $j = 0, 1, 2$, $R_{3,j}$ for $j = 0, 1, 2, 3$, and $R_{4,j}$ for $j = 0, 1$.

The case $a < 1$. We need to include $R_{i,j}$ for which $i - (1-a)j < 5$ and $0 \leq j \leq i$. That is, $i \leq 5/a$ and $(i-5)/(1-a) < j \leq i$.

An alternative way of obtaining the moments of Sections 2 and 3 is to use Theorems 2.2 and 2.3 of Withers and Nadarajah (2013).

Example 3.2 Let $F(x)$ be the gamma distribution function with density function $f(x) = x^{a-1}e^{-x}/\Gamma(a)$ on $[0, \infty)$, where $a > 0$. By equations (6.5.4) and (6.5.29) of Abramowitz and Stegun (1964),

$$x^{-a}F(x)\Gamma(a) = \sum_{n=0}^{\infty} (-x)^n / (a+n)n!.$$

So, (14) holds with $g_i = (-1)^n/(a+n)n!$, where $n = i - 1$. So, the distribution function of $Y_n = Nm_n$ is given by (18), (16) with

$$P_{n,i} = \sum_{j=0}^i P_{i,j}(y)\delta^j,$$

where

$$\begin{aligned} P_{1,0}(y) &= g_2 h_1 y^{a+1}, \quad P_{1,1}(y) = g_1^2 h_1 y^{2a}/2, \quad P_{2,0}(y) = g_3 h_1 y^{a+2} + g_2^2 h_2 y^{2a+2}, \\ P_{2,1}(y) &= g_2 (-y^{2a+1} + y^{3a+1}/2), \quad P_{2,2}(y) = g_1^3 h_1 y^{3a}/3 + g_1^4 h_2 y^{4a}/4, \\ P_{3,0}(y) &= g_4 h_1 y^{a+3} + 2g_2 g_3 h_2 y^{2a+3} + g_2^3 h_3 y^{3a+3}, \\ P_{3,1}(y) &= (g_1 g_3 + g_2^2/2) h_1 y^{2a+2} + (2g_1 g_2^2 + g_1^2 g_3) h_2 y^{3a+2} + 3g_1^2 g_2^2 h_3 y^{4a+2}/2, \\ P_{3,2}(y) &= g_2^2 h_1 y^{3a+1} + 5g_1^2 g_2 h_2 y^{4a+1}/3 + 35g_1^4 g_2 h_3 y^{5a+1}/4, \\ P_{3,3}(y) &= g_1^4 h_1 y^{4a}/4 + g_1^5 h_2 y^{5a}/3 + g_1^6 h_3 y^{6a}/8. \end{aligned}$$

The moments of $Y_n = Nm_n$ are given by Corollary 3.1.

4 More expansions for the sample maximum

Theorem 4.1 Suppose that

$$F(x) = 1 - \sum_{i=1}^{\infty} (-x)^{a-1+i} g_i$$

is a continuous distribution function on $[x_0, 0)$, where $a > 0$ and $g_1 > 0$. Then for $y > 0$ and N of (15), the distribution function, density function and moments of $Y_n = -NM_n$ are given by $P(Y_n > y) = P_n$ of (16), (20) and Corollary 3.1.

Proof: Relabel the sample $\{X_j\}$, maximum M_n and distribution function $F(x)$ as $\{X'_j\}$, M'_n and $F(x)'$, respectively. Then $X_j = -X'_j$ has distribution function

$$F(x) = 1 - F(-x)' = \sum_{i=1}^{\infty} x^{a-1+i} g_i$$

on $(0, -x_0]$ with minimum $m_n = -M'_n$. Now apply Theorem 3.2, Theorem 3.3 and Corollary 3.1. $\square$

Theorem 4.2 Suppose that

$$F(x) = 1 - \sum_{i=1}^{\infty} (x_1 - x)^{a-1+i} g_i \tag{21}$$

is a distribution function on $[x_0, x_1]$, where $a > 0$ and $g_1 > 0$. Then for $y > 0$ and N of (15), the distribution function, density function and moments of $Y_n = N(x_1 - M_n)$ are given by $P(Y_n > y) = P_n$ of (16), (20) and Corollary 3.1.

Proof: Follows from Theorem 4.1 by a location change. $\square$

If $a = 1$ then (21) is just (1) with $g_i = -f_i$ so that we can apply Section 2.

Finally, we revisit Example 2.3.

Example 4.1 Take $F(x) = x^\theta$ on $[0, 1]$, where $\theta > 0$. Then (21) holds with $a = 1$, $g_i = (-1)^{i-1} \binom{\theta}{i}$ and $Y_n = \theta n (M_n - 1)$.

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