STERILE ANTINEUTRINO SEARCH WITH THE MINOS AND MINOS+ EXPERIMENT

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy in the Faculty of Science and Engineering

2019

By

RUI CHEN

School of Physics and Astronomy
## Contents

Declaration 7

Copyright 8

Acknowledgements 9

1 Introduction 11

2 Theory of neutrino physics 14
   2.1 The history of the Neutrino 14
   2.2 The Solar and Atmospheric Neutrino Problems 15
   2.3 Three flavour paradigm 21
   2.4 Sterile Neutrinos 22
      2.4.1 LSND experiment 22
      2.4.2 MiniBooNE experiment 23
      2.4.3 3+1 Simple Model 24
      2.4.4 Reactor Anomaly 26
      2.4.5 Gallium Anomaly 27
      2.4.6 Sterile neutrino searches in different experiments 29

3 The MINOS and MINOS+ experiment 34
   3.1 NuMI Beam 34
   3.2 Data Quality Monitoring 40
   3.3 The MINOS Detectors 46
      3.3.1 Near detector 47
      3.3.2 Far detector 49
      3.3.3 Calibration detector 51
      3.3.4 Steel 51
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3.5</td>
<td>Scintillator strips and modules</td>
<td>51</td>
</tr>
<tr>
<td>3.3.6</td>
<td>Readout and PMTs</td>
<td>52</td>
</tr>
<tr>
<td>3.3.7</td>
<td>Magnetic field</td>
<td>52</td>
</tr>
<tr>
<td>3.3.8</td>
<td>Light Injection System</td>
<td>52</td>
</tr>
<tr>
<td>3.3.9</td>
<td>Triggering</td>
<td>54</td>
</tr>
<tr>
<td>4</td>
<td>Calibration and Reconstruction</td>
<td>55</td>
</tr>
<tr>
<td>4.1</td>
<td>Energy calibration</td>
<td>55</td>
</tr>
<tr>
<td>4.1.1</td>
<td>Drift calibration</td>
<td>56</td>
</tr>
<tr>
<td>4.1.2</td>
<td>Linearity calibration</td>
<td>57</td>
</tr>
<tr>
<td>4.1.3</td>
<td>Strip-to-Strip Calibration</td>
<td>57</td>
</tr>
<tr>
<td>4.1.4</td>
<td>Intra-Strip Attenuation Calibration</td>
<td>57</td>
</tr>
<tr>
<td>4.1.5</td>
<td>Inter-Detector Calibration</td>
<td>58</td>
</tr>
<tr>
<td>4.1.6</td>
<td>Summary and result</td>
<td>60</td>
</tr>
<tr>
<td>4.1.7</td>
<td>Absolute calibration</td>
<td>60</td>
</tr>
<tr>
<td>4.2</td>
<td>Simulation Software</td>
<td>64</td>
</tr>
<tr>
<td>4.2.1</td>
<td>NuMI beam simulation</td>
<td>64</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Neutrino interaction simulation</td>
<td>66</td>
</tr>
<tr>
<td>4.2.3</td>
<td>Detector simulation</td>
<td>66</td>
</tr>
<tr>
<td>4.3</td>
<td>MINOS and MINOS+ Reconstruction</td>
<td>67</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Digit formation</td>
<td>67</td>
</tr>
<tr>
<td>4.3.2</td>
<td>De-multiplexing</td>
<td>69</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Strip formation</td>
<td>69</td>
</tr>
<tr>
<td>4.3.4</td>
<td>Slices</td>
<td>69</td>
</tr>
<tr>
<td>4.3.5</td>
<td>Track reconstruction</td>
<td>69</td>
</tr>
<tr>
<td>4.3.6</td>
<td>Calorimetric shower reconstruction</td>
<td>70</td>
</tr>
<tr>
<td>4.4</td>
<td>Shower Energy Estimator</td>
<td>70</td>
</tr>
<tr>
<td>4.4.1</td>
<td>$k$NN shower energy</td>
<td>71</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Shower energy systematic uncertainties</td>
<td>71</td>
</tr>
<tr>
<td>5</td>
<td>Event Selection</td>
<td>74</td>
</tr>
<tr>
<td>5.1</td>
<td>Preselection</td>
<td>75</td>
</tr>
<tr>
<td>5.2</td>
<td>Low-energy $\bar{\nu}_\mu$-enhanced beam NC selection</td>
<td>75</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Near detector specific cut</td>
<td>75</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Far detector specific cut</td>
<td>77</td>
</tr>
<tr>
<td>5.3</td>
<td>Low-energy $\bar{\nu}_\mu$-enhanced beam antineutrino CC selection</td>
<td>80</td>
</tr>
</tbody>
</table>
5.4 Low-energy $\nu_\mu$-dominated beam antineutrino CC selection . . . . 84
5.5 Medium-energy $\nu_\mu$-dominated beam antineutrino CC selection . . 89

6 Systematic uncertainties 93
6.1 Covariance matrices . . . . . . . . . . . . . . . . . . . . . . . . . . 93
6.2 Correlation matrices . . . . . . . . . . . . . . . . . . . . . . . . . . 95
   6.2.1 Normalisation . . . . . . . . . . . . . . . . . . . . . . . . . 96
   6.2.2 Energy calibration . . . . . . . . . . . . . . . . . . . . . . . 97
   6.2.3 Background and flux uncertainties . . . . . . . . . . . . . . . 97
   6.2.4 Cross section systematics . . . . . . . . . . . . . . . . . . . 101
   6.2.5 Detector acceptance . . . . . . . . . . . . . . . . . . . . . . 104
   6.2.6 Hadron production . . . . . . . . . . . . . . . . . . . . . . . 107
   6.2.7 Beam optic systematics . . . . . . . . . . . . . . . . . . . . 109
   6.2.8 Detector cleaning systematics . . . . . . . . . . . . . . . . . 111
6.3 summary . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 114

7 The MINOS and MINOS+ Sterile Antineutrino Analysis 122
7.1 Sterile Neutrino Search in MINOS . . . . . . . . . . . . . . . . . . . 123
7.2 Separating the components of error matrices . . . . . . . . . . . . . 125
7.3 Decorrelation process . . . . . . . . . . . . . . . . . . . . . . . . . 126
7.4 Test Statistics and Optimisation . . . . . . . . . . . . . . . . . . . . 134
7.5 Dual Detector Technique . . . . . . . . . . . . . . . . . . . . . . . . 136
7.6 Data limit and Sensitivity Comparison . . . . . . . . . . . . . . . . 136
7.7 Feldman-Cousins procedure . . . . . . . . . . . . . . . . . . . . . . 143

8 Phenomenological Study of the Sensitivity 148
8.1 Toy Studies of Asimov Sensitivity . . . . . . . . . . . . . . . . . . . . 148
   8.1.1 Toy MC setup . . . . . . . . . . . . . . . . . . . . . . . . . . 149
   8.1.2 Use of Absolute Covariance Matrix . . . . . . . . . . . . . . . 150
   8.1.3 Use of Relative Covariance Matrix . . . . . . . . . . . . . . . 151
8.2 Impact of Biases in the Covariance Matrix . . . . . . . . . . . . . . 153
   8.2.1 Mock Data study of the Asimov Sensitivity . . . . . . . . . . . 153
   8.2.2 Comparison with MINOS Fit Framework . . . . . . . . . . . . . 153
   8.2.3 Impact of Biases and Correlations in the Relative Covari-
        ance Matrices . . . . . . . . . . . . . . . . . . . . . . . . . . . 155
   8.2.4 Impact of the the Correlations . . . . . . . . . . . . . . . . . 158
Sterile Antineutrino Search with the MINOS and MINOS+ Experiment

Rui Chen
A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy
March 15, 2019

Abstract
This thesis presents a sterile antineutrino search performed using the full MINOS data sample and part of the MINOS+ sample. MINOS is a two-detector on-axis experiment consisting of two detectors separated by 734 km. The near detector is located 1 km downstream of the NuMI neutrino target, which is on the Fermilab site. The far detector is located 735 km downstream of the NuMI neutrino target at the Soudan Underground Laboratory in Northern Minnesota. This thesis first presents a study of the data stability across all MINOS and MINOS+ data-taking periods. This is followed by the development of a selection method for $\bar{\nu}_\mu$ charge-current interactions from the $\nu_\mu$-dominated beam and a study of all sources of systematic uncertainties affecting the sterile antineutrinos search. Using both the charge-current and neutral-current samples we have probed the 3+1 sterile oscillation parameters and set constraints on $\Delta \bar{m}_{41}^2$ and $\bar{\theta}_{24}$ values. We have explored the seven orders of magnitude in $\Delta \bar{m}_{41}^2$ space from $10^{-4}$ eV$^2$ to $10^3$ eV$^2$. For most of the parameter space, we have excluded $\sin^2 \bar{\theta}_{24}$ values above around $10^{-2}$. Finally, a study of the differences between the Asimov sensitivity and the fluctuated sensitivity band is presented.
Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.
Copyright

i. The author of this thesis (including any appendices and/or schedules to this thesis) owns certain copyright or related rights in it (the “Copyright”) and s/he has given The University of Manchester certain rights to use such Copyright, including for administrative purposes.

ii. Copies of this thesis, either in full or in extracts and whether in hard or electronic copy, may be made only in accordance with the Copyright, Designs and Patents Act 1988 (as amended) and regulations issued under it or, where appropriate, in accordance with licensing agreements which the University has from time to time. This page must form part of any such copies made.

iii. The ownership of certain Copyright, patents, designs, trade marks and other intellectual property (the “Intellectual Property”) and any reproductions of copyright works in the thesis, for example graphs and tables (“Reproductions”), which may be described in this thesis, may not be owned by the author and may be owned by third parties. Such Intellectual Property and Reproductions cannot and must not be made available for use without the prior written permission of the owner(s) of the relevant Intellectual Property and/or Reproductions.

iv. Further information on the conditions under which disclosure, publication and commercialisation of this thesis, the Copyright and any Intellectual Property and/or Reproductions described in it may take place is available in the University IP Policy \(^1\), in any relevant Thesis restriction declarations deposited in the University Library, The University Library’s regulations \(^2\) and in The University’s policy on presentation of Theses.

---

\(^1\)http://documents.manchester.ac.uk/DocuInfo.aspx?DocID=24420
\(^2\)http://www.manchester.ac.uk/library/aboutus/regulations
I would like to take this opportunity to express my gratitude towards the people that have helped me throughout this process.

First of all, I would like to thanks my supervisor Justin Evans for his continuous support throughout my last four years as a PhD student. It is his encouragement and help that made this work possible.

In the MINOS+ collaborations, I would like to thanks Leigh Whitehead, Adam Aurisano, Adam Schreckenberger and Alex Sousa for many interesting conversations and helps towards the challenges we are facing in the field. I would also like to give a special thanks to Jenny Thomas and Karol Lang for many helpful ideas and comments they give. I like to thanks Ashley Timmons for his help when I first joined the collaboration, I will not be able to learn everything so fast without him. I want to thanks Arthur Kreymer, Robert Hatcher, Robert Plunkett, Donatella Torretta and Alberto Marchionni for their expert help in their fields. Finally, I would like to thanks Jacob Todd, Simon De Rijck, Tom Carroll, Paul Sail, Junting Huang, Anna Holin and many others. It is the conversation, their hard work and helps that made this thesis possible. I have learnt so much and enjoyed every conversation we had.

I want to thank everyone I met while I was in Chicago, particularly, Aaron Bercellie for his help during a very difficult part of my life. I also owe a big thanks to Zhen Hu, PengFei Ding, Wanwei Wu and Ao Liu for their help when I first arrived at Chicago.

I would like to thanks the Manchester HEP department for their help and support during my entire PhD period. In particular, I would like to thanks Stefan Sldner-Rembold for his help and suggestions with the visa issues. Frederick Loebinger for all his help in the past eight years. In the meantime, I would like to thanks Thomas Bird, Kevin Maguire, Fabian Wilk, Dominik Mller and Bobby Murrells for many deep conversations about the software and coding issues I
had. Finally, I would like to thanks all the members of the Manchester Neutrino Groups for their help and suggests.

I would also like to thanks all my friends and teachers back in Manchester, Norwich and Taiyuan, in particular, Dawn Wilkinson and Jinpu Ren, it is their encouragement that made me decide to pursue my degree in physics. It would be impossible for me to achieve it without them.

Finally, I would like to thanks all my family members and my girlfriend for their continuous love, support and encouragement through this long process.
Chapter 1

Introduction

Despite being the subject of significant, ongoing investigation many properties of neutrinos are still not well understood 80 years after their proposal. Through decades of different experiments, we have confirmed the flavour and oscillation pattern of the neutrinos. Today, many unknowns are still under investigation, including the CP phase, the mass hierarchy and of course the existence of the sterile neutrino.

The main difficulty for observing and studying neutrinos is the lack of statistics which is caused by the weakly interacting nature of neutrinos. To overcome this problem experiments have to increase the intensity of the neutrino source, increase the size of the detectors, take data over a longer period of time, or any combination of these. The oscillatory nature of neutrinos also requires any detector to be located hundreds of meters to hundreds of kilometres from the neutrino source. Based on these principles, many experiments have been built to study neutrinos. These pioneer experiments have explored many known features of neutrinos including interaction cross-sections and oscillation parameters. However, experiments like LSND and MiniBooNE, and the reactor anomaly also raised questions about our current model of neutrinos: these observed anomalies can be explained by considering sterile neutrinos. In this thesis I will present a detailed study of muon antineutrinos in MINOS and MINOS+ using both CC and NC channels, and use this data sample to search for sterile neutrinos.

In chapter 2, I will briefly discuss the theory of neutrinos, and how they were first observed and postulated. Following this, I will discuss the recent experimental anomalies which cannot be explained by the existing model.
In chapter 3, I will explain the details of the MINOS and MINOS+ experiment setup. Since the MINOS experiment uses a $\nu_\mu$ and $\bar{\nu}_\mu$ beam, I will briefly explain the beamline setup and the expected flux from different running modes. In this chapter, I will also include a service task I have conducted during the MINOS+ running, which is monitoring the beam condition by using the MINOS near detector data. Finally I will describe the MINOS detector technology.

In chapter 4, I will focus on the calibration and reconstruction of the events. The result presented in this thesis relies on the well-understood measurement of neutrino energies.

In the chapter 5, I will describe the event selection. The magnetic fields of the MINOS detectors provide a basic charge-sign selection tool. However, when performing an explicit measurement, one needs more specific cuts to ensure the selected event samples are both efficient and pure. I have first re-validated and updated the selection of antineutrinos in the MINOS low-energy $\nu_\mu$-dominated beam sample. I have then revalidated the selection both of NC interactions and of $\bar{\nu}_\mu$-CC interactions in the MINOS low-energy $\bar{\nu}_\mu$-enhanced beam. Finally, I have developed a new selection for CC $\bar{\nu}_\mu$ interaction in the MINOS+ medium-energy $\nu_\mu$-dominated beam. These various selected event samples form the basis of this thesis.

In chapter 6, I will detail all the systematic uncertainties that impact this analysis. I have re-assessed the uncertainty in the downstream production of neutrinos in the NuMI beam. I have implemented the PPFX corrections and uncertainties provided by the MINER$\nu$A collaboration to account for hadron production systematic uncertainties. I have also imposed an additional antineutrino cross-section systematic in response to the MINER$\nu$A $\nu/\bar{\nu}$ cross section measurement. Finally I have calculated and combined all the systematic covariance matrices.

In chapter 7, I will present the final result of the sterile antineutrino search obtained both from a grid scan method and from a full Feldman-Cousins procedure. The first part explains the reason why MINOS still has sensitivity at very large $\Delta m^2$, and I have validated our sensitivity by signal injection. There then follows explanation of why $\Delta \chi^2$ is a valid test statistic. Then I explain why the diagonal part of the covariance matrix is not representative of the errors on the energy spectra, and then present a new way of showing the energy spectra and their uncertainties. The final result is then presented and the need for a Feldman-Cousin procedure is also explained.
In chapter 8, I will perform an important phenomenological study of the sensitivity showing how the Asimov sensitivity is not always representative of the median sensitivity. The High Energy Physics community has history of using the Asimov sensitivity to represent experimental strength. It is important to understand when Asimov approach may not actually be representative.

In the final chapter, I summarise the findings in the thesis, and suggest future improvements that can be done to further improve the final result.
Chapter 2

Theory of neutrino physics

The theory of neutrinos is built on different experimental evidence discovered through the last century. A lot of neutrino properties which were discovered later on have impacted the Standard Model, in particular the mass of the neutrinos. This provides an excellent chance for physicists to peek beyond the Standard Model and understand the nature better.

2.1 The history of the Neutrino

At the beginning of the 20th century, three types of radioactivity had been observed, alpha, beta and gamma. Beta radioactivity, among all these radioactivities, was the least well understood. Initially, it was believed Beta radioactivity only emitted an electron in which case the emitted energy spectrum should produce a sharp peak. However, in 1914 James Chadwick observed that the Beta decay energy spectrum was a continuous spectrum [1]. This suggested that energy was not conserved in Beta decay. A lot of theories were proposed to account for this inconsistency. The theory of the electron neutrino was first postulated and mentioned in Pauli’s letter at the Gauverein meeting. The particles, “neutrons”, were mentioned in this letter and used to save the conservation of energy in beta decay [2]. However, the actual neutron was discovered two years later by Chadwick [3]. Only later, when Fermi built his beta decay model in 1934, was the neutrino properly introduced and named and its properties hypothesized [4]. It was impossible to directly measure the properties of the neutrino at that time.

In 1956, Cowan and Reines finally made the first direct observation of a free
2.2. THE SOLAR AND ATMOSPHERIC NEUTRINO PROBLEMS

The signal they were looking for was two gamma rays with 0.511 MeV energy from positron annihilation and a delayed gamma ray from neutron capture in the detector. Shortly after this, discovery of the parity violation [6] was discovered in the weak interaction. The neutrino was then demonstrated to be left-handed (i.e. the directions of its spin and momentum are antiparallel) by Maurice Goldhaber et al. [7] around 1958. This was achieved by observing the decay of Eu$^{152m}$ as follows:

$$\text{Eu}^{152m} \rightarrow \text{Sm}^{*152} + \nu_e \rightarrow \text{Sm}^{152} + \gamma + \nu_e,$$

(2.2)

where the decayed photon and electron neutrino have the same helicity. It was proven that neutrinos are only left-handed.

In 1962, the muon-neutrino was discovered in an experiment at the Brookhaven AGS [8]. This was believed to be a neutrino with a different flavour from the electron neutrinos studied by Reines and Cowan as, if it was the same flavour, it would have produced electrons and muons equally. The experiment detected 51 events, and as shown by the interaction of the produced particles, they were indeed muons that was produced rather than electrons. At the time, the theory then seemed to be complete, as there are both muon; muon neutrino pairs and electron; electron neutrino pairs. Everything changed with the discovery of a new lepton, $\tau$ [9]. One would simply expect a neutrino with tau flavour to exist as well. The LEP experiments later on showed that there are three light neutrinos that couple to the $Z^0$ boson [10], which also confirms the possible existence of the tau flavour neutrino. Finally, the tau neutrino was observed by the DONUT experiment in 2000 [11].

2.2 The Solar and Atmospheric Neutrino Problems

In the 1960s, Ray Davis’s Homestake experiment was the first to measure the flux of neutrinos from the sun. The experiment detected a deficit in the expected flux compared with the prediction of John N. Bahcall’s standard solar model [12]. This deficit is called the solar neutrino problem. Many other experiments also
confirmed this deficit including the Kamiokande, the Super-Kamiokande and the Sudbury Neutrino Observatory (SNO). In 1998, the Super-Kamiokande experiment present its atmospheric neutrinos data from a water Cherenkov detector [13]. Followed this, in 2001 and 2002 SNO release its measurement of the solar neutrinos also using a water Cherenkov detector [14] [15]. Together, this brought the first clear evidence that neutrinos change flavour (i.e. oscillate into different flavours) as they travel.

The flavour oscillations of the neutrinos suggest they have masses and the mass eigenstates are different from the flavour eigenstates [16]. Like the quark sector, one can describe neutrinos in terms of a mixing matrix called the PMNS matrix:

$$U = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix},$$

where 1, 2 and 3 denote different mass eigenstates, and $e$, $\mu$ and $\tau$ denote different flavour eigenstates. The mixing matrix can be parametrised with three mixing angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and a Dirac CP phase $\delta_{CP}$ [17]:

$$U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13}
\end{pmatrix},$$

where $c_{ij} \equiv \cos(\theta_{ij})$ and $s_{ij} \equiv \sin(\theta_{ij})$. It is possible to write this matrix by separating the different mixing angles:

$$U = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix} \begin{pmatrix}
c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\
0 & 1 & 0 \\
-s_{13}e^{-i\delta_{CP}} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}.$$

Without using the explicit form of the PMNS matrix, one can write any flavour state as a superposition of the mass eigenstates:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i}^*|\nu_i\rangle,$$  \hspace{1cm} (2.3)

where $\alpha$ represent different flavour eigenstates [18] and $i$ represent different mass eigenstates. Now considering a realistic case, assume a neutrino of initial flavour
2.2. THE SOLAR AND ATMOSPHERIC NEUTRINO PROBLEMS

\( \alpha \) at time 0 and distance 0, travelled for a distance \( L \), was detected with flavour \( \beta \) at time \( t \). One can calculate the oscillation probability amplitude:

\[
\langle \nu_\beta | \nu_\alpha(t) \rangle = \sum_j \sum_i U_{\alpha i}^* U_{\beta j} e^{-i(E_j t - p_j x)} \langle \nu_\beta | \nu_i \rangle \\
= \sum_j \sum_i U_{\alpha i}^* U_{\beta j} e^{-i(E t - p x)} \delta_{ij} \\
= \sum_j U_{\alpha i}^* U_{\beta i} e^{-i(E t - p x)}. \tag{2.4}
\]

Here the assumption is that the neutrino obeys the time-dependent Schrodinger equation, i.e. \( |\nu_\alpha(x,t)\rangle = \sum_i U_{\alpha i}^* e^{-i(E_i t - p_i x)} |\nu_i\rangle \). Making use of natural units and the fact neutrinos are relativistic (\( t = L \)):

\[
E_i t - p_i x = (p_i^2 + m_i^2)^{1/2} t - p_i L \\
\approx p_i (1 + \frac{m_i^2}{2p_i^2}) t - p_i L \\
= \frac{m_i^2 L}{2p_i}. \tag{2.5}
\]

where one can make the approximation that \( p_i \approx E \), and \( E \) is the average over \( E_i \). With this can calculate the oscillation probability using the amplitude formula:

\[
P(\nu_\alpha \rightarrow \nu_\beta) = \left( \sum_i U_{\alpha i}^* U_{\beta i} e^{-i \frac{m_i^2 L}{2p_i}} \right) \left( \sum_j U_{\alpha j} U_{\beta j} e^{i \frac{m_j^2 L}{2p_j}} \right) \\
= \sum_i \sum_j U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j} \frac{e^{i \Delta m^2_{ij} L}}{2p_i}. \tag{2.6}
\]

Now we separate the equation into three parts. The first part is when \( i = j \), such that the \( e^{i \frac{\Delta m^2_{ij} L}{2p_i}} = 1 \). The second part when \( i \neq j \), is the real part of \( e^{i \frac{\Delta m^2_{ij} L}{2p_i}} \) can be re-written into \( \cos \left( \frac{\Delta m^2_{ij} L}{2E} \right) \) which then can be transformed into \( 1 - 2 \sin^2 \left( \frac{\Delta m^2_{ij} L}{4E} \right) \).
The third part is the imaginary part of \( e^{\frac{\Delta m_{31}^2 L}{2E}} \) can be re-written into \( i \sin(\frac{\Delta m_{31}^2 L}{2E}) \).

\[
P(\nu_\alpha \rightarrow \nu_\beta) = \sum_{i=j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* + \sum_{i \neq j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left\{ \begin{array}{l} 1st \\
- 2 \sum_{i \neq j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin^2(\frac{\Delta m_{ji}^2 L}{4E}) \\
+ i \sum_{i \neq j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin(\frac{\Delta m_{ji}^2 L}{2E}) \end{array} \right. \right. 
\]

\[\text{Making use of the unitarity natural of the mixing matrix [18]:}
\]

\[
UU^\dagger = \begin{pmatrix}
\sum_i U_{e i} U_{e i}^* & \sum_i U_{e i} U_{\mu i}^* & \sum_i U_{e i} U_{\tau i}^* \\
\sum_i U_{\mu i} U_{e i}^* & \sum_i U_{\mu i} U_{\mu i}^* & \sum_i U_{\mu i} U_{\tau i}^* \\
\sum_i U_{\tau i} U_{e i}^* & \sum_i U_{\tau i} U_{\mu i}^* & \sum_i U_{\tau i} U_{\tau i}^*
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
i.e. \( \sum_i U_{\alpha i} U_{\beta j} = \delta_{\alpha\beta} \), therefore the first term in equation 2.7 can be simplified into \( \sum_i U_{\alpha i} U_{\beta i} \sum_j U_{\alpha j} U_{\beta j}^* = \delta_{\alpha\beta} \). The second term in equation 2.7 can be written in the following form:

\[
2^{nd} = \sum_{i \neq j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin^2(\frac{\Delta m_{ji}^2 L}{4E})
\]
\[
= \sum_{i \neq j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin^2(\frac{\Delta m_{ji}^2 L}{4E}) + \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin^2(\frac{\Delta m_{ji}^2 L}{4E})
\]
\[
= \sum_{i > j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin^2(\frac{\Delta m_{ij}^2 L}{4E}) + \sum_{i > j} U_{\alpha i}^* U_{\alpha j} U_{\beta j} U_{\beta i}^* \sin^2(\frac{\Delta m_{ij}^2 L}{4E})
\]
\[
= \sum_{i > j} (U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* + U_{\alpha j} U_{\beta j} U_{\alpha i} U_{\beta i}^*) \sin^2(\frac{\Delta m_{ij}^2 L}{4E})
\]
\[
= \sum_{i > j} (U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* + (U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta i}^*)^\ast) \sin^2(\frac{\Delta m_{ij}^2 L}{4E})
\]
\[
= 2 \sum_{i > j} \Re(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin^2(\frac{\Delta m_{ij}^2 L}{4E}).
\]

\[\text{Here we made use of the following facts: } U + U^* = 2 \Re(U), \sin^2(-\alpha) = \sin^2(\alpha)\]
and the fact we can swap and indices \( i \) and \( j \). The third term in equation 2.7 can be written in the following form:

\[
3^d = \sum_{i \neq j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin(\frac{\Delta m^2_{ji} L}{2E})
\]

\[
= \sum_{i > j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin(\frac{\Delta m^2_{ij} L}{2E}) + \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin(\frac{\Delta m^2_{ji} L}{2E})
\]

\[
= -\sum_{i > j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin(\frac{\Delta m^2_{ij} L}{2E}) + \sum_{i > j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \sin(\frac{\Delta m^2_{ji} L}{2E})
\]

\[
= \sum_{i > j} (U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* - U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin(\frac{\Delta m^2_{ij} L}{2E})
\]

\[
= -2i \sum_{i > j} \Im(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin(\frac{\Delta m^2_{ij} L}{2E}).
\]

Here we made used of the facts: \( U^* - U = -2i \Im(U) \) and \( \sin(-\alpha) = -\sin(\alpha) \). Now substituting all parts back to equation 2.7, the oscillation equation becomes:

\[
P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha \beta}
\]

\[
-4 \sum_{i > j} \Re(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin^2(\frac{\Delta m^2_{ij} L}{4E})
\]

\[
+2 \sum_{i > j} \Im(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin(\frac{\Delta m^2_{ij} L}{2E}).
\]

Table 2.2 gives a summary of all the PDG published neutrino oscillation parameters. Combining this equation and other parameters given in table 2.2, it is possible to formulate the approximated oscillation probability for any neutrino flavours.

Consider the case when \( \nu_\alpha = \nu_\beta = \bar{\nu}_e \). For a reactor experiment where \( L \approx 1 \text{ km}, E \approx 1 \text{ MeV} \) such that \( L/E \) is small. One can make an approximation that \( \frac{\Delta m^2_{31} L}{E} \ll 1 \) and \( \Delta m^2_{31} \approx \Delta m^2_{32} \approx \Delta m^2_{21} \) as \( \Delta m^2_{21} \) is much smaller than the other two mass splittings. This implies the oscillation probability of such search can be
Table 2.2: The PDG values for different neutrino oscillation parameters, where the $\theta_{23}$ and $\Delta m_{32}^2$ assumes Inverted Hierarchy and Lower Octant. Since the PDG didn’t give the $\delta_{cp}$ value this is left empty. All the values are taken from [17].

<table>
<thead>
<tr>
<th>Variables</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin^2(\theta_{12})$</td>
<td>$0.307^{+0.013}_{-0.012}$</td>
</tr>
<tr>
<td>$\Delta m_{21}^2$</td>
<td>$7.53 \pm 0.18 \times 10^{-5}$ eV$^2$</td>
</tr>
<tr>
<td>$\sin^2(\theta_{23})$</td>
<td>$0.421^{+0.035}_{-0.025}$</td>
</tr>
<tr>
<td>$\Delta m_{32}^2$</td>
<td>$-2.56 \pm 0.04 \times 10^{-3}$ eV$^2$</td>
</tr>
<tr>
<td>$\sin^2(\theta_{13})$</td>
<td>$2.12 \pm 0.08 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

considered as follows:

$$P(\nu_e \to \nu_e) = P(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \sin^2(2\theta_{13}) \sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right). \quad (2.11)$$

For the reactor experiments that are observing the disappearance of reactor antineutrinos [19] [17] (such as Daya Bay [20], Double Chooz [21] and RENO), $\theta_{13}$ and $\Delta m_{31}^2$ are the parameters they are sensitive to.

For the case of $\nu_\alpha = \nu_\mu$ and $\nu_\beta = \nu_e$, for $L/E \approx 10^3$ km/GeV the oscillation probability can be written as follows:

$$P(\nu_\mu \to \nu_e) \approx \sin^2(2\theta_{13}) \sin^2(\theta_{23}) \sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right). \quad (2.12)$$

Accelerator neutrino experiments with muon neutrino beams that are looking for the electron neutrino (antineutrino) appearance (such as T2K [22], NOvA [23] etc.), are sensitive to $\theta_{13}, \theta_{23}$ and $\Delta m_{31}^2$. These experiments can also look at the $\nu_\mu$ disappearance where $\nu_\alpha = \nu_\beta = \nu_\mu$. The oscillation probability in this case can be written as:

$$P(\nu_\mu \to \nu_\mu) = P(\bar{\nu}_\mu \to \bar{\nu}_\mu) = 1 - \cos^2(\theta_{13}) \sin^2(2\theta_{23}) \sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right)$$
$$+ \sin^4(\theta_{23}) \sin^2(2\theta_{13}) \sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right) \quad (2.13)$$
$$\approx 1 - \sin^2(2\theta_{23}) \sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right), \quad (2.14)$$

where the approximation here is that the $\theta_{13}$ is small. In general, for the long-baseline $\nu_\mu$ or $\bar{\nu}_\mu$ disappearance searches (such as MINOS [24], NOvA [25] etc.),
2.3. THREE FLAVOUR PARADIGM

$\theta_{23}$ and $\Delta m_{31}^2$ are the dominate parameters they can probe. Similarly one can also looking for $\nu_\tau$ appearance (such as OPERA [26] experiment), the oscillation probability can be written as following:

$$P(\nu_\mu \rightarrow \nu_\tau) \approx \cos^4(\theta_{13}) \sin^2(2\theta_{23}) \sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right). \quad (2.15)$$

The $\nu_\tau$ appearance channel is also sensitive to the $\theta_{13}$, $\theta_{23}$ and $\Delta m_{31}^2$ parameters.

For the studying of the $\theta_{12}$ and $\Delta m_{21}^2$ parameters, one has to built a detector satisfying $\frac{\Delta m_{21}^2 L}{4E} \sim 1$. This also would imply that $\frac{\Delta m_{21}^2 L}{4E} \gg 1$ and therefore oscillations driven by the longer mass splitting are averaged out. The solar and long-baseline $\nu_e$ or $\bar{\nu}_e$ disappearance experiments (such as KamLAND [27], SNO [28], etc.) have been developed to study these long-wavelength oscillation and the oscillation probability can be simplied to:

$$P(\nu_e \rightarrow \nu_e) = P(\bar{\nu}_e \rightarrow \bar{\nu}_e) \approx 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\Delta m_{21}^2 L}{4E}\right) \quad (2.16)$$

This shows that these experiments can probe $\theta_{12}$ and $\Delta m_{21}^2$. The full derivation of all the oscillation equation can be found in appendix A.

These experiments provide solid evidence for the neutrino oscillation and the building blocks for the three flavour neutrino oscillation paradigm.

2.3 Three flavour paradigm

As shown in last section, there are three mixing angles, two independent mass splittings and a CP-violating phase that is used to describe the neutrino oscillations. As mentioned in equations 2.11, 2.12, 2.15 and 2.14, existing experiments have established the three flavour paradigm with measurements of the mass splittings and mixing angles. However, there are still some unsolved problems. To begin with, solar electron neutrino disappearance confirms the positive value of the mass splitting $\Delta m_{21}^2$, that the sign of the mass splitting $\Delta m_{31}^2$ is still undetermined. This implies the mass eigenstates could be ranked $m_1 < m_2 < m_3$ or $m_3 < m_1 < m_2$, which are referred to the normal or inverted mass hierarchies.

One way to determine the sign of the mass splitting $\Delta m_{31}^2$ is by making use of the matter effects. Future experiments are aiming to determine the mass hierarchy by making use of this fact.
Secondly, it is unclear whether $\theta_{23}$ is larger than, smaller than or equal to $\frac{\pi}{4}$ which is known as “Octant” problem. As shown in equation 2.14 the disappearance probability leading term is only sensitive to $\sin^2(2\theta_{23})$. However, as shown by equation 2.12 precision measurement of the $\nu_e$ appearance channel is sensitive to the octant of $\theta_{23}$.

Thirdly, the $\delta_{\text{CP}}$ phase is very hard to measure as it only affects the sub-leading terms in the long-baseline $\nu_e$ appearance channel.

Finally, there are still a bunch of questions that can’t be answered at moment, for example, whether the neutrino is a Dirac particle or Majorana particle, and the absolute mass of the mass eigenstates. As these questions cannot be answered by oscillation experiments, it will not be discussed in this thesis.

2.4 Sterile Neutrinos

The light sterile neutrino was first postulated to explain dark matter and the solar neutrino deficit [29]. Tensions start to arise among the oscillation experiments as the Liquid Scintillator Neutrino Detector (LSND) experiment unveiled its results from a $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ search. Later on the MiniBooNE experiment confirmed the observations from LSND with a combined 6.1 $\sigma$ significance of a $\nu_e$ appearance excess [30]. More experiments started to look at the possible existence of sterile neutrinos and experiments such as MicroBooNE have been built recently to investigate the anomalies.

2.4.1 LSND experiment

The LSND experiment was designed mainly to search for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations from the $\mu^+$ decay. They also looked at $\nu_\mu \rightarrow \nu_e$ oscillation from $\mu^-$ decay. Figure 2.1 shows a schematic diagram of the LSND experiment. The detector is filled with liquid scintillator, which consists mainly of mineral oil, and it is located 30 m downstream of the neutrino source. The low energy $\bar{\nu}_\mu$ or $\nu_\mu$ source is provide by colliding a 798 MeV proton beam with a target. Pions and muons are then stopped in a carbon beam stop where they decay to muon antineutrinos and neutrinos. After travelling for 30 m, these neutrinos or antineutrinos interact in the LSND detector. In the case of antineutrinos, if the $\bar{\nu}_\mu$ has oscillated into a $\bar{\nu}_e$ it will interact through $\bar{\nu}_e C \rightarrow e^+ N$ which produces an $e^+$ with an energy of 52 MeV and a correlated photon with an energy of 2.2 MeV from neutron capture.
on a free proton. The final results presented come from $\bar{\nu}_e$ events selected in the $20 \text{ MeV} < E_\nu < 200 \text{ MeV}$ energy range and $\nu_e$ events in the $60 \text{ MeV} < E_\nu < 200 \text{ MeV}$ energy range as background dominates the rest of the energy range.

For the $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillation study, the dominate background comes from beam $\mu^- \rightarrow e^- \nu_e \bar{\nu}_\mu$. Another important background is when $\bar{\nu}_\mu p \rightarrow \mu^+ n$, where if the produced $\mu^+$ is low in energy this could be misidentified as $e^+$. For the $\nu_\mu \rightarrow \nu_e$ oscillation study, the dominate background comes from $\mu^+ \rightarrow e^+ \bar{\nu}_\mu \nu_e$ and $\pi^+ \rightarrow e^+ \nu_e$. The $\bar{\nu}_e$ appearance search results in a total excess of $87.9 \pm 22.4 \pm 6.0$ events above the predicted background, and the $\nu_e$ appearance search results in a total excess of $8.1 \pm 12.2 \pm 1.7$ events above the predicted background [31]. These events are consistent of the neutrino that oscillation driven by a mass splitting above $0.4 \text{ eV}^2/c^2$ and provide a first hint towards a fourth flavour of neutrino.

### 2.4.2 MiniBooNE experiment

The MiniBooNE experiment was built to confirm or refute the anomaly observed by the LSND experiment. This experiment used the Booster Neutrino Beam (BNB) at Fermilab which collided $8 \text{ GeV}$ protons with a beryllium target which was placed inside a magnetic focusing horn. Pions focused by the horns decayed to produce a muon neutrino/antineutrino beam. By changing the magnetic focusing horn polarity one could change between $\nu_\mu$ and $\bar{\nu}_\mu$ beams. In neutrino mode, one could produce a beam consisting of $93.5\% \nu_\mu$. In antineutrino mode, it could
produce a beam consisting of 83.7% $\bar{\nu}_\mu$. Figure 2.2 shows a schematic diagram of the MiniBooNE experiment. Similarly to the LSND detector, the MiniBooNE detector is also filled with mineral oil. It is located 541 m downstream of the target.

The MiniBooNE experiment accumulated a total of $11.27 \times 10^{20}$ Protons-On-Target (POT) in neutrino mode and $12.84 \times 10^{20}$ POT in antineutrino mode. 1959 and 478 electron-like events were observed in neutrino and antineutrino events respectively. After removing all the backgrounds this corresponds to a total of $381.2 \pm 85.2$ and $79.3 \pm 28.6$ excess electron-like events in the $200 \text{ MeV} < E_\nu < 1250 \text{ MeV}$ energy range for neutrino and antineutrino modes respectively. Figure 2.3 shows the $\nu_e$ and $\bar{\nu}_e$ event excesses with respect to the background predictions. The MiniBooNE experiment suffers background from the beam $\bar{\nu}_e$ and $\nu_e$ background. There are also additional $\gamma$ backgrounds from neutral-current (NC) $\pi^0$ production and $\Delta \rightarrow N\gamma$ decay.

The MiniBooNE excess alone is equivalent to a 4.8 $\sigma$ excess in the $200 \text{ MeV} < E_\nu < 1250 \text{ MeV}$ range.

### 2.4.3 3+1 Simple Model

As these observation cannot be explained by our Standard Model, one solution would be introducing an additional neutrino in a way that does not conflict with LEP measurements. The simplest model is 3+1 model where three active neutrino
2.4. STERILE NEUTRINOS

Figure 2.3: The MiniBooNE events with both neutrino and antineutrino mode data where the red and blue dots are the data collected in neutrino and antineutrino mode respectively. The red and blue dashed lines are the best fits assuming standard two neutrino oscillation. Diagram is taken from [30].

flavours and one sterile neutrino flavour are considered. The four flavour PNMS matrix can be written as follows:

\[
U_4 = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} & U_{e4} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} & U_{\mu4} \\
U_{\tau1} & U_{\tau2} & U_{\tau3} & U_{\tau4} \\
U_{s1} & U_{s2} & U_{s3} & U_{s4}
\end{pmatrix}
\]

This would modify the oscillation probability and introduce one more mass splitting \(\Delta m_{41}^2\), three mixing angles \(\theta_{14}\), \(\theta_{24}\) and \(\theta_{34}\) and two more CP violating phases \(\delta_{14}\), and \(\delta_{24}\). The full 3+1 oscillation probability for all different channels can be found in appendix B. For experiments like MicroBooNE and LSND, which are
looking for four flavour electron neutrino/antineutrino appearance, the probability can be written:

\[ P(\nu_\mu \rightarrow \nu_e) = 4|U_{\mu 4}|^2|U_{e 4}|^2 \sin^2\left(\frac{\Delta m_{41}^2 L}{4E}\right) \]

\[ \equiv \sin^2(2\theta_{\mu e}) \sin^2\left(\frac{\Delta m_{41}^2 L}{4E}\right), \] (2.17)

where \( \sin^2 2\theta_{\mu e} \equiv 4|U_{\mu 4}|^2|U_{e 4}|^2 \).

For experiment that are long baseline and looking for \( \nu_\mu \) disappearance with the presence of sterile neutrinos:

\[ P(\nu_\mu \rightarrow \nu_\mu) = 1 - 4|U_{\mu 3}|^2(1 - |U_{\mu 3}|^2 - |U_{\mu 4}|^2)\sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right) \]

\[ + 4|U_{\mu 4}|^2(1 - |U_{\mu 3}|^2 - |U_{\mu 4}|^2)\sin^2\left(\frac{\Delta m_{41}^2 L}{4E}\right) \]

\[ + 4|U_{\mu 4}|^2|U_{\mu 3}|^2\sin\left(\frac{\Delta m_{43}^2 L}{4E}\right). \] (2.18)

Figure 2.4 shows a comparison made between several different experiments. It is clear that the MiniBooNE and LSND experiments have similar allowed region and the MiniBooNE best fit point falls within LSND allowed regions.

### 2.4.4 Reactor Anomaly

The antineutrino fluxes emitted from nuclear reactors are important for reactor neutrino oscillation experiments. Knowledge on the expected \( \bar{\nu}_e \) flux is critical for these oscillation measurements. An old \( \bar{\nu}_e \) flux measurement used data from experiments at distances \( < 100 \text{ m} \) yielded a measured to observed rate ratios of \( 0.976 \pm 0.024\% \) [34]. This flux prediction only used phenomenological models of 30 effective beta-branches [34]. A new improved method has incorporated the knowledge of the decays of thousands of fission products, which yields a \(+3\%\) normalisation shift compared to the old measurement [35]. This new approach has increased the flux by 3.5\% which changes the measured to observed rate ratio to \( 0.943 \pm 0.023\% \). This is called the “Reactor Anomaly”. Figure 2.5 illustrates the reactor anomaly, by comparing to the 3 flavour oscillation model and a model with an extra mass state. As suggested by figure 2.5, the data seems to favour the solution with an extra mass state.
2.4. STERILE NEUTRINOS

Figure 2.4: LSND [31], KARMEN2 [33], OPERA [26] and MiniBooNE [30] results. For KARMEN2 and OPERA, regions to the right are ruled out. For LSND and MiniBooNE, the closed regions are allowed regions. Figure taken from [30].

2.4.5 Gallium Anomaly

Some experiments measure solar neutrinos through the inverse beta decay of Ga\textsuperscript{71}. In particular, SAGE [36] and GALLEX [37] aimed to measure solar neutrinos with an energy threshold of 233 keV. To calibrate their detectors, they chose to use an artificial Cr\textsuperscript{51} source placed near or inside their detector as this will produce
Figure 2.5: The experimental results for different reactor experiments indicated with black dots. The red line represents the model assuming 3 active neutrino mixing solutions; the blue line assumes solutions with one sterile neutrino mass state such that $|\Delta m^2_{\text{sterile}}| \approx 1 \text{ eV}^2$. Figure taken from [34].

large amount of $\nu_e$ through electron capture [38]. They both detect the neutrinos through the inverse beta decay process:

$$\nu_e + \text{Ga}^{71} \rightarrow \text{Ge}^{71} + e^-.$$  

Through the analysis of the calibration data they discovered their measured CC $\nu_e$ interaction cross-sections are consistent with each other and both lower than the predicted cross-section value [36] [37]. Later on the SAGE experiment performed a new study of an Ar$^{37}$ source in 2005 [39], and GALLEX performed a re-analysis of their results [40]. The result are summarised by [41]:

$$R_{S1}^{Cr} = 0.95 \pm 0.12,$$
$$R_{G1}^{Cr} = 0.953 \pm 0.11,$$
$$R_{S2}^{Ar} = 0.791^{+0.084}_{-0.078},$$
$$R_{G2}^{Cr} = 0.812^{+0.10}_{-0.11},$$

where $R \equiv \frac{\sigma_{\text{measured}}}{\sigma_{\text{theoretical}}}$ is the ratio of the measured cross-section to the theoretical predicted value. The Cr and Ar denote the calibration source Gr$^{51}$ and Ar$^{37}$ respectively. S1, S2 and G1, G2 represents the two SAGE and two GALLEX experiments respectively. Again these anomalies can also be explained by an
extra sterile mass state such that $\Delta m_{\text{new}}^2 \approx 1 \text{ eV}^2$. These observations are called the “Gallium Anomaly”.

2.4.6 Sterile neutrino searches in different experiments

Inspired by the LSND, MiniBooNE, reactor and gallium anomalies, a lot of experiments have started to look for sterile neutrinos. MINOS and IceCube have produced two of the most constraining limits on the mixing angle $\theta_{24}$. In 2016, the MINOS experiment placed a limit on sterile neutrinos as shown in figure 2.6 [42]. To allow a direct comparison between the MINOS, LSND and MiniBooNE results one has to combine the MINOS result with the $\bar{\nu}_e$ disappearance data from Daya Bay and Bugey. The combined limit is shown in figure 2.7 [23]. The 90% limit set by the combined result has eliminated the majority of the sterile parameter space allowed by LSND and MiniBooNE. The IceCube experiment has also placed a limit on the possible existence of sterile neutrinos by looking for the disappearance of muon neutrinos produced in the Earth’s atmosphere [43]. All of these results are compatible with three-flavour oscillation model. Alternatively one could also

![Figure 2.6: MINOS’s limit on sterile neutrinos. Regions to the right of the lines are excluded at 90% C.L.. The MINOS exclusion region is corrected by the Feldman-Cousins method [44]. The CHDS, CCFR and SciBooNE + MiniBooNE results are taken from [45], [46] and [47] respectively. Figure taken from [42].](image)
search for sterile neutrinos from cosmological measurements where, if a sterile neutrino is indeed present, it would modify the Cosmic Microwave Background (CMB) power spectrum \[51\]. Traditionally, CMB measurements from Planck \[52\] can only constrain the \(N_{\text{eff}}\) and the \(m_{\text{sterile}}\) parameters, and oscillation experiments can only set constrains on the mixing angle \(\theta\) and mass splitting \(\Delta m^2\).

However, using the method provided in \[51\], a result \[53\] has been released to map the \(N_{\text{eff}}\) and \(m_{\text{sterile}}\) parameter space to the \(\theta\) and \(\Delta m^2\) parameter space. This allows one to set constrains using the Planck 2015 data in comparison with other experiments shown in figure 2.8. These result again are compatible with three-flavour oscillation model.
2.4. STERILE NEUTRINOS

Figure 2.8: The top graph shows the exclusion from Planck at 95% C.L. [52] on sterile neutrinos in the $\Delta m_{21}^2$ and $\theta_{24}$ parameter space, together with MINOS [23], IceCube [43] and a forecast of the Fermilab Short-Baseline Neutrino (SBN) program [54]. The bottom graph shows the same exclusion regions mapped from the $\Delta m_{21}^2$ and $\theta_{24}$ to $N_{\text{eff}}$ and $m_{\text{sterile eff}}^2$ parameter space. Both figures are taken from [53].

Recently, the MINOS experiment has produced a newer limit using both MINOS and MINOS+ data and a new analysis technique. Figure 2.9 shows this
latest result from MINOS and MINOS+ in comparison to other experiments. It is clear that a significant amount of the parameter space has been excluded.

Figure 2.9: MINOS’s recent limit on sterile neutrinos. Regions to the right of the lines are excluded 90% C.L.. The MINOS exclusion region are corrected by Feldman-Cousins method [44]. The IceCube, Super-K, CDHS, CCFR, SciBooNE + MiniBooNE and Gariazzo’s global fit results are taken from [43], [22], [45], [46], [47] and [55]. Figure taken from [56].

The MINOS experiment has also collected a significant number of $\bar{\nu}_\mu$ events. As both the LSND and MiniBooNE results show their most significant excesses in their antineutrino channel, it is therefore important to perform a search with the $\bar{\nu}_\mu$ data collected by MINOS and MINOS+. In this thesis, I will present an
analysis to search for $\bar{\nu}_\mu$ disappearance assuming a 3 + 1 sterile neutrino model.
Chapter 3

The MINOS and MINOS+ experiment

In the late 1990s, while neutrino oscillation theory was gradually formed, the MINOS experiment was proposed. As one of the pioneers in long-baseline experiment, the MINOS experiment that consists of a beam of neutrinos, and two detectors that are 734 km apart. From 2005 to 2016, the MINOS and MINOS+ have completed its tasks and produced a number of the world’s best neutrino oscillation parameter measurements.

The MINOS experiment uses a neutrino beam produced at the Fermilab National Accelerator Laboratory and two steel-scintillator sampling calorimeter detectors. The neutrinos are produced by the Neutrinos at the Main Injector (NuMI) beam facility. These neutrinos are firstly detected by the the near detector, and then by far detector. Figure 3.1 shows the geographic location of the MINOS experiment, where the beam travels 735 km underground from Fermilab to south Soudan mine. With the help of the magnetic field it can identify the neutrino and antineutrino on a event by event basis. In this chapter, we will discuss the NuMI beam first. An additional Data Quality Monitoring task which I have performed is also mentioned in this chapter, followed by descriptions of the MINOS detectors.

3.1 NuMI Beam

The NuMI neutrino beam was built mainly to deliver neutrinos for the MINOS experiment. It has later on, mainly been used to provide neutrinos for
the MINER$\nu$A, ArgoNeuT, NO$\nu$A and MINOS+ experiments [58]. Figure 3.2 shows the NuMI beam setup where the $H^-$ ions are first accelerated by the Linac to 400 MeV. These ions are then delivered to Booster, stripped of electrons and the remaining protons accelerated to 8 GeV. The Booster then injects the protons into the Main Injector for further acceleration. As the circumference of the Main Injector is 7 times bigger than the size of the Booster, the Main Injector can store and accelerate 6 Booster batches [58]. The remaining slot is mainly used for the pulse kicker rise time. In the Main Injector the protons are accelerated to 120 GeV and then directed towards the 1 m long graphite target. Figure 3.3 shows the Proton-On-Target (POT) from the beginning of MINOS until the end of the MINOS+ runs. In this thesis, all the MINOS run periods data which includes $10.56 \times 10^{20}$ and $3.36 \times 10^{20}$ POT data and MINOS+ $5.80 \times 10^{20}$ POT processed data are used. From start of the MINOS+ runs, the beam group have started to focusing on increase the power of the beam, as the priority have been shifted to off-axis NO$\nu$A experiment. This have also changed the energy spectrum for the MINOS+ experiment. Figure 3.4 shows the energy spectra comparison for
Figure 3.2: Fermilab accelerator complex. Diagram is taken from [58].

Figure 3.3: The POT information during the entire MINOS and MINOS+ running period. Diagram is taken from [59].
MINOS, NO\(\nu\)A and MINOS+, where the MINOS+ and NO\(\nu\)A are subject to the same beam with different angle.

Figure 3.5 shows the detailed setup of NuMI beam. The collisions between the protons and the graphite target produces a set of secondary particles including both \(\pi\) and \(K\). These charged particle will be either focused or defocused (depending on the charge of the particle and the alignment of the magnetic field) by the magnetic focusing horns. Then these particle will travel and decay through a 675 m long decay pipe. Particles that have not yet decayed, will be absorbed by the absorber, and only decayed neutrinos will pass through the rock to the detector.

Before the proton beam hits the target, it goes through a 1.5 m long collimating graphite baffle that protects the downstream components of the beamline. The 6.4 \(\times\) 15 \(\times\) 940 mm target is segmented longitudinally into 47 segments and the beam spot size at the target is approximately 1.2–1.5 mm. These fins are 6.4 mm wide, 15 mm tall and 20 mm long with 0.3 mm spacing and are water
cooled through the stainless steel lines at top and bottom. An additional fin will be mounted to help alignment of the beam. It is also worth noting that the target assembly is mounted on a rail-drive system which allows for the horn-target separation to vary with a maximum of 2.5 m. Modifying the neutrino beam energy requires the changing of this separation. The dominated decay chains of the secondary particles are shown below:

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \quad (\pi^- \rightarrow \mu^- + \bar{\nu}_\mu),$$

$$K^+ \rightarrow \mu^+ + \nu_\mu \quad (K^- \rightarrow \mu^- + \bar{\nu}_\mu).$$

The two magnetic horns that focus the particles produced in the target are pulsed with a 200 kA current, yielding a maximum 3 T toroidal field. The parabolic-shaped inner conductors in the horns allow them to act as a focusing lens for particles. The second horn location can also be changed when changing the beam energy. Figure 3.6 shows the simulated neutrino energy spectra for different beam configurations. As mentioned above, by reversing the current, the horns can be used to focus either negatively or positively charged particles, to produce either $\nu_\mu$ or $\bar{\nu}_\mu$ dominated beams. Figure 3.7 shows a simulated near detector comparison between different beam modes. These two configurations are called Forward Horn Current (FHC) and Reversed Horn Current (RHC) modes.

Figure 3.6: The left plot shows the locations of the target and the second focusing horn for different beam configurations. The right plot shows the resulting neutrino energy spectra. Diagram is taken from [61].
3.1. NUMI BEAM

Figure 3.7: Beam composition of different running modes, where the left figure is for the FHC mode and right figure is for the RHC mode, figures are taken from [62].

respectively. For FHC mode, beam measured by the near detector consists of \(91.7\% \nu_\mu\), \(7.0\% \bar{\nu}_\mu\) and \(1.3\% (\nu_e + \bar{\nu}_e)\). The RHC mode, beam measured by the near detector consists of \(58.1\% \nu_\mu\), \(39.9\% \bar{\nu}_\mu\) and \(2.0\% (\nu_e + \bar{\nu}_e)\). The \(\bar{\nu}_\mu\) in the figure FHC mode and the \(\nu_\mu\) in RHC mode, mainly come from pions and kaons that have travelled through the centres of the focusing horns.

Figure 3.8 shown the detail of the NuMI target. The target will have a finite lifetime provided it is not seriously damaged through, for example, water leak. The degradation of the target would cause the neutrino flux to drop, which in turn would impact the overall event rate observed.

Figure 3.8: Technical drawing of the NuMI target. Diagram is taken from [63]
As the beam and target condition will affect the events MINOS detects, it is therefore important for MINOS to monitor and understand the behaviour of the beam and the target.

### 3.2 Data Quality Monitoring

The MINOS near detector Data Quality Monitoring (DQM) is established to monitor the behaviour of the NuMI beam and to check that the MINOS near detector is fully operational. The run conditions for the MINOS running period are well described in [42] and [58]. The Data Quality Monitoring task present in this thesis mainly covers MINOS+ period which is running from 2013 till 2016. The MINOS group uses run numbers to denote a specific period of running and this method is also adapted for the MINOS+ running period. The MINOS low-energy $\nu_\mu$-dominated configuration (also known as FHC) cover from Run 1,2,3,5,6
3.2. DATA QUALITY MONITORING

Figure 3.10: DQM of the MINOS+ runs 11 and 12, where the top figure is for Run 11 and bottom figure is for Run 12.

and 10. The MINOS low-energy $\bar{\nu}_\mu$-enhanced configuration (also known as RHC) covers from Run 4, 7 and 9 and the MINOS+ Medium-energy $\nu_\mu$-dominated configuration cover from Run 11,12 and 13.

Figure 3.9 shows a summary of the MINOS-era DQM summary, where the top figure shows all the neutrino runs and the bottom figure shows all the antineutrino
runs. The solid line here represent the POT-weighted average data spectrum over the entire period. The drop in Run 2 and 3 is mainly caused by target degradation. Starting from Run 3 the decay pipe had added helium, which also caused a drop in the event rate. It is also worth noting that Runs 5 and 6 had a new target. During the $\bar{\nu}_\mu$-enhanced run, the drop in Run 7 was caused by a water leak in the target. The lower rate in Run 9 was caused by the reuse of the degraded target [58].

Figure 3.11: Proton beam batch structure for January 2015 and March 2015, where the top figure covers the January 2015 period, the bottom figure covers the March 2015 period.
Different from the MINOS runs, the MINOS+ runs have a continuous increase in the beam intensity and energy. Run 11 and 12 both which are in the $\nu_\mu$-dominated beam mode are shown in figure 3.10. It is clear from the figure that comparing with Run 12, the Run 11 is much stabler across the full run. The Run 12 suffers an event drop start from March 2015. Figure 3.11 shows a comparison of the structure of the proton batches in the beam between January 2015 and March 2015. Detailed in [63], the beam group started to implement the “slipstacking” technique to increase the spill intensity [58]. The March 2015 period is when they implement the “6+2” operational mode where the first two batches have much higher intensity than rest of the batches as shown in figure 3.11. This resulted in an overall increase in spill intensity. The event rate drop observed in the near detector mentioned above is believed to be caused by the increase of the beam intensity. As more proton neutrinos arrive in the near detector simultaneously, merging of the events can cause failures in the reconstruction process, which in term decreases the observed event rate in the near detector. Figure 3.12 shows a comparison of the daily event rate across the entire Run 12 period against the average spill intensity. The trend can be seen clearly from
CHAPTER 3. THE MINOS AND MINOS+ EXPERIMENT

Figure 3.13: Run 12 event rate for different batches. The top figure represent the events have energy below 8 GeV, the bottom figure represent the event that have energy above 8 GeV. Batches are grouped together, where the blue dots represent the first two batches, black dots are the third and four batches and red dots are fifth and sixth batches.

The events with energy below 8 GeV range. Figure 3.13 shows the same graph where the information is split into the different proton batches as it is possible to tag the timing information of each proton batch. Prior to March 2015, all six batches have relatively similar event rates. Straight after the implementation of the “slip-stack” the event rates in the first two batches start to drop. This provide extra confident towards the finding, which indicate only the more intense batches will have a lower event rate.

A lot of studies have been performed to understand the intensity effects for different running period [64–66]. These studies have all confirmed the linearity of the intensity effects for different energy bins and shown that these intensity effects can be mapped and corrected. The Run 13 studies in contrast are more challenging.

The MINOS+ Run 13 period covers October 2015 to June 2016, which marks the end of the MINOS+ runs. Figure 3.14 shows the Run 13 near detector $\nu_\mu$ CC event rate for different batches. From January 2016, the beam group have started to use the “6+4” slip-stacking mode. In April 2016, a fire occurred at a transformer which impacted the beam performance during the middle of the run,
forcing the intensity of the beam to decrease [67]. Finally, starting from June 2016, the beam started to operate in “6+6” slip-stacking mode. Furthermore, a beam scan performed in October 2016, which also discovered that the first focusing horn was tilted. The intensity change can be mapped by using MC simulated at different intensities or by applying correcting factor. The biggest difficulty is to identify and correct for the horn tilt. The first obstacle is to understand if the horn tilt occurred instantly or gradually. Figure 3.15 shown the batch 5–6 event rate, where the black line represents a flat line fit with zero gradient, and the light green line represents a straight line fit with a gradient. Several observations can be made from the top graph in figure 3.15. June 2016, in comparison with the May 2015 period, confirms that a ∼ 5% event rate drop is equivalent to a ∼ 75% intensity increase. The May 2016 intensity is similar to that in November 2015, however the event rate differs by ∼ 5%. The fit confirms the overall decrease trend, which is affected both by the horn tilt and the “6+6” slip-stacking mode in June 2016. Roughly starting from December 2015, the event rate has decreased gradually and plateaued out around March 2016. These
observation suggest the horn tilt might have been a gradual effect over the course of three months. A more detailed investigation of the horn tilt and solutions for correcting for it are still in progress. Therefore in this thesis only data prior to Run 13 will be used for the analysis.

### 3.3 The MINOS Detectors

The two MINOS detectors are designed to be as similar as possible. This design allow the dependence on the Monte Carlo to be reduced, through the cancellation of certain systematic uncertainties that affect both detectors in similar ways. The MINOS near detector is placed 1 km downstream of the NuMI target, inside the Fermilab site, which provides a good measurement of the neutrinos before they oscillate. The far detector, in contrast, is placed 735 km downstream of the NuMI target, and it is located 705 m underground at the Soudan Mine in northern Minnesota.

Both detectors consist of alternating layers of 1 cm thick polystyrene plastic scintillator strips and 2.54 cm thick steel planes shown in figure 3.16. For each
3.3. THE MINOS DETECTORS

adjacent plane the scintillator strips are aligned orthogonally to allow the 3-D reconstruction of the event in the detectors shown also in figure 3.16. These alternating planes are called U and V plane, and they are both 45 degrees from the vertical axis. The steel provides a target for neutrino to interact with; charged particles produced in the neutrino interactions then deposit energy in the scintillator that is converted to light. Each steel plane is magnetized by current-carrying coils. The resulting magnetic field allows charge-sign identification of muons and measurement of the energies of muons that do not stop in the detector.

Each scintillator strip is 4.1 cm wide and has a groove for the placement of a fibre. The outer layer is coated with a TiO$_2$ to improve the light collection. WaveLength-Shift (WLS) fibres are glued into the groove, which allow the scintillated blue light to be absorbed and re-emitted as green light.

The WLS fibres guide the light to the readout PhotoMultiplier Tubes (PMTs). These PMTs are stored in light-isolated boxes. The collected information are then parsed by the Data Acquisition (DAQ) system and recorded on disk. The information is then reconstructed offline to recreate the event topology information from the data.

3.3.1 Near detector

The MINOS near detector has 282 steel and scintillator planes. Each plane is 4.8 m wide and 3.8 m tall. The near detector weighs 0.98 kt. Figure 3.17 shows the beam spot, the coil and the dimension of the MINOS near detector. The beam centre and the coil hole are 1.48 m apart. This arrangement offers a good
measurement of the hadronic shower energy without losing a significant numbers of events.

The near detector consists of four functional parts and it is shown in figure 3.18. As the beam is coming from the left, the veto part will first eliminate the background from neutrino interaction in the rock in front of the detector, and it consists of 0.5 m of steel. Neutrinos that interact in the target part are those for which we can make the best energy measurement. This target section consists of 1.0 m of steel. Considering the longitudinal profile of the shower, the hadronic shower section is designed to be 1.5 m long to contain the full showers of all neutrino interactions that occurred in the target region. These three parts make up the calorimeter section of the near detector and are made of 120 planes. Only the steel on the beam-side of the coil hole is instrumented with scintillator in
3.3. THE MINOS DETECTORS

this calorimeter section, except for every fifth plane which is fully instrumented. Figure 3.19 shows the 4 different UV arrangement of the detector scintillator modules. The partially instrumented calorimeter section covers 2.8 m by 2.8 m transversely, and is enough to cover the shower transverse profile which is $\sim 0.5$ m. The muon spectrometer region consists of 4 m of steel, which is equivalent to 162 planes, and it is mainly used to measure the momenta of the muons from the curvature. For spectrometer region, only every fifth plane is instrumented with scintillator, but these instrumented planes are always fully instrumented.

3.3.2 Far detector

As shown in figure 3.20, the far detector consists of two supermodules. The two supermodules are separated by a 1.15 m gap. Each supermodule has its own independent magnet coil. The first supermodule consists of 249 planes and is 14.78 m long. The second supermodule consists of 237 planes and has a length of 14.10 m. The far detector planes are much larger than the near detector planes and are 8 m wide regular octagons. Together the entire far detector weights 5.4 kt.
Different from the near detector, the entire far detector is fully instrumented as shown in figure 3.21. The scintillator covers 99% of the steel planes.

The far detector also has a veto shield which provide an extra suppression of cosmic muon events. It covers the top and partially east and west side of the detector and is made of three layers of scintillator strips for better tagging of the cosmic events.
3.3.3 Calibration detector

In addition to near and far detector, there was also a calibration detector which was exposed to a CERN test beam for the determination of the detector response. This detector was made of 60 unmagnetised steel and scintillator planes of 1 m². Each scintillator plane contained 24 1 m long scintillator strips. Both near and far detector electronics were used in the calibration detector to acquire data. This helps us to understand the different detector electronics to the same particles, and helped the development of the calibration chain (detailed in chapter 4).

3.3.4 Steel

The near detector steel thickness variations were found to be $\sim 0.3\%$. The average steel thickness for calibration detector is determined to be 2.5 cm.

3.3.5 Scintillator strips and modules

Figure 3.22 shows the design of the MINOS scintillator strips. The scintillation

![MINOS scintillator strips](image)

Figure 3.22: Cutaway drawing of the MINOS scintillator strips. Diagram is taken from [70].

light produced by ionizing particles may be reflected multiple times and eventually
absorbed by Wave-Length Shifting (WLS) fibre. The WLS fibre will then re-emit light isotropically, much of which is trapped by total internal reflection and routed to the PMTs. Scintillator strips are glued into groups of 16 or 28 to form scintillator modules, as shown in figure 3.19 and 3.21 [69].

3.3.6 Readout and PMTs

The WLS fibres are connected to clear fibre cables, and the light is transmitted by these clear optical fibre to the PMTs. The top diagram in figure 3.23 shows a schematic drawing of the readout for a scintillator module. These signals are read out by Hamamatsu 64-anode PMTs for near detector and Hamamatsu 16-anode PMTs for the far detector. The PMT mounting assembly is shown in the bottom diagram of figure 3.23.

In total, 194 PMTs are used for the near detector. In contrast, the far detector has a total 1452 PMTs that read out signals from both ends of each scintillator strip. There are an additional 64 PMTs used in the cosmic-ray veto shield [70].

3.3.7 Magnetic field

The magnetic fields are designed to provide a measurement of muon momenta using the curvature information as well as allowing the charge of the muons or antimuons to be determined. The design resolution is $\sigma_P \sim 12\%$ for muons with energies larger than 2 GeV. The magnetic fields, averaged over the fiducial volume, are 1.28 T in the near detector, and 1.42 T in the far detector [70].

These magnetic fields also allow the MINOS experiment to distinguish between neutrino and antineutrino CC interactions on an event-by-event basis [71]. This has made many measurement possible [72], [73] including the analysis presented in this thesis.

3.3.8 Light Injection System

The light injection system is mainly used to monitor the PMT non-linearities and gain, and it is heavily used in the calibration process. The light is injected from a pulser box which is made of group of 20 or 40 Ultra-Violet (UV) Light-Emitting Diodes (LEDs). The light is emitted directly onto the WLS fibres after they leave the scintillators. This is designed to simulate the scintillator output, and
3.3. THE MINOS DETECTORS

Figure 3.23: The top diagram shows the MINOS readout system, and the bottom diagram shows the PMT mounting assembly. Both diagrams are taken from [70].

the injected light is directly monitored by both the Positive Intrinsic Negative (PIN) diodes and PMTs.
3.3.9 Triggering

As mentioned above the NuMI beam deliver neutrinos in a short time period called a “spill”. The readout of both detectors is triggered 100\,\mu s\ [71] around the spill. The far detector suffers much noise from the spontaneous light emission in the fibres\ [74]. An additional trigger gate is therefore implemented to eliminate their low-level noise see details in chapter 5.
Chapter 4

Calibration and Reconstruction

To achieve precise measurements of the neutrino oscillation parameters, it is essential to measure the energies of the particle we observe precisely. A good energy calibration at both MINOS detectors is required. In the first part of this chapter, the calibration process in both MINOS detectors will be explained. We will then discuss the simulation software we used in our experiment; this is very important for any oscillation study, as it can provide a baseline model for any measurement. Reconstruction is another important process. It aims to convert raw data into meaningful neutrino-event related quantities and will be discussed in the third part of this chapter. Shower energy reconstruction is a dominating limiting factor for the performance of the reconstruction process. An improved shower energy reconstruction algorithm using a $k$-Nearest-Neighbour algorithm will be discussed in the final part of this chapter.

4.1 Energy calibration

Energy calibration is a crucial process for any analysis that looks for an energy-dependent signature, which includes but is not limited to an oscillation analysis. Essentially, for a given oscillation search, one is mainly looking for a difference between the neutrino energy spectra at the two different detectors. For a precise measurement of the three-flavour oscillation parameters, any inaccuracy in the neutrino energy measurement would introduce a bias in the measured oscillation parameters [61]. Sterile oscillation parameters would vary over a much broader range of the mass-splitting space; a detailed understanding of the shapes of the entire energy spectra in both detectors is therefore vital.
When a charged particle travels through a MINOS detector, it will deposit energy in the scintillator. This deposited energy will excite the scintillator to generate light. It is then collected by photomultiplier tubes and finally recorded as an electronic pulse which is often referred to as a ‘hit’. Calibration is required to correct the data so that the same measured pulse height in both data and Monte Carlo (MC) simulation describe the same energy deposition. The raw pulse height $Q_{\text{raw}}$ can be calibrated using the following equation:

$$Q_{\text{corr}} = Q_{\text{raw}}(d, s, t, x) \times D(d, t) \times L(s, d, Q_{\text{raw}}) \times S(d, s, t) \times A(d, s, x) \times M(d)$$

where $Q_{\text{corr}}$ is energy after this calibration process, $D(d, t)$ is a drift calibration, $L(s, d, Q_{\text{raw}})$ is a linearity calibration, $S(d, s, t)$ is a strip-to-strip calibration, $A(d, s, x)$ is an attenuation of light calibration and $M(d)$ is an energy scale factor. $s$, $x$, $t$ and $d$ represent a strip, the position along a strip, a given time, and a detector, respectively. To achieve a uniform energy response over an entire detector, a light injection system and cosmic muons are used for calibration.

### 4.1.1 Drift calibration

Drift calibration aims to calibrate the change in the detector response over time. Response changes are caused by temperature fluctuations, changes in voltage supplies, ageing of the PMTs and degrading scintillator and fibre light-output [75, 76]. To calibrate the combination of these effects, cosmic muons are used to look at the responses of each detector over time, as cosmic muons are considered as a constant source of ionising particles in both detectors. 24-hour periods are determined as the optimal granularity with which to perform this calibration [61]. The calibration factor $D(d, t)$ is then computed as a correction:

$$D(d, t) = \frac{m(d, t_0)}{m(d, t)}$$

where $m(d, t_0)$ is the daily median response (pulse height per plane) at an arbitrary reference date $t_0$ for a given detector $d$ and $m(d, t)$ is the measured daily median response at current time $t$ for the same detector. This factor is then applied to each individual hit in that detector to calibrate out the detector response drift [70].
4.1. **Energy Calibration**

4.1.2 **Linearity calibration**

The PMT response becomes non-linear once the light signal exceeds 100 photo-electrons \([77,78]\). This becomes problematic when we have an event that deposits a lot of energy in a small number of strips. By injecting light at different intensities with a light injection system, one can essentially map this non-linearity. At the near detector, the size of the light injection pulse is measured by the PIN diodes which are attached directly to the light injection box. The PIN diodes are known to be linear within \(\approx 1\%\) up to signal sizes of 100 photoelectrons, and within 2\% above that range.

4.1.3 **Strip-to-Strip Calibration**

Differences in scintillator light output, fibres and connections could cause each scintillator strip to give a different response to the same energy deposition. At both detectors, strip response within a given dataset can have variations up to 30\% \([70]\). The aim of the strip-to-strip calibration is to remove these variations using cosmic muons. Depending on the hit position in a strip and the incident angle, cosmic muons can have different responses in a detector. To overcome these differences, the mean strip response to a cosmic muon is defined when a cosmic muon hits the centre of the strip travelling normal to the plane. The cosmic muon energy response is corrected by applying linearity, attenuation and path length corrections to calculate the mean strip response and including the effects of gaps in tracks. The strip-to-strip calibration factor \(S(d, s, t)\) for a given strip can then be determined using the following equation:

\[
S(d, s, t) = \frac{\text{Mean Detector Response}(d, s, t)}{\text{Mean Strip Response}(d, s, t)}.
\]  

4.1.4 **Intra-Strip Attenuation Calibration**

Inside the WLS fibre, light will attenuate as it travels. Depending on the hit position, the attenuation will be different. These effects inside the strip need to be calibrated. During the construction stage of the detector, every scintillator strip was exposed to a well-defined photon source at various points along the scintillator strip. These collected light response data, known as mapper data, are then fitted to a double exponential empirical formula, and the best fit parameters are used for calibrating the attenuation effects. The attenuation correction factor
Figure 4.1: This graph shows the comparison between cosmic muon data and the double exponential best fit curve using mapper data for a typical strip at near detector. The graph is taken from [79].

\[ A(d,s,x) = A_1(d,s)e^{-x/L_1} + A_2(d,s)e^{-x/L_2} \]  

(4.4)

where \( x \) is the track position along the strip, and \( L_1 \) and \( L_2 \) are two attenuation lengths. This method is used mainly at the far detector. At the near detector, cosmic muons are used to calibrate the attenuation effects instead, as both methods provide similar results. Figure 4.1 shows the good agreement between the cosmic muon data and the fit result from the mapper data at the near detector.

### 4.1.5 Inter-Detector Calibration

Finally, one would want to calibrate the detector response to an known energy deposition to be the same between each detectors. Depending on the energy and momentum of the muons, the \( dE/dx \) is different and it is mainly described by the Bethe-Bloch equation in the energy region we are interested in. Figure 4.2 shows an example of the stopping power of a muon with different momenta travelling through copper. Stopping cosmic muons are used for this task as this allows us to calculate the energy of the muon, and therefore the \( dE/dx \), at every point along
4.1. ENERGY CALIBRATION

Figure 4.2: This graph shows the $dE/dx$ of muons with different momentum as travelling through copper, where the 0.005 to 50 GeV/$c$ momentum range is mainly described by the Bethe-Bloch process. Graph is taken from [80].

Instead of using the full range of the track, we only use muons when their energy is between approximately 0.5 and 1.1 GeV. Using this results in 0.2% uncertainty in the energy deposition [75]. The graph in figure 4.3 shows how the stopping power varies as a function of muon momentum; combining with figure 4.2 it is clear that the $dE/dx$ varies slowly in the chosen energy range. The cut-off at 0.5 GeV removes the rapid increase in the ionisation at the end of the track.

The result of the inter-detector calibration is expressed by the Muon Energy Unit (MEU). 1 MEU equals the detector response to a perpendicular 1 GeV muon traversing 1 plane of scintillator. For each detector it is calculated by:

$$\text{MEU} = \text{Median}\left(\frac{1}{N_p} \sum_{i=1}^{N_p} \frac{S_i}{L_i}, \ldots, \frac{1}{N_p} \sum_{i=1}^{N_p} \frac{S_i}{L_i}\right)$$

where $S_i$ is total detector response in plane $i$, $L_i$ is the path length through that plane, $N_p$ is the number of planes that is in the track window, and $N$ is the number of stopping muons used in this calibration. The median is used as muons can undergo stochastic energy losses, and the median eliminates the effect of rare high energy losses [70].
CHAPTER 4. CALIBRATION AND RECONSTRUCTION

4.1.6 Summary and result

Figure 4.4 summaries the relative calibration chain where input raw pulse height ADC is calibrated into MEU. Figure 4.5 shows the result of the calibration. One can see the response is uniform across the whole detector.

4.1.7 Absolute calibration

After the relative energy calibration process mentioned above, both detectors have uniform responses to all hadronic and electromagnetic showers. The final step is to convert these responses into a measured deposited energy. This was achieved by exposing a calibration detector to a test beam at CERN consists of muons, electrons and hadrons with momenta in the range 0.2 – 10 GeV/c. These measurements are then used to scale the Monte Carlo simulations and establish the uncertainty on the hadronic and electromagnetic energy scales [70]. Figure 4.6 shows the measured response to pions and electrons compared to simulation. The simulated detector response to electrons agrees with data to better than 2%, and to pion and proton induced showers to better than 6%. The energy resolutions for hadronic showers and electrons were determined to be 56%/\sqrt{E} \oplus 2\%, and 21.4%/\sqrt{E} \oplus 4%/E respectively [70].
4.1. ENERGY CALIBRATION

Figure 4.4: This figure shows the summary of the entire relative calibration chain, where ADC is our raw pulse height input, SigLin is ADC after Drift and Linearity calibration, SigCor is after strip-to-strip calibration, SigMap is after attenuation calibration and MEU is after the Inter-Detector calibration. The figure is taken from [81].
Figure 4.5: These graphs show the result of the calibration chain on a U plane, where the graphs on the left are the raw detector response, and the graphs on the right are the response after inter-detector calibration. The top and bottom graphs are the near and far detector response respectively. The figure is taken from [82].
Figure 4.6: These graphs show the MINOS calorimetric responses to pions (top) and electrons (bottom) at three momenta. The data was calibrated using the calibration detector and is compared with MC simulation. The graphs are taken from [70].
CHAPTER 4. CALIBRATION AND RECONSTRUCTION

4.2 Simulation Software

Simulation software is vital for any experimental measurement. For the exclusion search presented in this thesis we are setting limits based on the difference between measured and simulated neutrino energy spectra. An imperfect simulation model could introduce many potential problems including false signals. Instead of simulating everything in a single package, the simulation is broken down into three major branches. In the beginning, the beam simulation is used to simulate the collision between the NuMI proton beam and the target. Target geometry, magnetic horn focusing and subsequent decay processes are all simulated in this package. A major difficulty of the beam simulation is to predict the neutrino flux. A flux correction using a priori information will be discussed briefly in this section. With the information from the beam simulation, the neutrino interaction simulation can simulate the different interaction process for these neutrinos in the detectors. This information is then passed to detector response simulation, which will also perform the inverse-calibration process. This simulated data will then be calibrated and reconstructed in exactly the same way as real data.

4.2.1 NuMI beam simulation

Beam simulation is performed by using a Monte Carlo generator FLUGG [83,84]. This program incorporates the geometry, which is simulated by GEANT4 [85], and the hadron production, which is simulated by FLUKA [86]. The FLUGG simulation starts by simulating the 120 GeV Main Injector protons incident on the graphite target. The production of the secondary mesons, particles travelling through the magnetic horns, further interactions and decay inside the decay pipe are also simulated. The information stored in this process includes the ancestor particles’ position and momentum information, which offers a way to track down the hadron production source of the neutrino and antineutrino events. From this information one can also calculate the probability that a beam neutrino/antineutrino can reach a detector. Instead of selecting only the neutrinos that will reach a detector, one can just assume every neutrino is produced travelling towards a detector and use the calculated probability that the parent muon decayed such as to send the neutrinos in this direction to assign a “weight” to every neutrino event. This saves a significant amount of processing time. This weighted neutrino flux is then passed to the neutrino interaction simulation [87].
Hadron production corrections

For an oscillation measurement where there is only a physics signal at the far detector, near detector data can be used to constrain the hadron production processes [88]. A multiple beam configuration fit is performed using the BMPT inspired hadron production parametrisation [89] to improve the flux prediction and reduce the data to Monte Carlo simulation difference. More details on the fit can be found in [88]. The result of the fit is then converted to a weight called the SKZP weight which is applied to both the near detector and far detector Monte Carlo. Through this procedure the uncertainty of the hadron production is reduced and the hadron production prediction is improved. However, the sterile neutrino exclusion search conducted in this thesis considers a much wider mass splitting range. For sterile neutrinos with a mass splitting above $\sim 1 \text{ eV}^2$, oscillation could occur at the both MINOS detectors. A near detector data-driven flux correction would undermine the sterile signal in this mass splitting region, and is therefore not used.

For a sterile neutrino search using both near and far detector data, any potential deviation from data that is caused by hadron production would have a great impact on our limits. It could also fake a signal in some cases. Therefore one has to obtain an a priori source of hadron production corrections to reduce the deviation without introducing potential bias. In 2016, the MINERvA collaboration released a flux prediction package, Package for Predict the FluX (PPFX) [90,91], which offers a better prediction of the NuMI flux with reduced uncertainty. This flux prediction is obtained by using external data constraints from independent hadron production measurements for both thin and thick targets. More details can be found in [91]. The thin target data is particularly interesting as shows a better agreement in the energy range of interest than the thick target data [90] and is therefore used in our analysis. This data constraint is most accurate for the flux of neutrinos from $\pi$ parents as $\pi^+$ and $\pi^-$ yields are mainly used in forming the PPFX prediction. The $K^+$ and $K^-$ neutrino yields are generated by using NA49 $\pi$ data extrapolated with MIPP $K/\pi$ ratios. Further study with MINOS horn-off data suggests there are potential disagreements between data and Monte Carlo events in $K^\pm$ predicted region. In the MINOS simulation, the parent hadrons produced in the target are described using two kinematic parameters: the transverse momentum ($p_T$) and the longitudinal momentum ($p_Z$) defined relative to the direction of the initial proton beam. It has been shown that
the ratio of $\pi/K$ hadron production in the $pZ-\rho_T$ phase space of interest agrees reasonably well between data and simulation in MINOS. This suggests one can determine the parametrisation need for the kaon yield if the pion yield is known. Using the multiple beam configuration fit framework and this information, one can fit the MINOS simulation to the PPFX simulation in the $\pi$ dominated region to extract the equivalent MINOS simulation parametrisation to determine the $K$ parametrisation and therefore extrapolate the PPFX flux from the $\pi$-dominated to $K$-dominated region. More details can be found chapter 6. This provides an independent flux correction and greatly reduces the uncertainties on the neutrino flux.

### 4.2.2 Neutrino interaction simulation

Neutrino interactions are modeled by NEUGEN-v3 [92], which simulates both quasi-elastic and inelastic neutrino scattering from 100 MeV to 100 GeV. Low invariant-mass hadronization processes are simulated using the Andreopoulos-Gallagher-Kehayias-Yang (AGKY) model [93]. It uses a KNO-based model [94] at low invariant mass and the PYTHIA/JETSET model [95] at high invariant mass. The transitional region between the low and high invariant mass regions are modelled by gradually increasing the fraction of neutrino events that uses the PYTHIA/JETSET model. This provides a continuous transition between the two models. The final state interactions as hadrons are leaving the nuclei are performed using the intranuclear re-scattering package called INTRANUKE [96]. This addresses the treatment of pion elastic and inelastic scattering, single charge exchange and absorption.

### 4.2.3 Detector simulation

The GEANT3 [85] package is used for simulating the detector geometry. GMINOS, which is based on GEANT3, is also used to generate the energy deposition. Showers are simulated using the GCALOR [97] package, which is chosen as it has a good agreement with calibration detector shower data [98]. The magnetic field is simulated using bench measurements of the steel B-H curve via a finite element analysis method.

The simulated energy depositions in the scintillator are then fed into a C++ based Photon Transport program [99], which converts the energy depositions into
4.3 MINOS AND MINOS+ RECONSTRUCTION

It also simulates the photons travelling through the WLS fibres, and subsequent conversion of photons to photoelectrons in the PMTs. Photon Transport also includes a large amount of detail such as the behaviour of the PMTs and electronics, non-linearity effects, noise, cross-talk and triggering. This inverse-calibration process will convert everything we have simulated so far into a form that is equivalent to real data. This simulated data will then be calibrated using the calibration methods mentioned in section 4.1 to obtain simulated events that are as close to data as possible.

4.3 MINOS and MINOS+ Reconstruction

The reconstruction process in MINOS is an offline process, meaning it occurs after data collection. It consists of digit formation and de-multiplexing, strip formation, slicing, track reconstruction and shower reconstruction. During reconstruction the raw data, which includes energy deposition timing and topological information, will be interpreted as neutrino interaction products. The treatment for the near detector is special as there can be multiple events occurring during a beam spill window. This is addressed by applying a set of timing and topology cuts that create slices of hits that are close in both space and time. Track and shower reconstruction is very important for both CC and NC analysis. CC \( \nu_\mu \) and CC \( \bar{\nu}_\mu \) events usually consist of a muon track and a number of hadronic showers. In contrast NC events usually consist of a bunch of showers and possibly a short hadronic track. Figure 4.7 shows typical topologies of different events which are important for the sterile antineutrino analysis and figure 4.8 shows an occasion where mis-identification occurs due to the hard scattering. Here \( \nu_\mu \) and NC events are the main background for the antineutrino analysis presented in this thesis.

4.3.1 Digit formation

The basic reconstruction unit is a digit, which consists of a pulse height, a digitized measured time and a list of possible associated strip ends. The far detector suffers a strip ambiguity, and there are eight possible strip ends that can be associated with each far detector digit. This ambiguity is caused by the multiplexing which is described in chapter 3.
CHAPTER 4. CALIBRATION AND RECONSTRUCTION

Figure 4.7: This figure shows the MINOS event topologies for CC $\nu_\mu$ events (left), CC $\bar{\nu}_\mu$ events (middle) and NC events (right). The CC events are usually associated with a long muon track which will be bent in a different direction depending on the charge. NC events usually consist of a large hadronic shower. The figure is taken from [87].

Figure 4.8: This figure shows an occasion where a $\mu^-$ experienced hard scattering. The sharp bend would cause the overall track direction to be different from that expected from its curvature in the magnetic field. This is a major background for a charge sign cut selection. The figure is taken from [61].
4.3.2 De-multiplexing

To resolve this multiplexing issue at the far detector, one has to compare the possible strips on each side of the detector. The de-multiplexing algorithm first compares the digits from the two sides of the detector to produce all possible solutions in each plane. Timing information is next used to reduce the possible solutions. The algorithm then forms structures of strips that are unambiguously within a plane, by using the fact that there is only one pairing of strip ends that can place two digits on the same strip. This is then used to constrain and locate the event region.

4.3.3 Strip formation

The reconstruction analyses the list of digits in order to form a strip. ‘Strip’ here means a single energy deposition in a scintillator strip. For the far detector, a strip object is formed by combining the digits recorded at two ends of each scintillator strip. A strip object is formed by combining all the digits that are close in time and from the same scintillator strip.

4.3.4 Slices

A slice object is formed by combining all the strips that are close in both space and time. The near detector, as mentioned above, typically will detect multiple events within a spill gate. By using slice objects, events will be separated based on time and spatial information, thus improving the near detector reconstruction efficiency. The far detector typically will have zero or one events per spill, therefore zero or one slices per spill.

4.3.5 Track reconstruction

A muon track is a very important signature of CC interactions. It is therefore important to first identify the track, then determine the charge and momentum of the muon. To find the track, topological information is used. The algorithm will attempt to first search for track-like segments, which are groups of single strips that form a continuous line in adjacent planes. These segments are tagged and the algorithm will attempt to join them if they are unambiguously associated with each other. Additional joins are made by associating the longer segments and
any remaining segments will be used to seed further association, until no more segments are left. By combining these joined segments further, track objects will be formed.

These tracks object are then fit with a Kalman filter [100], to determine hit by hit if each hit is actually a part of the track. If a hit failed the track fit, it will be passed to the shower reconstruction. The track that passed the fit, with the earliest hit in time defined as the vertex, will be used to determine the $q/p$ of the muon, and the uncertainty $\sigma_{q/p}$ will also be computed. If the track ends inside the detector, a more accurate measurement of the momentum is made using the range of the track. The energy from range is measured by summing all the deposit energy, using the Groom [101] tabulation and the GEANT3 detector simulation. This range measurement gives agreement between data and Monte Carlo events of better than 2%. Comparison was made between the curvature measurement and range measurement, which suggests agreement is better than 3% [87] for curvature measurements.

### 4.3.6 Calorimetric shower reconstruction

Different from the track topology, the shower-reconstruction algorithm will attempt to search for shower clusters, which are groups of multiple strips in adjacent planes. These shower clusters are joined using a cluster window to form a shower object. More details can be found [102]. The energy of the shower is then reconstructed calorimetrically, i.e. by summing the pulse height of all the constituent hits.

### 4.4 Shower Energy Estimator

Calorimetric shower energy reconstruction does not maximise the energy resolution performance of the detectors. For CC samples, the neutrino energy consists of both track and shower energy; the neutrino energy resolution is limited mainly by the shower energy resolution. NC samples suffer even more as showers are the main parts of the NC events. An improved shower energy reconstruction is achieved by using additional shower information [103] and a $k$-Nearest-Neighbour ($k$NN) algorithm [104].
4.4. **SHOWER ENERGY ESTIMATOR**

4.4.1 *kNN shower energy*

The *kNN* shower energy estimator make use of the Monte Carlo simulation. By placing each data event into a multi-dimensional variable space, one can effectively quantify the similarity of the data event with a library of Monte Carlo simulated events using the Euclidean metric:

\[
d = \sqrt{\sum_{i} \left( x_i - y_i \right)^2 / \sigma_i^2}
\]  

where \(i\) is the variable, \(x_i\) is value of the \(i^{th}\) variable of the data event we want to determine the shower energy of, and \(y_i\) is value of the \(i^{th}\) variable of the Monte Carlo event. The shower energy of the data event will be determined as the average of the true energies of the \(k\) nearest Monte Carlo events. Many variables were considered, but only three shower features are in the end used as these are best in terms of improving the standard oscillation sensitivity \[105\]. These three variables are:

- The number of planes in the highest-energy shower.

- The deweighting energy within 1 m of the track vertex. The deweight procedure is applied to calorimetric energy to account for nonlinear detector response to lower energy events \[106\].

- The calorimetric energy in the first two showers if there is more than one reconstructed shower.

These three variables are used for all MINOS and MINOS+ near and far detector samples. Figure 4.9 shows a comparison of the calorimetric and *kNN* reconstructed shower energy over different true shower energy ranges. As one can see the improvement in resolution is significant in the low energy range \[81, 105\].

4.4.2 *Shower energy systematic uncertainties*

Shower energy measurement is sensitive to mis-modelling in the hadronic shower simulation. The main systematics arise from the energy calibration process and intranuclear re-scattering.

**Energy calibration process.** There are uncertainties for both absolute and relative calibration processes, which are 5.7% and 2.1% respectively.
Figure 4.9: This figure shows a distribution of reconstructed over true shower energy in different true shower energy ranges, where the black line is the calorimetric reconstructed shower energy and the red line is the $k$NN reconstructed shower energy. The figure is taken from [105].
4.4. SHOWER ENERGY ESTIMATOR

Figure 4.10: This figure shows a comparison between functional form of shower energy uncertainty (red curve) against the total error on the kNN shower energy estimator (black dots) from all sources. The figure is taken from [105].

Intranuclear rescattering with INTRANUKE. This uncertainty is calculated by varying the INTRANUKE parameters and reweighting the Monte Carlo to study the effects of these systematic variations.

Other sources This includes a large set of uncertainties in the modelling. More details can be found [81].

Combining all the sources above the final form of the systematic uncertainty takes the following form in $E_{\text{true}}$:

$$\sigma_{\text{shower,kNN}} = \sigma_{\text{offset}} + \sigma_{\text{scale}}e^{E_{\text{shower}}/1.44\text{GeV}}$$  \hspace{1cm} (4.7)

where $\sigma_{\text{offset}} = 6.6\%$ and $\sigma_{\text{scale}} = 3.5\%$. This form is estimated to mimic the shape of the calorimetric shower energy uncertainty, as shown in the figure 4.10, covers kNN shower energy uncertainty for all true energy range. It is therefore as an overestimated uncertainty on the shower energy.
Chapter 5

Event Selection

As mentioned in chapter 2, neutrinos and antineutrinos can interact via both CC and NC channels. Between the different beam configurations, the relative amount of each neutrino and antineutrino interaction types observed in the detector varies greatly. Depending on the beam configurations one would have very different beam compositions. For the both low-energy and medium-energy $\nu_\mu$-dominated beam, neutrino events are dominating the interactions at both CC and NC interaction channels. CC antineutrino events can be distinguished from neutrino events by looking at the direction the muon track bends in the magnetic field. However, if an antineutrino interacts via the NC channel, one cannot distinguish it from neutrino event. Therefore, in this analysis only CC samples are used for both the low-energy and medium-energy neutrino-dominated beam, as the NC samples are dominated by neutrino events. For MINOS $\bar{\nu}_\mu$-enhanced beam both CC and NC samples are used, as antineutrino interactions are a significant component of the NC samples. Table 5.1 summarises the samples used in different beam configurations.

The low-energy $\bar{\nu}_\mu$-enhanced beam CC selection is developed by [107] and optimised for purity. The low-energy $\bar{\nu}_\mu$-enhanced beam NC selection is developed as part of the neutrino analysis [42], and have been re-optimised for $\bar{\nu}_\mu$-enhanced beam. The low-energy $\nu_\mu$-dominated beam CC selection is also developed by [107] and optimised for purity. The medium-energy $\nu_\mu$-dominated beam CC selection is mainly developed and optimised by me and will be fully discussed in this chapter. In this chapter, the process of event selection will be discussed, followed by the detailed selection cuts for the different samples. For the low-energy $\bar{\nu}_\mu$-enhanced beam, the NC selection cut is applied before CC selection, as some events may
5.1 Preselection

The preselection is mainly used to ensure the basic quality of the beam events, thus applied to all samples. First, one have to remove the events that are not coming from the beam, in which case the light injection and cosmic ray contaminations have to be removed. Cosmic ray events are not used in this analysis therefore have to be removed. In order to remove the them, one would require the track and beam angle $\theta$ to satisfy $\cos \theta < 0.6$. This cut is rejecting most downwards and upwards going events. The light injection events will be recorded and removed based on the record. Second, the beam event that is recorded must have good quality, namely all the detector parts must operating well and consistently. These information are generally recorded as a binary answer and therefore will not be discussed in detail.

5.2 Low-energy $\bar{\nu}_\mu$-enhanced beam NC selection

5.2.1 Near detector specific cut

NC events are mainly composed of showers and occasionally tracks as we discussed in chapter 4. NC events in general are heavily affected by the poorly reconstructed events. These poorly reconstructed events are categorised into the following types [108]: split, vertexing failures and other low completeness events.
Split events occur when a single neutrino interaction results in two or more reconstructed events, these events would lead to double-counting and bad energy resolution of the events. Vertexing failures refers to the events which interact outside the fiducial volume that are reconstructed inside the fiducial volume, these events generally don’t have a complete energy. Other low completeness events are the remaining events that causes the low completeness. These poorly reconstructed events satisfy $E_{\text{shower\ reco}}/E_{\text{shower\ true}} < 0.3$ condition, where it would suggests the reconstructed shower energy are too different from the true shower energy. Since these events are mainly present in low energy range and would make a large background for NC events, it is crucial to remove them. Two main cuts are imposed at ND to remove them:

**Maximum Consecutive Plane $\geq 3$.** Shower that develops longitudinally inside the detector it would deposit energy continuously along the detector planes see left figure in figure 5.1.

**Fraction of Slice Pulse Height $> 0.5$.** As shower should be more close to both time and space window see right figure in figure 5.1.

If a neutrino interaction happens inside the detector but also develops outside the detector, the shower energy for this event will have a incomplete energy profile. Alternatively if neutrino event interact outside the detector but mistakenly reconstructed inside the detector, this would also leave a incomplete energy profile for event. These two type of failure can be removed using the following fiducial volume cuts:
Neutrino event vertex must be 0.5 m away from the edge of the detector plane including coil hole. This would remove all the events that are outside the detector plane.

Neutrino event vertex must be contained 1.7 m < z < 4.7 m. z = 0 is at the front of the first detector plane. This cut would remove all the events that is outside the calorimeter region or too close to the end of the detector see chapter 3 for more detail.

5.2.2 Far detector specific cut

FD preselections are imposed mainly to reduce the fibre noise, Light Injection (LI) background, cosmic muon background, split events, non-contained and leaking events.

During a spill, noise events can fake signal, these events are dominated by spontaneous light emission from the WLS fibre [74]. This actually contribute to large amount of the fake events. Since fibre noise event is very random, the fake events associated with a fibre noise will decrease exponentially as we increase the number of strips in a event. Also these noise events tends to have lower pulse height than MIP energy deposition. The following cuts are implemented considering the above facts:

For number of hit strips ≤ 8, pulse height ≥ 3750 ADCs. Aiming to remove low pulse height events at events with low number of hit strip.

For number of hit strips ≥ 8, pulse height ≥ 2000 ADCs. Aiming to remove again low pulse height events at events with high number of hit strip.

The main way to remove the LI events at far detector is to use the signal sent from trigger PMT hit, which would flag any events happened in this time should be removed. The effectiveness of this remove is > 99.99% [107]. However, as LI flashes 10^5 times per hours [109] from time to time, there will be some of fake LI events that contaminate the beam events. As mentioned in chapter 4, typically these LI events have following signatures: more energy at one end of the strip, higher energy deposition than standard MIP energy deposition and larger fraction of the strips in a single pulser box than typical beam events. Beam events typically are at the centre of the detector, where energy deposited will be
CHAPTER 5. EVENT SELECTION

symmetrical at both end of the strip. The following cuts are implemented and have to meet simultaneously to remove these LI events:

**Minimum summed pulse height from both side the reader smaller than** \(1.7 \times 10^6\) ADCs or the energy deposition asymmetry is smaller than 0.55. Unselected events are asymmetrical events or events with high energy deposition.

Split events occurs much less frequent in far detector than near detector as the event rate for FD is much lower than ND. However, these events can still contaminate the observation. One simply cut is implemented to remove them:

**remove event pulse height contains less than 75% of the spill pulse height.** It removes events that is splitting into two.

Finally fiducial volume cut is performed to remove the non-contained and leaking events as discussed also in ND fiducial volume cuts. Due to some detail difference in detectors, these cut will be slightly different from the ND fiducial volume cut. The following cuts are applied:

**Neutrino are more than 40 cm distant from the edge of the detector and more than 60 cm away from the coil.** This would remove all the events that is outside the detector plane.

**Neutrino event vertex must contained within** \(0.21 \, m < z < 13.72 \, m\) or \(16.25 \, m < z < 28.96 \, m\). \(z = 0\) is at the front of the first detector plane. This cut would remove all the events at the beginning and end of the each supermodule see chapter 3 for more detail.

With these detector specific cuts, the events remaining are considered to be a good NC candidates, and will be selected with same cuts for both detectors. Event length is defined by the number of planes which event deposit energy in. This cut requires Event Length \(<= 47\) Planes. For NC events with a track, event length and track extension are the two main cuts applied. For NC events without a track, only event length cut is applied. Figure 5.2 shows the effectiveness of this cut at different detector. Since muon generally travel further through the detector than the typical length of the hadronic shower from an NC events, the CC background here is dominating the events with higher number of planes.
Figure 5.2: This show the effectiveness of the event length cut for near detector (left) and far detector (right). The filled histograms represent the CC background which is the dominant background that contaminates the NC sample; the black dots and the red line are data and MC total of the NC samples respectively.

Figure 5.3: This shows the effectiveness of the track extension cut for near detector (left) and far detector (right). The filled histograms represent the CC background which is the dominate background that contaminates the NC sample; the black dots and the red line are the data and MC total of the NC samples respectively.

This cut alone would reject 55%, and 70% of the CC background at near and far detector respectively.

CC events can produce a shower, which would have similar feature as NC hadronic shower events. Making use of the fact that muon typically extend further from the shower than hadrons like protons or pions, one could effectively reduce the CC contamination. The track extension is defined as Track number of planes - Shower number of planes for a event. This cut requires Event extension <= 6 Planes. Figure 5.3 shows the effectiveness of this cut at different detectors. This cut in addition to event length cut, would reject 56% and 72% of the CC background at near and far detector respectively.
Combining all above cuts, the performance of this selector are measured by two quantities the efficiency and purity. The efficiency is defined as:

$$\frac{\text{Number of selected true signal events}}{\text{Total number of signal events before selection}},$$

and the purity is defined as:

$$\frac{\text{Number of selected true signal events}}{\text{Total number of selected events}},$$

Figure 5.4 shows the performance of the NC selector on the antineutrino beam NC samples. This selector achieved 79.9% and 87.8% efficiency and 71.4% and 74.0% purity for all NC $\nu$ events at near and far detector respectively. Due to the beam configuration difference as mentioned in chapter 3, the NC selected sample contains 56.6% and 53.9% of neutrino event and 43.4% and 46.1% of antineutrino event at the near and far detectors respectively.

5.3 Low-energy $\bar{\nu}_\mu$-enhanced beam antineutrino CC selection

To reduce the NC contaminations, events that pass NC selection are rejected first before CC selection is applied. As antineutrino are focused in $\bar{\nu}_\mu$-enhanced beam, when taking data with the antineutrino beam configuration, the magnetic fields in the detectors are oriented to focusing the positive muons. In the near detector, the coil hole is not well modelled, in order to reduce the systematic uncertainties, all the events with tracks that passes through the regions of the coil hole are rejected. The Particle IDentification (PID) we mainly used for all analysis are roID. This make uses of four different track variables to construct the multi-variate classification algorithm to classify the muon track. These four variables are number of track scintillator planes, mean pulse height of the track hits, signal fluctuation and transverse track profile. These four variables are used to form a single variable which represent the probability of how likely the event is a muon track [110].

**Number of track scintillator planes.** This variable is directly proportional to the length of the of the muon that leaves a track in our detector. As muons typically travel further through detector than hadrons. See top left
5.3. \( \bar{\nu}_\mu \)-ENHANCED BEAM CC

Figure 5.4: This shows the efficiency and purity of the low-energy \( \bar{\nu}_\mu \)-enhanced beam NC antineutrino selector for near detector (top) and far detector (bottom). The filled histograms are different backgrounds, where violet is CC \( \bar{\nu}_\mu \) background and blue is CC \( \nu_\mu \) background. The black and red lines are efficiency and purity of this selector respectively.
CHAPTER 5. EVENT SELECTION

Graph in figure 5.5.

Mean pulse height of track hits. This variable measures the average energy loss in the MINOS scintillator strips. Muons are Minimum Ionizing Particles (MIPs), therefore a muon track’s mean pulse height deposited per strip is well-defined and have sharp peaked at one unit of MIP. In contrast, hadronic shower tracks will have a mean pulse height distribution that is much broader. See top right graph in figure 5.5.

Signal fluctuation. Selected hits are sorted by pulse height. These hits are then divided into low and high pulse height group described in [110]. Signal fluctuation is defined as the ratio of the mean of the low pulse height hits over the mean of the high pulse height hits. Muon and none-muon tracks would have different shape and peak in these two regions and therefore can be separated. See bottom left graph in figure 5.5.

Transverse track profile. This variable measures the energy deposition in the transverse plane. A typical muon track would only hit a few strips on a scintillator plane. However, hadronic shower would have much broader profile over each scintillator plane [111]. By taking the ratio of the pulse height of the track hits to that of the event hits in each plane, one can separate track from hadronic shower. See bottom right graph in figure 5.5.

The roID value for each individual event is determined from the k-Nearest-Neighbour method, where a large collection of simulated events are plotted on these four multi-dimensional space. The actual events are classified based on the distance to the \( k \) numbers of closest simulated events. The \( k \)-values (i.e. number of the neighbour points used for classification) is determined from the signal acceptance and background rejection which is optimised to be 80 for MINOS neutrino beam and antineutrino beam samples. For both detectors, \( \text{roID} > 0.3 \) and \( q/p > 0 \) are required. Here, \( q \) is the measured of the muon charge sign, \( p \) is the measured momentum of the muon. As mentioned before, MINOS uses a Kalman filter to reconstruct the \( q \) and \( p \) of the muon, which returns the \( q/p \) value. \( q/p < 0 \) would signifying a reconstructed \( \mu^- \), and \( q/p > 0 \) a \( \mu^+ \). Figure 5.6 shows the roID cut for both near and far detector. \( q/p > 0 \) alone would remove 11.8% and 18.1% of \( \bar{\nu}_\mu \) signal, 93.7% and 95.2% of \( \nu_\mu \) wrong-sign background and 66.2% and 59.0% of NC background at the near and far detectors respectively.
Figure 5.5: The four figures represent four different kNN variables, where black dots are the data, red line is the expected MC and blue line is the NC background contribution. Four graphs from top left to right bottom are number of track scintillator planes, mean pulse height of track hits, signal fluctuation and transverse track profile respectively. All four figures are taken from [87].
CHAPTER 5. EVENT SELECTION

Figure 5.6: The black dots are data, the red line is the MC prediction, the blue filled histogram is the signal CC $\nu_\mu$ and the green lines represent the total background. These graphs show the effectiveness of the roID > 0.3 cut for the MINOS antineutrino beam antineutrino samples at the near detector (left) and far detector (right).

roID > 0.3 alone would cut 45.5% and 36.0% of $\nu_\mu$ wrong-sign background, 88.1% and 77.3% of NC background and only 1.6% and 1.6% of $\bar{\nu}_\mu$ signal at the near and far detectors respectively.

Figure 5.7 shows the performance of the CC selector on the $\bar{\nu}_\mu$-enhanced beam antineutrino CC samples. This selection achieved 51.6% and 94.9% efficiency and 95.2% and 95.7% purity at the near and far detectors respectively. The low efficiency at the near detector is mainly caused by the removal of events that are travelling through the near detector coil hole.

5.4 Low-energy $\nu_\mu$-dominated beam antineutrino CC selection

Different from the $\bar{\nu}_\mu$-enhanced beam, NC sample in $\nu_\mu$-dominated beam is dominated by the neutrino events therefore not used in this analysis. Three variables are heavily used to separate the wrong-sign neutrino and NC background events from the signal CC antineutrino events: $(q/p)/\sigma(q/p)$, $|\text{Relative Angle} - \pi|$ and roID.

As shown in [61] a simple $q/p > 0$ cut would reject 89.2% and 93.0% of the CC $\nu_\mu$ events, and only 3.3% and 3.8% of CC $\bar{\nu}_\mu$ events at near and far detector respectively. In this search, $(q/p)/\sigma(q/p)$ is used where $\sigma(q/p)$ represents the errors on the Kalman filter reconstruct $(q/p)$. The background are mainly caused by the multiple or hard scattering, which could cause the overall curvature to be
5.4. \( \nu_\mu \)-DOMINATED BEAM CC

Figure 5.7: This shows the efficiency and purity of the MINOS \( \bar{\nu}_\mu \)-enhanced beam antineutrino CC selector for the near detector (top) and the far detector (bottom). The filled histograms are different backgrounds, where violet is wrong-sign background, blue is NC background. Black and red lines are the efficiency and purity of this selector respectively.
opposite to the effect of magnetic field. See chapter 4.3 for more detail. For this sample, \((q/p)/\sigma_{(q/p)} > 2.3\) cut is applied. This cut alone would reject 39.9% and 46.4% of the remaining CC \(\nu_\mu\) wrong-sign background events, 40.8% and 43.1% of the remaining NC background events, and only 2.5% and 5.9% of CC \(\bar{\nu}_\mu\) signal events at the near and far detectors respectively. Figure 5.8 shows the cut performance in detail.

Relative angle measures the angle formed by the projection from the track vertex to the track end of plane, and same projection assume no magnetic field and matter in the detector. Due to the beam configuration, majority of the background wrong-sign and NC events will have smaller angular distribution than antineutrino events, as shown in figure 5.9. In our selection, \(|\text{Relative Angle} - \pi| > 2.0\) cut is applied. This cut along would reject 22.3% and 17.3% of the remaining CC \(\nu_\mu\) wrong-sign background events, 19.2% and 16.8% of the remaining NC background events, and only 1.2% and 0.7% of CC \(\bar{\nu}_\mu\) signal events at near and far detector respectively.

For this search, \(\text{roID} > 0.65\) is implemented for best efficiency and purity consideration which will be explained shortly. This cut alone would reject 73.4% and 76.0% of remaining CC \(\nu_\mu\) wrong-sign background events, 96.6% and 96.2% of the remaining NC background events, and only 2.9% and 3.8% of CC \(\bar{\nu}_\mu\) signal events at the near and far detectors respectively. Figure 5.11 shows the selector performance using two quantities defined above. As one can see the sample is primarily optimised for high purity, and the performance in the two detectors are
Figure 5.9: The black dots are data, the red line is the MC prediction, the blue filled histogram is the signal CC $\bar{\nu}_\mu$ and the green lines represent the total background. This plot shows the effectiveness of the $|\text{Relative Angle - } \pi| > 2.0$ cut for MINOS $\nu_\mu$-dominated beam CC antineutrino samples at the near detector (left) and the far detector (right).

Figure 5.10: The black dots are data, the red line is the MC prediction, the blue filled histogram is the signal CC $\bar{\nu}_\mu$ and the green lines represent the total background. This plot shows the effectiveness of the roID $> 0.65$ for MINOS $\nu_\mu$-dominated beam CC antineutrino samples at the near detector (left) and the far detector (right).
Figure 5.11: This shows the efficiency and purity of the MINOS neutrino beam antineutrino selector for near detector (top) and far detector (bottom). The filled histograms in these plots represent different backgrounds, where violet is wrong-sign background, blue is NC background. Black and red lines are efficiency and purity of this selector respectively.
5.5 Medium-energy $\nu_\mu$-dominated beam antineutrino CC selection

For the medium-energy $\nu_\mu$-dominated beam, only CC antineutrino samples is used. Due to the similarity between low-energy and medium-energy $\nu_\mu$-dominated beam, the selector cut variables used here are the same. Again $(q/p)/\sigma(q/p)$, roID and $|\text{Relative Angle} - \pi|$ variables are used for selecting antineutrino events, and the cut values are optimised to account for different energy spectrum and background. To obtain the cut values, one has to maximise the efficiency and purity simultaneously, following equation are selected for optimisation: efficiency $\times$ purity/$(2 - \text{purity})$ [107]. This allow one to prioritise purity without losing efficiency. All three variables are mapped onto three dimensional space, and only the maximum value of this equation is obtained and used for MINOS+ antineutrino event selection, which yields: $(q/p)/\sigma(q/p) > 2.5$, roID > 0.935 and $|\text{Relative Angle} - \pi| > 2.29$. Figure 5.12 shows the effectiveness of the $(q/p)/\sigma(q/p)$ cut. The $(q/p)/\sigma(q/p)$ cut alone would reject 46.3% and 50.4% of the $\nu_\mu$ wrong-sign background, 52.5% and 52.9% of NC background and only 3.7% and 7% of $\bar{\nu}_\mu$ signal at the near and far detectors respectively. Figure 5.13

Figure 5.12: The black dots are data, the red line is the MC prediction, the blue filled histogram is the signal CC $\bar{\nu}_\mu$ and the green lines represent the total background. The harsh cut-off at zero is due to the requirement of a positive curvature. This plot shows the effectiveness of the $(q/p)/\sigma(q/p) > 2.5$ cut for medium-energy $\nu_\mu$-dominated beam CC antineutrino samples at the near detector (left) and the far detector (right).

very similar. These cuts achieved 91% and 88% efficiency, 92% and 94% purity at the near and far detectors respectively.
Figure 5.13: The black dots are data, the red line is the MC prediction, the blue filled histogram is the signal CC $\bar{\nu}_\mu$ and the green lines represent the total background. This plot shows the effectiveness of the $\text{roID} > 0.935$ cut for medium-energy $\nu_\mu$-dominated beam CC antineutrino samples at the near detector (left) and the far detector (right).

shows the effectiveness of the roID cut. The roID cut alone will remove 90.8% and 96.1% of the $\nu_\mu$ wrong-sign background, 99.7% and 99.8% of the NC background and 18.1% and 17.6% of $\bar{\nu}_\mu$ signal events at the near and far detectors respectively. Finally, figure 5.14 shows the effectiveness of the $|\text{Relative Angle} - \pi|$ cut. The $|\text{Relative Angle} - \pi|$ cut alone would reject 29.6% and 21.2% of the $\nu_\mu$ wrong-sign background, 27.7% and 23.0% of the NC background and only 3.4% and 1.6% of $\bar{\nu}_\mu$ signal events at the near and far detectors respectively. Figure 5.15 shows the efficiency and purity after combining all the above selection cuts. This selector achieved 81.8% and 80.6% efficiency and 85.9% and 94.5% purity at the near and far detectors respectively.
Figure 5.14: The black dots are data, the red line is the MC prediction, the blue filled histogram is the signal CC $\bar{\nu}_\mu$ and the green lines represent the total background. This plot shows the effectiveness of the $|\text{Relative Angle} - \pi| > 2.29$ cut for medium-energy $\nu_\mu$-dominated beam CC antineutrino samples at the near detector (left) and the far detector (right).
Figure 5.15: This shows the efficiency and purity of the medium-energy $\nu_\mu$-dominated beam CC antineutrino sample for the near detector (top) and the far detector (bottom). The filled histograms are the backgrounds, where violet is $\nu_\mu$ background, blue is NC background. Black and red lines are efficiency and purity of this selector respectively.
Chapter 6

Systematic uncertainties

To perform a sterile antineutrino search, we are looking for modifications to the energy spectra of selected events in both the near and far detectors caused by oscillation-driven disappearance of muon neutrinos. There are many sources of systematic uncertainty that can also cause modifications of these spectra, in comparison to our expectation, that would be mistaken for an oscillation signal. In this chapter, we first discuss how our knowledge of these uncertainties will be incorporated into the sterile antineutrino search using covariance matrices. We then go on to explain in detail, and quantify, all the sources of systematic uncertainty that we consider in the analysis.

6.1 Covariance matrices

To describe the way that the systematic uncertainties affect the energy spectra of the various samples in the near and far detectors, we use covariance matrices. These allow us to include in our description the bin-to-bin correlations that quantify the possible shape changes to the energy spectra. They also enable us to capture the correlations between how each systematic uncertainty affects the energy spectra in the two detectors. For each source of systematic uncertainty, we therefore produce a covariance matrix for each event sample that has the form shown in figure 6.1, where the on-diagonal quadrants of the matrix represent correlations between energy bins in one detector and the off-diagonal quadrants represent correlations between energy bins in different detectors. For each source of systematic uncertainty, and each event sample, we begin by shifting the parameter that describes the uncertainty by $\pm 1\sigma$ in the Monte Carlo, and forming
a ±1σ error band, which may not be symmetric in the +1σ and −1σ directions, that shows how the selected \(\bar{\nu}_\mu\) CC reconstructed energy spectra change in each detector. We then form our covariance matrix by performing a random-sampling process using this error band.

We will call the value of the \(i^{th}\) bin of our +1σ error band \(e_i^+\) and the \(i^{th}\) bin of our −1σ error band \(e_i^-\), where \(i\) runs over bins of energy in both the far and near detectors. A series of random numbers \(\epsilon\) are drawn from a Gaussian of unit width. The error band is then symmetrized by calculating a fractional error on the \(i^{th}\) bin, \(\alpha_i\), using the following equation:

\[
\alpha_i = \left[\frac{1}{2} \epsilon (\epsilon - 1) \times e_{-\sigma}\right] + \left[\frac{1}{2} \epsilon (\epsilon + 1) \times e_{+\sigma}\right]
\] (6.1)

This equation is only applied when \(\epsilon \leq 1\). Linear interpolation is applied if \(\epsilon \geq 1\). This process is repeated \(N\) times and the covariance matrix, \(V_{ij}\), can then be constructed using the equation below:

\[
V_{ij} = \frac{1}{N} \sum_{k=1}^{N} \alpha_{ki} \alpha_{kj}
\] (6.2)
where \( k \) is the \( k^{th} \) draw of the random number \( \epsilon \).

The diagonal components of the covariance matrix can be thought of as \( \sigma_i^2 \), which represents the square of the error on the \( i^{th} \) bin. Statistical uncertainties, which are uncorrelated between bins, are placed along the diagonal of the covariance matrix. The off-diagonal components of the covariance matrix can be thought of as \( \sigma_i \times \sigma_j \), and contain the information on the correlation between the \( i^{th} \) and \( j^{th} \) bin. These off-diagonal components are particularly important as they quantify how sources of systematic uncertainty can cancel out when they are correlated between the near and far detectors.

The correlation matrix is incorporated into the analysis as part of the \( \chi^2 \) that is used to compare the data to the simulation:

\[
\chi^2 = (D - M)V^{-1}(D - M)^T, \tag{6.3}
\]

where \( D \) is the data spectrum, \( M \) is the MC simulation spectrum. \( V^{-1} \) is the inverse of the covariance matrix. A \( \chi^2 \) is formed for each sample of events, and these individual \( \chi^2 \) values summed to form an overall \( \chi^2 \) value

\[
\chi^2 = \chi^2_{LE\nu CC} + \chi^2_{LE\bar{\nu} CC} + \chi^2_{LE\nu NC} + \chi^2_{ME\nu CC}, \tag{6.4}
\]

where the individual samples are, respectively, the low-energy \( \nu_\mu \)-dominated CC sample, the low-energy \( \bar{\nu}_\mu \)-enhanced CC sample, the low-energy \( \bar{\nu}_\mu \)-enhanced NC sample and the medium-energy \( \nu_\mu \)-dominated CC sample. The physics parameters (\( \Delta m^2 \) and \( \theta \)) are varied in the MC until the minimum value of the overall \( \chi^2 \) is obtained. The final allowed region for the physics parameters is calculated based on the \( \Delta \chi^2 \) to this global minimum.

### 6.2 Correlation matrices

Systematic uncertainties are separated into several different categories: normalisation, energy scale, cross-section, background, detector acceptance and hadron production systematics. The effect of each source of systematic is different for different samples, and some specific systematic uncertainties are valid only for certain samples. As mentioned above the size of the correlation is important, however the covariance matrix might not always show the correlation since the absolute scale of the systematic uncertainties across an energy spectrum could be
very different. One could also produce a correlation matrix in a similar way to that shown in figure 6.1:

\[ \text{Corr}(V_{ij}) = \frac{V_{ij}}{\sigma_i \times \sigma_j}, \tag{6.5} \]

where \( V_{ij} \) is the covariance matrix of the \( i^{th} \) and \( j^{th} \) bin and \( \sigma_i = \sqrt{V_{ii}}, \sigma_j = \sqrt{V_{jj}} \).

It is clear that for such a matrix, all the diagonal parts would be 1 and one can directly observe the size of the correlation on the off-diagonal components.

It is also worth noting that the positive correlation means two bins are changing in a similar way. For example, if a systematic is cause the first bin to have more events, a positive correlation between these two bins would suggest the second bin under effects of systematics is having more events. The negative correlation would suggest opposite trend between bins.

### 6.2.1 Normalisation

Normalisation systematics take into account uncertainties on the POT counting error, detector mass, steel thickness, scintillator thickness and reconstruction efficiency [112]. The relative near-to-far normalisation uncertainty is 2.6% for low-energy and medium-energy \( \nu_\mu \)-dominated antineutrino samples and 4% for low-energy \( \bar{\nu}_\mu \)-enhanced antineutrino samples at far detector.

A second set of normalisation uncertainties account for normalisation differences between samples. Figure 6.2 shows a study using the data during the period where NuMI magnetic focusing horns were switched off (horn-off). The fit suggests the neutrino data to MC ratios between the the medium-energy horn-off sample and the low-energy horn-off sample differ by 5.6 ± 1.0 %. For antineutrinos the same pair of ratios differ by 11.1 ± 1.9 %. This can be summarised as:

- Neutrinos: \( \frac{\text{Data}/\text{MC}_{\text{LE}}}{\text{Data}/\text{MC}_{\text{ME}}} = 1.056 \pm 0.01 \)
- Antineutrinos: \( \frac{\text{Data}/\text{MC}_{\text{LE}}}{\text{Data}/\text{MC}_{\text{ME}}} = 1.11 \pm 0.019 \)

This suggests the medium-energy samples in general have better data MC agreement for both neutrino and antineutrino, however this could be due to an accidental cancelling of systematic uncertainties in the medium-energy samples. Since
the biggest difference we see is 11%, we therefore apply a fully-correlated 11% normalization uncertainty to all samples. The medium-energy antineutrinos are the sample that have been least studied in MINOS+, and in which we therefore have the least confidence in the normalization. The ratios above suggest that, to match the low-energy data-MC differences, medium-energy MC must be scaled down by between 5.6% and 11%. Therefore we take the average of these values and shift the medium-energy antineutrino MC down by 8%. (The medium-energy neutrino MC is not used in the analysis.) To cover the possible scalings of between 5.6% and 11%, when apply an additional 3% normalization uncertainty to the medium-energy antineutrino sample. Figure 6.3 shows the covariance and correlation matrices for the combination of all the normalisation systematics for the different samples. Within each detector, normalisation systematic uncertainties are fully correlated bin-to-bin. The various levels of inter-detector correlations are also illustrated by the figure 6.3.

6.2.2 Energy calibration

Energy calibration systematics include all the systematics that arise from the calibration process as we discussed in section 4.1. Details can be found in section 4.4.2. For CC samples, these include: the uncertainty on the measurement of the track energy which is 2% from range and 3% from curvature; and the uncertainty on the measurement of the shower energy which includes systematic uncertainties from varying INTRANUKE parameters and the calibration process. For NC samples this includes: the uncertainty on the absolute calibration process which is a flat 5.7% uncertainty and the uncertainty on the relative calibration process which is 1.9% for the near detector and 0.9% for the far detector. Figure 6.4 shows the combination of all energy-measurement systematic uncertainties.

6.2.3 Background and flux uncertainties

For CC antineutrino samples, CC neutrino and NC events are the main backgrounds. For NC samples, CC events are the main background. In figure 6.5, the two graphs on the left show the evaluation of the CC neutrino and the NC backgrounds for the CC $\bar{\nu}_\mu$ sample from the medium-energy neutrino-dominated beam. It is evaluated allowing the method previously used for the low-energy beam samples, by using an alternative selection on the same dataset, where the
Figure 6.2: Graphs on the left are selected $\nu_\mu$ events, graphs on the right are selected $\bar{\nu}_\mu$ events. Graphs on the from first row are the data MC comparison, for the low-energy horn-off period. Graphs on the second row are the data MC comparison, for the medium-energy period with magnetic focusing horns switched off. Graphs on the bottom row are the Low-energy Medium-energy data to MC double ratio between low-energy and medium-energy horn-off samples.
Figure 6.3: The graphs on the left are the covariance matrices of the normalisation systematics for different samples. The graphs on the right are the correlation matrices of the normalisation systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample respectively. For each matrix, the bottom left quadrant represent the far detector region, the top right quadrant represent the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
CHAPTER 6. SYSTEMATIC UNCERTAINTIES

Figure 6.4: The graphs on the left are the covariance matrices of the energy calibration systematics for different samples. The graphs on the right are the correlation matrices of the energy calibration systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample respectively. For each matrix, the bottom left quadrant represent the far detector region, the top right quadrant represent the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
background is enhanced. The background is then assumed to be the cause of the data and simulated MC difference. The uncertainty on the CC neutrino and NC backgrounds are then determined to be 50% and 50% for the low-energy CC $\bar{\nu}_\mu$ sample from $\nu_\mu$-dominated beam, 30% and 50% for low-energy CC $\bar{\nu}_\mu$ sample from the $\bar{\nu}_\mu$-enhanced beam and 70% and 100% for medium-energy CC $\bar{\nu}_\mu$ sample from the $\nu_\mu$-dominated beam respectively.

In addition to these systematic uncertainties, the $\nu_\mu$-dominated beam also suffers flux uncertainties from the interaction between hadrons and the downstream materials [87]. This contribution is relatively small for neutrinos in the $\nu_\mu$-dominated beam but very significant for antineutrinos in the same beam mode. In figure 6.5 the two graphs on the right show the evaluation of the downstream production uncertainty for medium-energy neutrino-dominate sample. In this case we assume the entire difference between data and MC are caused by the downstream production. The uncertainty on the downstream production is estimated to be 100% for the CC $\bar{\nu}_\mu$ sample from the low-energy $\nu_\mu$-dominated beam and 150% for the CC $\bar{\nu}_\mu$ sample from the medium-energy $\nu_\mu$-dominated beam.

Figure 6.6 contains the combined background and flux systematic uncertainties for all samples. These systematics are considered fully correlated between bins and between both detectors.

6.2.4 Cross section systematics

Cross-section systematics have 10 different components to cover all the possible energy range and types of neutrino interaction. The first five are components are uncertainties that affect both neutrino and antineutrino cross section. In a few GeV range, Quasi-Elastic (QE) and Resonance (RES) are the dominant interactions. The uncertainties on the total QE and RES cross-section are 35% and 25% respectively, which cover all known data and MC discrepancies [113]. In a few GeV to 10 GeV range, the RES to Deep-Inelastic-Scattering (DIS) transitional region, interactions are controlled by the parameters $r_{ijk}$. Here $i$ describes the interaction type (CC or NC), $j$ describes the type of interaction (between $\nu$ or $\bar{\nu}$ and $p$ or $n$) and $k$ describes the multiplicity of the final states. The uncertainties on $r_{ij2}$ and $r_{ij3}$ are 0.1 and 0.2 which are determined from fits to the experimental data [113]. For 10–30 GeV, DIS interactions are dominant, and the uncertainty is calculated from the fits to data in 10–30 GeV range, and the error is determined to be 3.5% [113].
Figure 6.5: The top left figure shows the background-enhanced data MC comparison with the CC $\nu_\mu$ background in dashed blue and the NC background in dashed green. The bottom left figure shows the percentage difference of the $\text{data} - \text{MC Background}$, where blue is the CC $\nu_\mu$ background, green is the NC background. The top right figure shows the data MC from the CC $\bar{\nu}_\mu$ sample from the medium-energy $\nu_\mu$-dominated beam with the component from downstream production. The dashed red line represents the MC with maximum possible systematic shifts. The bottom right figure shows the percentage difference of the $\text{data} - \text{simulatedMC}_{\text{Downstream}}$ for different MC.
6.2. CORRELATION MATRICES

LE $\nu_\mu$-dominated CC $\bar{\nu}_\mu$

(a) Covariance matrix

LE $\bar{\nu}_\mu$-enhanced CC $\bar{\nu}_\mu$

(c) Covariance matrix

LE $\bar{\nu}_\mu$-enhanced NC

(e) Covariance matrix

ME $\nu_\mu$-dominated CC $\bar{\nu}_\mu$

(g) Covariance matrix

Figure 6.6: The graphs on the left are the covariance matrices of the combined background and downstream production systematics for different samples. The graphs on the right are the correlation matrices of the combined background and downstream production systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample respectively. For each matrix, the bottom left quadrant represents the far detector region, the top right quadrant represents the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
The next four components are antineutrino specific systematics. These systematics are assigned to the $\bar{\nu}$ sample to address the uncertainty on the measured $\bar{\nu}$ to $\nu$ cross section ratio. In the few GeV range, this is estimated by the following form:

$$\delta r = r_{\text{QE on free nuclei}} - r_{\text{QE on iron}}$$

(6.6)

where $r$ is the cross section ratio of neutrino to antineutrino, the first term on the right is for QE interactions on free nucleons and the second term on the right is for QE interactions on iron nuclei. This yields 8% additional cross section uncertainties for both $\bar{\nu}$ QE and RES interactions. In the few GeV to 10 GeV range a fit to global data returns uncertainties on the $r_{132}$ and $r_{142}$ to be 0.2. Finally in 10–30 GeV region, the uncertainty on the $\bar{\nu}_\mu$ DIS cross section is increased by an extra 4% [113].

The final component contains the systematics on the cross-section weights. This incorporate an additional 8% systematic for $\bar{\nu}$ due to the recent MINERvA measurement of the neutrino to antineutrino cross section ratio [114], which we assign on the antineutrinos, since the neutrino cross-sections in the MC are very well constrained by the data described in [17]. We have also corrected the antineutrino cross-section model in response to the MINER$\nu$A measurements which results in a maximum 11% weight on antineutrino CC event. The correction to the antineutrino cross-section model is shown in figure 6.7. These cross-section systematics contain important shape information, therefore are considered fully correlated across the energy spectrum in our study. Figure 6.8 shows the combined cross-section systematics for all different samples.

### 6.2.5 Detector acceptance

The detector acceptance systematics are designed to account for the effects of uncertainties from the detector geometry and reconstruction and selection efficiencies on the sample we are collecting. This is achieved by comparing the data and MC before and after making a change to our selection requirement. The systematic error bands are calculated using the following expression:

$$\frac{(\text{data/MC})_{\text{shifted}}}{(\text{data/MC})_{\text{nominal}}} \times \frac{\text{data}_{\text{shifted}}}{\text{data}_{\text{nominal}}}$$

(6.7)
Figure 6.7: The graph shows the MINOS neutrino and antineutrino cross-section model (shown by the blue and red solid curves respectively) in comparison with digitised PDG [17] neutrino and antineutrino data (shown by the blue and red triangles respectively). Since all the data are digitised the error bars are ignored. The red dashed line shows the antineutrino cross-section model after the reweight.
Figure 6.8: The graphs on the left are the covariance matrices of the combined cross-section systematics for different samples. The graphs on the right are the correlation matrices of the combined cross-section systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\nu_\mu$ sample respectively. For each matrix, the bottom left quadrant represents the far detector region, the top right quadrant represents the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
where the first component evaluates the change in the data-MC agreement arising from the applied change in the selection requirement and the second component makes this relative to samples of the shift we introduced. The selection changes we apply are, to an extent, arbitrary and the shape changes are not meaningful, therefore these uncertainties are considered as fully uncorrelated bin-to-bin. Figure 6.9 show the combined detector acceptance systematics for all different samples.

### 6.2.6 Hadron production

$\bar{\nu}_\mu$ collected for this analysis mainly come from the decay of $K^-$ and $\pi^-$. Uncertainties on the $\bar{\nu}_\mu$ flux could be very large and need to be assessed carefully. In this analysis, hadron production systematic are calculated from the MINERvA PPFX flux package as discussed in section 4.2.1. For each $\bar{\nu}_\mu$ sample 100 alternative flux models are generated using the systematic error band provided in the PPFX package to randomly draw systematically shifted fluxes. This is then used to form the covariance matrices for $\bar{\nu}_\mu$ samples using the equation below:

$$V_{ij} = \frac{1}{N} \sum_\alpha (n_{i\alpha} - \bar{n}_i)(n_{j\alpha} - \bar{n}_j)$$  \hspace{1cm} (6.8)

where $V_{ij}$ is the covariance matrix, $N$ represents the number of samples, $n_{i\alpha}$ represents the $i^{th}$ bin from the $\alpha$ sample and $\bar{n}_i$ represents the average $i^{th}$ bin contents for all samples and it is calculated from $\bar{n}_i = \frac{1}{N} \sum_\alpha n_{i\alpha}$.

As mentioned in chapter 4 the PPFX package is accurate for the pion dominated region, however we observe significant disagreements in the kaon dominated region. Further checks confirms that the PPFX package is only validated for neutrino energies $< 20$ GeV [115]. As described in section 4.2.1, MINOS have determined the effective functional form of the flux prediction in the $p_T$-$p_Z$ space through the beam fit process. This enabled us to describe the raw FLUKA simulation through a set of warping parameters. Figure 6.10 shown a variety of MC simulation in comparison with thin target $K/\pi$ ratios measured by MIPP. It is clear that the $K^-$ to $\pi^-$ ratios from the FLUKA simulation MC agree reasonably well with the MIPP data. We then use the beam fit framework to fit the PPFX prediction in 0–20 GeV range, this would provide the effectively modification to those warping parameters in order to make the raw FLUKA prediction agree with
CHAPTER 6. SYSTEMATIC UNCERTAINTIES

Figure 6.9: The graphs on the left are the covariance matrices of the combined detector acceptance systematics for different samples. The graphs on the right are the correlation matrices of the combined detector acceptance systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample respectively. For each matrix, the bottom left quadrant represent the far detector region, the top right quadrant represent the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
PPFX. These modifications are then propagated using the same underlying weight to the 20–40 GeV range, which provide a much better data MC agreement in the $K$ dominated region. Figure 6.11 shows an example of the fit we performed, where both horn-off and horn-on samples are fitted simultaneously to cover all the possible phase spaces. The red and blue histogram represent the difference before and after the modified PPFX prediction, one can see that for the antineutrino sample the dominant changes are in the high and low energy region. This provides an extra modification towards the event spectra for all samples. Finally, figure 6.12 shows the hadron production systematics for all samples.

6.2.7 Beam optic systematics

There are also some beam optic systematics that need to be considered in this study. The horn current mis-calibration, which described the uncertainty that is caused by the calibration of the horn current scale. A study is conducted and found that the horn current in the MC is off by 1% [88]. The horn current distribution, which describes the relationship between the horn magnetic field $B$ and the applied current $I$. In an ideal, all current would flow on the outer surface of the horn, however, in reality it is related to the skin depth $\delta$. This have large impact on the neutrino spectrum, especially the portion coming from pions which would spend a large amount of time traversing horn’s inner conductor, more details can be found [88]. We estimate the $1\sigma$ uncertainty by taking the true neutrino energy spectrum difference between $\delta = \infty$ and $\delta = 6$ mm. The horn-1 offset, which describes the transverse misalignment of the magnetic horns with respect to the target. The $1\sigma$ value was estimated by shifting the horn-1 from the beam line by 1 mm and the changes are taken to be systematic uncertainties [88]. The total uncertainty in the number of protons delivered on the target, which describes the number of proton in the beam and what fraction of protons are incident on the target. These systematics are affected by the resolution of NuMI toroids and beam spot size at target. The error on that are assessed to be 2% [88]. The baffle scraping systematic described the systematic that arises when small fraction of proton that hits the horn protection baffle. This is assessed by beam width measurement and baffle temperature measurement, which yields $\pm 0.25\%$ [88]. Figure 6.14 shows the full systematics for the combined beam optic systematics for all the samples.
Figure 6.10: These plots show a comparison between a variety of simulated MC models and MINOS fitted ratios. The simulated MC and fitted ratios are generated with the MINOS target, the measured ratios are coming from the thin target data. Plots are taken from [116].
6.2.8 Detector cleaning systematics

As mentioned in chapter 5, the poorly reconstructed events affect the NC sample significantly, and therefore the systematics on the poorly reconstructed component and the effects of the cleaning cuts need to be evaluated. The uncertainty on the poorly reconstructed component is evaluated through a similar method as used for the background systematics as discussed earlier. The uncertainty on the effects of the cleaning cuts for each bin is evaluated using the following expression [109]:

\[
\frac{N_{\text{modified}}}{N_{\text{nominal}}}
\]

(6.9)

where \(N_{\text{modified}}\) represents the number of events in the bin for an altered set of cut values whereby both data and simulated MC have the same portion of rejected events, and \(N_{\text{nominal}}\) represents the number of events of the original cut. Figure 6.14 shows the NC cleaning systematics for the NC sample from the low-energy \(\bar{\nu}_\mu\)-enhanced beam.

Table 6.1 summaries all the systematics used for all the CC samples. Table 6.2 summaries all the systematics used for all the NC samples. All the systematics are calculated independently and combined in quadrature, figure 6.15 shows the
Figure 6.12: The graphs on the left are the covariance matrices of the hadron production systematic for different samples. The graphs on the right are the correlation matrices of the hadron production systematic for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\nu_\mu$ sample respectively. For each matrix, the bottom left quadrant represent the far detector region, the top right quadrant represent the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
\subsection{Correlation Matrices}

The graphs on the left are the covariance matrices of the combined beam optic systematics for different samples. The graphs on the right are the correlation matrices of the combined beam optic systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample respectively. For each matrix, the bottom left quadrant represent the far detector region, the top right quadrant represent the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.

Figure 6.13: The graphs on the left are the covariance matrices of the combined beam optic systematics for different samples. The graphs on the right are the correlation matrices of the combined beam optic systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample respectively. For each matrix, the bottom left quadrant represent the far detector region, the top right quadrant represent the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
CHAPTER 6. SYSTEMATIC UNCERTAINTIES

(a) Covariance matrix  
(b) Correlation matrix

Figure 6.14: The graph on the left is the covariance matrix of the cleaning systematics. The graph on the right is the correlation matrix of the cleaning systematics. Both graphs are calculated from low-energy $\bar{\nu}_\mu$-enhanced beam NC sample. For each matrix, the bottom left quadrant represents the far detector region, the top right quadrant represents the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.

6.3 summary

Now that we have both the event selectors and all the systematic uncertainties, we can plot the energy spectra of all four samples, showing both the data and the Monte Carlo with the full systematic error band. Figure 6.16 shows the spectra for these four samples in both detectors. The error band shows only the diagonal part of the covariance matrices which is not fully representative of the bin-to-bin correlations. In the next chapter a method is used to incorporate the off-diagonal part of the covariance matrices which will offer a much accurate way of demonstrating the actual systematic uncertainties.
6.3. SUMMARY

Figure 6.15: The graphs on the left are the covariance matrices of all the systematics for different samples. The graphs on the right are the correlation matrices of all the systematics for different samples. From top to bottom, each row represents: low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam CC $\bar{\nu}_\mu$ sample, low-energy $\bar{\nu}_\mu$-enhanced beam NC sample and medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample respectively. For each matrix, the bottom left quadrant represent the far detector region, the top right quadrant represent the near detector region, the top left and bottom right quadrants represent the inter-detector correlations.
Figure 6.16: The figure summaries all the spectra plots for four different samples with systematic uncertainty plot on top of it. All the plots on the left are far detector spectra, all the plots on the right are the near detector spectra. The top row are the low-energy $\nu_\mu$-mode CC samples, the second and third row are low-energy $\bar{\nu}_\mu$-mode CC and NC samples respectively. The last row are the medium-energy $\nu_\mu$-mode CC samples.
### Summary

<table>
<thead>
<tr>
<th>Systematics</th>
<th>σRES ±11.0%</th>
<th>ME ν-dominated ±11.0%</th>
<th>LE ν-enhanced ±12.1%</th>
<th>LE ν-dominated ±10.6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross-Section</td>
<td>σQE +10.6%</td>
<td>+8.2%</td>
<td>+8.2%</td>
<td>+8.2%</td>
</tr>
<tr>
<td></td>
<td>σ12 +7.2%</td>
<td>+7.1%</td>
<td>+7.1%</td>
<td>+7.1%</td>
</tr>
<tr>
<td></td>
<td>σ13 +6.7%</td>
<td>+7.7%</td>
<td>+7.7%</td>
<td>+7.7%</td>
</tr>
<tr>
<td></td>
<td>σE (E) DIS +7.0%</td>
<td>+7.0%</td>
<td>+7.0%</td>
<td>+7.0%</td>
</tr>
<tr>
<td></td>
<td>σE QE +7.1%</td>
<td>+7.2%</td>
<td>+7.2%</td>
<td>+7.2%</td>
</tr>
<tr>
<td></td>
<td>σE RES +7.8%</td>
<td>+8.4%</td>
<td>+8.4%</td>
<td>+8.4%</td>
</tr>
<tr>
<td></td>
<td>σE (E) DIS +7.2%</td>
<td>+7.2%</td>
<td>+7.2%</td>
<td>+7.2%</td>
</tr>
<tr>
<td>Backgrounds</td>
<td>ν CC +6.0%</td>
<td>+12.1%</td>
<td>+10.6%</td>
<td>+12.1%</td>
</tr>
<tr>
<td></td>
<td>Downstream production +4.8%</td>
<td>+7.8%</td>
<td>+7.8%</td>
<td>+7.8%</td>
</tr>
<tr>
<td></td>
<td>Track energy range +20.9%</td>
<td>+4.8%</td>
<td>+4.8%</td>
<td>+4.8%</td>
</tr>
<tr>
<td></td>
<td>Track energy curvature +1.0%</td>
<td>+1.0%</td>
<td>+1.0%</td>
<td>+1.0%</td>
</tr>
<tr>
<td></td>
<td>Shower energy scale +6.1%</td>
<td>+6.1%</td>
<td>+6.1%</td>
<td>+6.1%</td>
</tr>
<tr>
<td></td>
<td>Shower energy offset +1.2%</td>
<td>+1.2%</td>
<td>+1.2%</td>
<td>+1.2%</td>
</tr>
<tr>
<td></td>
<td>Hadronic energy scale +3.3%</td>
<td>+3.3%</td>
<td>+3.3%</td>
<td>+3.3%</td>
</tr>
<tr>
<td></td>
<td>Hadronic production PEFX +5.4%</td>
<td>+5.4%</td>
<td>+5.4%</td>
<td>+5.4%</td>
</tr>
<tr>
<td></td>
<td>Normalisation Near detector</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

**Flux**

**Energy scale**

- Downstream production: +20.9% ±11.4%
- Track energy range: +1.0% ±1.0%
- Track energy curvature: +6.1% ±6.1%
- Shower energy scale: +3.3% ±3.3%
- Shower energy offset: +1.2% ±1.2%
- Hadronic energy scale: +3.3% ±3.3%
- Hadronic production PEFX: +5.4% ±5.4%
- Normalisation: N/A

**Backgrounds**

- ν CC: +6.0% ±6.0%
- Downstream production: +4.8% ±4.8%
- Track energy range: +1.0% ±1.0%
- Track energy curvature: +6.1% ±6.1%
- Shower energy scale: +3.3% ±3.3%
- Shower energy offset: +1.2% ±1.2%
- Hadronic energy scale: +3.3% ±3.3%
- Hadronic production PEFX: +5.4% ±5.4%
- Normalisation: N/A
### Systematic Uncertainties

<table>
<thead>
<tr>
<th>Systematics</th>
<th>LE $\nu$-dominated</th>
<th>LE $\bar{\nu}$-enhanced</th>
<th>ME $\nu$-dominated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Removing events with a track ending within 10 planes of the end of the ND.</td>
<td>$\pm$ 0.5%</td>
<td>$\pm$ 10.7%</td>
<td>$\pm$ 2.5%</td>
</tr>
<tr>
<td>The events originating only in the left and right half of the fiducial volume.</td>
<td>$\pm$ 1.6%</td>
<td>$\pm$ 1.3%</td>
<td>$\pm$ 1.4%</td>
</tr>
<tr>
<td>Turning on/off coil hole cut.</td>
<td>N/A</td>
<td>$\pm$ 18.5%</td>
<td>N/A</td>
</tr>
<tr>
<td>Turning on/off containment cut.</td>
<td>N/A</td>
<td>$\pm$ 2.7%</td>
<td>N/A</td>
</tr>
<tr>
<td>Removing all the events with a track ending within 10 planes of the start of the spectrometer.</td>
<td>$\pm$ 2.5%</td>
<td>$\pm$ 1.7%</td>
<td>$\pm$ 1.2%</td>
</tr>
<tr>
<td>Tightening the fiducial radius from 80 cm to 60 cm.</td>
<td>$\pm$ 0.7%</td>
<td>$\pm$ 1.1%</td>
<td>$\pm$ 0.4%</td>
</tr>
<tr>
<td>Tightening fiducial z cut from 4.07710 m to 2.5 m.</td>
<td>$\pm$ 1.1%</td>
<td>$\pm$ 1.0%</td>
<td>$\pm$ 1.1%</td>
</tr>
<tr>
<td>Horn current mis-calibration</td>
<td>$\pm$ 14.3%</td>
<td>$\pm$ 13.7%</td>
<td>$\pm$ 8.5%</td>
</tr>
<tr>
<td>Horn current distribution</td>
<td>$\pm$ 3.9%</td>
<td>$\pm$ 3.2%</td>
<td>$\pm$ 1.4%</td>
</tr>
<tr>
<td>Horn one offset</td>
<td>$\pm$ 0.9%</td>
<td>$\pm$ 0.6%</td>
<td>$\pm$ 2.1%</td>
</tr>
<tr>
<td>Beam position</td>
<td>N/A</td>
<td>N/A</td>
<td>$\pm$ 3.0%</td>
</tr>
<tr>
<td>Beam width</td>
<td>N/A</td>
<td>N/A</td>
<td>$\pm$ 2.6%</td>
</tr>
<tr>
<td>Target position</td>
<td>$\pm$ 2.2%</td>
<td>$\pm$ 2.0%</td>
<td>$\pm$ 3.7%</td>
</tr>
<tr>
<td>POT counting</td>
<td>$\pm$ 2.0%</td>
<td>$\pm$ 1.8%</td>
<td>$\pm$ 2.0%</td>
</tr>
<tr>
<td>Material error</td>
<td>N/A</td>
<td>N/A</td>
<td>$\pm$ 2.2%</td>
</tr>
</tbody>
</table>
Table 6.1: Full systematics table summarizing every systematic uncertainty considered in this search, for all the CC samples. All the values here are evaluated based on the maximum fractional change to the energy spectrum, as the shape of individual systematics are different.
### Systematic Uncertainties

<table>
<thead>
<tr>
<th>Systematics</th>
<th>LE $\bar{\nu}$-enhanced NC</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cross-Section</strong></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{QE}$</td>
<td>± 12.1%</td>
</tr>
<tr>
<td>$\sigma_{RES}$</td>
<td>± 26.8%</td>
</tr>
<tr>
<td>$r_{ij2}$</td>
<td>± 5.8%</td>
</tr>
<tr>
<td>$r_{ij3}$</td>
<td>± 1.8%</td>
</tr>
<tr>
<td>$\sigma (E)$ DIS</td>
<td>± 4.7%</td>
</tr>
<tr>
<td><strong>Backgrounds</strong></td>
<td></td>
</tr>
<tr>
<td>CC</td>
<td>± 5.1%</td>
</tr>
<tr>
<td><strong>Energy scale</strong></td>
<td></td>
</tr>
<tr>
<td>Shower calibration</td>
<td>± 9.4%</td>
</tr>
<tr>
<td>Absolute hadronic calibration scale</td>
<td>± 11.0%</td>
</tr>
<tr>
<td>Absolute hadronic calibration offset</td>
<td>± 11.3%</td>
</tr>
<tr>
<td>Relative hadronic calibration at far detector</td>
<td>± 0.9%</td>
</tr>
<tr>
<td>Relative hadronic calibration at near detector</td>
<td>± 1.9%</td>
</tr>
<tr>
<td><strong>Hadronic production</strong></td>
<td></td>
</tr>
<tr>
<td>PPFX</td>
<td>± 19.2%</td>
</tr>
<tr>
<td><strong>Normalisation</strong></td>
<td></td>
</tr>
<tr>
<td>Far detector</td>
<td>± 2.2%</td>
</tr>
<tr>
<td><strong>Detector acceptance</strong></td>
<td></td>
</tr>
<tr>
<td>Tightening fiducial Z cut from 4.07710 m to 2.5 m</td>
<td>± 1.0%</td>
</tr>
<tr>
<td>The events originating on the left and right half of the fiducial volume.</td>
<td>± 0.6%</td>
</tr>
<tr>
<td><strong>Cleaning</strong></td>
<td></td>
</tr>
<tr>
<td>Far detector cleaning</td>
<td>± 1.7%</td>
</tr>
<tr>
<td>Far detector cosmic cleaning</td>
<td>± 2.5%</td>
</tr>
<tr>
<td>Near detector cleaning</td>
<td>± 6.1%</td>
</tr>
</tbody>
</table>
### Table 6.2: Full systematics table summaries every systematics considered in this search for NC samples. All the values here are evaluated based on the maximum fractional change to the energy spectrum, as the shape of individual systematics are different.

<table>
<thead>
<tr>
<th>Systematics</th>
<th>LE $\bar{\nu}$-enhanced NC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horn current mis-calibration</td>
<td>± 4.5%</td>
</tr>
<tr>
<td>Horn current distribution</td>
<td>± 0.1%</td>
</tr>
<tr>
<td>Horn one offset</td>
<td>± 0.1%</td>
</tr>
<tr>
<td>Target position</td>
<td>± 0.6%</td>
</tr>
<tr>
<td>POT counting</td>
<td>± 1.5%</td>
</tr>
</tbody>
</table>
Chapter 7

The MINOS and MINOS+ Sterile Antineutrino Analysis

In this chapter, I will first review sterile neutrino oscillations and then describe the search for a sterile antineutrino in MINOS and MINOS+. As mentioned in last chapter, the diagonal part of the covariance matrix is not very representative of the total systematic uncertainty. So firstly, I have introduced an alternative method to quantify different parts of a covariance matrix. This enables one to separate the error contributions according to shape, normalisation or mixed components. However, as I described before, the correlation term can lead to cancellation of the systematics between detectors. The above procedure doesn’t make any statement on the cancellation effects. Therefore I will introduce a new method called the decorrelation method, which will take the detector-to-detector and inter-detector correlated part of the systematics into account. This gives a better graphical representation of the energy spectra and their uncertainties.

Upon generating and analysing the systematics, the Asimov sensitivity is generated using the three-flavour energy spectrum which assuming no sterile oscillations. As the neutrino 2018 far-over-near result [115] suggests the actual limit is very far from the Asimov sensitivity, especially in the region where $\Delta m_{41}^2 \geq 10\text{eV}^2$. A new approach is adapted in which case the fluctuated sensitivity band is created to better represent the sensitivity range of a typical experiment. Finally the contour is computed by the use of the Feldman-Cousins procedure [44]. The final 90% C.L. contour is presented.
7.1 Sterile Neutrino Search in MINOS

As mentioned in chapter 3 the MINOS experiment was originally designed to measure the three-flavour atmospheric oscillation parameters $\Delta m_{32}^2$ and $\theta_{23}$, with $\frac{L}{E} \sim 700 \text{ km/GeV}$. MINOS has completed its designed mission and now moved on to measure additional oscillation parameters and search for new physics. As shown in chapter 2, through different channels, MINOS can also obtain sensitivity to the $\theta_{13}$ and $\delta_{CP}$ parameters. Figure 7.1 shows the oscillation probabilities for the different MINOS detectors. As one can see the near detector will not be able to see any oscillations assuming standard three-flavour neutrino oscillation.

In this thesis, $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$ are the two oscillation channels used. The muon disappearance in MINOS is measured via both the CC and NC interaction channels [117] [118]. It is also worth noting that in this thesis only the vacuum oscillation is assumed, as the matter effects have a negligible impact on the sterile oscillation probability [42].

For the search used in this thesis the full 3+1 oscillation probability is used;
more details can be found in appendix B. However, for the purpose of illustrating the phenomenology one can make a simple approximation by assuming that $\Delta m_{41}^2 \ll \Delta m_{32}^2$ and that $\Delta m_{21}^2$ is small; the muon antineutrino survival probability can then be written as:

$$P(\bar{\nu}_\mu \to \bar{\nu}_\mu) = 1 - \sin^2 2\theta_{23} \cos 2\theta_{24} \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) - \sin^2 2\theta_{24} \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right).$$

This equation suggests the MINOS CC channel is sensitive to mixing angles $\theta_{23}$ and $\theta_{24}$ and the mass splittings $\Delta m_{32}^2$ and $\Delta m_{41}^2$. The terms involving mixing angles $\theta_{14}$ and $\theta_{34}$ are much smaller and are ignored. Since the both disappearance channels are also sensitive to $\Delta m_{32}^2$, it is therefore constrained as described in section 7.4.

One can make a similar approximation for the NC channel:

$$1 - P(\bar{\nu}_\mu \to \bar{\nu}_s) = 1 - P(\nu_\mu \to \nu_s) \approx 1 - \cos^2 \theta_{14} \cos^2 \theta_{34} \sin^2 2\theta_{24} \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) - \sin^2 \theta_{34} \sin^2 2\theta_{23} \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) - \frac{1}{2} \sin \delta_{24} \sin^2 \theta_{24} \sin^2 2\theta_{34} \sin 2\theta_{23} \sin^2 \left( \frac{\Delta m_{31}^2 L}{2E} \right)$$

in addition to the parameters to which the CC channel is sensitive, the NC channel is also sensitive to $\theta_{34}$, $\theta_{14}$ and $\delta_{24}$. For the combined fit we free the mixing angle $\theta_{34}$. As shown in [42] the $\delta_{24}$ and all the other CP-violating phases all have negligible impact and are set to zero. The $\theta_{14}$ have been previously constrained in [119], which suggests the value should be very small. Therefore $\theta_{14}$ is fixed to be 0.

Table 7.1 shows the complete list of parameters used and the corresponding inputs or constraints.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Status</th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta m_{41}^2$</td>
<td>Free</td>
<td>$[10^{-4}, 10^3]$ eV$^2$</td>
</tr>
<tr>
<td>$\Delta m_{32}^2$</td>
<td>Free</td>
<td>$2.5 \times 10^{-3}$ eV$^2$</td>
</tr>
</tbody>
</table>
7.2 Separating the components of error matrices

In the previous chapter, we looked at how the systematic uncertainties are calculated and combined. However, we still don’t have a tool to visualise the different contributions. Later on in section 7.5, we will show that at $\Delta m^2_{41} \geq 10 eV^2$, the dominant constraints on the oscillation parameters come from only the normalisation effects. It is therefore important to know the effect to the systematic uncertainty on the normalisation [120] provides an excellent tool to achieve this.

Assuming a histogram of $n$ bins, the bin content of the $i^{th}$ bin is then $N_i$. The covariance matrix $V$ can be separated into three components: a shape component $V_{\text{shape}}$, a normalisation component $V_{\text{norm}}$ and a mixed shape/normalisation component $V_{\text{mixed}}$.

$$V_{ij} = V_{ij}^{\text{shape}} + V_{ij}^{\text{mixed}} + V_{ij}^{\text{norm}}.$$  \hspace{1cm} (7.3)

Each component can be written as follows:

$$V_{ij}^{\text{shape}} = V_{ij} - \frac{N_i}{N_T} \sum_{k=1}^{n} V_{ik} - \frac{N_j}{N_T} \sum_{k=1}^{n} V_{kj} + \frac{N_i N_j}{N_T^2} \sum_{kl} V_{kl},$$

$$V_{ij}^{\text{mixed}} = \frac{N_i}{N_T} \sum_{k=1}^{n} V_{ik} + \frac{N_j}{N_T} \sum_{k=1}^{n} V_{kj} - 2 \frac{N_i N_j}{N_T^2} \sum_{kl} V_{kl},$$

$$V_{ij}^{\text{norm}} = \frac{N_i N_j}{N_T^2} \sum_{kl} V_{kl},$$

$$N_T = \sum_i N_i.$$  \hspace{1cm} (7.4)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Constraint</th>
<th>Value [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta m_{21}^2$</td>
<td>Fixed</td>
<td>$7.54 \times 10^{-5} \ eV^2$</td>
</tr>
<tr>
<td>$\theta_{12}$</td>
<td>Fixed</td>
<td>0.554 rad</td>
</tr>
<tr>
<td>$\theta_{13}$</td>
<td>Fixed</td>
<td>0.149 rad</td>
</tr>
<tr>
<td>$\theta_{14}$</td>
<td>Free</td>
<td>0 rad</td>
</tr>
<tr>
<td>$\theta_{23}, \theta_{34}, \theta_{24}$</td>
<td>Free</td>
<td>$[0, \frac{\pi}{2}]$ rad</td>
</tr>
<tr>
<td>$\delta_{14}, \delta_{24}, \delta_{13}$</td>
<td>Fixed</td>
<td>0 rad</td>
</tr>
</tbody>
</table>

Table 7.1: The parameters used in the fit and their constraints.
where $V^\text{shape}$ describes the uncertainty that morphs the shape of the energy spectrum but does not change the total numbers of events. $V^\text{mixed}$ describes the uncertainty that changes each bin differently but also change the total number of events. $V^\text{norm}$ describes the uncertainty that changes each bin in the same way, thus changing only the total number of events and having the shape unchanged. Using the information given in equations 7.4, one can calculate the different matrix components from the full covariance matrix given in chapter 6. In figure 7.2 the three graphs on the left show the full, shape and mixed covariance matrices of the low-energy selected CC antineutrino sample from the $\nu_\mu$-dominated beam. The calculation gives the normalisation components of this sample which is 28.7% and further decomposition suggests it is mainly coming from the cross-section systematics. This procedure is performed for each sample and the shape and mixed components of each samples are plotted in figures 7.2 and 7.3 for all the samples. The normalisation components for these samples are 19.8% for the low-energy CC antineutrino sample from the $\bar{\nu}_\mu$-enhanced beam, 12.5% for the low-energy NC sample from the $\bar{\nu}_\mu$-enhanced beam and 33.3% for the medium-energy CC antineutrino sample from the $\nu_\mu$-dominated beam.

### 7.3 Decorrelation process

The problem with the statements made to quantify the systematic uncertainties in the last chapter and the spectra shown with systematic error bands is that they ignore the correlation terms, in particular the correlations between detectors that allow systematic uncertainties to cancel. When using a $\chi^2$ function to perform a fit with a covariance matrix, this implies that the bins contents are distributed according to a multivariate Gaussian [115]. Traditionally, an uncertainty band formed from the diagonal components of the covariance matrix is used to indicate the actual systematic uncertainties. However, the information provided by the diagonal component doesn’t reflect your knowledge of correlations between bins. In reality when one has information on the first bin, with the help of the correlation terms, we can infer significant information about uncertainties on the rest of bins. Therefore one needs a better way to visualise the systematics.

This is done using the decorrelation procedure [115]. Assume a two-bin multivariate Gaussian distribution $\alpha = (\alpha_1, \alpha_2)$ with expectation values $\mu = (\mu_1, \mu_2)$
7.3. DECORRELATION PROCESS

Low-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample

Medium-energy $\nu_\mu$-dominated beam CC $\bar{\nu}_\mu$ sample

Figure 7.2: The three graphs on the left are for the low-energy CC antineutrino sample from the $\nu_\mu$-dominated beam. The three plots on the right are from the medium-energy CC antineutrino sample from the $\nu_\mu$-dominated beam. From top to bottom, the three graphs show the breakdown of the full systematics, where the top plot is the total covariance matrix, the middle graph is the shape component and the bottom graph is the mixed component. The normalisation components for each sample is not shown as it is just a flat covariance matrix with fixed value.
Low-energy $\bar{\nu}_{\mu}$-enhanced beam CC $\bar{\nu}_{\mu}$ sample

Low-energy $\bar{\nu}_{\mu}$-enhanced beam CC $\bar{\nu}_{\mu}$ sample

(a) Total

(b) Total

(c) Shape

(d) Shape

(e) Mixed

(f) Mixed

Figure 7.3: The three graphs on the left are for the low-energy CC antineutrino sample from the $\bar{\nu}_{\mu}$-enhanced beam. The three plots on the right are from the low-energy NC sample from the $\bar{\nu}_{\mu}$-enhanced beam. From top to bottom, the three graphs show the breakdown of the full systematics, where the top plot is the total covariance matrix, the middle graph is the shape component and the bottom graph is the mixed component. The normalisation components for each sample is not shown as it is just a flat covariance matrix with fixed value.
and a correlation matrix:

\[ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \]

If one makes a measurement on second bin where \( x_2 = \beta \), we then have a new conditional expectation value for \( x_2 \). We can now write our new conditional expectation and covariance matrix as follows [121]:

\[
\bar{\mu} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (\beta - \mu_2) \\
\bar{\Sigma} = \Sigma_{11} + \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}
\]

(7.5)

Using this approach, effectively after computing the conditional distribution, one have removed the correlation between \( x_1 \) and \( x_2 \). For a fit with more than two bins, this can be applied recursively based on the measured data. Figure 7.4, 7.5, 7.6 and 7.7 show the comparison between the conventional spectra and decorrelated spectra. The error bands on the top spectra are generated with only the diagonal components of the covariance matrices; the error bands on the bottom spectra are generated using the decorrelation method. It is worth to noting that figure 7.6 is a good deception, where the diagonal of the covariance matrices suggests insufficient coverage of the data MC differences. However, when one takes the correlations into account the true data MC agreement is much better than it initially appeared. As one is taking into account the correlations, both systematics and MC energy spectra change according to the correlation. In fact when we take the systematics into consideration the MC becomes much more like the data.

It is worth to noting that this decorrelation process needs a first bin where everything kept fixed. This choice is made arbitrarily to be the last bin of each near detector spectrum. These decorrelated spectra and systematics are not used in the fit, they are instead a tool for better visualising the sizes of the systematic uncertainties relative to the data-MC differences. The near detector spectrum of almost all samples have huge reduction of the systematics after the decorrelation due to the large normalisation components of the systematics as shown in section 7.2.
Figure 7.4: These figures are the MINOS low-energy selected CC antineutrino sample energy spectra from the $\nu_\mu$-dominated beam. The top figures show the spectra with only the diagonal components of the covariance matrix is used. The bottom figures show decorrelated spectra and covariance matrices. All the figures on the left are far detector energy spectra; all the figures on the right are near detector energy spectra.
Figure 7.5: These figures are the MINOS low-energy selected CC antineutrino sample energy spectra from the $\bar{\nu}_\mu$-enhanced beam. The top figures show the spectra with only the diagonal components of the covariance matrix is used. The bottom figures show decorrelated spectra and covariance matrices. All the figures on the left are far detector energy spectra; all the figures on the right are near detector energy spectra.
CHAPTER 7. STERILE ANTINEUTRINO ANALYSIS

Figure 7.6: These figures are the MINOS low-energy selected NC sample energy spectra from the $\bar{\nu}_\mu$-enhanced beam. The top figures show the spectra with only the diagonal components of the covariance matrix is used. The bottom figures show decorrelated spectra and covariance matrices. All the figures on the left are far detector energy spectra; all the figures on the right are near detector energy spectra.
Figure 7.7: These figures are the MINOS medium-energy selected CC antineutrino sample energy spectra from the $\nu_\mu$-dominated beam. The top figures show the spectra with only the diagonal components of the covariance matrix is used. The bottom figures show decorrelated spectra and covariance matrices. All the figures on the left are far detector energy spectra; all the figures on the right are near detector energy spectra.
7.4 Test Statistics and Optimisation

The statistical method we are using is the log-likelihood method. Under the assumption of the Gaussian statistics, this is equivalent to the least square method. Assuming \( N \) independent Gaussian tests have been performed, the true parameter we want to measure \( \mu \) manifests as a function \( \lambda(\mu) \) with variance \( \sigma_i \) on \( i^{th} \) measurement. The measured value on \( i^{th} \) test is \( y_i \). This function is equivalent to the log-likelihood function if one drops the terms that do not depend on the parameters [122]:

\[
\chi^2(\mu) = -2 \ln L(\mu) = \sum_{i=1}^{N} \frac{(y_i - \lambda(\mu))^2}{\sigma_i^2} \quad (7.6)
\]

In general if one accounts for correlation, this will then transform into equation 6.3. One can Taylor expand the log-likelihood function around the minimum \( \mu_{\text{min}} \) which gives the following form:

\[
\ln L(\lambda) = \ln L(\lambda(\mu_{\text{min}})) + \left[ \frac{\partial \ln L}{\partial \lambda} \right]_{\lambda=\lambda(\mu_{\text{min}})} (\lambda(\mu) - \lambda(\mu_{\text{min}})) + \frac{1}{2!} \left[ \frac{\partial^2 \ln L}{\partial \lambda^2} \right]_{\lambda=\lambda(\mu_{\text{min}})} (\lambda(\mu) - \lambda(\mu_{\text{min}}))^2 + ..., \quad (7.7)
\]

where the higher order terms are ignored as they are small. Since the first order terms at minimum is zero, the likelihood can be reduced to:

\[
\ln L(\lambda) \approx \ln L(\lambda_{\text{min}}) + \frac{1}{2} \left[ \frac{\partial^2 \ln L(\lambda)}{\partial \lambda^2} \right]_{\lambda=\lambda(\mu_{\text{min}})} (\lambda(\mu) - \lambda(\mu_{\text{min}}))^2. \quad (7.8)
\]

Substitute the expression for \( \ln L \) back into the second term:

\[
\left[ \frac{\partial^2 \ln L(\lambda)}{\partial \lambda^2} \right]_{\lambda=\lambda(\mu_{\text{min}})} (\lambda(\mu) - \lambda(\mu_{\text{min}}))^2 = -\frac{1}{2\sigma^2} \left[ \frac{\partial^2 (y - \lambda(\mu))^2}{\partial \lambda^2} \right]_{\lambda=\lambda(\mu_{\text{min}})} (\lambda(\mu) - \lambda(\mu_{\text{min}}))^2 = -\frac{1}{\sigma}. \quad (7.9)
\]

This can then be expressed as:

\[
\ln L(\lambda(\mu)) \approx \ln L(\lambda(\mu_{\text{min}})) - \frac{(\lambda(\mu) - \lambda(\mu_{\text{min}}))^2}{2\sigma^2}. \quad (7.10)
\]
7.4. TEST STATISTICS AND OPTIMISATION

By changing the physics parameters until λ moves from its optimal value by σ using above equation:

\[
\ln L(\lambda(\mu_{\text{min}}) \pm \sigma) = \ln L(\lambda(\mu_{\text{min}})) - \frac{(\lambda(\mu_{\text{min}}) - \lambda(\mu_{\text{min}}) \pm \sigma^2)^2}{2\sigma^2}
\]

\[
= \ln L(\lambda(\mu_{\text{min}})) - \frac{1}{2}.
\]  

(7.11)

This can be interpreted as a shift of one σ from minimum value of λ being equivalent to an increase in the likelihood function of \(e^{-\frac{1}{2}}\).

Alternatively one can translate this to the least chi-square function by:

\[
-2\ln L(\lambda(\mu_{\text{min}}) \pm \sigma) \approx \chi^2(\lambda(\mu_{\text{min}}) \pm \sigma) \approx \chi^2_{\text{min}} + 1.
\]  

(7.12)

One can make further studies for multiple parameters or for different significance levels. The specific values are calculated and given in [123].

This thesis uses a global scan over a grid of possible values of \(\Delta m^2_{41}\) and \(\theta_{24}\), which is based on the method of least squares. As described in equation 6.3, the \(\chi^2\) is first computed and optimised using the MINUIT package [124] at each point in the grid. For the best possible minimisation result, the fit is performed four times in different mass hierarchy and \(\theta_{23}\) Octant. The minimum \(\chi^2\) result is thus obtained at each grid point. The \(\Delta \chi^2\) value is then computed by taking the \(\chi^2\) difference between each grid point and the global minimum. This provides a baseline for setting a two-sided confidence interval.

The final \(\chi^2\) consists of five terms:

\[
\chi^2 = \chi^2_{\text{LE}e_{\text{CC}}} + \chi^2_{\text{LE}e_{\text{NC}}} + \chi^2_{\text{ME}e_{\text{CC}}} + \left(\frac{|\Delta m^2_{32}| - \Delta m^2}{\sigma_{\Delta m^2}}\right)^2,
\]

(7.13)

where the first four terms represent goodness of fit for the low-energy selected CC antineutrino sample from the \(\nu_\mu\)-dominated beam, the low-energy selected CC antineutrino sample from the \(\bar{\nu}_\mu\)-enhanced beam, the low-energy selected NC sample from \(\bar{\nu}_\mu\)-enhanced beam and the medium-energy selected CC antineutrino sample from \(\nu_\mu\)-dominated beam respectively. The final term is the penalty term on the atmospheric mass splitting of \(\Delta m^2_{32}\).

This penalty term has been implemented to penalise any significant deviations during the minimisation of the \(\chi^2\) function. The width of the penalty term
is $\sigma_{\Delta m^2_{32}} = 0.5 \times 10^{-3}\text{eV}^2$, which is much bigger than the precision with which MINOS can measure $\Delta m^2_{31}$. This penalty term therefore has not been implemented to drive the fitted $\Delta m^2_{32}$ parameter towards any particular value, it is instead there to stop $\Delta m^2_{41}$ and $\Delta m^3_{32}$ from swapping their roles in the fit. The central value of the penalty term is $\Delta m^2_{32} = 2.5 \times 10^{-3}\text{eV}^2$.

### 7.5 Dual Detector Technique

Different from the far detector extrapolation method [125], and the far-over-near method [42], this analysis adopts the approach where the data and MC energy spectra from both detectors are used in the fit. Since the sterile mass splitting as indicated by the MiniBooNE and LSND experiments can lie well above 1 eV$^2$, if one is using the near detector spectra, which would be affected by the sterile oscillation, to extrapolate to the far detector spectra, this would significantly undermine the far detector prediction. The far-over-near method is very robust over much of the parameters space. However as shown in top two plots in figure 7.8, when considering the region where $\Delta m^2_{41}$ is large, the sterile oscillation would affect energy spectra in both detectors in a similar way. This would make the far and near spectrum for three-flavour and four-flavour to be indistinguishable as shown in bottom figure in figure 7.8. This is the main reason why 2017 sterile neutrino results shown in figure 2.9 in the high $\Delta m^2_{41}$ are much stronger than 2016 sterile neutrino result in figure 2.6.

Depending on where we are in the sterile parameter space, the different parameter values will produce very different spectra seen by the two detectors. Figure 7.9 shows the disappearance probability for a range of sterile oscillation parameter values, along with a comparison to the three-flavour case. Figure 7.10 shows the similar disappearance probability that affects the NC channel. Both suggests, even at $\Delta m^2_{41} \geq 7\text{eV}^2$, there are shape effects which affect the far detector region. By using both detector spectra, it is possible to setting limits by making use of both normalisation and shape components.

### 7.6 Data limit and Sensitivity Comparison

Using all four samples and two channels, and with all the systematic uncertainties, it is possible to compute first the sensitivity using the three-flavour MC spectrum
Figure 7.8: The top left and right plots show the far and near energy spectrum between the three-flavour and four-flavour energy spectrum, where four-flavour are generated at $\theta_{24} = 0.4$ rad $\Delta m^2_{41} = 800$ eV$^2$. The bottom plot shows the far over near ratio for both three-flavour and four-flavour cases. The clear discrepancy between the three-flavour and the four-flavour energy spectra observed in both far and near detector, are much subtler in far over near case.
Figure 7.9: The top figure shows $\nu_\mu$ disappearance probability for a range of values of $\Delta m^2_{41}$. The bottom figure shows the ratio to the three-flavour oscillation probability.
as the fake data input which we refer to as our Asimov sensitivity. This is shown by the red line in figure 7.11. One can also take an approach in which our fake data can be affected by systematic and statistical uncertainties. This can in fact be achieved by using the systematic and statistical uncertainties to fluctuate the three-flavour energy spectra. Assume a covariance matrix $V$ contains all the statistical and systematic uncertainties; it is a Hermitian and positive-definite matrix. One can therefore decompose that into two matrices $L$ and $L^T$ as $V = L \times L^T$ [126]. Assuming our three-flavour MC energy spectrum is $Y_{\text{nominal}}$ with $N$ bins, one can then generate a row of $N$ numbers, $Y_{\text{random}}$, using a Gaussian random number generator with $\mu = 0$ and $\sigma = 1$. The new fluctuated energy spectrum $Y_{\text{fluc}}$ can then be computed using the following expression:

$$Y_{\text{fluc}} = L \times Y_{\text{random}} + Y_{\text{nominal}}.$$  

By repeating this process many times and performing a global fit for each fluctuated energy spectrum, we can plot the 90% C.L. exclusion contour for each fluctuated spectrum and then selected the central 68% of these contours or the central 95% of these contours as our 1$\sigma$ and 2$\sigma$ fluctuated sensitivity bands respectively. The 1$\sigma$ fluctuated sensitivity band is shown as the green region in the

---

**Figure 7.10:** The figure shows the sterile neutrino appearance probability for a range of values of $\Delta m^2_{41}$. 

[Diagram showing sterile neutrino appearance probability with various $\Delta m^2_{41}$ values and confidence levels.]
The 90% C.L. exclusion contour is computed where $\Delta \chi^2 = 4.61$, which is corresponds to two-sided 90% C.L. for a $\chi^2$ with two degrees of freedom. It is a classical interpretation of the confidence interval and needs corrections that will be detailed later.

The data can now be used to compute and construct the limit shown by the black line in figure 7.11. The data limit is well-represented by the fluctuated sensitivity band, but is almost everywhere better than the Asimov Sensitivity. As shown in figure 7.11, the Asimov sensitivity are not fully covered by the fluctuated sensitivity band. This will be discussed further discussion in chapter 8.

Figure 7.12 shows the detailed structure of the data $\chi^2$ surface. The best fit point shown in the figure is very shallow, with $\chi^2_{\text{best fit}} - \chi^2_{\text{three-flavour}} = -0.002$. This suggests that by using all the antineutrino samples there is no deviation from the three-flavour model. The best fit energy spectra for all samples are identical to those shown in figures 7.4, 7.5, 7.6 and 7.7.

There are some interesting features in the contour, particularly in the region of $\Delta m_{41}^2 < 10^{-2}\text{eV}^2$. When $\Delta m_{41}^2 \approx 2\Delta m_{31}^2$, this corresponds to the island at the bottom of the contour. This implies that $\Delta m_{43}^2 \approx \Delta m_{31}^2$ which suggests the sterile oscillation dip would occur at the same energy as the three-flavour atmospheric oscillation dip. When, in addition, $\theta_{24} \approx \pi/4$, it is possible for both $\theta_{34}$ and $\theta_{23}$ to be at $\pi/2$. This would allow the sterile oscillation to look like three-flavour oscillation, thus the sterile oscillation cannot be distinguished or ruled out. Figure 7.13 shows the comparison between the sterile oscillation and standard oscillation probabilities in this case.

When $\Delta m_{41}^2 \approx \Delta m_{31}^2$, this corresponds to the region where the sensitivity drops significantly. This drop in sensitivity is also caused by the sterile oscillation dip coinciding with the three-flavour oscillation dip. When $\theta_{24} < \pi/4$, oscillations could be absorbed by the three-flavour oscillation parameter $\theta_{23}$, making sterile and atmospheric oscillation indistinguishable.

For the region where $\Delta m_{41}^2 \approx 10^{-3}$ and $\theta_{24} \approx \pi/4$, any combination of $\theta_{24}$ and $\theta_{34}$ would modify the oscillation probability and shift the three flavour oscillation dip, therefore can be ruled out.

Finally, for the region where $\Delta m_{41}^2 \ll \Delta m_{31}^2$, this implies the $\Delta m_{43}^2 \approx \Delta m_{31}^2$. Only when $\theta_{24} > \pi/4$, and both $\theta_{23}$ and $\theta_{34} \approx \pi/2$, will the four-flavour oscillation probability appear different from the three-flavour oscillation probability.
Figure 7.11: Sterile antineutrino exclusion contour for all MINOS and MINOS+ $\bar{\nu}_\mu$ events. The red line indicates a sensitivity contour using the nominal three flavour MC, the green band is the $1\sigma$ fluctuated sensitivity band generated by using 80 fluctuated spectra and selected the central 68% of the contour, the black contour uses actual data. The contour represents the 90% C.L. exclusion using global scan method, where region to the right of the contour are excluded.
Figure 7.12: Sterile antineutrino exclusion contour for all MINOS and MINOS+ $\bar{\nu}_\mu$ events. The black contour uses actual data 90% C.L. the color indicates the $\chi^2$ values at each grid point. The best fit point are indicated with yellow star. The best fit point is located at $\Delta m_{41}^2 = 0.0023$ eV$^2$ and $\theta_{24} = 0.012$ rad. At this point the $\chi^2_{\text{best fit}} - \chi^2_{\text{three-flavour}} = -0.002$. 
7.7 Feldman-Cousins procedure

As presented in [44], the Feldman-Cousins (FC) procedure is mainly used to provide a method of generating a confidence interval with correct, frequentist coverage. The frequentist interpretation of the confidence interval $[\mu_1, \mu_2]$ is just a member of a set [44], which satisfies

$$P(\mu \in [\mu_1, \mu_2]) = \alpha,$$

where $\mu$ is the true parameter we want to know, $\alpha$ is the confidence level, and $\mu_1$, $\mu_2$ are measured parameters which are functions of the measurable. The classical confidence interval construction ensures that if we repeat identical experiment many times, the fraction of intervals $[\mu_1, \mu_2]$ containing $\mu$ would approach $\alpha$. In this analysis we are setting intervals in the $\Delta m_{41}^2$ and $\theta_{24}$ parameter spaces with $\alpha = 90\%$ C.L.. The result presented in figure 7.11 uses $\Delta \chi^2 = 4.61$ for constructing such an interval, however, as shown in [44] and [42], this sometimes

![Figure 7.13: The CC region where $\Delta m_{41}^2 \approx 2\Delta m_{31}^2$. The sterile oscillation dip coincides with the atmospheric oscillation dip, which makes the sterile oscillation indistinguishable from the three flavour oscillation.](image-url)
does not give proper coverages. A simple example would be when we are consid-
ering a fluctuation in the high energy of the energy spectrum. This could yield
disappearance which can be misidentified as a signal oscillation at a high $\Delta m^2_{41}$
values, thus leading to the wrong $\chi^2_{\text{min}}$ and thus the wrong interval.

The FC procedure eliminates this problem by performing many pseudo-
experiments at each possible true parameter values. In this search, pseudo-
experiments are generated with the fluctuation method mentioned in chapter
7.6 at a specific $\Delta m^2_{41}$ and $\theta_{24}$ point. One can then compute the $\Delta \chi^2_{FC}$ for this
point using the following expression:

$$\Delta \chi^2_{FC} = \chi^2_{\text{Profile}} - \chi^2_{\text{Best}},$$

where $\Delta \chi^2_{\text{Profile}}$ is the $\chi^2$ value calculated at this point where $\Delta m^2_{41}$ and $\theta_{24}$
are fixed, and other nuisance parameters are minimised to give minimum $\chi^2$
values. The $\chi^2_{\text{Best}}$ value is obtained by finding the global minimum of the full
range of $\Delta m^2_{41}$ and $\theta_{24}$ for this pseudo-experiment. By repeating this process
many times, one would establish a $\Delta \chi^2_{FC}$ distribution. Ideally, we would expect
the $\Delta \chi^2 = 4.61$ interval to cover 90% of this distribution. Figure 7.14 shows a
clear case of under-coverage, where $\Delta \chi^2 = 4.61$ only covers 62% of the pseudo-
experiments. To cover 90% of pseudo-experiments, one would need $\Delta \chi^2 = 7.21$
at this point.

We can then do this at every point around the data contour shown in figure
7.12 and construct a FC surface of $\Delta \chi^2$ values shown in figure 7.15. After
correcting the data contour with the FC surface, one can finally generate the
proper 90% C.L.. The fully corrected 90% C.L. limit on the sterile antineutrinos
in comparison with other experiments is shown in figure 7.16. This result covers
almost seven orders of magnitude in the $\Delta \bar{m}^2_{41}$ parameter space. The limit is very
strong and competitive with other experiments, particularly in the region where
$\Delta \bar{m}^2_{41} > 10 \text{ eV}^2$. As shown in figures 7.4, 7.5, 7.6 and 7.7 this is largely due to
the fact that the systematic uncertainties after decorrelation is small in the near
detector energy spectrum. For the region where $\Delta \bar{m}^2_{41} < 1 \text{ eV}^2$, our limit is again
very strong, with no deviation from three-flavour oscillation model is observed.

Using the analysis method described in this chapter, we have set a robust 90%
C.L. exclusion. There are more questions towards the relationships between the
Asimov sensitivity, fluctuated sensitivity band and data limits.
Figure 7.14: An example case where $\Delta\chi^2_{FC}$ is calculated from 197 different fluctuated experiments generated using the covariance matrix at $\Delta m^2_{21} = 4.8 \text{ eV}^2$, $\theta_{24} = 0.17 \text{ rad}$. The red line marks the 4.61 value expected for the two-side 90% C.L.. The blue line marks the true $\Delta\chi^2_{FC}$ value needed to cover 90% of pseudo-experiments.
Figure 7.15: The Feldman-Cousins correction generated at each point in the $\Delta m^2_{41}$ and $\theta_{24}$ spaces, where the color indicates true 90\% C.L. corresponded $\Delta \chi^2$. 

\[ \theta_{24} \]
Figure 7.16: The black contour is the Feldman-Cousin corrected 90% C.L. in comparison with CCFR [46] and SciBooNE + MiniBooNE [47] result. The CCFR and SciBooNE + MiniBooNE results does presented here are only shown up to 100 eV$^2$. 
Chapter 8

Phenomenological Study of the Sensitivity

The result presented in the previous chapter raises questions about the relationships between the Asimov sensitivity, the contour from data and the fluctuated sensitivity band. In this chapter, I address these questions with a phenomenological approach by invoking toy MC, which can be used to describe the relationships between Asimov and fluctuated sensitivities.

The systematic uncertainties assessed in chapter 6 are incorporated through a relative covariance matrix, thus the covariance matrix needs to be scaled according to the four-flavour prediction at every four-flavour grid point. Alternatively, one could use an absolute covariance matrix scaled according to three-flavour prediction and using this fixed matrix for every four-flavour point.

The first part of this chapter uses toy MC to compare the performance of the relative covariance matrix and absolute covariance matrix. The second part of the chapter presents are bias tests, which first ensure the MINOS fit framework is robust, and then shows the toy MC does reflect the behaviour of the MINOS fit framework.

8.1 Toy Studies of Asimov Sensitivity

It is very time-consuming to perform fits with the full MINOS framework. Toy MC is therefore established to allow studies of the behaviour of the fitting framework to be carried out.

The dominant discrepancies between the Asimov sensitivity and fluctuated
8.1. **TOY STUDIES OF ASIMOV SENSITIVITY**

sensitivity band are observed in the $\Delta m^2_{41} > 100 \text{eV}^2$ region, where the near detector energy spectrum would only observe a normalisation effect if a sterile neutrino exists. One can use the simple toy MC setup to test this effect by setting $\Delta m^2_{41} = 1000 \text{eV}^2$. Here we are only looking at the disappearance channel for the near detector: the four-flavour energy spectrum in this case is just a downward normalisation-shifted three-flavour energy spectrum. Assume we have a three-flavour energy spectrum $\text{MC}_3$, a four-flavour energy spectrum $\text{MC}_4$ and a fluctuated energy spectrum $\text{MC}_{Fluc}$ which uses the fluctuation method as described in section 7.6. Instead of computing the $\Delta \chi^2$ between the test point and the global minimum, we can approximate this by calculating the $\Delta \chi^2$ between this test point and the three-flavour point. Then we can compute $\Delta \chi^2_{Asimov}$ using the equation below:

$$\Delta \chi^2_{Asimov} = \chi_{\text{MC}_3}^2 - \chi_{\text{MC}_4}^2 - \chi_{\text{MC}_3}^2.$$ \hspace{1cm} (8.1)

We can also compute the $\Delta \chi^2_{Fluc}$ using the expression below:

$$\Delta \chi^2_{Fluc} = \chi_{\text{MC}_{Fluc}}^2 - \chi_{\text{MC}_4}^2 - \chi_{\text{MC}_{Fluc}}^2 - \chi_{\text{MC}_3}^2.$$ \hspace{1cm} (8.2)

The values obtained from above two equations will be good indications of the relationships between the Asimov sensitivity and fluctuated sensitivity band. Several observations can be made directly using this setup: the $\Delta \chi^2_{Asimov}$ by definition is positive. If $\Delta \chi^2_{Fluc} > 0$ the fluctuated energy spectrum is a better fit to the three-flavour energy spectrum than the four-flavour energy spectrum.

### 8.1.1 Toy MC setup

The three-flavour energy spectrum used for toy MC is just a 20-bin energy spectrum from 0–40 GeV with uniform bin width. The energy spectrum is peaked at 5 GeV to simulate the MINOS near detector MC energy spectrum. The four-flavour energy spectrum as described above is just a constant downward-shifted three-flavour energy spectrum as we are only looking at $\Delta m^2_{41} = 1000 \text{eV}^2$. For each case, 10000 fluctuated energy spectra are generated using the method mentioned in section 7.6. This would generate a $\Delta \chi^2_{Fluc}$ distribution which will be representative of our fluctuated sensitivity band. The relationships between the
8.1.2 Use of Absolute Covariance Matrix

As mentioned in the last chapter, the covariance matrices used for calculating the $\Delta \chi^2$ and generating fluctuated energy spectra are identical. For this part of the test, a 90% bin-to-bin correlated covariance matrix with 15% errors on each bin is firstly scaled with the three-flavour energy spectrum to obtain an absolute covariance matrix. This matrix is then decomposed and used to generate 10000 fluctuated energy spectra using the method mentioned in section 7.6. This matrix is then used to calculate both $\Delta \chi^2_{\text{Asimov}}$ and $\Delta \chi^2_{\text{Fluc}}$ using the equation 8.1 and 8.2. One can plot the $\Delta \chi^2_{\text{Asimov}}$ value and $\Delta \chi^2_{\text{Fluc}}$ distribution for 10000 different
8.1. TOY STUDIES OF ASIMOV SENSITIVITY

Figure 8.2: The blue distribution is the $\Delta \chi^2_{\text{Fluc}}$ distribution calculated by using the 10000 fluctuated energy spectra as fake data input; the red dashed line is the $\Delta \chi^2_{\text{Asimov}}$ calculated by using the three-flavour energy spectrum as fake data input. Here we use the same absolute covariance matrix for both MC$^3$ and MC$^4$.

fluctuations as shown in figure 8.2. The $\Delta \chi^2_{\text{Asimov}}$ overlaps with the median of the $\Delta \chi^2_{\text{Fluc}}$ distribution which suggests the biases observed in the last chapter are not present in this case.

8.1.3 Use of Relative Covariance Matrix

One can also perform a similar toy MC study with the relative covariance matrix. Again a 90% bin-to-bin correlated covariance matrix with 15% errors on each bin is used. In this case the matrix is scaled with the four-flavour energy spectrum. This matrix is again used to generate 10000 fluctuated energy spectra and to
calculate both $\Delta \chi^2_{\text{Asimov}}$ and $\Delta \chi^2_{\text{Fluc}}$. A $\Delta \chi^2$ graph as shown in figure 8.3, is made for comparison with figure 8.2 from the previous section. It is clear that in this case $\Delta \chi^2_{\text{Asimov}}$ does not overlap with the median of the $\Delta \chi^2_{\text{Fluc}}$ distribution. If toy MC is a true representation of the real fit and one is constructing a two-sided 90% fluctuated sensitivity band using this matrix, we would expect the Asimov sensitivity to fall outside of the fluctuated sensitivity band. This suggests the relative covariance matrix is the main cause of the bias observed we made in the previous chapter.
8.2 Impact of Biases in the Covariance Matrix

Before one can reach any conclusion, it is important to test if the MINOS fit framework is robust and if the toy MC test is a true reflection of the behaviour of the MINOS fit framework. One can perform similar tests with the toy MC using biased inputs to test this.

8.2.1 Mock Data study of the Asimov Sensitivity

To ensure the MINOS fit framework is robust, one has to make sure that it is actually capable of identifying any sterile neutrino signal when one is presented. This is achieved by performing a Mock Data study, where fake data is generated using sterile oscillation parameters at points in the Asimov excluded region, i.e. to which our data should be sensitive. If the framework is able to return the injection point as the best fit point, this would be a strong indication that the framework is indeed capable of identifying different 3+1 sterile oscillation signals within the claimed region of sensitivity. In this thesis, three points of interest have been investigated: a high sterile mass splitting point where $\theta_{24} = 0.4$, $\Delta m^2_{41} = 100 \text{ eV}^2$, labelled as point 1; a medium sterile mass splitting point where $\theta_{24} = 0.3$, $\Delta m^2_{41} = 1 \text{ eV}^2$, labelled as point 2; and a low sterile mass splitting point where $\theta_{24} = 0.7$, $\Delta m^2_{41} = 0.02 \text{ eV}^2$, labelled as point 3. As shown in the right figure in figure 8.4, these points span the full range of the Asimov sensitivity space.

The bottom three figures in figure 8.4 show the different recovered best fit points and 90% C.L. allowed regions for each individual case. The recovered best fit points overlap nicely with the injected sterile oscillation parameters, which provides us with confidence in the framework.

8.2.2 Comparison with MINOS Fit Framework

With the confidence in the MINOS fit framework, we can start to compare the toy MC behaviour with the behaviour of the MINOS fit framework. With the exact same toy MC setup as described in section 8.1.1, three sets of different relative covariance matrices are used and compared with the results from the MINOS fit framework. These matrices are 10%, 50% and 90% bin-to-bin correlated relative covariance matrices with 15% errors on each bin. The MINOS fit only uses the near detector energy spectrum from the low-energy $\nu_\mu$-dominated $\bar{\nu}_\mu$ sample, with
Figure 8.4: Allowed regions for different Mock Data studies. The top graph shows the location of the different injection points. The bottom three graphs shows the recovered minimum points and allowed regions for the three different injection points.
the same relative covariance matrices used in the fit as are used to generate the toy MC. Due to the limited computing power, only 30 fluctuated energy spectra are used in each case. Figure 8.5 shows one-to-one comparisons between the toy MC and the fit results from the MINOS framework. From top row to bottom row both toy MC and fit results from MINOS framework show similar behaviour, especially when one is comparing the Asimov sensitivity and the fluctuated sensitivities for $\Delta m_{41}^2 \approx 1000 \text{ eV}^2$. An independent test has been carried out in [127] which confirms the toy MC finding using an absolute covariance matrix. Figure 8.5 also suggests that increasing the level of correlation will impact the relationship between the Asimov sensitivity and the fluctuated sensitivity band.

### 8.2.3 Impact of Biases and Correlations in the Relative Covariance Matrices

As mentioned in section 7.6, in MINOS the fluctuated energy spectra are generated with the same covariance matrix that is used for calculating both $\Delta \chi^2_{\text{Fluc}}$ and $\Delta \chi^2_{\text{Asimov}}$ values. Changing the correlation in the covariance matrix that is used to generate the fluctuated energy spectra could result in a change in the $\Delta \chi^2_{\text{Fluc}}$ distribution. As also shown in the section 8.2.2, changing the correlation in the covariance matrix that is used to calculate both $\Delta \chi^2_{\text{Fluc}}$ and $\Delta \chi^2_{\text{Asimov}}$ affects both the $\Delta \chi^2_{\text{Fluc}}$ distribution and the $\Delta \chi^2_{\text{Asimov}}$ value. Therefore we can separate the covariance matrices that are used to generate the fluctuated energy spectra and to calculate both the $\Delta \chi^2_{\text{Fluc}}$ distribution and the $\Delta \chi^2_{\text{Asimov}}$ value to study the impact of the biased relative covariance input.

**Fixing the matrix used to calculate $\chi^2$ value** For this part of test, we use a 10% bin-to-bin correlated relative covariance matrix with 15% errors on each bin to calculate the $\Delta \chi^2_{\text{Fluc}}$ distribution and the $\Delta \chi^2_{\text{Asimov}}$ value. As shown in figure 8.6 the Asimov sensitivity is still within the central region of the fluctuated sensitivity band, and the toy MC again captures the fit result from the MINOS framework. This suggests that when biases occurred while generating fluctuated energy spectra, this is insufficient to cause the Asimov sensitivity to fall outside the fluctuated sensitivity band.

**Fixing the matrix used to generate fluctuated energy spectra.** A similar study can be performed by keeping the relative covariance matrix that is used
Figure 8.5: The left three graphs show a set of the $\Delta \chi^2_{\text{Fluc}}$ distributions and the $\Delta \chi^2_{\text{Asimov}}$ values made with the toy MC setup. The blue distribution is the $\Delta \chi^2_{\text{Fluc}}$ distribution calculated using 10000 fluctuated energy spectra as fake data input; the red dashed line is the $\Delta \chi^2_{\text{Asimov}}$ calculated using the three-flavour energy spectrum as fake data input. The right three graphs are results from the MINOS fit framework using the near detector CC $\nu_\mu$ sample from the $\nu_\mu$-dominated beam. The 30 blue contours are generated by using the fluctuated energy spectra as fake data input, and the red contours are generated by using the three-flavour energy spectrum as fake data input. From top to bottom, the bin-to-bin correlation in the relative covariance matrices used to generate the fluctuated energy spectra and to calculate the $\chi^2$ are 10%, 50% and 90% respectively.
8.2. IMPACT OF BIASES IN THE COVARIANCE MATRIX

Figure 8.6: The left three graphs show a set of the $\Delta \chi^2_{\text{Fluc}}$ distributions and the $\Delta \chi^2_{\text{Asimov}}$ values made with the toy MC setup. The blue distribution is the $\Delta \chi^2_{\text{Fluc}}$ calculated using 10000 fluctuated energy spectra as fake data input; the red dashed line is the $\Delta \chi^2_{\text{Asimov}}$ calculated using the three-flavour energy spectrum as fake data input. The right three graphs are results from the MINOS fit framework using the near detector CC $\bar{\nu}_\mu$ sample from the $\nu_\mu$-dominated beam. The 30 blue contours are generated by using the fluctuated energy spectra as fake data input, and the red contours are generated by using the three-flavour energy spectrum as fake data input. The $\chi^2$ are calculated with 10% bin-to-bin correlated covariance matrix with 15% errors on each bin. From top to bottom the fluctuated energy spectra are generated using 90%, 50% and 10% bin-to-bin correlated covariance matrices with 15% uncertainty on each bin.
to generate the fluctuated energy spectra fixed. Figure 8.7 shows one-to-one comparisons between the toy MC and the fit result from the MINOS framework. Again toy MC captures the trend in the real fit results. This suggests that when biases are present in the relative covariance matrix that is used to calculate the $\Delta \chi^2$, one will also observe the Asimov sensitivity to fall outside the fluctuated sensitivity band. This would lead to similar observations as we made in chapter 7 but with a different cause.

### 8.2.4 Impact of the the Correlations

Summarising the observations above, but focusing on the impact of the correlations, table 8.1 shows how correlation in relative covariance matrices impacts the relationships between the $\Delta \chi^2_{Fluc}$ distribution and the $\Delta \chi^2_{Asimov}$ value. Here, the correlation in the covariance matrix that is used to generate the fluctuated energy spectra would intuitively impact the shape of the fluctuated energy spectrum. The correlation in the covariance matrix that is used to calculate $\chi^2$ values would affect the preferred shape of the uncertainties that are mentioned in section 7.3. The balance between these two factors will affect the detailed relationships between the fluctuated sensitivity band and the Asimov sensitivity; one would therefore need a more theoretical approach to understand this process completely. However, the observation made in this chapter suggests it is common for the disappearance channel to result in an Asimov sensitivity that does not overlap with the median of the fluctuated sensitivity band.

### 8.3 Discussion

Summarising all the observations above, the fact that the Asimov sensitivity is not covered by the fluctuated sensitivity band is caused by the relative covariance matrix. (In the relative covariance matrix the absolute values of the covariance matrix scale with the MC prediction at each point in four-flavour parameter space). The toy MC studies also suggest that this is a common property of any disappearance search. The behaviour of these sensitivity contours requires us to rethink how to correctly represent the experiment sensitivity.

The Asimov sensitivity is generated by using the three-flavour energy spectrum as the fake data. This would be an ideal energy spectrum if our data were not subject to any fluctuations caused by systematic or statistical effects. The
8.3. DISCUSSION

Figure 8.7: The left three graphs show a set of the $\Delta \chi^2_{Fluc}$ distributions and the $\Delta \chi^2_{Asimov}$ values made with the toy MC setup. The blue distribution is the $\Delta \chi^2_{Fluc}$ distribution calculated using 10000 fluctuated energy spectra as fake data input; the red dashed line is the $\Delta \chi^2_{Asimov}$ calculated using the three-flavour energy spectrum as fake data input. The right three graphs are results from the MINOS fit framework using the near detector CC $\bar{\nu}_\mu$ sample from the $\nu_\mu$-dominated beam. The blue contours are generated by using the fluctuated energy spectra as fake data input, and the red contour are generated using the three-flavour energy spectrum as fake data input. The fluctuated energy spectra are generated using a 10% bin-to-bin correlated relative covariance matrix with 15% errors on each bin. From top to bottom the $\Delta \chi^2$ are calculated using 10%, 50% and 90% bin-to-bin correlated covariance matrices with 15% errors on each bin.
Fluctuated energy spectra are 10%, 50% and 90% bin-to-bin correlated with 15% uncertainty per bin. From left to right the covariance matrices that are used to calculate the theoretical energy spectra as fake data input. From top to bottom the covariance matrices that are used to calculate the ∆χ² distribution calculated using 10000 fluctuated energy spectra as fake data input. The red dashed line is the ∆χ² calculated using 4flavour-3flavour values made with the MC setup. The blue distribution is the ∆χ² distribution calculated using 4flavour-3flavour values made with the MC setup.

Table 8.1: Each graph shows the ∆χ² distributions and the ∆χ² distribution calculated using 10000 fluctuated energy spectra as fake data input. The red dashed line is the ∆χ² calculated using 4flavour-3flavour values made with the MC setup. The blue distribution is the ∆χ² distribution calculated using 4flavour-3flavour values made with the MC setup.

<table>
<thead>
<tr>
<th>Correlation for Energy Fluctuation</th>
<th>90%</th>
<th>50%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

calculated ∆χ²
fluctuated sensitivity band is generated by using the fluctuated energy spectra as the fake data. These are a more realistic set of energy spectra containing the systematic and statistical fluctuations. The data limit is generated by using the data energy spectrum which is likely to be impacted by the systematic and statistical fluctuations we studied in chapter 6. Under the assumption mentioned above, it is very likely for the data limit to be more consistent with the fluctuated sensitivity band, and this is indeed what we observed. Summarising all the observations from chapter 7 and this chapter, for our purpose the Asimov sensitivity does not represent true sensitivity of an experiment [115]. [128] has also suggested that fluctuated sensitivity bands and median sensitivity contours should be used to represent true experimental sensitivities.

8.3.1 Limitations of the Toy MC

This toy MC study has its own limitations. The point of this test is to provide a quick, easy and robust test tool for detailed examination of the relationships between the fluctuated sensitivity band and Asimov sensitivity. However, it has simplified many problems. First of all, this is not a fit as we are not profiling over many nuisance parameters to find the minimum $\chi^2$ at a particular point in sterile parameter space. These nuisance parameters, as shown in previous chapter, might have a small impact on the relationships between the Asimov sensitivity and the fluctuated sensitivity band.

The actual fit calculates the $\Delta \chi^2$ by subtracting the global minimum $\chi^2$ from the $\chi^2$ value at each point in sterile parameter space. The toy MC setup only subtracts the $\chi^2$ value at the three-flavour point, which is closer to the raster scan method.

Nevertheless, bearing these differences in mind, the toy MC does provide a good indication of the behaviour of the real MINOS real fit. This also suggests the relationships between the Asimov sensitivity and the fluctuated sensitivity band are in fact governed by underlying mathematics; the observations made in section 7.6 are indeed reasonable and valid.
Chapter 9

Conclusion and Outlook

This thesis presents the first MINOS and MINOS+ sterile antineutrino search performed using the antineutrino sample collected from all of the MINOS and part of the MINOS+ runs. Throughout this thesis I have demonstrated the robustness of the analysis methods and performed studies of the statistical procedure to show that the strong claims of the exclusion contour are indeed valid. The final chapter, which explores the sensitivity and limit setting, will benefit the future experiments. There is also a huge amount of potential work that could have been carried out which will help to consolidate and improve the result.

To begin with, the horn tilt issue is continuing to be explored during the write up of this thesis. Once the issue is resolved, and the correction is available, this would allow more data, the Run 13 data, to be included in the analysis.

For most of the samples, the neutrino background is still one of the dominant backgrounds. The PPFX package has provided flux predictions for both neutrinos and antineutrinos. One could treat the systematics on the samples more carefully by including the corrections and uncertainties for both signal antineutrino and background neutrinos separately. This would provide a much more accurate statement of the flux and hadron production corrections and uncertainties.

The study of the relationship between the sensitivity band and Asimov sensitivities is fundamental to the statistical interpretation of the physics measurement. The phenomenological study performed in this thesis is just a first attempt to address this issue. Since other groups and experiments have also observed similar effects, it would be productive to see a collaborative investigation of these issues, including statistics experts.
An study has been performed while this thesis was being completed that suggests the horn-one mis-calibration parameters could be very different from those previously calculated. These would be potentially contribute to the slight shift of the MC peak location in the neutrino samples. For the antineutrinos this would manifest as normalisation shifts. These would provide a further improvement to the Data-MC agreement.

In summary, this thesis has presented an exclusion of the existence of sterile antineutrinos, covering seven orders of magnitude in $\Delta m^2_{41}$ parameter space. This is in conflict with observations from LSND and MiniBooNE, however the comparison of the disappearance measurements presented here and the appearance signals presented by LSND and MiniBooNE require model assumptions. It will be very exciting to see further appearance searches that come out of projects such as the Fermilab Short-Baseline Programme.
Appendix A

Two-flavour Neutrino Oscillation Approximation

For $\nu_e$ disappearance search in short-baseline neutrino experiments, beginning with equation 2.10, one could have:

$$P(\nu_e \to \nu_e) = P(\bar{\nu}_e \to \bar{\nu}_e) = 1 - 4s_{12}^2 c_{13}^2 c_{12}^2 s_{13}^2 \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E}\right)$$

$$- 4s_{13}^2 s_{12}^2 \sin^2 \left(\frac{\Delta m_{32}^2 L}{4E}\right)$$

$$- 4s_{13}^2 c_{13}^2 c_{12}^2 \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E}\right)$$

$$= 1 - 4s_{13}^2 c_{13}^2 (s_{12}^2 + c_{12}^2) \sin^2 \left(\frac{\Delta m_{32}^2 L}{4E}\right)$$

$$= 1 - \sin^2(2\theta_{13}) \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E}\right), \quad (A.1)$$

this suggests these experiments are very sensitive to $\theta_{13}$ and $\Delta m_{31}^2$. For $\nu_e$ appearance search with accelerator experiment using $\nu_\mu$ beam, ignoring the CP-violating phase one could have:

$$P(\nu_\mu \to \nu_e) \approx -4s_{23}^2 c_{13}^2 c_{12} (s_{12} c_{23} - c_{12} s_{23} s_{13})$$

$$- 4s_{23}^2 s_{12}^2 (c_{12} c_{23} - s_{12} s_{23} s_{13}) \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E}\right)$$

$$= 4s_{13}^2 s_{23}^2 c_{13} (c_{12}^2 + s_{12}^2) \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E}\right)$$

$$= \sin^2(2\theta_{13}) \sin^2(\theta_{23}) \sin^2 \left(\frac{\Delta m_{31}^2 L}{4E}\right), \quad (A.2)$$
this suggests these experiments are sensitive to $\theta_{13}$, $\theta_{23}$ and $\Delta m^{2}_{31}$, it is particularly sensitive to $\theta_{23}$ octant. For $\nu_{\tau}$ appearance search with accelerator experiment using $\nu_{\mu}$ beam, ignoring the CP-violating phase one could have:

\[
P(\nu_{\mu} \rightarrow \nu_{\tau}) \approx -4(s_{23}c_{23}c_{13}(s_{12}s_{23} - c_{12}c_{23}s_{13})(-s_{12}c_{23} - c_{12}s_{23}s_{13}) \nonumber \\
+ s_{23}c_{23}c_{13}(-c_{12}s_{23} - s_{12}c_{23}s_{13})(c_{12}c_{23} - s_{12}s_{23}s_{13})) \sin^{2}(\frac{\Delta m^{2}_{31}L}{4E}) \nonumber \\
= -4s_{23}c_{23}c_{13}(c_{12}^{2}c_{23}s_{23} + s_{12}s_{13}s_{23}c_{12} - c_{12}s_{12}c_{23}s_{13} + s_{12}^{2}s_{13}c_{23}s_{23} \nonumber \\
- s_{12}^{2}c_{23}s_{23} + c_{12}c_{23}c_{12}s_{13} - c_{12}s^{2}_{23}s_{13}s_{12} + c_{12}s^{2}_{13}c_{23}s_{23}) \sin^{2}(\frac{\Delta m^{2}_{31}L}{4E}) \nonumber \\
= 4s_{23}c_{23}c_{13}(c_{23}s_{23} - s_{12}^{2}s_{23}c_{23}) \nonumber \\
= 4s_{23}^{2}c_{23}c_{13}(1 - s_{13}^{2}) \nonumber \\
= 4s_{23}^{2}c_{23}c_{13}^{4} \nonumber \\
= \cos^{4}(\theta_{13}) \sin^{2}(2\theta_{23}) \sin^{2}(\frac{\Delta m^{2}_{31}L}{4E}). \quad (A.3)
\]

this suggests these experiments are sensitive to $\theta_{13}$, $\theta_{23}$ and $\Delta m^{2}_{31}$. For $\nu_{\mu}$ disappearance search with long-baseline experiment, one could have:

\[
P(\nu_{\mu} \rightarrow \nu_{\mu}) = 1 - 4s_{23}^{2}c_{13}(c_{12}c_{23} - s_{12}s_{23}s_{13})(c_{12}c_{23} - s_{12}s_{23}s_{13}) \nonumber \\
+ (-s_{12}c_{23} - c_{12}s_{23}s_{13})(-s_{12}c_{23} - c_{12}s_{23}s_{13})) \sin^{2}(\frac{\Delta m^{2}_{31}L}{4E}) \nonumber \\
= 1 - 4s_{23}^{2}c_{13}(c_{12}^{2}c_{23} + s_{12}^{2}s_{23}^{2}s_{13} - 2c_{12}c_{23}s_{12}c_{23}s_{13} \nonumber \\
+ s_{12}^{2}c_{23}^{2} + c_{12}^{2}s_{23}^{2}s_{13}^{2} + 2c_{12}c_{23}s_{12}s_{23}s_{13}) \nonumber \\
= 1 - \cos^{2}(\theta_{13}) \sin^{2}(2\theta_{23}) \sin^{2}(\frac{\Delta m^{2}_{31}L}{4E}) \nonumber \\
+ \sin^{4}(\theta_{23}) \sin^{2}(2\theta_{13}) \sin^{2}(\frac{\Delta m^{2}_{31}L}{4E}) \nonumber \\
\approx 1 - \sin^{2}(2\theta_{23}) \sin^{2}(\frac{\Delta m^{2}_{31}L}{4E}), \quad (A.4)
\]

this suggests these experiments are mainly sensitive to $\theta_{23}$ and $\Delta m^{2}_{31}$.
Appendix B

3+1 Neutrino Oscillation Approximation

As we are introducing an additional neutrino flavour, the PMNS matrix now are 4×4 form:

\[
U_4 = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} & U_{e4} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\
U_{s 1} & U_{s 2} & U_{s 3} & U_{s 4}
\end{pmatrix}.
\]

Each amplitude element is a function of the mixing angles and phases the electron flavours are:

\[
\begin{align*}
U_{e1} &= \cos \theta_{12} \cos \theta_{13} \cos \theta_{14}, \\
U_{e2} &= \cos \theta_{13} \cos \theta_{14} \cos \theta_{12}, \\
U_{e3} &= \cos \theta_{14} \sin \theta_{13} e^{-i\delta_{13}}, \\
U_{e4} &= \sin \theta_{14} e^{-i\delta_{14}}.
\end{align*}
\]
The muon flavours are:

\[
U_{\mu 1} = - \cos \theta_{12} \cos \theta_{24} \sin \theta_{13} \sin \theta_{23} e^{-i\delta_{13}} \\
- \cos \theta_{12} \cos \theta_{13} \cos \theta_{14} \cos \theta_{24} e^{-i(\delta_{24} - \delta_{14})} \\
- \cos \theta_{23} \cos \theta_{24} \sin \theta_{12},
\]

\[
U_{\mu 2} = \cos \theta_{12} \cos \theta_{23} \cos \theta_{24} \\
- \sin \theta_{12} \cos \theta_{24} \sin \theta_{13} \sin \theta_{23} e^{i\delta_{13}} \\
- \sin \theta_{12} \cos \theta_{13} \sin \theta_{14} \cos \theta_{24} e^{-i(\delta_{24} - \delta_{14})},
\]

\[
U_{\mu 3} = \cos \theta_{13} \cos \theta_{24} \sin \theta_{23} - \sin \theta_{13} \cos \theta_{14} \cos \theta_{24} e^{-i(\delta_{13} - \delta_{14} + \delta_{24})},
\]

\[
U_{\mu 4} = \cos \theta_{14} \sin \theta_{24} e^{-i\delta_{24}}.
\]

The tau flavours parts:

\[
U_{\tau 1} = \sin \theta_{12} \cos \theta_{34} \sin \theta_{23} + \sin \theta_{12} \cos \theta_{23} \sin \theta_{24} \sin \theta_{34} e^{i\delta_{24}} \\
- \cos \theta_{12} \cos \theta_{13} \cos \theta_{24} \sin \theta_{14} \sin \theta_{34} e^{i\delta_{14}} \\
- \cos \theta_{12} \sin \theta_{13} \cos \theta_{23} \cos \theta_{34} e^{i\delta_{13}} \\
+ \cos \theta_{12} \sin \theta_{13} \sin \theta_{23} \sin \theta_{24} \sin \theta_{34} e^{i(\delta_{13} + \delta_{24})},
\]

\[
U_{\tau 2} = \cos \theta_{12} \cos \theta_{34} \sin \theta_{23} + \sin \theta_{12} \cos \theta_{23} \sin \theta_{24} \sin \theta_{34} e^{i\delta_{24}} \\
- \sin \theta_{12} \cos \theta_{13} \cos \theta_{24} \sin \theta_{14} \sin \theta_{34} e^{i\delta_{14}} \\
- \sin \theta_{12} \sin \theta_{13} \cos \theta_{23} \cos \theta_{34} e^{i\delta_{13}} \\
+ \sin \theta_{12} \sin \theta_{13} \sin \theta_{23} \sin \theta_{24} \sin \theta_{34} e^{i(\delta_{13} + \delta_{24})},
\]

\[
U_{\tau 3} = \cos \theta_{24} \sin \theta_{13} \sin \theta_{14} \sin \theta_{34} e^{i(\delta_{13} - \delta_{14})} \\
+ \cos \theta_{13} \cos \theta_{23} \cos \theta_{34} \\
- \cos \theta_{13} \sin \theta_{23} \sin \theta_{24} \sin \theta_{34},
\]

\[
U_{\tau 4} = \cos \theta_{14} \cos \theta_{24} \sin \theta_{34}.
\]
The sterile flavours in particular are:

\[ U_{s1} = -\sin \theta_{12} \sin \theta_{34} \sin \theta_{23} + \sin \theta_{12} \cos \theta_{23} \cos \theta_{34} \sin \theta_{24} e^{i \delta_{24}} \]
\[ - \cos \theta_{12} \cos \theta_{13} \cos \theta_{24} \cos \theta_{34} \sin \theta_{14} e^{i \delta_{24}} \]
\[ + \cos \theta_{12} \sin \theta_{13} \cos \theta_{23} \sin \theta_{34} e^{i \delta_{13}} \]
\[ + \cos \theta_{12} \sin \theta_{13} \cos \theta_{34} \sin \theta_{23} \sin \theta_{24} e^{i (\delta_{13} + \delta_{24})}, \]

\[ U_{s2} = \cos \theta_{12} \sin \theta_{34} \sin \theta_{23} - \cos \theta_{12} \cos \theta_{23} \cos \theta_{34} \sin \theta_{24} e^{i \delta_{24}} \]
\[ - \sin \theta_{12} \cos \theta_{13} \cos \theta_{24} \cos \theta_{34} \sin \theta_{14} e^{i \delta_{24}} \]
\[ + \sin \theta_{12} \sin \theta_{13} \cos \theta_{23} \sin \theta_{34} e^{i \delta_{13}} \]
\[ + \sin \theta_{12} \sin \theta_{13} \cos \theta_{34} \sin \theta_{23} \sin \theta_{24} e^{i (\delta_{13} + \delta_{24})}, \]

\[ U_{s3} = - \cos \theta_{24} \cos \theta_{34} \sin \theta_{13} \sin \theta_{14} e^{-i (\delta_{13} - \delta_{14})} \]
\[ - \cos \theta_{13} \cos \theta_{23} \sin \theta_{34} \]
\[ - \cos \theta_{13} \sin \theta_{23} \sin \theta_{24} \cos \theta_{34}, \]

\[ U_{s4} = \cos \theta_{14} \cos \theta_{24} \sin \theta_{34}, \]

using this matrix together with equation 2.10 one can make a similar approximation for different channel.

### B.1 \( \nu_\mu \to \nu_e \) Channel

For the \( \nu_\mu \to \nu_e \) channel using equation 2.10:

\[
P(\nu_\mu \to \nu_e) = -4 \sum_{i>j} \Re \left( U_{\mu i}^* U_{ei} U_{\mu j}^* U_{ej} \right) \sin^2 \left( \frac{\Delta m_{i j}^2 L}{4E} \right) + 2 \sum_{i>j} \Im \left( U_{\mu i}^* U_{ei} U_{\mu j}^* U_{ej} \right) \sin \left( \frac{\Delta m_{i j}^2 L}{2E} \right)
\]
where the real part become:

\[
\sum_{i>j} \Re \left( U_{\mu i}^* U_{\mu j} U_{\mu j}^* U_{\nu e j}^* \right) \sin^2 \left( \frac{\Delta m_{ij}^2 L}{4E} \right) \\
= \Re \left( U_{\mu 1}^* U_{\nu e 1} U_{\mu 2} U_{\nu e 2}^* \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 1}^* U_{\nu e 1} U_{\mu 3} U_{\nu e 3}^* \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 2}^* U_{\nu e 2} U_{\mu 3} U_{\nu e 3}^* \right) \sin^2 \left( \frac{\Delta m_{32}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 1}^* U_{\nu e 1} U_{\mu 4} U_{\nu e 4}^* \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 2}^* U_{\nu e 2} U_{\mu 4} U_{\nu e 4}^* \right) \sin^2 \left( \frac{\Delta m_{42}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 3}^* U_{\nu e 3} U_{\mu 4} U_{\nu e 4}^* \right) \sin^2 \left( \frac{\Delta m_{33}^2 L}{4E} \right). 
\]

For a sterile signal suggested by the LSND and MiniBooNE experiment, \( \Delta m^2 \gg 1 \text{ eV}^2 \) one could make an approximation such that \( \Delta m_{41}^2 \approx \Delta m_{42}^2 \approx \Delta_{43} \), and the \( L/E \) one can ignore all the other mass splitting terms involving \( \Delta m_{31}^2 \), \( \Delta m_{21}^2 \) and \( \Delta m_{32}^2 \) therefore the above equation can be simplified to:

\[
\sum_{i>j} \Re \left( U_{\mu i}^* U_{\nu e i} U_{\mu j} U_{\nu e j}^* \right) \sin^2 \left( \frac{\Delta m_{ij}^2 L}{4E} \right) \\
\approx \left( U_{\mu 1}^* U_{\nu e 1} U_{\mu 4} U_{\nu e 4}^* \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 2}^* U_{\nu e 2} U_{\mu 4} U_{\nu e 4}^* \right) \sin^2 \left( \frac{\Delta m_{42}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 3}^* U_{\nu e 3} U_{\mu 4} U_{\nu e 4}^* \right) \sin^2 \left( \frac{\Delta m_{33}^2 L}{4E} \right) \\
= \Re \left( U_{\mu 1}^* U_{\nu e 1} \left( U_{\nu e 1}^* U_{\mu 1} + U_{\nu e 2}^* U_{\mu 2} + U_{\nu e 3}^* U_{\mu 3} \right) \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
= -\Re \left( U_{\mu 1}^* U_{\nu e 4} U_{\nu e 4}^* U_{\mu 4} \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
= -|U_{\mu 4}|^2 |U_{\nu e 4}|^2 \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right). 
\]
Similarly one can also workout the imaginary part with same approximation which yields:

\[
\sum_{i>j} \Im \left( U^*_{\mu i} U_{\alpha j} U^*_{\mu j} U_{\alpha i} \right) \sin \left( \frac{\Delta m^2_{ij} L}{2E} \right) = \Im \left( |U^*_{\mu 4}|^2 |U_{e 4}|^2 \right) \sin \left( \frac{\Delta m^2_{41} L}{2E} \right) = 0.
\]

For the $\nu_\mu \rightarrow \nu_e$ channel with this approximation the equation effectively reduces to:

\[
P(\nu_\mu \rightarrow \nu_e) = 4|U_{\mu 4}|^2|U_{e 4}|^2 \sin^2 \left( \frac{\Delta m^2_{41} L}{4E} \right).
\]  

(B.1)

The parameter space these experiment is probing are $4|U_{\mu 4}|^2|U_{e 4}|^2$ and $\Delta m^2_{41}$ and the first term often are noted as $\sin^2 \theta^{}_{\mu e}$.

### B.2 $\nu_\mu \rightarrow \nu_\mu$ Channel

For the $\nu_\mu \rightarrow \nu_\mu$ channel using equation 2.10:

\[
P(\nu_\mu \rightarrow \nu_\mu) = 1 - 4 \sum_{i>j} \Re \left( U^*_{\mu i} U_{\mu i} U^*_{\mu j} U_{\mu j} \right) \sin^2 \left( \frac{\Delta m^2_{ij} L}{4E} \right)
\]

For a long-baseline accelerator based experiment that is looking for $\nu_\mu$ disappearance, such as MINOS CC channel. $L/E$ is too small to probe $\Delta m^2_{21}$, we can make an approximation such that $\Delta m^2_{21} \approx 0 \text{eV}^2$, $\Delta m^2_{42} \approx \Delta m^2_{41}$ and $\Delta m^2_{32} = \Delta m^2_{31}$.
the above equation can be expressed as following:

\[
P(\nu_\mu \rightarrow \nu_\mu) = 1 \\
- 4 \Re \left( U_{\mu 1}^* U_{\mu 1} U_{\mu 2} U_{\mu 2}^* \right) \sin^2 \left( \frac{\Delta m_{21}^2 L}{4E} \right) \\
- 4 \Re \left( U_{\mu 1}^* U_{\mu 1} U_{\mu 3} U_{\mu 3}^* \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
- 4 \Re \left( U_{\mu 2}^* U_{\mu 2} U_{\mu 3} U_{\mu 3}^* \right) \sin^2 \left( \frac{\Delta m_{32}^2 L}{4E} \right) \\
- 4 \Re \left( U_{\mu 1}^* U_{\mu 1} U_{\mu 4} U_{\mu 4}^* \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
- 4 \Re \left( U_{\mu 2}^* U_{\mu 2} U_{\mu 4} U_{\mu 4}^* \right) \sin^2 \left( \frac{\Delta m_{42}^2 L}{4E} \right) \\
- 4 \Re \left( U_{\mu 3}^* U_{\mu 3} U_{\mu 4} U_{\mu 4}^* \right) \sin^2 \left( \frac{\Delta m_{43}^2 L}{4E} \right) \\
= 1 \\
- 4 |U_{\mu 3}|^2 |U_{\mu 1}|^2 \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
- 4 |U_{\mu 3}|^2 |U_{\mu 2}|^2 \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
- 4 |U_{\mu 4}|^2 |U_{\mu 1}|^2 \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
- 4 |U_{\mu 4}|^2 |U_{\mu 2}|^2 \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
- 4 |U_{\mu 4}|^2 |U_{\mu 3}|^2 \sin^2 \left( \frac{\Delta m_{43}^2 L}{4E} \right) \\
= 4 |U_{\mu 3}|^2 (1 - |U_{\mu 3}|^2 - |U_{\mu 4}|^2 \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
- 4 |U_{\mu 4}|^2 (1 - |U_{\mu 3}|^2 - |U_{\mu 4}|^2 \sin^2 \left( \frac{\Delta m_{43}^2 L}{4E} \right) \\
- 4 |U_{\mu 4}|^2 |U_{\mu 3}|^2 \sin^2 \left( \frac{\Delta m_{43}^2 L}{4E} \right). \quad (B.2)
\]

Using the explicit matrix component given initially and the approximation given above, it is easy to find that the parameter space this channel probes are mainly \( \theta_{24} \), \( \theta_{14} \) and \( \Delta m_{41}^2 \) (\( \Delta m_{43}^2 \) can be converted to \( \Delta m_{41}^2 \)). As the \( \theta_{14} \) have been constrained to very small value effectively we are only probing \( \theta_{24} \) and \( \Delta m_{41}^2 \) parameters.
B.3 $\nu_\mu \rightarrow \nu_s$ Channel

For $\nu_\mu \rightarrow \nu_s$ channel, equation can be written as following:

$$P(\nu_\mu \rightarrow \nu_s) = -4 \sum_{i>j} \Re (U_{\mu i}^* U_{\mu j}^U_{\mu j}) \sin^2 \left( \frac{\Delta m_{ij}^2 L}{4E} \right)$$

$$+ 2 \sum_{i>j} \Im (U_{\mu i}^* U_{\mu j}^U_{\mu j}) \sin \left( \frac{\Delta m_{ij}^2 L}{2E} \right)$$

Again one can make an approximation assuming long-baseline accelerator experiment looking for $\nu_s$ appearance such as MINOS NC channel. We can make an approximation such that $\Delta m_{21}^2 \approx 0 \text{eV}^2$, $\Delta m_{42}^2 \approx \Delta m_{41}^2$ and $\Delta m_{32}^2 = \Delta m_{31}^2$ the real part of the equation can be expressed as following:
\[
\sum_{i>j} \Re \left( U_{\mu i}^* U_{e i} U_{\mu j}^* U_{e j}^* \right) \sin^2 \left( \frac{\Delta m_{ij}^2 L}{4E} \right) \\
= \Re \left( U_{\mu 1}^* U_{e 1} U_{\mu 3} U_{e 3}^* \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 2}^* U_{e 2} U_{\mu 3} U_{e 3}^* \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 1}^* U_{e 1} U_{\mu 4} U_{e 4}^* \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 2}^* U_{e 2} U_{\mu 4} U_{e 4}^* \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 3} U_{e 3} U_{\mu 4} U_{e 4}^* \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right)
\]

\[
= \Re \left( U_{\mu 3} U_{e 3}^* \left( U_{\mu 1}^* U_{e 1} + U_{\mu 2}^* U_{e 2} \right) \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 4} U_{e 4}^* \left( U_{\mu 1}^* U_{e 1} + U_{\mu 2}^* U_{e 2} \right) \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 3} U_{e 3} U_{\mu 4} U_{e 4}^* \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right)
\]

\[
= -\Re \left( U_{\mu 3} U_{e 3}^* \left( U_{\mu 1}^* U_{e 1} + U_{\mu 2}^* U_{e 2} \right) \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \\
- \Re \left( U_{\mu 4} U_{e 4}^* \left( U_{\mu 1}^* U_{e 1} + U_{\mu 2}^* U_{e 2} \right) \right) \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) \\
+ \Re \left( U_{\mu 3} U_{e 3} U_{\mu 4} U_{e 4}^* \right) \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right)
\]

\[
= \Re \left( U_{\mu 3} U_{e 3} U_{\mu 4} U_{e 4}^* \right) \left( \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) - \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right) - \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) \right)
\]

\[
- |U_{\mu 3}|^2 |U_{e 3}|^2 \sin^2 \left( \frac{\Delta m_{31}^2 L}{4E} \right) - |U_{\mu 4}|^2 |U_{e 4}|^2 \sin^2 \left( \frac{\Delta m_{41}^2 L}{4E} \right).
\]

The imaginary part can be expanded similarly, however, only the part without \(|U_{\mu i}|^2 |U_{\mu j}|^2\) will survive as it’s real:

\[
\sum_{i>j} \Im \left( U_{\mu i}^* U_{e i} U_{\mu j}^* U_{e j}^* \right) \sin \left( \frac{\Delta m_{ij}^2 L}{2E} \right)
\]

\[
= \Im \left( U_{\mu 3} U_{e 3} U_{\mu 4} U_{e 4}^* \right) \left( \sin^2 \left( \frac{\Delta m_{43}^2 L}{2E} \right) - \sin^2 \left( \frac{\Delta m_{41}^2 L}{2E} \right) - \sin^2 \left( \frac{\Delta m_{31}^2 L}{2E} \right) \right).
\]
Therefore the full expression can be simplified to:

\[
P(\nu_\mu \rightarrow \nu_s) = 4|U_{\mu 3}|^2|U_{s3}|^2 \sin^2\left(\frac{\Delta m^2_{31} L}{4E}\right) + 4|U_{\mu 4}|^2|U_{s4}|^2 \sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right) - 4 \Re(U^*_{\mu 3} U_{s3} U_{\mu 4} U^*_{s4}) \left(\sin^2\left(\frac{\Delta m^2_{34} L}{4E}\right) - \sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right) - \sin^2\left(\frac{\Delta m^2_{43} L}{4E}\right)\right) + 2 \Im(U^*_{\mu 3} U_{s3} U_{\mu 4} U^*_{s4}) \left(\sin^2\left(\frac{\Delta m^2_{34} L}{2E}\right) - \sin^2\left(\frac{\Delta m^2_{41} L}{2E}\right) - \sin^2\left(\frac{\Delta m^2_{43} L}{2E}\right)\right)
\]

We can also expand this equation further with the matrix element provide above. By inspecting the matrix element involved in this equation, in addition to the $\theta_{24}$ and $\Delta m^2_{41}$ parameters, it suggests this channel also have some sensitivity towards the $\theta_{34}$ mixing angle.
Bibliography


http://eprints.gla.ac.uk/1330/.


[54] LAr1-ND, ICARUS-WA104, MicroBooNE Collaboration, M. Antonello et al., *A Proposal for a Three Detector Short-Baseline Neutrino Oscillation Program in the Fermilab Booster Neutrino Beam*,


[70] D. Michael et al., *The magnetized steel and scintillator calorimeters of the MINOS experiment*, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and


[83] M. Campanella, A. Ferrari, P. R. Sala, and S. Vanini, First Calorimeter Simulation with the FLUGG Prototype,

[84] M. Campanella, A. Ferrari, P. R. Sala, and S. Vanini, Reusing Code from FLUKA and GEANT4 Geometry,

[85] R. Brun, F. Bruyant, F. Carminati, S. Giani, M. Maire, A. McPherson, G. Patrick, and L. Urban, GEANT Detector Description and Simulation Tool,


http://lss.fnal.gov/cgi-bin/find_paper.pl?thesis-2010-44.


