PERFORMANCE OF MULTI-DRIVE SYSTEMS

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SCHOOL OF ELECTRICAL AND ELECTRONIC ENGINEERING
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<th>Description</th>
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<tbody>
<tr>
<td>AC</td>
<td>Alternating current</td>
</tr>
<tr>
<td>B2B</td>
<td>Business-to-Business</td>
</tr>
<tr>
<td>Back EMF</td>
<td>Back Electro-Motive Force</td>
</tr>
<tr>
<td>CCS-MPC</td>
<td>Continuous Control Set MPC</td>
</tr>
<tr>
<td>CNC</td>
<td>Computer Numerical Control</td>
</tr>
<tr>
<td>DC</td>
<td>Direct Current</td>
</tr>
<tr>
<td>DOF</td>
<td>Degrees of Freedom</td>
</tr>
<tr>
<td>EMPC</td>
<td>Explicit Model Predictive Control</td>
</tr>
<tr>
<td>EPL</td>
<td>Eclipse Public License</td>
</tr>
<tr>
<td>EU</td>
<td>European Union</td>
</tr>
<tr>
<td>FCS-MPC</td>
<td>Finite Control Set MPC</td>
</tr>
<tr>
<td>GPC</td>
<td>Generalized Predictive Control</td>
</tr>
<tr>
<td>IoT</td>
<td>Internet of Things</td>
</tr>
<tr>
<td>LAM</td>
<td>Laser-Assisted Machining</td>
</tr>
<tr>
<td>LCA</td>
<td>Life Cycle Assessment</td>
</tr>
<tr>
<td>LMD</td>
<td>Linear Motor Drive</td>
</tr>
<tr>
<td>LUT</td>
<td>Lookup Table</td>
</tr>
<tr>
<td>MMF</td>
<td>Magnetomotive Force</td>
</tr>
<tr>
<td>MPC</td>
<td>Model Predictive Control</td>
</tr>
<tr>
<td>MRR</td>
<td>Material Removal Rate</td>
</tr>
<tr>
<td>MT</td>
<td>Machine Tool</td>
</tr>
<tr>
<td>MTPA</td>
<td>Maximum Torque Per Ampere</td>
</tr>
<tr>
<td>MTPV</td>
<td>Maximum Torque Per Voltage</td>
</tr>
<tr>
<td>NC</td>
<td>Numerical Control</td>
</tr>
<tr>
<td>OE</td>
<td>Optimal Energy</td>
</tr>
<tr>
<td>OECD</td>
<td>The Organization for Economic Cooperation and Development</td>
</tr>
<tr>
<td>OT</td>
<td>Optimal Time</td>
</tr>
<tr>
<td>PID</td>
<td>Proportional–Integral–Derivative</td>
</tr>
<tr>
<td>PKM</td>
<td>Parallel Kinematic Machine</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
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<tr>
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<td>--------------------------------------------------</td>
</tr>
<tr>
<td>PM</td>
<td>Permanent Magnet</td>
</tr>
<tr>
<td>PMSM</td>
<td>Permanent Magnet Synchronous Motor</td>
</tr>
<tr>
<td>PWM</td>
<td>Pulse Width Modulation</td>
</tr>
<tr>
<td>QP</td>
<td>Quadratic Programming</td>
</tr>
<tr>
<td>SD</td>
<td>Screw Drive</td>
</tr>
<tr>
<td>SEC</td>
<td>Specific Energy Consumption</td>
</tr>
<tr>
<td>SME</td>
<td>Small and Medium-Sized Enterprises</td>
</tr>
<tr>
<td>SMPMSM</td>
<td>Surface-Mounted Permanent Magnet Synchronous</td>
</tr>
<tr>
<td></td>
<td>Motor</td>
</tr>
<tr>
<td>SOCP</td>
<td>Second-Order Cone Program</td>
</tr>
<tr>
<td>SPWM</td>
<td>Sinusoidal Pulse Width Modulation</td>
</tr>
<tr>
<td>SVPWM</td>
<td>Space Vector Pulse Width Modulation</td>
</tr>
<tr>
<td>TOMPC</td>
<td>Time Optimal Model Predictive Control</td>
</tr>
<tr>
<td>UK</td>
<td>United Kingdom</td>
</tr>
<tr>
<td>VSD</td>
<td>Variable-Speed Drive</td>
</tr>
<tr>
<td>VSI</td>
<td>Voltage Source Inverter</td>
</tr>
<tr>
<td>ZOH</td>
<td>Zero-Order Hold</td>
</tr>
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</table>
# Notations

In this thesis matrices and columns are represented by bold letters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{fr}$</td>
<td>Viscous friction coefficient, $\frac{Nm}{rad/s}$</td>
</tr>
<tr>
<td>$i_{a_{max}}$</td>
<td>Armature maximum current, [A]</td>
</tr>
<tr>
<td>$i_{s_{max}}$</td>
<td>Peak stator maximum current, [A]</td>
</tr>
<tr>
<td>$J_e$</td>
<td>Rotor shaft inertia, [$kgm^2$]</td>
</tr>
<tr>
<td>$L_a$</td>
<td>Armature inductance, [H]</td>
</tr>
<tr>
<td>$L_s$</td>
<td>Stator inductance, [H]</td>
</tr>
<tr>
<td>$Q_{nl}$</td>
<td>CORIN no load energy consumption, [J]</td>
</tr>
<tr>
<td>$Q_{sp}$</td>
<td>CORIN stand-by energy consumption, [J]</td>
</tr>
<tr>
<td>$Q_{tp}$</td>
<td>CORIN transfer phase energy consumption, [J]</td>
</tr>
<tr>
<td>$R_a$</td>
<td>Armature resistance, [$\Omega$]</td>
</tr>
<tr>
<td>$R_s$</td>
<td>Stator resistance, [$\Omega$]</td>
</tr>
<tr>
<td>$T_e$</td>
<td>Torque developed by a motor drive, [Nm]</td>
</tr>
<tr>
<td>$T_s$</td>
<td>Discretization time, [s]</td>
</tr>
<tr>
<td>$v_{a_{max}}$</td>
<td>Armature maximum voltage, [V]</td>
</tr>
<tr>
<td>$V_{dc}$</td>
<td>DC-link voltage, [V]</td>
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<tr>
<td>$v_{s_{max}}$</td>
<td>Peak stator maximum voltage, [V]</td>
</tr>
<tr>
<td>$i_a$</td>
<td>Armature current, [A]</td>
</tr>
<tr>
<td>$i_{d_s}$</td>
<td>Stator current, $d$-axis rotational reference frame, [A]</td>
</tr>
<tr>
<td>$i_{q_s}$</td>
<td>Stator current, $q$-axis rotational reference frame, [A]</td>
</tr>
<tr>
<td>$t_f$</td>
<td>Final time</td>
</tr>
<tr>
<td>$v_a$</td>
<td>Armature voltage, [V]</td>
</tr>
<tr>
<td>$v_{d_s}$</td>
<td>Stator voltage, $d$-axis rotational reference frame, [V]</td>
</tr>
<tr>
<td>$v_{d_s}$</td>
<td>Stator voltage, $d$-axis stationary reference frame, [V]</td>
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<td>$v_{q_s}$</td>
<td>Stator voltage, $q$-axis rotational reference frame, [V]</td>
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<tr>
<td>$v_{q_s}$</td>
<td>Stator voltage, $q$-axis stationary reference frame, [V]</td>
</tr>
<tr>
<td>$J_e$</td>
<td>OE performance measure/optimization criteria</td>
</tr>
<tr>
<td>$J_t$</td>
<td>OT performance measure/optimization criteria</td>
</tr>
<tr>
<td>$\theta_r$</td>
<td>Rotor electrical position, [rad]</td>
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### Notations

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<th>Description</th>
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<tbody>
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<td>$\theta_{rm}$</td>
<td>Rotor mechanical position, [rad]</td>
</tr>
<tr>
<td>$\lambda_{pm}$</td>
<td>Back EMF constant, $\left[\frac{V}{rad/s}\right]$</td>
</tr>
<tr>
<td>$\omega_r$</td>
<td>Rotor electrical speed, [rad/s]</td>
</tr>
<tr>
<td>$\omega_{rm}$</td>
<td>Rotor mechanical speed, [rad/s]</td>
</tr>
<tr>
<td>$A$</td>
<td>Acceleration limit, [rad/s$^2$]</td>
</tr>
<tr>
<td>$V$</td>
<td>Speed limit, [rad/s]</td>
</tr>
<tr>
<td>$p$</td>
<td>Number of poles</td>
</tr>
<tr>
<td>$pp$</td>
<td>Number of pole pairs</td>
</tr>
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**Subscript:**

<table>
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<th>Symbol</th>
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<tr>
<td>$A_a$</td>
<td>Armature</td>
</tr>
<tr>
<td>$A_s$</td>
<td>Stator</td>
</tr>
<tr>
<td>$A_{rm}$</td>
<td>Rotor mechanical (rad/s)</td>
</tr>
<tr>
<td>$A_r$</td>
<td>Rotor electrical (rad/s)</td>
</tr>
<tr>
<td>$A_{dc}$</td>
<td>Direct current</td>
</tr>
<tr>
<td>$A_k$</td>
<td>Column on the $k$th step</td>
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**Superscript:**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A^*$</td>
<td>Reference value</td>
</tr>
<tr>
<td>$\hat{A}$</td>
<td>Estimated/measured value</td>
</tr>
<tr>
<td>$\dot{A}$</td>
<td>Derivative</td>
</tr>
<tr>
<td>$\ddot{A}$</td>
<td>Vector</td>
</tr>
<tr>
<td>$</td>
<td>A</td>
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Abstract

In this thesis, a novel control strategy has been developed to reduce industrial electrical energy use. This solution requires little or no investment, and will not compromise production targets, product quality or environmental impact. It can be used to promote Industry 4.0 via smart information exchange on the lowest device/unit process manufacturing level. By allowing communication between motors in a multi-drive system the energy consumption can be reduced through smart scheduling of motor operation times and loading.

The positioning system in metal-cutting machine tools provides an example application, where the separate axes are powered through variable-speed drives. The multi-axis positioning is a series kinematics system, where the motor drives are indirectly mechanically coupled through the tool tip/end-effector trajectory. 2D systems are considered for simplicity, where the drives’ dynamics are independent from each other. The proposition of this thesis is that overall electrical energy consumption can be reduced, by controlling the drive with the longer axis’ movement distance for minimum time, and the drive with the shorter distance for optimal energy. The latter is required to finish its move by the time the former reaches its end point, therefore an ability to compute and exchange expected finishing times between axes is required. The resulting end-effector trajectory is nonlinear, so the method is proposed for point-to-point moves.

Two different approaches have been studied for optimal control of the positioning system: Variational and Model Predictive Control (MPC). Both these methods were tested to control the motor drive within its current and voltage constraints. The former is shown to be suitable for a simple plant model approximation, where the analytical open-loop solution for both control problems can be obtained. However, this approach is of limited use or even not applicable for the derivation of a close-loop control law. The variational technique is useful for reference profile generation, where the energy use is dominated by stand-by or support needs, for example, in robotics applications. This method was demonstrated on a mobile six-legged robot CORIN, where an open-loop optimal time profile was generated for each joint’s motor control, leading to the more than 20% energy saving, whilst remaining within all constraints.

In the MPC approach, the control is split into two stages. First, the optimal reference state trajectory is generated off-line. It is shown that a linear MPC formulation can be used for brushed DC motor drives whereas a non-linear MPC formulation is necessary for the Surface Mounted Permanent Magnet Synchronous Motors (SMPMSM). Then the optimal state trajectory is used as reference and feed-forward to either a conventional nested-loop controller or to a MPC on-line reference tracking controller. The proposed control has been compared with conventional G00 and G01 trajectories. For the DC motor drive under test, experimental results show 13% and 8% efficiency rise compared to the G00 and G01 respectively. For the SMPMSM control, simulation results reveal 16% energy consumption reduction compared to the G00 case, and almost the same energy consumption as the G01 case, due to the field-weakening operation of the drive in the proposed control method. Since the proposed algorithm reduces the operational time, this still results in lower stand-by energy consumption, which is shown to be dominant.
Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.
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To my mother, who has always been my active listener.
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The following publications came out from his PhD research so far:

Chapter 1  Introduction

1.1  Fourth Industrial Revolution concept

The concept of the Fourth Industrial Revolution or Industry 4.0 was initially introduced in Germany in April 2011 at the Hanover Fair. After the announcement by the German government of a “High-Tech Strategy 2020” action plan in March 2012 this concept started spreading across the globe. In October 2013, the United Kingdom announced the “Future of Manufacturing: a new era of opportunity and challenge for the UK” report looking at the long-term picture of the manufacturing, where the term “Industry 4.0” was not explicitly used, but similar directives to support the growth of advanced manufacturing sector were included. Also major industrial companies Siemens, ABB, Schneider Electric etc. are already heavily involved in the development of Industry 4.0 projects [1-3] as well as research institutions [4].
Generally, the term Industry 4.0 covers the development of manufacturing technologies in the Cyber-Physical System framework. Applying a concept similar to the Internet of Things (IoT) to the factory environment will lead to substantial improvement in industrial processes by interconnecting the equipment into the single network, which is integrated with the higher hierarchical levels of product design, operation management and even sales. Industry 4.0 is not just about complete factory automation but more about finding smarter ways to work, being more efficient in the utilization of equipment and resources through the smart information exchange coming from different sources and decision-making layers, being more flexible and agile, creating new services especially for B2B (business-to-business).

According to [4] in order to successfully implement Industry 4.0 work in eight key areas is required.

- Standardization and reference architecture;
- Managing complex systems;
- A comprehensive broadband infrastructure for industry;
- Safety and security;
- Work organization and design;
- Training and continuing professional development;
- Regulatory framework;
- Resource efficiency.

These areas require research into technology and management fields as well as in industrial policy, decision making and legislation from the government.

The number of international conferences debating the Fourth Industrial Revolution has increased from 2013 to 2015 by more than ten times [5]. Leading manufacturing companies are already using the benefits of this novel approach, for example for energy consumption reduction [6]. The question arises, what are the greatest challenges that are blocking the ubiquitous widespread use of Industry 4.0? It is clear that the newly announced revolution is not based on new physical principles and most technology is already available [7], however as summarized in [4] the three major obstacles are standardization, work organization and product availability. The first, to set up connections across a range of different manufacturers' equipment, was also mentioned as the greatest problem in [8], especially for small and medium enterprises (SME), which are facing a lack of resources and probably technical expertise [9]. Moreover, Industry 4.0 requires either hardware modification, or proprietary software utilization,
or both simultaneously, which could be a problem in terms of lack of the capital available for investments [10],[11].

Due to the aforementioned issues, to increase the migration speed from now to the world of Industry 4.0, its immediate benefits have to be transferred “from vision level to reality level” [9]. In other words, a migration strategy to promote Industry 4.0 implementation solutions and reveal its benefits should be developed [4]. Moreover, this solution should require either no or very small investment, while not compromising production targets, product quality or environmental impact.

The industrial sector accounts for 37% of world’s energy consumption according to [12] or even for 51% [13]. Of course, these numbers differ from country to country, but even for OECD countries characterized by a non-industrial type of economy the industrial energy consumption could be very considerable. For example, in the UK the electrical energy use in manufacturing industry accounts for 19.2% or 27% according to [12] or [14] respectively. Due to this the efficient use of electrical energy in the industrial sector should have a high priority and would reduce carbon dioxide emissions, relax the electrical energy demand issue and reduce electricity bills for the manufacturers. The development of an innovative energy saving solution (a part of “Resource efficiency” key area) based on Industry 4.0 framework will support its widespread and adoption.

According to [15], manufacturing production, especially for discrete parts (vehicles, electronics, consumer goods), has a hierarchical and distributed nature, where different processes are characterised by different timescales and decision-making layers. Following the concept of Industry 4.0, it would be beneficial to collect the data from all the layers and perform plant-level energy optimization, however, such manufacturing data is not readily available and such an approach faces all the previously mentioned problems. In order to provide a smooth mitigation strategy it is possible to perform the energy optimization at the lowest production cell level. In literature an example application has been shown [4], where the energy consumption is reduced by systematically powering down inactive machinery of a vehicle body assembly line. This is important because a huge portion of energy is consumed in stand-by mode, moreover to transfer from the stand-by mode to the production mode takes time, and so smart scheduling is required. It is claimed that around 12% of total energy consumption reduction is achievable if robots are powered down during un-productive periods [4].
1.2 Multi-drive systems optimization

This thesis proposes to go further down through the manufacturing layers and apply the Industry 4.0 approach for the separate electric motor drives. According to the International Energy Agency [16], electric motors account for two thirds of industrial power consumption, so the significance of motor drive energy efficiency maximization is crystal clear. By allowing the communication between the motor drives in a single system the energy consumption could be reduced by smart scheduling and planning the motor operation times and loading.

In July 2014 during the National Centre for Power Electronics Conference professor J.W. Kolar among others defined one challenge or opportunity for the further research, the system-oriented analysis, where the study of multi-drive systems is believed the biggest potential for optimization [17]. This approach could be seen as an application of the Industry 4.0 framework on the lowest production level, performing the energy efficiency optimization of systems comprising of several interacting motor drives. Brushed DC motors together with the permanent magnet synchronous motors (PMSM) are chosen as a target for the research.

While talking about multi-drive systems two different architectures could be considered. First, a multi-drive network with mechanically coupled drives and, second, a multi-drive network with electrically coupled drives. Or in other words, two types of interaction between drives are possible: mechanical interaction through the shaft and electrical interaction through the AC or DC bus. This thesis will focus on the first type of interaction.

Also, there are two possible ways mechanical coupling can occur in the multi-drive system. First, the drives could be coupled directly through a belt and pulley or through gears; second, indirectly coupled, where the mechanical load is unequally shared between the drives over time. In [18] possible applications of both coupling types were discussed in details. Directly mechanically coupled drives find a wide application in numerous industrial fields like steel milling industry and in railway traction or electric propulsion. For example, dual-motor systems are used instead of a single motor due to the power rating limitations [19], [20]. Another reason to use the dual-motor configuration is to increase the overall system reliability, where the system could still operate at reduced power rating in the case of a single motor failure. For electric propulsion applications such a configuration is used when the existing motors’ power
range doesn’t cover the need of a particular vehicle. In order to reduce energy consumption, it is critical to introduce the most energy efficient way to control each single motor considering the operation of another one. In other words, a load torque sharing scheme between the motors must be developed, since the speed relationship between the drives is determined by the gearbox. Different solutions are introduced in the literature, two different motors with different rated power are to be coupled to increase the performance in [21], a hybrid control method suppressing the mechanical resonances is discussed in [22], a fuzzy-logic controller for the maximum efficiency control is discussed in [19], etc. All these papers deal only with the suppressing of mechanical imperfections in the motor coupling (motor shafts misalignment, different shaft stiffness and damping, etc), however in [23] it was shown that for the perfectly equal systems in the mechanical sense the reference torque should be equally shared between two permanent magnet (PM) motors or one motor should be switched off, by a clutch. This happens due to the presence of iron loss in the PM motors, even at the no load condition.

Due to the aforementioned reasons, directly mechanically coupled drives are not the target object for this thesis research. Consideration of mechanical imperfections or hardware modifications like mechanical clutch can’t be treated as a general solution to improve the energy consumption of the mechanically coupled systems. Accordingly, indirectly mechanically coupled motor drives are chosen for the control with the reduced energy consumption. The energy consumption is optimized by a proper control of each drive taking in account the operation of other drives.

1.3 Aims and objectives

The aim of this research is to reduce the electrical energy consumption in the industrial system comprising of multiple motor drives within the Industry 4.0 framework, i.e. via the smart information exchange between the drives.

The objectives are as follows:

- Through the thorough literature review identify the industrial system with relatively large electrical energy consumption and containing a number of interacting motor drives.
- In this system investigate the possible mechanical drive interactions.
- Develop the drive’s control procedure (energy-optimised control) which will benefit from the information regarding another drives in a system.
• The proposed control method should not require hardware modifications, while not compromising the production targets.

• Perform the simulation and experimental verification of the proposed control algorithm. The obtained energy efficiency improvement should be at least of 5%.

1.4 Thesis overview

The thesis is organised as follows. First, aims and objectives of the research are formulated. In Chapter 2, by a comprehensive literature review, the positioning system of the metal cutting machine tool is chosen as a target example of the indirectly mechanically coupled system suitable for the energy consumption reduction strategy development followed by the main control strategy idea. In Chapter 3 and 4, the conventional variational approach is implemented to find the optimal control strategy for the simplified system for open- and close-loop cases respectively. Then in Chapter 5 the model predictive control (MPC) is introduced to solve the optimization task. Two types of MPC for motor drives are considered: off-line, where the optimal reference state trajectory is identified and feed-forwarded to the conventional nested loop control and the combination of the off-line MPC for the reference trajectory and in Chapter 6 on-line MPC for the trajectory following task. The feasibility of these control strategies is checked through the simulations and experiment.
In Chapter 1 the large industrial sector contribution to the electrical energy consumption has been identified. Therefore, the electrical energy consumption reduction has been chosen as a target for the research within the Industry 4.0 framework. In the past, energy saving techniques were not actively developed by industry mainly due to the small electrical energy cost compared to the installation cost. However, these days governments and organizations are trying to reduce the energy consumption by imposing several regulations and initiatives like ECODESIGN Directive 2009/125/EC [24] adopted by EU which insists on energy labelling of industrial systems particular machine tools (MT) [25]. Under this directive CECIMO the European Machine Tool Builder association presented a self-regulatory initiative for the more efficient MT [26]. The CO₂PE! initiative analyses and improves the environmental footprint of manufacturing [27]. Several standards have been developed
as well. For example both ISO 14955-1:2017 “Machine tools — Environmental evaluation of machine tools” [28] and ISO 20140-1:2013 “Automation systems and integration - Evaluating energy efficiency and other factors of manufacturing systems that influence the environment” analyse the environmental impact of machine tools and develop the recommendations for its reduction.

It is possible to identify three major drivers for the more-efficient industrial energy consumption technologies development:

- Ecological issues;
- Electrical power demand;
- Electricity price.

Ecology is a major focus of many researchers, for example in China manufacturing industries are responsible for about 72% of carbon dioxide emissions [29], so the efficiency improvement is a clear way to prevent the climate change and preserve the environmental sustainability. Electrical power demand is an emerging factor as well. As reported in [29] in the last 30 years energy consumption has increased by more than 35%, with predicted continued growth especially in non-OECD countries so the development of energy saving technologies in manufacturing leading to the increase of their efficiency becomes an important issue. Increased power demand produce a pressure on power generation and transmission infrastructure especially in OECD countries with legacy infrastructure. As it was mentioned, for a long time, electricity prices have not been a priority for industry due to the high up-front/installation/capital cost of the equipment, however considering the long life-span, typically 10 years [30], [31], of MT and the increasing electricity price it starts to motivate energy saving procedures application.

2.1 Why is machining chosen as multi-drive optimization target?

In the introduction to this chapter several regulations and initiatives have been mentioned particular focused on MT. In this thesis metal-working MTs have been chosen as a target for the multi-drive optimization due to the number of reasons. First, according to [32] the third most significant, from the energy consumption point of view, subsector of manufacturing industry is metal processing, accounting for about 10% of energy consumption. Second, according to [33] 63% of total energy consumed by metal processing machinery in the EU accounts for a metal-working computer numeric control (CNC) machining which corresponds to the 410 PJ/year. Third, in order to be
able to increase the efficiency through drive optimization the specific energy consumption requirements (J/cm³) per material process rate (cm³/s) should be as small as possible for a particular MT type. In [34] the electricity requirements for different machining processes was compared and machining shows almost the best performance, Fig. 2.1. Aforementioned factors make CNC metal-working machine the best option to be chosen as an example system.

While developing energy efficient metal-cutting MT different levels of analysis
could be considered. In [35], a manufacturing analysis scale is placed against a temporal decision scale and five distinct levels of analysis are introduced: 1) Process control, 2) Microplanning, 3) Macroplanning, 4) Production planning & scheduling, 5) Supply chain/Enterprise asset management, Fig. 2.2. A similar approach based on splitting all energy minimization activities in five different categories based on manufacturing organization stages is introduced in [36]:

- Enterprise/global supply chain;
- Multi-factory system;
- Facility;
- Line/Cell/Multi-machine system;
- Device/unit process.

The latter classification approach is very helpful in reviewing the research contributions already made in a field. Classification starts from the top global level where the full supply chain, with all supporting infrastructure, is considered and goes down to the lowest level where a single MT performs the discrete part production. In all these optimization levels, energy saving through the motor drive control optimization could be achieved, for example, in [37] the “Facility” level optimization is done, where the purpose was to demonstrate the applicability of improved management practices to increase the efficiency of metal processing plants. The major focus was on water and chemical consumption reduction. The efficiency was increased by reducing the drag-out losses, rinsing water consumption and evaporation losses. All these measures require minor plant modifications. Also, water reuse was proposed in addition to obtaining it from ground water pumping; this leads to considerable (36%) electrical energy saving by eliminating the necessity of continuous water pumping. The water pump is powered by an electric motor, so it is possible to say that the energy consumption is reduced through the drive control optimization (shortening the run-time).

In [36] and [38] the superiority of optimization on higher level of organization strategies is shown, however in many developed countries more than 90% of all manufacturing companies are Small and Medium Enterprises (SME) [39] and higher level optimization is not applicable for them. Moreover “Factory”, “Multi-factory” and “Enterprise” levels of optimization require confidential information regarding the production processes which are generally not available. Also, the motor drive optimization on such optimization levels would usually apply only the shortening of the
run-time strategies, as it was in the aforementioned example, which has nothing to do with the topic of this research where the energy consumption supposed to be reduced by considering the interaction between drives in a multi-drive system.

Due to the nature of multi-drive systems research, only the “Device/Unit process” optimization level is considered, in other words, the reduction of energy consumption strategies in a single MT are studied.

2.2 Classification of low level energy-saving methods

In the literature, a huge number of different optimization approaches have been presented within the “Device/Unit process” optimization level. For a better understanding of these methods several different classification strategies have been developed. In [40] it is suggested to split all the methods into two groups: 1) Reduction of the component-power demand, 2) Reduction of power integration time. The first group of methods is related to the hardware design optimization, where more efficient devices are used in the MT. For example, more efficient motors could be used as pumps or variable speed drives (VSD) could be adopted for a MT fans. Changes to cutting technology methods are included in this group as well. For example, Laser-Assisted Machining (LAM) could considerably reduce the power consumption of a spindle motor drive by softening of the work-piece using a laser. The second group of methods is either reducing the idle/stand-by times or shutting down the devices which are not continually in use. However, such classification is not comprehensive and doesn’t directly consider, for example, methods of energy optimization by modification of process parameters.

Similar to [40] classification has been suggested in [13]. The authors split all the optimization methods into 3 groups: 1) Optimized machine tool design, 2) Optimized process control, 3) Process/machine tool selection. Such an approach widens the methods covered compare to the previous one, plus it introduces the new group of methods, where for example efficiency could be increased by proper sizing of the MT capacity.

The most complete classification of MT energy saving methods is given in [12]. The hierarchical approach with the six stages is introduced in Fig. 2.3:

- Assessment and modelling;
- Software-based optimization;
- Control improvement;
• Cutting improvement;
• Hardware based optimization;
• Design for environment.

The stages are ordered considering the effort for the supposed energy minimization methods implementation. Strictly speaking, the first group has nothing to do with the energy reduction. It covers the energy consumption monitoring methods, which is a mandatory stage in the decision-making process, where the most energy demanding process could be defined. The last three stages are mostly considering hardware modifications of MT and do not deal with the motor drive control optimization. “Software based optimization” and “Control improvement” are the group of strategies of interest, where the MT energy consumption is reduced by the modified control of motor drives. Further different approaches from mostly the second and third stages introduced in literature are set out in some details below.

In [41] the aforementioned energy monitoring procedure for machine tools is presented. In this paper it is confirmed that in real machining a very small amount of the overall power consumption is used for the material removing (2-8%), while most of the power is consumed in idle mode, highlighting the importance of idle time reduction in manufacturing planning. Moreover, high power consumption of coolant pumps, controllers and tool changers was also revealed, proving the importance of process time minimization. Nine machining centres were studied and in order to compare their power...
consumption a standardized start-up procedure, spindle and axis move profiles together with the standard test piece were developed. [41] Proposes standardized energy consumption tests to enable energy labelling of machine tools which could potentially boost competition between the machine tool manufacturers on the energy efficiency maximization.

[42] Doesn’t directly deals with the MT, but shows the improved control strategy for a positioning system, which is the part of MT. Authors describe the difference between the contouring and tracking control of multi-axis systems, where the axis’s linear actuators are driven by electric motors. The end-effector should follow some predetermined geometrical path. If the path is time-dependent than it is a tracking control, whereas if the path is not time-dependent than it is a contouring control. Therefore in contouring control one additional degree of freedom appears – the path-following speed, moreover the positioning error is more naturally defined in a contouring control as a minimum distance between the desired path and current position than in tracking control, where each axis positioning error is minimized independently from each other. This paper shows, that it is possible to reduce the positioning system operation time by increasing the path following speed while maintaining the positioning accuracy.

[43] Is one of a series of papers from the Toyohashi University of Technology where the energy efficiency of 2D, 3D and 5D machine tools is improved. It is also done using the contouring control, where in contrast to the conventional tracking error minimization of each feed drive, the contour error is minimized. By introducing such an error, [43] shows that it is possible to reduce the controller bandwidth while keeping the performance or positioning accuracy at the same level. The reduction of controller bandwidth leads to the reduction of higher frequency noise component and hence to the reduction of electric power consumption.

In [44] a novel hybrid feed drive for each positioning system axis is introduced. The authors propose to use two different drives for a single axis control. One drive is a linear motor drive (LMD) which is suitable for higher speed operation and able to achieve higher accelerations, whereas the other one is a screw drive (SD) driven by a brushless DC motor which has higher efficiency at low speeds and better dynamic stiffness. They present a mechanism to connect and disconnect the SD from the axis. The LMD is used for off-cut operations without the SD, while during the in-cut mode, the axis is powered primarily by SD and LMD is used for vibration compensation and
cutting errors minimization. By doing so, it is possible to reduce the overall system power consumption by reducing the auxiliary system operation time during the off-cut mode without compromising the performance, and even improving it due to the reduced vibration. The obvious drawback of such approach is the necessity of MT hardware modification with bulky and not very reliable toggle mechanism for SD connection/disconnection.

In [45] a method of industrial manipulator control reducing the energy consumption is presented. It is based on developing optimal energy trajectory using the predefined end-effector time-independent path with the defined end time. This trajectory is obtained by describing the manipulator through the Euler-Lagrange equations, with limits on joint torques. Further, this description is modified through the nonlinear variable transformation in order to obtained a second-order cone program (SOCP) (a type of convex optimal control problem) which is solved using the open-access tool YALMIP. It is shown that by compromising the trajectory duration (rise of end time) it is possible to considerably reduce the energy consumption compared to the time optimal case. For a given example, the 20% end time rise results in 65% energy reduction. The major drawback of the method is the necessity to solve the SOCP online, which could be too difficult for the modern DSPs used for a manipulator control. Also, the method does not consider the joint motors dynamics, reducing the accuracy of the optimization results.

Not only university researchers are working on improving the energy efficiency of machine tools. In [46] the conclusions made by Mori Seiki Co., Ltd. a leading Japanese machine tools manufacturer are presented. Three different measures are presented in this paper; the first is related to the modification of cutting conditions and the last two deals with the energy consumption minimization in specific operation instances. The comparison of the conventional efficiency method with the proposed methods is done using a performance index similar to the Specific Energy Consumption (SEC) and depends on energy consumption per material removal rate. Proposed improvement measures are supported by a huge number of experiments. Obviously by increasing the feed rate and cutting speed the performance index is improved (more material is removed in a shorter time leading to the reduced cycle time, and less energy is consumed by auxiliary power consumers), however the authors show that such harsh conditions lead to unacceptable levels of tool wear, so the feed rate and cutting speed should be reduced in order to determine the optimal cutting conditions acceptable for
MT users. A second performance improvement method is related to deep hole drilling operations. The possibility to reduce the power consumption by using an adaptive pecking drilling technique is shown. In this method, in contrary to the fixed position pecking, the drill is not extracted regularly, but only when the torque on the spindle drive exceeds a certain level due to chip accumulation in a hole. This method reduces the number of extraction events and consequently cycle time, so energy consumption reduction of around 20% was reported. The last case studied is related to the situation when a feed drive needs to move the tool from, for example, tool change position to the machining position. This is done in a rapid traverse mode. However, in order to start machining, the spindle should accelerate to the rated speed and the time of this acceleration for high-speed cutting could far exceed the time required by the positioning system. So authors suggest to reduce the traverse speed in order to match the axis travel time and spindle acceleration time. Applying this control strategy, 10% energy consumption reduction was reported.

In [47] the specific case of redundant actuation in machine tools is studied. It is shown that for the machine tools with more than a single actuator (motor) feeding the single axis, it is possible to reduce the energy consumption of the overall positioning system following the given end-effector path by smart assignment of the motion distance for each actuator. This motion distance is determined by the solution of the optimization task, where the motor drives loss minimization (including both copper and iron loss) is taken as a cost function. Optimal control theory or more specifically calculus of variations is used to solve the optimization task. No experiment verification has been done, however, simulation reveals promising results in energy consumption reduction.

The research presented in [48] is motivated by the fact that typically MTs are in production for only around 70% of time, whereas during the unproductive times the MT is not switched off [49]. As it is stated by many authors [41], [31], [14] in most cases the stand-by energy consumption is dominant and reduction of it could considerably reduce the overall energy consumption of MT. Authors in [48] separate the operation cycle of MT into 5 groups or states: production, warm-up, energy-saving mode [40], machine ready, machine off. The example of running equipment at each state is shown on Fig. 2.4. The energy consumption is reduced by modifying the state trajectory, i.e. the transition between the states of the MT considering the technological constraints and required piece quality. It is suggested to use the Dijkstra algorithm for the optimal state
path modification, where not only the each state’s power consumption is given in each node but also the transition times between the states are indicated on graph edges. Different boundary and supplementary conditions (for example the equality of the start and desired states) could be considered. The experimental verification shows the 5% energy consumption reduction for the 5kW milling machine working under the typical industrial scenario. However in order to implement the suggested strategy the knowledge or an estimation of MT stand-by time should be available in order not to sacrifice its performance.

In [50] the parallel kinematic machine (PKM) is taken as an example for energy optimization. It relies on low-level optimization of auxiliary machine functions and doesn’t take in consideration the energy required for actual material removing during the drilling action. At first, the energy consumption of PKM modelled with respect to the machine tool operation based on Euler-Lagrange equations is derived. This model is time based and represents the torques and forces of the joint actuators. The model is redundant so the possibility to reduce the energy consumption in each particular moment through the tool posture is shown. The experimental verification proves the overall system 6.6%-9.9% energy consumption reduction which involves no machining quality degradation or any hardware modifications. The presented idea is similar to the energy consumption reduction technique for the legged walking robots presented in [51].

In [52] the influence of end-effector speed and acceleration on the robot-manipulator energy consumption was studied. The total energy consumed by manipulator was considered as comprising of three parts.
The energy associated with the through the air movement of the point mass against the gravity force in a form $dE = Fv dt$, where $E$ is the energy required for such movement, $F$ is the required force, $v$ is the movement speed and $t$ is the time of operation.

The energy associated with the energy consumption of the electrical components of manipulator, such as control unit or auxiliary power consumers. The power consumption of electrical components is assumed to be constant, i.e. energy consumption of these components is a linear function of manipulator operation time.

The energy related to the friction force of ambience obtained through the Bernoulli equation.

It is shown that it is possible to identify the optimal manipulator speed for the linear steady movement which minimizes the energy consumption or in other words the optimal operation time for a single cycle of operation could be deduced, which depends only on travel distance, friction coefficient and auxiliary power consumption. Optimal acceleration for the steady accelerated movement could be deduced as well. The theoretical trends in energy consumption obtained for the steady linear and accelerated movements were checked through experiment with the help of IRB 4400/60 robot. It was revealed that at speeds of around 35-40% and at accelerations of around 30% from the maximum, the robot consumes the smallest amount of energy for the fixed movement distance. However, these results are not easily applicable in practice. Authors show the dependency of the consumed energy on the speed and acceleration, but do not provide the method to predict the optimal speed and acceleration values. Moreover, even if these speed and acceleration are experimentally obtained, they will change with manipulator posture and load surface area.

In [30] a MT system model is constructed which predicts the power and energy profiles based on the MT process information. With the help of this model two options to reduce the electrical power consumption have been presented:

- Braking energy storage, where a battery pack is introduced to store the recuperation energy from spindle and axes drives. This system reduces the peak power requirements and allows power converter to be used with reduced input power and higher efficiency.
• Reactive power compensation system installed in the input of a grid side converter, which reduces the reactive power consumption. These measures were analysed not only from the energy efficiency perspective but also considering life-cycle costing. With the average life span of MT center of 10 years [53], [31] braking energy system appeared to be not advantageous due to the high cost of energy storage system, whereas the reactive power compensation appeared to both economically profitable and eco-friendly. However, as authors also mentioned, these calculations are done considering current (2012) Germany electricity energy storage prices. The major disadvantage of the proposed methods is the requirement to introduce hardware modifications to the existing MT, which is not always possible.

In [54] the control of a single actuator in an electro-mechanical system is optimized from the energy efficiency point of view is proposed. At first, for a given duration of motion, with no restrictions on the motion speed the optimal point-to-point motion profile is derived applying Pontryagin’s minimum principle. Then with the restrictions on maximum speed and for the same motion time as in the original case, the modified parabolic velocity profile is deduced and compared with conventional variable-rate and exponential profiles. Easy-to-use expression for the required input energy are deduced for all three studied cases and the superiority in the energy consumption of the proposed one is shown. However, the profile evaluation strategy considers fixed final/motion time and fixed maximum speed, letting the acceleration be arbitrary. As a result, the proposed strategy has a higher peak acceleration than, for example, a simple trapezoidal velocity profile. In the electromechanical systems, acceleration is generally proportional to the motor current, which has a fixed maximum value. So in order to compare different motion profiles, maximum speed and acceleration should be fixed, whereas final time is free.

2.3 MT inventorization

In the previous section the overview of some existing measures to improve the efficiency of MT on the device level was presented. In order to develop the new energy saving strategy which will consider the multi-drives nature of a MT it is possible to use the methodology described in [30] and in CO2PE! framework [27], [36], where the so called life cycle assessment (LCA) is described. Further the procedure inspired by LCA is used to identify the promising area for the multi-drive energy optimization applied for MT.
At first, the goal and scope of the optimization should be defined. The current research topic is multi-drive optimization, so the electrical system operation of a single MT unit is considered and the main purpose is to reduce the electrical energy consumption. With such scope and goal choices it is important to consider that the energy consumed in raw materials preparation (aluminium or steel production), cutting fluid preparation, tool preparation, cleaning, transportation and disposal are omitted, though for example as it is shown in [31] that the aluminium smelting uses 40 to 120 times more energy than its processing (material removing) in a MT. In [55] it is shown that the operation cost dominates over the installation cost in a long run, though it is not always true. The installation/up-front cost of the MT has the lowest priority in current research.

When the scope is defined, the process inventorisation should be performed. The electrical energy consumption could be identified by integrating the MT power consumption over the predefined time period. Moreover, it is important to define the energy consumption in each electrical component of MT [30] to develop the energy reduction strategy accordingly. It is significant because a little power consumption reduction of the most-energy consuming component will considerably reduce the overall energy consumption.

It is possible to calculate the theoretical value of energy required for a particular material removal rate (MRR). Table 2.1 summarizes the approximate theoretical energy requirements for cutting different materials, the material hardness is also included. However, such an energy consumption approach will generally give an underestimated value, due to the neglect of all auxiliary processes’ energy consumption of MT. Moreover it shows only the total energy consumption, but some decomposition is required to perform the multi-drive optimization.

Table 2.1 Energy requirement for different material cutting. Redrawn from [8]

<table>
<thead>
<tr>
<th>Material</th>
<th>Specific energy, [Ws/mm$^3$]</th>
<th>Brinell hardness, [HB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnesium alloys</td>
<td>0.4-0.6</td>
<td>50-90</td>
</tr>
<tr>
<td>Aluminium alloys</td>
<td>0.4-1.1</td>
<td>40-130</td>
</tr>
<tr>
<td>Copper alloys</td>
<td>1.4-3.3</td>
<td>50-420</td>
</tr>
<tr>
<td>Cast irons</td>
<td>1.6-5.5</td>
<td>130-450</td>
</tr>
<tr>
<td>Steels</td>
<td>2.7-9.3</td>
<td>120-650</td>
</tr>
<tr>
<td>Titanium alloys</td>
<td>3.0-4.1</td>
<td>-</td>
</tr>
<tr>
<td>High-temperature alloys</td>
<td>3.3-8.5</td>
<td>-</td>
</tr>
<tr>
<td>Nickel alloys</td>
<td>4.9-6.8</td>
<td>110-380</td>
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According to ISO 14955-1 [28] a MT diagram describing the functionality of each system with the associated power consumption could be the first step for the process inventorisation. On this diagram the processes, technology and design are not considered. On Fig. 2.5 the example of such diagram is presented, for the energy flow of a milling centre. It could be noted that MT servo drive unit is the highest power consumer and among these units the axis servo drives are the most powerful part. The obvious drawback of such an approach is that it is relying on the datasheet power, which is either rated or maximum due to the physical limitations. Due to the varied nature of MT manufacturing, steady operation in the rated power regime is not likely; MT power

![Milling centre energy flow diagram](image)

Fig. 2.5 Milling centre energy flow diagram [30]
demand is very dynamic. So the energy consumption could not be accurately estimated purely based on MT wiring diagram, though this approach is suggested in [40] for an approximate energy consumption evaluation.

In the general case single power and energy flow diagram is insufficient. The most precise way to determine the energy consumption distribution among the MT systems would be connecting the power meter to the each component and integrate the resultant power over the operation time to determine the energy consumption. However, such an approach is hardly applicable in practice due to the equipment limitations and impossibility of direct energy consumption measurement in some systems, for example, due to the absence of available measuring point. Moreover, in the general case the total energy consumption is product specific so the energy distribution between the axis drives and spindle drive could vary according to the exact manufacturing process (for example for drilling and milling cases) and components produced. Also, the operation time influences the energy consumption a lot.

The most widely used approach for the MT process inventorization is splitting the consumed energy into groups considering the particular MT operation. One of the earliest energy assessment and modelling analysis is performed by T.G. Gutowski, et al. [31], where the total energy consumed by typical MT was split into two parts:

1) Machining;
2) Constant.

The former represents the actual energy needed for the material removing and the latter, the energy consumed by the auxiliary equipment (cutting fluid handling equipment, workpiece handling equipment, chip handling equipment, tool changers, computers and so on). By measuring the overall MT energy consumption during the cutting test it was revealed that only around 15% of energy is used for the actual machining, whereas all other energy is used by the auxiliary equipment. Moreover, the trend for MT is to move to more complicated systems with an increased number of auxiliary components and to increase their portion in the overall energy consumption.

Further, in [14] the total energy consumption of MT was studied considering three different “states”:

1) Basic;
2) Ready;
3) Cutting.
The “basic state” describes the energy required for MT activation to bring it to the level, when it could perform the cutting action. Start-up, computers, cooling fans, lighting and etc. are included here. The “ready state” describes the energy required to bring the spindle to the cutting position and accelerate it to the necessary speed, run the chip conveyor and tool changer. “Cutting state” represents the actual energy required for cutting. Comparing to [31] both “machining energy” and “cutting” are describing the energy consumption of the same nature, whereas in fact in [14] the “constant energy” from [31] is studied in more details by splitting it into two categories “basic” and “ready”. The summation of these three states gives the total energy consumption by a MT. Based on such analysis, the energy consumption model has been presented and validated through the experiment. A power meter was used to measure the consumed energy of three different MT: 1) MHP CNC Lathe, 2) MAC-V2 Takisawa Milling Machine 3) Mikron HSM 400 High Speed Milling Centre. By performing the experiments without actual metal cutting (“cutting energy is zero”) the authors were able to compare the “basic” and “ready energy consumption” and have shown the dominance of the former one, however, the difference is not huge, slightly more than 20% in average. Based on the results the areas for the improvement were identified as: machine start-up, spindle acceleration/deceleration, spindle movement, pumping.

In [40] a similar approach for the energy estimation is presented. It is based on MT power measurement and splits the results into four categories:

1) Fixed power (power consumption of components responsible for the MT readiness, for example, computers, fluid and cooling pumps);
2) Operational power (power consumption of components in off-cut/air-cut mode, for example, axis drives and spindle rotation);
3) Tool tip power (power required for a material removing, i.e. spindle and axis drives);
4) Unproductive power (heat generation during the cutting operation).

Further, the energy consumption is estimated based on integration of each category power demand. Authors are dealing with the constant energy consumption, which includes the “fixed” and “operational” powers only, two other categories “tool tip” and “unproductive” powers are beyond the scope of their research. Using six different machine tools the dominance of hydraulic and coolant/lubrication systems’ energy consumption was revealed experimentally for the average case, though the servo drives
and auxiliary systems are also shown to be important with the 17% and 19% of energy consumption respectively.

Aforementioned classifications are all very similar to each other, whereas each new classification specifies some part of the previous one. Thus in [14] the “Constant” energy from [31] is split into two parts: “Basic” and “Ready”. Further on in [40] the “Cutting” energy is composed by two parts: “Tool tip power” and “Unproductive power”. Other energy decomposition methods available in literature do more or less the same by introducing some more details in these decompositions. For the purpose of multi-drive energy minimization the most convenient one is presented in [56], where the energy is decomposed into three groups:

1) Basic energy;
2) Momentum energy;
   a) $E_{stage}$
   b) $E_{spindle}$
3) Machining energy.

These three groups are in fact mimicking the [14] classification, the only difference is “Momentum” energy, which is split into two elements: $E_{stage}$ representing the spindle/table movement and $E_{spindle}$ representing the spindle acceleration/deceleration and off-cut operation. This is more convenient due to the motor drive oriented nature of the current research.

All the mentioned schemes rely on the direct power/energy measurements at MT terminals. The only difference is in the results processing and evaluation. In the literature some other methods for the energy consumption estimation are presented. In [57] the authors are still performing the direct energy measurements, but the results are evaluated automatically, using an event streaming technique. The importance of the automation of energy consumption analysis is rising with the MT’s complexity. In [58], [59] the energy consumption estimation approach, which is using neither energy nor power measurement, is shown. It is based on numerical control (NC) programs analysis, since these NC codes contained all the detailed information regarding the MT operation. However, for both measurement and analysis the trend in investigating the most energy demanding processes is maintained.

To conclude a MT inventorization, the method based on splitting the consumed energy into groups representing the MT operation is chosen and groups classification
from [56] is adopted. According to it, “Basic” energy is the dominant part of overall energy consumption according to the majority of researches. It is independent on particular process and depends only on MT operation time. There are two ways to reduce the “Basic” energy: either by hardware modification, which is beyond the scope of this research, or by reducing this operation time by either minimizing the idle time or increasing the process rate (rise of a spindle speed). The idle time is beyond the scope of this research since it deals with scheduling, not motor drive control optimization. The process rate results in increased tool wear, which is more expensive than electrical energy in most cases [60], so the overall manufacturing cost is increased. Moreover increased process rate results in a reduction of product quality, so the spindle speed must be carefully chosen.

“Machining” energy is reported to have a little impact on overall energy consumption. It could be as small as only 1% [56] for micro drilling, though most researchers report around 15% value. This energy could also be reduced by increasing the spindle speed, which results not only in process time reduction, but also in torque reduction or by proper choice of mutual position of tool and product during the machining. However, these measures are also beyond the scope of multi-drive optimization, since they again affect production quality.

“Momentum” energy reflects the movements in the MT, either axis movements or the spindle rotation. This energy could be even larger than “Basic” for a specific cases [56] (58% vs. 41%), but generally it has a smaller, but still very considerable value. The movements in the MT are powered by the electrical motor drives, which perform the common production task, making the “Momentum” energy consumption reduction a promising area for the multi-drive performance optimization research. However, both spindle rotation and positioning system (axis) are included in this group. According to [31] the positioning system during the off-cut operations consumes around 15% of total energy consumption, i.e. approximately the same amount of energy as material removing operations making it an ideal candidate for multi-drive optimization with the focus on energy saving.

2.4 Research area. Research proposition

This thesis addresses positioning system optimization for multi-drive machine tools. According to the available research in literature the positioning system in the typical MT during off-cut operation accounts for around 15% of total MT energy
consumption. It was shown earlier that metal cutting MTs account for around 63% of all machinery tools. Considering the huge contribution of industrial electrical energy consumption within the total electrical energy consumption (up to 70% in China), the energy optimization of positioning system is an important area for the research.

In order to identify the particular strategy to optimize the positioning subsystem of the MT, the information regarding the operation of NMV8000 and NT5400 by DMG MORI Company has been obtained from the Advanced Manufacturing Research Centre (AMRC), Sheffield.

To verify the contribution of the positioning system to total energy consumption the processes inventorization following the ISO 14955-1 has been performed. On Fig. 2.6 the high level wiring diagram with the rated power consumption of each block of the NT5400’s MT is shown. In a contrast to [30] here the servo-drive unit (servomotors) is not the highest power consumer, but still the considerable amount of power is consumed by it (up to around 40%). It could be seen that as well as powerful servo- and spindle motors there are a lot of uncontrolled AC motors (coolant and oil pressure pumps), most of them are always in operation during the work in the presented MT. This correlates with the discussion in [14], [31], [41] where it was stated that stand-by energy consumption is dominant and that by reducing the machining time, energy consumption is also reduced, due to the decrease of operational time of these auxiliary energy consumers. Moreover for MT’s the production throughput is a priority for the manufacturer to maximize return on investment of high value equipment. Due to the aforementioned factors, the restriction on positioning system optimization appears: The operation time \textit{(time to move the end-effector from the initial to the final point)} must not increase.

For illustrative purposes, the NMV8000 vertical machining centre is presented on Fig. 2.7, where the positioning system moves its spindle motor in three different directions x, y and z. And control system for the positioning system of this MT could be split on three distinct layers (lowest to highest):

1. Reactive layer $\rightarrow$ Stabilization, trajectory following tasks;
2. Tactical layer $\rightarrow$ Trajectory planning;
3. Strategical layer $\rightarrow$ Behaviour plan.
Chapter 2 Motivation

Fig. 2.6 NT5400 Machining centre electrical circuit diagram
On the highest strategical layer it is decided what the machining centre is intended to do (for example make a hole with the given geometry). On the lower tactical level tool trajectory is generated in order to be able to perform the desired task (in an aforementioned example, movement trajectory of a spindle motor is generated, to bring it to the desired point). On the lowest reactive level control commands for servo motors are generated for tool to follow the trajectory. Due to such an approach, the motor is considered as a “black box”, receiving the control commands and performing the desired movement. According to [62] control requirements for a servo drive are limited to fast response time, zero overshoot, smooth transient response and minimum steady state error, but these can conflict, so the importance of each depends on the application. This is done by precise tuning of a servo motor PI control system, which is designed to follow the sophisticated speed profile [63].

However, in the aforementioned control procedure, the interaction between motors is not considered, whereas servo motors controlling the \(x\)-axis and \(y\)-axis positions (two-dimensional problem is studied for simplicity) are performing the common task opening there a room for energy consumption improvement.

This research proposes to minimise the off-cut energy use of the positioning system comprising of two servo motors. A lot research is done [64],[65] to optimize the trajectory of tool path or making the motor to follow it precisely. However, it matters only when the spindle motor is loaded, i.e. the machine tool is performing some action.
and cutting metal. Whilst cutting, the movement affects the quality of cut, which is beyond the scope of this research. During the overall operation of MTs, for a huge portion of time spindle motors are unloaded (this is supported by the data received from ARMC, where for particular example the spindle motors are unloaded for around 58% of time also similar conclusions are presented in [14]). In this time machining centre perform some positioning job, for example moving the spindle to the position where a new hole is going to be made. In this case precise following of the linear trajectory is not required and **it is possible to control servo motor which is intended to path longer distance in a time optimal way, while a servo motor with a shorter required path could be controlled in an energy efficient way.** So this research considers point-to-point operation (off-cut movements) with no restriction on geometrical path. It can also be applied to 3D-printers, robots etc. If the proposed control strategy is implemented the overall operation time would not be changed, however, the overall energy consumption would be reduced. The proposed control principle could be readily applicable for the greater number of motors, whereas in this case the single motor with the longest movement path should be controlled in the optimal time way and all others are controlled in the optimal energy way.

The spindle movement across the x- and y-axes of NMV8000 is presented on Fig. 2.8. On this figure only the off-cut positioning operation is presented, i.e. from the overall operational profile all the periods with the active spindle (i.e. spindle motor

![Fig. 2.8 x, y-axes position dynamics, with no spindle motor torque](image-url)
torque not equal to zero) are omitted. It could be seen that frequently during the operation, $x$- and $y$-axes servo motors are simultaneously in use, although the distances they cover are different. For such cases the optimization process is described in this report.

The idea of controlling positioning servos in a different way considering their required operation time correlates to the one, presented in [46], where it was suggested to reduce the axis speed, while the spindle is accelerating. However, the control principle described here is wider and deals not only with the maximum speed reduction, but with the whole drive trajectory modification. In addition, the method in [46] deals only with the spindle and servo-drive, whereas this thesis proposes a drive control which is scalable and could be applied for the whole MT.
Chapter 3 Optimal control theory. Variational approach

In the previous chapter the concept of the optimal from the energy point of view control of the MT’s positioning system has been introduced. For simplicity in this thesis only the 2D case will be studied. In this chapter the assumptions and limitations of the suggested control will be discussed in detail. The proposed control approach is based on controlling the linear actuator’s motor drives in two different modes: optimal time (OT), for a drive with the longer reference path, and optimal energy (OE), for a drive with the shorter reference path, given that they must finish their work simultaneously. In this chapter, some assumptions and possible model simplifications are described in sections 3.1 and 3.2. Then some basic definitions and mathematical description of the optimal control problem are presented for a general form linear time-invariant system, section
3.3. Further the motor is modelled in a double integrator form and the open-loop optimal control solution is obtained for such a case, section 3.4. Finally, to test the theoretical results, the variational approach is used in control of a six-legged mobile robot, CORIN, where the target is to reduce the energy consumption, section 3.5.

3.1 Assumptions

In this work several assumptions and simplifications are made. First, the study focuses on the 2D case only, i.e. the energy consumption reduction of a positioning system on the x-y plane is considered. However, the approach is readily applicable for a higher number of motor drives working simultaneously. In this case the drive, whose task requires the longest operational time is controlled in the optimal time mode, whereas all other drives are controlled in optimal energy mode, given that they finish their move at the same time as the first one.

Another assumption is that electric drives are comprised of equal electric motors, connected to the AC grid through the separate inverters. The drives are coupled through the end-effector trajectory and the principal diagram of such connection is shown on Fig. 3.1.

Losses in the electrical drive have different physical causes and depend on drive’s torque (current), speed (frequency) and mechanical characteristics of the feed drive mechanism. For the sake of simplicity the loss associated with the torque generation, i.e. copper loss is assumed to be dominant as this is generally true for MT drives. Torque developed by the servo drive can be expressed as in (3.1).

$$T_e = J_e \frac{d\omega_{rm}}{dt} + B_f \omega_{rm} + \left( T_{gf} + T_{lf} \right),$$  \hspace{1cm} (3.1)
where $B_{fr}$ is a viscous friction coefficient, $\omega_{rm}$ – mechanical speed. In the positioning system, the angular motion of the servo drive is transferred to the linear motion. For illustrative purposes, the diagram of a guideway with a ball screw drive used to realize such transfer in NMV8000 is shown on Fig. 3.2. Following the discussion in [62] all the loads which the feed drive faces (required torque) are either static or dynamic. The first, considering only positioning action, without cutting, comprises of friction in the slideways ($T_{gf}$), and in the bearings ($T_{lf}$). The friction in plain lubricated guideways could be big. The friction in the bearings is caused by the preload force, which is required to suppress backlash. The second types of loads come from the acceleration torque during the speed variations. It is proportional to the speed change and the inertia of the shaft ($J_e$). The servo motor is chosen to be able to deliver torque covering both these loads. Neither static load nor dynamic inertia depend on the motor drive speed [66], so both could be modelled as a constants, static load – as a constant load torque, dynamic load – as a constant inertia. In this thesis only the dynamic load and the viscous friction are considered in optimal control generation. This is mainly because of the experimental facility properties, where the feed drive mechanism is represented by a separate motor. However this assumption doesn’t influence the final result, because the static torque model could be simply added to the system model and control algorithm would take it into account.

In the feed servo drive an encoder is used to identify the rotor position. It is considered that the coupling between the motor drive and load has an infinite stiffness, so by knowing the geometry of a system it is possible to directly and exactly match the linear motion distance to the encoder count.

![Fig. 3.2 NMV8000 feed drive mechanism [61]](image-url)
3.2 Single drive optimization potential

To control the positioning system the mathematical model representing its dynamical behaviour should be known; particularly the influence of one axis movement on another axis movement should be deduced. To do so the Lagrange-Euler formulation, which is commonly used to cover a manipulator dynamics is used. The derivation follows a procedure similar to [67], where first kinetic and potential energy equations are derived and then substituted to the Lagrange’s equation.

A typical positioning system and its geometry are shown in Fig. 3.3(a). Assuming that the system has holonomic constraints and there is no energy dissipation for this system, the Lagrange’s equation of motion is formulated as following.

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau, \tag{3.2}
\]

where \( q \) is an \( n \)-dimentional vector of generalized coordinates (joint angles in robotics or \( x,y \) coordinate changes for the system of interest), \( \tau \) is an \( n \)-dimensional vector of generalized forces and \( L \) is a Lagrangian, which represents the difference between the kinetic (\( K \)) and potential (\( P \)) energies. Variables representing vectors are shown in bold and defined in (3.3) as 2-dimensional vectors, for the 2D system (\( n=2 \)).

\[
q = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix}, \quad \tau = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{3.3}
\]

The 2D positioning system is located on a surface, it’s diagram is on Fig. 3.3(b), therefore there is no change of its potential energy and only the kinetic energy describes the movement of the end-effector. If \( r = \sqrt{r_1^2 + r_2^2} \) is the distance of end-effector displacement on \( x-y \) plane and \( m_1,m_2 \) are feed drives masses, the kinetic energy of

Fig. 3.3 Positioning system: a) FESTO solution [68] b) 2D system diagram.
such linear motion is given as follows.

\[ K = \frac{1}{2} (m_1 + m_2) \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 \]  \hspace{1cm} (3.4)

Using (3.4) the Lagrangian is derived in (3.5).

\[ L = K - P = K = \frac{1}{2} (m_1 + m_2) \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 \]  \hspace{1cm} (3.5)

By substituting (3.5) to the Lagrange’s equation of motion (3.2) one will obtain the following:

\[ \frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{r}_1} \\ \frac{\partial L}{\partial \dot{r}_2} \end{bmatrix} = \begin{bmatrix} (m_1 + m_2) \dot{r}_1 \\ m_2 \dot{r}_2 \end{bmatrix} \]  \hspace{1cm} (3.6)

\[ \frac{\partial L}{\partial q} = 0 \]  \hspace{1cm} (3.7)

Finally, the equations describing the dynamics of the positioning system are formulated in (3.8).

\[ \begin{cases} (m_1 + m_2) \ddot{r}_1 = f_1 \\ m_2 \ddot{r}_2 = f_2 \end{cases} \]  \hspace{1cm} (3.8)

From (3.8) it is clear that the dynamics of two different axes are independent from each other and there is no linking between them through inertia, Coriolis or gravity components of the Euler-Lagrange equation. This result complies with the discussion in [47], where the completely decoupled behavior of the two motion axes of an x-y table are shown. Due to this, the optimization task for each axis (OT and OE) could be solved independently.

### 3.3 Optimal control

In order to find optimal control of a servo drive (in either energy or time sense) general control theory methods may be applied. In this section the major control theory equations will be presented following [69]. Here the optimal tracing problem solution will be discussed, where for the reference signal \( r(t) = X \) (required final drive position) and \( t \geq 0 \) controller should provide that output \( y(t) \) approaches \( r(t) = X \) for the minimum time or using the minimum energy [70]. Optimal plant input \( u^*(t) \) is a stator voltage, or a PWM reference signal. The block diagram of such control for a linear plant is shown on Fig. 3.4, where the plant represents the dynamic model of a servo drive. Output \( y(t) \) is equivalent to one of the state functions \( x_i(t) \). An open loop solution will be considered first, whereas in the latter chapters the close-loop solution will be discussed.
Chapter 3 Optimal control theory. Variational approach

An automatic control system which optimises a chosen performance measure over a required movement of a plant is called optimal. Control \( u^* \) which provides such movement is called optimal control. The plant state trajectory during the optimal movement, \( x^*(t) \) is called the optimal state. Optimal movement implies movement of a plant from initial state \( x(t_0) = x_0 \) to the final state \( x(t_f) = x_f \).

The optimal control problem statement is to find an admissible control \( u^* \), which minimizes the performance measure whilst fulfilling any constraints. This can be presented in a mathematical form:

- Plant description – unconditional constraints/state equations
  \[
  \dot{x}_i(t) = f_i(x, u, t), \quad i = 1, 2, \ldots, n
  \]

- Boundary conditions
  \[
  x(t_0) = x_0; \quad x(t_f) = x_f
  \]

- Physical constraints
  \[
  u \in \Omega_u \text{ or } |u_j| \leq \beta_j, \quad j = 1, 2, \ldots, m
  \]
  \[
  x \in \Omega_x \text{ or } |x_i| \leq \alpha_i, \quad i = 1, 2, \ldots, n
  \]

- Joint constraints
  \[
  Q_z(x, u) \leq \gamma_z, \quad z = 1, 2, \ldots, k
  \]

- Optimization criteria/performance measure
  \[
  J(x, u)
  \]

Find \( u^*(t) : \arg \min_u \left\{ J(x, u) \mid \dot{x}_i(t) = f_i(x, u, t); x(t_0) = x_0; x(t_f) = x_f; u \in \Omega_u; x \in \Omega_x; Q_z(x, u) \leq \gamma_z \right\} \),

where \( J(x, u) \) is a functional, \( n \) is the number of state variables, \( m \) is the number of control variables, \( k \) is the number of joint constraints, \( f_i \) and \( Q_z \) are an arbitrary functions and \( \alpha_i, \beta_j, \gamma_z \) are an arbitrary constants.
Optimization problems may be classified by different parameters, further for illustrative purposes different approaches for the optimization problem classification are introduced.

A) By the formulation of optimization criteria:

1. Integral optimization criteria (Lagrange problem) \( J(x, u) = \int_{t_0}^{t_f} g(x, u, t) dt \);

2. Boundary conditions dependent (Mayer problem) \( J(x(t_0), x(t_f)) = h(x(t_0), x(t_f), t_0, t_f) \);

3. Bolza problem (combination of former two) \( J(x, u) = \int_{t_0}^{t_f} g(x, u, t) dt + h(x(t_0), x(t_f), t_0, t_f) \);

4. Minimum-time problem \( J_t = \int_{t_0}^{t_f} dt = T \to \min \)

B) By the form of boundary conditions:

1. With final state specified. \( t_f \) is given or not, \( x(t_0) = x_0 \) and \( x(t_f) = x_f \) are given.

2. With final state free. \( x(t_f) \) is arbitrary, \( t_f \) is specified or not.

3. With final state lying on the surface defined by \( \eta_l(x(t_f), t_f) = 0 \quad l = 1, 2, \ldots, p, \)

   where \( p \leq n \).

C) By how constraints are set:

1. Separate constraints on \( u \) and \( x \)

2. Joint constraints \( Q_j(x, u, t) = 0 \quad j = 1, 2, \ldots, k \)

3. Integral constraints \( Y_j = \int_{t_0}^{t_f} g(x, u, t) dt \leq \beta_j \quad j = 1, 2, \ldots, k \)

It is possible to describe the motor drive control tasks according to the classification presented. In Chapter 2 two ways to control the motor drive were considered. Time-optimal control: is a minimum time problem (A4); loss minimum control: is a Lagrange problem (A1). For both problems, the final state is specified (the drive should move the end-effector to the defined position and stop) and the final time \( t_f \) is arbitrary (B1). For both tasks constraints are separate for the state and the control (C1).

The conventional solutions of the aforementioned optimal control problems using the variational approach and Pontryagin’s theorem have been extensively studied in [69]. The key definitions, theorems and findings of [69] are summarised in Appendix 1.
3.4 Application of optimal control methods to the linear motion

As mentioned in Chapter 2 while controlling the servo motors in a machine tool, different cutting tool velocity profiles are used for each axis. These profiles represent the linear motion of an object. To start with, consider a situation when a constant force, $\vec{F}$, acts on a rigid body of mass, $m$, shown on Fig. 3.5. According to the Newton’s second law such force causes body acceleration, $\vec{a}$, at a constant rate. Omitting the friction the resultant acceleration could be represented as follows.

$$\vec{a} = \frac{\vec{F}}{m}$$  \hspace{1cm} (3.14)

Where the force $\vec{F}$ could be considered as a control action, or the torque developed by the axis’ servomotor. Therefore, as a preliminary stage of the optimal motor control, the optimal control of a second order linear motion system described by (3.15) and Fig. 3.6 is discussed in this section.

The state space representation of such system, where states $x_1$ and $x_2$ represent position ($x$) and velocity ($v$) respectively and control $u$ – is an acceleration ($a$), is shown in (3.15).

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u,
\end{align*}$$  \hspace{1cm} (3.15)

In a conventional matrix representation this system or plant is described as:

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u$$  \hspace{1cm} (3.16)

It is required to move the object from the initial position to some predefined distance $X$. Initial and final speeds are zero, so $x_1(t_0) = 0$, $x_2(t_0) = 0$, $x_1(t_f) = X$, $x_2(t_f) = 0$. For simplicity initial time $t_0$ is set as zero, final time $t_f$ is not specified.
Optimal minimum time and minimum energy solutions will be obtained for the unconstrained case, for the constrained control case (acceleration) and for the constrained control and second state (velocity). In a DC machine acceleration is assumed to be proportional to the armature current and armature copper loss is proportional to the square of the stator current. So for the optimal energy use, the performance measure is chosen as \( J_e = \frac{1}{2} \int_{t_0}^{t_f} u^2 \, dt \rightarrow \min \). The optimal time performance measure is conventional for minimum time problem \( J_t = \int_{t_0}^{t_f} dt \rightarrow \min \).

3.4.1 Unconstrained movement

In this section, these two optimal control tasks will be solved. For clarity, the full problem formulation for both optimal energy and optimal time control problems are given in Table 3.1 and Table 3.2 respectively. These optimisation tasks are solved based on variational approach, summarised in Appendix 1.

A) Minimum energy loss

Table 3.1 Unconstrained minimum energy problem

<table>
<thead>
<tr>
<th>State equations</th>
<th>Boundary conditions</th>
<th>Physical constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x}_1 = x_2 )</td>
<td>( x(t_0) = [0]_0 ), ( x(t_f) = [X]_0 )</td>
<td>( t_f ) - free</td>
</tr>
<tr>
<td>( \dot{x}_2 = u )</td>
<td>Final time ( t_f ) - free</td>
<td>-</td>
</tr>
</tbody>
</table>

In order to obtain the optimal control, which moves the system from initial state \((0; 0)\) to the final state \((X; 0)\) system of equations (A.1.6) should be solved. For the unconstrained minimum energy problem\(^1\)

**Hamiltonian function:**

\[
H(x(t), u(t), p(t)) = \frac{1}{2} u^2(t) + p_1(t)x_2(t) + p_2(t)u(t)
\]

**State equations:**

\[
\dot{x}_1^*(t) = \frac{\partial H}{\partial p_1} = x_2^*(t)
\]

\(^1\) For the variables definition refer to Appendix 1
\[
\dot{x}_2^\ast(t) = \frac{\partial H}{\partial p_2} = u^\ast(t)
\]

Costate equations:

\[
\begin{align*}
\dot{p}_1^\ast(t) &= - \frac{\partial H}{\partial x_1} = 0 \\
\dot{p}_2^\ast(t) &= - \frac{\partial H}{\partial x_2} = -p_1^\ast(t)
\end{align*}
\]

\[\Rightarrow \begin{align*}
p_1^\ast(t) &= c_1 \\
p_2^\ast(t) &= -c_1 t + c_2
\end{align*}\]

where \(c_1\) and \(c_2\) are arbitrary constants of integration

Algebraic equation (sometimes called control stationary equation):

\[u^\ast(t) + p_2^\ast(t) = 0\]

Simultaneous solving of state, costate and stationary equations results in the following system of equations for state trajectories

\[
\begin{align*}
x_1^\ast(t) &= \frac{c_1}{6} t^3 - \frac{c_2}{2} t^2 + c_3 t + c_4 \\
x_2^\ast(t) &= \frac{c_1}{2} t^2 - c_2 t + c_3
\end{align*}\]

and for the optimal control function

\[u^\ast(t) = c_1 t - c_2\]

Arbitrary constants \(c_1 \ldots c_4\) are found from the initial and final state values as

\[
[c_1, c_2, c_3, c_4]^T = \left[-\frac{12X}{t_f^3}, -\frac{6X}{t_f^2}, 0, 0\right]^T
\]

Substituting (3.19) into (3.18) the optimal control function minimizing the loss is obtained as (3.20).

\[u^\ast(t) = -\frac{12X}{t_f^3} t + \frac{6X}{t_f^2}\]

Final time \(t_f\) is not specified and it should be calculated by the equation (A.1.7), after substituting the values of \(c_1 \ldots c_4\) to the Hamiltonian function, algebraic equation (3.21) is obtained.

\[
H\left(x^\ast(t_f), u^\ast(t_f), p^\ast(t_f)\right) = 0 = \frac{1}{2} \left[u^\ast(t_f)\right]^2 - \frac{6X}{t_f^2} \left[-\frac{1}{2} \frac{12X}{t_f^3} t_f^2 + \frac{6X}{t_f^2} t_f\right] - u^\ast(t_f)u^\ast(t_f)
\]

From (3.21) the value of optimal control function at the final time, \(u^\ast(t_f)\), must be equal to zero, i.e. \(t_f \to \infty\). In other words, the longer the process, the lower are the losses, note that friction is not considered in (3.15). If the final time, \(t_f\), is specified the position-speed-acceleration profile is shown on Fig. 3.7.
B) Minimum time

Table 3.2 Unconstrained minimum time problem

<table>
<thead>
<tr>
<th>State equations</th>
<th>Boundary conditions</th>
<th>Physical constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x}_1 = x_2 )</td>
<td>( x(t_0) = [0 \ 0]^T, x(t_f) = [X \ 0]^T )</td>
<td>Final time ( t_f ) - free</td>
</tr>
</tbody>
</table>

The minimum time problem formulation is summarized in Table 3.2. However due to the unconstrained control, the solution is simply a delta function, \( u^*(t) = \delta(t) \), with the zero final time, \( t_f = 0 \).

3.4.2 Control constrained movement

In the previous section problems were solved with an unconstrained input, however as discussed in section A.1.1, there are always physical constraints imposed on a control vector. This section addresses a limit on acceleration \( A \) corresponding to a motor current limit. In this section the implementation of Pontryagin’s minimum principle on the minimum energy and time problems is shown.
A) Minimum energy

Similarly to the section 3.4.1 the minimum energy problem is discussed first. The problem formulation is given in Table 3.3.

Table 3.3 Control constrained minimum energy problem

<table>
<thead>
<tr>
<th>State equations</th>
<th>Boundary conditions</th>
<th>Physical constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{x}_1 = x_2$</td>
<td>$x(t_0) = [0]$, $x(t_f) = [X]_0$</td>
<td>$</td>
</tr>
<tr>
<td>$\dot{x}_2 = u$</td>
<td>Final time $t_f$ - free</td>
<td></td>
</tr>
</tbody>
</table>

Performance measure

$$J_e = \frac{1}{2} \int_{t_0}^{t_f} u^2 dt$$

Similarly to the previous section, in order to obtain the optimal control, which moves the system from initial state $(0; 0)$ to the final state $(X; 0)$ system of equations (A.1.8) should be solved. For the control constrained minimum energy problem:

Hamiltonian function:

$$H(x(t), u(t), p(t)) = \frac{1}{2} u^2(t) + p_1(t)x_2(t) + p_2(t)u(t)$$

State and costate equations are the same as in 3.4.1 and they are repeated in (3.22).

$$\begin{align*}
\dot{x}_1^*(t) &= x_2^*(t) \\
\dot{x}_2^*(t) &= u^*(t) \\
p_1^*(t) &= 0 \\
p_2^*(t) &= -p_1^*(t)
\end{align*}$$

(3.22)

To determine the optimal control trajectory equation (A.1.8) should be used. If, in the Hamiltonian function all the states containing $u(t)$ are separated then

$$\frac{1}{2} [u^*(t)]^2 + p_2^*(t)u^*(t) \rightarrow \min$$

or

$$u^*(t) = -p_2^*(t)$$

(3.23)

However the maximum acceleration has a final value $A$, $|u| \leq A$ so the optimal control trajectory is defined as the following.

$$u^*(t) = \begin{cases} 
-A, & \text{for } p_2^*(t) > A \\
c_1 t - c_2, & \text{for } -A \leq p_2^*(t) \leq A \\
A, & \text{for } p_2^*(t) < -A
\end{cases}$$

(3.24)

Constants $c_1$ and $c_2$ can be found from the boundary conditions. Expression (3.24) is basically the saturation function. However in 3.4.1 it was shown that the optimal final
time, $t_f$ tends to infinity, causing the optimal control function to stay within the saturation limits making the solution of constrained control without specified final time the same to the unconstrained one, expressed by (3.20).

B) Minimum time

For the unconstrained control, the minimum time problem had no physical sense, whereas here the problem will be solved for a limited control action. The problem formulation is given in Table 3.4. As for the minimum energy problem, the solution follows the procedure described in section A.1.1. For the minimum time transfer from the initial state $(0; 0)$ to the final state $(X; 0)$ the system of equations (A.1.8) is solved.

Table 3.4 Control constrained minimum time problem

<table>
<thead>
<tr>
<th>State equations</th>
<th>Boundary conditions</th>
<th>Physical constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{x}_1 = x_2$</td>
<td>$x(t_0) = [0; 0], x(t_f) = [X; 0]$</td>
<td>$</td>
</tr>
<tr>
<td>$\dot{x}_2 = u$</td>
<td>Final time $t_f$ - free</td>
<td></td>
</tr>
</tbody>
</table>

**Performance measure**

$$J_t = \int_{t_0}^{t_f} dt \rightarrow \min_u$$

**Hamiltonian function:**

$$H(x(t), u(t), p(t)) = 1 + p_1(t)x_2(t) + p_2(t)u(t)$$

**State equations:**

$$\dot{x}_1^*(t) = \frac{\partial H}{\partial p_1} = x_2^*(t)$$

$$\dot{x}_2^*(t) = \frac{\partial H}{\partial p_2} = u^*(t)$$

**Costate equations:**

$$\begin{align*}
\dot{p}_1^*(t) &= -\frac{\partial H}{\partial x_1} = 0 \\
\dot{p}_2^*(t) &= -\frac{\partial H}{\partial x_2} = -p_1^*(t)
\end{align*}$$

$$\implies p_1^*(t) = c_1, \quad p_2^*(t) = -c_1 t + c_2$$

where $c_1$ and $c_2$ are arbitrary constants of integration.

Applying the Pontryagin’s minimum principle from (A.1.8), the optimal control trajectory should be dependent on the Lagrange multiplier function $p_2(t)$. The expression for the optimal control function is presented in (3.25).
\[ u^* = \begin{cases} -\text{sign}(p_2^*(t)) & \text{if } p_2^*(t) \neq 0 \\ A \text{ or } -A & \text{if } p_2^*(t) = 0 \end{cases} \quad (3.25) \]

According to the costate equation, \( p_2^*(t) \) is a linear function of time, which can’t cross zero more than once, moreover the “Number of switchings” theorem from [69], states that for the second order state matrix (3.16) there is at most one switching of the control function. So the control function could have one of the four a) to d) forms from the expression below.

\[
u^*(t) = \begin{cases} (a) & A \\ (b) & -A \\ (c) & \begin{cases} A & \text{for } 0 \leq t \leq t_1 \\ -A & \text{for } t_1 \leq t \leq t_f' \end{cases} \\ (d) & \begin{cases} -A & \text{for } 0 \leq t \leq t_1 \\ A & \text{for } t_1 \leq t \leq t_f \end{cases} \end{cases} \quad (3.26)\]

where \( t_1 \) the an arbitrary time where switching of the control trajectory happens. The solution of the constrained control problem is well established in literature. Following the discussion in [69] and [71] the optimal control function should follow the option c), with \( t_1 = \frac{t_f}{2} \).

The optimal final time \( t_f \), or in other words the minimum time of a process could be found using the equation (A.1.9), with the optimal control function defined by the option c) of equation (3.26). To transfer the plant from the initial state \((0; 0)\) to the final state \((X; 0)\), with the limited control action \( \pm A \), the final time \( t_f \) is found as (3.27).

\[ t_f = 2 \sqrt{\frac{X}{A}} \quad (3.27) \]

As discussed in section Chapter 2, with a 2-axis system, the servo motor (motor 1) that is intended to move the spindle motor or table for the longer path is controlled in a minimum time mode. The time for this process is decided by (3.27), then this time is used as a final value for the optimal energy controlled servo motor (motor 2), which moves the spindle or table for the shorter pass. Considering this fact the problem formulated by Table 3.1 and Table 3.3 should be redefined to have the fixed final time, resultant from the minimum time problem solution. Due to the presence of the fixed final time \( t_f \), equation (A.1.9) for both energy minimum problems doesn’t exist and for the unconstrained problem the optimal trajectory (3.20) is now redefined by substituting expression (3.27). However, it should be stressed that for minimum time and energy problems, the final position of state \( x_1 \) is not generally the same. As discussed earlier,
time minimum control is implemented for the drive with a longer path (motor 1), with energy optimal control for the shorter (motor 2). So the final boundary condition for the state $x_1$ in the time minimal control problem is $X_1$ and in the energy minimum control problem is $X_2$, where $X_1 \geq X_2$. Then the resultant minimum energy, unconstrained control would be expressed in form of (3.28).

$$u^*(t) = -\frac{3X_2A}{2X_1}\left(\frac{A}{X_1}t - 1\right)$$ (3.28)

For the constrained control case, when the final time is specified, integral time constants $c_1$, $c_2$ must be determined to find the optimal solution. It is done by substituting the initial and final state coordinates to the solution of the constrained energy minimum problem. Two cases should be considered, first when $t_f \geq \sqrt{\frac{6X_2}{A}}$, then the solution of both constrained and unconstrained problems are the same and expressed by (3.28). Second, when $t_f < \sqrt{\frac{6X_2}{A}}$, the optimal control trajectory is saturated and described by (3.24). After applying the constants of integration optimal control is defined as follows.

$$u^*(t) = \begin{cases} -A, & \text{for } t > t_f - t_1 \\ -\frac{2A}{3\sqrt{2}}t + \frac{At_f}{\sqrt{3t_f^2 - 12X_2^2}}, & \text{for } t_1 \leq t \leq t_f - t_1, \\ \frac{A}{2}, & \text{for } t < t_1 \end{cases}$$ (3.29)

where $t_1 = \frac{t_f - \frac{1}{2}}{2} \sqrt{3t_f^2 - 12\frac{X_2}{A}}$. For this case fixing the final time from the time minimum problem solution and using the redefined boundary conditions $X_1$, $X_2$ for a state $x_1$, as was done for the unconstrained problem, one would obtain the expression for the minimal energy control trajectory.

$$u^*(t) = \begin{cases} -A, & \text{for } t > t_f - t_1 \\ -\frac{A}{2\sqrt{2(L_1 - L_2)}} \left(t - \frac{X_1}{A}\right), & \text{for } t_1 \leq t \leq t_f - t_1, \\ \frac{A}{2}, & \text{for } t < t_1 \end{cases}$$ (3.30)

where $t_1 = \frac{X_1}{A} - \frac{3}{A}(X_1 - X_2)$. For $X_1 \geq X_2$ optimal time and energy profiles are shown on Fig. 3.8.
3.4.3 State and control constrained movement

To find the optimal time and energy control for a system having both state and control constraints, the procedure described in section A.1.2 should be implemented. The problems are formulated in Table 3.5 and Table 3.6 for a limit in speed of $\pm V$.

Firstly the state constraints $|x_2| \leq V$ should be expressed in the form of (A.1.10). The absolute value function is replaced by two non-negative functions $L_1, L_2$.

$$L_1(x_2(t)) = x_2(t) + V \geq 0$$
$$L_2(x_2(t)) = V - x_2(t) \geq 0$$

(3.31)

Then for both problems a new state variable is $x_3(t)$ is defined as

$$\dot{x}_3(t) = [x_2 + V]^2h(-x_2 - V) + [V - x_2]^2h(x_2 - V),$$

(3.32)

Table 3.5 State and control constrained minimum time problem

<table>
<thead>
<tr>
<th>State equations</th>
<th>Boundary conditions</th>
<th>Physical constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{x}_1 = x_2$</td>
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<td>$</td>
</tr>
<tr>
<td>$\dot{x}_2 = u$</td>
<td>Final time $t_f$ - free</td>
<td>$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Performance measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_t = \int_{t_0}^{t_f} dt \rightarrow \min_{u}$</td>
</tr>
</tbody>
</table>
Table 3.6 State and control constrained minimum energy problem

<table>
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<tr>
<th>State equations</th>
<th>Boundary conditions</th>
<th>Physical constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x}_1 = x_2 )</td>
<td>( x(t_0) = [0] ), ( x(t_f) = [X] )</td>
<td>Final time ( t_f ) - free</td>
</tr>
<tr>
<td>( \dot{x}_2 = u )</td>
<td></td>
<td>(</td>
</tr>
<tr>
<td>( x_2 \leq V )</td>
<td></td>
<td>(</td>
</tr>
</tbody>
</table>

Performance measure

\[
J_e = \frac{1}{2} \int_{t_0}^{t_f} u^2(t) dt \rightarrow \min_{u}
\]

where \( h(L_i) \) is the Heaviside function defined in (A.1.12). Including this new state variable, Hamiltonian functions for optimal time, \( H_t \) and energy, \( H_e \) are introduced in (3.33) and (3.34).

\[
H_t(x(t), u(t), p(t)) = 1 + p_1(t)x_2(t) + p_2(t)u(t)
\]

\[
+ p_3(t) ([x_2 + V]^2 h(-x_2 - V) + [V - x_2]^2 h(x_2 - V))
\]

(3.33)

\[
H_e(x(t), u(t), p(t)) = \frac{1}{2} u^2(t) + p_1(t)x_2(t) + p_2(t)u(t)
\]

\[
+ p_3(t) ([x_2 + V]^2 h(-x_2 - V) + [V - x_2]^2 h(x_2 - V))
\]

(3.34)

Apart with the new Hamiltonian functions, definitions for the state and costate equations are the same. Applying the Pontryagin’s minimum principle for both problems, the necessary conditions for time optimal control (with free final time \( t_f \)) are

\[
\dot{x}_1^* = x_2^*(t)
\]

\[
\dot{x}_2^* = u^*(t)
\]

\[
\dot{x}_3^* = [x_2 + V]^2 h(-x_2 - V) + [V - x_2]^2 h(x_2 - V)
\]

\[
p_1^* = 0
\]

\[
p_2^* = -p_1^* - 2p_3^*(t)(x_2(t) + V)h(-x_2(t) - V)
\]

\[
+ 2p_3^*(t)(V - x_2(t))h(x_2(t) - V)
\]

(3.35)

\[
p_3^* = 0
\]

\[
u^*(t) = \begin{cases} -\text{sign}(p_2^*(t)), & \text{if } p_2^*(t) \neq 0 \\ A \text{ or } -A, & \text{if } p_2^*(t) = 0 \end{cases}
\]

\[
1 + p_2^*(t_f)u^*(t_f) = 0
\]

and for energy optimal control with fixed final time obtained from the time minimization problem.
\[
\begin{align*}
\dot{x}_1^*(t) &= x_2^*(t) \\
\dot{x}_2^*(t) &= u^*(t) \\
\dot{x}_3^*(t) &= [x_2 + V]^2 h(-x_2 - V) + [V - x_2]^2 h(x_2 - V) \\
p_1^*(t) &= 0 \\
p_2^*(t) &= -p_1^*(t) - 2p_3^*(t)(x_2(t) + V)h(-x_2(t) - V) \\
&\quad + 2p_3^*(t)(V - x_2(t))h(x_2(t) - V) \\
p_3^*(t) &= 0 \\
u^*(t) &= \begin{cases} 
-A, & \text{for } p_2^*(t) > A \\
-p_2^*(t), & \text{for } -A \leq p_2^*(t) \leq A \\
A, & \text{for } p_2^*(t) < -A 
\end{cases}
\end{align*}
\] (3.36)

The boundary conditions of such problems at initial and final time are updated due to the presence of additional state \(x_3\). New conditions are

\[
x(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad x(t_f) = \begin{bmatrix} X \\ 0 \\ 0 \end{bmatrix}
\] (3.37)

Both systems of equations (3.35) and (3.36) could be solved numerically. However, using the inbuilt MATLAB function \(bvp4c\), the solution was not obtained due to the Jacobian singularity problem. This problem occurs, when the new state variable, \(x_3\), has been introduced using the exact Heaviside/step function as well as its approximation (A.1.15), (A.1.16). With this level of numerical complexity for an ideal double integrator system, the approach was deemed unsuitable for a high order AC motor system.

However the solution of the minimum time problem in analytical form is presented in the literature. In [71] for a second order system, the time optimal control for a state and control constrained problem was obtained through the analysis of the system trajectory behaviour of \(x_2-x_3\) on the state space plane. According to the obtained results optimal control as a function of time is shown in (3.38).

\[
u^*(t) = \begin{cases} 
-A, & \text{for } t > t_f - t_1 \\
0, & \text{for } t_1 \leq t \leq t_f - t_1, \\
A, & \text{for } t < t_1 
\end{cases}
\] (3.38)

where final time \(t_f = \frac{X}{V} + \frac{V}{A}\) and \(t_1 = \frac{V}{A}\). The optimal minimum time profile for the bounded acceleration and speed is shown on Fig. 3.9.
3.4.4 Optimal control conclusion remarks

In section 3.4 optimal control for a linear moving object, described by a second order system of ODE was discussed. Different solutions have been obtained for the different performance measures. The resultant control functions for energy and time optimal controls are shown in Table 3.7 for unconstrained and constrained cases. Final time, $t_f$ is decided by the drive controlled in minimum time mode, with the longer path. Then this $t_f$ is substituted in optimal energy calculations limiting the available duration of a process. It is clearly shown that the optimal acceleration profiles are different, meaning that the optimal speed (feed) profiles are different as well and that a single profile should not be used for both servo drives in a positioning system.
Chapter 3 Optimal control theory. Variational approach

Table 3.7 Summary of optimal energy and optimal control methods

<table>
<thead>
<tr>
<th>Constraints</th>
<th>( J_t = \int_{t_0}^{t_f} dt \to \min ) optimal time</th>
<th>( J_c = \frac{1}{2} \int_{t_0}^{t_f} u^2 dt \to \min ) optimal energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>No constraints</td>
<td>( u^*(t) = a(t) )</td>
<td>Numerical solution of ODE system (3.36)</td>
</tr>
<tr>
<td>Constrained control (</td>
<td>u</td>
<td>\leq A</td>
</tr>
<tr>
<td>State and control constrains (</td>
<td>u</td>
<td>\leq A</td>
</tr>
</tbody>
</table>

When a spindle motor or a table are simply moved from one position to another, without any cutting or drilling action, conventional CNC systems perform a linear motion. However if servo drives are controlled in a manner described in a Table 3.7, then the trajectory of a spindle motor or table is shown on Fig. 3.10 versus the conventional linear trajectory on a surface plane. It shows the movement from the starting point \( P_s(x_s, y_s) \) to the end point \( P_e(x_e, y_e) \). Note that the path represented by \( x-axis \) is longer.

![Fig. 3.10 Linear vs. optimal in overall energy sense trajectories](image-url)
When considering only different feed profiles in CNC systems there is no way to consider the motor dynamics by itself. However, applying optimal control methods it is possible to include motor state equations in the problem formulation and improve the performance not only in the linear motion domain but also considering the motor itself.

CNC systems are very vibration-sensitive applications. The speed profiles with limited Jerk are developed in literature [62], [72] in order to suppress the vibration. However, there is no direct connection of limited Jerk with the system vibration. Vibration in a motor drive system is caused by the high frequency torque variation, which in its turn is caused by rapid current variation. If the motor equations are added in the optimized plant state equations, than it is possible to directly limit the maximum current variation, i.e. there is a way to directly suppress the motor vibrations. Moreover, current variation should be bounded due to the limited bandwidth of the current controller.

### 3.5 Example of optimal control application for the legged mobile robot

In the previous sections the detailed discussion of the variational approach to optimal control problem was presented. It was shown that electrical motors, which are used to drive the linear actuators in positioning systems, could be modelled as a double integrator with acceleration as an input and position as an output. Chapter 4 will move on to discuss the close-loop optimal control, but this section will check the aforementioned theoretical results in practice. For such purposes the system where the positioning accuracy is not very critical should be chosen. In this research it is suggested to implement the obtained results for a six-legged mobile robot, CORIN. The photo and 3D model of this robot is presented on Fig. 3.11. CORIN has been developed
in the University of Manchester for the remote inspection of nuclear power plants. Before the optimal control implementation CORIN kinematics and design together with the conventional control approach are briefly discussed following the [73] and [74].

3.5.1 CORIN conventional control, design and kinematics

In order to make the robot to move the tripod gait inspired by the biological gaits of insects is used. The diagram explaining the legs movement is presented on Fig. 3.12, where the black colour represents the amount of time the particular leg is touching the ground (leg is in a support phase according to [73]) and white colour – amount of time, when the leg is in the air (transfer phase [73]). To control the single leg movement each of its joints is controlled in a Reflex model fashion, shown on Fig. 3.13, where the feedback signal of joint’s position and speed is continuously compared with the reference signals in order to generate an input to the conventional PID controller. The reference signal is obtained as follows. First, the initial and final positions and speeds of a robot’s foot during a single gait are defined (“end-effector positions and speeds” using the manipulator notations). Second, the continuous cubic polynomial trajectory is generated, given that it must pass through the initial and final points. Third, from the obtained Cartesian space leg movement trajectory the reference signal to control each joint’s motor is obtained through the inverse kinematic equations. It is worth noting that if only two points are set, the feet will drag across the floor, so the intermediate waypoints are added. The waypoints are setting the step height. To illustrate the
waypoints influence on foot trajectory, two different cases are shown for some arbitrary step size and height on Fig. 3.14.

Forward and inverse kinematics equations are a conventional tool to put the end-effector Cartesian position in correspondence with the joint angles and vice versa. For the CORIN leg geometry, shown on Fig. 3.15, the forward kinematics is derived in [73] through the Denavit-Hartenberg parameter (3.39).

\[
\begin{bmatrix}
{x_e} \\
{y_e} \\
{z_e}
\end{bmatrix} = \begin{bmatrix}
\cos \theta_1 [l_3 \cos(\theta_2 + \theta_3) + l_2 \cos \theta_2 - l_d \sin(\theta_2 + \theta_3)] \\
\sin \theta_1 [l_3 \cos(\theta_2 + \theta_3) + l_2 \cos \theta_2 - l_d \sin(\theta_2 + \theta_3)] \\
l_1 + l_3 \sin(\theta_2 + \theta_3) + l_2 \sin \theta_2 + l_d \cos(\theta_2 + \theta_3)
\end{bmatrix}
\]

(3.39)

The equation (3.39) is applied to get the actual foot position using the encoder readings of each joint motor. To solve the inverse task or inverse kinematics, i.e. to identify the joint angles based on the Cartesian leg position the geometrical considerations could be applied. For the given geometrical parameters, summarized in Table 3.8 and shown on Fig. 3.15 the joint’s reference signal could be deduced from the given Cartesian space trajectory as in (3.40), where the reference point is denoted by \(0\) on the figure.

![3D diagram of different leg trajectories according to number of waypoints](image)

**Fig. 3.14 3D diagram of different leg trajectories according to number of waypoints**

**Table 3.8 Corin Design Parameters**

<table>
<thead>
<tr>
<th>Geometrical parameters</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_1), [m]</td>
<td>0.04475</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(l_2), [m]</td>
<td>0.192</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(l_3), [m]</td>
<td>0.192</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(l_d), [m]</td>
<td>0.028</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Joint mass, [kg]</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_1)</td>
<td>0.072</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_2)</td>
<td>0.126</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_3)</td>
<td>0.072</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(M)</td>
<td></td>
<td></td>
<td>4.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Full body weight, [kg]</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(M)</td>
<td>4.7</td>
</tr>
</tbody>
</table>
\[ \theta_1 = \tan^{-1} \frac{y_e}{x_e} \]

\[ \epsilon = \tan^{-1} \frac{l_1 - x_e}{a} \]

\[ \beta = \cos^{-1} \left( \frac{l_2^2 - l_3^2 - h^2}{2l_2l_3} \right) \]

\[ \theta_2 = \beta - \epsilon \]

\[ l'_3 = \sqrt{l_2^2 + l_3^2} \]

\[ c = \tan^{-1} \frac{a}{l_3} \]

\[ a = \sqrt{x_e^2 + y_e^2} \]

\[ h = \sqrt{a^2 + (z_e + l_3)^2} \]

\[ \alpha = \cos^{-1} \left( \frac{h^2 - l_2^2 - l_3^2}{2l_2l_3} \right) \]

\[ \theta_3 = \frac{3\pi}{2} - \alpha - c \] (3.40)

Applying the equation (3.40) it is possible to identify the joint’s rotational angles knowing the Cartesian position of a foot, however during the optimal control law development in section 3.4 the motor acceleration was used as an input to the positioning system and its shape was optimized. It is clear that in the conventional CORIN control case no acceleration shape optimization is done. The joint acceleration trajectory is dictated by the desired end-effector movement profile. Therefore in order to be able to perform the comparison between the energy optimized CORIN movement...
acceleration profile, considering its leg as an example of a positioning system, with the conventional acceleration profile based on cubic polynomial leg trajectory in the Cartesian space equations describing the connection between the Cartesian velocity/acceleration and joint angular velocity/acceleration should be derived.

Conventionally the Jacobian matrix is used to describe the relationship between the angular velocity (joint angular speed) and Cartesian velocity. For the single leg and three degrees of freedom (DOF) the Jacobian is as a 3x3 matrix defined in (3.41).

\[
J_{ac} = \begin{bmatrix}
\frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\
\frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\
\frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \frac{\partial z}{\partial \theta_3}
\end{bmatrix}
\]  \hspace{1cm} (3.41)

Substituting (3.39) into (3.41) the Jacobian matrix for the single leg is obtained as follows.

\[
J_{ac} = [J_1 \quad J_2 \quad J_3]
\]  \hspace{1cm} (3.42)

where \( J_1 = \begin{bmatrix}
\sin \theta_1[l_d \sin \theta - l_2 \cos \theta_2 - l_3 \cos \theta] \\
\cos \theta_1[l_3 \cos \theta + l_2 \cos \theta_2 - l_4 \sin \theta] \\
0
\end{bmatrix}, \]

\( J_2 = \begin{bmatrix}
-\cos \theta_1[l_3 \sin \theta + l_2 \sin \theta_2 + l_d \cos \theta] \\
-\sin \theta_1[l_3 \sin \theta + l_2 \sin \theta_2 + l_d \cos \theta] \\
l_3 \cos \theta + l_2 \cos \theta_2 - l_d \sin \theta
\end{bmatrix}, \)

\( J_3 = \begin{bmatrix}
-\cos \theta_1[l_2 \sin \theta + l_d \cos \theta] \\
-\sin \theta_1[l_3 \sin \theta + l_d \cos \theta] \\
l_3 \cos \theta - l_d \sin \theta
\end{bmatrix} \)

and \( \theta = \theta_2 + \theta_3 \). Knowing the Jacobian matrix the joint angular speed of each joint is obtained as (3.43) knowing the Cartesian velocity \( v = [v_x; v_y; v_z]^T \).

\[
\omega = \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} = J_{ac}^{-1} \begin{bmatrix}
v_x \\
v_y \\
v_z
\end{bmatrix}
\]  \hspace{1cm} (3.43)

In a similar way the column of joint angular accelerations \( a_{ang} \) could be calculated through the Cartesian acceleration and a Jacobian matrix (3.44).

\[
a_{ang} = J_{ac}^{-1} (a - \frac{\partial J_{ac}}{\partial t} \omega),
\]  \hspace{1cm} (3.44)

where \( a \) and \( \omega \) are the column vectors of Cartesian speeds and acceleration. The time derivative of a Cartesian matrix could be expressed through the matrix \( K = \frac{\partial J_{ac}}{\partial t} = \frac{\partial J_{ac}}{\partial \theta} \frac{\partial \theta}{\partial t} \) \((\theta = [\theta_1; \theta_2; \theta_3]^T)\), which consists of nine elements (3.45).

\[
K = \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\]  \hspace{1cm} (3.45)

For the ease of calculation each element of matrix \( K \) is defined as \( K_{ij} = \sum_{k=1}^{3} \frac{\partial J_{ac(k)}}{\partial \theta_k} \omega_k \).

The final value of each element is shown in Appendix 2.
Applying the transformations introduced by (3.40), (3.43) and (3.44) the joint space angular acceleration, speed and position trajectories for the conventional CORIN single leg movement could be obtained and shown on Fig. 3.16.

3.5.2 **CORIN joints optimal energy control**

The conventional approach used to control the movement of the robot single leg has been introduced above. It could be described as a Cartesian space-oriented approach, where at first the desired leg end point trajectory is set in the Cartesian space and later on using the inverse kinematics transformations the reference trajectory in the joint space is obtained. According to [75], [74] such an approach is convenient to use in obstacle avoidance tasks, however in most cases there are no sensors capable to measure the leg position in Cartesian space. In order to identify the leg coordinates the information from the position sensors installed on joint shafts is used (applying the forward kinematics). Taking this into account another control approach arises naturally, using the joint-space orientation. It generates the trajectory in the joint space providing that the leg end-point will pass through the defined intermediate waypoints. Additionally using the joint space approach physical limitations of the actuator motors could be taken in account. In section 3.4 the optimal energy control of a system described as a double integrator in the presence of constraints was shown. So the same approach for the robot’s leg trajectory generation is chosen.

According to [74] CORIN’s actuators are powered by the DC-motors. More specifically, MX-series actuators with Maxon motors are used. First and third joints are powered by a less powerful MX-28T and the second joint is powered by the MX-64T. The actuator parameters are shown in Table 3.9.
Table 3.9 CORIN actuators parameters [74]

<table>
<thead>
<tr>
<th>Actuator</th>
<th>Nominal voltage, [V]</th>
<th>Torque constant, $\lambda_{pm1,3}$ [Nm/A]</th>
<th>11.2e-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>MX-28T</td>
<td>12</td>
<td>Rotor inertia, $[\text{kgm}^2]$</td>
<td>9.93e-8</td>
</tr>
<tr>
<td></td>
<td>Nominal current, [A]</td>
<td>0.413</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Terminal resistance, $R_{1,3}$ [Ω]</td>
<td>11.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Terminal inductance, [mH]</td>
<td>0.343</td>
<td></td>
</tr>
<tr>
<td>MX-64T</td>
<td>9</td>
<td>Torque constant, $\lambda_{pm2}$ [Nm/A]</td>
<td>8.55e-3</td>
</tr>
<tr>
<td></td>
<td>Nominal current, [A]</td>
<td>0.72</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Terminal resistance, $R_2$ [Ω]</td>
<td>2.21</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Terminal inductance, [mH]</td>
<td>0.085</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Gear ratio, $N_1$</td>
<td>193:1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Gear ratio, $N_2$</td>
<td>200:1</td>
</tr>
</tbody>
</table>
reformulation of the control constrained minimum energy problem given in section 3.4.2, where $X$ represents the required rotation angle in rad to pass through the intermediate waypoint\(^3\). As it was explained in [74], purely for convenience issues the final rotational angle is set to zero, so the initial one is defined as $-X$. Both initial and final speeds of a motor are zero as well. In this optimization task initial time is set as $t_0 = 0$, and final time $t_f$ has some fixed value, which is set to be equal to the single leg step movement time applying the conventional control strategy in order to have a fair comparison. In Chapter 2 it was suggested to control the single motor in the positioning system in the optimal time mode, whereas others are controlled in the optimal energy mode, given that they finish their operation by the time the first one finishes. So the operation time of motor controlled in optimal time mode is setting the overall system’s operation time. However, in the CORIN case, this time is set to match the conventional operation time in such a manner the optimization problem formulation in Table 3.10 is the same for all three joints and the state variables are defined in the joint space. The difference is the initial position, $x_i(t_0)$, value for each of three joints.

Applying the control law developed in section 3.4.2 for each joint the new reference trajectory of a single joint is obtained and by feed-forwarding this trajectory to the CORIN actuator an experiment has been performed. CORIN was set to move in a straight line for 2 minutes and the power consumption for both conventional and suggested control strategies were measured during the whole locomotion period. The power measurements were done by a Yokogawa WT3000 power analyser and later on were integrated over the 2 min period to compare the consumed energy. The energy

---

\(^3\) In the conventional case two intermediate waypoints were introduced. However when the joint space approach is used it is more convenient to introduce a single intermediate waypoint. To make a fair comparison of the control algorithms this waypoint is set right at the top of the leg end point trajectory, Fig. 3.14.
consumption was found to reduce by around 1%, which is much smaller than the expected energy consumption reduction.

The reason for such a negligible energy consumption improvement is described in [74]. Earlier in this chapter when the assumptions were discussed, it was mentioned that for the positioning system in off-cut operation mode load torque could be neglected and only the torque which is needed for the actual end-effector position change is optimized. However, for CORIN this assumption is not valid. It is possible to identify three sources of the joint torque due to the CORIN movement:

- Torque required to maintain the CORIN’s body in the upright position during the support phase of operation ($Q_{sp}$);
- Torque required to hold the CORIN’s leg during the transfer phase of operation ($Q_{tp}$);
- Torque required to change the joint speed in a presence of inertia during both transfer and support phases, considering no-load operation ($Q_{nl}$).

It is possible to estimate the relative importance of each torque source by comparing the energy consumed by each part during the single step ($Q_{sp}, Q_{tp}, Q_{nl}$). The $Q_{nl}$ is obtained as a weighted integration of acceleration profile, whereas $Q_{sp}$ and $Q_{tp}$ could be roughly approximated as a time integration of gravity force multiplied by the amplitude of the arm vector. For example, for the second joint the energy consumed in the support phase is approximated as follows.

$$Q_{sp} = \int_{t_0}^{t_f} \left( \frac{M_{123}g}{N_{23}pm_2} \right)^2 R_2 dt, \quad (3.47)$$

where it is assumed that the mass is equally shared between all three legs in the support phase. For the $Q_{tp}$ instead of CORIN body mass the single leg mass should be used.

Considering the CORIN design parameters from Table 3.8 and Table 3.9 the portion of each torque in the overall energy consumption is roughly 95%, 4% and 1%. It is clear that the most energy is used to maintain robot body position, i.e. $Q_{sp}$ is dominant. Moreover the portion of $Q_{nl}$ is negligible, which is supported by the aforementioned experiment results, where the trajectory modification has resulted only in minor improvement in the amount of consumed energy. Hence the assumption of zero static torque used for the positioning systems can’t be applied for CORIN. To deal with this issue in [74] it was suggested to modify the cost function in the optimization problem formulation from the Table 3.10. A time independent parameter $\mu$ is
Chapter 3 Optimal control theory. Variational approach

introduced, which represents the $Q_{sp}$, notice that $\mu \gg u^2(t), \forall t$. Then the modified cost function is defined in (3.48).

$$\mathcal{J}_e = \frac{1}{2} \int_{t_0}^{t_f} u^2(t) dt \rightarrow \mathcal{J}_e = \frac{1}{2} \int_{t_0}^{t_f} (\mu + u^2(t)) dt \rightarrow \mathcal{J}_t = \mu \int_{t_0}^{t_f} dt = \int_{t_0}^{t_f} dt \rightarrow \min_{u(t)}$$  \hspace{1cm} (3.48)

It could be seen that the modified cost function is now not in the optimal energy form, but optimal time one. The solution of such a problem (Table 3.10 with the cost function defined in (3.48)) is derived in section 3.4.2B and the resultant joint trajectories are shown on Fig. 3.17. The resultant Cartesian space leg trajectory in the $x$-$z$ plane (the step sway in the $y$ plane is zero for the conventional case and negligible in the proposed one) is shown on Fig. 3.18. It could be noted that the single step time is changed from 4

---

**Fig. 3.17** Proposed single leg joint space moving profile: a) Position b) Speed c) Acceleration

**Fig. 3.18** Leg trajectory in the Cartesian space [74]
seconds in the conventional case to the 3 seconds in the proposed. Due to this it is not fair to directly compare the energy consumption so another criteria is needed. In [74] it is suggested to use a dimensionless number $\mathcal{R}$ -specific resistance [76] as the comparison criterion, which is defined as a ratio of the total energy consumed to travel to the Certance distance to that distance multiplied by CORIN mass and gravitational acceleration. Three sets of experiments were performed in order to compare the proposed control with the conventional one and to separate each part of the energy consumption. First, the stand-by energy was measured, $Q_{sb}$. It is the energy consumed by joint motor-control PCBs and the main control PCB when joint motors are not powered. Obviously it is a linear time-dependent value, which rises with the rise of time. Second, the robot was placed on a support bar, so that the legs are not touching the ground. In such operation mode no energy is consumed to maintain the body position ($Q_{sp} = 0$), but the energy, $Q_{tp}$, is consumed to hold the leg. Total energy consumption is increased around 50%. And last, the energy consumption for the three meter robot motion was measured. The results of all three experiments in terms of specific resistance are summarized in Table 3.11. It is not very correct to talk about the specific resistance value for the first two experiments, but value was calculated as if robot were moving for comparison purposes.

Table 3.11 Specific resistance, $\mathcal{R}$, for conventional and proposed control.

<table>
<thead>
<tr>
<th></th>
<th>Original</th>
<th>Proposed</th>
<th>Saved energy, %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mathcal{R}$</td>
<td>%</td>
<td>$\mathcal{R}$</td>
</tr>
<tr>
<td>Stand-by</td>
<td>3.23</td>
<td>30.6</td>
<td>2.43</td>
</tr>
<tr>
<td>No support phase</td>
<td>4.78</td>
<td>45.3</td>
<td>3.65</td>
</tr>
<tr>
<td>CORIN 3m locomotion</td>
<td>10.55</td>
<td>100</td>
<td>8.06</td>
</tr>
</tbody>
</table>

From Table 3.11 it could be seen that the CORIN stand-by energy consumption is very considerable. Returning back to the cost function formulation, the high stand-by energy consumption increases $\mu$ even more, making it very important to reduce the robot operation time. This result emphasises the discussion in Chapter 2, where it was mentioned by many authors that the stand-by energy consumption of a machine tool is generally very high and the most efficient way to reduce it is to reduce the MT operation time.

For the straight locomotion across the flat undistorted terrain the total energy consumption was reduced more than 20%. This result is important for the mobile robots
due to the possibilities either to increase the robot operation range or to reduce the battery size and weight by reducing the energy consumption.

3.6 Conclusion

In this chapter the assumptions imposed on this research have been introduced, based on these assumptions it was shown that for the 2D positioning systems the drive feeding each axis could be treated independently (there is no cross coupling between the axis actuators). Further on, two separate optimization tasks, called optimal time and optimal energy, have been formulated and solved using the classical approach based on variational methods and Pontryagin’s minimal principle assuming that the drive feeding the axis in the positioning system could be modelled as a double integrator. The obtained open-loop solution has been applied to CORIN six-legged mobile robot in order to check the applicability of the developed control. Experimentally more than 20% energy saving has been obtained.

The obtained optimal control generates the optimal reference state trajectories, which are further feed-forward to the drive. However, these trajectories are not very precise mainly due to the imperfection of double integrator model. Moreover, there is difficult to consider possible disturbances which are influencing the optimal state trajectory.
Chapter 4 Optimal Control under Uncertainty

In Chapter 3 the optimal acceleration profiles for OT and OE problems have been obtained for the linear motion system described by a double integrator (3.15). However, this solution has a very limited application in practice due to the neglect of model parameter variation and possible disturbances or any other uncertainties in general. The optimal solution in the closed form is required, where the optimal control input $u^*$ would be a function of the system states ($x_1$ and $x_2$ for a double integrator), not time. The block diagram illustrating the required control is shown on Fig. 4.1, which is essentially different from the one studied in Chapter 3 and presented on Fig. 3.4 ($u^*(x)$ v.s. $u^*(t)$).
Optimal control problem has been actively studied in past and the wide discussion is presented in [69],[71],[77]. However, it is worth summarizing the major results here, based on [69]. This is done due to the following reason: in [69] it is stated that the procedure to find the OT control law for the second order system (double integrator) could be extended to the general $n$th-order, stationary, linear system. Therefore this chapter investigates how much the complexity of the OT solution increases with the rise of system order (to be able to consider more detailed electric motor model) and makes a conclusion regarding applicability of the discussed method.

The close-loop optimal energy solution for the problem formulation introduced in Chapter 3 is not presented in the literature and this chapter gives an explanation for why is it impossible.

It is of little interest to obtain the solution for the unconstrained movement, because in practice the control action is always limited by the source maximum output power, so this section will start with the control-constrained movement for the optimal time case.

### 4.1 Optimal time control

In section 3.4.2 the open-loop optimal control with a bounded input ($|u^*| \leq A$) was obtained using Pontryagin’s minimum principal as

$$u^*(t) = \begin{cases} -A \text{sign}(p^*_2(t)), & p^*_2 \neq 0 \\ \pm A, & p^*_2 = 0 \end{cases},$$

(3.25)

where Lagrange multipliers were defined with the arbitrary constants $c_1$ and $c_2$ as follows

$$p^*_1(t) = c_1$$

$$p^*_2(t) = -c_1 t + c_2,$$

(4.1)

therefore the optimal control $u^*(t)$ could take one of the $\pm A$ values, with a limit of a single switching event according to “Number of switchings” theorem. The intervals of
the state trajectories with the $u^*(t) = \pm A$ could be found by integrating the state equations (3.15).

$$
x_2(t) = \pm At + c_3
$$

$$
x_1(t) = \frac{1}{2}At^2 + c_3t + c_4,
$$

where $c_3$ and $c_4$ are the arbitrary constraints as well, and the upper sign corresponds to the positive control and bottom to negative. By a simple mathematical manipulation to eliminate $t$ as described in [69] system states can be expressed as a function of each other as follows.

$$
x_1(t) = \frac{1}{2A}x_2^2(t) + c_5
$$

$$
x_1(t) = -\frac{1}{2A}x_2^2(t) + c_6,
$$

where the first equation corresponds to the positive control, and the second one to the negative and $c_5$ and $c_6$ are arbitrary constants of integration. Equations (4.3) represent two families of parabolas in $x_1$-$x_2$ plane obtained by varying the constants, which are shown on Fig. 4.2(a). Red curves represent the $u(t) = +A$ control, whereas blue curves represent $u(t) = -A$ control.

For the convenience of discussion the minimum time problem boundary conditions formulated in Table 3.4 are redefined to have $-X$ position at time $t = t_0$ and zero position at time $t = t_f$ and allow the initial speed, $x_2(t_0)$, to have an arbitrary value. Final speed, $x_2(t_f)$, is kept zero. If the initial point $x(t_0)$ lies somewhere on segment $D$-$0$ of the $x_1$-$x_2$ plane, then the $u^*(t) = +A$ control is applied and the system reaches the origin in a minimum time. Similarly, if the initial point $x(t_0)$ lies on segment $B$-$0$, then $u^*(t) = -A$ and systems reaches the origin. In all other points of the $x_1$-$x_2$ plane,
in order to reach the origin first some control should be applied in order to reach one of the segments D-0 or B-0, where the previously mentioned control strategy is applied. It is clear from Fig. 4.2(a) that for any initial condition from the left hand side of D-0-B curve the only way to reach on of the aforementioned segments is to apply $u^*(t) = +A$; if so the segment B-0 is reached, where the optimal control should change the sign to $u^*(t) = -A$ and state slides to the origin. For the initial states from the right hand side of D-0-B curve the same strategy is applied but with opposite sign and sliding to the origin is performed along the D-0 segment. Hence this control is often referred to as “sliding mode” control.

Thus, the line D-0-B is known as a switching curve, where the optimal control changes its sign. It is obtained by setting both constants $c_5, c_6$ to zero, so the equation for the switching curve is shown in (4.4) and the optimal state trajectories for the random set of initial points are shown on Fig. 4.2(b).

$$x_1(t) = -\frac{1}{2A}x_2(t)|x_2(t)|$$

(4.4)

To summarize the final control strategy, it is convenient to introduce the switching function $s(x(t))$, which is defined as follows.

$$s(x(t)) \triangleq x_1(t) + \frac{1}{2A}x_2(t)|x_2(t)|$$

(4.5)

Then the close-loop optimal time control law for a double integrator with the control constraints is

$$u^*(x(t)) = \begin{cases} 
-A, & \forall x(t): s(x(t)) > 0 \\
+ A, & \forall x(t): s(x(t)) < 0 \\
-A, & \forall x(t): s(x(t)) = 0 \text{ and } x_2(t) > 0 \\
+ A, & \forall x(t): s(x(t)) = 0 \text{ and } x_2(t) < 0 \\
0, & \text{for } x(t) = 0.
\end{cases}$$

(4.6)

This control scheme relies on accurate knowledge of a system model and limiting values. In practice, multiple sampling is required to track the switching curve, resulting in chattering.

4.2 Optimal energy control

In section 3.4.2 the open-loop optimal energy control with a bounded input ($|u^*| \leq A$) was obtained using Pontryagin’s minimum principal for two cases: first, when the input limit is not reached and, second, when the optimal control is touching
the limits. In both cases final time $t_f$ is specified. The obtained control trajectories are as follows.

$$
u^*(t) = c_1 t + c_2
$$

$$
u^*(t) = \begin{cases}
-A, & p_2^*(t) > A \\
c_1 t + c_2, & -A \leq p_2^*(t) \leq A \\
+A, & p_2^*(t) < -A
\end{cases}
$$

(4.7)

In equation (4.7) for both first and second cases the arbitrary constants $c_1$ and $c_2$ are to be determined by the boundary conditions and final time $t_f$.

In the literature, the close-loop solution for the fixed final time case is not generally presented, but it is possible to follow the procedure applied in the optimal time control section, by starting plotting the state trajectories in $x_1$-$x_2$ plane. Here the results of the open-loop control (sections 3.4.1, 3.4.2) are used to plot these trajectories. Substituting the control trajectories from (4.7) to the state equations, state trajectories could be obtained. Since the situation, when the input limits are not reached is a subcase of the more general solution, when the control limits are activated, only the last one is studied.

For three different values of $p_2(t)$ state trajectories are obtained as shown in Table 4.1.

<table>
<thead>
<tr>
<th>Lagrange multiplier $p_2$</th>
<th>$p_2 \leq -A$</th>
<th>$-A \leq p_2 \leq A$</th>
<th>$p_2 \geq A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>State equations</td>
<td>$\dot{x}_1 = x_2$</td>
<td>$\dot{x}_1 = x_2$</td>
<td>$\dot{x}_1 = x_2$</td>
</tr>
<tr>
<td></td>
<td>$\dot{x}_2 = A$</td>
<td>$\dot{x}_2 = c_1 t + c_2$</td>
<td>$\dot{x}_2 = -A$</td>
</tr>
<tr>
<td>Resultant trajectory</td>
<td>$x_1 = \frac{1}{2A}x_2^2 + c_5$</td>
<td>$x_1 = \frac{1}{6}c_1 t^3 + \frac{1}{2}c_2 t^2 + c_3 t + c_4$</td>
<td>$x_1 = -\frac{1}{2A}x_2^2 + c_6$</td>
</tr>
<tr>
<td></td>
<td>$x_2 = \frac{1}{2}c_1 t^2 + c_2 t + c_3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The resultant trajectory in all three variants of multiplier $p_2$ values contains the constants of integration $c_1 ... c_6$, which are dependent on the boundary conditions and the final time $t_f$.

Similar to the previous section, boundary conditions formulated in Table 3.3 are redefined to have $-X$ position at time $t = t_0$ and zero position at time $t = t_f$. The initial speed, $x_2(t_0)$, is equal to zero. Given the results of open-loop control obtained in 3.4.2 the state trajectory in the $x_1$-$x_2$ plane can be plotted, Fig. 4.3. Two initial points are shown, depending on the sign of the desired position value. For the initially positive boundary value of the first state, defined in Table 3.3, trajectory lies in the left $x_1$-$x_2$
semi plane and for the negative, vice versa. The discussion is focused on the left hand side trajectory; for the right one it is the same up to the sign. According to (4.7) for zero initial speed the trajectory is symmetrical and finishes on the switching curve derived for the optimal time case, \(B - 0\), which corresponds to the \(u^* = -A\) control and is part of trajectory represented by a thick blue line on Fig. 4.3. The starting part of the trajectory is also similar to the minimal time control case, where the \(u^* = +A\) and is shown by a thick red line. In-between the areas of \(u^* = \pm A\) control, there is a part of a state trajectory which represents the linear control region \(u^*(t) = c_1 t + c_2\) and it is shown by a thick pink line. To determine the moment when the trajectory should switch from the one which corresponds to the \(u^* = +A\) control, to the linear control \(u^*(t) = c_1 t + c_2\), switching curves have been obtained. These curves are shown by black dots on Fig. 4.3 and are expressed in (4.8).

\[
\begin{align*}
    x_{1,sw1}(t) &= -\frac{1}{2A}x_2(t) - A \left( \frac{c_2}{c_1} \right)^2 + \frac{A^3}{3c_1^2} + 2A\frac{c_3}{c_1} + c_6 \\
    x_{1,sw2}(t) &= \frac{1}{2A}x_2(t) + A \left( \frac{c_2}{c_1} \right)^2 - \frac{A^3}{3c_1^2} - 2A\frac{c_3}{c_1} + c_5
\end{align*}
\]

where constants \(c_1 \ldots c_6\) are dependent on the initial position \(X\), final time \(t_f\) and time \(t_1\) found for the open-loop control. These curves were obtained by an observation that the speed at the transition between trajectories is the same, \(x_2(t_1) = x_2(t_f - t_1)\). Constants \(c_1 \ldots c_6\) could be determined as follows.
\[ c_1 = \frac{-2A \cdot \text{sign}(X)}{\sqrt{3t_f^2 - 12|X|}} \]
\[ c_2 = \frac{t_f A \cdot \text{sign}(X)}{\sqrt{3t_f^2 - 12|X|}} \]
\[ c_3 = -\frac{1}{2} c_1 t_1^2 + (A \cdot \text{sign}(X) - c_2) t_1 \]
\[ c_4 = -\frac{1}{6} c_1 t_1^3 + \frac{1}{2} (A \cdot \text{sign}(X) - c_2) t_1^2 - c_3 t_1 - X \]
\[ \begin{cases} c_5 = -X & c_6 = 0, \quad X \geq 0 \\ c_5 = 0 & c_6 = -X, \quad X < 0 \end{cases} \]

From (4.8), (4.9) it is clear that with the rise of final time, \( t_f \), switching curves tend to move toward the boundary of the area, limited by the OT state trajectories. If the curves pass through \((\pm X; 0)\) and \((0; 0)\) points in the \(x_1-x_2\) plane respectively, than the control limits are not reached. In that case the control action is the same as for an unconstrained problem, the subcase of the studied one. The corresponding state trajectory is shown in the dashed pink line on Fig. 4.3, which is determined by the equations in the second interval of \(p_2(t)\) variation from Table 4.1.

In section 4.1 for the optimal time case, for all initial points on the \(x_1-x_2\) plane except those lying on the \(D-B\) curve some form of control is maintained till the trajectory reaches the switching curve, where the sign of control is changed and further the trajectory is sliding to the origin. The switching curve in that case is independent of time or any other condition and depends purely on the relative combination of \(x_1\) and \(x_2\) coordinates. However, in the optimal energy case, switching curves in (4.8) are depending on final time \(t_f\) and are derived with the use of \(t_1\) time, obtained in the open-loop section and representing the time when switching from the constant control to the linear decreasing/increasing one occurs. It is impossible to derive the switching curves in \(x_1-x_2\) plane solely as a function of \(x_1\) and \(x_2\) because the final time \(t_f\) is set by the OT case solution, so algebraic/stationary equation (A.1.7) from Appendix 1 is not valid any more.

Due to impossibility of deriving a switching curve for a close loop case, another strategy to identify the switching instant between the OE trajectory sections should be found. For the ease of explanation, the desired system position is set positive (i.e. after the redefinition \(x_1(0) = -X < 0\)), so we are dealing with the left hand side of a \(x_1-x_2\) plane shown at Fig. 4.3. For such initial position initial optimal control \(u^*[0] = A\) is used, whenever the case when the control boundary is reached is studied. Then the task is to identify the moment of time \(t_1\) (it should be determined solely based on \(x_1,x_2\) and \(t_f\) values) when the control law should be switched to the linear one. The control is
performed in a discrete manner, so the problem of identification, should be understood as finding the number of discretization intervals, \( n \), after which the control is switched to the linear one, see Fig. 4.4.

Thus, at the first step \( u^*[0] = A \). In order to find a second/next step optimal control value, the feedback information \( x_1[1], x_2[1] \) (speed and position) is analysed. It was mentioned that, at the transition instant between the control regions (from constant to the linear and vica versa) the speed values should be the same \( x_2(t_1) = x_2(t_2) \), then knowing \( x_1[1], x_2[1] \) the system model can be used to predict if it is possible to move the system to \((X - x_1[1])\), in a given time \((t_f - T_s)\), where \( T_s \) is a discretization time by applying the linear control law. If it is possible, then the linear control law \( u^*[1] = c_1T_s + c_2 \) is applied as it is shown by the black dotted line on Fig. 4.4. If it is impossible to reach the desired position \( u^*[1] = A \) is applied and the procedure is repeated for the further steps. Finally after \( n \) steps, it becomes possible to move the system to \((X - x_1[n])\) in a given time \((t_f - nT_s)\) (the resultant control is shown by the red line), which means that the switching curve is reached and the optimal control is switched from \( u^*[n] = A \) to \( u^*[n] = c_1nT_s + c_2 \). Further, when following the linear control region, coefficients \( c_1 \) and \( c_2 \) should be adjusted at every step according to the feedback state information, desired position and final time.

To conclude, the OE control of the double-integrator system with the specified final time requires a prediction of future system behaviour at every time step for the whole remaining time period. Based on this prediction the decision regarding the optimal control action should be made.

Fig. 4.4 Optimal energy control
4.3 Optimal time control with both state and control constraints

Equation (4.6) presents the close-loop solution for the optimal time problem, however, it considers only the control constraints so the solution for the state (limited speed) should be obtained as well. In [71] a strict mathematical derivation is given, which is not based on Pontryagin’s minimum principle, instead the authors use a study of the phase trajectories of (3.15). Dividing the first equation from (3.15) by the second one it is possible to obtain

\[
\frac{dx_1}{dx_2} = \frac{x_2}{u(x_2)}
\]

(4.10)

By integrating both parts of (4.10) phase trajectories corresponding to control \( u(x_2) \) are shown in (4.10).

\[
x_1 = \mathcal{X}(x_2) + c
\]

(4.11)

where \( \mathcal{X}(x_2) = \int_0^{x_2} \frac{x_2}{u(x_2)} dx_2 \) and \( c \) is an arbitrary integration constant, which depends on the initial conditions (similarly to section 4.1 \( x(t_f) = 0 \), and the desired position is obtained by the offset for the initial position value). By analyzing different initial conditions, considering the maximum absolute value of speed, \( |x_2| \leq V \), authors in [71] obtain the optimal phase trajectories, which are shown on Fig. 4.5.

In the same way as for the problem that was only control constrained, the whole \( x_1 - x_2 \) plane is divided into two parts of positive and negative control by the switching curve, see Fig. 4.5, where the trajectories corresponding to \( u^* = +A \) control are shown in red, whereas the trajectories corresponding to \( u^* = -A \) control are in blue. However, due to the presence of the state constraints two more trajectories are added, corresponding to \( u^* = 0 \), when the system reaches its maximum speed. These

![Fig. 4.5 x1-x2 plane switching curve and optimal state trajectories](image-url)
trajectories are shown in black. According to Fig. 4.5, any complete optimal phase trajectory moving the system from an arbitrary initial point to the origin has at maximum three segments, representing the three different controls $u^* = 0, \pm A$. However it could contain fewer segments; two – if the maximum speed is not reached before the intersection with the switching curve, or one – if the initial point lies on the switching curve itself.

For the state and control constrained case the close loop solution is the same as for the control constrained one, but one condition is added: if the speed, $x_2$, reaches its maximum value $V$ the control is switched to $u^* = 0$, for all other cases control law stays the same, refer (4.6).

### 4.4 Optimal time control of the triple integrator system, with the control constraint

A very simple close–loop optimal time solution (4.6) has been obtained in the previous section for the OT control of double-integrator system. However, as mentioned in section 3.4, the double-integrator is a rough approximation of a real motion system, where the actuator is powered by some type of servomotor (DC or AC), which are inherently higher order systems than just double integrator. So it is worth checking how the complexity of the close-loop solution changes with the order of a system.

System (4.12) describes the linear motion, where control input, $u$, is jerk, $x_3$ is system acceleration, $x_2$ is a system speed and $x_1$ is system position. The block diagram of such system is presented on Fig. 4.6. It is worth noting that generally in the mechanical systems the jerk (system input) cannot change immediately, however, in the motor drive systems the voltage across the motor stator winding serves as a system input, which could be changed almost immediately (semiconductor switching speed) comparing to the other rest system’s time constants. Therefore it is worth exploring the close-loop solution for the system, where the control input or jerk is able to switch its value immediately.

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 & |u| \leq 1 \\
\dot{x}_3 &= u
\end{align*}
\]  

(4.12)

In [69] it is stated that the same procedure as described in section 4.1 could be used in order to find an optimal time solution of general $n$th-order system, however, the
solutions for the systems of higher than the second order are rather rare in literature. In [71] a detailed solution of the third order system optimal time problem is presented, where the problem is formulated as follows: Given the third order system (triple integrator), presented in (4.12), with the bounded input $|u(t)| \leq 1$, find the control law, $u^* = f(x_1, x_2, x_3)$, transferring the system from the initial state $x_1 = x_{10}, x_2 = x_{20}, x_3 = x_{30}$ to $x_1(t_f) = 0, x_2(t_f) = 0$ in a minimum possible time $t_f$.

In such problem formulation, the third state is not required to be zero at the final time $t_f$, however as $x_3$ is a system acceleration, finally it will be zero after applying closed-form control $u^* = f(x_1, x_2, x_3)$, which makes both speed and position to be zero at final time.

Utilizing the Pontryagin’s minimum principle in a similar way as was done for a double integrator system it is possible to find the minimum time solution in a form (4.13).

$$u^*(t) = \begin{cases} \sigma, & t \in (0, \theta_1) \\ -\sigma, & t \in (\theta_1, t_f) \end{cases}$$

(4.13)

where $\sigma = \pm 1$ and $\theta_1$ depends on the initial conditions $(x_{10}, x_{20}, x_{30})$. Knowing the initial conditions it is easy to find the switching instant $\theta_1$ and the sign of $\sigma$, but this would be an open-loop solution, which is not robust to even minor disturbances and model imperfections, therefore the close-loop solution is required. The full derivation of such solution is presented in Appendix 3. The solution is rather bulky and is applicable only for a given problem formulation, therefore only the final close-loop optimal time control law for a triple integrator system is presented in the main text.

At the first step, given the state feedback $x_1, x_2, x_3$ corresponding to the system position, speed and acceleration respectively, two new state variables are introduced, $\xi = x_1/x_3, \eta = \frac{x_2}{x_3|x_3|}$. Further on, these newly introduced state variables are used for control law formulation. At the second step, to introduce two new functions $\varphi(\eta) = \begin{cases} \Phi(\eta, 1, \frac{1}{2}), & \eta \leq \frac{\varphi}{4} \\ \Phi(\eta, -1, \frac{1}{2}), & \eta > \frac{\varphi}{4} \end{cases}$ and $\Phi(\eta, \alpha, C) = \alpha \eta - \frac{1}{2} + C |1 - 2\alpha \eta|^{3/2}$ are introduced.

Then for $x_3 > 0$ the control law is formulated as follows.
Chapter 4 Optimal Control under Uncertainty

\[ u^*(\xi, \eta) = \begin{cases} 
1, & \text{if } \xi < \varphi(\eta) \\
1, & \text{if } \xi = \Phi(\eta, 1, \frac{1}{3}), \ \eta \leq 0 \\
-1, & \text{in all other points of } \xi - \eta \text{ plane} 
\end{cases} \quad (4.14) \]

For \( x_3 < 0 \)

\[ u^*(\xi, \eta) = \begin{cases} 
-1, & \text{if } \xi < \varphi(\eta) \\
-1, & \text{if } \xi = \Phi(\eta, 1, \frac{1}{3}), \ \eta \leq 0 \\
1, & \text{in all other points of } \xi - \eta \text{ plane} 
\end{cases} \quad (4.15) \]

And for \( x_3 = 0 \), additional function need to be introduced \( \Psi(x_1, x_2, 0) = 3x_1 + 2x_2 \sqrt{|2x_2|} \) and

\[ u^*(x_1, x_2, 0) = \begin{cases} 
-\text{sign}\Psi(x_1, x_2, 0), & \text{if } \Psi \neq 0 \\
\text{sign}(x_1), & \text{if } \Psi = 0 
\end{cases} \quad (4.16) \]

Equations (4.14), (4.15) and (4.16) fully describe the control procedure for any feedback values \( x_1, x_2, x_3 \) and the control decision is made solely based on these three values, so the close-form solution of the problem is derived.

In Fig. 4.7 the example of an optimal trajectory in \( (x_1, x_2, x_3) \) plane after applying the developed control procedure is show. The green lattice represents the \( x_3 = 0 \) plane, and the phase trajectory is shown in red and blue colours depending on the control applied. Applying the developed control law the system is finally moved to the origin. It could be seen that as distinct from the conclusions in previous sections, the sign of \( x_3 \) is changed more than two times. This happens due to non-zero acceleration, \( x_3 \), at final

![Fig. 4.7 Example of \((x_1, x_2, x_3)\) space phase trajectory](image-url)
time $t_f: x_1(t_f) = 0, x_2(t_f) = 0$. So when the origin is reached on the $(x_1, x_2)$ plane, non-zero acceleration pushes the system away from zero point and the control procedure is repeated. Such repetition is responsible for an increased number of $x_3 = 0$ plane crossings.

4.5 Summary

The variational approach to the optimal control problem, exploiting Pontryagin’s minimum principle, is a classical way to solve optimization tasks, starting from the late 1950’s. It results in very simple close-loop solutions for simple tasks, which is perfect for a hardware realization using analog devices. However, as was mentioned in section 4.4, the process of searching for such a solution could be very difficult for even slightly higher order minimum-time control problems. The derivation becomes even more complicated when state constraints are involved. Also, it was shown in section 4.2, that the variational approach is not applicable for the minimum-energy problem, because it is necessary to predict the future behaviour of the system over all remaining operation time at every step of the closed-loop algorithm.

To conclude, for both OT and OE problems, some other optimal control algorithm is required. It should deal with both state and control constraints and for the OE case, have an ability to predict the behaviour of the system over some horizon. The choice of such algorithm and its implementation for the positioning systems will be shown in the following chapters.
Chapter 5 Offline OT/OE MPC for a Brushed DC Motor

In the previous chapters the solution of OT and OE problems using the variational approach applied to the simple motor models (double and triple integrator) was shown. However, such an approach is not suitable for more detailed and complex motor models. In this chapter the control of the positioning system powered by actuators based on DC motors is discussed. First, the conventional linear time-invariant DC motor model is developed following the discussion in [78]. Next, the differences with the double integrator representation are discussed and the necessity of the developed model for accurate positioning control is shown. The conventional approach for the 2D positioning systems' control based on G-codes and the possible improvements presented in the literature are introduced. And finally, the control strategy based on
simultaneous OT and OE control of real brushed DC motors using the Model Predictive Control (MPC) is developed. The feasibility of such a control scheme is proved both through the simulation results and experiments. The content of this chapter is based on the paper that has been presented at the 2017 IEEE International Electric Machine & Drives Conference (IEMDC).

5.1 Voltage and electromagnetic torque equations in machine variables

A practicable 2-pole permanent magnet (PM) brushed DC machine with the rotor (or armature) equipped with two parallel lap windings $a$ and $A$ is shown schematically (both windings consist of four coils) in Fig. 5.1. The constant field flux is produced by

![Diagram of DC machine](image1)

**Fig. 5.1 A DC machine a) $\theta_{rm} = 0$ b) $\theta_{rm} \approx 22.5^\circ$ counter-clockwise**

![Diagram of idealized DC machine](image2)

**Fig. 5.2 Idealized representation of the DC machine with uniformly distributed rotor windings during the commutation**
two permanent magnets fixed on the stator. Rotor coils are connected to the isolated copper segments forming a commutator (shown in white), mounted on the rotor shaft. Stationary carbon brushes (shown in dark grey) with a spring riding upon the copper segments are connecting the motor to the external DC voltage source. For the instant shown in Fig. 5.1(a) the top brush short-circuits the $A_4 - A'_4$ coil and the bottom one short-circuits the $a_4 - a'_4$ coil. The current from the DC-voltage source is flowing through the top brush into $a_1$ out $a'_1$, into $a_2$ out $a'_2$, into $a_3$ out $a'_3$ to the bottom brush closing the loop. Similarly in the parallel path current circulates through the top brush$-A_3 - A'_3 - A_2 - A'_2 - A_1 - A'_1$ -bottom brush. On a conductor path of length $l$ carrying the electrical current $i_a$ in a uniform magnetic field with magnetic induction $\vec{B}$ according to the Lorentz’s law, magnetic force, $F_A$, applies.

$$\vec{F}_A = i_a [\vec{l} \times \vec{B}]$$  \hspace{1cm} (5.1)

By definition, the north pole is the one from which the flux issues into the air gap, then in Fig. 5.1 the magnetic induction direction is from the top to the bottom of the stator and applying the (5.1) to each coil of the winding $a$ and $A$, the resultant magnetic force will make rotor to rotate in the counter clockwise direction. In Fig. 5.1(b) the further instant of the rotor position is shown. For such a rotor position none of the coils are short-circuited and the current from the DC-voltage source is flowing through the top brush into $A'_4$ out $A_4$, into $a_1$ out $a'_1$, into $a_2$ out $a'_2$, into $a_3$ out $a'_3$ to the bottom brush. Similarly in the parallel pass current circulate through the top brush$-A_3 - A'_3 - A_2 - A'_2 - A_1 - A'_1$ -bottom brush. By analysing the current path from the positive to the negative terminal of the DC-voltage source flowing through the rotor windings it is possible to show that for any rotor position the current flows into the top and out of the bottom rotor coils, so the commutator makes the rotor coils appear as a stationary winding. In practice the number of coils is usually considerably bigger than four, so for any rotor position the idealized permanent magnet DC machine is shown in Fig. 5.2, where the rotor coils are forming an equivalent uniformly distributed winding with a magnetic axis orthogonal to the PM one.

The system is assumed to be a linear magnetic system, so no saturation is considered, also the armature reaction is considered to be perfectly cancelled out and the voltage drop in the brushes is ignored. With these assumptions the DC machine voltage equation is as follows.
Chapter 5 Offline OT/OE MPC for a Brushed DC Motor

\[ v_a = R_a i_a + L_a \frac{d i_a}{dt} + e, \]  

(5.2)

where \( v_a \) stands for the terminal voltage, \( R_a \) and \( L_a \) are an equivalent armature winding resistance and inductance respectively and \( e \) is the voltage induced in the rotor coils due to the rotation of coil in the stationary magnetic field, which is commonly referred as a back electromotive force (back EMF). In [78], [79], [80] it is shown that the back EMF and the rotor shaft torque, \( T_e \), are related to the rotor speed, \( \omega_{rm} \), and armature current through the same constant usually referred as back EMF constant, \( \lambda_{pm} \left[ \frac{V}{\text{rad/s}} \right] \) or \( \left[ \frac{\text{Nm}}{A} \right] \).

\[ e = \lambda_{pm} \omega_{rm} \]
\[ T_e = \lambda_{pm} i_a \]  

(5.3)

Substituting (5.3) into (5.2) and considering (3.1) with zero static torque, \( T_{st} \), component the equations describing the electro-mechanical dynamics of a permanent magnet DC motor could be obtained as follows.

\[ v_a = R_a i_a + L_a \frac{d i_a}{dt} + \lambda_{pm} \omega_{rm} \]
\[ T_e = \lambda_{pm} i_a = J_e \frac{d \omega_{rm}}{dt} + B_{fr} \omega_{rm} \]  

(5.4)

\[ \omega_{rm} = \frac{d \theta_{rm}}{dt} \]

Motor equations in (5.4) can be rearranged to formulate linear time-invariant DC-motor model in a conventional matrix form.

\[
\begin{bmatrix}
\frac{d i_a}{dt} \\
\frac{d \omega_{rm}}{dt} \\
\frac{d \theta_{rm}}{dt}
\end{bmatrix}
= A \begin{bmatrix}
i_a \\
\omega_{rm} \\
\theta_{rm}
\end{bmatrix}
+ B \begin{bmatrix}1 \\
0 \\
0
\end{bmatrix}

v_a, \quad u
\]  

(5.5)

where the vector \( x^T = [x_1, x_2, x_3]^T = [i_a, \omega_{rm}, \theta_{rm}]^T \) represents the state variables vector and the control variable, \( u \), is equal to the armature voltage, \( v_a \). The linear constraints are formulated in (5.6).

\[ i_a \leq I_{a}^{\text{max}} \]
\[ i_a \geq -I_{a}^{\text{max}} \]
\[ v_a \leq V_{a}^{\text{max}} \]
\[ v_a \geq -V_{a}^{\text{max}} \]  

(5.6)
where the $I_{a}^{\text{max}}$ and $V_{a}^{\text{max}}$ are the maximum available armature current and voltage respectively. According to [79] the thermal limit is the reason for the current constraint. It could be decided either by inverter maximum current passing ability dictated by power semiconductor losses or by DC machine itself. Generally for higher power machines the machine thermal time constant is way higher than the inverter one, making the latter dominant for the current constraint value identification. The voltage constraint originates from the maximum available DC-link voltage. For unipolar switching in a four-quadrant DC-DC converter the maximum armature voltage is equal to the DC-link voltage, $V_{dc}$.

A control block diagram of a permanent magnet brushed DC-machine, representing the mathematical description (5.5) is shown in Fig. 5.3 and will be used for the controller development.

In the previous chapters the movement of a single axis of a MT positioning system was modelled using the double integrator representation. Here the state space representation from the third chapter is repeated.

$$
\frac{dv}{dt} = \frac{dx_2}{dt} = a = u
$$
$$
\frac{dx}{dt} = \frac{dx_1}{dt} = v = x_2
$$

where $x, v, a$ are the linear position, speed and acceleration respectively. It was assumed that the coupling between the motor drive and the load has an infinite stiffness, so the angular drive position and speed from (5.5) are proportional to the load linear displacement and speed. Due to this, the last equations in (5.5) and (5.7) are the same from the system dynamics point of view. In addition, if the viscous friction coefficient is relatively small, then the second element of the second row of matrix $A$ from (5.5) could be ignored and the second equation in (5.5) system will be dynamically equivalent to the first one in (5.7). That means that the major difference between the
motion double-integrator representation and the practical representation with the actuator powered by the DC-motor is the existence of an equation describing the current dynamics (first row in (5.5)). According to it, the current cannot be changed immediately, the rate of its change is described by the electrical time constant, \( \tau_{elect} = \frac{L_a}{R_a} \). The electrical time constant is generally smaller than the mechanical one, \( \tau_{mech} = \frac{R_ale}{J_{pm}^2} \), but still the immediate acceleration change required by the optimal time control of the double integrator (3.38) is impossible. The existence of acceleration dynamics is not the only difference between the (5.5) and (5.7) descriptions. When the system was described through the (5.7), simple linear state and control constraints were taken in account, i.e. \( |u| \leq A \) and \( |v| \leq V \), where \( A \) and \( V \) were representing maximum speed and acceleration values. Taking in account the (5.5) description these limits could be approximated through the motor parameters as follows.[68]

\[
A = \left. \frac{d\omega_{rm}}{dt} \right|_{max} = \frac{\lambda_{pm} l^a_{max}}{J_e} \\
V = \omega_{rm}^{max} = \frac{V^a_{max}}{\lambda_{pm}}
\]

(5.8)

However (5.8) is obtained with the assumption of zero current at maximum speed and obviously cannot fully represent the constraints (5.6) in the speed-acceleration plane. The first equation in (5.5) not only introduces the current dynamics, but also couples speed-acceleration constraints. In order to use the double integrator description of the single axis motion powered by the DC-motor one more constraint should be added.

\[
-\frac{V^a_{max} + R_a i_a}{\lambda_{pm}} \leq \omega_{rm} \leq \frac{V^a_{max} - R_a i_a}{\lambda_{pm}}
\]

(5.9)

(5.9) is obtained by ignoring the current dynamics (\( L_a \frac{di_a}{dt} \) is small). Due to (5.9) the original rectangular range describing the acceleration and speed constraints for the double integrator system is modified as shown in Fig. 5.4, where the blue area represents the range of the original double integrator constrained control and the red one is representing the modification of this area due to the DC-motor operation consideration, where \( V_c = \frac{V^a_{max} - R_a i_a}{\lambda_{pm}} \). Clearly, the range of possible system’s speed-acceleration combinations is reduced due to the voltage drop in the stator resistance. The effect is less significant for the higher power machines.
Due to the aforementioned drawbacks, specifically the neglect of current dynamics and the reduction of the possible system’s speed-acceleration range, the double integrator description is not able to accurately describe the positioning system behaviour and the system description (5.5) with the constraints (5.6) is used in this chapter to develop the optimal time and energy control for the axes movements of the MT’s positioning system.

5.2 Conventional control strategies for the 2D positioning system

In Fig. 5.5 a diagram of typical positioning solution by FESTO, which has already been presented in Chapter 3 is shown. Its working head (end-effector) is located next to the z-axis motor and can perform effective work (drilling, cutting, handling, etc.). It can be moved in three different directions x, y and z, however following the assumptions introduced in section 3.1 only x- and y-axis movements are considered. In the discussion of feed drive control two major aspects should be covered: first, the positioning control of a DC-motor powering a single axis actuator, second, the real-time end-effector trajectory interpolation methods. Further on, both these aspects are discussed in some detail.

5.2.1 Servo motor positioning control

Conventionally, according to [62], for a given single axis trajectory feed drive, the servo is controlled in a cascaded manner, the block diagram of such control is shown in Fig. 5.6. In order to enhance the performance of the motor drive, speed and acceleration references could be fed forward to precisely tuned speed and current PID controllers respectively. As it was mentioned earlier control requirements for a positioning drive
are limited to fast response time, zero overshoot, smooth transient response and minimum steady state error, where the importance of each requirement depends on the application. This is done by precise tuning of a servo motor PID control system, which is designed to follow the sophisticated speed profile.

The “Discrete Positioning Command” serving as a position reference in Fig. 5.6 is obtained by applying one of the trajectories widely presented in literature [81]. Probably the most common reference generation profile is the so-called “trapezoidal velocity”, which is shown in Fig. 5.7, where the maximum speed and acceleration are limited to the values denoted by \( V \) and \( A \) respectively. As it could be seen, the trapezoidal velocity profile is duplicating the optimal time control for a double-integrator system with state and control constraints (refer to (3.38)). It is simple to implement, does not require big computational power and used for low cost applications. However, because of the discontinuous change of acceleration, i.e. infinite jerk, the resultant torque or force acting on the ball screw contains high frequency components with significantly large amplitudes causing undesirable vibrations. In order to generate the smoother
acceleration profile jerk-limited profiles are introduced, for example “trapezoidal acceleration”, however in this thesis, for the sake of simplicity, the “trapezoidal velocity” profile is used as a conventional one for comparison with the proposed control method.

5.2.2 CNC real time interpolation methods

Positioning systems of the MT are designed to be able to follow the complex trajectories. In general trajectories are constructed from linear, circular or spline segments. However, for the point-to-point (off-cut) operations where the trajectory is not critical, two different approaches (known as G-codes for FANUC and similar systems [82]) are adopted to control the 2D positioning system movement. G00 is a rapid positioning mode, where both x- and y-axis motors are controlled at their maximum capability, which yields different end times of operation for each drive. G01 is a linear interpolation mode, where the drives are controlled to follow a linear trajectory connecting the 2D start and end points.

Fig. 5.7 Trapezoidal velocity profile [68]
When drives are controlled in G00 mode with a trapezoidal velocity profile, the end point of a trajectory is reached by the tool in minimum time, but this results in highly non-linear end-effector/spindle motion and the highest energy consumption. When drives are controlled in G01 mode, the end point is reached more slowly, due to the reduction of the maximum speed limit (feed rate). To illustrate the positioning system...
operation in the G00 and G01 modes, the drives trajectories in both modes are plotted in Fig. 5.8. The feed rate (speed limit) for the $x$-axis motor in G01 mode is chosen to be equal to the maximum possible feed rate corresponding to the G00. This is done to match the operation times in both modes, however, in real applications the feed rate in G01 is generally smaller.

The proposed control suggests that the axis’s servo drive with the longest reference trajectory is controlled in optimal time mode, whereas the drive with the shorter reference trajectory is controlled in optimal energy mode. Due to this, a new interpolation method is invented, which is as fast as G00, the resultant 2D trajectory is only slightly nonlinear, but it keeps the loss at a minimum level.

### 5.3 Proposed reduced energy control of the positioning system

In this section the novel control algorithm based on the proposed reduced energy control of the positioning system is developed. Two control tasks were formulated: optimal time (OT) and optimal energy (OE). In section 4.2 it was demonstrated that it is impossible to solve the OE task through the variational approach with given state and control constraints and final time even for a simple double integrator case. The reason is the requirement to predict the future system behaviour at each control step for the whole remaining period. Due to this, another algorithm to solve the optimization task is required. As soon as the prediction of system’s behaviour based on the system model is required, Model Predictive Control [83] is the obvious candidate to solve this optimization problem. MPC is one of so-called advanced control techniques (compared to the conventional proportional-integral-differential (PID) control), which has been successfully implemented in industry, especially chemical industry [84],[85]. In the past decade the application of MPC for the electric motor drives control [86] and power converters [87] has been reported as well. The major advantage of MPC is the inherent ability to deal with the constraints. MPC solves an optimization problem every sampling time in a receding-horizon fashion [88], where the design of the controller is based on a mathematical model of the plant [89]. It is worth noting that term “MPC” does not specify any exact controller design, rather it is describing the batch of control methods, where by using the plant model, the input control signal is defined by minimizing some objective function over a future horizon. The algorithm is repeated continuously in receding horizon fashion every sampling time, applying only the first calculated input signal and obtaining the updated state information.
5.3.1 Existing solutions of positioning systems control based on MPC

It is worth noting that no energy saving methods for the point-to-point operation of a positioning system based on MPC have been identified in the literature. However, there are a number of researchers who are applying MPC for biaxial systems (X-Y table or series kinematic systems) and it is worth mentioning these applications, though their major focus is accurate contouring control, i.e. minimization of the contouring error. The difference between contouring control and conventional tracking control is illustrated in Fig. 5.9, Fig. 5.10. The major difference is that in contouring control the path to be followed is not time-dependent. For example, in Fig. 5.9(a) the arbitrary 2D positioning system end-effector reference trajectory in x-y plane is shown. The

![Figure 5.9 Difference between the contouring and tracking control: a) Contouring control b) Tracking control](image)

![Figure 5.10 Contouring and tracking errors](image)
contouring error at arbitrary time instant, $t_k$, is defined as the normal deviation of the $x(t_k)$ and $y(t_k)$ displacements from the desired path and shown in Fig. 5.10 as $e_x$ and $e_y$ respectively [90]. However, contouring control is not directly applicable for the motor drives. In order to control the feed drive it is necessary to transform the time independent trajectory in the $x$-$y$ plane to the time dependent path (trajectory) for each feed drive, provided that feed drives are able to follow the obtained path. It is possible to achieve it in many different ways, however here the solutions applying the system model and MPC are going to be discussed in brief.

In [91] it is shown that for the desired path-following accuracy, the contouring error is more important than each individual drives tracking error, however, it is not easy to identify the contouring error on-line in each step of the control algorithm implementation. Instead of pure contouring error, $\varepsilon_c$, its approximation is used. A new reference frame is introduced connected to the reference position, with the abscissa in the tangential direction and axis of ordinates aligned with the normal direction in this point. Error component $e_n$ is used as an approximation of contouring error. To reduce the contouring error the MPC controller is designed, taking in account the system model describing the mechanical dynamics (DC-motor dynamics are not considered). The performance index is a weighted sum of the $e_n, e_t$ and the input armature voltages square values. Through the set of experiments authors prove the importance of the normal error component (therefore its weighting factor in the performance index is increased) and increased length of the prediction horizon.

MPC for contouring control is further developed in [92], [42]. The MPC cost function in this research represents the relative importance of contouring error versus path speed. In other words the path speed is automatically adjusted at the lower level of control in order to maintain the desired accuracy. The optimization task is solved online in a receding horizon fashion, subject to actuator and state constraints. In order to improve the accuracy of the $X$-$Y$ table model, a two-mass-system dynamic model is chosen to describe the mechanical part, while the motor dynamics are still not considered and the proposed controller output is used as an input to the conventional PI current controller of a motor drive. The control scheme was checked experimentally, showing the improved performance in terms of accuracy compared to the conventional PI and tracking controller. The controller is particularly useful for trajectories
containing huge number of tight angles due to the ability to reduce the path velocity around the angle.

The work in [92], [42] has inspired the research in Professor Ralph Kennel’s laboratory, where so called model predictive contouring control (MPCC) was further improved [93]. The controller is using the same contouring error approximation, which is the major component of the linear cost function. The feed drives are controlled using a dual step process. At the first step the time-dependent position reference for each drive is obtained from the original time independent 2D path considering that the path velocity is constant. These position references are used as an input to the reference governor block, which is using the standard MPC with the linear cost function and linear constraints (current and voltage limits of the motor drives) generates the updated desired trajectories (time-dependent position references). This is done off-line, therefore the computationally intense work is omitted during the online control, and the major difference from [92], [42] is motor dynamics consideration (current dynamics or voltage equation), which is important for small high speed CNC systems where the mechanical and electrical time constants are close to each other. At the second step the obtained references are sent to the unconstrained linear state-feedback controller. The controller is easy to tune, due to the cost function linearity and the presence of the full motor model (considering the finite stiffness between the motor and load) improves the path following accuracy. In [94] the ideas were further developed where the path following velocity is also included in the cost function, therefore the tradeoff between the accuracy and productivity could be achieved. Authors present some minor control approach modifications in [95], where the obtained trajectory tracking is carried out by the dynamic feed-forward control, which improves the tracking quality even during the dynamic operation.

However, in this thesis the application of MPC to the energy reduction is discussed. In further sections the mathematical problem description is given and the simulation and experimental results are presented.

5.3.2 Quadratic optimization problem

The obtained DC-motor model (5.5) is linear time-invariant, moreover in section 3.1 it was assumed that the copper losses are dominant (this is generally true for the DC-motor case, whereas for the permanent magnet synchronous motor the iron loss become comparable with the copper loss at higher speeds [96], but due to MT
positioning system’s motors operation profile copper loss domination is a valid assumption as well), which, as it has been already mentioned, is proportional to the square of the stator current and stator current in its turn is one of three state variables in DC-motor behaviour description (5.5). Due to this the Linear MPC is adopted for the OE controller development. It assumes a quadratic performance index (convex) corresponding to the copper loss, a linear system model (which are treated as equality constraints) and linear inequality constraints (physical limitation on the armature voltage and currents (5.6)).

Linear MPC is using the quadratic performance index due to the convenience of copper loss representation, in addition with the symmetric positive defined Hessian matrix in the objective function the necessary and sufficient conditions for optimality are always satisfied [97]. The general formulation of quadratic programming problem is as follows.

\[
\min_{\theta} f(\theta) = \frac{1}{2} \theta^T \mathcal{H} \theta + g^T \theta
\]

\[
s.t. \quad \mathcal{F} \theta = b
\]

\[
C \theta \leq d,
\]

where \( \theta \in \mathbb{R}^n \) is a decision vector, or vector of free variables for the optimization, \( \mathcal{H} \in \mathbb{R}^{n \times n} \) is the Hessian matrix, \( g \in \mathbb{R}^n \) representing the linear part of the objective function and \( \mathcal{F} \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^m, d \in \mathbb{R}^p \) are describing the linear equality and inequality constraints. The task is to find a decision vector \( \theta \) which minimizes the quadratic performance index, \( f(\theta) \), subject to constraints.

Loosely speaking, the solution of (5.10) is achieved when the gradient of \( f(\theta) \) could be represented as a linear combination of the gradients of both constraints, as long as they are satisfied, i.e. that the solution lies on the feasible domain of the equalities and inequalities constraints. In more rigorous mathematical form the necessary conditions for the (5.10) optimization problem can be described as the Karush-Kuhn-Tucher (KKT) conditions [89],[98].

\[
\nabla f + \lambda^T \nabla h + \mu^T \nabla \omega = 0
\]

\[
h = 0
\]

\[
\omega \leq 0
\]

\[
\mu^T \omega = 0
\]

\[
\lambda \neq 0, \mu \geq 0,
\]

[57x795]Chapter 5 Offline OT/OE MPC for a Brushed DC Motor

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where \( h = F\theta - b \) and \( \omega = C\theta - d \), \( \lambda \) and \( \mu \) are Lagrange multipliers of the equality and inequality constraints respectively. In literature several numerical methods were developed in order to solve the systems of equations (5.11). For example, an Active Set Method, where in each step of the algorithm the active set of inequality constraints is identified (inequality constraint \( C_i\theta \leq d_i \) is called active if \( C_i\theta = d_i \)) and then the algorithm is sliding on the surface defined by active set to the optimal point, the Primal-Dual Method, systematically identifies inactive constraints and leading to the simple programming procedure [89]. In this work the Interior-Point Method (IPM) is used to solve the optimization problem. The problem is solved by reducing it to the sequence of linear equality constrained problems, which are further solved applying the Newton’s method [99]. The discussion on pros and cons of each method and their implementation is beyond the scope of this research. To solve the optimization problem existing mathematical packages are used (see the further discussion). The major focus here is to set up the MPC formulation for the plant (5.5) with the inequality constraints (5.6) into the quadratic optimization problem (5.10).

### 5.3.3 Problem formulation in the MPC form

The formulation of OT and OE problems for the positioning system control is given in Table 5.1. The task is to express this problem into the MPC form, which is further expanded to the form of the QP problem and solved numerically.

#### A. Optimal energy control (OE)

The OE control is defined as follows. Given an initial state \( x_0 \), compute a finite-horizon, state variable sequence \( X = (x_0, x_1, ..., x_N) \) and an input sequence \( U = (u_0, u_1, ..., u_{N-1}) \) which minimize the quadratic performance index while guaranteeing that all the constraints (equalities and inequalities) are satisfied over the prediction horizon \( i \in 0, 1, ..., N \). The prediction horizon \( N \) corresponds to the given process final time \( t_f \). Therefore final position and speed are specified for the end of horizon \( N \), i.e \( x_N = x(t_f) \).

---

\(^4\) Increasing the prediction horizon to cover full process time is a necessary measure to be able to perform boundary constraints optimization task. The method to solve this task in the dissertation is called MPC, although the moving horizon window is an intrinsic characteristic of MPC, therefore, such a name is not really correct. However, it is used in Chapter 5 and Chapter 6 for the sake of simplicity, where the numerical algorithms used to solve the off-line optimization task are also developed for the on-line MPC as proposed in the future work on-line reference tracking.
Basically, there are two possible types of Linear MPC formulation:

- Dense MPC, where the decision vector $\theta$ contains only the input sequence, resulting in a dense Hessian matrix $\mathcal{H}$ formulation, i.e. most of its elements are non-zero;
- Sparse MPC, where the decision vector $\theta$ contains both state and input variable sequences, resulting in a sparse Hessian matrix $\mathcal{H}$, i.e. most of its elements are zero, while the non-zero elements are confined to the diagonal band of the matrix.

In this thesis the sparse formulation is preferred, because the sparse formulation of the Hessian matrix $\mathcal{H}$ is easier for the numerical calculation, moreover this formulation is easier for the boundary conditions incorporation.

To begin with, a more relaxed control problem, where the final state is free, i.e. there is no second boundary condition or $x(t_f)$ is undefined, is described. Later on the modifications corresponding to the presence of the second boundary condition are shown. The mathematical description of free-final states problem defined similarly to one in Table 5.1 for MPC is described in (5.12).

$$
\mathcal{J}_e = \arg\min_{x,u} x_N^T P x_N + \sum_{i=0}^{N-1} (x_i^T Q x_i + u_i^T R u_i) \\
\text{s.t. } x_{i+1} = A x_i + B u_i \\
x_{low} \leq x_i \leq x_{high} \\
u_{low} \leq u_i \leq u_{high}
$$

Table 5.1 Optimal problem formulation

<table>
<thead>
<tr>
<th>State Equations</th>
<th>Boundary Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} \frac{di_a}{dt} \ \frac{d\omega_{rm}}{dt} \ \frac{d\theta_{rm}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} &amp; \frac{\lambda_p}{L_a} &amp; 0 \ \frac{\lambda_p}{J_e} &amp; -\frac{B_f}{J_e} &amp; 0 \ 0 &amp; 1 &amp; 0 \end{bmatrix} \begin{bmatrix} i_a \ \omega_{rm} \ \theta_{rm} \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \ \theta_{rm} \end{bmatrix} v_o \begin{bmatrix} u \end{bmatrix}$</td>
<td>$x(t_0) = x_0 = \begin{bmatrix} 0 \ 0 \ 0 \end{bmatrix}, x(t_f) = \begin{bmatrix} 0 \ 0 \ X/Y \end{bmatrix}$</td>
</tr>
<tr>
<td>$t_f$ is not fixed for OT</td>
<td>$t_f$ is fixed for OE</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Physical Constraints</th>
<th>Performance Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = i_a \leq I_a^{max}$</td>
<td>$\mathcal{J}<em>t = \int</em>{t_0}^{t_f} dt \to \min_{v_a}$</td>
</tr>
<tr>
<td>$x_1 = i_a \geq -I_a^{max}$</td>
<td>$\mathcal{J}<em>e = \int</em>{t_0}^{t_f} i_a^2(t) dt \to \min_{i_a}$</td>
</tr>
<tr>
<td>$u = v_a \leq V_a^{max}$</td>
<td></td>
</tr>
<tr>
<td>$u = v_a \geq -V_a^{max}$</td>
<td></td>
</tr>
</tbody>
</table>
In (5.12) the equality constraints represent the system dynamics, where matrices $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$ are constant (time-invariant system) state matrices, defined in Table 5.1 (it is clear that the continuous-time model should be discretized first with some specified sampling interval $T_s$). $x_i \in \mathbb{R}^{n_x}$ is the state variable (armature current, rotor speed and position) with sequence defined earlier as $X = (x_0, x_1, ..., x_N)$; $u_i \in \mathbb{R}^{n_u}$ is the input variable (armature voltage) with the aforementioned sequence $U = (u_0, u_1, ..., u_{N-1})$. Affine inequality constraints represent the system physical limitations and include the minimum and maximum values allowed for the state and input variables, i.e. $l_{a}^{\text{max}}$ and $v_{a}^{\text{max}}$ for the DC-motor case. The quadratic performance index has three weighting matrices to define the relative importance of each variable, $P \in \mathbb{R}^{n_x \times n_x}$ for the final state $x_N$, $Q \in \mathbb{R}^{n_x \times n_x}$ for the state variable sequence $(x_0, x_1, ..., x_{N-1})$ and $R \in \mathbb{R}^{n_u \times n_u}$ for the input variables sequence $(u_0, u_1, ..., u_{N-1})$. The matrices $P$ and $Q$ are assumed to be positive semidefined and symmetric, while $R$ is just a strictly positive constant because the dimension of input variable is one. To reflect the copper loss domination in the performance index the first element of matrices $P$ and $Q$, relating to $x_1 = i_a$ has the highest weighting.

The task is to expand the MPC problem formulation (5.12) into the form of the quadratic problem defined in (5.10). In a sparse formulation the decision vector is defined as $\theta = [x_0^T, x_1^T, ..., x_N^T; u_0^T, u_1^T, ..., u_{N-1}^T]^T$, $\theta \in \mathbb{R}^{((N+1)n_x + Nn_u)}$. Then the Hessian Matrix together with the matrices describing equality and inequality constraints are defined as following.

- **Performance index**

$$\mathcal{H} = \begin{bmatrix} Q & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & Q & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & P & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & R & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & R \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (5.13)$$

where the dimensions are: $\mathcal{H} \in \mathbb{R}^{((N+1)n_x + Nn_u)\times((N+1)n_x + Nn_u)}$, $g \in \mathbb{R}^{(N+1)n_x + Nn_u}$

- **Equality constraints**

$$\mathcal{F} = \begin{bmatrix} -I & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ A & -I & 0 & \cdots & 0 & B & 0 & \cdots & 0 \\ 0 & A & -I & \cdots & 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A & -I & 0 & 0 & \cdots & B \end{bmatrix}, \quad b = \begin{bmatrix} x_0 \\ \vdots \\ 0 \end{bmatrix} \quad (5.14)$$

where the dimensions are: $\mathcal{F} \in \mathbb{R}^{((N+1)n_x \times (N+1)n_x + Nn_u)}$, $b \in \mathbb{R}^{(N+1)n_x}$
• Inequality constraints

\[ C = \begin{bmatrix} \omega & \cdots & 0 & 0 & \phi & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \omega & 0 & 0 & \cdots & \phi \\ 0 & \cdots & 0 & \omega & 0 & \cdots & 0 \end{bmatrix}, \quad d = \begin{bmatrix} v \\ \vdots \\ v \\ v \end{bmatrix} \tag{5.15} \]

where \( \omega = [-I, I, 0, 0]^T \), \( \phi = [0, 0, -I, I]^T \), \( v = [-x_{l\text{ow}}, x_{h\text{igh}}, -u_{l\text{ow}}, u_{h\text{igh}}]^T \) and \( I \) is an identity matrix. The matrices dimensions are \( \omega \in \mathbb{R}^{2(n_x+n_u)} \), \( \phi \in \mathbb{R}^{2(n_x+n_u)} \), \( v \in \mathbb{R}^{2(n_x+n_u)} \), \( d \in \mathbb{R}^{2(N+1)(n_x+n_u)} \), \( C \in \mathbb{R}^{2(N+1)(n_x+n_u) \times (N+1)n_x+n_u} \), \( d \in \mathbb{R}^{2(N+1)(n_x+n_u)} \), \( I \in \mathbb{R}^{n_x \times n_x} \).

After the substitution of (5.13), (5.14) and (5.15) to the QP problem defined in (5.10) the open-loop optimal energy control solution for the single actuator movement problem during a specified time \( t_f \) can be solved numerically by any standard numerical quadratic programing solver. However, in the original task described in Table 5.1 there was a requirement of specific distance \((X, Y)\) movement in a specified time \( t_f \) therefore the MPC formulation in (5.12) should be modified. In (5.16) the MPC formulation for the constrained states optimization problem is given. This formulation is important for any robotics application, where the controller is required to provide a specific system final position with zero velocity.

\[
J_e = \text{argmin}_{x,u} x_N^TPx_N + \sum_{i=0}^{N-1} (x_i^TQx_i + u_i^TRu_i)
\]

s. t. \( x_{i+1} = Ax_i + Bu_i \)
\( I_Nx_N = s_N \)
\( x_{l\text{ow}} \leq x_i \leq x_{h\text{igh}} \)
\( u_{l\text{ow}} \leq u_i \leq u_{h\text{igh}} \)

(5.16)

The only difference between the (5.12) and the (5.16) is one additional equality constraint (apart from the system model), which corresponds to the system final value of the states, i.e. at the end of prediction horizon, \( N \), or at final time \( t_f \). In (5.16) \( x_N \in \mathbb{R}^{n_x} \) is the state variable at the end of the horizon \( N \), \( I_N \in \mathbb{R}^{n_x \times n_x} \) is an identity matrix and \( s_N \in \mathbb{R}^{n_x} \) is a vector representing the second boundary condition, i.e. the defined states values at the end of prediction horizon \( N \). Due to the presence of the additional equality constraint, the only modification of the aforementioned process of MPC formulation transformation into the QP form happens in equation (5.14), where the additional “row” is added to both \( F \) and \( b \) matrices (5.17).
The dimensions of the newly obtained matrices are \( \mathcal{F} \in \mathbb{R}^{(N+2)n_x \times [(N+1)n_x+Nn_u]} \), 
\( \mathbf{b} \in \mathbb{R}^{(N+2)n_x} \).

With these modifications the original optimal energy control problem formulated in Table 5.1 could be solved numerically with \( x_{low} = -I_a^{\max} \), \( x_{high} = I_a^{\max} \) and \( u_{low} = -V_a^{\max} \), \( u_{high} = V_a^{\max} \). The result of these calculations forms the optimal control vector, \( \mathbf{U}^* = (u_0^*, u_1^*, ..., u_{N-1}^*) \), and the optimal states reference trajectories, \( \mathbf{X}^* = (x_0^*, x_1^*, ..., x_N^*) \), for the whole prediction horizon \( N \), corresponding to the process final time, \( t_f \). Two different methods are tested for such a numerical solution. First the matrices (5.13), (5.15), (5.17) of the corresponding QP defined above are manually implemented in MATLAB and then the standard numerical QP solver quadprog.m is used with the default “interior-point-convex” option. The results of this method were compared with the results of YALMIP, a free MATLAB Toolbox developed for a rapid prototyping of optimization problems [100], where instead of complex formulation in (5.13), (5.15), (5.17) it is possible to explicitly describe the optimization problem in the form (5.16). This Toolbox is also utilizes the quadprog.m solver, therefore the obtained results (the simulation and experimental results of DC-motor control could be found further in this chapter) exactly match each other.

### B. Optimal time control (OT)

The optimal energy solution for the DC-motor positioning control is discussed above, however the presented approach is not applicable for the optimal time control for a number of reasons. First the cost function is not quadratic, and in addition the process final time, \( t_f \), or in other words the prediction horizon, \( N \), is not given, therefore another approach is required. In literature many different methods to control the drive in the optimal time way are presented. For example, in [101] the modification for the conventional PI current control in the inner control loop of the electrical motor drive is suggested. The proposed method optimizes the transient of the current response and reduces the settling time by adding some simple current reference modifier. In [102], [103] the cascade-free controller is proposed, where the MPC is used to follow the state trajectory obtained by approximation of the motor dynamics through the solution of the
optimal time control problem applied for the double/triple integrator system (similar to the Chapter 4). Loosely speaking, the sliding mode controller with some additional tuning parameters to approximate the switching curve and get rid of chattering problem is used to obtain the reference values for the motor state variables, which are later feed-forwarded to the reference tracking MPC controller.

Both aforementioned approaches are not applicable for the multidrive control case. The major reason is the requirement to obtain the final process time, \( t_f \), before the implementation of OT control. This is because both drives of the positioning system, running in OT and OE modes, must start and finish their operation simultaneously, therefore the OT controller which predicts the final process time is required. Possible solution is described in [104] and called Time Optimal MPC (TOMPC). In TOMPC the optimization problem is formulated in the same way as (5.16) with some arbitrary prediction horizon \( N \). The feasibility of the solution is checked; if feasible, the prediction horizon is reduced and procedure repeats. After finding the unfeasible solution, the last feasible one is considered as time optimal; control and state reference trajectories are saved and last feasible \( N \) is considered as the process final time, \( t_f \). In this algorithm the objective from the OE case is implemented, but in practice the objective is not important, because the process time is gradually reduced until the only one feasible solution is left, no matter what the objective function is. The detailed optimization algorithm is as follows.

**Algorithm 1.** Optimization procedure. Redrawn from [104].

```
input: \( x(t_0), x(t_f) \)
output: input sequence \( U^* = (u_0^*, u_1^*, ..., u_{N-1}^*) \), state variables sequence \( X^* = (x_0^*, x_1^*, ..., x_N^*) \)
start with initial guess for \( N \)
solve QP problem defined in (5.16)
if (5.16) is feasible then
  while (5.16) was feasible do
    store \( U^*(N), X^*(N) \)
    \( N = N - 1 \)
  end while
else
  while (5.16) was infeasible do
```
\[ N = N + 1 \]

end while

store \( U^*(N), X^*(N) \)

end if

This procedure is time consuming due to the requirement of multiple (depending on the initial guess, \( N \)) QP numerical solution. However it is done off-line, therefore it is not that critical. The major advantage of the proposed method is that it explicitly provides the prediction horizon, \( N \), value, which can be substituted in to the energy optimal problem.

5.3.4 Close-loop OT and OE control

The OT and OE controllers proposed above provide the open-loop solutions which cannot be directly used in practice due to motor parameter variations and unpredictable external influences. Therefore the drives should be forced to follow the predefined reference trajectories [105]. This is achieved by feed-forwarding the MPC-calculated states to a conventional cascaded controller shown in Fig. 5.11. Feed-forwarding of the MPC-calculated states makes it possible to find the QP solution off-line, so on-line computational effort is profoundly reduced. The cascaded control with the feedforward has been chosen due to the excessive computational burden resulting from the on-line QP calculation, where, for example, with 10kHz valley sampling PWM the available control time is only 100us including the sampling time as well.

![Cascaded motor control diagram](image.png)

Fig. 5.11 Cascaded motor control [68]
5.4 Simulation results for the proposed control strategy

The proposed control method of the multi-motor system was checked by MATLAB simulation. The rapid positioning mode G00, linear interpolation mode G01 and the new one were compared according to the energy consumption for the case where the x-axis distance was 30% longer than the y-axis (9 and 12 motor shaft rotations respectively). The conventional resultant tool trajectories together with the one obtained by the proposed method are shown in Fig. 5.12, where \(x_e - x_s = X\) and \(y_e - y_s = Y\). The proposed trajectory is fairly close to G01, whereas the G00 mode control yields a so-called “hockey-stick” motion, which could be a problem for obstacle avoidance.

Experimental setup parameters presented in Table 5.2 are used for the simulation. For all three cases the motor was controlled by a 10kHz switching inverter and trapezoidal velocity profiles were assumed for drives in G00 and G01 modes. Following the above discussion the maximum drive speed was set as \(V = \frac{v_{a \text{max}}}{\lambda_{pm}}\) and maximum acceleration was set as \(A = \frac{\lambda_{pm} l_{a \text{max}}}{J_e}\). To make a fair comparison, the feed rate for the drive with a longer path in the G01 mode was kept on its maximum level. Feed-forward values for all three modes were used for the cascaded controller shown on Fig. 5.11. PD-P-PI

| \(R_a [\Omega]\) | 0.8 | \(J_e [\text{kg} \cdot \text{m}^2]\) | 14e-4 |
| \(L_a [\text{mH}]\) | 0.42 | \(v_{a \text{max}} [\text{V}]\) | 24 |
| \(\lambda_{pm} \frac{V}{\text{rad/s}}\) | 0.095 | \(l_{a \text{max}} [\text{A}]\) | 8 |

Table 5.2 DC motor parameters M642TE, 150[W]

![Fig. 5.12 Resultant tool trajectories](image)
controllers for position, speed and current respectively have been chosen (the controller tuning procedure is described in Appendix 4). The differential component is added to the position controller in order to suppress the vibration close to the set point value. The respective bandwidths are 3.5Hz, 35Hz and 350Hz. In order to make a fair comparison, gain tuning is the same for all modes of operation.

Simulation shows that for the given example, when \(x\)-axis path is equal to the 12 rotations and controlled in optimal time way (OT) and \(y\)-axis path is 9 rotations and controlled in optimal energy way (OE), the net energy consumed by both motor drives controlled in the proposed way is reduced by around 25% compared to mode G00 and 14% compare to G01. The simulation state and control trajectories of both drives controlled in the proposed way are shown on Fig. 5.13. Both drives fulfil the task in 0.36s and their states do not exceed the maximum values, however, the armature current of \(y\)-axis has lower peak magnitude and smoother current variation, making the system copper loss lower.

![Simulated state and control variables trajectories](image)

Fig. 5.13 Simulated state and control variables trajectories [68]
It is worth noting that during the MATLAB simulation the optimal control problem (5.16) was formulated in both (5.10) form through the (5.13), (5.15), (5.17) and directly from (5.16) using the YALMIP interface. For both approaches the exact match of the optimal state and control trajectories was achieved.

5.5 Experimental results

To verify the obtained results experimentally, the experimental setup for one of the motor drives was assembled. The appearance is shown in Fig. 5.14. Motors are controlled using a commercially-available Texas Instruments board equipped with a digital controller based on the TMS320F28379D DSP and PS21765 intelligent power module. The board is powered through the external DC power supply. In order to measure the electrical energy consumption a Yokogawa WT3000 power analyser is adopted. It is connected between the DC power supply and control board for accurate comparisons. The sampling rate of power analyser is 200kS/s. Precise positioning information is required by the proposed method so the rotor shaft is equipped with the 4000ppr incremental encoder.

Fig. 5.15 and Fig. 5.16 show the experimental results of the motor working in minimum time and minimum energy fashion. Experiments that reproduce the simulation conditions by tracking the MPC pre-calculated trajectories are presented. For both cases, the motor’s states follow the pre-calculated trajectory, but fail to reach the desired position at 0.46s. This happens due to the errors in motor parameters estimation and not
very precise cascaded controller tuning. However, the same phenomena is observed on conventional G00 and G01 control and does not influence the relative energy consumption.

For the proposed method the net energy consumption after 20 cycles of repetitions by each motor was 107mWh, whereas for G00 it was 123mWh and 116mWh – for G01\(^5\). So the improvement is 13% compared to the former and 8% compared to the latter. These results are lower than obtained by MATLAB simulation due to the neglect of friction and inaccuracies in the motor mechanical models in the simulation.

5.6 Discussion and possible control scheme modifications

In this chapter following the assumptions introduced in the Chapter 3 the static torque component is omitted from the system model (5.5) mostly due to the experimental setup limitations. However, this component could be noticeable, therefore it is worth discussing the way to incorporate the static torque component in the optimization task formulation. Static torque component, \( T_{st} \), could be augmented as a

\(^5\) WT3000 power analyser voltage measurement range is 15-1000[V] and current measurement range is 0.5-30[A] with the basic power accuracy of ±0.02% of reading.
state variable under the assumption that it is not changing during the operation, \( \frac{dT_{st}}{dt} = 0 \). Then the system model (5.5) will change as follows.

\[
\frac{d}{dt} \begin{bmatrix} i_a \\ \omega_{rm} \\ \theta_{rm} \\ T_{st} \end{bmatrix} = \begin{bmatrix} -R_a/L_a & -\lambda_{pm}/L_a & 0 & 0 \\ \lambda_{pm} / J_e & -B_{fr} / J_e & 0 & -1/J_e \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_{rm} \\ \theta_{rm} \\ T_{st} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_{gs} \tag{5.18}
\]

The remaining algorithm is kept unchanged. In the presence of the load torque the reference trajectory shape becomes more asymmetrical with respect to the system acceleration and deceleration, because the static torque “helps” the system to decelerate. To illustrate the optimal trajectory modification in the presence of static torque the section 5.4 simulation is repeated with the \( T_{st} = 0.15 \text{Nm} \) and the results are shown on Fig. 5.17. With a non-zero static torque, the process time increases and the period of negative current saturation decreases.

![Fig. 5.17 Reference state variables](image-url)
On Fig. 5.17 the optimal state trajectories resulted from the variational control method (Chapter 3) are shown by a green dotted line as well. It can be clearly seen that double integrator system model approximation is able to consider neither viscous friction nor static torque leading to underestimation of the required process time.

The top level block diagram of the proposed control scheme is shown in Fig. 5.18. The coordinates of starting point, \((x_s, y_s)\), and end point, \((x_e, y_e)\), during point-to-point operation are used as an input for the “Path generation” block. In this block, based on the obtained coordinates, the feed drive with longer path is identified and the OT state trajectory for this drive is calculated as it is described in section 5.3.3B. Together with the OT state trajectory the process final time, \(t_f\), is identified and using this final time the OE energy problem is solved for a second drive as it is described in section 5.3.3A. The obtained optimal state trajectories, \(X_x^*\) and \(X_y^*\), are transferred to the “Path tracking control”, where they are used as a feed-forward values in the conventional nested loop control described in section 5.3.4. The optimal armature voltages, \(v_{ax}^*\) and \(v_{ay}^*\), resulting from such kind of control are used as inputs to the conventional PWM modulators. Therefore, the proposed control scheme decouples the computationally intense path generation and the motor control itself.

It is worth noting that the proposed algorithm could be seen as a modification of the contouring control mentioned in 5.3.1, where the objective function is changed from the

![Fig. 5.18 Proposed control block diagram](image)
contouring error minimization to the energy consumption minimization and the input is changed from the time independent contour in x-y plane to the initial and final end-effector positions in the x-y plane. Loosely speaking, in the proposed algorithm the energy consumption is reduced by relaxing the requirement of the desired contour to follow (since this is not critical for the point-to-point operation).

It was mentioned earlier that in general the term “MPC” does not specify any particular controller design, but just a control principle, therefore it is worth noting the place of the suggested controller in the “MPC family” and the reasons why this particular realization is preferred. In the literature, two major MPC control methods are developed for the power converters: Finite Control Set MPC (FCS-MPC) and Continuous Control Set MPC (CCS-MPC) [106],[107]. The former one was used a lot to increase the current control bandwidth in high power applications, where the switching frequency is relatively small [108],[109], however these days it is successfully implemented to control various power electronics and drive applications as well [87],[110], [111]. FCS-MPC exploits the discrete-time nature of digital controller and motor inverter, where the inverter has a finite number of switching states. At each sampling time the system behaviour is identified (predicted) according to each switching state. The unfeasible solutions (exceeding the constraints) are omitted, then for the remaining states the cost function is evaluated and the optimal switching state is chosen, which minimizes the cost function. Clearly, this control strategy does not require an external modulator, and generates the variable switching frequency (only a single voltage vector is applied during the single switching period, which could be the same for a several consequent switching periods [106]). But the major drawback of this method for the positioning system optimization task solution is the prediction horizon limitation. Each possible switching state is checked; due to this when the horizon is sufficiently larger than one, some exhaustive search algorithm is required to check the performance of all possible sequences. The computational burden of such an algorithm grows exponentially with the growth of prediction horizon [112]. The optimization procedure for OE and OT control described above, which performs the optimization over whole prediction horizon, corresponding to the process final time, \( t_f \), makes the FCS-MPC inappropriate for OT and OE algorithm implementation.

In CCS-MPC, in contrast to FCS-MPC, the optimization results in continuous-time control signals, which are sent to PWM modulator, consequently a fixed switching
frequency is produced. There are two common optimization techniques in the power conversion field which use the CCS-MPC. The first, is the Generalized Predictive Control (GPC), which is introduced by Clarke in late 80s [113] and described in [86] in detail. GPC uses a transfer function model determining an analytical optimization problem solution at every sampling time in receding horizon fashion. However, this control strategy is generally applied for the unconstrained optimization problems, whereas the presence of constraints is the major issue in the OT and OE control of the positioning system feed drives. The second, is Explicit Model Predictive Control (EMPC) [114]. At the first step it solves the optimization problem parametrically and offline. The result is an explicit solution depending on measured states and control parameters, which is stored in the lookup table (LUT) for all possible states. During the on-line implementation step EMPC searches for the optimal control action in this LUT and applies the optimal control at every switching input. The algorithm is usually applied for the systems requiring the long prediction horizon, due to the limited computational burden during the online implementation if, for example, a binary search-tree algorithm is used. The major drawback is the necessity to store the LUT in a controller memory, whereas the memory size rises significantly with the rise of optimization problem size. In the proposed positioning system control online implementation is not required, therefore there is no need for the EMPC implementation.

Due to the aforementioned reasons the CCS-MPC formulation of the optimization problem was chosen in this thesis. Particularly, as it was shown in section 5.3, the Interior Point Method (IPM) is used as an optimization algorithm. For the offline implementation of IPM the conventional MATLAB quadprog.m could be used, whereas recently the efficient online IPM solvers were reported applicable for high speed implementations [115] and even for brushless PM motors [116].

In the proposed method, the constraints (current and voltage limits) were considered to be time-invariant, however, in [117] the active thermal management for the motor control is introduced. In that research unlike the discussion in section 5.1 it is assumed that the stator windings and rotor magnet (interior permanent magnet synchronous machine were used for experiments) temperatures determine the peak current choice. Then the method to modify the torque (current dependent) limit dynamically is introduced. It is also based on MPC, where the motor thermal model is used to predict the future motor temperatures and the limit is adjusted not to exceed the
Chapter 5 Offline OT/OE MPC for a Brushed DC Motor

motor temperature limits. Such a maximum current value (constraint) modification method could be applied for the proposed positioning systems’ energy reduction technique, leading to the reduction of process final time, $t_f$, especially at the start of MT operation, when the motors are not warmed-up. The reduction of positioning system operation time will lead to the whole MT operation time reduction which in its turn will result in the reduction of so called “Basic energy” consumption of MT (refer to the second chapter). The way to incorporate the “Active thermal management” block improving the performance is shown in the dotted line in Fig. 5.18. It uses the measured motor temperature, $temp_{x,y}$, to calculate the modified torque (current) limit for the “Path generation block”. However, the application of active thermal management is beyond the scope of current research.

5.7 Conclusion

In this chapter, the novel control strategy for the multi-drive systems is proposed. The approach is based on Model Predictive Control, where the full DC motor model is used to generate the OT and OE trajectories, which are further feed-forwarded to the conventional nested loop controller. The process of OE problem expansion to the conventional QP form is shown and the so called time optimal MPC (TOMPC) concept for the OT control is presented. The major advantage of TOMPC is the ability to predict the final process time, which is required for the OE task formulation.

The superiority of the introduced algorithm is supported by both MATLAB simulation and experiment. The energy consumption of the proposed method was compared with the conventional G00 and G01 methods, revealing the considerable (up to 13%) energy consumption reduction when the proposed algorithm applied.

The major drawback of the proposed method is that it could be seen as an open-loop control. The optimal trajectory for the motor drives is generated off-line based on the motor model. When the generated references are further used as a feedforward values to the cascaded motor control, the inaccuracy in the model parameters during the trajectory generation step could cause speed and position oscillations.

In the following chapter the cascaded control with the reference generated by MPC would be substituted with the on-line reference point tracking MPC, moreover, the application of more advanced and robust permanent magnet motor as a feed drive will be discussed.
Chapter 6  PMSM Control

In the previous chapter, the application of the proposed concept to control the positioning system was shown with the axes actuators powered by the DC-motors. However, these days the DC-motor is rarely used, mostly due to maintenance and reliability issues. Therefore, in modern MT mostly Surface Mounted Permanent Magnet Synchronous Machines (SMPMSM) are used to power the axes actuators. In this chapter the application of the proposed control concept to the positioning system with SMPMSMs is studied. At first, for the sake of readability the voltage and electromagnetic torque equations of the SMPMSM are introduced and by applying reference frame theory, the machine equations for stator variables in the rotor reference frame are derived (the detailed derivation could be found in [78], [79] and [118]). Further on the model nonlinearities are discussed together with the conventional approaches to deal with them. A novel way to generate the OT and OE trajectories
based on nonlinear plant and linearized constraints is proposed and a simulation example using these trajectories as an input for the reference tracking MPC is presented.

6.1 Voltage and electromagnetic torque equations in machine variables

An equivalent model of a 3-phase wye-connected notional SMPMSM with distributed windings and an isolated neutral point is shown in Fig. 6.1. This machine is used for the voltage equations development following the procedure described in [79],[78] and [118]; later these equations are modified by considering an arbitrary pole number, \( p \). The stator windings \( a_s, b_s, c_s \) are balanced (same impedance in each phase), spatially displaced from each other by 120°, generate ideal sinusoidal air gap flux and have an equivalent resistance, \( R_s \). The rotor is considered to be non-salient with two pole-pairs and magnets’ field flux, \( \lambda_{pm} \). The system is assumed to be a linear magnetic system, so no saturation is considered. The air gap between the stator and rotor is uniform.

In Fig. 6.1 \( \theta_r \) is the angle from the \( \alpha \)-phase magnetomotive force (MMF) of the stator axis to the rotor main flux direction (rotor direct or “\( d \)” axis). \( \theta_r \) is also known as an electrical angular displacement which is related to the mechanical rotational angle \( \theta_{rm} \) through the following relation.

\[
\theta_{rm} = \frac{\theta_r}{pp},
\]

where \( pp \) is a pole pair number, which is equal to the half of actual pole number, \( p \). Mechanical speed, \( \omega_{rm} \), is a time derivative of \( \theta_{rm} \) in rad/s. With the aforementioned
assumptions, the stator phase voltage equations can be represented as follows.

\[ v_{as} = R_s i_{as} + \frac{d\lambda_{as}}{dt} \]
\[ v_{bs} = R_s i_{bs} + \frac{d\lambda_{bs}}{dt} \]
\[ v_{cs} = R_s i_{cs} + \frac{d\lambda_{cs}}{dt} \]

(6.2)

Magnetic flux linking the \( as \) stator winding (flux linkage \( \lambda_{as} \)) consists of a three components: flux generated by current passing through the winding itself, flux generated by other stator windings and the rotor magnet flux. Part of the \( as \) winding generated flux is linking only with itself and generally called leakage flux; another flux part is linking the rest of stator windings and generally called main flux. For the two identical windings with the same current and their axes aligned in space the magnetic flux generated by the first winding and linking the second one is identical to the flux generated by the second winding and linking the first one. However, in the given case the windings are 120° apart so the magnetic flux change is proportional to the cosine of the shift angle [119]. The same principle applies for the flux generated by the rotor magnet. The resultant PM flux linkage is shifted (due to the rotor rotation) from the \( as \) winding axis through \( \theta_r \) degrees and its peak amplitude is denoted by \( \lambda_{pm} \), which is known as a back EMF constant. For the linear magnetic system, flux is linearly proportional to the current passing through the winding and for the \( as \) phase it could be written as follows.

\[ \lambda_{as} = L_{ls} i_{as} + L_{ms} i_{as} + L_{ms} i_{bs} \cos(120^\circ) + L_{ms} i_{cs} \cos(-120^\circ) \]
\[ + \lambda_{pm} \cos \theta_r, \]

(6.3)

where \( L_{ls} \) represents the leakage inductance of the stator winding and \( L_{ms} \) stands for a mutual inductance. The same principle is applied for the \( bs \) and \( cs \) stator windings. Representing the currents and flux linkages from (6.2) in matrix form, where the bold \( \lambda_{abcs} \) stands for a vector \([\lambda_{as}, \lambda_{bs}, \lambda_{cs}]^T\) and bold \( i_{abcs} \) for a current vector \([i_{as}, i_{bs}, i_{cs}]^T\) the flux linkages are derived as in (6.4).

\[
\begin{bmatrix}
L_{ls} + L_{ms} & \frac{1}{2} L_{ms} & \frac{1}{2} L_{ms} \\
\frac{1}{2} L_{ms} & L_{ls} + L_{ms} & \frac{1}{2} L_{ms} \\
\frac{1}{2} L_{ms} & \frac{1}{2} L_{ms} & L_{ls} + L_{ms}
\end{bmatrix}
\begin{bmatrix}
i_{abcs} \\
i_{abcs} \\
i_{abcs}
\end{bmatrix}
+ \begin{bmatrix}
\cos \theta_r \\
\cos \left( \theta_r - \frac{2\pi}{3} \right) \\
\cos \left( \theta_r + \frac{2\pi}{3} \right)
\end{bmatrix}
\lambda_{pm},
\]

(6.4)
where each \( i,j \) component of the inductance matrix \( L_s \) represents the inductance between the \( i \) and \( j \) windings, it is clear because the circuit is reciprocal \( L_{s_{i,j}} = L_{s_{j,i}} \).

Substituting (6.4) into (6.2) the voltage equation of the SMPMSM in matrix form can be obtained.

\[
v_{abc} = R_s i_{abc} + L_s \frac{d i_{abc}}{dt} + \lambda_{pm} \frac{d K}{dt},
\]

(6.5)

where \( v_{abc} \) is a stator voltage vector \([v_{as}, v_{bs}, v_{cs}]^T\). The last component in the equation (6.5) makes it rotor-position dependent (for the matrix \( K \) definition refer the equation (6.4)) or in other words (6.5) is time-varying, leading to difficulties in analysis and control law development. In order to get rid of the time-varying term and remarkably reduce the complexity of voltage equations the variable transformation was introduced by R.H. Park [120]. The variable transformation from the stationary reference frame to the one attached to the rotor may be expressed using matrix algebra (matrix \( K_r^s \) is a transformation matrix from stationary to rotor reference frame), which is more convenient for the computer simulation as follows.

\[
f_{dq0}^r = K_r^s f_{abc} = \frac{2}{3} \begin{bmatrix}
    \cos \theta_r & \cos \left( \theta_r - \frac{2\pi}{3} \right) & \cos \left( \theta_r + \frac{2\pi}{3} \right) \\
    -\sin \theta_r & -\sin \left( \theta_r - \frac{2\pi}{3} \right) & -\sin \left( \theta_r + \frac{2\pi}{3} \right) \\
    1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}
\begin{bmatrix}
    f_a \\
    f_b \\
    f_c
\end{bmatrix},
\]

(6.6)

where \( f_{abc} \) and \( f_{dq0}^r \) are an arbitrary columns of 3-phase variables in stationary and rotor reference frame and \( \theta_r \) is the angle between the magnetic axis of \( a \)-phase and rotor \( d \)-axis and is derived as

\[
\theta_r(t) = \int_{t_0}^{t} \omega(\xi) \, d\xi + \theta_r(t_0),
\]

(6.7)

where \( \theta_r(t_0) \) is an initial time displacement which is set as zero.

Applying the variable transformation defined in (6.6) to the SMPMSM stator equations (6.5) they are expressed in the rotor reference frame.

\[
K_r^s v_{abc} = R_s K_r^s i_{abc} + K_r^s L_s (K_r^s)^{-1} \frac{d i_{dq0}^r}{dt} + K_r^s L_s \frac{d (K_r^s)^{-1}}{dt} i_{dq0}^r
\]

(6.8)

Using the MATLAB Symbolic toolbox [121] the matrices in (6.8) could be calculated as following.

\[
K_r^s L_s (K_r^s)^{-1} = \begin{bmatrix}
    L_{ls} + \frac{2}{3} L_{ms} & 0 & 0 \\
    0 & L_{ls} + \frac{3}{2} L_{ms} & 0 \\
    0 & 0 & L_{ls}
\end{bmatrix} \frac{d (K_r^s)^{-1}}{dt} = \omega_r \begin{bmatrix}
    -\sin \theta_r & -\cos \theta_r & 0 \\
    -\sin \left( \theta_r - \frac{2\pi}{3} \right) & -\cos \left( \theta_r - \frac{2\pi}{3} \right) & 0 \\
    -\sin \left( \theta_r + \frac{2\pi}{3} \right) & -\cos \left( \theta_r + \frac{2\pi}{3} \right) & 0
\end{bmatrix}
\]

(6.9)
Due to the assumption of the symmetrical PMSM winding with an isolated neutral point there is no neutral or zero sequence current, since \(i_{as} + i_{bs} + i_{cs} = 0\), and the last row in (6.8) does not need to be considered, consequently only \(dq\) voltage components are calculated. Substituting (6.9) into (6.8) simple stator voltage equations are obtained.

\[
\begin{bmatrix}
    v^r_{ds} \\
    v^r_{qs}
\end{bmatrix} = 
\begin{bmatrix}
    R_s + L_s p & -\omega_r L_s \\
    \omega_r L_s & R_s + L_s p
\end{bmatrix}
\begin{bmatrix}
    i^r_{ds} \\
    i^r_{qs}
\end{bmatrix} + 
\begin{bmatrix}
    0 \\
    \omega_r \lambda_{pm}
\end{bmatrix},
\]

where \(v^r_{ds}, v^r_{qs}\) and \(i^r_{ds}, i^r_{qs}\) are stator voltages and currents in the \(dq\) reference frame, \(L_s = L_{ts} + \frac{3}{2} L_{ms}\) and \(p\) is the differential operator. Unlike the (6.2) equations (6.10) are time-invariant (for a constant speed), moreover if the balanced 3-phase voltages are applied to the stator windings the resultant \(dq\) steady state stator currents have pure DC values making the machine model easier from control point of view. However the resultant system is nonlinear (multiplication of current component and the rotor speed) and becomes linear only in a steady state.

To describe the mechanical dynamics of SMPMSM, i.e. rotor position and speed the electromagnetic torque, \(T_e\), equation is required, where the particular stator current is associated with the torque on the motor’s shaft. The torque can be evaluated based on electromagnetic energy conversion principle considering the power balance. The motor output power (multiplication of the electromagnetic torque and rotor mechanical speed) is obtained by subtraction of the resistive loss and rate of change of stored magnetic energy in the coupling field from the input electrical power.

\[
P_{in} = i^T_{abc} v_{abc} = \left[ (K^T_s)^{-1} i^T_{dq} \right] \left[ (L^T) R_s \right] = \frac{3}{2} (\dot{\omega}_r)^T \lambda_{pm} = \frac{3}{2} \left( \dot{\omega}_r \right)^T \omega_r = \frac{3}{2} \left( \dot{\omega}_r \right)^T \lambda_{pm}
\]

where \(p\) denotes the matrix transpose, \([L]\) is matrix of multipliers of the derivative operator \(p\) coefficients and \([G]\) coefficients of electrical rotor speed, \(\omega_r\). The first two components in (6.11) correspond to resistive loss and rate of change of stored energy and do not contribute to the electromechanical torque. Due to this the output power could be calculated as follows.

\[
P_{out} = T_e \omega_{rm} = \frac{3}{2} (\dot{\omega}_r)^T \lambda_{pm} = \frac{3}{2} \left( \dot{\omega}_r \right)^T \omega_r = \frac{3}{2} \left( \dot{\omega}_r \right)^T \lambda_{pm}
\]

From (6.12) and (6.10) the resultant electromagnetic rotor torque is obtained as
Chapter 6 PMSM Control

\[ T_e = \frac{3p}{2} \lambda_{pm} i_{qs}^r \] (6.13)

Before the full SMPMSM model formulation, it is important to note that the phase magnitude invariant method of variable transformation has been chosen, so the voltage and current constraints in \( dq \) reference frame do not change their magnitude and the final nonlinear time-invariant mathematical model which is going to be used for further analysis is obtained as

\[ \frac{di_{ds}^r}{dt} = \frac{1}{L_s} (-R_s i_{ds}^r + \omega_r L_s i_{qs}^r + v_{ds}^r) \]
\[ \frac{di_{qs}^r}{dt} = \frac{1}{L_s} (-R_s i_{qs}^r - \omega_r L_s i_{ds}^r - \omega_r \lambda_{pm} + v_{qs}^r) \]
\[ \frac{d\omega_r}{dt} = \frac{1}{J_e} \left( \frac{3p^2}{2} \lambda_{pm} i_{qs}^r - B_{fr} \omega_r - \frac{T_L p}{2J_e} \right) \]
\[ \frac{d\theta_r}{dt} = \omega_r \]

where the \( J_e \) is the rotor shaft inertia, \( B_{fr} \) is a viscous friction coefficient and \( T_L \) is a static torque, which is set to zero in this work. In (6.14) \( x = [i_{ds}^r, i_{qs}^r, \omega_r, \theta_r]^T \) is considered as a state variables vector and \( u = [v_{ds}^r, v_{qs}^r]^T \) is a control variables vector.

The constraints applied for the system are similar to the one introduced earlier in Chapter 5, i.e. motor/inverter thermal limit is dictating the maximum phase peak current limit, \( I_{s \text{max}} \), whereas the DC-link voltage defines the maximum phase peak voltage limit, \( V_{s \text{max}} \). However though the physical nature of the constraints is the same, several changes are apparent, because the system now has three phases and the Park/Clarke variable transformation is applied during the motor model development. Indeed, the per-phase current limit is generally available in the datasheets and applying the variable transformation (6.6) the current constraints are defined as follows.

\[ |i_{abc}| = \begin{bmatrix} |I_s \sin(\omega_r t)| \\ |I_s \sin(\omega_r t - \frac{2\pi}{3})| \\ |I_s \sin(\omega_r t + \frac{2\pi}{3})| \end{bmatrix} \leq \begin{bmatrix} I_{s \text{max}} \\ I_{s \text{max}} \\ I_{s \text{max}} \end{bmatrix} \Rightarrow K_S^r |i_{abc}| = \begin{bmatrix} |I_s \sin(\omega_r t - \theta_r)| \\ |I_s \cos(\omega_r t - \theta_r)| \end{bmatrix} = \begin{bmatrix} I_{s \text{max}} \\ 0 \end{bmatrix} \] (6.15)

From (6.15) it is clear that \( d-q \) currents are shifted 90 degrees apart. In order to ensure that phase currents are not exceeding the limit, the \( d-q \) currents should satisfy the
inequality (6.16), which describes the circle of radius $I_s^{max}$ in the $d$-$q$ current reference plane.

$$(i_{ds}^r)^2 + (i_{qs}^r)^2 \leq (I_s^{max})^2 \quad (6.16)$$

Unlike the DC-motor case, the maximum phase voltage of SMPMSM is not equal to the DC-link voltage. For a conventional inverter feeding the motor windings from the DC-link capacitor the maximum available phase voltage is defined by the modulation strategy used for the inverter control. The typical two-level, three-phase voltage source inverter (VSI) is shown in Fig. 6.2, where $n$ is a conceptual neutral point, $S_a, S_b, S_c$ are the switching functions with their complementary $\overline{S}_a, \overline{S}_b, \overline{S}_c$ and $e_{as}, e_{bs}, e_{cs}$ are back EMF voltages. The amplitude of motor phase voltages $v_{abcs}$ could be controlled through the pole voltages $v_{abcn}$ and offset voltage $v_{sn}$, where $v_{abcn} = v_{abcs} + v_{sn}$. If the offset voltage is constant, $v_{sn} = \frac{v_{an} + v_{bn} + v_{cn}}{3}$ and each pole voltage is changed once in a period from its maximum value, $\frac{V_{dc}}{2}$, to the minimum value, $-\frac{V_{dc}}{2}$, then the so called “six step operation” is achieved, where the phase voltage can take values between $\frac{2}{3}V_{dc}$ to $-\frac{2}{3}V_{dc}$ and varies six times for a period. For the “six step operation” the maximum achievable peak phase voltage is equal to $\frac{2}{3}V_{dc}$ with the fundamental harmonic peak amplitude equal to $\frac{2}{\pi}V_{dc}$. This is the maximum peak voltage amplitude available from the VSI shown in Fig. 6.2, but it is fixed (not controllable) and rarely used for the VSD. If the offset voltage is set to zero, the phase and pole voltages are equal to each other, so the motor phase voltage could be easily controlled by so called sinusoidal pulse width modulation (SPWM). The switches are controlled according to the comparison of a

![Fig. 6.2 Three phase voltage source inverter](image-url)
triangular carrier wave with $\frac{V_{dc}}{2}$ amplitude and the reference pole voltage equal to the reference phase voltage. In that case the maximum achievable peak phase voltage magnitude is limited to value $\frac{V_{dc}}{2}$. If the offset voltage at each sampling instant is set to the average value of the maximum and minimum reference phase voltages multiplied by -1, i.e. $v_{sn} = -\frac{v_{abc,max} + v_{abc,min}}{2}$, than the so called offset PWM operation is achieved, which is able to feed the motor with the voltages up to $\frac{V_{dc}}{\sqrt{3}}$ for the maximum peak amplitude. The space vector pulse weight modulation (SVPWM) uses a different principle for the phase voltage generation. At every sampling instant the space vector is formulated using the reference phase voltages in the abc frame through the following expression.

$$V^* = \frac{2}{3}(v_{as}^* + av_{bs}^* + a^2v_{cs}^*),$$  \hspace{1cm} (6.17)

where $a = e^{j\frac{2\pi}{3}}$. Then this reference voltage in space vector form is synthesized using two adjacent active vectors ($V_1 \sim V_6$) and two zero vectors ($V_0, V_7$) for a switching period, in other words the periods of time when each switching function ($S_\alpha, S_\beta, S_\gamma$) is equal to one is calculated, then the maximum phase voltage is located in the yellow area shown in Fig. 6.3. The phase voltage limit is shown on the stationary or $\alpha$-$\beta$ plane, so generally the SVPWM voltage vector domain is approximated by the circle inscribed in the hexagon to consider the space vector rotation. It could be noted that in this case the
maximum peak voltage produced by SVPWM is equal to the maximum peak voltage of the offset PWM and equals to $\frac{V_{dc}}{\sqrt{3}}$. The phase voltage limits for all discussed methods are shown in the figure as well.

Similarly to the current constraints the presented voltage constraints are defined in the stationary or $\alpha$-$\beta$ reference frame, whereas the control variables vector $\mathbf{u}$ in (6.14) is given in the rotor reference frame, so the input constraints should be specified in this reference frame as well. If the SVPWM is used, than the voltage constraints are dynamically changing according to the rotation angle and could be defined as in (6.18).

$$f(v_{ds}^r, v_{qs}^r, \theta_r) \leq g(V_{dc}),$$  \hfill (6.18)

where $f(\cdot)$ and $g(\cdot)$ are some complex functions describing the stationary hexagon in the rotation frame. As mentioned earlier, the space vector domain is approximated as a circle in the stationary frame, which is transformed to a circle with the same radius in the rotary reference frame and the voltage constraints for both SVPWM and offset PWM are formulated in (6.19).

$$v_{ds}^r)^2 + (v_{qs}^r)^2 \leq (V_s^{\text{max}})^2,$$  \hfill (6.19)

where $V_s^{\text{max}}$ is equal to $\frac{V_{dc}}{\sqrt{3}}$.

### 6.2 Model nonlinearities. Field weakening control

In the previous section, the state space model of the SMPMSM has been derived in the $d$-$q$ reference plane, together with the current and voltage constraints. It was mentioned that the model is nonlinear, furthermore, the constraints are nonlinear as well. In this section the model nonlinearities and their consequences are discussed.

The model (6.14) is nonlinear due to the multiplication of the states in the first and second differential equations ($\omega_r i_{qs}^r$, $\omega_r i_{ds}^r$). These nonlinear terms are generally called cross-couplings, as a result of the second state’s value, $i_{qs}^r$, influence on the dynamics of the first state, $i_{ds}^r$, and vice versa. The presence of the cross-coupling terms leads to the so called field weakening regime of operation of SMPMSM, making it possible to reach higher speed under the same current and voltage constraints. The ability to reach the higher speed is critically important for the optimal time control, therefore the origins of field weakening are briefly summarized here following the discussion in [79].

Both voltage and current constraints described above (6.16), (6.19) are given in the $d$-$q$ reference frame, however, the former is presented in a current plane, whereas the
latter is in a voltage plane. In order to consider simultaneously both constraints they are depicted at the same plane, where the current one is the natural choice due to the electromagnetic torque dependency on the motor current. Substituting the $d-,q$- voltages from (6.14) into (6.19) and considering the steady state operation ($\frac{dt_ {\text{qs}}}{dt} = 0$) the voltage constraints in the current plane are represented by the inner area of the following circle.

\[
\left( i_{ds}^r + \frac{\omega_r L_s \lambda_{pm}}{R_s^2 + \omega_r^2 L_s^2} \right)^2 + \left( i_{qs}^r + \frac{\omega_r R_s \lambda_{pm}}{R_s^2 + \omega_r^2 L_s^2} \right)^2 \leq \left( \frac{V_s^\text{max}}{R_s^2 + \omega_r^2 L_s^2} \right)^2
\]  

(6.20)

For simplicity the stator resistance could be neglected, especially if operation at high speeds is discussed, then (6.20) is simplified.

\[
\left( i_{ds}^r + \frac{\lambda_{pm}}{L_s} \right)^2 + \left( i_{qs}^r \right)^2 \leq \left( \frac{V_s^\text{max}}{\omega_r^2 L_s^2} \right)^2
\]  

(6.21)

(6.21) describes a circle in the $i_{ds}^r, i_{qs}^r$ plane with its center at $[-\lambda_{pm} / L_s; 0]$ and with the radius continuously reducing with the rise of rotor speed. Depending on whether the $\frac{\lambda_{pm}}{L_s}$ value is bigger or smaller than the current limit, $I_{s}^\text{max}$, two major types of SMPMSM machines are distinguished: Finite-speed drive systems and Infinite-speed drive systems. Clearly the “Infinite-speed drive” describes the case where there is no theoretical speed limit during the field weakening operation. The operation regions of both types of machine drives together with the maximum torque per ampere (MTPA) and maximum torque per volt (MTPV) are shown in Fig. 6.4. According to the (6.21) the dotted circles, describing the voltage constraints, shrink with the rise of speed, i.e.
\( \omega_{r1} > \omega_{r2} > \omega_{r3} \), where the \( \omega_{r1} = \omega_b \) is called the base speed and is derived as follows.

\[
\omega_b = \frac{V_s^{\text{max}}}{\sqrt{\lambda_{pm}^2 + (L_s I_s^{\text{max}})^2}} \tag{6.22}
\]

Above base speed the motor speed cannot be increased whilst maintaining the same torque level (constant torque levels are shown in dash-dotted lines, where \( T_{e1} > T_{e2} \)). In order to further increase the speed, part of the stator current should be used to produce a magnetic field in the opposite direction from the rotor magnetic flux, i.e. the field should be weaken or \( i_{ds}^r \) should become negative. In such a regime of operation, higher speeds could be achieved at the expense of the reduced maximum torque. For the finite-speed drive system the maximum theoretical speed of the rotor is limited to the

\[
\omega_{\text{max}} = \frac{V_s^{\text{max}}}{\lambda_{pm} - L_s I_s^{\text{max}}} \tag{6.23}
\]

From (6.21) the extended field weakening is achieved when the rotor magnet flux is comparable to the \( L_s I_s^{\text{max}} \), or flux generated by current passing through the stator windings. For the infinite-speed drive system theoretically infinite speed is achievable, where above a critical speed \( \omega_c = \frac{V_s^{\text{max}}}{\sqrt{(L_s I_s^{\text{max}})^2 - \lambda_{pm}^2}} \) only voltage constraints are working so the regime is called maximum torque per voltage (MTPV).

The current trajectories in Fig. 6.4 do not represent the current trajectory during the motor acceleration, rather they show the optimal steady state current values for any particular maximum available load torque/speed combination. In this research the focus will be on the finite-speed drives, though all the conclusions and control procedure is applicable for the infinite-speed drive, with some minor modifications. The transient instant between the “normal” motor operation and the start of field weakening for the single drive in the positioning system, for the case when there is no static torque and the motor is required to generate the torque to overcome the viscous bearing friction only, is shown in Fig. 6.5. It is considered that at the initial time the motor is rotating at \( \omega_{r1} \) speed (omitting the discussion how this speed has been reached), which is higher than the base speed, \( \omega_b \), due to the requirement to overcome only viscous friction torque \( T_{e1} = B_{fr} \omega_{r1} \). In this case the stator current is \( I_{s1} \), which lies on the MTPA curve. It is required to increase the speed to \( \omega_{r2} \), within the same voltage and current constraints. After the transient, the stator current will be \( I_{s2} \), which is overcoming a slightly higher
friction torque $T_{e2} = B_f r \omega_{r2}$ and cannot follow the MTPA curve, due to the voltage constraints, therefore the motor is running in the field weakening region. There are an infinite number of possible transient trajectories between these two points depending on the motor control strategy (shown in red in Fig. 6.5), however all these trajectories must follow the following logic.

Assuming steady state operation at the initial point and negligible effect of voltage drop across the stator windings resistance at $\omega_{r1}$ speed, the stator voltage and torque equations could be derived from (6.14) as

$$v_{d1}^r = -\omega_{r1} L_s i_{s1}$$
$$v_{q1}^r = \omega_{r1} \lambda_{pm}$$
$$\omega_{r1} = \frac{J_e}{B_f} K_T I_{s1}, \quad (6.24)$$

where $K_T = \frac{1}{J_e} \frac{3}{2} \frac{p^2}{4} \lambda_{pm}$. Using (6.24) the $d$-$q$ voltages at the initial point could be derived, and from the assumption that $\omega_{r1}$ is the maximum achievable speed without the field weakening, these voltages must lie at the voltage constraint curve, i.e. $(v_{d1}^r)^2 + (v_{q1}^r)^2 = (V_s^{max})^2$. According to the third equation from (6.14) in order to further increase the speed, the $q$- current component must be increased.

$$\frac{d\omega_{r}}{dt} = K_T i_{qs}^r - \frac{B_f r}{J_e} i_{ds}^r > 0 \Rightarrow \frac{di_{qs}}{dt} = \frac{v_{qs}}{L_s} - \omega_{r} i_{ds}^r - \omega_{r} \frac{\lambda_{pm}}{L_s} > 0 \quad (6.25)$$

From (6.25) the $q$- current component could be increased in two ways. First, by reducing (making it more negative) the $d$- axis current and using the cross coupling term, $-\omega_{r} i_{ds}^r$. Second, by increasing the $v_{qs}^r$ voltage. Further both these speed rise scenarios are discussed in some details.
Reduction of $d$- component:

The differential equation describing the variation of the $d$- component is as follows and it should be negative.

$$\frac{d^2 i_{ds}^r}{dt^2} = \frac{v_{ds}^r}{L_s} + \omega_r i_{qs}^r < 0$$ \hspace{0.5cm} (6.26)

The second term of (6.26) cannot be reduced due to the assumption of viscous friction in the system, which is proportional to the speed. Therefore, the $d$- stator voltage component should be reduced. It is important to note, that according to (6.24) $v_{ds}^r$ is negative, so the reduction of it means a rise of its magnitude, therefore, considering that the voltage limit has already been reached the positive $q$- axis voltage, $v_{qs}^r$ should be reduced simultaneously with the reduction of $v_{ds}^r$. In other words, on voltage $d$-$q$ reference frame the voltage vector is following the circular voltage limit in the counter clockwise direction. Thus, the reduction of $i_{ds}^r$ is leading to decrease of the first and third components of the $q$- current differential equation ($v_{qs}^r$ and $-\omega_r \lambda_{pm}$ respectively) and increase of the second component $-\omega_r i_{ds}^r$.

Increase of $q$- voltage component:

The equation describing the $q$- voltage component is as follows. $v_{qs}^r$ is positive and should increase.

$$v_{qs}^r = L_s \frac{di_{qs}^r}{dt} + \omega_r L_s i_{ds}^r + \omega_r \lambda_{pm} > 0$$ \hspace{0.5cm} (6.27)

The second component of (6.27) is negative and rising, therefore the voltage rise should be secured by the rise of derivative term ($L_s \frac{di_{qs}^r}{dt}$) and relatively big back EMF voltage, $\omega_r \lambda_{pm}$, value, which will compensate the reduction of cross-coupling component $\omega_r L_s i_{ds}^r$.

The exact motor speed rise scenario, when the voltage limit is reached, is determined by the motor parameters and operation point, where for example, in the studied case, Fig. 6.5, the first one is not applicable due to the zero $d$- current component in the initial operation point and speed would rise via the increase of the $q$- voltage component, i.e. the voltage vector will rotate clockwise in the $d$-$q$ voltage frame. Nevertheless, in both aforementioned scenarios the presence of the nonlinear terms is essential and nonlinear terms in the motor model make it possible to increase the motor speed by weakening the magnet field while keeping the winding currents and voltages within the constraints. In its turn, the increased maximum motor speed allows a
reduction in the positioning system operation time, i.e. improve the OT control performance. Therefore nonlinear terms in the SMPMSM could be seen as an advantage compared with the pure linear DC-motor model. However, from the controller perspective it is hard to deal with the nonlinearities, therefore in the further sections the conventional ways to deal with model’s and constraints’ nonlinearities are discussed.

6.3 Dealing with the nonlinearities

In Chapter 5 the application of Linear MPC was suggested to control the DC-motors of the feed-drive system, which is suitable for the systems with a linear model only. In this section different approaches to apply Linear MPC for nonlinear systems with nonlinear constraints are discussed with a particular focus on field weakening control ability.

6.3.1 State space model linearization

First, the conventional ways to deal with the SMPMSM model nonlinearities are discussed together with their applicability for the OT and OE control of the positioning system.

In [122] it is assumed that the current time constant is much smaller than the mechanical one, therefore the speed is considered to be constant during the whole optimization horizon making the model (6.14) linear time-invariant, i.e. the Linear MPC is applicable. A similar approach is used in [123], where in order to get rid of nonlinearities it is suggested to simply omit them in the controller design. Both these approaches offer implementation simplicity but are not able to guarantee the motor field weakening operation, moreover both are predominantly applicable for the cases where the motor is operating in the steady state at some prior known speed for the most of the operational time.

In the Electric Drives Laboratory of the University of Padova headed by professor Silverio Bolognani a model linearization approach, which has already become a classical one, has been developed based on the ideas presented in [124]. In [125],[126] the MPC implementation for the SMPMSM was studied where due to the assumption of nonsalient rotor structure the $q$- voltage cross-coupling term, $\omega_r i_q^r$, is neglected considering the small value of direct current, $i_d$, during the operation. $d$- axis cross-coupling term cannot be ignored, however both $i_q^s$ and $\omega_r$ are measured during the motor drive operation, therefore their multiplication is considered as a measured
disturbance [127], \( u_{cc_d} = \tilde{\omega}_r r_{qs} \), with zero dynamics, i.e. \( \frac{d}{dt} u_{cc_d} = 0 \). The “\( \tilde{\cdot} \)” denotes the measured value, i.e. the cross coupling term \( u_{cc_d}(k) = \omega_r(k) i_{qs}(k) \). Applying these assumptions to the motor model (6.14) the linear time-invariant state space representation is obtained in a conventional matrix form.

\[
\begin{bmatrix}
\frac{di_{ds}}{dt} \\
\frac{di_{qs}}{dt} \\
\frac{du_{cc_d}}{dt} \\
\frac{d\omega_r}{dt} \\
\frac{d\theta_r}{dt}
\end{bmatrix} =
\begin{bmatrix}
\frac{R_s}{L_s} & 0 & 1 & 0 & 0 \\
0 & -\frac{R_s}{L_s} & 0 & -\frac{\lambda_{pm}}{L_s} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
i_{ds} \\
i_{qs} \\
u_{cc_d} \\
\omega_r \\
\theta_r
\end{bmatrix} +
\begin{bmatrix}
\frac{1}{L_s} & 0 \\
0 & \frac{1}{L_s} & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_{ds} \\
v_{qs}
\end{bmatrix}
\tag{6.28}
\end{equation}

It is worth noting that in their later work the same authors have extended the controller design to be able to operate with the IPMSM [128], however, the approach to deal with the cross-coupling terms is the same. Though this approach is quite extensively used in literature [129], [130] it is not able to deal with field weakening operation. At each switching instant the controller is obtaining the information regarding the cross-coupling term based on the previous step’s measurement, where the dynamics of this cross-coupling term is neglected, therefore there is no way for the controller to perform field weakening because the cross-coupling term or back EMF disturbance is considered to be constant over the whole prediction horizon.

The MPC controller design presented in [131] is based on the aforementioned solution, where the nonlinear terms are also treated as measured disturbances, however, it splits the overall motor speed region into a number of regions characterised by the particular speed, \( \Omega_{mi} \), and develop a multiple MPC controllers for each of the regions. The switching between the controllers is performed according to the actual motor speed, \( \omega_r \). In order to improve the controller accuracy an additional external voltage controller is developed, which calculates a compensation voltage \( u_{sdq}^{cmp} = [u_{sd}^{cmp}, u_{sq}^{cmp}]^T \) based on the difference between the actual rotor speed and the selected region speed, \( \Omega_{mi} \). This compensation voltage is added to the stator voltage reference calculated by the MPC controller. With the proposed technique, field weakening operation is possible, however the accuracy of the prediction depends on the number of speed regions, in other words on the number of MPC controllers. Increasing the number of these controllers increases
the accuracy as well, but seriously complicates the control algorithm. Compensation voltages do not influence the prediction accuracy since they are added outside the MPC controller.

While developing the MPC controller for the motor drive it is possible to deal with its nonlinearities by retaining the conventional cascaded control structure. For example, in [132] the conventional current controller is replaced by the MPC one, whereas in [133] both speed and current controllers have an MPC structure. If the cascaded structure is applied it is possible to change the current references outside the MPC current controller, taking into account the necessity of field weakening. For instance in [133] for the SMPMSM, the \( i_{ds}^r \) current is maintained to zero for speeds lower than the base speed, \( \omega_{b} \), and for the higher speeds the \( i_{ds}^r \) current reference is changed according to the speed value. The approach to deal with nonlinearities inside the MPC current controller is similar to [122]: assuming that the electrical time constant is much smaller than the mechanical one, the speed is considered as a time varying parameter and all the terms from the voltage equations containing this time varying parameter are collected in a single auxiliary variable, \( \boldsymbol{\omega} \), which is defined as follows.

\[
\boldsymbol{\omega} = \omega_r \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i_{ds}^r \\ i_{qs}^r \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\lambda_{pm}}{L_s} \right) \tag{6.29}
\]

Due to the assumption of the slower mechanical dynamics the (6.29) is considered to be constant during whole prediction horizon and the value of \( \boldsymbol{\omega} \) is updated every sampling period. Though such an approach is applicable in practice it suffers from the lack of proper constraints handling and due to the presence of two separate controllers the control objectives are not easily achieved.

In the Chapter 5 the finite control set MPC was mentioned, it could be seen as another possibility to deal with the model nonlinearities and field weakening ability [110], [134].

However, all the aforementioned approaches, even those having the ability to deal with the field weakening are not applicable for the proposed positioning system control, where the prediction horizon is equal to the whole operation time, \( t_f \), since the speed is varying from zero to maximum and back to zero and cannot be seen as a constant or in other words the cross coupling in the voltage equations is not negligible. The major issue with the optimal time position control is the necessity to identify the moment when the motor needs to start deceleration to be able to stop the motor at the desired
position, but the aforementioned methods are either unable to predict the system
dynamics from the maximum speed to the full stop or such prediction will take too
much computational resources, as it was mentioned earlier for the FCS-MPC.

6.3.2 Constraints linearization

The current and voltage constraints (6.16), (6.19) are nonlinear, whereas Linear
MPC is able to deal with linear states and input constraints only.

\[(i_{ds}^r)^2 + (i_{qs}^r)^2 \leq (I_s^{max})^2\]  \hspace{1cm} (6.16)

\[(v_{ds}^r)^2 + (v_{qs}^r)^2 \leq (V_s^{max})^2,\]  \hspace{1cm} (6.19)

The conventional way to consider these constraints is to linearize them, i.e. the circular
constraints in the \(d\)-\(q\) current and voltage reference frame are approximated by the
affine equations or in other words, by some rectangular, hexagonal, octagonal or other
polygoned region. Some current and voltage constraints’ approximation techniques
presented in literature together with their mathematical description are summarized in
Table 6.1.
Table 6.1 Current and voltage constraints affine approximation

<table>
<thead>
<tr>
<th>Ref</th>
<th>Mathematical model</th>
<th>Region shape in $d − q$ reference frame</th>
<th>Current constraints</th>
<th>Voltage constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>[ \begin{bmatrix} 1 &amp; 0 \ -1 &amp; 0 \ -1 &amp; \beta \ -1 &amp; -\beta \end{bmatrix} \begin{bmatrix} i'<em>{ds} \ i'</em>{qs} \end{bmatrix} \leq \begin{bmatrix} 0 \ -1 \min \beta i_{d}^{max} \ \beta i_{q}^{max} \end{bmatrix} ]</td>
<td>![Region shape in d−q reference frame]</td>
<td>![Current constraints]</td>
<td>![Voltage constraints]</td>
</tr>
<tr>
<td>2.</td>
<td>[ \begin{bmatrix} 1 &amp; \frac{1}{\sqrt{3}} \ 0 &amp; 1 \ -1 &amp; \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} i'<em>{ds} \ i'</em>{qs} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\sqrt{3}} \max \end{bmatrix} ]</td>
<td>![Region shape in d−q reference frame]</td>
<td>![Current constraints]</td>
<td>![Voltage constraints]</td>
</tr>
<tr>
<td>3.</td>
<td>[ \begin{bmatrix} 1 &amp; 0 \ -1 &amp; 0 \ 0 &amp; 1 \ 0 &amp; -1 \end{bmatrix} \begin{bmatrix} i'<em>{ds} \ i'</em>{qs} \end{bmatrix} \leq \begin{bmatrix} \epsilon_{1}^{max} \ \epsilon_{2}^{max} \end{bmatrix} ]</td>
<td>![Region shape in d−q reference frame]</td>
<td>![Current constraints]</td>
<td>![Voltage constraints]</td>
</tr>
<tr>
<td>4.</td>
<td>[ \begin{bmatrix} 1 &amp; -1 \ 0 &amp; 1 \ -1 &amp; 1 \ 1 &amp; 1 \end{bmatrix} \begin{bmatrix} i'<em>{ds} \ i'</em>{qs} \end{bmatrix} \leq \begin{bmatrix} 0 \ \frac{1}{\sqrt{3}} \max \end{bmatrix} ]</td>
<td>![Region shape in d−q reference frame]</td>
<td>![Current constraints]</td>
<td>![Voltage constraints]</td>
</tr>
</tbody>
</table>
Different constraint approximation techniques in Table 6.1 have been numbered and they are discussed further, using this number as a reference. In 1. the current constraints area is developed, considering the loss minimization control of a motor drive; both iron loss and copper loss are taken in account, so the lowest direct current value could be derived from the machine loss model as follows. [122]

\[
    i_d^{\text{min}} = -\frac{L_s \lambda_{\text{pm}}}{L_s^2 + R_s (\omega_b k_{Fe})^{-1}}
\]  

(6.30)

where \(k_{Fe}\) is an iron loss constant which is determined experimentally. This minimum direct current could be seen as the fraction of the maximum stator current, \(i_{s}^{\text{max}}\), i.e. 

\[
i_d^{\text{min}} = -\alpha i_{s}^{\text{max}}
\]

where \(\alpha > 0\) is some positive coefficient depending on the motor parameters. Taking in account (6.30) and defining a new positive coefficient \(\beta = \)
\[
\frac{\alpha}{(1-\sqrt{1-\alpha^2})},
\]
the affine equations describing the current constraints are presented in the second column of Table 6.1. Voltage constraints are presented there as well, the smaller rectangular area describes the steady state voltages range for the min/max current and maximum speed. This area is defined as follows.

\[
\begin{align*}
-L_s \omega_b I_s^{max} & \leq u_{ds}^r \leq L_s \omega_b I_s^{max} \\
-R_s I_s^{max} - \lambda_{pm} \omega_b & \leq u_{ds}^r \leq R_s I_s^{max} + \lambda_{pm} \omega_b
\end{align*}
\] (6.31)

However during transients voltages exceed the area described by (6.31), and in order to consider this, the area is expanded to touch the voltage limit.

In 2. for both current and voltage constraints’ circular areas are approximated via hexagons. The hexagon approximation is perhaps the most widespread technique presented in literature. [131]

In 3. the \(d\)- and \(q\)- current limits are independent from each other. Due to an assumption of no saliency, the MTPA curve follows the \(q\)-axis, therefore the full stator current range is used as a quadrature current range, while the range of direct axis current is limited to some small fraction of the, \(I_s^{max}\). In [126] the maximum \(d\)-current amplitude is set as 20\% of the maximum available stator current, \(I_s^{max}\). Clearly such current constraints could end up with the currents exceeding the circle defined by (6.16), but for applications where there is no need to push system to its limits this is somehow acceptable. The voltage constraint is approximated as an octagon inscribed in the voltage limits (6.19). It is worth noting that the octagon’s orientation, is not defined in [126], the position, when two octagon corners align with the \(d\)- and \(q\)- is chosen for the reasons which are discussed further in this chapter.

In 4. the constraints linearization technique is chosen considering the trade-off between the approximation accuracy and the number of additional inequality constraints added to the plant model, therefore the voltage constraints are represented by the octagon, whereas for SMPMSM the \(d\)- current is never positive, therefore the octagon is reduced by half to make the model simpler. [135]

In 5. authors are dealing with interior permanent magnet synchronous motor (IPMSM), therefore both \(d\)- and \(q\)- currents are used to formulate the MTPA curve, and the current constraints are chosen to be approximated as the area containing the MTPA curve. Voltage constraints are approximated by an octagon, where the voltage limiting circle is inscribed. Such an approximation could result in the controller asking for voltages, which the inverter cannot produce, however for all CCS-MPC methods the
reference voltages are sent to the modulator first, where some over modulation strategy is applied, therefore such an approach is valid for most applications. [128]

In 6. the approach for the voltage constraints approximation is very similar to the one in 5., however, the size of octagon is bigger. For SVPWM the voltage constraints in $\alpha-\beta$ frame form a stationary hexagon, shown in Fig. 6.3 as discussed earlier. This hexagon in the $d-q$ reference frame will result in the hexagon with the inscribed circular voltage limit, rotating with the rotor frequency. This rotating hexagon could be inscribed into a new circle area, with a circle radius of $\frac{2\sqrt{3}}{3}V_{dc}$ and the desired octagon is inscribed outside it. Over modulation is mandatory here as well. [133]

7. presents the “ideal” case, where for each control algorithm step the voltage constraints are dynamically changing according to the rotor rotational speed. The case where the prediction horizon, $N_p$, is equal to 3 is presented in a picture and the constraints’ mathematical description is presented in the second column, where $R \in \mathbb{R}^{2 \times 2}$ and denotes the orthonormal rotation matrix. This method allows the optimization algorithm to fully utilize the DC-link voltage, however the implementation is rather tricky and for example EMPC is not able to deal with such constraints description, which are linear, but time-varying. [136]

All aforementioned constraints approximation techniques result from trade-off between the accuracy of approximation, or in other words, how well the DC-link voltage and thermal limits are utilized, and the complexity of constraints’ implementation. The best solution should express the constraints using the minimum number of linear equations, while covering the maximum area of the current/voltage limiting circles. Moreover, the proposed positioning system control requires off-line calculation of the reference states for the whole prediction horizon, therefore approximation solutions which rely on over-modulation are not acceptable.

For SMPMSM, where the $d$-current is non-zero only during the field weakening operation, therefore current approximation as in 1. is preferred. It requires only four additional inequality constraints and covers the current variation area well enough for the most applications. For the voltage constraints, the solution 1. could not be adopted, due to the poor utilization of the DC-link voltage, where the voltage variation is much higher (from the circle constraint area utilization point of view) than the current one, the inscribed hexagon or octagon should be preferred.
To compare the hexagon and octagon approximations, their maximum coverage areas are plotted in Fig. 6.6 in the $\alpha$-$\beta$ stationary reference plane for the different inverter modulation techniques. Only the first quadrant is shown for clarity. Depending on the respective values of $v_{ds}^r$ and $v_{qs}^r$ the maximum available voltage could be inside the areas denoted as “Octagon approximation” and “Hexagon approximation” in Fig. 6.6. For the hexagon approximation the maximum voltage vector in $\alpha$-$\beta$ reference frame (obtained as a combination of $v_{ds}^r$ and $v_{qs}^r$) has a magnitude in between the maximum voltage vector magnitudes for SPWM and SVPWM. For the octagon approximation the lower bound is increased and is equal to $\frac{\sqrt{2}(1+\sqrt{2})}{2\sqrt{3}}V_{dc}$.

In order to illustrate the difference between the octagon and hexagon approximations maximum voltage vector values, a numerical example is proposed. For the conventional 3-phase bridge rectifier connected to the conventional 3-phase 400V, 50Hz AC grid the average DC-link voltage value could be calculated through the following expression, assuming zero grid-side inductance and resistive load.

$$V_{dc} = \frac{3}{\pi} \sqrt{2\sqrt{3}}V_{ph,\text{rms}}$$

where $V_{ph,\text{rms}}$ is the grid phase voltage, rms value, which equals to 230V in the UK. With the aforementioned supply system parameters the DC link voltage is equal to $V_{dc} \equiv 540\text{V}$, then the maximum voltage vector magnitude for the hexagon approximation lies in the [270-311]V interval, whereas for the octagon approximation this interval is reduced to [288-311]V. The difference between the lowest possible

![Fig. 6.6 Approximated voltage constraints range](image-url)
voltage vector magnitudes of this approximation options is 18V, which is around 6% of maximum phase voltage corresponding to the Offset PWM. What is more important is the interval value, which is more than 13% for the hexagon approximation and around 7% for the octagon.

The OT control supposes that all the state variables are brought to their limits, making it very important to bring the resultant approximation area as close as possible to the original circular constraints, therefore the octagon approximation looks more attractive, though the dimension of the resultant control problem is increased. However the maximum possible voltage vector magnitudes are the same for both strategies. In addition, during the positioning system OT operation the maximum voltage is required in two cases, first, at motor start-up when the maximum voltage is required for the motor winding to create the current in the stator winding and have rapid acceleration, second, during the field weakening operation, where the voltage is following the constraint’s circle.

Schematically the ideal voltage trajectory with nonlinear constraints during the OT positioning operation is shown by the red line in Fig. 6.7(a). Starting from the [0;0] point the voltage vector rises following the q-axis until the voltage limit is reached (region is denoted by ①), this is required to provide the maximum possible acceleration at the initial point. Later on when the current reaches its maximum value the voltage

![Voltage trajectory during the OT positioning control: a) Conventional constraints approximation b) Proposed approximation](image-url)
drops almost to zero, from where both $d$- and $q$- voltage components start rising (note that $v_{ds}'$ is negative and the rise of amplitude is meant) following the inclined line with the inclination angle, $\xi_d$, (region is denoted by ②) as the speed increases. The rise is continued until the voltage limit is reached again. There is no way to further increase the speed whilst keeping torque at the same level so the field weakening regime is started (region is denoted by ③). During field weakening the voltage vector magnitude is kept constant, but rotating counter clock-wise (note that the voltage vector is rotating in the $d$-$q$ reference frame, which is rotating with the rotor speed, $\omega_r$) in order to reduce the $d$-current (make it more negative)\(^6\). The electric motor could be seen as an inductive loading, where it is impossible to impose a step change in current, therefore before motor starts to decelerate ($i^r_{qs}$ become negative) the voltage vector must change the rotational direction from the counter clock-wise to clock-wise and the deceleration is started when the voltage vector reaches $[0;V_s^{\text{max}}]$ point. Further on the deceleration trajectory is the same (no friction is considered for simplicity), but mirror reflected around the $q$- axis. In Fig. 6.7(a) the aforementioned operation steps are illustrated by a dark grey dashed line.

The voltage vector trajectory could be characterized by two parameters: inclination angle, $\xi_d$, and maximum $d$- voltage amplitude, $v_{ds}^{\text{min}}$. Both could be calculated analytically as following.

\[
\xi_d \cong \text{atan}\left(\frac{\lambda_{pm}}{L_s I_s^{\text{max}}}\right) \tag{6.33}
\]

\[
v_{ds}^{\text{min}} \cong -\omega_{i_d}^{\text{min}} L_s \sqrt{(I_s^{\text{max}})^2 - (I_d^{\text{min}})^2} \tag{6.34}
\]

The region ② is representing the constant acceleration area, where $i^r_{ds} = 0$, while $i^r_{qs}$ is kept at maximum level. In order to provide the speed rise the stator $q$- voltage is gradually increasing to compensate the rising back EMF voltage, $\lambda_{pm} \omega_r$, whereas the $d$- component should be increased as well to compensate the cross coupling term $-\omega_r L_s i^r_{qs}$. If the stator resistance term is omitted, the expression (6.33) describing the inclination angle is obtained. It was mentioned that ③ is describing the field weakening area, where the minimum value of the $d$- voltage could be estimated considering the

\(^6\) Note that the first mechanism of field-weakening control, described in section 6.2, is taken as an example.
current constraint approximation as in case 1. from Table 6.1. Roughly speaking the voltage $v_{ds}^{min}$ is reached when the minimum $d$- current, $i_{ds}^{min}$ is reached, after this the $d$-current stays constant (for the OT operation), therefore the derivative term in the voltage equation is zero and $v_{ds}^{min}$ can be approximated as (6.34). In (6.34) variable $\omega_{i_d}^{min}$ denotes the speed corresponding to the maximum voltage vector and maximum current vector, with the $i_d^r = i_{ds}^{min}$ and could be calculated using the voltage constraint equation (6.21) substituting the minimum $d$- current value into (6.35).

$$\omega_{i_d}^{min} = \frac{V_s^{max}}{\sqrt{(L_s i_s^{max})^2 + 2L_s i_d^{min} \lambda_{pm} + \lambda_{pm}^2}}$$  (6.35)

The inclination angle, $\xi_d$, depends on the relative value of back EMF value and the inductance and maximum current product. For the finite speed drive $\lambda_{pm} \geq L_s i_s^{max}$ and considering a normally small inductance of SMPMSM due to the wide airgap, the inclination angle is relatively big and for the typical motor parameters is more than 70°.

The conventional hexagon and octagon areas are compared with the theoretical voltage trajectory in Fig. 6.7(a). It was mentioned earlier that the orientation of the octagon is not always specified in the papers, for example in [135] the octagon corners are aligned with the $d$- and $q$- voltage axes as in Fig. 6.7(a), whereas in [126] the exact orientation is not specified. Comparing the octagon type approximation with the theoretical trajectory, clearly, the sketched octagon position is preferable. The hexagon approximation does not cover the voltage variation area as effectively, however, the coverage area could be improved by rotating the hexagon in such a way as to match its corner with the $q$- axis instead of $d$- axis.

The minimum $d$- voltage defines how deep the motor can go into the field weakening area. For the chosen current constraint approximation it depends on the $i_{ds}^{min}$ value. If this voltage magnitude is less than $\frac{1-\sqrt{2}}{2-\sqrt{2}} V_s^{max}$, a new voltage constraints’ approximation technique can be used as shown in Fig. 6.7(b). The constraints are hexagonal, but the sides of this hexagon are not equal to each other, and the corner of the proposed hexagon is aligned with the $q$- axis, whilst the sides touch the voltage constraint circle at $v_{ds}^{min}$ and $-v_{ds}^{min}$ points. Applying this approximation, the number of additional voltage constraints in the optimization task is kept the same as for the hexagon case, whereas the coverage of the required voltage area variation is even better.
than for the octagon approximation. The mathematical description of such voltage constraints approximation is given in (6.36).

\[
\begin{bmatrix}
1 & 0 \\
-1 & 0 \\
-c_1 & 1 \\
-c_1 & -1 \\
c_1 & 1 \\
c_1 & -1
\end{bmatrix}
\begin{bmatrix}
v_{ds}^r \\
v_{qs}^r
\end{bmatrix}
\leq
\begin{bmatrix}
-v_{ds}^{min} \\
v_{ds}^{min} \\
V_s^{max} \\
V_s^{max} \\
V_s^{max} \\
V_s^{max}
\end{bmatrix},
\] (6.36)

where the constant \( c_1 = \sqrt{\left(\frac{V_s^{max}}{v_{ds}^{min}}\right)^2 - \left(\frac{v_{ds}^{min}}{v_{ds}^{min}}\right)^2} - v_{ds}^{max} \).

To conclude, in this thesis it is proposed to use the current constraints approximation method introduced in [122] (denoted as i. in Table 6.1). The mathematical description of such an approximation is repeated in (6.37). Whereas for the voltage constraints a novel approximation method is proposed, which covers the voltage variation area better than the conventional octagon approximation, whilst requiring the same number of additional constraints as the hexagon approximation. It is worth noting that the method was derived under the assumption of OT control, however it is perfectly applicable for the OE control, since the OT is the extreme case for motor control, requiring the maximum possible voltages and currents, therefore the voltage variation area in the OE case will be always smaller than the one for the OT case. In Fig. 6.7(b) the hexagon of proposed approximation is stretched along the \( q \)-axis and covering both positive \( v_{qs}^r \) and negative \( v_{qs}^r \) areas, though the sketch of the OT voltage trajectory variation also presented in a plot does not go to the negative \( q \)-voltage semicircle. This is done to be able to consider the possibility for \( \theta_r(t_f) \) to have a negative value.

\[
\begin{bmatrix}
1 & 0 \\
-1 & 0 \\
-1 & \beta \\
-1 & -\beta
\end{bmatrix}
\begin{bmatrix}
i_{ds}^r \\
i_{qs}^r
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
\alpha I_s^{max} \\
\beta I_s^{max} \\
\beta I_s^{max}
\end{bmatrix},
\] (6.37)

### 6.4 Nonlinear MPC controller

Earlier in this chapter it was shown that the conventional MPC control strategies are not applicable for OT position control due to impossibility to consider the cross-coupling for the whole prediction horizon, where the prediction horizon corresponds to the whole operation time, \( t_f \). However, in the control strategy proposed in Chapter 5 the trajectory generation is done off-line, therefore it is possible to step back from the
system linearization techniques development and directly use the nonlinear model (6.14).

In Chapter 5 “c2d” MATLAB function was used for a system discretization, where the default zero-order hold (ZOH) discretization method was applied. However, such a method is not applicable for nonlinear system discretization. For simplicity, in this work the forward Euler differentiation method is preferred, whereas more advanced discretization techniques, like a Taylor series expansion could be also applied as discussed in [137]. The basics of the Euler method are summarized below.

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
\dot{x}(t_k) &\approx \frac{x(t_{k+1}) - x(t_k)}{T_s} \quad (6.38) \\
x_{k+1} &= x_k + T_s f(x_k, u_k),
\end{align*}
\]

where \(x(t)\) is the state vector of a nonlinear system described by \(f(\cdot), t = kT_s, k = 0, 1, \ldots\) Current and future time steps are denoted as \(t_k\) and \(t_{k+1}\) respectively and \(T_s = t_{k+1} - t_k\) is a sampling time. The Euler method (6.38) is applied for discretization of the nonlinear model (6.14) and the discrete time representation of a SMPMSM positioning system could be derived as follows. Note that the load torque was considered to be zero, so the term \(T_L(k)\) is added for the model completeness.

\[
\begin{align*}
i_{ds}^r(k + 1) &= (1 - T_s \frac{R_s}{L_s}) i_{ds}^r(k) + T_s \omega_r(k) i_{qs}^r(k) + T_s \frac{1}{L_s} v_{ds}^r(k) \\
i_{qs}^r(k + 1) &= (1 - T_s \frac{R_s}{L_s}) i_{qs}^r(k) - T_s \omega_r(k) i_{ds}^r(k) - T_s \frac{\lambda_{pm}}{L_s} \omega_r(k) \\
+ T_s \frac{1}{L_s} v_{qs}^r(k) \quad (6.39) \\
\omega_r(k + 1) &= \omega_r(k) + T_s \frac{1}{2} \frac{3 p^2}{4} \lambda_{pm} i_{qs}^r(k) - T_s \frac{B_{fr}}{J_e} \omega_r(k) - T_s \frac{T_L(k)p}{2J_e} \\
\theta_r(k + 1) &= \theta_r(k) + T_s \omega_r(k)
\end{align*}
\]

In (6.39), notations similar to the one used in Chapter 5 are applied, where the discrete time state vector is defined as \(x_k = [i_{ds}^r(k), i_{qs}^r(k), \omega_r(k), \theta_r(k)]^T \in \mathbb{X} \subset \mathbb{R}^{7n} \forall k = 0 \ldots N\) and the control variables vector \(u_k = [v_{ds}^r(k), v_{qs}^r(k)]^T \in \mathbb{U} \subset \mathbb{R}^{2n} \forall k = 0 \ldots N - 1\), state variable sequence \(\mathbb{X} = (x_0, x_1, \ldots, x_N)\) and input sequence \(\mathbb{U} = (u_0, u_1, \ldots, u_{N-1})\) are also the same as in Chapter 5.

State and input constraints (6.36), (6.37) are applied for the whole operation time or prediction horizon, then in the discrete form these nonlinearities are written as follows.
Chapter 6 PMSM Control

\[
\begin{bmatrix}
1 & 0 \\
-1 & 0 \\
-c_1 & 1 \\
-c_1 & -1 \\
c_1 & 1 \\
c_1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
v_{ds}^r(k) \\
v_{qs}^r(k) \\
\end{bmatrix}
\leq
\begin{bmatrix}
-v_{ds}^{min} \\
v_{ds}^{min} \\
v_s^{max} \\
v_s^{max} \\
v_s^{max} \\
v_s^{max} \\
\end{bmatrix}, \forall k = 0 \ldots N-1 \tag{6.40}
\]

\[
\begin{bmatrix}
1 & 0 \\
-1 & 0 \\
-1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
i_{ds}^r(k) \\
i_{qs}^r(k) \\
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
\alpha I_s^{max} \\
\beta I_s^{max} \\
\beta I_s^{max} \\
\end{bmatrix}, \forall k = 0 \ldots N \tag{6.41}
\]

Applying the new discrete-time nonlinear plant model (6.39), the mathematical formulation of the optimal control problem is obtained in (6.42).

\[
J_{t,e} = \arg \min_{x,u} \quad \sum_{i=0}^{N-1} (x_i^TQx_i + u_i^TRu_i)
\]

\[\text{s. t. } \quad (6.39)\]

\[\text{s. t. } \quad (6.40)\]

\[\text{s. t. } \quad (6.41)\]

\[I_N x_N = s_N,\]

where \(J_t\) and \(J_e\) are the optimal time and optimal energy cost functions respectively. Optimal time cost function, \(J_t\), corresponds to the \(\int_{t_0}^{t_f} dt \rightarrow \min_{v_{ds}v_{qs}}\) continuous time performance measure and the optimal energy cost function, \(J_e\), corresponds to the \(\int_{t_0}^{t_f} [(i_{ds}^r)^2 + (i_{qs}^r)^2] dt \rightarrow \min_{v_{ds}v_{qs}}\) continuous time performance measure, where the copper loss are assumed to dominate. The optimization procedure is kept the same as described in section 5.3.3A for the OE and as in section 5.3.3B for the OT. \(N\) is the prediction horizon length corresponding to the final time \(t_f\), i.e. \(N = \frac{t_f}{T_s}\). \(I_N\) and \(s_N\) are an identity matrix and vector representing the final time boundary conditions respectively. The symmetrical matrices \(P\) and \(Q\) from the cost problem formulation are positive semidefinite, whereas the matrix \(R\) is symmetrical and positive definite. In order to penalize the currents for the OE control, the first two elements of the main diagonal of \(Q\) are chosen to be big positive integers, whereas the other diagonal elements corresponding to the other states are kept as one.\(^7\)

\(^7\) The choice of \(P,Q,R\) matrix for a particular example is demonstrated in section 6.7, where the MPC control simulation of a laboratory SMPMSM is presented.
The optimization problem (6.42) cannot be solved using the *quadprog.m* solver from MATLAB, due to the nonlinear equality constraints, i.e. it is not a quadratic problem anymore. However, the YALMIP toolbox could be still used, where instead of *quadprog.m*, the *IPOPT* (“Interior Point OPTimizer”) solver [138] suitable for the nonlinear optimization problems is used. It is developed in Carnegie Mellon University, based on the Interior Point Method and is released under the Eclipse Public License. This solver is relatively slow with typical solution times of the order of 1 s, however, the trajectory optimization task is considered to be solved offline, therefore such a method is applicable. In the next section, simulation results for both OE and OT (6.42) problem solution using the *IPOPT* solver are presented and discussed.

### 6.5 State and control trajectories for the OT and OE problems

In order to highlight the advantages of the proposed method it is simulated in MATLAB with the notional machine parameters given in Table 6.2. These atypical parameters (with artificially high inductance, \( L_s \)) are chosen to increase the field weakening area. This section is intended to show the machine state and control variables’ trajectories during the positioning operation, therefore there will be no comparison between the energy consumption of the proposed and conventional G00/G01 methods, which is done later for the real SMPMSM machine from the university laboratory. The motoring operation regions of this machine are shown in Fig. 6.8, where the SVPWM is considered. The theoretical MTPA+field weakening steady state trajectory is shown in red, where the minimum \( d \)-current, \( i_d^\text{min} = \frac{v_s}{\lambda} \) (an arbitrary choice).

![Fig. 6.8 Notional SMPMSM operation regions](image)

### Table 6.2 Notional machine parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_s ), [( \Omega )]</td>
<td>0.7</td>
</tr>
<tr>
<td>( L_s ), [H]</td>
<td>4.3e-2</td>
</tr>
<tr>
<td>( \lambda_{pm} ), ( \frac{V}{\text{rad/s}} )</td>
<td>0.356</td>
</tr>
<tr>
<td>( p )</td>
<td>6</td>
</tr>
<tr>
<td>( J_e ), [kg m^2]</td>
<td>4.3e-3</td>
</tr>
<tr>
<td>( B_{fr} ), ( \frac{Nm}{\text{rad/s}} )</td>
<td>0</td>
</tr>
<tr>
<td>( i_{s,\text{max}} ), [A]</td>
<td>5</td>
</tr>
<tr>
<td>( V_{dc} ), [V]</td>
<td>250</td>
</tr>
</tbody>
</table>
The simulation results for the state and control trajectories of this machine during the OT and OE control are shown in Fig. 6.9 and Fig. 6.10 respectively. Similarly to the Chapter 5 discussion, the distance for the drive in the OE mode is reduced by 30% compared to the one in the OT mode of operation (\(\theta_{r}^{\text{ref}} = 70\text{rad} \) and \(\theta_{r}^{\text{ref}} = 100\text{rad} \) respectively) The PWM switching frequency is \(f_{\text{sw}} = 10\text{kH} \), where single sampling per PWM period is considered. Note that for this simulation the viscous friction coefficient, \(B_{fr} \), is set to zero in order to have symmetrical acceleration and deceleration periods during the positioning operation.

In Fig. 6.9 the state and control trajectories for the OT control are presented. The results are shown against time and against each other together with the characteristic points denoted by ①, ②, ③, ④, ⑤ for all three plots. The initial or starting time \(t_{0} \) is denoted by ①, where the maximum voltage is applied in order to reach the maximum current and accelerate the system. Due to the small electrical time constant the maximum current is reached almost immediately, this time instant is denoted as ②.

From ② to ③ the motor speed is gradually increasing with the \(q\)-current kept at the maximum value, \(I_{s}^{\text{max}} \), and the \(d\)-current is set to zero, i.e. ①-②-③ is MTPA region. At ③ the voltage limit is reached and the field weakening regime of operation starts, continuing till ⑤, where the deceleration starts. At ④ the minimum \(d\)-current is reached and further from ④ to ⑤ the \(d\)-current is fixed and \(q\)-current is reducing till the maximum speed is reached.

In Fig. 6.10 the state and control trajectories for the OE control are presented. During the OE operation the required number of rotations is reduced, where the operation time is kept the same as for the OT case, therefore the constraints are not active at most of the time and motor is able to operate in the MTPA mode. Characteristic points denoted by ①, ②, ③, ④ are also shown on all three plots (state and control against time and two state variables against each other together with two control inputs against each other), where in ② the current constraint is reached and in ④ the deceleration is started. The results are obtained considering the reduction of copper losses, however in the AC servomotors the iron loss could be significant, especially at high speeds. Iron loss could be considered by increasing the weight in the \(Q \) matrix, associated with the speed. The weighting factor would be different depending on the particular machine design, or in other words the ratio between the copper and iron loss. For the notional machine under study the iron loss has been considered in
The resultant trajectories are shown by the dotted lines on Fig. 6.10 as well. It could be seen that the presence of iron loss increases the acceleration speed of the machine, but reduces the maximum speed it reaches. In the further discussions iron loss will be omitted.

![Diagram showing trajectories and control regions for SMPMSM OT control](image-url)
Fig. 6.10 Notional SMPMSM OE control
According to the simulation results the minimum time needed for the drive to reach the reference position of 100 rad is 0.2825 s. If field weakening ability is not used (the same simulation parameters but the $d$-current is set strictly to zero) the same position could be reached in 0.3212 s. Therefore the proposed method allows a reduction in operation time for this particular example for around 9%. If the “movement distance” is increased, the difference will be more considerable as drive will rotate at higher speed for a longer period of time.

For the sake of completeness the OT problem is solved, where instead of the linear constraints (6.40) and (6.41) the discrete analogues of the original nonlinear constraints (6.16) and (6.19) are applied. The results of this simulation are presented in Fig. 6.11. Due to the better utilization of the DC-link voltage, and available current, enabling operation closer to the windings’ thermal limit the process final time is reduced, making it possible to reach $\theta_r^{ref} = 100$ rad in 0.276 s, i.e. the operation time is further reduced by around 2%. However, it is worth noting that the simulation time increases significantly. Moreover the nonlinear solver is not able to solve the optimization task when the sampling time is reduced to $T_s = 0.1 \cdot 10^{-3}$ s, so $T_s = 0.1 \cdot 10^{-2}$ s was used instead. The IPOPT running on a 3.7 GHz quadcore processor with 32 GB memory needs around 1.3 s to solve the OT problem, if the nonlinear plant with linear constraints at $T_s = 0.1 \cdot 10^{-3}$ s is considered (the problem involving 4 variables and 10 constraints). For both nonlinear plant and constraints with the sampling time $T_s = 0.1 \cdot 10^{-2}$ s the same problem is solved in around 700 s, making such a formulation impractical. Even though the operation time is decreased (i.e. the MT energy consumption is reduced) when the nonlinear equality and inequality constraints are simultaneously considered in the optimization task formulation, the difference with the nonlinear equality and linear inequality constraints formulation is insignificant making the latter the better choice for the practical applications, whereas the former could be used for states and control variables variation area check and to find the theoretical absolutely minimal time of operation.

For the OE case the presence of nonlinear constraints does not influence the result because the constraints are not active for this regime of operation, therefore there is no need to perform time-demanding computations and linear inequality constraints could be always used. Regarding the non-active constraints it is worth noting that the maximum $q$-current, $i_{q,s}^r \leq I_s^{max}$ constraint is active in the OE mode of operation, but
due to the MTPA operation $d$-current is kept zero and the nonlinear quadratic constraints are equal to the linear approximation. If the position reference for the drive controlled in the OT mode is only slightly larger than the drive’s position reference controlled in the OE mode, both voltage and current constraints are active in the OE
task for a considerable amount of time, however the time when the constraints are active in the OE case is rapidly reducing with the rise of drives’ reference position difference.

6.6 MPC for the reference tracking problem

In Chapter 5 a version of close-loop control has been presented, where the reference state trajectories, are feedforwarded to a conventional cascaded controller, refer Fig. 5.11. A similar approach could be used for the SMPMSM control as well, with minor modifications for the 3-phase synchronous motor control as shown in Fig. 6.12, Fig. 6.13. In the cascaded control structure the output of the speed controller is the required torque, which is directly proportional to the armature current in the DC-motor case, however, in the SMPMSM case the stator current has two components $d$- and $q$-, where only $q$- component is resulting in torque generation, whereas the $d$- current component is needed for field-weakening operation. In the first stage of the proposed control strategy the off-line reference trajectory is generated. Therefore, it is possible to either feed-forward both direct and quadrature currents to the two separate current controllers or calculate the torque reference trajectory from the combination of $d$- and $q$- currents and feed it forward to the conventional current controller to assure the field weakening control. The control block diagrams for both variants are presented in Fig. 6.12 and Fig. 6.13 respectively. It is worth mentioning that in the former variant a reference current saturation block should be added to the current controllers, because if the feed-
forwarded speed, $\omega_r^*$, is not equal to the real one, $\omega_r$, measured by the incremental encoder than the speed controller will generate some additional current reference and the current limit could be exceeded.

Despite its simplicity and clarity, the cascaded control scheme suffers from different issues. For instance, the PID controllers of the nested loops are tuned in assumption that the controlled system is linear, however, the major focus of this research is the constraint optimization, where, especially during the OT control, state and control constraints are active during the motor operation. In other words the system becomes highly nonlinear due to the presence of nonlinear saturation terms. It is hard to tune the PID control system in such a case while providing the desired control quality (overshoot, settling time, steady state error and so on), therefore it is proposed to implement a MPC controller for online reference trajectory tracking as well, the block

![Cascaded controller with the field-weakening ability](image)

**Fig. 6.13** Cascaded controller with the field-weakening ability

![MPC controller structure](image)

**Fig. 6.14** MPC controller structure
diagram of such controller is shown on Fig. 6.14. The controller is for online implementation, therefore the nonlinear system model \(6.39\) cannot be used for the optimization, due to calculation time limitations. Note that for a 10kHz inverter switching frequency with peak or valley sampling, the time interval available for optimization problem solution is equal to 100us. In this period of time apart from the control signal calculation (reference \(d,q\)-voltages), currents and rotor position sampling should be performed, together with the PWM implementation and variable transformations. In order to reduce the calculation time, the system model is linearized and the prediction horizon is limited to \(N=5\). This value is chosen taking in account that due to the digital delay the change in the input signals, \(v_{qs}^r\) and \(v_{ds}^r\), in the time instant \(k\) will influence the real stator voltage only at the next time step \(k+1\), therefore currents will be affected at \(k+2\), speed at \(k+3\) and finally position is affected only at \(k+4\), making \(N=5\) the minimal prediction horizon value applicable for the positioning system control. The control is implemented in the receding horizon fashion.

The linear time-invariant model used for the reference tracking is similar to one suggested in [128], where both cross-coupling terms are added to the state vector and considered as measured disturbances. This model is discretized using the forward Euler method and presented in \(6.43\).

\[
\begin{align*}
    i_{ds}^r(k+1) &= \left(1 - T_s \frac{R_s}{L_s}\right) i_{ds}^r(k) + T_s \frac{1}{L_s} v_{ds}^r(k) + T_s u_{cc_d}(k) \\
    i_{qs}^r(k+1) &= \left(1 - T_s \frac{R_s}{L_s}\right) i_{qs}^r(k) - T_s \frac{\lambda_{pm}}{L_s} \omega_r(k) + T_s \frac{1}{L_s} v_{qs}^r(k) - T_s \frac{\lambda_{pm}}{L_s} \omega_r(k) + T_s \frac{1}{L_s} v_{qs}^r(k) \\
    u_{cc_d}(k+1) &= u_{cc_d}(k) \\
    u_{cc_q}(k+1) &= u_{cc_q}(k) \\
    \omega_r(k+1) &= \omega_r(k) + T_s \frac{3p^2}{J_e} \frac{1}{2} \lambda_{pm} i_{qs}^r(k) - T_s \frac{B_f r}{J_e} \omega_r(k) \\
    \theta_r(k+1) &= \theta_r(k) + T_s \omega_r(k)
\end{align*}
\]

The state vector is defined as \(x_k = \left[ i_{ds}^r(k), i_{qs}^r(k), u_{cc_d}(k), u_{cc_q}(k), \omega_r(k), \theta_r(k) \right]^T \in \mathcal{X} \subset \mathbb{R}^6 \ \forall k = 0 \ldots N\), where \(u_{cc_d}(k) = \omega_r(k) i_{qs}^r(k)\) and \(u_{cc_q}(k) = \omega_r(k) i_{ds}^r(k)\) and their values are updated at each sampling instant, but considered as constant for the whole prediction horizon \(N=5\). The control variables vector together with the state variables sequence and input sequence are the same as in section 6.4. The constraints are also applied in the same way.
The reference state trajectories at time instant $k$ calculated at the offline stage are denoted as $x_k^*$, where the reference cross couplings are calculated as $u_{ccd}^*(k) = \omega^r(k)i_{qs}^r(k)$ and $u_{ccq}^*(k) = \omega^r(k)i_{ds}^r(k)$. The controller must force the plant to follow the predefined reference trajectory. It is achieved by introducing the precalculated reference trajectory into the quadratic performance index by means of minimizing the difference between the state variables $x_k$ and the reference trajectory, $x_k^*$. The optimization problem is formulated in (6.44).

$$J_{e,t} = \arg\min_{x,u} (x_N - x_N^*)^T P (x_N - x_N^*)$$

$$+ \sum_{i=0}^{N-1} ((x_i - x_i^*)^T Q (x_i - x_i^*) + u_i^T R u_i)$$

s.t. (6.43)

(6.40)

(6.41)

It is worth noting that with such optimization problem formulation and controller structure, the system is able to reach the state associated with the field-weakening region due to the regular update of cross-coupling/nonlinear voltage terms, therefore the external field-weakening controller is not required.

In order to improve the stability of the presented controller, inequality constraints (6.41) are modified to consider not the whole prediction horizon, $N=5$, but only the current value at the end of the prediction horizon, i.e. instead of $\forall k = 0 \ldots N$ in (6.41), $k$ is just equals to $N$. This will not only improve the stability but also reduce the computational burden during the online implementation, due to the reduced number of constraints.

6.7 Simulation example

In the section 6.5 the operation of a notional machine was studied. It is useful for the illustrative purposes, however, in the test SMPMSM the back EMF constant value is relatively low leading to a small field-weakening area. In this section the OT and OE energy operation of the real machine is studied and the energy consumption for the positioning operation is compared with the conventional G00 and G01 trajectories similarly to Chapter 5. In addition this chapter is checking the reachability of the obtained optimal trajectories via the online MPC implementation discussed in section 6.4. Only simulation is performed due to the computational limitations of modern
embedded controllers. The machine parameters together with its operation regions plot are given in Table 6.3 and Fig. 6.15 respectively. From the Fig. 6.15 field-weakening operation allows an increase in the speed of only ~10%, whereas if the absolute value of $i_d^{min}$ is chosen to be a half of $I_s^{max}$, the possible speed rise is even smaller, only ~5%.

![Fig. 6.15 B18 SMPMSM operation regions](image)

Table 6.3 Lab SMPMSM machine parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_s$, [$\Omega$]</td>
<td>0.86</td>
</tr>
<tr>
<td>$L_s$, [H]</td>
<td>6.65e-3</td>
</tr>
<tr>
<td>$\lambda_{pm}$, [$\frac{V}{rad/s}$]</td>
<td>0.356</td>
</tr>
<tr>
<td>$p$</td>
<td>6</td>
</tr>
<tr>
<td>$J_e$, [$kgm^2$]</td>
<td>0.011</td>
</tr>
<tr>
<td>$B_{fr}$, [$\frac{Nm}{rad/s}$]</td>
<td>4.65e-4</td>
</tr>
<tr>
<td>$I_s^{max}$, [$A$]</td>
<td>5</td>
</tr>
<tr>
<td>$V_{dc}$, [V]</td>
<td>250</td>
</tr>
</tbody>
</table>

It could be shown that the control scheme (6.44) is able to track any reachable position trajectory without an offset [127] if a perfect match of the real SMPMSM parameters with the model parameters is assumed. The resultant OT and OE state trajectories are shown in Fig. 6.16 together with their reference values. In the simulation of the controller closed-loop behaviour it was assumed that during the reference trajectory generation the inertia was 10% overestimated, moreover the current measurements are noisy with the noise having a normal distribution with zero ampere mean and standard deviation, $\sigma = 0.05A$. In (6.44) the reference trajectories, which include the augmented states, $u_{cc_d}$ and $u_{cc_q}$, are used, however in the open-loop optimal control problem (6.42) these states do not exist, therefore they are calculated using the obtained reference speed and current values, $\omega^*_r(k)$, $i_{d}^{r*}(k)$, $i_{q}^{r*}(k)$. The matrices $P$ and $Q$ in (6.44) are set equal to each other with large values corresponding to the currents and speed, and an even higher value corresponding to the position. The weight corresponding to the cross-coupling terms is zero, because the cross-coupling variation is not considered in the (6.43) model. The matrices are introduced in (6.45).
\[ P = Q = \begin{bmatrix} 10^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^6 \end{bmatrix}, \quad R = 0.08 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \] (6.45)
Fig. 6.16 Positioning system state trajectories: a) Optimal time control b) Optimal energy control (the measured are values denoted by "^" and the reference values are denoted by "^∗")

The reference position for the drive powering the actuator with the longer desired movement (OT control) distance is set to $\theta^*_r = 140\text{rad}$, whereas for the one for the shorted distance (OE) is set 30% shorter, i.e. $\theta^*_r = 98\text{rad}$ (both are “electrical” values). In this example, the energy required to move the end-effector in the 2D positioning system is estimated as 303mWh after 20 cycles of repetitions by each motor, where the time required for single cycle movement is $t_f = 0.525\text{s}$, with the maximum speed reaching the value of 426 rad/s during the OT operation, which is close to the predicted one $\omega_{i_{\text{dv}} \min} = 424\text{rad/s}$, shown in Fig. 6.15. To compare the energy consumption of the proposed method with the conventional G00 and G01 modes, these modes of operation were simulated assuming a trapezoidal velocity profile. The maximum drive speed was set as $\omega_b$ and maximum acceleration was set as $\frac{1}{J_e} \frac{3}{2} \lambda_{pm} I_{\text{max}}^m$, corresponding to
\[ i_{qs}^r = I_{s}^{max} \]. Similarly to the DC-motor case, the feed rate for the drive with the longer path in the G01 mode was kept on its maximum level. The obtained reference trajectories were feedforwarded to the cascaded controller Fig. 6.13(a), where the reference \( d \)-current was set to zero, \( i_{ds}^r = 0 \), therefore all the available current is used to produce torque. It is clear that the field-weakening operation is impossible in such a control strategy, therefore the operation time is increased, \( t_f = 0.532s \). The net energy consumed by both motor drives is approximately 361mWh and 302mWh in the G00 and G01 cases respectively. These results are summarized in Table 6.4 together with the maximum achievable speed and time required to accomplish the task.

For this particular example, the proposed algorithm improves the energy consumption by approximately 16% for G00, whereas compared to the G01 case the energy consumption is even slightly increased (less than half of percent). The rise of energy consumption of the proposed method compared to the G01 mode of operation appears due to the field-weakening operation, where the negative \( d \)-current is not generating torque. However, the proposed algorithm reduces the positioning system operation time. It was shown in Chapter 2 that the positioning system of the MT is generally responsible for 15% of the total energy consumption whereas 40% is accounted for the idle energy, which is constant and depends only on operation time. Therefore, improvement of the positioning system speed results in the considerable reduction of energy consumption even compared to the G01 mode of operation. Note, that generally in G01 mode the maximum drive speed used for the trapezoidal profile generation is lower than the maximum speed of the single drive, therefore the time difference and therefore the total energy consumption is reduced even more by the proposed scheme.

Note that in the presence of large model-parameter error and measured signal disturbances it is possible to improve the performance of (6.44) by including an integral action in the control loop (offset-free tracking). A number of different formulations have been developed in the literature, which are mostly some variant of the general

<table>
<thead>
<tr>
<th>( \omega_{max}, \text{[rad/s]} )</th>
<th>( t_f, \text{[s]} )</th>
<th>Energy, [mWh]</th>
</tr>
</thead>
<tbody>
<tr>
<td>G00 404</td>
<td>0.532</td>
<td>361</td>
</tr>
<tr>
<td>G01 404</td>
<td>0.532</td>
<td>302</td>
</tr>
<tr>
<td>Proposed 426</td>
<td>0.525</td>
<td>303</td>
</tr>
</tbody>
</table>
disturbance model approach [127]. For example, the velocity form model could be implemented to prevent an offset in the position tracking, however, these control strategies are beyond the scope of the research.

6.8 Conclusion

In this chapter the proposed multi-drive system control approach was further extended to consider the control of a SMPMSM motor. It is shown that as distinct from the DC-motor case the plant model, state and control constraints are nonlinear for the SMPMSM, whereas in Chapter 5 only linear time-invariant systems were considered. However, this nonlinearity is shown to be important as it allows the motors to run in the field-weakening regime, leading to the further increase in the motor speed and corresponding reduction in the positioning time without the constraints violation.

It is proposed to use the nonlinear MPC formulation for off-line path generation operation, where the IPOPT solver released under the EPL software license is used to solve the optimization control problem having the nonlinear plant. In order to reduce the calculation time the nonlinear constraints are approximated by polygons, where a novel constraints approximation method was suggested, which is able to reduce the complexity of the resulting controller, while also improving the accuracy compared with the conventional ones.

The resultant off-line reference trajectory could be used as a feed-forward for the conventional cascaded control, however, it is suggested to implement the MPC controller for the reference tracking as well. The plant is linearized using the augmented state approach in order to be able to implement the control on-line in the system with the kHz level sampling.

The proposed approach has been verified through the MATLAB simulation, using the drive parameters available in the University of Manchester. The simulation results reveal a considerable reduction of the proposed method energy consumption compared to the G00 mode of operation. Compared to the G01 case the energy consumption is almost the same or even worse for the proposed algorithm, however, this is compensated by the operation time reduction, leading to the reduction of the idle energy, which is generally much larger than the positioning system energy consumption.
Chapter 7 Conclusion

In this thesis a multi-drive system control approach is presented with strategies to decrease the overall energy consumption. It is intended to increase the speed of industrial transition towards the Fourth Industrial Revolution or Industry 4.0. Industry 4.0 communications allow better monitoring of industrial energy use, with the potential to significantly reduce energy consumption. Energy management provides an uncontroversial application of Industry 4.0, that does not threaten jobs. As such, it can encourage manufacturing companies to participate in the fourth industrial revolution. In this chapter the summary of the research undertaken is presented, together with the key contributions and findings. Areas for future research are identified and discussed.

7.1 Review of presented work and significant findings

7.1.1 Motivation
A thorough literature review has been carried out in order to identify the major target for multi-drive optimization. The reduction in energy consumption has been chosen due to its ecological, economical and power demand impacts. Metal-working machine tools are identified as a prospective candidate for such optimization through the intelligent control of the MT’s electric motor drives, which have an indirect interaction between each other. In the literature, a wide number of device-level optimization methods are available. However, this thesis addresses the benefits if the drive interactions are specifically considered.

In this research the energy consumption of the auxiliary processes (such as material preparation, cleaning, transportation and so on) is omitted; only the reduction of electrical energy precisely for the cutting itself is considered. In order to identify the most energy-demanding processes, MT power and energy inventorization has been performed. Both the systems’ power rating and particular processes’ energy consumption are studied in the literature, where the former has been shown to have less importance for determining the area of possible energy savings. The study of different processes’ energy consumption during the MT operation confirms the dominance of the so-called “Basic” energy which is similar to the stand-by energy consumption and depends only on the MT operation time; therefore it is possible to reduce the energy consumption by increasing the process rate.

The increase of the MT process rate could be achieved by reduction of the positioning system’s operation time during the off-cut mode of operation. The positioning system, by itself, is typically responsible for the around 15% of total energy consumption.

The central idea of the thesis is that the energy consumption of the MT can be reduced by advanced control of its positioning. A 2D example is studied where new move trajectories are proposed: the servo-motor with a longer required path is controlled in optimal time mode, whereas the servo-motor with the shorter path is controlled in the optimal energy mode, with the constraint that it should complete its move by the time the first one finishes. Such a control strategy simultaneously reduces the total operation time and therefore dominant “Basic” energy consumption, whilst also reducing the still considerable energy required for the positioning system operation. The strategy requires low-level communication between the drives, where the one in OT mode of operation defines the operation time of the one in OE mode.
7.1.2 *Optimal control theory. Variational approach*

This thesis proposes to optimize the off-cut mode of operation, therefore the static torque component is neglected. Moreover, it is shown that the drive dynamics of both axis in the 2D positioning system are independent from each other, therefore the optimization task is split into two independent tasks: optimal time control for one drive and optimal energy control for another.

The optimization task is formulated for three different cases depending on the constraints: no constraints, control constraints, state and control constraints. The drives are performing linear motion, therefore, as a first approximation the control plant is modelled as a simple double integrator system, with the acceleration as an input. For such a plant the conventional variational approach for optimal control problems (for both OT and OE) is employed, where all three types of constraints are considered. The Pontryagin’s minimal principle is used for the determination of open-loop trajectory profiles.

The modified drives trajectory profiles lead to the novel interpolation method, which is further compared with the conventional G00 and G01 methods. The proposed control method results in a slightly nonlinear tool trajectory, which is acceptable for the off-cut mode of operation, where only initial and final tool positions matter. However, the obtained open-loop solution has very limited application in practice due to inability to deal with the model parameters uncertainties and external disturbances; therefore the close-loop solution is required. Nevertheless, in order to test the proposed concept the derived open-loop control and state trajectories based on the double-integrator assumption have been implemented to control the mobile robot platform CORIN. It contains 18 DC motors, one for each robot joint. Chapter 3 shows that the dominant part of the overall energy consumption arises from the maintaining the robot body position against the gravity, therefore by controlling the drives in the OT mode it is possible to reduce the energy required to move the robot to the specified location. More than 20% energy consumption reduction has been achieved for the robot locomotion across the flat surface. This result would also apply to a MT where the stand-by or “Basic” energy consumption was assumed to dominates.

7.1.3 *Implementation of close-loop sliding-mode control for the linear motion system*

As it was mentioned, close-loop control solution is required, to suppress the parameters variation and model uncertainties. The analytical closed-loop solution is
obtained for the OT case for both state and control constraints. However, it is shown that the solution is elegant and simple only for the low order systems. If the system order is increased (triple integrator instead of double integrator) the derivation of optimal control laws becomes very complicated. For the more advanced drive models where the dynamics of the electrical system is considered in the presence of voltage and current constraints, the variational approach becomes too complicated (if possible). For the OE case the analytical solution even for the simple double integrator model is impossible, therefore for both control problems it is shown that some control algorithm predicting the future system behavior at each sampling instant is desired.

7.1.4 Permanent magnet brushed DC motor model predictive control

It is shown that the double integrator model is not able to accurately represent the drive’s behavior in the positioning system of a MT, although optimization using this model provides a basis for the conventional trapezoidal velocity profile, which is used in MTs with real time interpolation methods known as “rapid positioning” (G00) and “linear interpolation” (G01) for the point-to-point tool tip motion.

For the conventional linear time-invariant DC-motor plant model the novel two step procedure is suggested. At the first step, the Model Predictive Control technique is used to generate the open-loop trajectory profile off-line, where full DC-motor dynamics including the electrical dynamics are considered. The trajectories are generated separately for each drive with the longer and shorter movement distance. Linear MPC is used due to the linear plant model with the constraints represented by linear inequalities as well. It is assumed that the copper loss dominates, therefore it is convenient to use the quadratic performance index for the cost function formulation. The procedure of bringing this quadratic cost function together with the linear time-invariant equality constraints, i.e. drive model, and linear inequality constraints, i.e. current and voltage limits, to the general formulation of a quadratic programing (QP) problem is discussed in detail.

For the drive operating in the OE mode, the QP problem could be immediately used to generate the OE trajectory, whereas for the one in the OT mode, an additional procedure known as Time Optimal MPC or TOMPC is suggested. In order to solve the obtained QP for both OT and OE tasks any conventional quadratic problem solver could be adopted, for example *quadprog.m* in MATLAB.
At the second step, the close-loop control is performed, where the previously obtained optimal drive trajectories are used as feed-forward reference values. In order to implement such control for the DC-motor case it is convenient to use the conventional nested-loop control structure, where the slowest position control loop is an outer loop, the fastest current control loop is an inner one and the speed control loop is in-between.

A test platform has been developed in order to compare the simulation and experimental results. The ability to reduce the consumed energy has been justified and the possible control scheme modifications considering the variable current constraint level, depending on the predicted motor temperature, are suggested.

7.1.5 Permanent magnet synchronous motor control

The proposed control strategy has been extended to a Surface Mounted Permanent Magnet Synchronous Machine (SMPMSM) which is generally used instead of DC-motor to power the linear actuator of each axis of the MT positioning system. Since the SMPMSM model is nonlinear and the voltage and current constraints are nonlinear as well, this thesis shows that the previous procedure to formulate the QP problem for the DC-motor case is not valid for the SMPMSM. In spite of this clear drawback, a control algorithm was derived, allowing the SMPMSM to be controlled in so-called field-weakening region, where the machine speed could be increased above the base speed at the expense of the reduced torque. The higher maximum speed is shown to reduce the positioning system’s operation time and therefore reduce the MT’s stand-by loss.

A novel voltage constraint linearization method has been developed, to enlarge the portion of available DC-link voltage usage without increasing the number of inequality voltage constraints in the optimization problem. A new nonlinear MPC formulation, exploiting the proposed novel linear constraints is used to formulate the open-loop reference state trajectories for the drives working in both OT and OE modes.

In case of SMPMSM it is difficult to use the feed-forward control scheme for the close-loop control as it has been done for the DC-motor due to the cross-couplings between the direct and quadrature machine axis. A novel MPC formulation to follow the open-loop reference trajectories on-line has been developed and the superiority of the proposed method has been validated by comparison with the conventional control methods through the simulation.
7.2 **Contribution**

A novel control algorithm to reduce the energy consumption in indirectly mechanically-coupled drives has been developed. An implementation for the MT positioning system was demonstrated in the laboratory, considering the 2D case, where the energy is saved by controlling the motor drives with respect to each other.

Two different model predictive control schemes are proposed to generate the optimal state trajectory for a drive depending on the length of the required movement path. In order to follow the optimal trajectory two possible schemes have been considered in detail: feed-forward to the conventional nested-loop control and reference tracking MPC. The former has been fully implemented. To apply the reference-tracking MPC to a SMPMSM, a new approach to constraints linearization has been formulated, allowing field-weakening to be incorporated within the MPC framework.

Applying the proposed control method an energy saving of around 10% was confirmed experimentally using the DC-motors to power the positioning system’s actuators, where the operation time hasn’t been changed from the conventional one. For the SMPMSM case, the energy consumption could be reduced only compare to the G00 mode of operation, however the operation time is reduced to more than 1%, which leads to the “Basic” energy reduction, which is shown to be dominant.

The developed control algorithm does not require any hardware modifications and could be readily implemented in the real factory environment, boosting the overall take-up of the Industry 4.0 concept across the industry. The methods are applicable to industrial and mobile robots as well as MTs.

7.3 **Future work**

- Further investigation and experiments on the reduction of electrical energy consumption in the multi-drive systems need to be carried out. The proposed control strategy needs to be tested using the SMPMSM applying the on-line MPC reference tracking algorithm on the conventional floating point DSP used for the motor control.
- Since the speed and currents of the optimal trajectory during the OT control are not changing their value when the maximum speed is reached it is possible to simplify the off-line algorithm part described in Chapter 5 and Chapter 6. If the movement distance is long enough for a drive to reach its maximum speed, then instead of
solving the optimization problem of the OT trajectory generation this trajectory could be constructed using the pre-calculated acceleration and deceleration profiles and intermediate interval of constant currents and constant maximum speed of the appropriate length. Such an algorithm modification will reduce the calculation time, while not compromising the quality of the obtained reference trajectory.

- Further modification of the on-line reference tracking algorithm. In order to increase robustness of the proposed in Chapter 6 control to the parameters uncertainty and measurement noise the development of a proper algorithm extension is required. A velocity form model combined with the state observer or Kalman filter in the feedback loop is a possible candidate [139], provided it is fast enough.

- The application of MPC for the reference tracking problem is a computationally intense task, therefore the investigation of the different processing units, and architectures is desired. Modern trends to combine DSP and FPGA components are particularly interesting.

- In this thesis and particularly in Chapter 5 the Continuous Control Set MPC (CCS-MPC) was justified for the off-line trajectory generation, therefore a similar control strategy was adopted in Chapter 6 for the reference tracking problem, however additional investigation is required comparing the CCS-MPC with the Finite Control Set MPC (FCS-MPS) in terms of computational efficiency and accuracy of the constraints compliance.

- Currently the research scope is restricted to the DC-motor and SMPMSM cases, which are generally used in the positioning systems; however it could be expanded to the Interior Permanent Synchronous Motor (IPMSM) case as well. From a control point of view the major difference between the SMPMSM and IPMSM is the presence of the reluctance torque and consequently the different shape of the MTPA curve. Due to the modified MTPA shape the current and voltage trajectories in current and voltage $d$-$q$ reference planes are changed and the constraints approximation strategy would needed to be modified.

- In Chapter 5 the existing research regarding active thermal management was mentioned [117]. The algorithm keeps the motor winding temperature within safe limits, therefore if the maximum current value could be increased while temperature constraints are not violated, the minimum possible operation time
could be reduced and consequently the total energy consumption of the MT. This requires an algorithm capable of detecting the real time motor winding temperature in order to perform the constraints adaptation technique. The energy efficiency of the MT could be improved even more if the new process duration prediction, taking in account the modified current constraints, is transferred to the supervisory control level to adapt the overall control of production cell minimizing the energy consumption of inactive machinery.

- To put the multi-drive energy consumption reduction algorithm into practice by performing the tests on real hardware, considering the metal-working machine tool environment.
Appendix 1

Basics of the optimal control theory

In this section some basics of variational approach for the optimal control problem are presented. The discussion is started with several definitions and theorems following the [69].

Definition 1

“A functional $J$ is a rule of correspondence that assigns to each function $x$ in a certain class $\Omega$ a unique real number. $\Omega$ is called the domain of the functional, and the set of real numbers associate with the functions in $\Omega$ is called range of the functional.”

In other words, a functional is a function of a function. In order to find the extremum of a functional (3.13) the definitions of functional increment and its variation are given as well.

Definition 2

“If $x$ and $x + \delta x$ are functions for which the functional $J$ is defined, then the increment of $J$, denoted by $\Delta J$, is

$$\Delta J(x, \delta x) \triangleq J(x + \delta x) - J(x)$$

Definition 3

“The increment of a functional can be written as

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + y(x, \delta x) \cdot \|\delta x\|,$$

where $\delta J$ is linear in $\delta x$ and $y$ is remaining non-linear variation. If

$$\lim_{\|\delta x\| \to 0} y(x, \delta x) = 0$$

then $J$ is said to be differentiable on $x$ and $\delta J$ is the variation of $J$ evaluated for the function $x$.”

With these definitions it is possible to formulate the fundamental theorem of the calculus of variations. All the optimal control analysis in Chapter 3 and 4 is based on this theorem. According to [69], “Let $x$ be a vector function of $t$ in the class $\Omega$, and
Let \( J(x) \) be a differentiable functional of \( x \). Assume that the functions in \( \Omega \) are not constrained by any boundaries. Then the fundamental theorem is:

**Theorem 1**

"If \( x^* \) is an extremal (a value of \( x^* \) that maximises/minimises \( J \) as desired), the variation of \( J \) must vanish on \( x^* \); that is,

\[
\delta J(x^*, \delta x) = 0 \text{ for all admissible } \delta x.
\]

Using the result of Theorem 1 it is possible to formulate the necessary conditions for \( u^*(t) \) to be an extremal. The problem is to find an admissible control \( u^*(t) \) for the minimum time problem and for the Lagrange problem. This optimal control would cause the system in a form (3.9) to minimize the performance criteria in a form (3.13) to follow an admissible optimal trajectory \( x^*(t) \) with a specified final state admissible optimal trajectory \( x^*(t) \) with a specified final state \( x(t_f) \) and arbitrary final time \( t_f \).

\[
J(u) = \int_{t_0}^{t_f} g(x, u, t) dt
\]

(A.1.1)

At first, it is assumed that both the admissible state and control functions are not bounded. For motor applications the initial state \( x(t_0) = x_0 \), final state \( x(t_f) = x_f \) and initial time \( t_0 \) are specified. \( x \) is an \( n \times 1 \) state vector and \( u \) is an \( m \times 1 \) control vector.

To find the optimal control \( u^* \), system state equations (3.9) must be included to the functional (A.1.1) to form an augmented functional \( J_a \).

\[
J_a = \int_{t_0}^{t_f} \left[ g(x, u, t) + p^T [f(x, u, t) - \dot{x}] \right] dt,
\]

(A.1.2)

where \( p^T(t) \) is a vector of Lagrange multipliers. It should be noted that if the plant equations are satisfied, then \( J_a = J \) for any function \( p \).

According to Theorem 1, to find the optimal \( u^*(t) \) the variation of \( J_a \) through the variations \( \delta x, \delta u, \delta p, \delta t_f \) should be determined and set to zero. It is convenient to define new function, which is the integrant of (A.1.2)

\[
g_a(x, \dot{x}, u, p, t) = g(x, u, t) + p^T [f(x, u, t) - \dot{x}]
\]

(A.1.3)

Then for (A.1.1) the variation of augmented functional is defined as follows.
\[\delta J_a(u^*) = 0 = \left[ g_a(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f \right] \\
- \left[ \frac{\partial g_a}{\partial x} 
\left( x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f \right) \right]^T \dot{x}^*(t_f) \right] \delta t_f \\
+ \int_{t_0}^{t_f} \left\{ \left[ \frac{\partial g_a}{\partial x} 
\left( x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f \right) \right]^T \delta x(t) \\
- \frac{d}{dt} \left[ \frac{\partial g_a}{\partial x} 
\left( x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f \right) \right]^T \delta u(t) \\
+ \left[ \frac{\partial g_a}{\partial p} \left( x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f \right) \right]^T \delta p(t) \right\} dt \]

(A.1.4)

Variation \(\delta t_f\) is arbitrary, so in order to equalize the variation of the augmented functional to zero, members of a sum (A.1.4) must equal zero simultaneously. Considering this and introducing the Hamiltonian function

\[ H(x(t), u(t), p(t), t) \equiv g(x(t), u(t), t) + p^T(t) [f(x(t), u(t), t)] \]

the necessary conditions for the optimal control are stated in (A.1.6) and (A.1.7).

\[ \dot{x}^*(t) = \frac{\partial H}{\partial p} \left( x^*(t), u^*(t), p^*(t), t \right) \]  
\[ \dot{p}^*(t) = -\frac{\partial H}{\partial x} \left( x^*(t), u^*(t), p^*(t), t \right) \]  \( t \in [t_0, t_f] \)  
\[ 0 = \frac{\partial H}{\partial u} \left( x^*(t), u^*(t), p^*(t), t \right) \]  
\[ H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) = 0 \]  

(A.1.6a)  
(A.1.6b)  
(A.1.6c)  
(A.1.7)

It is important to mention that necessary conditions (A.1.6) consists of a set of \(2n\) first order, generally nonlinear, ordinary differential equations ODE (\(n\) state equations (A.1.6a) and \(n\) costate equations (A.1.6b) with split boundary values and \(m\) algebraic relations (A.1.6c). The solution of these \(2n\) differential equations would lead to appearance of \(2n\) constants of integrations. These constants are identified through \(2n\) equations \(x^*(t_0) = x_0, \dot{x}^*(t_f) = x_f\). Finally the time \(t_f\) is found from the boundary condition (A.1.7).

In the derivation of (A.1.6), (A.1.7) presented in [69] admissible curves for the optimal state \(x^*(t)\) and control \(u^*(t)\) are assumed to be continuous and have a continuous first derivatives, i.e. admissible trajectories are smooth.
A.1.1 Control constraints. Pontryagin’s Minimum Principle

So far in the development of optimal control task (3.9)-(3.13) it was considered that the admissible control functions are not constrained. However, in a motor drive system, the voltage (which is the control input function) is limited by the maximum DC link value and some type of PWM used to control the inverter. In order to consider the effect of control constraints, (A.1.6) should be modified. This is done by the implementation of Pontryagin’s minimum principle [140].

\[
\begin{align*}
\dot{x}^*(t) &= \frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) \\
\dot{p}^*(t) &= -\frac{\partial H}{\partial x}(x^*(t), u^*(t), p^*(t), t) \\
H(x^*(t), u^*(t), p^*(t), t) &\leq H(x^*(t), u(t), p^*(t), t) \\
&\text{for all admissible } u(t)
\end{align*}
\]

\[A.1.8\]

\[
H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) = 0
\]

\[A.1.9\]

In the previous section a strict requirement was imposed on the state and control functions, which must be smooth. However Pontryagin’s minimum principle enlarges the class of admissible curves \(\Omega\) to include piecewise-continuous curves for the control functions \(u^*(t)\) and curves, that have piecewise-continuous first derivatives for the state \(x^*(t)\). Also it must be emphasized that generally it is not possible to calculate the optimal control trajectory by the (A.1.6) and later saturate it when the control curve touches the boundary.

A.1.2 State Variable Constraints

In motor drive systems it is not only the voltage that is limited, but also the maximum current. Maximum current is generally decided by the thermal time constant of either the inverter or the motor itself. In motor drive control, the current is considered as a state variable, so the state variable constraints should be included in the optimal control problem derivation. Following [69] the state variable constraints are assumed to be in the form

\[
L(x(t), t) \geq 0
\]

\[A.1.10\]

where \(L\) is a “\(r\)-vector function” [69] with continuous first and second partial derivatives. In order to get the solution of such a problem a new state variable should be introduced in addition to the \(n\) states in (3.9).
\[ \dot{x}_{n+1}(t) \triangleq [L_1(x(t), t)]^2 h(-L_1) + [L_2(x(t), t)]^2 h(-L_2) + \ldots + [L_r(x(t), t)]^2 h(-L_r) \quad (A.1.11) \]

where \( h(-L_i) \) is a Heaviside step function defined by
\[
h(-L_i) = \begin{cases} 
0, & \text{for } L_i \geq 0 \\
1, & \text{for } L_i < 0 
\end{cases} \quad (A.1.12)\]

According to (A.1.11) and (A.1.12), \( \dot{x}_{n+1}(t) = 0 \) only when all the constraints are satisfied, and also \( x_{n+1}(t_0) = 0 \) and \( x_{n+1}(t_f) = 0 \). Thus with a new state function

Hamiltonian function is defined as
\[
H(x(t), u(t), p(t), t) \triangleq g(x(t), u(t), t) + \sum_{i=1}^{n} p_i(t) f_i(x(t), u(t), t) + p_{n+1}(t) \{ [L_1(x(t), t)]^2 h(-L_1) + \ldots + [L_r(x(t), t)]^2 h(-L_r) \} \quad (A.1.13)
\]

and applying the procedure analogues to one at the beginning of section 3.3 a set of equations defining the necessary condition for the optimality is given as
\[
\begin{align*}
\dot{x}^*(t) &= \frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) \\
\dot{p}_i^*(t) &= -\frac{\partial H}{\partial x_i}(x^*(t), u^*(t), p^*(t), t) \\
& \quad \vdots \\
\dot{p}_{n+1}^*(t) &= -\frac{\partial H}{\partial x_{n+1}}(x^*(t), u^*(t), p^*(t), t) \\
H(x^*(t), u^*(t), p^*(t), t) &\leq H(x(t), u(t), p(t), t) \\
&\text{for all admissible } u(t) \quad (A.1.14)
\end{align*}
\]

Notice that now \( x(t) \) and \( p(t) \) are \( n + 1 \) vectors and (A.1.14) is a set of \( 2n + 2 \) differential equations, where additional two constants of integration are found by applying \( x_{n+1}(t_0) = 0 \) and \( x_{n+1}(t_f) = 0 \).

Consideration of inequality constraints increases the dimension of the system of differential equations (A.1.14), moreover a highly non-linear function (Heaviside) is introduced in (A.1.11) making it difficult to solve the problem not only analytically, but numerically as well. In order to reduce the computational burden, the Heaviside function and the saturation function (which is often a result of a control-constrained problem) could be approximated by smooth varying continuous functions as [141]
\[
S_{app}(x, \nu) = \frac{1}{2} \left( \sqrt{\nu + (x + 1)^2} - \sqrt{\nu + (x - 1)^2} \right), \quad (A.1.15)
\]

for a saturation function \( S_{app}(x) = \begin{cases} 
 x, & \text{for } |x| \leq 1 \\
 \text{sign}(x), & \text{for } |x| > 1
\end{cases} \). And
Appendix 1

\[ h_{app}(x, \nu) = \frac{1}{2} \left( 1 + \frac{2}{\pi} \text{atan}\left(\frac{x}{\nu}\right) \right), \]  \hspace{1cm} (A.1.16)

for a Heaviside function. For both (A.1.15) and (A.1.16) \( \nu \) is an arbitrary small positive number. By reducing parameter \( \nu \) the approximation error is also reduced.
Appendix 2

Angular acceleration in the Cartesian space

In section 0 matrix $K$ has been introduced, which contains the nine elements representing the Jacobian time derivative terms in a form $K_{i,j}$. For the single CORIN leg this terms are listed as follows.

\[
K_{11} = \cos \theta_1 (l_d \sin \theta - l_2 \cos \theta_2 - l_3 \cos \theta) \omega_1 \\
+ \sin \theta_1 (l_3 \sin \theta + l_2 \sin \theta_2 + l_d \cos \theta) \omega_2 \\
+ \sin \theta_1 (l_3 \sin \theta + l_d \cos \theta) \omega_3 \\
K_{12} = \sin \theta_1 (l_d \cos \theta + l_2 \sin \theta_2 + l_3 \sin \theta) \omega_1 \\
+ \sin \theta_1 (l_d \sin \theta - l_2 \cos \theta_2 - l_3 \cos \theta) \omega_2 \\
+ \sin \theta_1 (l_d \sin \theta - l_3 \cos \theta) \omega_3 \\
K_{13} = \sin \theta_1 (l_d \cos \theta + l_3 \sin \theta) \omega_1 \\
+ \cos \theta_1 (l_d \sin \theta - l_3 \cos \theta) \omega_2 + \cos \theta_1 (l_d \sin \theta - l_3 \cos \theta) \omega_3 \\
K_{21} = \sin \theta_1 (l_d \sin \theta - l_2 \cos \theta_2 - l_3 \cos \theta) \omega_1 \\
- \cos \theta_1 (l_3 \sin \theta + l_2 \sin \theta_2 + l_d \cos \theta) \omega_2 \\
- \cos \theta_1 (l_3 \sin \theta + l_d \cos \theta) \omega_3 \\
K_{22} = -\cos \theta_1 (l_d \cos \theta + l_2 \sin \theta_2 + l_3 \sin \theta) \omega_1 \\
+ \sin \theta_1 (l_d \sin \theta - l_2 \cos \theta_2 - l_3 \cos \theta) \omega_2 \\
+ \sin \theta_1 (l_d \sin \theta - l_3 \cos \theta) \omega_3 \\
K_{23} = -\cos \theta_1 (l_d \cos \theta + l_3 \sin \theta) \omega_1 \\
+ \sin \theta_1 (l_d \sin \theta - l_3 \cos \theta) \omega_2 + \sin \theta_1 (l_d \sin \theta - l_3 \cos \theta) \omega_3 \\
K_{31} = 0 \\
K_{32} = -(l_d \cos \theta + l_2 \sin \theta_2 + l_3 \sin \theta) \omega_2 -(l_d \cos \theta + l_3 \cos \theta) \omega_3 \\
K_{33} = -(l_d \cos \theta + l_3 \sin \theta) \omega_2 -(l_d \cos \theta + l_3 \sin \theta) \omega_3 \\
\]

where $\theta = \theta_1 + \theta_2$. 

(A.2.1)
Appendix 3

In Chapter 4 the optimal time control problem of a triple integrator system has been formulated. In the following sections the optimal control law for that problem is derived.

A.3.1 State variable redefinition

In (4.13) the open-loop OT solution using the Pontryagin’s minimum principle has been obtained.

\[ u^*(t) = \begin{cases} \sigma, & t \in (0, \theta_1) \\ -\sigma, & t \in (\theta_1, t_f) \end{cases} \]  

(4.13)

In order to find a closed-form solution of the minimal time problem it is possible to define the time interval, where \( \sigma \) is applied as \( \theta_1 \) and the time interval where \( -\sigma \) is applied as \( \theta_2 \), then \( \theta_1 + \theta_2 = t_f \). Also it is convenient to introduce one more variable

\[ \lambda = -\frac{\theta_2}{\theta_1 + \theta_2} \]

If \( \lambda = 1 \rightarrow \theta_1 = 0 \rightarrow u(t) = -\sigma, \forall t \) and if \( \lambda = 0 \rightarrow \theta_2 = 0 \rightarrow u(t) = \sigma, \forall t \).

Integrating the initial system (4.12) with the initial conditions \( (x_{10}, x_{20}, x_{30}) \) and considering the boundary conditions \( x_1(t_f) = 0, x_2(t_f) = 0 \), one would obtain the dependency of different control intervals duration from the initial conditions (A.3.1). However, while developing the close-loop control, the initial values \( x_{10}, x_{20}, x_{30} \) could be seen as the current values of the state variables \( x_1, x_2, x_3 \) so in the following equations the subscript “0” is omitted.

\[ \begin{align*}
  x_1 &= \frac{1}{2}x_3 t_f^2 - \frac{1}{3} \sigma (\theta_1^3 + 3\theta_1^2 \theta_2 - \theta_2^3) \\
  x_2 &= -x_3 t_f - \frac{1}{2} \sigma (\theta_1^2 + 2\theta_1 \theta_2 - \theta_2^2) \\
  \theta_1 + \theta_2 &= t_f
\end{align*} \]

(A.3.1)

(A.3.1) is a system of two nonlinear and one linear equation with four unknowns \( (\theta_1, \theta_2, t_f, \text{sign}(\sigma)) \) and can’t be solved explicitly.

In order to synthesize the control law, variable substitutions are introduced as follows [71].
Appendix 3

\[ \xi = \frac{x_1}{x_3}, \quad \eta = \frac{x_2}{x_3}, \quad \zeta = \text{sign}(x_3), \quad s = \frac{t_f}{|x_3|}, \quad \lambda = \frac{\theta_2}{t_f} \]  
(A.3.2)

\[ X_1(\lambda) = \frac{1}{3}(1 - 3\lambda^2 + \lambda^3), \quad X_2(\lambda) = \lambda^2 - \frac{1}{2} \]

This variable substitution reduces the order of the state space and allows the control law for the optimal time problem to be defined in a \( \xi - \eta \) plane. Applying the variable substitutions, (A.3.2) is written as

\[ \zeta \left( \xi s^{-3} - \frac{1}{2}s^{-1} \right) = \sigma X_1(\lambda) \]
\[ \zeta (\eta s^{-2} + s^{-1}) = \sigma X_2(\lambda) \]  
(A.3.3)

for \( x_3 \neq 0 \). If \( x_3 = 0 \), then (A.3.1) with the same variable substitutions will be written as (A.3.4).

\[ x_1 = \sigma X_1(\lambda) t_f^3 \]
\[ x_2 = \sigma X_2(\lambda) t_f^2 \]  
(A.3.4)

As mentioned earlier the optimal time control law is developed in a form \( u^* = f(x_1, x_2, x_3) \). Together with (4.13) that means that it is enough to find switching surfaces in a 3D state space \( (x_1, x_2, x_3) \) where the control change its sign: \( u^* = \pm 1 \). On these surfaces one of the intervals \( \theta_1 \) or \( \theta_2 \) is zero, or in other words \( \lambda = 0 \) or \( \lambda = 1 \). According to (A.3.2) these values of \( \lambda \) correspond to \( X_1(\lambda) = \pm \frac{1}{3} \) and \( X_2(\lambda) = \mp \frac{1}{2} \) respectively and substituting these into (A.3.3) will define the conditions that should be met by the switching curves in \( \xi - \eta \) plane (A.3.5).

\[ \zeta \left( \xi s^{-3} - \frac{1}{2}s^{-1} \right) = \pm \frac{1}{3} \sigma \]
\[ \zeta (\eta s^{-2} + s^{-1}) = \mp \frac{1}{2} \sigma \]  
(A.3.5)

However, if the final time \( t_f \left( s = \frac{t_f}{|x_3|} \right) \) is not known, equations (A.3.5) cannot fully define the switching curves in \( \xi - \eta \) plane. More detailed analysis is required.

A.3.2 Control law preparation

With the newly introduced variables, the control problem should be redefined. Using the equations (A.3.3), the problem is formulated as follows; at every control step find the value \( \sigma = \pm 1 \) which corresponds to the solution of (A.3.3) with fixed \( \xi, \eta, \zeta \) with the smallest possible \( s > 0 \), where \( \xi, \eta, \zeta \) are obtained by a state feedback, also it is worth noting that by definition \( \lambda \in [0,1] \).

To solve the optimal time problem, two cases are considered. To begin with, the case when \( x_3 = 0 \) is discussed, later on the \( x_3 \neq 0 \) will be discussed.
a) \( x_3 = 0 \)

As shown earlier, on the switching curve when the acceleration is zero, variables \( X_1(\lambda) \) and \( X_2(\lambda) \) become \( \pm \frac{1}{3} \) and \( \mp \frac{1}{2} \) respectively. So the system (A.3.4) is written as

\[
\begin{align*}
x_1 &= \pm \frac{1}{3} \sigma t_f^3 \\
x_2 &= \mp \frac{1}{2} \sigma t_f^2
\end{align*}
\] (A.3.6)

From the second equation, \( t_f = \sqrt{|2x_2|} \) because \( \sigma = \pm 1 \) and \( t_f \) must be positive real number. By summing the first and second equations from (A.3.6) and substituting the value for \( t_f \), the switching curve in the \( x_3 = 0 \) plane is obtained.

\[
\Psi(x_1, x_2, 0) = 3x_1 + 2x_2\sqrt{|2x_2|} = 0
\] (A.3.7)

Considering (A.3.7), the optimal time close-loop control law for the zero acceleration case could be obtained as (A.3.8).

\[
u^*(x_1, x_2, 0) = \begin{cases} 
- \text{sign}(\Psi(x_1, x_2, 0)), & \text{if } \Psi \neq 0 \\
\text{sign}(x_1), & \text{if } \Psi = 0
\end{cases}
\] (A.3.8)

The resultant control law is shown on Fig. A.3.1, which represents the \( x_3 = 0 \) state plane, where \( D-B \) is a switching curve \( \Psi \). On the left hand side from the \( D-B \) curve the optimal control law is \( u^* = 1 \), the area is shown in red, and on the right hand side from the \( D-B \) curve the optimal control law is \( u^* = -1 \) and the area is shown in blue. On the switching curve itself, the \( D-0 \) part corresponds to the positive control whereas the \( B-0 \) part corresponds to the negative control.

![Fig. A.3.1 Switching curve, \( x_3 = 0 \)](image)

b) \( x_3 \neq 0 \)

Starting from the original system state model (4.12) and using the variable substitutions (A.3.2), a reduced order state space representation, \( (x_1, x_2, x_3) \rightarrow (\xi, \eta) \), shown in (A.3.9).
\[ \dot{x}_1 = (\xi x_3^3)' = \xi x_3^3 + 3\xi x_3^2 u = x_2 = \eta x_3 x_3 \] \[ \dot{x}_2 = (\eta x_3 x_3)' = \eta x_3 x_3 + 2\eta x_3 \dot{x}_3 \zeta = x_3 \rightarrow \dot{\eta} = |x_3^{-1}|(1 - 2\eta u \zeta) \] (A.3.9)

It was shown previously that the optimal control \( u^* \) could only be \( \pm 1 \), also the value of \( \zeta \) could be only \( \pm 1 \) as well, so it is possible to introduce a new variable \( \alpha = u \zeta = \pm 1 \). Using that newly introduced variable and dividing the first state equation from (A.3.9) by the second one, a linear differential equation with respect to \( \xi \) could be obtained (A.3.10).

\[ \frac{d\xi}{d\eta} = \frac{\eta - 3\xi \alpha}{1 - 2\eta \alpha} \] (A.3.10)

Equation (A.3.10) represents the system dynamics in the \( \xi - \eta \) plane. While moving along the optimal trajectory \( (u^* = \pm 1) \), which is not crossing the \( x_3 = 0 \) surface, the sign of \( \alpha \) does not change. So it is possible to integrate (A.3.10) with the constant parameter \( \alpha \). The result is the equation describing the system dynamics in the \( \xi - \eta \) plane in a form \( \xi = \Phi(\eta, \alpha, C) \), where \( C \) is an arbitrary constant. The equation (A.3.10) is solved under the consideration that parameter \( \alpha \) is constant and could take the value \( \pm 1 \). In that case, equation (A.3.10) can be reduced to homogeneous differential equation. To prove this, the fraction in the right hand side is expanded as in (A.3.11).

\[ \frac{d\xi}{d\eta} = \frac{\eta - 3\xi \alpha}{1 - 2\eta \alpha} = \frac{a_1 \eta + b_1 \xi + c_1}{a_2 \eta + b_2 \xi + c_2} \] (A.3.11)

where \( a_1 = 1, b_1 = -3\alpha, c_1 = 0 \) and \( a_2 = -2\alpha, b_2 = 0, c_2 = 1 \). Then the determinant (A.3.12) is to be examined.

\[ \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{vmatrix} 1 & -3\alpha \\ -2\alpha & 0 \end{vmatrix} = -6\alpha^2 \neq 0 \] (A.3.12)

Since the determinant is not equal to zero it is possible to reduce the equation (A.3.10) to a homogeneous differential equation. In order to do this, as a first step a system of linear equations with \( x, y \) unknowns is constructed.

\[ x - 3\alpha y = 0 \]
\[ 1 - 2\alpha x = 0 \] (A.3.13)

From (A.3.13) the values \( x = \frac{1}{2\alpha}, y = \frac{1}{6\alpha^2} \) are deduced. Using the obtained values, a variable substitution is performed.

\[ \xi = \dot{\xi} + y = \dot{\xi} + \frac{1}{6\alpha^2} \]
\[ \eta = \dot{\eta} + x = \dot{\eta} + \frac{1}{2\alpha} \] (A.3.14)
By substituting the modified variables to the original equation (A.3.10) the conventional homogeneous differential equation with respect to $\xi, \eta$ is obtained.

$$\frac{d(\xi + \frac{1}{6\alpha^2})}{d(\eta + \frac{1}{2\alpha})} = \frac{d\xi}{d\eta} = \frac{\eta - 3\alpha \xi}{-2\alpha \eta}$$ (A.3.15)

Equation (A.3.15) is solved by a conventional variable substitution. A new variable is introduced, $\hat{\xi} = \rho \hat{\eta}$, where $\rho$ is a function of $\hat{\eta}$ as well. Than the equation (A.3.15) is solved as following

$$\frac{d\hat{\xi}}{d\hat{\eta}} = \frac{d\rho}{d\hat{\eta}} \hat{\eta} + \rho = \frac{\hat{\eta} - 3\alpha \rho \hat{\eta}}{-2\alpha \hat{\eta}} = \frac{1 - 3\alpha \rho}{-2\alpha}$$ (A.3.16)

The variable separation is applied

$$\frac{d\rho}{1 - \alpha \rho} = \frac{d\hat{\eta}}{-2\alpha \hat{\eta}}$$ (A.3.17)

By integrating left and right parts of (A.3.17) one would obtain

$$\int \frac{d\rho}{1 - \alpha \rho} = \frac{-1}{2\alpha} \ln|1 - \alpha \rho| = \frac{-1}{2\alpha} \ln|\hat{\eta}| - \frac{1}{\alpha} \ln|C_1|$$

$$\ln|1 - \alpha \rho| = \ln|\hat{\eta}|^{\frac{1}{2\alpha}} + \ln|C_1| = \ln|C_1| \sqrt{\hat{\eta}}$$ (A.3.18)

where $C_1$ is an arbitrary positive constant. By omitting the natural logarithm the equation (A.3.18) is simplified to (A.3.19).

$$|1 - \alpha \rho| = |C_1| \sqrt{\hat{\eta}}$$ (A.3.19)

After the return substitution $\rho = \frac{\hat{\xi}}{\hat{\eta}}$ followed by the return substitution $\hat{\xi} = \xi - \frac{1}{6\alpha^2}$ and $\hat{\eta} = \eta - \frac{1}{2\alpha}$ the equation in the original $\xi, \eta$ variables is obtained in (A.3.20).

$$\left|1 - \alpha \frac{\hat{\xi}}{\hat{\eta}}\right| = |C_1| \sqrt{\hat{\eta}}$$

$$\left|\hat{\eta} - \alpha \hat{\xi}\right| = |C_1| |\hat{\eta}|^{\frac{3}{2\alpha}}$$

$$\left|\eta - \alpha \xi - \frac{1}{3\alpha}\right| = \frac{|C_1| |2\alpha \eta - 1|^{\frac{3}{2}}}{2^{\frac{3}{2}} |\alpha|^{\frac{3}{2}}}$$ (A.3.20)

$$\frac{\left|\alpha \eta - \xi - \frac{1}{3}\right|}{|\alpha|} = \frac{|C_1| |1 - 2\alpha \eta|^{\frac{3}{2}}}{2^{\frac{3}{2}} |\alpha|^{\frac{3}{2}}}$$

$$\left|\alpha \eta - \xi - \frac{1}{3}\right| = C_2 |1 - 2\alpha \eta|^{\frac{3}{2}}$$
where $C_2$ is an arbitrary positive constant. By substituting $C_2$, which is strictly positive by an arbitrary constant $C$ it is possible to expand the modulus in the left hand side of (A.3.20). So finally the dependency of $\xi$ from $\eta$ could be expressed as follows.

$$\xi = \Phi(\eta, \alpha, C) = a\eta - \frac{1}{3} + C|1 - 2a\eta|^{3/2}$$  \hspace{1cm} (A.3.21)

While developing the control law for the triple integrator it is important to note that with the simultaneous change of signs of $\xi$ and $\sigma$ in (A.3.3) (both are multipliers in left and right hand side of equations respectively), both equations in (A.3.3) are still valid. So when $\xi$ changes its sign, or in other words, when acceleration, $x_3$, changes its sign, the target value (control), $\sigma$, should change its sign as well, making it possible to derive the control law for the positive $\xi$ and arbitrary value of $\xi, \eta$ and then just change the sign in the obtained dependency for $\xi = -1$. Due to this, the control law for positive values of acceleration ($\xi = 1$) only will be developed.

### A.3.3 Control law

Equations (A.3.3) are an indirect representation of the control time intervals dependency on the state feedback values and it is possible to use these equations in order to find the rule for moving the system in a minimum time to the origin\(^9\). From the second equation of (A.3.3) the value of $\lambda$ could be deduced.

$$\lambda = \sqrt{\frac{1}{2} + \sigma(\eta s^{-2} + s^{-1})}$$ \hspace{1cm} (A.3.22)

Further, this value is substituted into the first equation of (A.3.3) and for the two possible values of $\sigma = \pm 1$, it is possible to deduce the dependency of state variable $\xi^\pm$ from $s, \eta$, where “+” in the superscript of $\xi$ denotes the $\sigma = 1$ and “−” denotes the $\sigma = -1$.

$$\xi^\pm = \mp \frac{1}{6}s^3 - \frac{1}{2}s^2 - s\eta \pm \frac{1}{3}\left(\frac{s^2}{2} \pm \eta \pm s\right)^{3/2}$$  \hspace{1cm} (A.3.23)

It is possible to analyze (A.3.23) in the whole range of $s$ and $\eta$. The set of curves corresponding to $\sigma = \pm 1$ would be obtained and by plotting in the $s$-$\xi$ plane the line corresponding to the $\xi = \text{const}$, it is possible to deduce the minimum value of $s$ ($s$ is proportional to the process final time $t_f$), by intersection of this line and the curves defined by (A.3.23). The value of $\sigma = \pm 1$ corresponds to the minimum value of $s$ that will define the control law $u^* = \sigma$ with a given $\xi, \eta$ and positive $\xi$.

---

\(^9\) According to problem formulation only $x_1(t_f) = 0$, $x_2(t_f) = 0$, whereas $x_3(t_f)$ is not fixed. So here “origin” is a right term only for two state coordinates.
According to the definition of $\lambda$ the expression under the square root in (A.3.22) should be in $[0;1]$ interval $\forall s \geq 0 \in \mathbb{R}$, moreover the expression in the brackets in (A.3.23) should be strictly non-negative as well. These lead to the following conditions defining the area, where $\xi^\pm(s)$ exists.

\[ 0 \leq \frac{1}{2} \pm \eta s^{-2} + s^{-1} \leq 1 \rightarrow \begin{cases} \sigma = +1 : s \in [s_3; +\infty), s \in [0; s_2] \cup [s_1; +\infty) \\ \sigma = -1 : s \in [0; s_2] \cup [s_1; +\infty), s \in [s_3; +\infty) \end{cases} \]

\[ 0 \leq \frac{s^2}{2} \pm \eta s \pm s \rightarrow \begin{cases} \sigma = +1 : s \in [s_3; +\infty) \\ \sigma = -1 : s \in [0; s_2] \cup [s_1; +\infty) \end{cases} \tag{A.3.24} \]

where $s_1 = 1 + \sqrt{1 + 2\eta}$, $s_2 = 1 - \sqrt{1 + 2\eta}$, $s_3 = -1 + \sqrt{1 - 2\eta}$. All the conditions from (A.3.24) must be valid simultaneously, however the condition $\forall s \geq 0 \in \mathbb{R}$ is not fulfilled for $s_1 \ldots s_3$ for the whole of $\eta$ variation range. So for $\eta \geq 0$ the function $\xi^\pm(s)$ is defined on $s \in [s_1; +\infty)$, for $-\frac{1}{2} \leq \eta < 0$ the function $\xi^\pm(s)$ is defined on $s \in [s_3; s_2] \cup [s_1; +\infty)$ and for $\eta < -\frac{1}{2}$ the function $\xi^\pm(s)$ is defined on $s \in [s_3; +\infty)$.

To find the position of $\xi^+(s)$ and $\xi^-(s)$ curves with respect to each other their intersection points are defined first by equalizing the expressions for $\xi^+(s)$ and $\xi^-(s)$. Analysis of this equation states that $s_1 \ldots s_3$ are the only intersection points of these curves. Also $\xi^\pm(s_3) > \xi^\pm(s_1)$ only for $\eta \in (-\frac{\pi}{4}, 0]$. So functions $\xi^\pm(s)$ are plotted on Fig. A.3.2 for the four different intervals of $\eta$ variation.
Appendix 3

In [71] it is mathematically proven that the shape and relative positions of $\xi^\pm(s)$ curves would stay the same as is shown on Fig. A.3.2 for all values of $\eta$. So the minimal time control for the each region of $\eta$ variation could be identified.

For $\eta \geq 0$, shown on Fig. A.3.2(a) the minimal time is reached on $\xi^+(s) \forall \xi < \xi(s_1)$ and on $\xi^-(s) \forall \xi > \xi(s_1)$.

For $-\sqrt{3}/4 < \eta < 0$ shown on Fig. A.3.2(b) the minimal time is reached in the same way as the previous case, i.e. on $\xi^+(s) \forall \xi < \xi(s_1)$ and on $\xi^-(s) \forall \xi > \xi(s_1)$. Since the small isolated curve lies higher that $\xi(s_1)$ it does not influence the control strategy.

For $-1/2 \leq \eta \leq -\sqrt{3}/4$ shown on Fig. A.3.2(c) the small isolated curve does influence the control action for a minimal time performance. So $\sigma = +1$ is chosen $\forall \xi < \xi(s_3)$ and $\sigma = -1 \forall \xi > \xi(s_3)$.

For $\eta < -1/2$ shown on Fig. A.3.2(d) the minimal time is reached in the same as for $-1/2 \leq \eta \leq -\sqrt{3}/4$. i.e. on $\xi^+(s) \forall \xi < \xi(s_3)$ and on $\xi^-(s) \forall \xi > \xi(s_3)$.

For all four cases the optimal control choice ($\sigma = \pm 1$) is shown by bold lines, where the red one corresponds to $\sigma = +1$ control and blue one to $\sigma = -1$. The missing part is a choice of a control action along the $\xi^\pm(s_1(s)), \xi^\pm(s_2(s)), \xi^\pm(s_3(s))$ curves in $\xi-\eta$ plane. According to (A.3.22) at $s = s_1$ and $\sigma = -1$ parameter $\lambda$ is equal to 0, in other words the duration of $u^* = -\sigma = 1$ is zero. That means that on a $\xi^\pm(s_1(s))$ curve $u^* = -1$ and this curve is a switching curve in $\xi-\eta$ plane for all $\eta > -\sqrt{3}/4$. Similarly at $s = s_2$ and $\sigma = -1$ parameter $\lambda = 0$, i.e. $u^* = -1$ along whole $\xi^\pm(s_2(s))$ curve, but this is not a switching curve. At $s = s_3$ and $\sigma = 1$ parameter $\lambda = 0$, so on a $\xi^\pm(s_3(s))$ curve $u^* = 1$ and this curve is a switching curve in $\xi-\eta$ plane for all $\eta \leq 0$.

A.3.4 Switching curves

In the previous section the control principle to minimize the operation time was deduced. It was shown that after introducing the new state variables it is possible to reduce the order of a system (with a fixed sign of the third state, $x_3 > 0$). For that reduced order system, and for the new state variables $x, \eta$ the control principle is as follows. For a current state $\eta$ value, which is bigger than $-\sqrt{3}/4$, the value of a current state value $x$ is smaller than $\xi^\pm(s_1(s))$, then the control should be positive, $u^* = 1$, for other values of $x$ the optimal control is negative, $u^* = -1$. In the same way if current state $\eta$ is less or equal than $-\sqrt{3}/4$ and the current state $x$ is smaller $\xi^\pm(s_3(s))$, then the control
should be positive, \( u^* = 1 \), for other values of \( \xi \) the optimal control is negative, \( u^* = -1 \). And finally if for the non-positive \( \eta \) value \( \xi \) is equal to \( \Phi(\eta, 1, \frac{2}{3}) \), the optimal control is positive, \( u^* = 1 \).

The control law has now been developed, but for convenience of usage it is possible to introduce in a \( \xi-\eta \) plane switching curves, where the control law changes its sign. Previously the system dynamics were described by a function \( \Phi \) in a form \( \xi = \Phi(\eta, \alpha, C) \). \( \xi^- (s_1(\eta)) \) and \( \xi^+ (s_3(\eta)) \) have been shown to be the switching curves, so it is possible to define the constant \( C \) for both cases. By equalizing the equation (A.3.11) with \( \alpha = -1 \) and any \( \eta > -\frac{\sqrt{3}}{4} \) and equation (A.3.23) with the same \( \eta \), constant \( C \) is equal to \(-\frac{1}{3}\). So the function \( \Phi(\eta, -1, -\frac{1}{3}) \) corresponds to the \( \xi^- (s_1(\eta)) \) switching curve. In the same way it is possible to find the function \( \Phi(\eta, 1, \frac{1}{3}) \) corresponding to the \( \xi^+ (s_3(\eta)) \) switching curve for all values of \( \eta \leq -\frac{\sqrt{3}}{4} \).

\[
\begin{align*}
\xi^- (s_1(\eta)) &\rightarrow \xi = \Phi\left(\eta, -1, -\frac{1}{3}\right) = -\eta - \frac{1}{3} - \frac{1}{3} |1 + 2\eta|^{3/2}, & \eta > -\frac{\sqrt{3}}{4} \\
\xi^+ (s_3(\eta)) &\rightarrow \xi = \Phi\left(\eta, 1, \frac{1}{3}\right) = \eta - \frac{1}{3} + \frac{1}{3} |1 - 2\eta|^{3/2}, & \eta \leq -\frac{\sqrt{3}}{4}
\end{align*}
\]  

Equations (A.3.25) corresponds to the switching curves in a \( \xi-\eta \) plane with a positive \( x_3 \) and they are plotted by bold curves on \( \xi-\eta \) plane in Fig. A.3.3, where the red one corresponds to \( u^* = 1 \) control and the blue one to \( u^* = -1 \). In Fig. A.3.3, phase

Fig. A.3.3 \( \xi-\eta \) plane switching curves and state trajectories, adapted from [71]
trajectories are shown by thin lines with the arrows showing the movement direction, color coding is the same.

According to Fig. A.3.3, similarly to the discussion in [71], if the \((\xi; \eta)\) point is lying on the \(D - K - 0\) curve then the system is sliding along this curve with \(u^* = 1\) towards the origin. This curve corresponds to \(\xi^\pm(s_3(\eta))\). All other optimal trajectories also reach the origin along this curve with the one exception: if the \((\xi; \eta)\) point lies on the \(R - 0\) curve, then the system reaches the origin with the \(u^* = -1\) control.

If the \((\xi; \eta)\) point lies in the \(R - D - 0\) sector, than the optimal trajectory consists of section of a \(u^* = -1\) control followed by the section of \(u^* = 1\) control, when trajectory reaches the \(K - 0\) curve.

If the \((\xi; \eta)\) point lies in a \(D - K - B\) sector, than the movement is started with the \(u^* = 1\) control till it reaches the switching curve \(K - B\) (corresponds to \(\xi^-(s_1(\eta))\)), where the control changes it sign, \(u^* = -1\), and the systems slides along this curve to infinity. Infinity in the \(\xi-\eta\) plane corresponds to \(x_3 = 0\), so at the infinity \(x_3\) changes its sign and so does the control law. The phase trajectory is continued with the \(u^* = -1\) and along the \(\xi^\pm(s_3(\eta))\) curve to reach the origin. It is noted in [71], that the transfer through the infinity is done without the change of control sign and takes a finite amount of time.

For all other points from the \(\xi-\eta\) plane at the initial stage the control is \(u^* = -1\), which moves the system to infinity, where \(x_3\) changes its sign and phase trajectory returns to the \(D - K - B\) sector and with the \(u^* = -1\) reaches the \(K - B\) switching curve, where the control sign is changed to \(u^* = 1\) and the system moves to infinity again, where the acceleration (\(x_3\)) becomes positive again and along the \(\xi^\pm(s_3(\eta))\) curve with the \(u^* = 1\) slides to origin.

For any initial position of the \((\xi; \eta)\) point, the \(x_3\) sign changes not more than two times and the control switch at most once. This result agrees with a number of switchings theorem presented in [69].
Appendix 4

Nested loop controllers and initial Bode plot tuning

The nested loop controller is used to control the brushed DC-motor in Chapter 5. In this section the procedure for the initial gain tuning (before the experiment) is described for each single loop, starting with the inner one (current regulator). The control block diagram of a PM brushed DC machine was presented earlier in Chapter 5.

Fig. 5.3 Control block diagram of a permanent magnet brushed DC machine

The electrical part of the plant is considered to be much faster than the mechanical one, or in other words the electrical time constant is much smaller than the mechanical one allowing the position, speed and current controllers to be tuned independently.

A.4.1 Current regulator

P and PI controllers are generally used for the motor control. The differential part, D, is normally omitted due to measurement noise problems. The P current regulator for

Fig. A.4.1 P current regulator: (a) Control block diagram (b) Bode plot (magnitude)
the electrical part of the plant together with its Bode plot is shown in Fig. A.4.1. The open-loop transfer function, \( W_{OL}(s) \), of the resultant system is as follows.

\[
W_{OL}(s) = \frac{K_{pc}}{L_a s + R_a} = \frac{K_{pc}}{R_a} \frac{R_a}{s + 1} = \frac{K}{T s + 1},
\]

(A.4.1)

where \( K = \frac{K_{pc}}{R_a}, T = \frac{L_a}{R_a} \) and the resultant current controller bandwidth is obtained as \( \omega_{cc} = \frac{\sqrt{K^2 - 1}}{T} \). Therefore the desired bandwidth could be set in a current control system by defining the proportional gain \( K_{pc} = \sqrt{R_a^2 + (\omega_{cc} L_a)^2} \). It is worth checking the steady state error, \( \Delta_{st} \), of P current regulator, when the step change of the reference signal with magnitude \( a \) is applied.

\[
\Delta_{st} = \lim_{s \to 0} \frac{a}{s} W_\delta(s) = \lim_{s \to 0} \frac{a}{1 + W_{OL}(s)} = \frac{a}{1 + \frac{K_{pc}}{R_a}}
\]

(A.4.2)

From (A.4.2) the steady state error is always exists in a system. It is possible to reduce it by increasing the proportional gain, \( K_{pc} \), however the controller bandwidth will rise is this case leading to undesired high frequency noise in the commanded armature voltage. Therefore, an integral component is required to eliminate the steady state error, i.e. PI controller is used instead of P.

The PI current regulator for the electrical part of the plant is shown in Fig. A.4.2 together with its Bode plot for the specific choices of \( K_{pc}, K_{ic} \) given below in (A.4.4).

The open-loop transfer function (\( W_{OL}(s) \)) of the resultant system is as follows.

\[
W_{OL}(s) = \frac{K_{pc} s + K_{ic}}{s} \left( \frac{1}{L_a s + R_a} - \right) \frac{1}{\frac{K_{ic}}{R_a} \left( \frac{K_{pc}}{R_a} s + 1 \right)} = \frac{K_{ic}}{R_a} \left( \frac{K_{pc}}{R_a} s + 1 \right)
\]

(A.4.3)

If in (A.4.3) the proportional gain is set to \( K_{ic} = R_a \omega_{cc} \) and the integral gain is set...
\( K_{pc} = L_a \omega_{cc} \), then the open-loop transfer function is reduced to \( W_{OL}(s) = \frac{\omega_{cc}}{s} \). The Bode plot magnitude for such case is shown in Fig. A.4.2(b). It is clear that steady state error of such system is zero, which is important for motor current control. Therefore in this work the PI regulator is preferred for the current control loop, with the proportional and integral set as follows.

\[
K_{pc} = L_a \omega_{cc} \\
K_{ic} = R_a \omega_{cc} \tag{A.4.4}
\]

Applying the parameters from (A.4.4) the close-loop transfer function of the current control loop could be obtained as in (A.4.5).

\[
W_{cc}(s) = \frac{\omega_{cc}}{s + \omega_{cc}} \tag{A.4.5}
\]

### A.4.2 Speed regulator

The speed regulator of a motor drive is shown in Fig. A.4.3, where the current regulator is as presented as in (A.4.5) and the friction is ignored, \( B_{fr} = 0 \).

\[
\begin{align*}
\omega_{rm}^* &\rightarrow \omega_{rm} \rightarrow \text{Speed controller} \\
&\rightarrow \frac{\omega_{cc}}{s + \omega_{cc}} \rightarrow i_n \rightarrow \lambda_p m \rightarrow \frac{1}{J_e s} \rightarrow \omega_{rm}^*
\end{align*}
\]

Fig. A.4.3 Speed regulator control block diagram

Similar to the current controller case two, different variants, P and PI, for the speed controller are considered.

For the PI controller with the proportional gain \( K_{ps} \) and integral gain \( K_{is} \) the open-loop transfer function, \( W_{OL}(s) \), of the resultant speed control is

\[
W_{OL}(s) = \frac{K_{ps} s + K_{is}}{s + \omega_{cc}} \frac{\lambda_p m}{J_e s} = \frac{\lambda_p m K_{is}}{s} \frac{K_{ps}}{1 + \frac{1}{\omega_{cc}} s} = \frac{K(T_2 s + 1)}{s^2(T_1 s + 1)} \tag{A.4.6}
\]

where \( K = \frac{\lambda_p m K_{is}}{J_e} \), \( T_1 = \frac{1}{\omega_{cc}} \) and \( T_2 = \frac{K_{ps}}{K_{is}} \). The Bode plot for such transfer function is shown in Fig. A.4.4.
From Fig. A.4.4 it is possible to derive the desired proportional and integral gain following the conventional procedure proposed by the control theory. These gains are derived as a function of the speed controller bandwidth, \( \omega_{sc} \), which is set far apart of the current controller bandwidth, \( \omega_{cc} \), i.e. \( \omega_{cc} \gg \omega_{sc} \). In addition, the difference between the \( \frac{1}{T_2} \) and \( \omega_{sc} \) is also set to be significant, normally around \((0.1\text{-}0.9)\text{dec}\); in this work the distance between these frequencies is set to be 0.5dec. Due to the aforementioned relative frequency positions at the speed controller cutoff frequency the magnitude of \( |\omega_{cc} + j\omega_{sc}| \approx 1 \) and \( |K_p + \frac{K_i}{j\omega_{sc}}| \approx K_p \), i.e. the magnitude of the full open-loop transfer function at this frequency is \( |W_{OL}(j\omega_{sc})| \approx \frac{K_p\lambda_{pm}}{f_{e\omega_{sc}}} \) and it should be equal to one.

Therefore the proportional gain of the speed controller could be derived as \( K_p = \frac{f_{e\omega_{sc}}}{\lambda_{pm}} \).

Similarly if the \( \frac{1}{T_2} = \frac{\omega_{sc}}{5} \), then the integral gain of the speed controller is equal to \( K_i = \frac{K_p}{5} \omega_{sc} \).

The close-loop transfer function of the motor drive with the PI speed control loop could be derived as follows.

\[
W_{sc}(s) = \frac{W_{OL}(s)}{1 + W_{OL}(s)} = \frac{KT_2}{T_1} s + K \left( \frac{1}{T_1} s^2 + \frac{KT_2}{T_1} s + \frac{K}{T_1} \right) \tag{A.4.7}
\]

However, if the system’s response to the speed reference is studied, i.e. the frequencies of interest are \( \ll \omega_{cc} \), then the close-loop transfer function could be simplified. From (A.4.7), considering that \( T_1 \) is small the simplified transfer function could be obtained.
\[ W_{sc}(s) = \frac{KT_2s + K}{s^2 + KT_2s + K} = \frac{\lambda_{pm}K_{ps}}{J_e}s + \frac{\lambda_{pm}K_{is}}{J_e} \]

(A.4.8)

(A.4.8) is a second order system; with the gain parameters defined earlier the damping coefficient is \( \frac{\sqrt{5}}{2} \), therefore the speed response is considered to be overdamped, however due to the zeros in the transfer function the speed response will have an overshoot while following the feed-forwarded values, which is undesirable in the positioning system.

The issue with the presence of overshoot could be solved by substituting the PI controller with the P controller in the speed regulator. The resultant open-loop transfer function, \( W_{OL}(s) \), is

\[ W_{OL}(s) = K_{ps} \frac{\omega_{cc}}{s + \omega_{cc} J_e s} = \frac{\lambda_{pm}K_{ps}}{J_e} \frac{1}{s \left( \frac{1}{\omega_{cc}} s + 1 \right)} = \frac{K}{s(T_1 s + 1)} \]

(A.4.9)

where \( K = \frac{K_{ps}\lambda_{pm}}{J_e} \) and \( T_1 = \frac{1}{\omega_{cc}} \). With the same proportional controller setting, \( K_{ps} = \frac{J_e\omega_{sc}}{\lambda_{pm}} \) the bandwidths of both P and PI controllers will be equal to each other and the close-loop transfer function is as follows.

\[ W_{sc}(s) = \frac{W_{OL}(s)}{1 + W_{OL}(s)} = \frac{K}{T_1} \frac{1}{s^2 + \frac{1}{T_1}s + K \frac{1}{T_1}} = \frac{K_{ps}\lambda_{pm}}{J_e} \frac{\omega_{cc}}{s^2 + \omega_{cc}s + \frac{K_{ps}\lambda_{pm}}{J_e} \omega_{cc}} \]

(A.4.10)

With the aforementioned parameter setup, the damping coefficient of (A.4.10) is equal to \( \frac{1}{2 \sqrt{\frac{\omega_{sc}}{\omega_{cc}}} } \) and considering that \( \omega_{cc} \gg \omega_{sc} \) the response to the step input as the speed reference is always overdamped, therefore in this work the P controller is used as a speed regulator. It is worth noting that in contrast to current regulator the open-loop transfer function contains an integrator, therefore the steady state error is eliminated. To illustrate the difference between the P and PI controllers used as a speed regulator their input step responses together with the Bode plots are shown in Fig. A.4.5.
Appendix 4

A.4.3 Position regulator

The transfer function between the reference and actual speed, \( \omega_{rm}^* \) and \( \omega_{rm} \) respectively, is given by (A.4.10). For the purpose of position regulator development the speed controlled drive could be approximated as the first-order low-pass filter with the cutoff frequency equal to the speed controller bandwidth, \( \omega_{sc} \). With such assumptions the positioning system control block diagram is shown in Fig. A.4.6, where the conventional P controller is used.

\[
W_{pc}(s) = \frac{K_{pp}\omega_{sc}}{s^2 + \omega_{sc}s + K_{pp}\omega_{sc}} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{A.4.11}
\]

(A.4.11) represents the second-order low-pass filter with the natural frequency \( \omega_n \) and damping coefficient \( \xi \). The overshoot is unacceptable in the MT positioning system, therefore the damping coefficient, \( \xi \), is set greater than 1. In this thesis it is set as \( \frac{3}{2} \).
therefore the position controller proportional gain $K_p = \frac{\omega_{sc}}{g}$. With such parameter choice the overall position regulator transfer function is derived as

$$W_{pc}(s) = \frac{\omega_{sc}^2}{g} \frac{1}{s^2 + \omega_{sc}s + \frac{\omega_{sc}^2}{9}}$$  \hspace{1cm} (A.4.12)
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