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Indirect Disjunctive Belief Rule Base Modeling using Limited Conjunctive Rules: Two Possible Means

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Abstract: A traditional Belief Rule Base (BRB) is constructed under the conjunctive assumption (conjunctive BRB), which requires covering the traversal combinations of the referenced values for the attributes. Consequentially, a traditional conjunctive BRB may have to face the combinatorial explosion problem when there are too many attributes and/or referenced values for the attributes. It is difficult or at least expensive to construct a complete conjunctive BRB, while it is easy to derive one or several conjunctive rules. Comparatively, a BRB under the disjunctive assumption (disjunctive BRB) requires only covering the referenced values for the attributes instead of the traversal combination of them. Thus, the combinatorial explosion problem can be avoided. However, it is difficult to directly obtain a disjunctive BRB from either historical data or experts' knowledge. To combine the advantages of both conjunctive and disjunctive BRBs, a new approach is proposed to construct a disjunctive BRB using a limited number of conjunctive rules (insufficient to construct a complete conjunctive BRB). In the new disjunctive BRB modeling approach, each disjunctive rule is derived by quantifying its correlation with one or multiple conjunctive rules. To do so, two means for belief generation are proposed, namely, equal probability and self-organizing mapping (SOM). Two cases are studied for validating the efficiency of the proposed approach. The results by the disjunctive BRB show consistency with those derived by the conjunctive BRB as well as other approaches, which validates the efficiency of the proposed approach considering that the disjunctive BRB is constructed with only a limited number of conjunctive rules.

Keywords: disjunctive belief rule base (BRB); indirect modeling; limited conjunctive rules; equal probability; Self-Organizing Map (SOM).

1 Introduction

The Belief Rule Base (BRB) is comprised of multiples belief rules in a single belief structure [29] [30]. As an expert system approach based on the D-S evidence theory [10] [11], BRB has shown superior performance in representing, transforming and integrating multiple types of information under uncertainty, which is quite useful in complex systems modeling [1] [15]-[17] [31]-[34]. Moreover, it is also a white box that provides direct access to experts and decision makers [4] [6] [32].

Under different assumptions, it can be classified as a conjunctive BRB or disjunctive BRB [6] [8] [9] [29]. Normally, there is no ambiguity in complex system modeling on whether a conjunctive or disjunctive model should be applied. If the antecedent attributes are conjunctively correlated, then a conjunctive model should be adopted, and vice versa. However, practical systems are far more complex in nature. Under many practical conditions, it can be rather difficult to have a clear understanding of the complex correlations between the antecedent attributes. Consequently, people's perception plays an important role in determining how the model should be constructed.

Take medical diagnosis as an example. In the diagnosis of a patient, one doctor may believe that a patient can only be diagnosed when all of the key symptoms have appeared, while some other doctor may believe that any (or a subset) of the symptoms can guarantee the diagnosis. A rule given in the former condition would be conjunctive, while the latter would produce a disjunctive rule. To be more exact, a conjunctive rule stands if and only if all of the attributes are activated, while a disjunctive rule stands if at least one attribute is activated.

Furthermore, the two types of BRB do not have the same requirements.

The conjunctive BRB is much easier to understand: the conclusion of a conjunctive rule is the natural inference result when all of its attributes are activated. The challenge is the combinatorial explosion problem: the traversal combinations of the referenced values for the attributes must be covered, which would lead to the combinatorial explosion problem if there are too many attributes and/or referenced values for the attributes [4] [7]. In other words, it is hard to gather all of the conjunctive rules to form a complete conjunctive BRB, although it is easier to derive one or several conjunctive rules.

The disjunctive BRB requires far fewer rules in comparison with a conjunctive BRB in the same belief structure (with the same reference values for the same attributes) [6] [8] [9]. Thus, the combinatorial explosion problem can be avoided.

However, it is difficult to directly produce a disjunctive rule since it is not completely consistent with people's direct judgment. To summarize, it is hard to directly produce a disjunctive rule, although a disjunctive BRB can help avoid the combinatorial explosion problem, which has caused a great challenge for the conjunctive BRB.

A rational thought would be to combine the advantages of the two types of BRBs and avoid their disadvantages. Thus, a new BRB modeling approach is proposed that uses limited conjunctive rules to construct a disjunctive BRB.

The core of this new BRB modeling approach is to quantify the correlations between different types of BRBs, or rules, to be more specific. There are two major steps within the new approach.

First, how one/multiple conjunctive rule(s) is/are **correlated** to a disjunctive rule needs to be determined. In this study, they are correlated to each other if they have at least one shared attribute. Under practical conditions, a disjunctive rule and its correlated conjunctive rule(s) would be in the same belief structure (with the same referenced values for the same attributes) because a disjunctive BRB is normally constructed after the conjunctive rules are given.

Second, such **correlations** need to be quantified. If one/multiple conjunctive rule(s) is/are correlated to a disjunctive rule, then their correlations should be quantified using certain means. In this study, we propose using the equal probability (assuming multiple conjunctive rules are equally correlated to a designated disjunctive rule) [23] and self-organizing mapping (SOM) (assuming multiple conjunctive rules are correlated to a designated disjunctive rule by SOM) [13] [18] [19] to quantify this correlation to further generate beliefs for the scales in the conclusion part for the new disjunctive rules.

A numerical and a practical case are used for validation. The case results show that, with a limited number of conjunctive rules, a complete disjunctive BRB can be constructed. Moreover, the consistency of the disjunctive BRB is still maintained with a high stability, especially considering that the input information requirement is far reduced. The case study results are validated by comparison with multi linear regression (MLR) and artificial neural network (ANN).

The remainder of this study is organized as follows. Section 2 introduces the challenges of conjunctive and disjunctive BRB modeling. Section 3 discusses the correlation between different types of BRBs. Section 4 proposes the new indirect disjunctive BRB modeling approach via equal probability and SOM, respectively. Two cases are studied in Section 5, and this study is concluded in Section 6.

2 Problem demonstration

2.1 Basics of conjunctive and disjunctive BRBs

The BRB system is an expert system that can well handle different types of information under uncertainty. BRB is comprised of multiple belief rules in a single belief structure [29] [30]. Conventionally, BRB is constructed under the conjunctive assumption, conjunctive BRB. The k th rule in a conjunctive BRB is described as:

$$\begin{aligned} R_k : & \text{if } (x_1 \text{ is } A_1^k) \wedge (x_2 \text{ is } A_2^k) \wedge \cdots \wedge (x_M \text{ is } A_M^k), \\ & \text{then } \{(D_1, \beta_{1,k}), \dots, (D_N, \beta_{N,k})\} \\ & \text{with rule weight } \theta_k \end{aligned} \quad (1)$$

where $x_m (m=1, \dots, M)$ denotes the m th attribute, $A_m^k (m=1, \dots, M; k=1, \dots, K)$ denotes the referenced values for the m th attribute, M denotes the number of attributes, $\beta_{n,k} (n=1, \dots, N)$ denotes the belief for the n th degree, D_n , and N denotes the number of degrees. " $\wedge$ " denotes that the rule in Eq. (1) follows the conjunctive assumption.

Another type of BRB is the disjunctive BRB, in which each rule describes the condition when the attributes are disjunctively correlated with each other [6] [8] [9]. Such a disjunctive rule could be activated if at least one attribute is activated.

$$\begin{aligned} R_k : & \text{if } (x_1 \text{ is } A_1^k) \vee (x_2 \text{ is } A_2^k) \vee \cdots \vee (x_M \text{ is } A_M^k), \\ & \text{then } \{(D_1, \beta_{1,k}), \dots, (D_N, \beta_{N,k})\} \\ & \text{with rule weight } \theta_k \end{aligned} \quad (2)$$

A conjunctive or disjunctive rule describes the condition when the attributes are conjunctively or disjunctively correlated with each other. Comparatively, a conjunctive rule would be activated only when all of the attributes are activated simultaneously, whereas a disjunctive can be activated with at least one attribute being activated.

With either conjunctive or disjunctive belief rules, different types of information are modeled into the same belief structure for the convenience of further processing and inferencing. BRB systems have been successfully applied in many problems from different fields [1] [5] [21] [22] [33].

Example 1: There are two indicators (attributes in BRB) for selling a product, its price (x_1) and quality (x_2). Supposing that there are two reference values for each attribute, the four rules are as follows:

$$\begin{aligned} R_1 : & \text{if } (x_1 \text{ is Low}) \wedge (x_2 \text{ is High}), \text{ then } \{(\text{Sell}, 100\%), (\text{not Sell}, 0)\} \\ R_2 : & \text{if } (x_1 \text{ is Low}) \wedge (x_2 \text{ is Low}), \text{ then } \{(\text{Sell}, 50\%), (\text{not Sell}, 50\%)\} \\ R_3 : & \text{if } (x_1 \text{ is High}) \wedge (x_2 \text{ is High}), \text{ then } \{(\text{Sell}, 50\%), (\text{not Sell}, 50\%)\} \\ R_4 : & \text{if } (x_1 \text{ is High}) \wedge (x_2 \text{ is Low}), \text{ then } \{(\text{Sell}, 0), (\text{not Sell}, 100\%)\} \end{aligned} \quad (3)$$

Apparently, price (x_1) and quality (x_2) are two conjunctive attributes. A conjunctive BRB with Rules 1-4 is constructed as in (2).

Example 2: The following are four conditions in an epidemiological study for the diagnosis of Hepatitis A:

- x_1 : Consumption of unclean food or water in the 2-7 weeks before the incidence;
- x_2 : Intimate contact with a patient of acute Hepatitis A;
- x_3 : An outbreak or epidemic of Hepatitis A in the local area;
- x_4 : Travel to places with an outbreak or epidemic of Hepatitis A.

If any of the above conditions is verified, then the epidemiology for Hepatitis A is verified. Therefore, it is apparent that the above four attributes are disjunctively correlated. Such a disjunctive Rule 5 should be constructed disjunctively:

$$R_5 : \text{if } (x_1 \text{ is true}) \vee (x_2 \text{ is true}) \vee (x_3 \text{ is true}) \vee (x_4 \text{ is true}), \quad (4) \\ \text{then } \{(\text{epidemiology} = \text{verified}, 100\%)\}$$

Example 3: In more complex conditions, it is not so easy to differentiate the conjunctive and disjunctive conditions. Let the diagnosis of a cold be an example,

- x_1 : the temperature of the patient;
- x_2 : whether the patient has got a cough;

Supposing that there is one Doctor A who believes that the two factors should be considered conjunctively, the conjunctive Rules 6-9 are constructed as follows:

$$\begin{aligned} R_6 : & \text{if } (x_1 \text{ is } 39^\circ\text{C}) \wedge (x_2 \text{ is severe}), \text{ then } \{(\text{got a cold}, 100\%), (\text{not got a cold}, 0)\} \\ R_7 : & \text{if } (x_1 \text{ is } 39^\circ\text{C}) \wedge (x_2 \text{ is normal}), \text{ then } \{(\text{got a cold}, 80\%), (\text{not got a cold}, 20\%)\} \\ R_8 : & \text{if } (x_1 \text{ is } 37^\circ\text{C}) \wedge (x_2 \text{ is severe}), \text{ then } \{(\text{got a cold}, 40\%), (\text{not got a cold}, 60\%)\} \\ R_9 : & \text{if } (x_1 \text{ is } 37^\circ\text{C}) \wedge (x_2 \text{ is normal}), \text{ then } \{(\text{got a cold}, 0), (\text{not got a cold}, 100\%)\} \end{aligned} \quad (5)$$

Moreover, there is also one Doctor B who believes that the two factors should be considered disjunctively, and the disjunctive Rules 10-11 are constructed as follows:

$$\begin{aligned} R_{10} : & \text{if } (x_1 \text{ is } 39^\circ\text{C}) \vee (x_2 \text{ is severe}), \text{ then } \{(\text{got a cold}, 90\%), (\text{not got a cold}, 10\%)\} \\ R_{11} : & \text{if } (x_1 \text{ is } 37^\circ\text{C}) \vee (x_2 \text{ is normal}), \text{ then } \{(\text{got a cold}, 20\%), (\text{not got a cold}, 80\%)\} \end{aligned} \quad (6)$$

Remark 1: For system with a clear understanding of its inputs in correlation with its outputs, a conjunctive BRB should be constructed if its inputs are conjunctively correlated, and vice versa. For systems with more complexly correlated attributes, experts' knowledge and experiences, or even their perceptions and preferences, play an important role in determining which type of BRB to construct.

Remark 2: If a conjunctive BRB and a disjunctive BRB are in the same belief structure, a rational question is whose inference result is right and should be trusted. The answer depends on which assumption it is built upon. As long as the assumption

has been determined, the type of BRB is determined, and its inference result is unique (the other type of BRB and its result would then be nonexistent).

2.2 Advantages and challenges by different types of BRB

Both conjunctive and disjunctive BRBs have their own advantages and challenges. The conjunctive BRB has the following advantage:

(1) **It is easy to understand the conjunctive rules:** the conclusion is the natural inferencing result of the attributes being assigned with specific referenced values.

(2) By extension, **it is also easy to extract such conjunctive rules** from experts and historical archives.

However, the conjunctive BRB also has to face different challenges:

(1) **The combinatorial explosion problem**, when there are too many attributes and/or referenced values for the attributes [9];

(2) People or even experts only have a grasp of **partial** knowledge; historical data may also be **insufficient** to construct and train a complete BRB;

(3) It is **hard or at least very expensive** to gather certain rules;

(4) Some rules are practically **unnecessary** (inefficient, since they are rarely activated (such rules are hard to train and validate)) but theoretically indispensable or the BRB would be incomplete;

(5) Too many rules could result in **overfitting**, especially when the dataset is relatively small.

The disjunctive BRB has the following advantage:

(1) **It can help avoid the combinatorial explosion problem:** the size requirement for a disjunctive BRB is far smaller, which could help avoid the combinatorial explosion problem.

(2) **By extension, it can help avoid certain very expensive, unnecessary, or impossible rules** with a guarantee of the completeness of the disjunctive BRB.

However, the disjunctive BRB also has to face the following challenges:

(1) **It is inconsistent with human intuition and historical records;**

(2) By extension, **it is hard to directly gather disjunctive rules**. If the data source is historical archives, then the data are conjunctive to the system output; if they are from experts' knowledge, then it is intuitionistic from people's natural knowledge.

Example 4: Consider an extreme condition when more observations and tests have been done on a patient to make a diagnosis of whether the patient has got a cold,

- x_1 : the temperature of the patient;
- x_2 : whether the patient has got a cough;

- x_3 : whether the patient has got a runny nose;
- x_4 : the white blood cell count of the patient;

Doctor A gives the following conjunctive rule:

$$R_{12} : \text{if } (x_1 \text{ is } 39^\circ\text{C}) \wedge (x_2 \text{ is severe}) \wedge (x_3 \text{ is severe}) \wedge (x_4 \text{ is severe}), \quad (7)$$

$$\text{then } \{(\text{got a cold}, 100\%), (\text{not got a cold}, 0)\}$$

Such a conjunctive rule stands only if all of its attributes are activated. Apparently, it can be easily understood. However, suppose that there are three referenced values for each attribute and it would require $3^4=81$ rules to construct a complete conjunctive BRB.

Doctor B gives the following disjunctive rule:

$$R_{13} : \text{if } (x_1 \text{ is } 39^\circ\text{C}) \vee (x_4 \text{ is severe}), \quad (8)$$

$$\text{then } \{(\text{got a cold}, 90\%), (\text{not got a cold}, 20\%)\}$$

Such a disjunctive rule stands if at least one of its attributes is activated. This would significantly reduce the size requirement for the BRB; this is extremely important considering that certain rules are very expensive but rarely activated. However, it is difficult to directly construct such a disjunctive rule, especially the beliefs of scales in the conclusion part. As in (8), the beliefs should not be extreme numbers, i.e., 100% or 0, since a disjunctive rule is a combination of multiple conditions.

As discussed above and demonstrated by Example 4, both types of BRBs have their advantages as well as challenges. A new approach is devised to make full use of limited conjunctive rules to construct a disjunctive BRB to avoid the disadvantages of both the conjunctive and disjunctive BRBs.

3 The BRB space and correlations between conjunctive and disjunctive BRBs

To generate disjunctive rules from limited conjunctive rules requires quantifying their correlations. To do so, the mathematical basis for the BRB space is defined in Sections 3.1 and 3.2, based on which the correlation between the conjunctive and disjunctive BRBs is discussed in Section 3.3.

3.1 Mathematical basis of the BRB space

The attributes of the BRB are defined as an attribute measure space (as in **Definition 1**) [2] [14], and the referenced values for the attributes are defined as an attribute product measure space (as in **Definition 2**) [20].

Definition 1: attribute measure space

Let μ_x be the attribute measures in (A, B) , where $\mu_x(A)$ is used to measure the degree to which element x of the object space has attribute A . (A, B, μ_x) is an attribute measure space if the following restraints are satisfied.

- (1) Nonnegativity: $\mu_x(A) \geq 0, \forall A \in B$;
- (2) Regularity: $\mu_x(A) = 1$;
- (3) Countable additivity: $\mu_x(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_x(A_i)$, if $A_i \in B, A_i \cap A_j = \emptyset$

Definition 2: attribute product measure space

Let X be the object space, and let $(A_i, B_i, \mu_x^{(i)}), i = 1, 2, \dots, M$ be an attribute measure space on A_i . Thus, $(\prod_{i=1}^M A_i, \prod_{i=1}^M B_i, \prod_{i=1}^M \mu_x^{(i)})$ is the attribute product measure space, in which:

- (1) the multiple attribute space $\prod_{i=1}^M A_i = \{(a_1, a_2, \dots, a_M) | a_i \in A_i, i = 1, 2, \dots, M\}$;
- (2) the product σ algebra $\prod_{i=1}^M B_i = \sigma\{b_1 \times b_2 \times \dots \times b_M | b_i \in B_i, i = 1, 2, \dots, M\}$, in which “ $\times$ ” denotes that the intersected points in the BRB space are generated by the traversal combination of the referenced values of the attributes;
- (3) the measure function $\prod_{i=1}^M \mu_x^{(i)} = \{(\mu_x^{(1)}(b_1), \mu_x^{(2)}(b_2), \dots, \mu_x^{(M)}(b_M)) | \forall b_i \in B_i,$

$i = 1, 2, \dots, M\}$. The measure function is used to measure the value range of the referenced values of the attributes in the BRB.

The measure function $\prod_{i=1}^M \mu_x^{(i)}$ should satisfy three properties:

- (1) Nonnegative: $\mu_x(A) \geq 0, \forall A \in \prod_{i=1}^M B_i$;
- (2) Regularity: $\mu_x\left(\prod_{i=1}^M A_i\right) = 1$;
- (3) Countable additivity: $\mu_x\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_x(A_i)$, where

$A_j = (a_1^{(j)}, a_2^{(j)}, \dots, a_M^{(j)}) \in \prod_{i=1}^M B_i$, and $a_i^{(j)} \cap a_i^{(k)} = \emptyset, i = 1, 2, \dots, M$.

Example 5: Suppose that there are three referenced values for the temperature x of a patient, namely, $A_1 = 37^\circ\text{C}, A_2 = 39^\circ\text{C}, A_3 = 40^\circ\text{C}$, and suppose that the distribution among the three referenced values is even. Then, we have the following: for

nonnegativity, there is $\mu_x(A_1 = 37^\circ\text{C})=0, \mu_x(A_2 = 39^\circ\text{C})=2/3, \mu_x(A_3 = 40^\circ\text{C})=1/3$. For regularity, there is $\mu_x(\{A_1, A_2, A_3\} = \{37^\circ\text{C}, 39^\circ\text{C}, 40^\circ\text{C}\})=1$. For countable additivity, there is $\mu_x(\bigcup_{i=1}^{\infty} A_i) = \mu_x(\{A_1, A_2, A_3\}) = 1, \sum_{i=1}^{\infty} \mu_x(A_i) = \mu_x(\{A_1\}) + \mu_x(\{A_2\}) + \mu_x(\{A_3\}) = 1$.

If another factor, x_2 (whether the patient has got a cough), is also taken into account, x_2 has two referenced values $A_1 = no, A_2 = yes$. Thus, we have $\mu_{x_2}(A_1 = no)=0, \mu_{x_2}(A_2 = yes)=1$, which obviously satisfies the nonnegativity and the regularity and countable additivity properties.

Additional factors such as x_3 (whether the patient has got a runny nose) and x_4 (the white blood cell count of the patient) can also be included as attributes to assess whether a patient has got a cold.

Suppose that $(\prod_{i=1}^M A_i, \prod_{i=1}^M B_i, \prod_{i=1}^M \mu_x^{(i)})$ is an attribute product measure space and $[0, 1]^N$ is a vector space. Then, the following mapping relation is given:

$$R_k : \prod_{i=1}^M B_i \rightarrow [0, 1]^N.$$

The mapping relation can be regarded as the “THEN” part in the BRB model. More specifically, the belief for the N scales can be assigned based on the mapping relation when the referenced values of attributes are known.

In summary, the aforementioned product measure space is proposed based on the framework of the BRB. In comparison, for the BRB, X would be the object for the targeted problem, c would be the maximal attribute set for the BRB, the product σ algebra $\prod_{i=1}^M B_i$ would be constructed by some subsets of $\prod_{i=1}^M A_i$, and the measure function $\mu_x^{(i)}$ would be A_i^k under rule R_k , which correlates the attribute to its referenced values. Therefore, the BRB space can be recognized as an integration of rules. Furthermore, the number of referenced values for the attributes is finite when applying the BRB in practical problems. Hence, the size of the BRB space can be defined as the product of the number of referenced values for all attributes. To further study the disjunctive assumption, the base of the BRB space is defined in Section 3.2.

3.2 Base of the BRB space

The base of the BRB space is the combination of all the referenced values for the attributes in a BRB (as in **Definition 3**). The base of the BRB space is not unique.

Definition 3: the base of the BRB space

Let X be the object space and $\left(\prod_{i=1}^M A_i, \prod_{i=1}^M B_i, \prod_{i=1}^M \mu_x^{(i)}\right)$ be the attribute product measure space. The number of referenced values of attribute set A_i is $p(i)$ for $i = 1, 2, \dots, M$. Note that $N = \max_{i \in \{1, 2, \dots, M\}} p(i)$, which can be called the dimension of the BRB space. In addition, $\{x_1, x_2, \dots, x_N\}$ is a selected object set from the object space that satisfies the following conditions:

The corresponding measure of object x_k is $\mu_{x_k}(A) = (A_1^k, A_2^k, \dots, A_M^k)$ for $k = 1, 2, \dots, N$. For attribute set A_i , the rank of $\{A_i^k\}_{k=1}^N$ is $p(i)$ for $i = 1, 2, \dots, M$. Then, $\{(A_1^k, A_2^k, \dots, A_M^k)\}_{k=1}^N$ can be regarded as a base of the BRB space.

The above discussion on the base of the BRB space also directs the size requirement for different types of BRBs. Comparatively, the conjunctive BRB requires a traversal combination of the referenced values for the attributes $\sum_{m=1}^M p(m)$, while a disjunctive BRB requires at least the maximum number of referenced values for the attributes, $\max_{m=1}^M p(m)$, which is just its minimal base [8]. Table 1 shows the size comparison between the conjunctive and disjunctive BRBs.

Table 1 Size comparison between the conjunctive and disjunctive BRBs

assumption	No. attr.	No. ref. values for m th attr.	size of BRB	
			min	max
conjunctive	M	$p(m)$	$\prod_{m=1}^M p(m)$	
disjunctive			$\max_{m=1}^M p(m)$	$\prod_{m=1}^M p(m)$

Lemma 1: It is easy to find that the base of the BRB space is not unique as long as it covers all the referenced values for the attributes.

Proof: Without loss of generality, we suppose that the number of referenced values of attribute set A_i is N . One possible measure of object $\{x_1, x_2, \dots, x_N\}$ is in (9).

$$\begin{aligned}
 x_1 &: (A_1^1, A_2^1, \dots, A_M^1) \\
 x_2 &: (A_1^2, A_2^2, \dots, A_M^2) \\
 &\dots \\
 x_N &: (A_1^N, A_2^N, \dots, A_M^N)
 \end{aligned} \tag{9}$$

where $A_i^l \neq A_i^j, i \neq j, i, j = 1, 2, \dots, N$. For every one of the residual attribute sets, the total referenced values are assigned to the N location. Finally, the base of the BRB space can successfully achieve the goal of covering all the referenced values for the attributes.

Remark 3: Additionally, the following conclusions can be derived based on the concept of the BRB space:

- (1) The BRB space can be recognized as an integration of rules.
- (2) The minimal base of the BRB space is the set of rules that contains all referenced values for the attributes without repetition.
- (3) The minimal size of the BRB is the size of its base, and the maximal size of the BRB would be the product of the number of referenced values for all attributes.
- (4) Regardless of the size of the BRB, it is complete if and only if all referenced values for the attributes are covered to ensure that any input can be handled.

3.3 Correlation between conjunctive and disjunctive BRBs

Based on the definition of the BRB space, the attributes in either a conjunctive or a disjunctive BRB are the dimensions in the BRB space, and the referenced values for the attributes are the coordinates of the respective dimensions. In other words, a conjunctive rule is a crossing point in the BRB space, whereas a disjunctive rule represents all of the points that share the same referenced values for the attributes. Thus, a disjunctive rule would be correlated to a conjunctive rule with at least one shared referenced value for any attribute.

Definition 4: A conjunctive rule and a disjunctive rule are correlated with each other if they share at least one referenced value for at least one attribute, $(A_1^k = A_1^{k'}) \vee \dots \vee (A_M^k = A_M^{k'}) = \text{true}$.

Consider three rules under different assumptions as follows,

$$\left\{ \begin{array}{l} R_k : \text{if } (x_1 \text{ is } A_1^k) \wedge (x_2 \text{ is } A_2^k) \wedge \dots \wedge (x_M \text{ is } A_M^k), \\ \quad \text{then } \{(D_1, \beta_{1,k}), \dots, (D_N, \beta_{N,k})\} \text{ with rule weight } \theta_k \\ R_{k'} : \text{if } (x_1 \text{ is } A_1^{k'}) \vee (x_2 \text{ is } A_2^{k'}) \vee \dots \vee (x_M \text{ is } A_M^{k'}), \\ \quad \text{then } \{(D_1, \beta_{1,k'}), \dots, (D_N, \beta_{N,k'})\} \text{ with rule weight } \theta_{k'}. \\ R_{k^*} : \text{if } (x_i \text{ is } A_i^{k^*}), \\ \quad \text{then } \{(D_1, \beta_{1,k^*}), \dots, (D_N, \beta_{N,k^*})\} \text{ with rule weight } \theta_{k^*}. \end{array} \right. \quad (10)$$

Rule k' with two attributes in (10) is correlated to $p(m)$ rules for the m th attribute, $m = 1, \dots, M$. As there are M attributes in the rules in (10), the number of correlated conjunctive rules should be counted M times. As a consequence, for this specific case, there would be $\sum_{m=1}^M p(m)$ correlated conjunctive rules. A rule with the same referenced

value for all of the attributes, as in (10), would be counted M times. Therefore, any disjunctive rule is correlated to $\sum_{m=1}^M p(m) - (M-1)$ conjunctive rules that share at least one referenced value for the attributes.

Rule k'' in (10) with a single attribute is correlated to $p(i)$ rules for the i th attribute, $i = 1, \dots, M, i \neq m$. For the other $M-1$ attributes, the total correlated conjunctive rules are the product of the correlated rules for each attribute. Therefore, any disjunctive rule with a single attribute is correlated to $\prod_{i=1, i \neq m}^M p(i)$ conjunctive rules that share the same referenced value for one attribute.

To summarize, a disjunctive rule with multiple attributes is correlated to $\sum_{m=1}^M p(m) - (M-1)$ conjunctive rules in the same belief structure as the disjunctive rule. Comparatively, a disjunctive rule with a single attribute is correlated to $\prod_{i=1, i \neq m}^M p(i)$ conjunctive rules that share the same attribute.

Example 6: Suppose that there are two attributes and each attribute has three referenced values. Fig. 1 shows the possible BRBs under different assumptions and with different numbers of attributes (solid circles in Fig. 1 (a) and Fig. 1 (b) denote actual rules, and hollow circles denote projected rules; bold lines in Fig. 1 (c) denote actual rules, and hollow circles denote projected rules).

In Fig. 1(a), there are 9 rules, namely, Rules 1-9, in the complete conjunctive BRB, and each rule has two attributes. First, we consider the condition for a disjunctive BRB with two attributes, and each attribute has three referenced values. For this condition, if each disjunctive rule is assumed to have two attributes, a minimum of three rules would be sufficient to cover all the referenced values for the attributes as long as any referenced value for any attribute only appears once in a disjunctive rule. Furthermore, there would be six possible combinations of three disjunctive rules (see Fig. 5 in Section 5.1.1). Fig. 1(b) shows one possible combination of such a disjunctive BRB, namely, Rules 1'/5'/9'.

- R_1 : if $(x_1 \text{ is } A_1^1) \wedge (x_2 \text{ is } A_2^1)$, then $\{(D_1, \beta_{1,1}), \dots, (D_N, \beta_{N,1})\}$
 R_2 : if $(x_1 \text{ is } A_1^1) \wedge (x_2 \text{ is } A_2^2)$, then $\{(D_1, \beta_{1,2}), \dots, (D_N, \beta_{N,2})\}$
 R_3 : if $(x_1 \text{ is } A_1^1) \wedge (x_2 \text{ is } A_2^3)$, then $\{(D_1, \beta_{1,3}), \dots, (D_N, \beta_{N,3})\}$
 R_4 : if $(x_1 \text{ is } A_1^2) \wedge (x_2 \text{ is } A_2^1)$, then $\{(D_1, \beta_{1,4}), \dots, (D_N, \beta_{N,4})\}$
 R_5 : if $(x_1 \text{ is } A_1^2) \wedge (x_2 \text{ is } A_2^2)$, then $\{(D_1, \beta_{1,5}), \dots, (D_N, \beta_{N,5})\}$
 R_6 : if $(x_1 \text{ is } A_1^2) \wedge (x_2 \text{ is } A_2^3)$, then $\{(D_1, \beta_{1,6}), \dots, (D_N, \beta_{N,6})\}$
 R_7 : if $(x_1 \text{ is } A_1^3) \wedge (x_2 \text{ is } A_2^1)$, then $\{(D_1, \beta_{1,7}), \dots, (D_N, \beta_{N,7})\}$
 R_8 : if $(x_1 \text{ is } A_1^3) \wedge (x_2 \text{ is } A_2^2)$, then $\{(D_1, \beta_{1,8}), \dots, (D_N, \beta_{N,8})\}$
 R_9 : if $(x_1 \text{ is } A_1^3) \wedge (x_2 \text{ is } A_2^3)$, then $\{(D_1, \beta_{1,9}), \dots, (D_N, \beta_{N,9})\}$

- $R_{1'}$: if $(x_1 \text{ is } A_1^1) \vee (x_2 \text{ is } A_2^1)$, then $\{(D_1, \beta_{1,1'}), \dots, (D_N, \beta_{N,1'})\}$
 $R_{5'}$: if $(x_1 \text{ is } A_1^2) \vee (x_2 \text{ is } A_2^2)$, then $\{(D_1, \beta_{1,5'}), \dots, (D_N, \beta_{N,5'})\}$
 $R_{9'}$: if $(x_1 \text{ is } A_1^3) \vee (x_2 \text{ is } A_2^3)$, then $\{(D_1, \beta_{1,9'}), \dots, (D_N, \beta_{N,9'})\}$

- $R_{1''}$: if $(x_1 \text{ is } A_1^1)$, then $\{(D_1, \beta_{1,1''}), \dots, (D_N, \beta_{N,1''})\}$
 $R_{2''}$: if $(x_1 \text{ is } A_1^2)$, then $\{(D_1, \beta_{1,2''}), \dots, (D_N, \beta_{N,2''})\}$
 $R_{3''}$: if $(x_1 \text{ is } A_1^3)$, then $\{(D_1, \beta_{1,3''}), \dots, (D_N, \beta_{N,3''})\}$
 $R_{4''}$: if $(x_2 \text{ is } A_2^1)$, then $\{(D_1, \beta_{1,4''}), \dots, (D_N, \beta_{N,4''})\}$
 $R_{5''}$: if $(x_2 \text{ is } A_2^2)$, then $\{(D_1, \beta_{1,5''}), \dots, (D_N, \beta_{N,5''})\}$
 $R_{6''}$: if $(x_2 \text{ is } A_2^3)$, then $\{(D_1, \beta_{1,6''}), \dots, (D_N, \beta_{N,6''})\}$

Fig. 1 BRB under different assumptions (a: conjunctive BRB with 9 rules; b: disjunctive BRB with 3 rules; c: disjunctive BRB with 6 rules, where each rule has a single attribute)

In Fig. 1(c), there are 6 rules, namely, Rules 1''/2''/3''/4''/5''/6'', and each rule has only one attribute. Rules 1''/2''/3'' only have x_1 , while x_2 is missing, and Rules 4''/5''/6'' only have x_2 , while x_1 is missing.

Apparently, the number of rules in a disjunctive BRB, whether it has two attributes or only one attribute, is reduced in comparison with the conjunctive BRB in the same belief structure.

Fig. 2 shows the correlation between disjunctive and conjunctive rules. Fig. 2(a) shows that a disjunctive rule with multiple attributes is correlated to multiple conjunctive rules with the same referenced values for each attribute, while Fig. 2(b) shows that a disjunctive rule with a single attribute is also correlated to multiple conjunctive rules with the same referenced values for the attribute.

Fig. 2 Correlation between disjunctive and conjunctive BRBs

Table 2 shows the correlated conjunctive rules for the disjunctive rules with multiple/single attribute(s).

Table 2 Rule weights for the rules in Fig. 2

Fig. 2	disjunctive rules	correlated conjunctive rules concerning x_1	correlated conjunctive rules concerning x_2	correlated conjunctive rules
(a)	Rule 1'	1/2/3	1/4/7	1/2/3/4/7
	Rule 5'	4/5/6	2/5/8	2/4/5/6/8
	Rule 9'	7/8/9	3/6/9	3/6/7/8/9
(b)	Rule 1''	/	1/2/3	1/2/3
	Rule 2''	/	4/5/6	4/5/6
	Rule 3''	/	7/8/9	7/8/9
	Rule 4''	1/4/7	/	1/4/7
	Rule 5''	2/5/8	/	2/5/8
	Rule 6''	3/6/9	/	3/6/9

Based on Fig. 2 and Table 2, the conjunctive assumption can be reduced to the disjunctive assumption by eliminating the rules that contain the same referenced value(s) for the attributes. In retrospect, new disjunctive rules can be generated by projecting the referenced values for the attributes on each dimension to extend the disjunctive assumption to the conjunctive assumption.

The core problem lies in how to calculate the weights of the different conjunctive rules when integrating them into one disjunctive rule, regardless of whether the disjunctive rule has a single attribute or multiple attributes.

Example 7: Multiple disjunctive rules with the same referenced values for the same attribute are not contradictory.

If another disjunctive rule is introduced by another Doctor C, we specifically list the four rules in the same belief structure with the same referenced values as follows:

$$\begin{aligned}
 R_{14} &: \text{if } (x_1 \text{ is } 39^\circ\text{C}) \wedge (x_2 \text{ is severe}), \text{ then } \{(\text{got a cold}, 100\%), (\text{not got a cold}, 0)\} \\
 R_{15} &: \text{if } (x_1 \text{ is } 39^\circ\text{C}) \wedge (x_2 \text{ is normal}), \text{ then } \{(\text{got a cold}, 80\%), (\text{not got a cold}, 20\%)\} \\
 R_{16} &: \text{if } (x_1 \text{ is } 39^\circ\text{C}) \vee (x_2 \text{ is severe}), \text{ then } \{(\text{got a cold}, 90\%), (\text{not got a cold}, 10\%)\} \\
 R_{17} &: \text{if } (x_1 \text{ is } 39^\circ\text{C}) \vee (x_2 \text{ is normal}), \text{ then } \{(\text{got a cold}, 70\%), (\text{not got a cold}, 30\%)\}
 \end{aligned} \tag{11}$$

Consider that only the temperature of the patient, x_1 , has been taken, and it is 39°C . In other words, the referenced value for x_2 is missing. Under this condition, x_2 is assumed to be “*severe*” or “*normal*” with equal probabilities, or $\{(\text{severe}, 50\%), (\text{normal}, 50\%)\}$.

For a disjunctive BRB, Rules 16 and 17 would be activated with equal activated weights since the input for x_2 is missing. Integrating Rules 16 and 17 using the evidential reasoning (ER) algorithm [26] [29] [30], the inferencing result would be

$$\{(\text{got a cold}, 83.83\%), (\text{not got a cold}, 16.17\%)\}$$

For a conjunctive BRB, Rules 14 and 15 are also activated with equal activated weights. Also integrating Rules 14 and 15 using ER, the inferencing result would be

$$\{(\text{got a cold}, 92.86\%), (\text{not got a cold}, 7.14\%)\}$$

Although there are slightly differences in the beliefs, the results are consistent: (1) the scale with the largest belief by the different BRBs is the same, and (2) the largest beliefs by the different BRBs are close.

Note that even for the conjunctive rules, the activated rules are also “different from each other”. This is because certain input (x_2 in this example) information is missing, and all of the possible referenced values must be taken into consideration.

Remark 4: A natural question is raised on the equivalence between the two types of rules. To be more specific, would a disjunctive rule and its correlated conjunctive rules produce the same results with the same input?

The answer is no. This is because they are fundamentally constructed under different assumptions. A disjunctive rule would be used more often since it would be activated more frequently than a conjunctive rule in the same belief structure. Considering that a disjunctive rule is normally assumed to be less strict than a

conjunctive rule in the same belief structure (see (11) in Example 7)), the result produced by the disjunctive rule would tend to be more conservative (a smaller variance among the beliefs of the scales) if the result of the conjunctive rule is considered as the benchmark. Nevertheless, consistency between the different types of BRBs is still expected if they are constructed to model the same object.

Remark 5: Although a conjunctive and a disjunctive rule are correlated if they share at least one referenced value for one attribute (see **Definition 4**), this is a greatly relaxed requirement. Under practical conditions, one or more conjunctive rules are constructed first, and then the disjunctive rule(s) would be constructed. That is, a disjunctive rule and its correlated conjunctive rules are normally in the same belief structure (with the same referenced values for the same attributes).

Remark 6: It should be reemphasized that how a rule should be constructed, i.e., under which assumption, is primarily dependent on the actual correlations among the attributes. If their correlation is clear, then there is no confusion over whether to construct a conjunctive or a disjunctive rule.

Let **Examples 1** and **2** be recalled, as it is clear how the attributes are correlated with each other (the attributes in **Example 1** are conjunctively correlated, whereas they are disjunctively correlated in **Example 2**); conjunctive Rules 1-4 and disjunctive Rule 5 are constructed, respectively.

However, in more practical conditions, the correlation between the attributes is unclear. Under this circumstance, either assumption is applicable, and the selection of the assumption is mostly dependent on the decision maker's knowledge, experience and/or preference.

Let **Example 7** also be recalled. In Example 7, it is not clear or decisive how the attributes are correlated with each other. Therefore, the rules can be constructed under either assumption, mainly depending on the knowledge and experience of the doctors. As such, Rules 14/15 and 16/17 are constructed under different assumptions.

4 Disjunctive BRB modeling using limited conjunctive rules

4.1 New disjunctive BRB modeling approach

The below is six steps for the new disjunctive BRB modeling approach.

Step 1: Invite experts or use historic data to derive conjunctive rules;

In practical conditions, invite experts or use historic data to construct conjunctive rules.

For validation purposes, a complete BRB would be constructed first to provide a benchmark result as well as to further derive a disjunctive BRB using further steps.

Step 2: Identify the belief structure for the disjunctive BRB

For the disjunctive BRB to be constructed, it should be in the same belief structure (with the same attributes x and referenced values for the attributes A) and have the same scales $D(n)$ in the conclusion part as the conjunctive rules. Thus, only the beliefs β for the scales in the conclusion part in each rule need to be determined.

Step 3: Belief generation based on equal probability or self-organization mapping

With Steps 1 and 2, a disjunctive rule should be correlated with one or multiple conjunctive rules. Two means, namely, equal probability in Section 4.2 and SOM in Section 4.3, are proposed for belief generation in a new disjunctive rule (which further constructs a disjunctive BRB) using correlated conjunctive rules.

Step 3.1: Identify the correlated conjunctive rules

For each targeted disjunctive rule, gather correlated conjunctive rule(s) by identifying all of the correlated conjunctive rules for each referenced value for the attributes in the targeted disjunctive rule.

Step 3.2: Calculate the weights for the correlated conjunctive rules

There are two components for the weights for the correlated conjunctive rules: the initial weight for the conjunctive rules θ and the activated weights between the targeted disjunctive rule and the conjunctive rules $w_{activate}$.

Thus, the weights w for the correlated conjunctive rules would be

$$w = \theta * w_{activate} \quad (12)$$

in which θ would be assumed to be 1 for each rule if not given. The weights w are normalized before entering the next step for integration.

Step 3.3: Obtain the disjunctive rules by integrating the correlated conjunctive rules

The correlated L rules are integrated using ER as in Eqs. (13-14) [26],

$$\beta_n = \frac{\mu[\prod_{k=1}^L (w_k \beta_{n,k} + 1 - w_k \sum_{n=1}^N \beta_{n,k}) - \prod_{k=1}^L (1 - w_k \sum_{n=1}^N \beta_{n,k})]}{1 - \mu[\prod_{k=1}^L (1 - w_k)]} \quad (13)$$

$$\mu = [\sum_{n=1}^N \prod_{k=1}^L (w_k \beta_{n,k} + 1 - w_k \sum_{n=1}^N \beta_{n,k}) - (N-1) \prod_{k=1}^L (1 - w_k \sum_{n=1}^N \beta_{n,k})]^{-1} \quad (14)$$

where β_n represents the belief for the n th scale.

Step 4: Rule weight calculation

The rule weight for each new rule in the disjunctive BRB is determined by summing the rule weights of the correlated conjunctive rules.

Step 4.1: Identify the conjunctive rules correlated to the k 'th disjunctive rule. The k 'th disjunctive rule is correlated to $R_{k'}$ conjunctive rules, $k'=[1, \dots, K']$, $R_k=[1, \dots, K]$;

Step 4.2: Sum up the weights for the $R_{k'}$ correlated conjunctive rules;

$$\theta_{k'} = \sum_{r=1}^{R_{k'}} \theta_r \quad (15)$$

Step 4.3: Normalize the weight for the new disjunctive rule;

$$\theta_{k'} = \theta_{k'} / \sum_{k'=1}^{K'} \theta_{k'} \quad (16)$$

The normalized weight $\theta_{k'}$ is the weight for the k 'th disjunctive rule.

Step 5: Validation

Normally, the error between the results of the conjunctive and disjunctive BRBs is used for the validation. The mean absolute error (MAE) and the mean absolute percentage error (MAPE) are calculated by Eqs. (17-18).

$$\text{MAE} = \frac{1}{N} \sum_{n=1}^N \text{abs}(res_{\text{con},n} - res_{\text{dis},n}) \quad (17)$$

$$\text{MAPE} = \frac{1}{N} \sum_{n=1}^N \frac{\text{abs}(res_{\text{con},n} - res_{\text{dis},n})}{res_{\text{con},n}} \quad (18)$$

where $res_{\text{con},n}$ and $res_{\text{dis},n}$ denote the results by the conjunctive and disjunctive rules with the n th input, respectively, and N denotes the number of inputs.

Remark 7: As we have discussed that there are multiple means of calculating the weights for the disjunctive rules, other means may be plausible as well, i.e., Monte Carlo simulation. The selection of a specific means to calculate the rule weights should be determined by the characteristics of the specific problem.

Remark 8: In this study, the Evidential Reasoning (ER) algorithm is applied for integration. This is due to two main reasons: (1) ER has shown superior performance in handling multiple types of information under uncertainty, including complete and incomplete information [1] [15]-[17] [31]-[34], (2) to maintain consistency since many previous case studies have employed ER for integration, e.g., [27] [28] (for Case II in this study). However, it should also be clear that the use of ER is not compulsory, and integration algorithms of other forms are theoretically applicable as well.

4.2 Belief generation based on equal probability

The first means of belief generation is based on equal probability. That is, all conjunctive rules correlated to a disjunctive rule are assigned with equal probability.

First, assume that a BRB has M attributes and the m th attribute has $p(m)$ referenced values.

For a disjunctive BRB in which each rule has a single attribute, the weights of $\prod_{i=1, i \neq m}^i p(i)$ conjunctive rules correlated to such a disjunctive rule should be

$$w_{k'} = 1 / \prod_{i=1, i \neq m}^i p(i) \quad (19)$$

For a disjunctive rule with multiple attributes, as in (10), the equal probability principle assumes that it is correlated to $p(m)$ conjunctive rules for the m th attribute by equal probability. In total, there would be $\sum_{m=1}^M p(m)$ equal weights to be assigned. However, the conjunctive rule in the same belief structure as in (8) would be counted M times. Therefore, among the $\sum_{m=1}^M p(m) - (M - 1)$ conjunctive rules correlated to a disjunctive rule, the one in the same belief structure would be assigned the weight of $(M - 1) / \sum_{m=1}^M p(m)$, and the other $\sum_{m=1}^M p(m) - (M - 1)$ rules would be assigned a weight of $1 / \sum_{m=1}^M p(m)$.

The weight assigned to the k 'th conjunctive rule for deriving a disjunctive rule by probability is

$$w_{k'} = \begin{cases} 1 / \sum_{m=1}^M p(m) & \text{if } (A_1^k = A_1^{k'}) \vee \cdots \vee (A_m^k = A_m^{k'}) \vee \cdots \vee (A_M^k = A_M^{k'}) \\ (M - 1) / \sum_{m=1}^M p(m) & \text{if } (A_1^k = A_1^{k'}) \wedge \cdots \wedge (A_m^k = A_m^{k'}) \wedge \cdots \wedge (A_M^k = A_M^{k'}) \end{cases} \quad (20)$$

Example 6: As in Example 5, a system is assumed to have two attributes and each attribute has three referenced values. Under this condition, a disjunctive rule with two attributes is thus correlated to five conjunctive rules ($\sum_{m=1}^2 p(m) - (2 - 1) = 3 + 3 - 1 = 5$), and a disjunctive rule with a single attribute is correlated to three conjunctive rules ($\sum_{m=1}^1 p(m) - (1 - 1) = 3$).

Disjunctive **Rule 1'** with two attributes is correlated to conjunctive Rules 1/2/3/4/7 with weights of 2/6, 1/6, 1/6, 1/6, and 1/6, respectively.

Disjunctive **Rule 1**'' with a single attribute is correlated to conjunctive Rules 1/2/3 with weights of 1/3, 1/3, and 1/3, respectively.

4.3 Belief generation based on self-organization mapping

The second means of belief generation is based on SOM. SOM was originally used to map the input in a higher-dimensional space to a lower-dimensional space. SOM can be considered “a nonlinear version of PCA (principal component analysis)” [13]. SOM has been used for missing information imputation [24] [25], and it has been found that SOM has better performance than hot-desk and standard multilayer perception-based imputation [18] [19]..

In the SOM-based belief generation, each disjunctive rule is generated by quantifying its “distance” to the correlated conjunctive rules. More specifically, the distance is derived by summing up the differences between the referenced values for the attributes of the disjunctive and correlated conjunctive rules. They are then again used to inversely produce the weights for different rules.

Fig. 3 shows the SOM-based belief generation. The separate conjunctive rules in the upper right and lower left denote the conjunctive rules correlated to different disjunctive rules. The green lines denote their correlations, which are quantitatively measured as the weights of the correlated conjunctive rules.

Fig. 3 SOM-based belief generation

Calculate the distance between the k 'th disjunctive rule and a k th rule out of the $\sum_{m=1}^M p(m) - (M - 1)$ correlated conjunctive rules by Eq. (21),

$$w_D = \sqrt{\sum_{m=1}^M (A_m^k - A_m^{k'})^2} \quad (21)$$

The weight assigned to the k 'th conjunctive rule by SOM is

$$w_k = \frac{\max(w_D) - w_{D,k}}{\sum_{m=1}^M p(m) - (M-1) \sum_{i=1}^M (\max(w_D) - w_{D,i})} \quad (22)$$

Based on Eq. (22), the farthest rule would be assigned a weight of “0”, and the sum of the weights for all rules would be “1”.

Example 7: As in Example 5, suppose that Attributes 1/2 both have the referenced values of 5/8/10,

Disjunctive **Rule 1'** with two attributes and referenced values of “5/5” is correlated to conjunctive Rules 1/2/3/4/7 with weights of 0.5556, 0.2222, 0, 0.2222, and 0, respectively.

For disjunctive **Rule 1''** with a single attribute (e.g., Attribute 1), the SOM-based procedure is inapplicable, and only equal probability is applicable.

Remark 9: The Denoising Autoencoder Self-Organization Map (DASOM) has been developed to handle unsupervised learning that requests unlabeled data or clustering problems [12]. It has been very promising in successfully solving “a comprehensive series of experiments comprising optical recognition of text and images” [12]. However, the proposed approach in this study aims at modeling systems with labeled data, or supervised learning, so DASOM is inapplicable. Nevertheless, if the conditions are changed and unlabeled data are introduced, DASOM should be further tested.

5 Case study

5.1 Case I: Numerical case

5.1.1 Basics and inputs

The numerical example is to illustrate the proposed disjunctive modeling approach. Suppose that there are two attributes, namely, x_1 and x_2 , and each has three referenced values, namely, 5, 8 and 10. Consequently, 9 rules would be required to construct a complete conjunctive BRB, as shown in Fig. 4. The conjunctive BRB with two attributes is given in Table 3. For validation, 1000 inputs with x_1 and x_2 as attributes are randomly generated. The conjunctive BRB is used as the benchmark, and the MAE and MAPE are calculated by Eqs. (17-18).

Fig. 4 Complete conjunctive BRB with 9 rules for the numerical case

Table 3 Conjunctive BRB with two attributes for Case I

No.	IF		THEN	
	x_1	x_2	Good	AVERAGE
1	5	5	0.50	0.50
2	8	5	0.55	0.45
3	10	5	0.70	0.30
4	5	8	0.55	0.45
5	8	8	0.70	0.30
6	10	8	0.85	0.15
7	5	10	0.80	0.20
8	8	10	0.90	0.10
9	10	10	0.95	0.05

5.1.2 Numerical case study results

The disjunctive BRB with both attributes is first generated with six possible combinations of disjunctive rules, as shown in Fig. 5.

Fig. 5 Six possible disjunctive BRBs with two attributes for the numerical case

The first disjunctive Rule 1 in Fig. 5(a) with $x_1=5$ and $x_2=5$ would be correlated to Rules 1/2/3/4/7 with weights of 2/6, 1/6, 1/6, 1/6, and 1/6, respectively. Integrate them by ER, and a disjunctive rule is derived as follows:

$$\text{if } (x_1 \text{ is } 5) \vee (x_2 \text{ is } 5), \text{ then } \{G = 61.93\%, A = 38.07\%\}$$

Table 4 shows the disjunctive BRBs in the same belief structure as the conjunctive BRB in Table 3 by equal probability (BRB_ep) and SOM (BRB_SOM).

Table 4 BRB_ep and BRB_SOM for Case I

No.	IF		THEN (BRB_ep)		THEN (BRB_SOM)	
	x_1	x_2	Good	AVERAGE	Good	AVERAGE
1	5	5	0.6193	0.3807	0.5236	0.4764
2	8	5	0.6789	0.3211	0.6242	0.3758
3	10	5	0.7488	0.2512	0.7176	0.2824
4	5	8	0.6882	0.3118	0.6533	0.3467
5	8	8	0.7486	0.2514	0.7905	0.2095
6	10	8	0.8143	0.1857	0.8591	0.1409
7	5	10	0.7961	0.2039	0.7869	0.2131
8	8	10	0.8482	0.1518	0.8909	0.1091
9	10	10	0.9005	0.0995	0.9385	0.0615

Similarly, we would illustrate how to derive a disjunctive rule R1' when the referenced value for x_1 is 5 and x_2 is missing (x_2 could be 5, 8, or 10 with equal probabilities). Then, Rule R1' would be correlated to Rules 1/4/7 in Table 2 with equal weights of 1/3. Integrate them by ER, and Rule R1' is derived as follows:

$$\text{if } (x_1 \text{ is } 5), \text{ then } \{G = 64.27\%, A = 35.73\%\}$$

Similarly, other disjunctive rules with only one attribute (BRB_single) are given in Table 5.

Table 5 Disjunctive BRB with a single attribute for Case I

No.	IF		THEN (BRB_single)	
	x_1	x_2	Good	AVERAGE
1	5		0.6427	0.3573
2	8		0.7574	0.2426
3	10		0.8731	0.1269
4		5	0.6021	0.3979
5		8	0.7385	0.2615
6		10	0.9156	0.0844

5.1.3 Result comparison and discussion

As discussed in Fig. 5, there are 6 possible disjunctive BRBs with two attributes, and each has 3 referenced values. Therefore, 6 disjunctive BRBs (each with 3 disjunctive rules) are used for further comparison in Section 5.1.3.

The MAE and MAPE results of the different disjunctive BRBs are given in Table 6. The first column “rules” of Table 6 show the combination of the disjunctive rules.

Table 6 MAEs and MAPEs of results in 6 disjunctive BRB structures for Case I

No.	rules	Fig. 5	MAE (10^{-3})		MAPE (10^{-3})	
			BRB_ep	BRB_SOM	BRB_ep	BRB_SOM
1	1-5-9	(a)	73.6725	153.9086	117.4791	247.9063
2	2-4-9	(d)	77.5460	125.1851	122.6304	197.4034
3	1-6-8	(b)	97.6927	166.0029	154.3118	267.4116
4	3-5-7	(e)	115.5569	136.9188	183.2134	219.0351
5	2-6-7	(c)	115.2020	150.4778	182.1889	242.2386
6	3-4-8	(f)	100.8259	122.8086	159.1906	192.8282

Table 7 further summarizes the minimum and average MAEs and MAPEs. It shows that the BRB_ep and BRB_SOM with Rules 1/5/9 produce rather close results.

Table 7 Result comparison for Case I

	MAE (10^{-3})			MAPE (10^{-3})		
	min	average	variance (10^{-6})	min	average	variance (10^{-6})
BRB_single	77.1562			124.1320		
BRB_ep	73.6725	96.7493	322.6085	117.4791	153.1690	797.7518
BRB-SOM	122.8086	142.5503	292.9752	192.8282	227.8039	881.1232

Based on Table 7, it could be found that the result is very stable on all parameters (min/avg/var of MAE/MAPE). Only a slight decrease in the min/avg values has been identified, and the variance of the MAE even decreased. Overall, all of the indicators are maintained at rather stable levels.

Of the two calculation methods, BRB_ep produces the optimal result, while that produced by BRB_SOM has less satisfactory results. This is probably caused by the weights assigned to different rules. Comparatively, the weights assigned to rules by BRB_ep are highly “attached” to each rule. The less satisfactory performance by

BRB_SOM is most likely caused by the overinclusion of correlated rules (only the farthest rules would be assigned a weight of “0”).

The combination of the disjunctive BRB should be carefully chosen. With a good combination of disjunctive rules, very consistent results can be obtained with far reduced input information. Even considering an average level, the results are rather satisfactory---values of 0.073/0.096 for the min/avg. MAE and 0.117/0.153 for the min/avg. MAPE in 1000 runs. The results are very stable, as the variances of the MAEs/MAPEs in 1000 runs are very small.

Note that the differences in the case study results by different methods could also be case-specific. That is, they could also be caused by different settings of a specific case. For different cases, different methods could produce varied results and performances. In fact, it is believed that the selection of a specific method for belief generation should be case-specific, as different methods have different features and therefore are fit for different problems. There thus is not and should not be a universal method for all problems.

5.2 Case II: Consumer preference prediction

5.2.1 Background

The new product development problem attempts to model consumers’ needs and preferences for a new product, which is very important for a company to launch a new product [27] [28]. In the modeling process, there are multiple factors that need to be taken into consideration. Therefore, many researchers use the Multi-Attribute Decision Analysis (MADA) model for the new product development problem. In this study, the BRB is used to solve this problem.

In [27] [28], six factors are considered, namely, Package size (A1), Appearance (A2), Convenience (A3), Aroma (A4), Taste (A5), and Brew color (A6). Then, principal component analysis (PCA) is used to integrate them into two principal components, namely, packaging (comprising factors A1-A3) and quality (comprising of factors A4-A6). The constructed “*packaging*” and “*quality*” are used to infer the consumers’ preferences. In other words, a two-layer model is constructed with three sub-BRBs to avoid the combinatorial explosion problem if the six factors are directly used as attributes in constructing a conjunctive BRB.

Fig. 6 shows the model structure for the consumer preference prediction problem.

Fig. 6 Consumer preference prediction evaluation structure

Even taking only the upper-level model under consideration, a total of 25 rules are required to construct a complete conjunctive BRB if “*quality*” and “*packaging*” are assumed to have five referenced values (namely, H1-H5).

Next, this specific case is taken as an example to validate the proposed approach by using reduced conjunctive rules to construct a complete disjunctive BRB.

5.2.2 Constructing the disjunctive BRB

Step 1: Construct the conjunctive BRB.

For consistency, the original conjunctive BRB from [28] is used as the initial conjunctive BRB, as shown in Table A.1 in the Appendix. In this conjunctive BRB, each rule has two attributes, namely, *packaging* and *quality*, and each attribute has five referenced values. For *packaging*, they are -2.910, -0.621, 0.063, 0.510, and 2.696, and for *quality*, they are -2.587, -0.700, 0.177, 0.817, and 1.913. The conclusion part is assumed to have five scales.

Step 2: Identify the belief structure for the disjunctive BRB.

For consistency, the belief structure of the disjunctive BRB is the same as that of the conjunctive BRB. Therefore, the disjunctive BRB also has 5 rules, with each rule comprising one referenced value for each attribute. The conclusion part is again assumed to have five scales.

Step 3: Belief generation.

Disjunctive BRBs using only half of the original conjunctive rules are derived. Consequently, the equal probability and SOM techniques are used.

Step 3.1: Identify the original rules.

To avoid confusion, the above 25 rules are divided into two halves, with the first half comprising Rules 1/3/5/.../21/23/25 and the second half comprising Rules 2/4/6/.../20/22/24. In other words, only half of the original information is used to construct a new disjunctive BRB.

Step 3.2: Identify correlated conjunctive rules.

For a targeted disjunctive rule, identify its correlated conjunctive rules. Take the first disjunctive rule with *packaging* as -2.910 and *quality* as -2.587 as an example. For *packaging*, it is related to conjunctive Rules 1/3/5, and it is related to Rules 1/11/21 for *quality*. Therefore, Rules 1/3/5/11/21 are correlated for such a disjunctive rule with *packaging* as -2.910 and *quality* as -2.587. The correlated conjunctive rules for other disjunctive rules can also be identified using this procedure.

Step 3.3: Calculate the weights for the correlated conjunctive rules.

Both equal probability and SOM are used to calculate the weights for the correlated conjunctive rules. Next, the equal probability-based procedures are used to illustrate the steps. Again, take the disjunctive rule with *packaging* as -2.910 and *quality* as -2.587 as an example to show the detailed calculation procedures to calculate the weights.

First, the weights for Rules 1/3/5/11/21 in Table 7 are 2/6, 1/6, 1/6, 1/6, and 1/6, since Rule 1 and the disjunctive rule have the same referenced values for both *packaging* and *quality*.

Then, considering that the initial weights for Rules 1/3/5/11/21 (also in Table 7) are 0.01, 1.00, 0.90, 0.00, and 0.84, the integrated weights for the five rules are 0.0033, 0.1667, 0.1500, 0, and 0.1400, respectively.

Finally, they are normalized as 0.0072, 0.3623, 0.3261, 0, and 0.3043, respectively.

Step 3.4: Belief integration using ER.

Consequently, the beliefs of such a disjunctive rule are derived as follows by integrating the beliefs of the correlated conjunctive rules using ER.

$$\{0.2306, 0.1307, 0.3123, 0.2569, 0.0695\}$$

The rest of the disjunctive rules can be derived using similar procedures.

Step 4: Rule weight calculation.

The weight for this rule is derived as $2.75(=0.01+1+0.9+0+0.84)$, and the weights for the other rules are calculated as 3, 2.25, 2.96, and 4.46, respectively. These weights are normalized to 0.1783, 0.1946, 0.1459, 0.1920, and 0.2892, respectively.

Tables 8-11 show the obtained disjunctive BRB with the first/second half rules and following the equal probability/SOM assumption, namely, BRB_ep_1/2 and BRB_SOM_1/2. Note that the initial rule weight calculation is not compulsory. Thus, Tables 8-11 assume initial weights of “1” (equal weights).

When *packaging* and *quality* are used as attributes in this case, they are also inferencing results by taking the package size/appearance/convenience and

aroma/taste/brew color as attributes, respectively. Similarly, different disjunctive BRBs can be constructed using the proposed approach as it is applied in this case. However, the data were not presented in [28] and is thus unavailable for this study.

Table 8 Disjunctive BRB with the first half rules and equal probability (BRB_ep_1) for Case II

No.	weights		inputs		belief degrees				
	initial	summed	packaging	quality	D ₁	D ₂	D ₃	D ₄	D ₅
1	1	0.1783	-2.910	-2.587	0.2306	0.1307	0.3123	0.2569	0.0695
2	1	0.1946	-0.621	-0.700	0.0194	0.0035	0.1185	0.2071	0.6515
3	1	0.1459	0.063	0.177	0.1769	0.0661	0.0381	0.2306	0.4883
4	1	0.1920	0.510	0.817	0.0451	0.0184	0.0071	0.2989	0.6305
5	1	0.2892	2.696	1.913	0.1655	0.0638	0.0053	0.2034	0.5619

Table 9 Disjunctive BRB with the second half rules and equal probability (BRB_ep_2) for Case II

No.	weights		inputs		belief degrees				
	initial	summed	packaging	quality	D ₁	D ₂	D ₃	D ₄	D ₅
1	1	0.1266	-2.910	-2.587	0.0504	0.1301	0.0187	0.1264	0.6744
2	1	0.2346	-0.621	-0.700	0.0572	0.0870	0.0576	0.2886	0.5096
3	1	0.1827	0.063	0.177	0.0015	0.0200	0.0513	0.4286	0.4986
4	1	0.2654	0.510	0.817	0.0522	0.0259	0.0031	0.2516	0.6672
5	1	0.1907	2.696	1.913	0.1092	0.0364	0.0105	0.2243	0.6196

Table 10 Disjunctive BRB with the first half rules and SOM (BRB_SOM_1) for Case II

No.	weights		inputs		belief degrees				
	initial	summed	packaging	quality	D ₁	D ₂	D ₃	D ₄	D ₅
1	1	0.1783	-2.910	-2.587	0.0689	0.0948	0.7030	0.1078	0.0255
2	1	0.1946	-0.621	-0.700	0.0036	0.0000	0.1958	0.0501	0.7505
3	1	0.1459	0.063	0.177	0.1148	0.0463	0.0621	0.2247	0.5521
4	1	0.1920	0.510	0.817	0.0441	0.0257	0.0011	0.1288	0.8003
5	1	0.2892	2.696	1.913	0.1489	0.0536	0.0000	0.1113	0.6862

Table 11 Disjunctive BRB with the second half rules and SOM (BRB_SOM_1) for Case II

No.	weights		inputs		belief degrees				
	Initial	summed	packaging	quality	D ₁	D ₂	D ₃	D ₄	D ₅
1	1	0.1266	-2.910	-2.587	0.0869	0.3220	0.0476	0.1051	0.4384
2	1	0.2346	-0.621	-0.700	0.0248	0.0527	0.0902	0.2854	0.5469
3	1	0.1827	0.063	0.177	0.0000	0.0377	0.0367	0.4073	0.5184
4	1	0.2654	0.510	0.817	0.0403	0.0315	0.0054	0.3521	0.5707
5	1	0.1907	2.696	1.913	0.1830	0.0510	0.0000	0.0973	0.6688

5.2.3 Result comparison and discussion

Table 12 gives the results of the conjunctive BRB, BRB_ep_1/2 and BRB_SOM_1/2 as well as their MAPEs. Table 12 shows that BRB_ep_1 outperforms

the rest, as it produces the smallest MAPE with or without the consideration of initial rule weights. Moreover, it is very stable, since the variance of the absolute percentage error is very small.

Table 12 Results by disjunctive BRB for 10 testing cases for Case II

Case No.	actual result	conjunctive BRB	disjunctive (no weight)				disjunctive (with weight)			
			BRB_ep_1	BRB_ep_2	BRB_SOM_1	BRB_SOM_2	BRB_ep_1	BRB_ep_2	BRB_SOM_1	BRB_SOM_2
1	7.9	8.39	9.11	9.19	9.32	9.30	9.16	9.14	9.34	9.28
2	8.8	8.24	9.44	9.22	9.58	9.29	9.28	9.16	9.44	9.28
3	8.5	8.82	8.37	9.36	8.77	9.36	7.62	7.63	7.63	7.63
4	8.6	8.56	9.44	9.21	9.58	9.28	8.90	9.15	9.14	9.06
5	8	7.45	8.74	9.30	8.93	8.99	8.23	9.09	8.36	8.91
6	8.3	8.16	8.92	9.20	9.18	9.12	8.87	9.26	9.12	9.11
7	7.1	7.99	9.20	9.22	9.36	9.17	8.83	9.19	9.08	9.04
8	9.5	8.89	9.29	9.35	9.48	9.25	8.76	9.22	9.02	9.00
9	8.9	8.67	9.31	9.37	9.50	9.31	8.85	9.28	9.12	9.12
10	9.5	8.51	9.25	9.35	9.45	9.26	9.25	9.36	9.45	9.29
MAPE		0.0573	0.0896	0.1037	0.1030	0.1024	0.0803	0.0998	0.0925	0.0969
vara.		0.0014	0.0070	0.0075	0.0088	0.0066	0.0053	0.0070	0.0067	0.0060

Table 13 shows the comprehensive result comparison of the conjunctive BRB, multi linear regression (MLR) and artificial neural network (ANN) [28].

Table 13 Result comparison for Case II

	conjunctive BRB	BRB_ep_1	MLR	ANN
accuracy	94.27%	91.97%	87.76%	93.23%
MAPE	0.0573	0.0803	0.1224	0.0677
variance	0.007012	0.005283	0.003074	0.006729
information	100%	52%	100%	100%

Since [28] initially used the conjunctive BRB, it is used as the benchmark in this practical case. Therefore, the comparison is only conducted among the disjunctive BRB (BRB_ep_1), MLR and ANN. It shows that, with only 52% of the original information, its modeling accuracy is rather close to the best results (conjunctive BRB), slightly inferior to ANN and outperforms MLR. Another indicator is the variance of the outputs of the 10 cases. It shows that MLR produces the most stable results with the smallest variance (however, its accuracy is inferior). Comparatively, the disjunctive BRB produces superior results in comparison with the conjunctive BRB and ANN, which validates its high stability with a relatively small variance. This is partially because each disjunctive rule is the integration of multiple original conjunctive rules, which tends to be more comprehensive, or even conservative for this specific case.

Remark 10: As demonstrated by the case study results, a disjunctive BRB can produce quite consistent results while carrying less information in comparison with a conjunctive BRB if a rule can be recognized as information. This is consistent with practical perceptions: making a decision normally does not need full and complete prior knowledge because it could be very expensive or even impractical to do so. Furthermore, it may be not necessary at all. In many practical conditions, only grasping the key knowledge is sufficient to make a relatively sound or at least acceptable decision. In this study, a complete conjunctive BRB would require too many rules, especially when there are too many factors, whereas a disjunctive BRB can be more focused and still be able to model the complexity of the targeted problem.

5.3 Discussion

To summarize the case study results, the following can be drawn:

(1) With limited conjunctive rules (insufficient to construct a complete conjunctive BRB), a complete disjunctive BRB can be constructed. This is especially useful in conditions when certain rules are very expensive or even improbable to extract;

(2) The constructed disjunctive BRB can produce rather close results to those by the conjunctive BRB. This is especially important considering that the input requirement has been reduced.

(3) High robustness has been shown among the results of the disjunctive BRB for both cases, as demonstrated by the very small variances in comparison with the min/avg. error, which proves the high reliability of the proposed disjunctive BRB modeling approach.

(4) Considering that some very expensive rules which are barely used but also indispensable for a complete conjunctive BRB are not compulsorily required in the first place, this new approach is of special meaning from an engineering and practical perspective.

(5) The traditional training and learning approach for conjunctive BRBs and disjunctive BRBs may also be applicable for the new disjunctive BRB modeling approach, pending revisions being made to the characteristics of the targeted problem.

We would also like to address the ability of the proposed approach to handle incomplete information from two perspectives:

(1) The evidence theory has the ability to handle incomplete information, mainly through the ER algorithm considered in this study.

We have discussed this topic in a more detailed fashion in [3], which shows that ER can well handle incomplete information, and it could also help reduce the scope of the incomplete information through an integration process under many circumstances.

(2) The disjunctive BRB can also help produce satisfactory results with less information as inputs in comparison with the conjunctive BRB.

As we discussed in the case study part, the proposed approach has been proven to be effective in producing matching results with less input. This ability can be further extended to conditions when there is incomplete information in the input, although different types of disjunctive BRBs may show different performances in handling the incomplete information. This discussion should be fully addressed in further studies in a more extensive and comprehensive fashion.

6 Conclusion

This paper proposed a new disjunctive BRB modeling approach by making full use of limited conjunctive rules. Moreover, to the best of our knowledge, this is also the first attempt among existent BRB-related researches to discuss the correlations between different types of BRBs from a mathematical perspective.

First, the mathematical basis for the new approach is given. Then, the new disjunctive BRB modeling approach is proposed using equal probability and self-organization mapping. With only limited conjunctive rules, a complete disjunctive BRB is constructed.

The new approach can make full use of both the conjunctive and disjunctive BRBs. First, it is easy to initialize the conjunctive rules. Furthermore, as they are integrated as a disjunctive BRB, the combinatorial explosion problem can be avoided. Note that the integrated disjunctive BRB, although derived from insufficient/limited conjunctive rules, is still complete since any input can be well handled.

The case study results show that a very consistent result can be drawn by the constructed disjunctive BRBs, regardless of the calculation method or sampling technique. Moreover, it has shown a good ability of producing stable results.

For future research, other techniques can be applied to more efficiently construct the disjunctive BRB with limited conjunctive rules. Moreover, using BRB training and learning techniques, the modeling ability can be improved.

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Appendix

Table A.1 Initial conjunctive rules for Case II

Rule ID	Rule weight	attributes		conclusion (belief degrees)				
		packaging	quality	D1	D2	D3	D4	D5
1	0.01	-2.91	-2.587	0.00	0.00	0.00	0.28	0.72
2	1.00	-2.91	-0.700	0.08	0.33	0.05	0.11	0.43
3	1.00	-2.91	0.177	0.08	0.11	0.80	0.02	0.00
4	1.00	-2.91	0.817	0.00	0.00	0.00	0.14	0.86
5	0.90	-2.91	1.913	0.00	0.00	0.04	0.77	0.20
6	0.16	-0.621	-2.587	0.46	0.15	0.00	0.00	0.39
7	1.00	-0.621	-0.700	0.00	0.00	0.21	0.00	0.79
8	0.88	-0.621	0.177	0.00	0.00	0.22	0.34	0.44
9	1.00	-0.621	0.817	0.05	0.02	0.02	0.27	0.65
10	0.91	-0.621	1.913	0.00	0.00	0.05	0.62	0.32
11	0.00	0.063	-2.587	0.00	0.00	0.00	0.37	0.63
12	0.66	0.063	-0.700	0.01	0.02	0.00	0.26	0.72
13	0.52	0.063	0.177	0.15	0.06	0.11	0.12	0.56
14	0.93	0.063	0.817	0.00	0.00	0.01	0.81	0.18
15	0.99	0.063	1.913	0.06	0.03	0.00	0.46	0.45
16	0.23	0.51	-2.587	0.04	0.00	0.00	0.44	0.52
17	1.00	0.51	-0.700	0.06	0.00	0.02	0.84	0.09
18	0.98	0.51	0.177	0.00	0.07	0.01	0.26	0.66
19	0.96	0.51	0.817	0.05	0.03	0.00	0.13	0.80
20	1.00	0.51	1.913	0.08	0.03	0.00	0.06	0.83
21	0.84	2.696	-2.587	0.68	0.30	0.00	0.01	0.01
22	0.82	2.696	-0.700	0.17	0.08	0.00	0.20	0.56
23	0.74	2.696	0.177	0.44	0.16	0.00	0.11	0.29

24	0.87	2.696	0.817	0.27	0.07	0.00	0.12	0.54
25	0.99	2.696	1.913	0.04	0.01	0.00	0.01	0.93