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Demand Side Management Using a Distributed Initialisation-free Optimisation in a Smart Grid

Yi Dong, Tianqiao Zhao, Zhengtao Ding*

School of Electrical and Electronic Engineering, University of Manchester, M13 9PL, Manchester, UK

* E-mail: zhengtao.ding@manchester.ac.uk

Abstract:

Due to the integration of the renewable generation and the distributed load that inherently uncertain and unpredictable, developing an efficient distributed management structure of such a complex system remains a challenging issue. Most of the existing works on the demand side management concentrate on the centralised methods or need a proper initialisation process; This paper proposed a demand side management strategy that can solve the optimisation problem in a distributed manner without initialisation. The objective of the designed demand management system is to maximise the social welfare of a smart grid by controlling the active power economically. The proposed optimisation strategy generates the optimal power references uses the neighbouring information while considering the local feasible constraints by using a projection operation. Furthermore, the optimisation algorithm is initialisation-free, which avoids any initialisation process when plugging-in new customers or plugging-out power units, such as demand loads, battery energy storage systems and distributed generators. Our strategy only uses the neighbouring information, so that the proposed approach is scalable and potentially applicable to large-scale smart grids. The effectiveness and scalability of the proposed algorithm are established and verified through case studies.

Nomenclature

Indices and sets

i, j	index of units
Ω	set of projection area
S_B	set of BESS units
S_C	set of controllable load units
S_D	set of demand units
S_G	set of generator units
S_R	set of renewable generator units

Variables	
$f_{B,i}$	the penalty function of i th BESS
$f_{G,i}^E$	emission penalty function of i th generator
$f_{G,i}^{O\&M}$	the operation and maintenance penalty function of i th generator
k	numbers of power generator units
λ_i, ξ_i	control variables of consensus algorithm
ϕ_i	communicated information of i th power unit between its neighbours
θ_i	the percentage of i th unit that knows the total power mismatch
C_i	the cost function of i th unit
m	numbers of power demand units
n	numbers of battery energy storage system units
p	the electricity price
$W_{B,i}$	the welfare function of i th BESS
$W_{D,i}$	the welfare function of i th demand unit
$W_{G,i}$	the welfare function of i th generator

Parameters

α_i, β_i	cost parameters of i th controllable demand unit
$\kappa_f, \kappa_c, \iota$	the fixed operation cost, the charging or discharging cost and the charging efficiency of i th BESS
a_i, b_i, c_i	cost parameters of i th generator
P_D	active power mismatch between supply and demand in a smart grid

P_i	active power output of i th power unit
P_i^{max}	maximum active power output of i th power unit
P_i^{min}	minimum active power output of i th power unit
$P_{B,i}$	active power output of i th BESS
$P_{C,i}$	active power output of i th controllable demand unit
$P_{D,i}$	consumed active power of i th demand unit
$P_{G,i}$	active power output of i th generator unit
$P_{R,i}$	active power output of i th renewable generator

1 Introduction

Due to the increasing energy demand, economic consideration and the emission concerns, the power grid is facing the challenges and opportunities of transforming the traditional power grid into a smart grid. With the integration of renewable energy resource, battery energy storage systems, and controllable loads, the power grid becomes distributed and complex [1]. The operation situation of the power grid may change frequently, and therefore the reasonable energy management strategy of the power grid is an important aspect in the smart grid research which is designed to meet the demand requirements at different time intervals and to realise the efficient operation of the smart grid [2]. Due to increasing power demand, economic consideration and emission concern, the smart grid is facing the challenges and opportunities of integrating renewable energy [1]. The percentage of energy derived from renewable sources has risen from 6.7% in 2009 to 29.4% in the UK, 2017 [3]. Due to the intermittency and unpredictability of renewable energy, the future smart grid inevitably integrates more dynamic elements. Meanwhile, the smart grid must be able to maintain the balance between the supply and demand [4, 5].

Demand Side Management (DSM) refers to all those strategies aiming at varying controllable load profiles to optimise the entire power system from the supply side to the demand side, optimising power allocation to obtain efficient and eco-friendly usage of electricity [6]. DSM can be implemented by additional equipment to reduce and shift consumption, such as BESS and smart control of EV. Unbalanced conditions resulting from the uncertain load changes and the renewable power generation affect the power quality, and may even damage customer equipment [7]. Therefore, it is crucial for DSM to have an effective and optimal strategy [8].

Typically, DSM focuses on centralised algorithms. For example, the authors in [9] propose a hierarchical control structure to maximise the economic benefits through a central controller, whereas it does not consider the inequality constraints, such as the maximum and the minimum power generation of Distributed Generation (DG) and the limitations of the charging/discharging power in BESSs. To keep the power units working in a feasible mode, the authors in [10] divide an inequality constraint into five intervals, which greatly increases the complexity of the algorithm. The penalty function is applied to eliminate the inequality constraints in [11], whereas the studied algorithm may still exceed the constraint area in some cases. In [12], a centralised second-order economic power dispatch system is utilised to minimise the consumption costs among the supply and demand profiles. The authors in [13] solve an optimal power and heat scheduling problem, which minimises the operation cost of the smart grid with the uncertain power market prices. In [14], a centralised optimisation strategy is applied in a Photovoltaic (PV) solar farm, which optimises the investment cost, operation and maintenance cost, fuel cost, emission cost and network losses cost. However, this centralised computation process is very complex, and it is time-consuming to calculate the optimal results. The centralised algorithms require a powerful control centre and a large data server centre to collect the global information and process the massive data [15], which are not conducive to the development and the upgrade of smart grids. Besides, the complexity of the centralised demand management system grows exponentially with the increasing number of power units [16].

Due to the uncertainty and the variability of renewable energy resources and demands, the topology of a smart grid may change frequently and suddenly. Therefore, the centralised algorithms are not suitable for the requirement of plug-and-play. Therefore, different flexible distributed control strategies are studied in a significant number of papers, e.g., [17–27]. In [17], the authors work on a social maximal welfare problem in a smart grid in a distributed manner. Hug *et al.* [18] investigate a consensus based distributed power management algorithm to solve the demand of a smart grid. In [19], the authors introduce a cooperative distributed demand management system based on Karush-Kuhn-Tucker conditions. Deng *et al.* in [20] present a distributed demand response problem and define sub-problems by decomposing the main optimization problem and solving each sub-problem locally. In [21], a three-layer strategy is used to control a power grid, which includes supervision, optimisation and execution process. Meanwhile, the authors in [23] employ a cooperative control strategy of multiple BESSs to maintain the active power balance and minimise the total active power loss associated with BESSs' charging/discharging inefficiency. In [25], the authors propose an optimal distributed solution for economic dispatch to minimise the operation cost. However, most of them are sharing users' information, e.g., the output power or the incremental cost, which would cause privacy concerns. Furthermore, the optimal solutions of the algorithms in [21, 23, 24, 28] can be only obtained when certain initial conditions, i.e., the sum of initial power allocations should equal to the system active power mismatch, are satisfied. In other words, the network resource constraint can be ensured only

if it satisfies the initial conditions. Hence, it is not compatible with the plug-and-play function.

In this paper, a distributed initialisation-free optimisation strategy is proposed to handle the optimal demand management under the uncertain renewable generation. To deal with the physical constraints and the initialisation problem, we combine the proportional-integral (PI) consensus dynamics [29] and the projection algorithm [30], which can solve the constraints problem by local power units. This strategy considers the costs of demand units, BESSs and DGs. Different from the existing results, the emission costs and the battery degradation costs are also considered in our DSM model. In addition, the proposed algorithm can handle DSM problem with any initial errors so that it can adapt the plug-and-play operation. In order to achieve the control objectives of DSM, each power unit generates an optimal power reference through the proposed algorithm by coordinating information with its neighbours. At the meantime, each power unit will meet the power reference through a local controller. The effectiveness of the proposed distributed algorithm is validated through simulation studies in an IEEE 14-bus system and a complex power grid with 40 generators, 15 BESSs and 200 loads.

The major contributions of the proposed distributed optimisation strategy for DSM are summarised as follows:

1. The power outputs of all units including controllable loads, BESSs and DGs are optimised according to their welfare functions while maintaining the supply-demand balance in various conditions, i.e., different prices, communication failures and time-varying active power mismatches.
2. The proposed strategy is distributed based on a distributed average estimator. Thus, the proposed strategy is scalable and could be applied to large-scale systems. Additionally, it only requires the exchange of one information variable that does not contain the information of the welfare function. The customer privacy could be protected during the information transmission process.
3. The practical implementation of a distributed algorithm would have initial mismatches. To this end, the proposed algorithm in this paper is initialisation-free, which has the advantage of solving the initial errors. Meanwhile, this feature enables the plug-and-play functions in DSM since it does not require any initialisation process.

The rest of this paper is organised as follows. The mathematical preliminaries used in this paper are summarised in Section 2. The welfare functions of power units in a smart grid are designed in Section 3, including DGs, BESSs and local users. A distributed projection-based consensus protocol is proposed in Section 4. Simulation results and corresponding analysis are presented in Section 5. Finally, Section 6 concludes this paper.

2 Preliminary

In this section, we recall some preliminaries related to the graph theory, the convex analysis and the projection. Let $\mathbb{R}^{n \times m}$ be the set of $n \times m$ real matrices and the superscript T means the transpose of real matrices. I_N denotes the identity matrix of dimension N and $\mathbf{1}_N$ represents a column vector with all entries being 1. $\mathbb{R}^{++}$ denotes the positive real numbers. $\|\cdot\|_2$ represents the 2-norm of the argument.

2.1 Graph Theory

Following [31], an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be used to describe the communication topology among the power units, where $\mathcal{V} = \{\nu_1, \dots, \nu_N\}$ is the vertex set and $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the edge set. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of

$\mathcal{G}(\mathcal{V}, \mathcal{E})$ is an $N \times N$ matrix, such that $a_{ij} = 1$ if $(\nu_j, \nu_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Define the degree matrix $D = \text{diag}\{\sum_{j=1}^N a_{1j}, \sum_{j=1}^N a_{2j}, \dots, \sum_{j=1}^N a_{Nj}\}$. A graph is connected if and only if every pair of vertices can be connected by a path, namely, a sequence of edges. In this paper, we assume that the graph is connected and undirected. The Laplacian matrix related to $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is defined as $\mathcal{L} = D - \mathcal{A}$, i.e.,

$$\mathcal{L} = \begin{cases} l_{ij} = -a_{ij}, i \neq j \\ l_{ii} = \sum_{i \neq j} a_{ij} \end{cases} \quad (1)$$

When $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a connected undirected graph, 0 is an eigenvalue of Laplacian $\mathcal{L}$ with the eigenvector $\mathbf{1}_N$ and all the other eigenvalues are positive. Then,

$$\mathcal{L}\mathbf{1}_N = 0_N, \quad \mathbf{1}_N^T \mathcal{L} = 0_N^T. \quad (2)$$

2.2 Convex Analysis and Projection

By [32], a set $\Omega \subseteq \mathbb{R}^n$ is convex if $\theta x_1 + (1 - \theta)x_2 \in \Omega$ for any $x_1, x_2 \in \Omega$ and $0 \leq \theta \leq 1$. For a closed convex Ω , the projection map $P_\Omega : \mathbb{R}^n \rightarrow \Omega$ is defined as

$$P_\Omega(x) = \arg \min_{y \in \Omega} \|x - y\|. \quad (3)$$

Then the following inequalities hold, $\exists x \in \mathbb{R}^n, y \in \Omega$

$$\begin{aligned} (x - P_\Omega(x))^T (P_\Omega(x) - y) &\geq 0, \\ \|x - P_\Omega(x)\|_2^2 + \|P_\Omega(x) - y\|_2^2 &\leq \|x - y\|_2^2, \end{aligned} \quad (4)$$

For $x \in \Omega$, the normal cone to Ω is

$$\mathcal{N}_\Omega(x) \triangleq \{v \in \mathbb{R}^n | v^T(y - x) \leq 0, \text{ for all } y \in \Omega\}. \quad (5)$$

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2)$ for any $x_1, x_2 \in \Omega$ and $0 \leq \theta \leq 1$.

3 Problem Formulation

In this section, we formulate the welfare functions of the power units involved in a smart grid, consisting of demand units, BESSs, and DGs. The objective of the proposed model is to maximise the total welfare associated with the demand units, BESSs and DGs, including operating costs [1, 5]. Also, we consider emission cost in our model. Therefore, we have

$$\max_{P_i} \left\{ \sum_{i \in S_D} W_{D,i}(P_i) + \sum_{i \in S_B} W_{B,i}(P_i) + \sum_{i \in S_G} W_{G,i}(P_i) \right\}, \quad (6)$$

To ensure that all power units work in the normal mode, the formulated problem should be subject to the power balance constraint and local power constraints that will be discussed later. Note that the power output of the battery can be positive or negative, depending on discharging and charging states, respectively.

3.1 Controllable Power Units

1. Welfare on Controllable Demand Units

The welfare function of the i th load is formulated as the level of consumer satisfaction, which is related to the power consumption of applications. For the demand customers, consuming more power will bring more satisfaction. Therefore,

similar to [5], the welfare function is defined as

$$W_{D,i}(P_{D,i}) := \begin{cases} -\alpha_i P_{D,i}^2 + \beta_i P_{D,i}, & P_i^{\min} \leq P_{D,i} \leq \beta_i / 2\alpha_i \\ \beta_i^2 / 4\alpha_i, & \beta_i / 2\alpha_i \leq P_{D,i} \leq P_i^{\max} \end{cases} \quad (7)$$

where the power utility satisfies $P_i^{\min} \leq P_{D,i} \leq P_i^{\max}$. Here, the satisfaction level of customer increases with the consumption of electrical power and will eventually get saturated.

2. Welfare on BESSs

To save electricity and balance the uncertain power generation, some BESSs are also installed in the smart grid. Referring to [33], we formulate the welfare function of BESSs as

$$W_{B,i}(P_{B,i}) := pP_{B,i} - f_{B,i}(P_{B,i}). \quad (8)$$

Because of the cost varies with the characteristics of BESSs, following the approximation in [33], both of the charging and discharging process will increase the DoD cost of BESSs. Hence, the cost function can be uniformly expressed as

$$f_{B,i}(P_{B,i}) := \kappa_f + \kappa_c(1 + \iota)|P_{B,i}|. \quad (9)$$

Furthermore, the power output satisfies $P_{B,i}^{\min} \leq P_{B,i} \leq P_{B,i}^{\max}$ with the minimum and maximum power output $P_{B,i}^{\min}, P_{B,i}^{\max}$. Here we assume that BESSs are eco-friendly [34], and therefore the emission cost of BESSs is zero.

3. Welfare on Generators

The welfare function for the power generators is usually formulated by the income minus the costs [1, 5]. The costs of DGs mainly include the operation and maintenance (O&M) costs [14, 35] which can be expressed as a quadratic function and a linear function of active power respectively. Generally, the O&M cost of DGs is expressed as:

$$f_{G,i}^{O\&M}(P_{G,i}) := a_i P_{G,i}^2 + b_i P_{G,i} + c_i, \quad (10)$$

$$i = 1, 2, \dots, k,$$

where the power output satisfies $P_i^{\min} \leq P_{G,i} \leq P_i^{\max}$ with $P_i^{\min}, P_i^{\max} \in \mathbb{R}^{++}$.

For comparison purposes, the total emission cost of various pollutants is generally expressed as [36–38]

$$f_{G,i}^E(P_{G,i}) := \alpha_i P_{G,i}^2 + \beta_i P_{G,i} + \gamma_i + \eta_i \exp(\varphi_i P_{G,i}), \quad (11)$$

$$i = 1, 2, \dots, k,$$

where $f_{G,i}^E(P_i)$ is the total pollution emission cost for the i th power generator.

Therefore, the welfare function is expressed as

$$W_{G,i}(P_{G,i}) := pP_{G,i} - f_{G,i}^{O\&M}(P_{G,i}) - f_{G,i}^E(P_{G,i}). \quad (12)$$

Basically, the welfare for the generation unit represents the benefit of selling power minus the cost of operation and emission. The first term in (12) means the income by selling energy and the other terms denote the cost caused by maintenance and pollution.

3.2 Uncontrollable Power Units

From [2, 14], renewable generators and users' loads are uncontrollable power units and their power output and consumption are related to the light intensity, illumination time, wind speed, and customers' habits, etc. Therefore, these

uncontrollable power units generates power by different conditions, and these uncontrollable power units are considered as undispachable in this paper.

In the smart grid, transmission losses are inevitable, accounting for around 5-7% of the total power load [39], which can be modelled by multiplying the load with this percentage. Overall, to maintain system stability, the active power balance between the supply and the demand side is described as

$$\sum_{i \in S_G} P_{G,i} + \sum_{i \in S_R} P_{R,i} = \sum_{i \in S_D} P_{D,i} + \sum_{i \in S_B} P_{B,i}. \quad (13)$$

Since the renewable source is non-dispatchable and the load is related to the customers' habits which are partial controllable, we rewrite the above constraint as

$$P_D = \sum_{i \in S_G} P_{G,i} + \sum_{i \in S_B} P_{B,i} + \sum_{i \in S_C} P_{C,i}. \quad (14)$$

Here, $P_{B,i}$ can be negative or positive.

3.3 Problem Reformulation

In this paper, our objective is to design a reliable demand side management system that can maximise the social welfare while maintaining active power balance under various conditions. To this end, an objective function is formulated by integrating the above welfare functions and subjecting to physical constraints. For notation convenience, we denote the power vector as $P = [P_1, P_2, \dots, P_m, P_{m+1}, P_{m+2}, \dots, P_{m+n}, P_{m+n+1}, P_{m+n+2}, \dots, P_{m+n+k}]^T \in \mathbb{R}^N$, where m, n and k denote the numbers of demand units, BESSs and DGs with $N = m + n + k$.

$$\begin{aligned} \min & \sum C_i(P_i), \quad i = 1, 2, \dots, N, \\ \text{s.t.} & \sum_i P_i = P_D, \\ & P_i^{\min} \leq P_i \leq P_i^{\max}, \end{aligned} \quad (15)$$

where $C_i(P_i) \triangleq -\sum_{i \in S} W_i(P_i)$, $S = S_D \cup S_B \cup S_G$ and $W_i(P_i)$ denotes the welfare of the i th unit. Notice that the cost function $C_i(P_i)$ of each unit is strictly convex and continuously differentiable.

Traditionally, the constrained optimisation problem can be solved using centralised methods, but those algorithms require a powerful control centre to collect data from the sub-systems and distribute control instruction to the units after calculation. In the following section, we design a distributed algorithm, where each subsystem is allocated with a low-price processor, by which the collection and calculation can be performed locally.

4 Distributed Solution

In this section, a projection-based gradient decent algorithm is developed, by which the inequality constraints can be tackled accordingly. We consider the Lagrangian function for each unit with the affine equality constraint, written as

$$L = \sum_{i=1}^N C_i(P_i) - \bar{\lambda} \left[\sum_{i=1}^N P_i - P_D \right], \quad (16)$$

where $\bar{\lambda}$ is the Lagrangian multiplier which is used to ensure the equality constraints are met during the optimisation process. The inequality constraints are not considered in the Lagrangian function since the inequality constraint can be solved by local units with projection algorithm.

The optimal solution can be obtained by using a well-known centralised saddle-point dynamics as

$$\begin{aligned} \frac{\partial L}{\partial P_i} &= \nabla C_i(P_i) - \bar{\lambda} = 0 \\ \frac{\partial L}{\partial \bar{\lambda}} &= \sum_{i=1}^N P_i - P_D = 0. \end{aligned} \quad (17)$$

Its equilibrium points (16) satisfy the Karush-Kuhn-Tucker (KKT) conditions (see, e.g., [32]). However, it collects the global information about the Lagrangian multiplier ($\bar{\lambda}$) of all the power units. In the modern smart grid, most loads are distributed, so that it is desired to design a distributed algorithm for DSM that solves the optimisation problem locally. To facilitate our design, a local copy $\bar{\lambda}_i$ of the global variable is used to estimate the global ($\bar{\lambda}$). As a result, the problem is solved when the local copy variables converge to the $\bar{\lambda}^*$. According to the KKT conditions, the following lemma is proposed.

Lemma 4.1. *The optimisation problem has an optimal solution P^* if and only if there exists a vector of Lagrangian multiplier $\bar{\lambda}^* \triangleq \bar{\lambda}^* \mathbf{1}_N \in \mathbb{R}^N$ such that*

$$\nabla C(P) = \bar{\lambda}^* \mathbf{1}_N, \quad (18)$$

where $C(P) \triangleq [C_1(P_1), C_2(P_2), \dots, C_N(P_N)]^T$ denotes cost function vector, and $\nabla C(P)$ represents its gradient.

4.1 Algorithm Design

Inspired by [29, 30], the objective of the distributed algorithm is developed in order to achieve a consensus of the Lagrangian multiplier and reach the optimal power output. Let $\Omega_i \triangleq [P_i^{\min}, P_i^{\max}]$ denote the feasible domain of the i th unit's power output, which is clearly a compact and convex set. The problem (15) can be solved by the following distributed algorithm, $\forall i = 1 \dots N$

$$\dot{P}_i = P_{\Omega_i} [P_i - \nabla C_i(P_i) + \phi_i] - P_i, \quad (19a)$$

$$\dot{\phi}_i = \sum_{j=1}^N a_{ij} (\phi_j - \phi_i) - \xi_i + (\theta_i P_D - P_i) + \lambda_i - \phi_i, \quad (19b)$$

$$\dot{\lambda}_i = -\lambda_i + \phi_i, \quad (19c)$$

$$\dot{\xi}_i = \sum_{j=1}^N a_{ij} (\phi_i - \phi_j), \quad \sum_{i=1}^N \xi_i(0) = 0, \quad (19d)$$

where $P_i \in \mathbb{R}$ is the power output of the i th power unit and $\phi_i, \xi_i \in \mathbb{R}$ are two auxiliary variables of the i th unit. The convergence analysis of (19) is detailed in Appendix.

Note that the proposed algorithm can be implemented respecting the local feasible constraints by projection operations. Furthermore, the algorithm (19) does not require any initialisation process, as proved in Appendix. Therefore, the system constraints can be satisfied without specifying initial conditions. Due to free of any control centre and initialisation process, the algorithm can work under a "plug-and-play" operation that improves the flexibility of future power systems.

The algorithm (19) is distributed, in this sense that the i th unit only needs the local data and the information which is shared with the neighbouring units. Therefore, (19) does not require any centre to process the data or coordinate the units. Since the local data do not need to be uploaded and downloaded from the centre, each unit can respond to local data changes rapidly which can quickly adapt the local decisions.

The algorithm can be understood concerning singular perturbation, where the third dynamic is on a faster scale than the second one. Hence, λ_i goes to ϕ_i , as t goes to infinity, and substituting this to the algorithm yields a saddle-point seeking algorithm. We will show the detailed proof in the next section. Different from the algorithm in [30], the proposed algorithm does not require each unit to know the load information, $\sum_{i \in S} P_i$ is not always equal to P_D . Furthermore, the algorithm (19) introduced the estimation ability $\theta_i \in [0, 1]$ to detect the total mismatch, which can be selected according to [40] and $\sum_i^N \theta_i = 1$. Therefore, this algorithm could be applied to a large grid.

Notably, the proposed algorithm does not require the initialisation process so that it is more compatible with the operation of a smart grid. The initialisation for the network resource constraint is quite restrictive for a sizeable dynamical grid because it is related to the global coordination and has to be performed whenever the network data/configuration changes. However, the communication network may change frequently and rapidly in a smart grid, such as when plugging in an electric vehicle and routine maintenance. Furthermore, it is not trivial to achieve the initialisation coordination with both the local feasibility and the network resource constraints.

4.2 Algorithm Implementation

The proposed distributed algorithm adopts a distributed average consensus estimator to measure the global mismatch information locally. The step-by-step algorithm for all power units is shown in Algorithm 1. To illustrate more clearly, a flow chart for solving distributed demand side management by proposed algorithm is shown in Fig. 2. As in [11], the communication structure can be designed independently of the power system in a cost-efficient way based on the location and the convenience of the smart grid. For example, a communication topology is designed as Fig. 1. DGs, BESSs and controllable loads are connected through a communication network as the power units in a smart grid. To obtain the optimal power outputs, each power unit only interacts with its neighbouring units to exchange the information through the communication network at the top level, and then each power unit performs the proposed algorithm accordingly.

Fig. 1: The two-level structure of optimisation algorithm.

5 Case Study

In this section, four cases are employed to validate the effectiveness and applicability of the proposed algorithm. At the beginning, two sub-cases in the first case study are used to

Fig. 2: The flowchart for i th power unit management.

Algorithm 1 Distributed Optimal Demand Side Management

Input: Power mismatch $\theta_i P_D$ for i th power unit.

Output: Optimal power reference for each power unit.

Initialisation:

For $i=1,2,\dots,N$

$P_i = P_i(0), \lambda_i = \lambda_i(0), \phi_i = \phi_i(0), \xi_i = \xi_i(0)$

Consensus Algorithm:

For i th power unit:

Communicate and get the parameter ϕ_j from its neighbour j ;

Calculate and updates its own parameters ϕ_i, ξ_i, λ_i and P_i according to 19(a)-19(d)

End if: Each power unit achieves the optimal values.

test the algorithm in a modified IEEE-14 bus system. First sub-case considers a constant power mismatch as 20MW, and the results obtained using the proposed algorithm are compared with the results based on previous work [28]. In the second sub-case, the optimisation algorithm is studied under different time-of-use prices. Next, Case 2 is studied to test the proposed algorithm under communication failures, which assumes that all communication links of the DG_4 fail to exchange information with neighbours. The plug-and-play adaptability of our algorithm has been tested in Case 3. Lastly, a large-scale power system is adopted to test the scalability of the proposed algorithm in Case 4. The parameters are chosen from [41]. Results of the four cases are discussed in the following section.

5.1 Case 1

In this study, we test the algorithm on an IEEE 14-bus system which consists of 4 DGs, 2 BESSs and 10 loads in Fig. 3, whose coefficients are shown in Table 1.

Table 1 Cost coefficients for simulation studies.

	a_i	b_i	α_i	β_i	η_i	φ_i	κ_f	κ_c	ι	$P^{\min}(\text{MW})$	$P^{\max}(\text{MW})$
DG ₁	0.008	33.83	2.3e-3	-1.5e-3	2.0e-4	2.857	-	-	-	0	70
DG ₂	0.0062	34.03	2.1e-3	-1.82e-3	5.0e-4	3.333	-	-	-	0	65
DG ₃	0.0075	33.93	2.2e-3	-1.249e-3	1.0e-6	8.0	-	-	-	0	70
DG ₄	0.0072	33.97	2.3e-3	-1.355e-3	2.0e-3	2.0	-	-	-	0	65
BESS ₁	-	-	-	-	-	-	0.025	3.41	95%	-40	40
BESS ₂	-	-	-	-	-	-	0.025	3.41	95%	-30	30
L ₁	-	-	0.072	8.25	-	-	-	-	-	0	50
L ₂	-	-	0.066	7.90	-	-	-	-	-	0	60
L ₃	-	-	0.070	7.55	-	-	-	-	-	0	30
L ₄	-	-	0.055	8.00	-	-	-	-	-	0	40
L ₅	-	-	0.075	7.75	-	-	-	-	-	0	40
L ₆	-	-	0.045	8.05	-	-	-	-	-	0	60
L ₇	-	-	0.058	7.85	-	-	-	-	-	0	60
L ₈	-	-	0.062	8.45	-	-	-	-	-	0	40
L ₉	-	-	0.070	7.25	-	-	-	-	-	0	50
L ₁₀	-	-	0.058	8.00	-	-	-	-	-	0	50

Fig. 3: Modified IEEE 14-bus system.

5.1.1 Constant Demand and Price: As a baseline test, we first consider the situation with a constant power mismatch and electricity price, where all the power units are connected. To reveal the effectiveness of the proposed strategy, our algorithm is first compared with another algorithm in [28]. Here, the electrical price is set to 40€/MWh, which is chosen from the UK electric price report [42], and the BESSs are distributed among the communication network. The electricity price and the supply-demand mismatch are assumed to be constant, and the system parameters are set to be the same to make the comparison study more convincing.

As shown in Fig 4, the power outputs of each power unit can converge to its optimal value with the proposed algorithm. Here, the negative/positive power value means the consumed/generated power. The BESSs can be charged or discharged depending on the electricity price and the power mismatch. Fig. 5 shows the power mismatch during the optimisation process. It indicates that the power is not balanced at the beginning and the power mismatch converges to zero within 5s, which reflects that our algorithm is capable of maintaining the power balance. Thus, the proposed strategy can address the DSM problem from any initial error.

In order to show the advantage of initialisation-free feature, we firstly compare the proposed algorithm with an existing work in [28]. In the comparison study, we assumed both

Fig. 4: The update of power outputs with the proposed algorithm. **Fig. 5:** The power mismatch with the proposed algorithm.

algorithms are initialised randomly and do not satisfy the initialisation conditions in [28]. The results are shown in Fig. 6. The power outputs using these algorithms can converge to stable values, but the results of the algorithm in [28] are not optimal since it needs the sum of all initial power output is equal to the power mismatch during the initialisation process. Then, we further compare the proposed algorithm with the algorithm in [27]. To make the comparison more clearly, the ideal results are obtained through a centralised solver in Matlab, and the comparison result is given in Table 2. Compared with the algorithms in [27] and [28], the proposed algorithm

Fig. 6: The update of power outputs with the algorithm in [28].

can achieve the optimal active power values even under initial errors.

Table 2 Comparison results.

	Algorithm (19)	Algorithm in [28]	Algorithm in [27]	Matlab (fmincon)
DG ₁	30.6819	22.9948	26.7769	30.6820
DG ₂	55.3153	43.9412	50.2758	55.3155
DG ₃	39.3940	30.2443	35.2270	39.3941
DG ₄	43.8133	35.7782	39.4714	43.8134
BESS ₁	-22.1363	-24.7803	-24.2210	-22.1362
BESS ₂	-24.8522	-25.8211	-26.4163	-24.8522
L ₁	-11.0477	-10.8696	-15.3938	-11.0477
L ₂	-9.4004	-6.4048	-14.1400	-9.4006
L ₃	-6.3632	-1.6057	-10.8304	-6.3634
L ₄	-12.1895	-4.7591	-17.8732	-12.1898
L ₅	-7.2724	-1.9037	-11.4392	-7.2725
L ₆	-15.4540	-7.9295	-22.3970	-15.4542
L ₇	-10.2661	-7.0456	-15.6520	-10.2661
L ₈	-14.4426	-14.7112	-19.4805	-14.4425
L ₉	-4.2206	-7.6875	-8.6827	-4.2205
L ₁₀	-11.5595	-19.4404	-16.9449	-11.5593

Remark 1. Note that the proposed distributed algorithm generates the optimal power reference for each unit in a communication network. Its convergence speed is related to the structure, namely the second smallest eigenvalue of the Laplacian matrix, of the communication network. Meanwhile, by tuning the algorithm parameters, the convergence speed can be adjusted according to different applications.

5.1.2 Different Prices: In the smart grid, the electricity price is changing over time, and different electricity prices bring different economic influence to customers, BESSs and DGs. Therefore, in the second sub-case, we test the algorithm with 24-hour time-of-use (ToU) prices shown as Fig. 7, which comes from the UK annual report [42].

The results are presented in Fig. 8. Note that the power outputs of each power unit are varying with ToU prices to maximise the social welfare. At low electricity prices periods, customers will consume more power and the battery will be charged to make sure it has enough energy to sell at high prices. When the electricity price rises, the algorithm will control the usage of these controllable loads, reduce their electricity consumption to save the electricity bills, such as reducing the load of air-conditioner and changing the EV charging time. At the same time, the battery will be discharged to increase the income. Therefore, the proposed algorithm

Fig. 7: Time-of-use prices.

Fig. 8: The update of power outputs with the proposed algorithm.

could maximise the social welfare according to the different prices, which could be a potential solution for DSM in a future smart grid.

5.2 Case 2

In the real power grid, the communication network may change by many causes. For instance, a generator is disconnected from the smart grid for maintenance and overhaul. In this case, it is assumed that there is a power generator DG₄ which loses its all communication links at 50s, and the links are repaired so that the DG₄ is reconnected at 150s. From (19), if the communication links are connected, then its neighbours have the information about the communication variable ϕ . It is assumed that the neighbour of DG₄ knows the last information ϕ when the communication links breaks. To better verify the effectiveness of the proposed algorithm under the communication mismatch failure, we further assume that the total active power mismatch P_D is increased from 20MW to 40MW at time $t = 100$ s.

It can be seen in Fig. 9 that the rest of power units still work at their optimal power states to hold the power balance when the communication network of DG₄ is failed. If the power mismatch changes during the broken period, the DG₄ will not response to the power mismatch changing and cannot be optimised because it loses all communication links. However, the rest of power units will converge to new optimal values according to the current mismatch condition using the proposed algorithm. After the links are repaired, the DG₄ will be reconnected into the system so that all power units can share and update their information to reach the new optimal values.

Fig. 9: All communication links of DG₄ are failed.

Fig. 10: The power mismatch with failed communication network.

As shown in Fig. 10, the total power demand and total active power generation can be balanced after a very short period. Therefore, the proposed algorithm can keep the active power balance even if some communication links are failed.

5.3 Case 3

To verify that the algorithm can adapt to the changes in the topology, the plug-and-play adaptability of our strategy is studied in this case.

5.3.1 Constant Power Mismatch: The plug-and-play adaptability of BESSs under a constant power mismatch is studied in this sub-case. For example, the BESS₂ is disconnected at 50s for daily maintenance, and then after 50s, the BESS is connected back to the smart grid. The total power mismatch at the beginning is assumed as 20MW. The results show that the proposed algorithm can handle the plug-in and plug-out operations.

As shown in Figs. 11 - 12, the power outputs of all power units quickly converge to their optimal values, and the total power mismatch converges to zero. When the BESS₂ is plugged out from the system at 50s, the rest of power units will fast converge to the new optimal values and keep the power balance at the same time. With plugging in a new battery at 100s, all power units converge back to the optimal values which are same as the previous optimal values.

5.3.2 Time-varying Power Mismatch: In fact, due to the intermittent of renewable energy sources and customer behaviours, the smart grid may face problems with uncertain power mismatch. To this end, we test the proposed strategy

Fig. 11: The update of power outputs during plug-and-play.

Fig. 12: The power mismatch during plug-and-play.

under a time-varying supply-demand condition. It is assumed that there is a time-varying uncontrollable power mismatch given by $P_{\text{total}} = 20 + 50 \sin(0.3t)$ (MW). The communication network and the other parameters are same as Case 1. To reflect the effectiveness of the proposed algorithm, we assume that a distributed generator is disconnected from the smart grid for daily maintenance at 50s and connected back to the smart grid at 100s.

Fig. 13: The power mismatch with time-varying demand.

As shown in Fig. 13, the optimal power references change with the time-varying power mismatch. However, the power

outputs are allocated to their optimal value under the time-varying power mismatch condition, and while the algorithm keeps the power balance under the time-varying power mismatch. Besides, the proposed algorithm can keep working when plugging in/out the DG₄ at 50/100s, respectively. Therefore, the plug-and-play adaptability is also guaranteed by our algorithm even under the time-varying supply-demand condition.

5.4 Case 4

In this case, to verify the scalability of the proposed algorithm, the algorithm is applied in a complex scale system, consisting of 40 generators, 15 BESSs and 200 controllable loads, which is introduced in [41]. Each power unit is connected to its adjacent 10 neighbours in the communication network.

Fig. 14: The update of power outputs with the proposed algorithm.

As shown in Fig. 14, the studied algorithm can guarantee the allocated power outputs to converge to their optimal values within 10s. Also, it shows that the fluctuation of our strategy is much less than the algorithm in [1]. Therefore, the proposed optimisation strategy can be applied in a large and complicated smart grid.

6 Conclusion

In this paper, a distributed algorithm for DSM has been proposed in the context of smart grids. It maximises the social welfare of participants, i.e., customers, BESSs and DGs, according to their different objectives while subjecting to the system active power constraint and the local physical constraints. The proposed algorithm could solve DSM in a distributed fashion to handle the problems caused by the centralised method, such as communication interruptions, computing burdens, and single-point failures. Additionally, without the requirement of certain initialisation processes, the proposed algorithm can be executed in an initialisation-free manner that can deal with initial errors and enables the plug-and-play functions in DSM. The effectiveness and the scalability of the proposed algorithm are demonstrated by several case studies.

7 References

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8 Appendix

The convergence analysis follows similar steps to [30]. To analysis the convergence of our algorithm, the algorithm (19) is rewritten in a compact form as

$$\begin{aligned}\dot{P} &= P_{\Omega}(P - \nabla C(P) + \Phi) - P, \\ \dot{\Phi} &= -\mathcal{L}\Phi - \Xi + \Theta P_D - P + \Lambda - \Phi, \\ \dot{\Lambda} &= \Phi - \Lambda, \\ \dot{\Xi} &= \mathcal{L}\Phi,\end{aligned}\quad (20)$$

where $P = [P_1, P_2, \dots, P_N]^T$, $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_N$, $\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_N]^T$, $\Theta = [\theta_1, \theta_2, \dots, \theta_N]^T$, $\Phi = [\phi_1, \phi_2, \dots, \phi_N]^T$ and $\Xi = [\xi_1, \xi_2, \dots, \xi_N]^T$. Firstly, we consider the equilibrium point $\bar{P}, \bar{\Lambda}, \bar{\Xi}$ and $\bar{\Phi}$, which can be obtained by

$$\begin{aligned}0 &= P_{\Omega}(\bar{P} - \nabla C(\bar{P}) + \bar{\Phi}) - \bar{P}, \\ 0 &= -\mathcal{L}\bar{\Phi} - \bar{\Xi} + \Theta P_D - \bar{P} + \bar{\Lambda} - \bar{\Phi}, \\ 0 &= \bar{\Phi} - \bar{\Lambda}, \\ 0 &= \mathcal{L}\bar{\Phi}.\end{aligned}\quad (21)$$

Left multiplying the second equation in (21) by 1_N^T yields

$$1_N^T \bar{\Xi} = 1_N^T (\Theta P_D - P). \quad (22)$$

With the algorithm (19), Λ will converge to Φ as $t \rightarrow \infty$ and ϕ_i will achieve consensus by the last equation. Furthermore, we have the first equation to make sure that $\Delta C(P)$ converges to Φ , which is equal to λ^* at $t \rightarrow \infty$. Therefore, with the Lemma 4.1, the equilibrium point satisfies

$$\bar{P} = P^*, \bar{\Phi} = \Phi^*, \bar{\Lambda} = \Lambda^*, \bar{\Xi} = \Xi^*, \quad (23)$$

where P^*, Λ^*, Ξ^* and Φ^* are the optimal solution of the allocation problem in (15).

We define the new variables for equilibrium point of (20),

$$\begin{aligned}\hat{P} &= P - P^*, \quad \hat{\Lambda} = \Lambda - \Lambda^*, \quad \hat{\Phi} = \Phi - \Phi^*, \\ \hat{\Xi} &= \Xi - \Xi^*, \quad S = [r \ R]^T \hat{\Phi}, \quad U = [r \ R]^T \hat{\Xi},\end{aligned}\quad (24)$$

where $[r \ R]^T$ is an orthonormal matrix, here we let $r \in \mathbb{R}^N$, $R \in \mathbb{R}^{N \times (N-1)}$, then we have

$$\begin{aligned}[r \ R]^T [r \ R] &= [r \ R][r \ R]^T = I_N, \\ r^T R &= 0_{N-1}^T, \quad R^T R = I_{N-1}, \\ R R^T &= I_N - r r^T,\end{aligned}\quad (25)$$

and we divide the new variables as $U = (u_1, U_{2:N})$ and $S = (s_1, S_{2:N})$

$$\begin{aligned}\dot{u}_1 &= 0, \\ \dot{U}_{2:N} &= R^T \mathcal{L} R S_{2:N}, \\ \dot{s}_1 &= -s_1 + r^T (\hat{\Lambda} - \hat{P}), \\ \dot{S}_{2:N} &= -R^T \mathcal{L} R S_{2:N} - U_{2:N} - S_{2:N} + R^T (\hat{\Lambda} - \hat{P}), \\ \dot{\hat{\Lambda}} &= -\hat{\Lambda} + [r \ R] S, \\ \dot{\hat{P}} &= P_{\Omega}(\hat{P} + P^* - \nabla C(\hat{P} + P^*) + \nabla C(P^*) + [r \ R] S) \\ &\quad - (\hat{P} + P^*).\end{aligned}\quad (26)$$

Let $Q(\hat{P}, S) = \nabla C(\hat{P} + P^*) - [r \ R] S - \nabla C(P^*)$ and define the Lyapunov function as

$$\begin{aligned}V &= -Q(\hat{P}, S)^T (P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)) \\ &\quad - \frac{1}{2} \|P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)\|_2^2 \\ &\quad + \frac{1}{2} \|\hat{P}\|_2^2 + \frac{1}{2} \|\hat{\Lambda}\|_2^2 + \frac{1}{2} \|S\|_2^2 + \frac{1}{2} U_{2:N}^T (R^T \mathcal{L} R)^{-1} U_{2:N}.\end{aligned}\quad (27)$$

With (4) and if $(\hat{P}(0) + P^*) \in \Omega$ then the parameter $(\hat{P}(t) + P^*) \in \Omega$. We have

$$\begin{aligned}V &= \frac{1}{2} \|Q(\hat{P}, S)\|_2^2 \\ &\quad - \frac{1}{2} \|Q(\hat{P}, S) + P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)\|_2^2 \\ &\quad + \frac{1}{2} \|\hat{P}\|_2^2 + \frac{1}{2} \|\hat{\Lambda}\|_2^2 + \frac{1}{2} \|S\|_2^2 + \frac{1}{2} U_{2:N}^T (R^T \mathcal{L} R)^{-1} U_{2:N} \\ &\geq \frac{1}{2} \|P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)\|_2^2 + \frac{1}{2} \|\hat{P}\|_2^2 \\ &\quad + \frac{1}{2} \|\hat{\Lambda}\|_2^2 + \frac{1}{2} \|S\|_2^2 + \frac{1}{2} U_{2:N}^T (R^T \mathcal{L} R)^{-1} U_{2:N}.\end{aligned}\quad (28)$$

Therefore, $V \geq 0$ and $V = 0$ if and only if $\hat{P} = 0$. With the Theorem 3.2 in [43], differentiating V yields

$$\begin{aligned}\dot{V} &= -(P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*))^T \nabla_{\hat{P}} Q(\hat{P}, S) \dot{\hat{P}} \\ &\quad + \|P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)\|_2^2 \\ &\quad + Q(\hat{P}, S)^T (P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)) \\ &\quad + \hat{P}^T (P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)) \\ &\quad + \hat{\Lambda} \dot{\hat{\Lambda}} + \hat{S} \dot{\hat{S}} + S_{2:N}^T U_{2:N}.\end{aligned}\quad (29)$$

From (4), we have $(\hat{P} + P^* - Q(\hat{P}, S) - P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)))^T (P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - P^*) \geq 0$. Hence,

$$\begin{aligned}&(P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*))^T (\hat{P} + Q(\hat{P}, S)) \\ &\leq -\|P_{\Omega}(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*)\|_2^2 - Q(\hat{P}, S)^T \hat{P}.\end{aligned}\quad (30)$$

Since $\nabla_{\hat{P}} Q(\hat{P}, S) = \nabla^2 C(\hat{P} + P^*) \geq 0$, $-(P_\Omega(\hat{P} + P^* - Q(\hat{P}, S)) - (\hat{P} + P^*))^T \nabla_{\hat{P}} Q(\hat{P}, S) \hat{P} \geq 0$. Consequently

$$\begin{aligned}
\dot{V} &\leq -Q(\hat{P}, S)^T \hat{P} \\
&\quad + \hat{\Lambda}^T \hat{\Lambda} + \dot{s}_1 s_1 + \dot{S}_{2:N}^T S_{2:N} + S_{2:N} U_{2:N} \\
&= -\hat{P}^T (\nabla C(\hat{P} + P^*) - [r \ R] S - \nabla C(P^*)) \\
&\quad - \hat{\Lambda}^T \hat{\Lambda} + \hat{\Lambda}^T [r \ R] S - s_1^2 + S^T [r \ R]^T (\hat{\Lambda} - \hat{P}) \\
&\quad - S_{2:N}^T R^T \mathcal{L} R S_{2:N} - U_{2:N}^T S_{2:N} + S_{2:N}^T U_{2:N} \\
&\leq -\hat{P}^T (\nabla C(\hat{P} + P^*) - \nabla C(P^*)) - S_{2:N}^T R^T \mathcal{L} R S_{2:N} \\
&\leq -\hat{P}^T (\nabla C(\hat{P} + P^*) - \nabla C(P^*)). \tag{31}
\end{aligned}$$

Because the Hessian matrix of $C(P)$ is positive definite, we have

$$\nabla C(\hat{P} + P^*) = \nabla C(P^*) + \int_0^1 \nabla^2 C(1 - \tau \hat{P})^T \hat{P} d\tau. \tag{32}$$

Therefore, $\dot{V} \leq 0$. Based on the LaSalle invariance principle and the Lyapunov stability theory. The system (20) converges to its equilibrium point if and only if $\dot{V} = 0$, and therefore $\hat{P}$ goes to 0, then $P = P^*$, which means the optimisation problem is solved.

The algorithm described in this paper uses projection operations to map the variables into their specified domains, and completes the optimisation of power dispatch in the smart grid.