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Linguistic terms with weakened hedges: A model for qualitative decision making under uncertainty

Hai Wang^{a,b}, Zeshui Xu^c, Xiao-Jun Zeng^b

^a*School of Government Audit, Nanjing Audit University, Nanjing, Jiangsu 211815, China*

^b*School of Computer Science, University of Manchester, Manchester, M13 9PL, UK*

^c*Business School, Sichuan University, Chengdu, Sichuan, 610064, China*

Abstract

When expressing the experts' opinions in qualitative decision making (QDM), linguistic hedges can be considered to modify the force expressed by a predefined linguistic term. If an expert is not sure to select one term, weakened hedges would be a natural way to express the uncertainty. This is usually implemented by using a hedge to modify the most possible term, like the expression "more or less good". To model the uncertainty implied by hedges in QDM, this paper presents a novel linguistic representational and computational model in which the linguistic expressions take the form of a weakened hedge and a linguistic term, which is named as linguistic term with weakened hedge (LTWH). The syntax of LTWHs is defined by a set of hedges and a set of linguistic terms. The semantics of a LTWH is determined, objectively, based on the semantics of the term and a similarity measure of the reference domain. Accordingly, the negation, order relations and some basic operations of LTWHs are defined. To illustrate the effectiveness of LTWHs in granular computing, the connection to some multi-granularity linguistic models is exploited and a process for unifying multi-granularity linguistic information is developed. The major contributions of this paper are: (1) The proposed model enables a new manner to express and operate uncertain linguistic information in QDM; (2) it possesses clear syntax and semantics and the computational results are very interpretable; and (3) the proposed solution of multi-granularity linguistic unification maintains the semantics of the original linguistic information.

Keywords: Decision making, linguistic hedges, linguistic term sets, multi-granularity linguistic decision making, semantics

1. Introduction

Computing with words (CWW) is very useful and effective for qualitative decision making (QDM) problems, especially if the decision information is not quantifiable due to its nature or too expensive to obtain precise quantitative information [28, 17]. Different from the common sense of soft computing, such as the techniques proposed in Refs. [1, 3, 32, 43], CWW focuses on processing words and expressions constructed by a natural or artificial language rather than precise numbers. Since was introduced by Zadeh [62, 63, 64], CWW has become more and more popular for representing and computing linguistic information during the recent decades. Two main concepts of CWW are linguistic variables and granules. Roughly, a linguistic variable is "variable whose values are not numbers but words or sentences in a natural or artificial language" and thus is less specific than numerical ones but closer to the human thinking and knowledge [62]. A granule is usually treated as a fuzzy set of points drawn together by similarity or resemblance [16, 65]. In this paper, we will propose and develop a new CWW model for QDM under uncertainty.

Frequently, all the possible values of a linguistic variable are a finite set of linguistic terms defined by linguistic descriptors and semantics. This set is referred as a linguistic term set (LTS) denoted by $S = \{\alpha | \alpha = 0, 1, \dots, \tau\}$. Given a non-empty domain U , the semantics of each $s_\alpha \in S$ is defined by a fuzzy membership function, such as linear trapezoidal membership function (or triangular membership function) [12], taking the form of fuzzy numbers defined in U . The terms serve as a fuzzy partition of U and the parameter $\tau + 1$ indicates the granularity of knowledge. Usually, three aspects should be defined to form the basis of a computational model, which are the order of terms, the negation operator and some basic operations. For example, a LTS with 7 linguistic terms defined in the interval $[0, 1]$ could be (as shown in Fig. 1): $S = \{s_0 = \text{nothing}, s_1 = \text{very low}, s_2 = \text{low}, s_3 = \text{medium}, s_4 = \text{high}, s_5 =$

very high, $s_6 = \text{perfect}$). Based on this kind of discrete LTSs, several extended versions have been proposed, such as the linguistic 2-tuple model [19] and the linguistic virtual model [60].

However, an expert often is not confident enough to use a certain linguistic term to represent his/her preference. For example, a university wants to evaluate several students to decide the winner of a scholarship. Then the scientific research potential of the students might be considered. Suppose that an expert uses the above-mentioned LTS to express the result of evaluation, then the assessment value for a student may be the following types of complex linguistic expressions: (1) between *medium* and *very high*; (2) *medium* or *high* or *very high*; (3) more or less *high*. It is apparent that these evaluations are close to each other and around $s_4 = \text{high}$. These expressions emerge because the granule of the expert's knowledge does not match the granule of the provided LTS. Two strategies are often employed for this situation. Firstly, it is interesting and useful to develop some tools to deal with these kinds of expressions directly because developing a LTS, which fits every expert involved, is often very difficult or impossible. Based on the existing techniques, the first expression can be modelled by the uncertain linguistic term (ULT) $[s_3, s_5]$ proposed by Xu [55]; the second one can be represented by the hesitant fuzzy linguistic term set (HFLTS) $\{s_3, s_4, s_5\}$ proposed by Rodríguez et al. [42]. In the third expression, a linguistic term s_4 is included and modified by a hedge "more or less". This expression is emerged because the uncertainty exists and the expert is not confident enough to use the single term s_4 . Clearly, it is quite possible that s_4 is the real value of the linguistic variable; also it is possible that some other linguistic terms close to s_4 , such as s_3 and s_5 , might be the actual value. But we cannot compute with this kind of linguistic expressions directly by the existing techniques. The second strategy is the employment of some techniques involved with multi-granularity linguistic decision making. In our example, if the LTS with 7 linguistic terms is not coarse enough, then a new LTS, whose cardinality is 5, can be defined as $S_1 = \{s_0 = \text{nothing}, s_1 = \text{low}, s_2 = \text{medium}, s_3 = \text{high}, s_4 = \text{perfect}\}$. However, for the given evaluation problem, another attribute, i.e., course grades, may be considered at the same time by the LTS: $S_2 = \{s_0 = \text{fail}, s_1 = \text{pass}, s_2 = \text{medium}, s_3 = \text{good}, s_4 = \text{very good}\}$. It is obvious that the cardinalities of S_1 and S_2 coincide but they are defined in different domain and their semantics are generally different. It is hard to compute with these two LTSs in one problem. In conclusion, computing with uncertain linguistic expressions is generally inevitable. Linguistic terms modified by hedges are prevalent due to the common way of human thinking. But existing techniques are unable to handle this kind of linguistic information. Thereby further models should be developed to address this problem.

Remark 1. *Note that, if linear trapezoidal fuzzy numbers corresponding to the semantics of linguistic terms is used directly, the two LTS S_1 and S_2 can work in such as multi-granularity linguistic setting by using the idea proposed in Ref. [23]. But as argued by Dubois [13], this kind of method is as quantitative as any standard number-crunching method.*

Therefore, the major objective of this paper is to develop and present a new computational model to deal with linguistic expressions taking the form of a linguistic term modified by a hedge. Our motivation is that linguistic hedges are a frequent manner to express the uncertainty of using single terms in natural languages. They are useful to include more than one term in an indirect way. Under the framework of fuzzy linguistic approach, a hedge maps a fuzzy set to another [10]. Generally, hedges can be classified into two categories, which are intensified hedges (such as "very") and weakened hedges (such as "more or less"), according to their modified power. In the inclusive interpretation, a hedge modifies a linguistic term to its superset or subset; whereas in the non-inclusive interpretation, a hedge moves one term to another [24]. As it is widely known, hedges with the non-inclusive interpretation are commonly used in QDM. But hedges with the inclusive interpretation, which just intensify or weaken the degree of a term, have not been considered in QDM. In this paper, we focus on modelling weakened hedges, such as "more or less" and "roughly", in QDM. The model would extend the domain of values of a linguistic variable for CWW. Besides ULTs and HFLTSs, one more type of complex linguistic expressions elicited by natural language, i.e., linguistic terms with weakened hedges (LTWHs), could be considered and modelled in QDM.

Note that LTWHs are a type of complex linguistic expressions which frequently emerge in natural languages. The main purpose of this study is to enable LTWHs in QDM rather than to improve or substitute HFLTSs which model another type of complex linguistic expressions. Similar to De Dock and Kerre [10], we model weakened hedges by defining similarity relations on the domain. Based on some simple assumptions, the semantics of LTWHs can be determined objectively by the semantics of the terms. Moreover, to clarify the effectiveness of LTWHs in granular computing, we will exploit the connection between LTWHs and some existing multi-granularity linguistic models, and then present a novel algorithm for multi-granularity unification. The contributions of this paper are as follows:

(1) A novel type of complex linguistic expressions in natural languages are modelled for the potential applications in the qualitative setting. The syntax and semantics of LTWHs, as well as the computational model, are defined based on the framework of CWW.

(2) The capability of the novel model in granular computing has been explored preliminarily by developing a new solution of multi-granularity linguistic decision making.

The rest of the paper is organized as follows: Section 2 presents the definition of the syntax and semantics of LTWHs. Then Section 3 completes the CWW model by developing the basic theory of computing with LTWHs. A comparable analysis with several existing CWW models is presented in Section 4. The connection to existing multi-granularity linguistic models is analyzed in Section 5 to serve as an example of applying the proposed CWW model. Finally, Section 6 draws some conclusions and raises some open issues and the directions for future work.

2. Linguistic terms with weakened hedges

To enable LTWHs in CWW, their syntax and semantic should be defined and clarified at first. In this section, we will firstly define the syntax of LTWHs and then present the semantics by means of similarity relations defined in the domain.

Generally, a linguistic variable is an approximation of characterization of phenomena that are too complex or too ill-defined to be amenable to describe by a numerical variable. Based on the concept of fuzzy sets, Zadeh [63] defined linguistic variables as follows.

Definition 1. A linguistic variable is characterized by a quintuple $(X, S(X), U, G, M)$, where X is the name of the variable; $S(X)$ (or simply S) denotes the term set of X with each term being a fuzzy variable denoted generically by s and ranging over the domain U which is associated with the base variable u ; G is a syntactic rule for generating the names, s , of values of X ; and M is a semantic rule for associating with each s its meaning, $M(s)$, which is a fuzzy set of U .

Moreover, three denotations, i.e., the name s , its meaning (semantics) $M(s)$ and its restriction $R(s)$ can be used interchangeably to avoid a profusion of symbols [63]. It is clear that a linguistic variable should be defined by the following two rules: (1) A syntactic rule to define the names of the values of a linguistic; and (2) A semantic rule to compute the meaning of each value of the linguistic variable.

Without loss of generality, we assume in this paper that the reference domain $U = [L, R]$, where L and R are real numbers.

2.1. The syntax of LTWHs

As a LTWH is generated by a linguistic term of a LTS and a weakened hedge, we assume that there is a predefined LTS, associated with semantics of each term, having the following form:

$$S^{(\tau)} = \{\alpha | \alpha = 0, 1, \dots, \tau\} \quad (1)$$

Each $s_\alpha \in S^{(\tau)}$ is considered as an atomic term or an original term. As an ordinal linguistic computational model, the following conditions are assumed as well:

- (1) The set is ordered: $s_i \geq s_j$ iff $i \geq j$;
- (2) The negation operator is defined: $neg(s_\alpha) = s_{\tau-\alpha}$.

Furthermore, for a QDM problem in hand, the set of all considered weakened hedges considered is denoted by:

$$H^{(\varsigma)} = \{h_i | i = 1, 2, \dots, \varsigma\} \quad (2)$$

such that hedge h_j has more weakening force than h_i if and only if $i < j$.

In practical evaluations in QDM, it is usually enough to consider a few weakened hedges, such as h_1 = more or less and h_2 = roughly.

Figure 1: A set of seven terms with uniformly distributed semantics.

Remark 2. Generally, the set of weakened hedges for a specific QDM problem should be collected based on linguistic knowledge and the language custom of the involved experts by several steps: (1) Collect available weakened hedges; and (2) Order them according to their weakening power and then encode them. Especially, the hedges with the same or very approximate weakening power could be considered as the same one, i.e., encoded by the same h_t . In this study, we use these two hedges in most of the examples because: (1) They are frequently used to represent uncertainties. In fact, some other weakened hedges [14, 25], such as “rather”, “possibly”, can also be considered if necessary; (2) It is straightforward that the weakening force of “roughly” is stronger than that of “more or less” [11]; (3) Using too many hedges may lead to the difficulty of distinguishing their semantics.

Then the generation of LTWHs can be defined as follows:

Definition 2. (The syntactic rule). Given a LTS $S^{(\tau)}$ and a weakened hedge set (WHS) $H^{(s)}$ defined as before, a LTWH, denoted by a 2-tuple $l = \langle h_t, s_\alpha \rangle$, is generated by the following rule:

$$\begin{aligned} \langle \text{weakened hedge} \rangle &:= h_t, h_t \in H^{(s)}; \\ \langle \text{atomic term} \rangle &:= s_\alpha, s_\alpha \in S^{(\tau)}; \\ \langle \text{LTWH} \rangle &:= \langle \text{weakened hedge} \rangle \langle \text{atomic term} \rangle \end{aligned}$$

Moreover, an atomic term s_α can be seen as a special case of LTWHs if the hedge “definitely” is used because “definitely” has no weakening force. That is, $s_\alpha = \langle \text{definitely}, s_\alpha \rangle$ for any $s_\alpha \in S^{(\tau)}$. Thus, we denote $h_0 = \text{definitely}$ and

$$\bar{H}^{(2)} = \{h_0 = \text{definitely}, h_1 = \text{more or less}, h_2 = \text{roughly}\} \quad (3)$$

which is frequently used in the sequel.

Example 1. Given a WHS $\bar{H}^{(2)}$ defined by Eq. (3) and a LTS $S^{(6)}$ defined by Fig. 1, then some LTWHs could be:

$$\begin{aligned} l_1 &= \langle h_1, s_5 \rangle = \text{more or less high}; \\ l_2 &= \langle h_2, s_2 \rangle = \text{roughly low}; \\ l_3 &= \langle h_0, s_3 \rangle = (\text{definitely}) \text{medium} \end{aligned}$$

2.2. The semantics of LTWHs

It is natural that the semantics of a LTWH $l = \langle h_t, s_\alpha \rangle$ is computed based on the semantics of s_α . As the piecewise linear membership functions are usually used and linear triangular fuzzy numbers (TriFNs) are frequently considered

to represent the vagueness of these linguistic assessments, we assume that the semantics of each s_α is depicted by TriFNs in this paper. Formally, a TriFN tf can be depicted by a tri-tuple $tf = (a, b, c)$ such that (for any $x \in U$):

$$\mu_{tf}(x) = \begin{cases} (x-a)/(b-a), & \max\{L, a\} \leq x \leq b \\ (c-x)/(c-b), & b < x \leq \min\{c, R\} \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

More specifically, we assume that the semantics of LTS $S^{(\tau)}$ is defined by: (1) insert $\tau - 1$ points $x_1, s_2, \dots, x_{\tau-1}$ into the domain U . The distances between two adjacent points are generally different; (2) let $x_0 = L$, $x_\tau = R$; (3) let $\{x_{-i} | i = 1, 2, \dots, \varsigma + 1\}$ and $\{x_{\tau+j} | j = 1, 2, \dots, \varsigma + 1\}$ be two sets of points which are used for the convenience of representation, where ς is the number of linguistic hedges; and (4) $x_i < x_j$ for any $i < j$. Notice that the points derived in (3) are out of the domain U and thus are named by virtual points. Based on the partition, the semantics of term s_α is represented by the TriFN $s_\alpha = (x_{\alpha-1}, s_\alpha, s_{\alpha+1})$, $\alpha = 0, 1, \dots, \tau$.

Remark 3. At the first look, the representations of s_τ and s_0 seem lacking of meaning. In fact, the rationality can be ensured by Eq. (4). For instance, let $U = [0, 1]$, $\tau = 6$, the LTS is shown in Fig. 1 and

$$s_0 = (-0.1667, 0, 0.1667) = \begin{cases} (x+0.1667)/(0-0.1667), & \max\{0, -0.1667\} \leq x \leq 0 \\ (0.1667-x)/(0.1667-0), & 0 < x \leq \min\{0.1667, 1\} \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

which is the same as shown in Fig. 1.

It is clear that U is divided into τ intervals. As can be seen in Eq. (4), the membership functions are linear in each interval. This implies that each interval can be regarded as uniformly distributed. Accordingly, the domain is said to be piecewise uniformly distributed. Especially, like the case of Fig. 1, if $|x_i - x_{i-1}| = |x_j - x_{j-1}| = \delta$ for any $i, j = 1, 2, \dots, \tau$, and the sets of virtual points satisfy $\{x_{-i} = L - i \cdot \delta | i = 1, 2, \dots, \varsigma + 1\}$ and $\{x_{\tau+j} = R + j \cdot \delta | i = 1, 2, \dots, \varsigma + 1\}$, where $\delta = (R - L)/\tau$, then U is uniformly distributed. In sum, during this paper, the domain U is either (overall) uniformly distributed or piecewise uniformly distributed.

To derive the semantic of a LTWH, we propose the analysis below based on the following premises:

(1) Given $s_\alpha \in S^{(\tau)}$, the following semantic entailment holds for any $h_t \in H^{(\varsigma)}$:

$$\langle h_1, s_\alpha \rangle \subseteq \langle h_2, s_\alpha \rangle \subseteq \dots \subseteq \langle h_\varsigma, s_\alpha \rangle \quad (6)$$

That is, for any $x \in U$,

$$\mu_{\langle h_1, s_\alpha \rangle}(x) \leq \mu_{\langle h_2, s_\alpha \rangle}(x) \leq \dots \leq \mu_{\langle h_\varsigma, s_\alpha \rangle}(x) \quad (7)$$

(2) For any $t \in \{1, 2, \dots, \varsigma - 1\}$, the gap of weakening force between h_t and h_{t+1} ($h_t, h_{t+1} \in H^{(\varsigma)}$) are equal. In this case, the semantics of LTWHs can be defined recursively, i.e., $\langle h_{t+1}, s_\alpha \rangle = \langle h_t, \langle h_1, s_\alpha \rangle \rangle$ for any $s_\alpha \in S^{(\tau)}$ and $t \in \{1, 2, \dots, \varsigma - 1\}$.

In fact, there may exist a hedge whose weakening power is between h_t and h_{t+1} . We ignore this hedge and consider only h_t and h_{t+1} to avoid unnecessary difficulty of both using and representing. That is, we reduce the number of available hedges subjectively to ease the application of decision makers.

Based on these premises, it is enough to compute the semantics of $\langle h_1, s_\alpha \rangle$ ($s_\alpha \in S^{(\tau)}$). Motivated by De Cock and Kerre [10], we analyze the similarity between any $x, y \in U$ at first. It is intuitive that x is $\langle h_1, s_\alpha \rangle$ if and only if x is similar to some y which are s_α . If the LTS $S^{(\tau)}$ is uniformly distributed in U , like the case of Fig. 1, then the similar relation can be defined according to $|x - y|$. However, if this is not the case, the positive correlation between the similarity relation and the value of $|x - y|$ may be false. Given a LTS with semantics, the similarity between x and y could be defined by recognizing the density of domain. Generally, a similarity measure can be defined by a mapping $Sim : U \times U \rightarrow [0, 1]$ satisfying the following properties:

- (1) $0 \leq Sim(x, y) \leq 1$;
- (2) If $x = y$, then $Sim(x, y) = 1$;
- (3) $Sim(x, y) > Sim(x, y')$ for any $x < y < y'$;

where $x, y, y' \in U$. Now we introduce the definition of semantics with a simple case.

2.2.1. Semantics of LTWHs based on uniformly distributed domain

As it can be found in many researches, LTSs defined in uniformly distributed domains are frequently considered. We start with considering the similarity measure in this case.

Take the LTS shown in Fig.1 for example. If $x = 0.1$ and $y = 0.2$, then $Sim(x, y)$ should be greater than 0 because they are both “very low” to a certain degree; if $z = 0.5$, then $Sim(x, z)$ should be equal to 0 because they do not belong to a term with a positive degree at the same time. Based on this nature, we define the similarity measure by the following definition.

Definition 3. Given a uniformly distributed U , $\forall x, y \in U$, the similarity between x and y is defined by:

$$Sim(x, y) = \begin{cases} 1 - |x - y|/\delta, & |x - y| < \delta \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

It is obvious that this definition satisfies the required properties of a similarity measure mentioned above.

As shown in Fig.1, given $x = 0.1$, we have $\mu_{s_2}(x) = 0$, but $\mu_{\langle h_1, s_2 \rangle}(x)$ should be greater than 0 because x is similar to some points, such as 0.2, which are s_2 . Thus, given $x \in U$, $\mu_{\langle h_1, s_\alpha \rangle}(x)$ could be defined by the semantics $\mu_{s_\alpha}(x)$ and the similarity between x and some y in the domain. Based upon the idea of upper approximation of rough fuzzy sets [14], we have the following definition.

Definition 4. Let $S^{(\tau)}$ be a LTS defined in a uniformly distributed U . For any $x \in U$, $\langle h_1, s_\alpha \rangle$ is defined by:

$$\mu_{\langle h_1, s_\alpha \rangle}(x) = \sup_{y \in U} \mathcal{T}(Sim(x, y), \mu_{s_\alpha}(y)) \quad (9)$$

where the function Sim is the similarity measure defined in Definition 3 and $\mathcal{T}$ is a triangular norm.

Apparently, Definition 4 computes the semantics of LTWHs objectively. In fact, given a LTS defined in a specific domain, the similarity measure only depends on the distribution of the domain, and then the semantics of a LTWH is fixed. We can illustrate Definition 4 by the following example.

Example 2. Let $S^{(6)}$ be the LTS shown in Fig. 1, $\mathcal{T}$ be the minimum operator such that $\mathcal{T}(x, y) = \min\{x, y\}$, then given $x = 0.3$ and $\mu_{s_3}(0.3) = 0$, we have

$$\mu_{\langle h_1, s_3 \rangle}(0.3) = \sup_{y \in U} \mathcal{T}(Sim(0.3, y), \mu_{s_3}(y)) = 0.40$$

The computational process can be shown visually in Fig. 2. Moreover, the semantics of $\langle h_1, s_3 \rangle$ can be found in Fig. 3.

As can be seen in Fig. 3, the derivation of semantics of $\langle h_1, s_\alpha \rangle$ is rational. If $0.3333 \leq x \leq 0.6667$, then $\mu_{s_3}(x) \geq 0$, if an expert uses “more or less” to state the uncertainty around s_3 , then it is rational that $\mu_{\langle h_1, s_3 \rangle}(x) \geq \mu_{s_3}(x)$. Similarly, if $0.1667 < x < 0.3333$, then $\mu_{s_3}(x) = 0$, however, x is similar to some y who are “medium”. Thus $\mu_{\langle h_1, s_3 \rangle}(x) > 0$. Finally, for any $x \leq 0.1667$, x is not similar to any y who is “medium”. In this case, $\mu_{\langle h_1, s_3 \rangle}(x)$ should be 0.

Remark 4. Liang and Mendel [30] suggested characterizing a type-2 fuzzy set by a pair of upper and lower membership functions, each of which is a membership function of a type-1 fuzzy set. The bounded area between the two functions represents the footprint of uncertainties. As shown in Fig. 3, an atomic term and a LTWH form a specific type-2 fuzzy set and the semantics of the LTWH acts as the type-2 upper membership function. In our case, thanks to the similarity measure which is defined objectively based on the distribution of the domain, the uncertainty of using a linguistic term is fixed if the hedge is given. Therefore, we can model the uncertainty implied by a hedge using only the upper membership function. Type-2 fuzzy sets, as well as relevant techniques such as Ref. [34, 40], are not necessary in this case.

It is interesting to notice that, as shown in Fig. 3, the semantics of a LTWH can be represented by a TriFN if the atomic term is represented by a TriFN. In fact, we have $s_3 = (0.333, 0.5, 0.667)$ and $\langle h_1, s_3 \rangle = (0.167, 0.5, 0.833)$. Generally, we have the following theorem.

Figure 2: An example of the derivation of the semantics of more or less *medium*.

Figure 3: The semantics of *medium* and more or less *medium*.

Theorem 1. Given a LTS $S^{(\tau)}$ defined in the uniformly distributed domain U and $\mathcal{T}(x, y) = \min\{x, y\}$, for any $s_\alpha = (x_{\alpha-1}, x_\alpha, x_{\alpha+1}) \in S^{(\tau)}$, the semantics of LTWH $\langle h_1, s_\alpha \rangle$ is:

$$\langle h_1, s_\alpha \rangle = (x_{\alpha-2}, x_\alpha, x_{\alpha+2}) \quad (10)$$

Proof. For any $x \in U$,

(1) If $x \leq \max\{L, x_{\alpha-2}\}$, then $\min\{Sim(x, y), \mu_{s_\alpha}(y)\} = 0$ holds for any $y \in U$. Based on Definition 4, this implies $\mu_{\langle h_1, s_\alpha \rangle}(x) = 0$.

(2) If $\max\{L, x_{\alpha-2}\} < x \leq \max\{L, x_{\alpha-1}\}$, like the case shown in Fig. 2, we have

$$\begin{aligned} \mu_{\langle h_1, s_\alpha \rangle}(x) &= \sup_{y \in U} \min\{Sim(x, y), \mu_{s_\alpha}(y)\} \\ &= \sup_{y \in [x-\delta, x+\delta]} \min\{1 - (y-x)/\delta, (y-x_{\alpha-1})/(x_\alpha - x_{\alpha-1})\} = (x - x_{\alpha-2})/2\delta \end{aligned}$$

(3) If $\max\{L, x_{\alpha-1}\} < x \leq x_\alpha$, we have

$$\begin{aligned} \mu_{\langle h_1, s_\alpha \rangle}(x) &= \sup_{y \in [x-\delta, x+\delta]} \min\{1 - |x-y|/\delta, \mu_{s_\alpha}(y)\} \\ &= \max\left\{ \sup_{y \in [x_{\alpha-1}, x]} \min\{1 - (x-y)/\delta, (y-x_{\alpha-1})/\delta\}, \right. \\ &\quad \sup_{y \in [x, x_\alpha]} \min\{1 - (y-x)/\delta, (y-x_{\alpha-1})/\delta\}, \\ &\quad \left. \sup_{y \in [x_\alpha, x+\delta]} \min\{1 - (y-x)/\delta, (x_{\alpha+1}+y)/\delta\} \right\} \\ &= \sup_{y \in [x, x_\alpha]} \min\{1 - (y-x)/\delta, (y-x_{\alpha-1})/\delta\} = (x - x_{\alpha-2})/2\delta \end{aligned}$$

Similar to (1), we have $\mu_{\langle h_1, s_\alpha \rangle}(x) = 0$ if $x \geq \min\{R, x_{\alpha+2}\}$. In addition, similar to (2) and (3), we can get $\mu_{\langle h_1, s_\alpha \rangle}(x) = (x_{\alpha+2} - x)/2\delta$ if $x_\alpha \leq x \leq \min\{R, x_{\alpha+2}\}$. The proof is complete by using Definition 4 and Eq.(4). $\square$

Moreover, it is easy to obtain $\langle h_1, \langle h_1, s_\alpha \rangle \rangle = (x_{\alpha-3}, x_\alpha, x_{\alpha+3})$ by repeating the procedure of the above proof. Based on the second premise, we can draw the following conclusion immediately.

Theorem 2. *Given a LTS $S^{(\tau)}$ defined in the uniformly distributed domain U and $\mathcal{T}(x, y) = \min\{x, y\}$, for any $s_\alpha = (x_{\alpha-1}, x_\alpha, x_{\alpha+1}) \in S^{(\tau)}$ and $h_t \in H^{(s)}$, the semantics of LTWH $\langle h_t, s_\alpha \rangle$ is*

$$\langle h_t, s_\alpha \rangle = (x_{\alpha-t-1}, x_\alpha, x_{\alpha+t+1}) \quad (11)$$

2.2.2. Semantics of LTWHs based on non-uniformly distributed domain

LTSs defined in non-uniformly distributed domains are frequently considered as well in many studies, such as [56, 57, 61]. TriFNs are usually used for the representation of semantics of linguistic terms. As has been illustrated at the beginning of Section 2.2, this kind of non-uniformly distributed domain is, actually, piecewise uniformly distributed. For instance, the first part of Fig. 4 shows the LTS defined in Xu [57]. In this case, the complicated issue of defining the semantics of a LTWH is to obtain the similarity between any two objects in the domain. To address it, a piecewise linear function could be employed to transform the piecewise uniformly distributed domain into a uniformly distributed one. Then the conclusions in the above subsection can be used directly. Formally, we represent this case as follows.

The derivation of semantics of a LTWH based on piecewise uniformly distributed domain can be divided into the following three steps:

Step 1. Employ the following piecewise linear function $f : [L, R] \rightarrow [0, 1]$ such that

$$f(x) = \frac{x - ((\alpha - 1)x_\alpha - \alpha x_{\alpha-1})}{\tau(x_\alpha - x_{\alpha-1})}, x \in [x_{\alpha-1}, x_\alpha] \quad (12)$$

to map each interval $[x_{\alpha-1}, x_\alpha]$ into $[(\alpha - 1)/\tau, \alpha/\tau]$, where $\alpha = 1, 2, \dots, \tau$. Thus the domain U is transferred to another uniformly distributed domain $\tilde{U} = [0, 1]$. Accordingly, linguistic term $s_\alpha = (s_{\alpha-1}, s_\alpha, s_{\alpha+1})$ corresponds to linguistic term $\tilde{s}_\alpha = ((\alpha - 1)/\tau, \alpha/\tau, (\alpha + 1)/\tau)$, $\alpha = 0, 1, \dots, \tau$.

Step 2. Calculate the semantics of the new LTWH $\langle h_1, \tilde{s}_\alpha \rangle$. According to Theorem 1, $\langle h_1, \tilde{s}_\alpha \rangle = ((\alpha - 2)/\tau, \alpha/\tau, (\alpha + 2)/\tau)$ for any $\alpha = 0, 1, \dots, \tau$.

Step 3. Calculate the semantics of the LTWH $\langle h_1, s_\alpha \rangle$ by mapping the TriFN derived in Step 2 into another TriFN in the domain U , using the inverse function f^{-1} . Apparently, the result coincides with the one of Theorem 1, i.e., $\langle h_1, s_\alpha \rangle = (x_{\alpha-2}, x_\alpha, x_{\alpha+2})$.

The process can be illustrated by the following example.

Example 3. Let $S^{(6)}$ be the LTS shown in the first part of Fig. 4 and $s_3 = (0.417, 0.5, 0.583)$ be the linguistic term “medium”. According to the Eq.(12), we get a new linguistic term $\tilde{s}_3 = (0.3333, 0.5, 0.6667)$ defined in the uniformly distributed domain $[0, 1]$. Because of Theorem 1, we have $\langle h_1, \tilde{s}_3 \rangle = (0.1667, 0.5, 0.8333)$, as shown in the second part of Fig. 4. Then the TriFN can be mapped into $(0.278, 0.5, 0.722)$. Thus we obtain the semantic of the LTWH (i.e., more or less medium): $\langle h_1, s_3 \rangle = (0.278, 0.5, 0.722)$, which is shown in the third part of Fig. 4.

Based on the second premise, Theorem 2 can be extended to a general form which is summarized in the next theorem.

Theorem 3. *(The semantic rule). Given a LTS $S^{(\tau)} = \{s_\alpha = (x_{\alpha-1}, x_\alpha, x_{\alpha+1}) | \alpha = 0, 1, \dots, \tau\}$ defined in the domain $U = [L, R]$ and $\mathcal{T}(x, y) = \min\{x, y\}$, for any $s_\alpha \in S^{(\tau)}$ and $h_t \in H^{(s)}$, the semantics of LTWH $\langle h_t, s_\alpha \rangle$ is*

$$\langle h_t, s_\alpha \rangle = (x_{\alpha-t-1}, x_\alpha, x_{\alpha+t+1}) \quad (13)$$

3. Linguistic computational model based on LTWHs

To enable computing with LTWHs in decision making, we should specify several concepts in advance, which are the negation, order relations and basic computational laws. We shall focus our attention on them in this section. The set of all LTWHs based on the sets $S^{(\tau)}$ and $H^{(s)}$ is denoted by $\mathcal{L}$ in the rest of this paper, i.e., $\mathcal{L} = \{l = \langle h_t, s_\alpha \rangle | h_t \in H^{(s)}, s_\alpha \in S^{(\tau)}\}$.

Figure 4: An example of deriving semantics of LTWHs based on a non-uniformly distributed domain.

3.1. Negation operator of LTWHs

The negation operator of a LTWH can be defined by the corresponding negation operator of its atomic linguistic term.

Definition 5. Given a LTWH $\langle h_t, s_\alpha \rangle \in \mathcal{L}$, the negation operator of $\langle h_t, s_\alpha \rangle$, denoted by $Neg(\langle h_t, s_\alpha \rangle)$, is defined by

$$Neg(\langle h_t, s_\alpha \rangle) = \langle h_t, neg(s_\alpha) \rangle \quad (14)$$

where neg is the negation operation defined in $S^{(\tau)}$.

To ensure that the negation of a LTWH is also a LTWH, it is necessary to let the function Neg be a one-one mapping. Based on the classical and rational version of function neg in Section 2.1, Eq. (14) can be rewritten as:

$$Neg(\langle h_t, s_\alpha \rangle) = \langle h_t, s_{\tau-\alpha} \rangle \quad (15)$$

Remark 5. It should be mentioned that function neg may be not a one-one mapping if the reference domain is extremely non-uniformly distributed. In this case, the generalized version of negation should be defined according to the semantics of linguistic terms. See Ref. [45] for more details.

3.2. Order relations of LTWHs

To compare any two given LTWHs, the order relations on set $\mathcal{L}$ should be defined. Inspired by the classical total order on $S^{(\tau)}$, we can develop the following partial order, for any $\langle h_{t_1}, s_{\alpha_1} \rangle, \langle h_{t_2}, s_{\alpha_2} \rangle \in \mathcal{L}$,

$$\langle h_{t_1}, s_{\alpha_1} \rangle \preceq_V \langle h_{t_2}, s_{\alpha_2} \rangle \Leftrightarrow (t_1 = t_2) \wedge (s_{\alpha_1} \leq s_{\alpha_2}) \quad (16)$$

The partial set (poset) defined by $\mathcal{L}$ and $\preceq_V$ is denoted by $(\mathcal{L}, \preceq_V)$. In practical applications, the partial order $\preceq_V$ may be not sufficient. Total orders are necessary to make sure that any two LTWHs are comparable. For example, given $\langle h_{t_1}, s_{\alpha_1} \rangle, \langle h_{t_2}, s_{\alpha_2} \rangle \in \mathcal{L}$, the relation $\preceq_{Lex}$ defined by

$$\langle h_{t_1}, s_{\alpha_1} \rangle \preceq_{Lex} \langle h_{t_2}, s_{\alpha_2} \rangle \Leftrightarrow (s_{\alpha_1} < s_{\alpha_2}) \vee ((s_{\alpha_1} = s_{\alpha_2}) \wedge (t_1 \geq t_2)) \quad (17)$$

is a total order on the set $\mathcal{L}$. The total order $\preceq_{Lex}$ refines the partial order $\preceq_V$ and is rational and acceptable for many cases. However, take the LTS in Fig. 1 for example, we obtain $\langle h_1, s_4 \rangle \preceq_{Lex} \langle h_0, s_4 \rangle$ according to Eq.(17), which means that “medium” is better than “more or less medium”. Thus the order $\preceq_{Lex}$ is based on the strategy of risk aversion. Because of the uncertainty involved in the linguistic hedges, it is inevitable to consider risk preferences when distinguishing two LTWHs. We may need total orders based on other strategies if the decision maker is risk neutral or risk loving. In the following definition, we present a general framework to develop admissible orders for specific applications.

Definition 6. Let $(\mathcal{L}, \preceq_V)$ be a poset. The order $\preceq$ is called an admissible order if

- (1) $\preceq$ is a total order on $\mathcal{L}$,
- (2) for any $\langle h_{t_1}, s_{\alpha_1} \rangle, \langle h_{t_2}, s_{\alpha_2} \rangle \in \mathcal{L}$, $\langle h_{t_1}, s_{\alpha_1} \rangle \preceq \langle h_{t_2}, s_{\alpha_2} \rangle$ if $\langle h_{t_1}, s_{\alpha_1} \rangle \preceq_V \langle h_{t_2}, s_{\alpha_2} \rangle$.

In fact, an admissible order is a total order which refines the partial order defined by Eq.(16). Hence, the order $\preceq_{Lex}$ is an admissible order.

3.3. Basic operational laws of LTWHs

Based on a total order $\preceq$ defined in $\mathcal{L}$, we can define two operations as follows.

Definition 7. Given $\langle h_{t_1}, s_{\alpha_1} \rangle, \langle h_{t_2}, s_{\alpha_2} \rangle \in \mathcal{L}$, $\langle h_{t_1}, s_{\alpha_1} \rangle \preceq \langle h_{t_2}, s_{\alpha_2} \rangle$,

- (1) Maximum operation: $\langle h_{t_1}, s_{\alpha_1} \rangle \cup \langle h_{t_2}, s_{\alpha_2} \rangle = \langle h_{t_2}, s_{\alpha_2} \rangle$;
- (2) Minimum operation: $\langle h_{t_1}, s_{\alpha_1} \rangle \cap \langle h_{t_2}, s_{\alpha_2} \rangle = \langle h_{t_1}, s_{\alpha_1} \rangle$.

When aggregating a collection of LTWHs, the weight of each LTWH may be taken into account. In this case, we have to operate both linguistic information and the associated weights. Operations which are more accurate than the maximum and minimum operations should be developed. In order to reach reasonable accuracy and acceptable interpretability, this kind of operations could be achieved by two procedures as follows

$$\mathcal{L}^n \xrightarrow{F} \tilde{F}(\mathbf{R}) \xrightarrow{app} \mathcal{L} \quad (18)$$

where $\mathcal{L}^n$ represents the n Cartesian product of $\mathcal{L}$, F is an aggregation operator, $\tilde{F}(\mathbf{R})$ is the fuzzy set representing the intermediate aggregation result, the function app transfers the intermediate result into a LTWH in $\mathcal{L}$. Specifically, these procedures can be implemented based on the virtual linguistic model [54, 60] and the function $round$.

Definition 8. Given two LTWHs $l_1 = \langle h_{t_1}, s_{\alpha_1} \rangle, l_2 = \langle h_{t_2}, s_{\alpha_2} \rangle \in \mathcal{L}$, their weights are denoted by w_1 and w_2 respectively, where $w_1, w_2 \in [0, 1]$, then the weighted averaging, denoted by $w_1 l_1 \oplus w_2 l_2$, is derived by the following two steps:

- (1) Aggregation: $w_1 l_1 \oplus w_2 l_2 = \langle h_{w_1 t_1 + w_2 t_2}, s_{w_1 \alpha_1 + w_2 \alpha_2} \rangle$;
- (2) Approximation: $w_1 l_1 \oplus w_2 l_2 = \langle h_t, s_\alpha \rangle$, where $t = round(w_1 t_1 + w_2 t_2)$, $\alpha = round(w_1 \alpha_1 + w_2 \alpha_2)$.

According to Definition 8, the first step obtains the accurate aggregation results based on the idea of the virtual linguistic model, but leads to the lack of interpretability. The second step ensures that the result is in $\mathcal{L}$ and thus can be interpreted by the semantics. It is clear that the approximation may result in loss of information. As will be seen in the next example, the two steps could provide satisfactory results in many cases. For simplicity, we can rewrite them as follows:

$$w_1 l_1 \oplus w_2 l_2 = w_1 \langle h_{t_1}, s_{\alpha_1} \rangle \oplus w_2 \langle h_{t_2}, s_{\alpha_2} \rangle = \langle h_{\text{round}(w_1 t_1 + w_2 t_2)}, s_{\text{round}(w_1 \alpha_1 + w_2 \alpha_2)} \rangle \quad (19)$$

However, if the approximate result is not sufficient and only the ranking of objects is necessary for some special cases, then the aggregation result in the first step can be used and considered as virtual linguistic terms with virtual hedges. The second step can be ignored. In these cases, the orders defined in the above section work as well.

Example 4. Given a WHS $\bar{H}^{(2)}$ defined by Eq.(3) and a LTS $S^{(7)}$ defined by Fig. 1, two linguistic expressions “roughly low” and “definitely very high” are denoted by $l_1 = \langle h_2, s_2 \rangle$ and $l_2 = \langle h_0, s_6 \rangle$, respectively. Their weights are denoted by w_1 and w_2 . If $w_1 = 0.8$ and $w_2 = 0.2$, then

$$\begin{aligned} \text{round}(0.8 \times 2 + 0.2 \times 0) &= 2; \text{round}(0.8 \times 0.2 + 0.2 \times 6) = 2.8 \\ w_1 l_1 \oplus w_2 l_2 &= 0.8 \langle h_2, s_2 \rangle \oplus 0.2 \langle h_0, s_6 \rangle = \langle h_2, s_3 \rangle \end{aligned}$$

the aggregating result is close to l_1 . Similarly, if $w_1 = w_2 = 0.5$, then $w_1 l_1 \oplus w_2 l_2 = \langle h_1, s_4 \rangle$, which can be considered as the mean of l_1 and l_2 ; if $w_1 = 0.1$, $w_2 = 0.9$, then $w_1 l_1 \oplus w_2 l_2 = \langle h_0, s_6 \rangle$, which results in the same value of l_2 .

Any specific operator can be defined according the problems in hand. For instance, the most commonly used operator, i.e., the weighted averaging operator, can be defined based on Definition 8. Given a set of n LTWHs $\{l_i = \langle h_{t_i}, s_{\alpha_i} \rangle \in \mathcal{L} | i = 1, 2, \dots, n\}$ with weighting vector $\mathbf{w} = (w_1, w_2, \dots, w_n)$ having $\sum_{i=1}^n w_i = 1$ and $w_i \geq 0$ ($i = 1, 2, \dots, n$), the weighted averaging operator of LTWHs (LTWHWA) is mapping $LTWHWA : \mathcal{L}^n \rightarrow \mathcal{L}$ such that:

$$LTWHWA_{\mathbf{w}}(l_1, l_2, \dots, l_n) = w_1 l_1 \oplus w_2 l_2 \oplus \dots \oplus w_n l_n = \langle h_t, s_{\alpha} \rangle \quad (20)$$

where $t = \text{round}(\sum_{i=1}^n w_i t_i)$, $\alpha = \text{round}(\sum_{i=1}^n w_i \alpha_i)$.

4. Comparative analyses on similar models

In order to show the characteristics of the proposed CWW model, this section will conduct the comparisons with some existing studies of modeling hedges and some similar linguistic models which consider uncertain linguistic terms.

4.1. Compared with the existing techniques of modeling hedges

We will recall the main idea of several techniques at first. The shifting hedge suggested by Lakoff [25] does not change the shape of membership function of an atomic term but shift it to a certain level. As shown in Fig. 1, the intensified hedge “very” modifies the term “high” to a new term “very high”. The powering hedge [63] would be the most widely acknowledged kind of hedges. In this sense, given a term s_{α} , the semantics of $\langle h_t, s_{\alpha} \rangle$ is defined by $(\mu_{s_{\alpha}}(x))^{\gamma}$, where γ is determined according to the weakening power of h_t . For weakened hedges, γ is fixed in $[0, 1]$. For example, Cordon et al. [8] let $\gamma = 1/2$ and $\gamma = 2$ to represent “more or less s_{α} ” and “very s_{α} ”, respectively, in a fuzzy rule-based classification system. Accordingly, the surface rule structure is changed by incorporating hedges. Later, Casillas et al. [2] improved the work of Cordon et al. [8] by tuning both surface rule structure and deep rule structure, where the deep rule structure is tuned by adjusting the parameters of membership functions of linguistic terms. De Cock and Kerre [10] presented the first approach to model hedges by fuzzy relations. Recently, Lewis and Lawry [29] represented linguistic hedges based on the label semantics framework. The approach is defined with both unmodified prototypes and differing prototypes. The result of the former is similar to the powering hedge and the latter changes the core of the atomic term.

We can see that the proposed model is not the first model to model hedges but the first one to model uncertainty of using single terms in QDM. Others focus on general definition of hedges, natural language processing or artificial intelligence. Accordingly, only weakened hedges are focused on. Instead of concentrating on specific hedges, we

present a general representational and computational framework to enable computing with a set of hedges satisfying the two premises in Section 2.2. The powering hedges are usually criticized because they are arbitrary and semantically ungrounded. The proposed model, the fuzzy relation-based model [10] and the label semantic model [29] have a clear semantic grounding. The label semantic model [29] highly depends on the threshold and its computational process is much more complex than that of the proposed model. The proposed model and the fuzzy relation-based model [10] utilize piece-wise linear membership functions for representing semantics and thus the application is simple. Theoretically, the proposed model could be a special case of the fuzzy relation-based model [10] because of the specification of domain and similarity relations. But, thanks to Theorem 3, the proposed computational model is much easier than others.

4.2. LTWHs vs. ULTs and HFLTSS

Let $S^{(\tau)}$ be a LTS defined by Eq.(1) and $H^{(s)}$ be the WHS defined by Eq.(2). Given $s_{\alpha-1}, s_{\alpha}, s_{\alpha+1} \in S^{(\tau)}$ and $h_1 \in H^{(s)}$, three distinct linguistic expressions with similar semantics can be constructed, which are ULT $l^{(I)} = [s_{\alpha-1}, s_{\alpha+1}]$, HFLTSS $l^{(H)} = \{s_{\alpha-1}, s_{\alpha}, s_{\alpha+1}\}$ and LTWH $l^{(L)} = \langle h_1, s_{\alpha} \rangle$. For any $x \in U$, we have $\mu_{l^{(I)}}(x) \geq 0 \Leftrightarrow \mu_{l^{(H)}}(x) \geq 0 \Leftrightarrow \mu_{l^{(L)}} \geq 0$. Thus it is sufficient to compare these techniques by the three linguistic expressions. Table 1 summarizes their syntactic and semantic.

Table 1: Comparisons among ULTs, HFLTSSs and LTWHs.		
Technique	Syntax	Semantics
ULT [55] ($[s_{\alpha-1}, s_{\alpha+1}]$)	Between... and ...
HFLTSS [42] ($\{s_{\alpha-1}, s_{\alpha}, s_{\alpha+1}\}$)	Greater than... Lower than ... Between... and ...
LTWH ($\langle h_1, s_{\alpha} \rangle$)	Weakened hedge + atomic term

The second column of Table 1 lists the forms of linguistic expression which are adopted to elicit the three types of linguistic values. ULTs are introduced for the case when the linguistic argument does not match any of the atomic terms but located between two distinct terms [55]. HFLTSSs are developed to suit the case when experts are thinking of several terms at the same time [42], and thus can be represented by several consecutive terms in a LTS. As can be seen in Table I, three forms of linguistic expression can be considered for eliciting HFLTSSs. It seems that the syntactic rules of ULTs and HFLTSSs are similar, but they are totally different. ULTs are on the basis of the virtual linguistic model. Thus when considering $[s_{\alpha-1}, s_{\alpha+1}]$, any virtual terms in the interval are included. Whereas in HFLTSSs, such as $\{s_{\alpha-1}, s_{\alpha}, s_{\alpha+1}\}$, only the listed atomic terms in the predefined LTS are involved. Different from ULTs and HFLTSSs, LTWHs begin with only one atomic term, and employ a weakened hedge to express the uncertainty around the term. For instance, when we say something is “more or less s_{α} ”, the fact is that s_{α} is the most possible term. The possibility of “the object is s_{α} ” is greater than “it is $s_{\alpha-1}$ or $s_{\alpha+1}$ ”. From this point of view, we would like to emphasize that the purpose of proposing LTWHs is neither to serve as a substitute of the existing tools nor to be a better technique.

LTWHs are proposed to represent another way of human thinking. And this way is very natural but has not been systematically studied.

The third column in Table 1 shows the semantics of the techniques. It can be derived from the corresponding syntactic rules. Clearly, the semantics are quite different. But it is not necessary to highlight the difference unless you are going to compute with their semantics.

There are also some relationships among the computational models of the three techniques. Their ideas of the order relation and negation operations are similar to each other. But other operations, especially the aggregation operators, are totally different. The operations of ULTs are defined based on the operation of virtual linguistic model, whose details can be found in Xu [55]. The operations of HFLTSs are more complex than those of ULTs [51]. One strategy is to compute based on their envelopes [5, 7, 42], i.e., ULTs. Thus, HFLTSs are transformed into ULTs for computing. Another strategy is to compute by using each component term in the HFLTS such as in Refs. [26, 27, 49, 68]. This strategy is closer to the idea of hesitation. However, the aggregation results may be not HFLTSs but extended HFLTSs (EHFLTSs) [47, 48]. The basic aggregation operation of LTWHs is implemented by two steps. The first step is similar to the operation in the virtual linguistic model, and the second step makes the results more coarse but interpretable.

4.3. Compared with other similar techniques

Besides, there are some other linguistic representational and computational models, such as the proportional linguistic terms (PpLTs) [52], linguistic distribution assessments (LDAs) [53, 66], discrete fuzzy numbers (DFNs) [41, 46], probabilistic linguistic term sets (PLTSs) [31, 35, 67]. Their main characteristics are shown in Table 2.

Table 2: Main features of the comparative techniques.

Technique	Characteristic	Required information	Computational strategy	Result
ULT [55]	An interval of linguistic terms	Two terms, as the interval boundary	Compute with its boundary	ULT
HFLTS [42]	A list of several consecutive terms	Two terms, as the boundary of the set	Compute with its envelope	ULT or EHFLTS
EHFLTS [47]	A list of several terms	Possible terms	Compute with possible terms	EHFLTS
PpLT [52]	A proportion of two terms	Two terms and probabilistic distribution	Compute with terms and probabilities	PpLT
LDA [66]	A probabilistic distribution of several terms	Several terms and probabilistic distribution	Compute with terms and probabilities	LDA
DFN [41]	Several terms with membership degrees	Several terms and membership degrees	Compute with membership degrees	DFN
PLTS [35]	A probabilistic distribution of several terms	Several terms and incomplete probabilities	Compute with terms and probabilities	PLTS or EHFLTS
LTWH	A term modified by a weakened hedge	A term and a weakened hedge	Compute with the term and hedge	LTWH

As can be seen in Table 2, most of these linguistic representation models include at least two linguistic terms for the purpose of modeling uncertainties. Some techniques need more information, such as complete or incomplete probabilistic distribution and membership degrees. However, the proposed LTWHs begin with only one linguistic term which is modified by a weakened hedge h_t . The uncertainty of using the linguistic term is represented by the hedge. The greater the value of t is, the bigger the uncertainty is.

Because hedges are usually used for the case when the granule of experts is coarser than that of the given LTS, there exists an inherent connection between the proposed model of LTWHs and the multi-granularity linguistic models. Basically, given a LTS, a LTWH $\langle h_1, s_\alpha \rangle$ might correspond to a linguistic term in the upper level of s_α . Therefore, a linguistic hierarchy could be constructed based on a bottom-up strategy. We will specify this issue in the coming section.

5. The connection to multi-granularity linguistic models

A multi-granularity linguistic decision making problem refers to evaluate a set of alternatives using several LTSs with different granularity and/or semantics. These LTSs can be denoted by $S^{MG} = \{S^{(\tau_q)} | q = 1, 2, \dots, Q\}$, where the q -th LTS is $S^{(\tau_q)} = \{s_\alpha^{(q)} | \alpha = 0, 1, \dots, \tau_q\}$. LTSs with different granularities are necessary because the granules of knowledge of experts are different and then uncertainty exists when expressing opinions by a single linguistic term. Moreover, certain hierarchical relationship often exists in the set of LTSs and thus hierarchical representational models are easy and effective for applications. A general version of hierarchical representational models, namely hierarchical tree [21], represents the set of LTSs by

$$HT = \cup_{q=1}^Q \{S^{(\tau_q)}\} \quad (21)$$

where $S^{(\tau_q)}$ is the LTS of level q with a granularity of $\tau_q + 1$. The hierarchical tree can be constructed by a top-down strategy. Given a LTS $S^{(\tau_q)}$ in the q -th level, the new LTS in the $(q + 1)$ -th level should satisfy: (1) $\tau_q < \tau_{q+1}$; and (2) there exists only one mapping which implements the semantics derivation of $S^{(\tau_{q+1})}$ from the previous level of LTS $S^{(\tau_q)}$. Especially, a frequently considered version, which is called linguistic hierarchy [15, 20], can be formed by preserving all modal points of the membership function of linguistic terms from the q -th level to the $(q + 1)$ -th level and then adding a new linguistic term between each pair of terms of the q -th level. Therefore, we have $\tau_{q+1} = 2\tau_q$ in this case. In this section, we focus on the hierarchical tree or its special case (linguistic hierarchy) to represent of multi-granularity linguistic information. For more representational models as well as the recent advance in MGLDM, please refer to Refs. [33, 59].

We will demonstrate that the use of weakened hedges presents an easy and direct way to represent and operate multi-granularity linguistic information in a hierarchical tree.

5.1. Quasi multi-granularity hierarchical tree

To clarify the connection between LTWHs and multi-granularity linguistic models, we construct a hierarchical tree based on a LTS whose granularity is sufficiently fine and a WHS. The constructed hierarchical tree is called quasi hierarchical tree because it includes only one LTS.

The existing models generate a hierarchical model to represent the values of a linguistic variable by a top-down strategy. The underlying reason of this strategy is that a LTS with the refined granularity is necessary if the previous LTS is too coarse to express the values accurately. Different from such a strategy, we construct a hierarchical model by the opposite strategy. This bottom-up strategy assumes that a LTS with coarse granularity is considered if the previous LTS is so refined that experts are not sure which term can be used. Simultaneously, if uncertainty exists, experts can express it by means of linguistic hedges. This idea drives us to the following recursive algorithm to construct a hierarchical tree with Q levels based on a predefined and refined enough LTS.

Algorithm 1. Construct a quasi hierarchical tree.

Input: The number of levels Q , the LTS of the Q -th level $S^{(\tau_Q)} = \{s_\alpha^{(Q)} | \alpha = 0, 1, \dots, \tau_Q\}$ with $\tau_Q = 2^{Q_0}$ (where the integer $Q_0 \geq Q$), a WHS $H^{(s)} = \{h_t | t = 1, 2, \dots, \varsigma\}$.

Output: A quasi linguistic hierarchy QHT.

Step 1: Initialization. Let $q = Q$, $\tau_q = \tau_Q$, $t = 0$, $S^{(\tau_q)} = \{\langle h_t, s_\alpha^{(q)} \rangle | \alpha = 0, 1, \dots, \tau_q\}$.

Step 2: If $q > 1$, then go to Step 3; else, go to Step 4.

Step 3: Construct the q -th level of the quasi hierarchical tree. *Input* t ; let $S^{(\tau_{q-1})} = \{l_\beta^{(q-1)} = \langle h_t, l_\alpha^{(q)} \rangle | h_t \in H^{(\varsigma)}, l_\alpha^{(q)} \in S^{(\tau_q)}, \alpha = 0, 2^t, 2 \cdot 2^t, \dots, \tau_q\}$; let $q = q - 1$. Go to Step 2.

Step 4: The derived quasi hierarchical tree is $QHT = \cup_{q=1}^Q \{S^{(\tau_q)}\}$.

In the second step of Algorithm 1, the value of hedge h_t is assigned according to the degree of uncertainty. For instance, given q , if $\tau_{q-1} = \tau_q/2$ is required, then we let $t = 1$. The condition $Q_0 \geq Q$ is given to make sure that a hierarchy with Q levels can be constructed. Moreover, t is fixed in each loop. Because $t \geq 1$, we have $\tau_q \geq 2\tau_{q-1}$. Especially, if $t = 1$ holds throughout the algorithm, then $\tau_q = 2\tau_{q-1}$, the derived quasi hierarchical tree reduces to a quasi linguistic hierarchy. In this case, it is sufficient to set $\tau_Q = 2^Q$, i.e., $Q_0 = Q$, the step of constructing LTS $S^{(\tau_{q-1})}$ can be rewritten as

$$S^{(\tau_{q-1})} = \{l_\beta^{(q-1)} = \langle h_1, s_\alpha^{(q)} \rangle | s_\alpha^{(q)} \in S^{(\tau_q)}, \alpha = 0, 2, 4, \dots, \tau_q\} \quad (22)$$

Algorithm 1 can be further illustrated by the next example.

Figure 5: An example of quasi linguistic hierarchy of three, five, and nine terms.

Example 5. Given the LTS shown at the 3rd level in Fig. 5, denoted by $S^{(8)}$, we will show how to generate the quasi linguistic hierarchy.

Step 1: Initialization. Let $q = 3$, $\tau_3 = 8$ and $S^{(\tau_3)} = S^{(8)}$.

Step 2: $q > 1$, go to Step 3.

Step 3: Let $t = 1$, then $\alpha = 0, 2, 4, 6, 8$, and

$$S^{(\tau_2)} = \{l_\beta^{(2)} = \langle h_1, s_\alpha^{(3)} \rangle | s_\alpha^{(3)} \in S^{(8)}, \alpha = 0, 2, 4, 6, 8\} = \{\langle h_1, s_0^{(3)} \rangle, \langle h_1, s_2^{(3)} \rangle, \langle h_1, s_4^{(3)} \rangle, \langle h_1, s_6^{(3)} \rangle, \langle h_1, s_8^{(3)} \rangle\}$$

There are 5 LTWHs in $S^{(\tau_2)}$, thus $\tau_2 = 4$. According to Theorem 3, the LTS is the one shown in the second level in Fig. 5. Let $q = 2$. Go to Step 2.

Step 2: $q > 1$, go to Step 3.

Step 3: Let $t = 1$, then $\alpha = 0, 4, 8$, and $S^{(\tau_1)} = \{l_\beta^{(1)} = \langle h_1, l_\alpha^{(2)} \rangle | l_\alpha^{(2)} \in S^{(4)}, \alpha = 0, 4, 8\} = \{\langle h_2, s_0^{(3)} \rangle, \langle h_2, s_4^{(3)} \rangle, \langle h_2, s_8^{(3)} \rangle\}$

There are 3 LTWHs in $S^{(\tau_1)}$, thus $\tau_1 = 2$. According to Theorem 3, the LTS is the one shown in the top level in Fig. 5. Let $q = 1$. Go to Step 2.

Step 2: $q = 1$, go to Step 4.

Step 4: The derived quasi hierarchical tree is the linguistic hierarchy in Fig. 5.

Moreover, in the case $q = 3$, if we let $t = 2$, then $\alpha = 0, 4, 8$, we can generate the LTS in the top level directly from the bottom level of Fig. 5.

Comparing with the existing hierarchical tree model, Algorithm 1 creates the set of LTSs by simulating the natural way of human thinking. In fact, it is always indispensable and valuable to describe information as accurate as possible. If one object cannot be evaluated by numerical values because of the limitation of knowledge and/or expertise, then fuzzy techniques, including linguistic terms, could be considered as an alternative; In the even worse situation where the opinions cannot be expressed with sufficient confidence by a linguistic term, then we shall seek for another LTS with a coarser knowledge granule. The proposed LTWHs can act as a natural way to make the granule coarser.

However, it should be clarified that the proposed Algorithm 1 does not aim at constructing linguistic hierarchy for multi-granularity linguistic decision making. On the contrast, given a linguistic hierarchy which can be constructed by the algorithm, we have drawn the inherent connection between the linguistic terms of different levels and LTWHs. This fact could lead to a simple and effective information unification process, which will be presented in the coming subsection.

5.2. Unifying multi-granularity linguistic information by using LTWHs

Suppose that the decision maker evaluates the set of alternative based on the hierarchical tree $HT = \cup_{q=1}^Q \{S^{(\tau_q)}\}$ which can be constructed by Algorithm 1. The unification of a given term $s_\alpha^{(q)} \in S^{(\tau_q)}$ in the q -th level can be processed by the following algorithm:

Algorithm 2. Unification of multi-granularity linguistic information from a hierarchical tree.

Input: Hierarchical tree $HT = \cup_{q=1}^Q \{S^{(\tau_q)}\}$, linguistic term $s_\alpha^{(q)} \in S^{(\tau_q)}$.

Output: The unified value l (i.e., a LTWH).

Step 1: Construct a quasi hierarchical tree QHT , which possesses the same structure of the given HT , by using Algorithm 1 associated with $S^{(\tau_Q)}$ and $H^{(s)}$. Especially, for any q in Algorithm 1, t is fixed by $\log_2 \tau_q / \tau_{q-1}$.

Step 2: Find the linguistic term $s_\beta^{(Q)} \in S^{(\tau_Q)}$ whose core coincides with the core of $s_\alpha^{(q)}$.

Step 3: Let $i = 0$, $h_{t_i} = h_0$.

Step 4: If $i < Q - q$, then go to Step 5; else go to Step 6.

Step 5: Let $h_{t_i} = h_{\log_2 \tau_q / \tau_{q-1}}$; $i = i + 1$. Go to Step 4.

Step 6: The unified value is $l = \langle h_{t_{Q-q-1}}, \langle h_{t_{Q-q-2}}, \dots, \langle h_{t_1}, \langle h_{t_0}, s_\beta^{(Q)} \rangle \rangle \rangle \rangle$.

Algorithm 2 forms a LTWH to represent a linguistic term in the q -th level of HT . Specifically, the term in the LTWH is the one in the Q -th level whose core is the same as the given term. The weakened hedge is derived by searching all the hedges which are employed to constructed the q -th, $(q + 1)$ -th, $\dots$, $(Q - 1)$ -th levels of QHT when using Algorithm 1. According to Section 2, the outcome of Step (5) in Algorithm 2 can be rewritten as $l = \langle h_{t_0+t_1+\dots+t_{Q-q-1}}, s_\beta^{(Q)} \rangle$. In application, the number of levels Q is usually not big, thus Algorithm 2 can be finished in a few loops. Especially, if $t = 1$ holds when constructing QHT , then Steps 3-6 can be reduced to

$$l = \langle h_{Q-q}, s_\beta^{(Q)} \rangle \quad (23)$$

The unification process can be further illustrated by the following example:

Example 6. The following linguistic decision matrix is derived from the non-financial performance of three private banks (a_1, a_2, a_3) with respect to pricing, differentiation, marketing, service delivery, and productivity [44] based on a linguistic hierarchy $HT = \{S^{(2)}, S^{(4)}, S^{(8)}\}$ whose structure is exactly the same as the one shown in Fig. 5:

$$M^{(MG)} = \begin{pmatrix} s_4^{(3)} & s_3^{(2)} & s_2^{(1)} & s_1^{(2)} & s_2^{(2)} \\ s_7^{(3)} & s_2^{(2)} & s_1^{(1)} & s_2^{(2)} & s_3^{(2)} \\ s_6^{(3)} & s_4^{(2)} & s_0^{(1)} & s_3^{(2)} & s_3^{(2)} \end{pmatrix} \quad (24)$$

To select the best bank, the original information should be unified and then the overall performances can be computed. According to Algorithm 2, we have:

Step 1: Based on $HT = \{S^{(2)}, S^{(4)}, S^{(8)}\}$, a quasi hierarchical tree can be constructed by using Algorithm 1 with $t = 1$.

Step 2: Take $s_3^{(2)}$ and $s_1^{(1)}$ for instance. $s_6^{(3)}$ is the term whose core coincides with that of $s_3^{(2)}$, and the term whose core coincides with that of $s_1^{(1)}$ is $s_4^{(3)}$.

Steps 3-6: According to Eq. (23):

$$s_3^{(2)} \rightarrow \langle h_1, s_6^{(3)} \rangle (\text{more or less } s_6^{(3)}), \quad s_1^{(1)} \rightarrow \langle h_{3-1}, s_4^{(3)} \rangle = \langle h_2, s_4^{(3)} \rangle (\text{roughly } s_4^{(3)})$$

The decision matrix in Eq. (24) can be unified as follows:

$$M^{(LTWH)} = \begin{pmatrix} s_4^{(3)} & \langle h_1, s_6^{(3)} \rangle & \langle h_2, s_8^{(3)} \rangle & \langle h_1, s_2^{(3)} \rangle & \langle h_1, s_4^{(3)} \rangle \\ s_7^{(3)} & \langle h_1, s_4^{(3)} \rangle & \langle h_2, s_4^{(3)} \rangle & \langle h_1, s_4^{(3)} \rangle & \langle h_1, s_6^{(3)} \rangle \\ s_6^{(3)} & \langle h_1, s_8^{(3)} \rangle & \langle h_2, s_0^{(3)} \rangle & \langle h_1, s_6^{(3)} \rangle & \langle h_1, s_6^{(3)} \rangle \end{pmatrix} \quad (25)$$

Furthermore, the proposed computational model in Section 3 can solve the multi-granularity linguistic decision making problem in Example 6. In fact, associated with the weights of criteria derived in Ref. [44] ($\mathbf{w} = (0.364, 0.131, 0.190, 0.062, 0.254)$), the overall performances of the three banks can be obtained. For instance, the overall performance of a_1 is:

$$l_1 = \langle h_{t_1}, s_{\alpha_1} \rangle = 0.364 \langle h_0, s_4^{(3)} \rangle \oplus 0.131 \langle h_1, s_6^{(3)} \rangle \oplus 0.19 \langle h_2, s_8^{(3)} \rangle \oplus 0.062 \langle h_1, s_2^{(3)} \rangle \oplus 0.254 \langle h_1, s_4^{(3)} \rangle$$

where $t_1 = \text{round}(0.364 \cdot 0 + 0.131 \cdot 1 + 0.19 \cdot 2 + 0.062 \cdot 1 + 0.254 \cdot 1) = \text{round}(0.827) = 1$, $\alpha_1 = \text{round}(0.364 \cdot 4 + 0.131 \cdot 6 + 0.19 \cdot 8 + 0.062 \cdot 2 + 0.254 \cdot 4) = \text{round}(4.902) = 5$. Thus $l_1 = \langle h_1, s_5^{(3)} \rangle$ (more or less $s_5^{(3)}$). Similarly, the overall performances of a_2 and a_3 are, respectively, $l_2 = \langle h_1, s_6^{(3)} \rangle$ (more or less $s_6^{(3)}$) and $l_3 = \langle h_1, s_5^{(3)} \rangle$ (more or less $s_5^{(3)}$). Note that, for l_3 , we have $t_3 = \text{round}(0.827)$ and $\alpha_3 = \text{round}(5.128)$. According to the total order defined in Eq. (17), $l_1 \preceq_{Lex} l_2$, $l_3 \preceq_{Lex} l_2$. Thus a_2 is the best one among the three.

As have been mentioned in Section 3.3, the approximation step, i.e., the function *round*, used in the linear operation of LTWHs may lead to a rough solution. In the above case, the overall performances of a_1 and a_3 are equal because of the approximation. If the decision maker persists in obtaining a total ranking of the two alternatives, then the results in the aggregation step can be used. Thus we can denote them by $l_1 = \langle h_{0.827}, s_{4.902}^{(3)} \rangle$ and $l_3 = \langle h_{0.827}, s_{5.128}^{(3)} \rangle$. The order $\preceq_{Lex}$ works well in this case. Based on $\preceq_{Lex}$, we have $l_1 \preceq_{Lex} l_3$. Note that the intermediate results can only be used for ranking because they are not interpretable.

5.3. Comparative analysis regarding the unification algorithm

To analyze the features of the proposed unification algorithm, we recall three unification approaches proposed by Herrera et al. [18], Herrera and Martínez [20] and Espinilla et al. [15], which are named by the fuzzy set-based approach, the linguistic 2-tuple-based approach and the extended hierarchical approach respectively.

The fuzzy set-based approach [18] does not bring any restriction on the LTSs in the set S^{MG} . To deal with the arbitrariness of semantic distributions of the LTSs, a so-called BLTS which is uniformly distributed in the domain is employed to unify multi-granularity linguistic information. Once the BLTS $S^{(\tau_T)} = \{s_\alpha^{(\tau_T)} | \alpha = 0, 1, \dots, \tau_T\}$ is chosen, a linguistic term $s_\alpha^{(q)}$ can be expressed by a fuzzy set defined in $S^{(\tau_T)}$ according to the following function $TF_{q,T}$:

$$\begin{aligned} TF_{q,T} : S^{(\tau_q)} &\rightarrow \mathcal{F}(S^{(\tau_T)}), \\ TF_{q,T}(s_\alpha^{(q)}) &= \{(s_\beta^{(\tau_T)}, \gamma_\beta^\alpha) | \beta = 0, 1, \dots, \tau_T\}, \text{ for any } s_\alpha^{(q)} \in S^{(\tau_q)}, \\ \gamma_\beta^\alpha &= \max_y \min(\mu_{s_\alpha^{(q)}}(y), \mu_{s_\beta^{(\tau_T)}}(y)), \end{aligned} \quad (26)$$

where $\mathcal{F}(S^{(\tau_T)})$ is the set of fuzzy sets defined in $S^{(\tau_T)}$. For the problem in Example 6, it is natural to let LTS $S^{(8)}$ in the third level act as the BLTS. Then each entry of the matrix $M^{(MG)}$ can be unified and represented by a fuzzy set defined in $S^{(8)}$. For instance, the unified result of $s_3^{(2)}$ is shown in the second column of Table 3.

Table 3: Examples of the unified results and the overall performances of the four comparative approaches.

Approach	Unified result of $s_3^{(2)}$	Overall performance of a_1
Fuzzy set-based approach [18]	$\{(s_0^{(3)}, 0), (s_1^{(3)}, 0), (s_2^{(3)}, 0), (s_3^{(3)}, 0), (s_4^{(3)}, 1/3), (s_5^{(3)}, 2/3), (s_6^{(3)}, 1), (s_7^{(3)}, 2/3), (s_8^{(3)}, 1/3)\}$	$\{(s_0^{(3)}, 0.021), (s_1^{(3)}, 0.041), (s_2^{(3)}, 0.147), (s_3^{(3)}, 0.393), (s_4^{(3)}, 0.720), (s_5^{(3)}, 0.515), (s_6^{(3)}, 0.330), (s_7^{(3)}, 0.239), (s_8^{(3)}, 0.234)\}$
Linguistic 2-tuple-based approach [20]	$(s_6^{(3)}, 0)$	$(s_5^{(3)}, -0.098)$
Extended hierarchical approach [15]	$(s_6^{(3)}, 0)$	$(s_5^{(3)}, -0.098)$
The proposed approach	more or less $s_6^{(3)}$	more or less $s_5^{(3)}$

The linguistic 2-tuple-based approach [20] aims at introducing a symbolic and precise approach for multi-granularity linguistic decision making based on the linguistic 2-tuple model. This approach begins with constructing a linguistic hierarchy, like the one shown in Fig. 5. In the unification phase, the following transformation function is defined to transform the linguistic 2-tuple $s_\alpha^{(q)} = (s_i^{(q)}, t^{(q)})$ in the q -th level to another linguistic 2-tuple $s_\beta^{(q')} = (s_j^{(q')}, t^{(q')})$ in the

q' -th level:

$$TF'_{q,q'}(s_i^{(q)}, t^{(q)}) = \Delta \left(\frac{\Delta^{-1}(s_i^{(q)}, t^{(q)}) \cdot (\tau_{q'})}{\tau_q} \right) \quad (27)$$

where Δ and Δ^{-1} are a pair of functions which are used to transform information between a real number and a linguistic 2-tuple [19]. For the purpose of comparison, the unified information of $M^{(MG)}$ is represented based on LTS $S^{(8)}$. For instance, linguistic term $s_3^{(2)}$ is transformed and shown in the second column of Table 3.

The extended hierarchical approach [15] is based on the extended hierarchical model which generalizes the linguistic hierarchy [20] by using $\tau_{q+1} > \tau_q$ instead of $\tau_{q+1} = 2\tau_q$. An additional LTS, serving as the position of BLTS, is employed so that the transformation between linguistic terms in any two levels can be conducted using the manner of Eq. (27). Specifically, the given linguistic term is transformed into a linguistic term of the BLTS at first, and then the resultant term can be transformed into the target level. Thus, linguistic information could be unified after using Eq. (27) twice. Because of the special linguistic hierarchy used in our case, the computational results coincide with the one of the linguistic 2-tuple-based approach.

Furthermore, the third column of Table 3 lists the overall performances of a_1 based on the four comparative approaches and the data in Example 6. Their forms are the same as that of the corresponding unified results. According to Table 3, it is apparent that the proposed approach possesses the best interpretability because the computational results are natural linguistic expressions, i.e., LTWHs. Based on the semantic rule of the linguistic 2-tuple model, the linguistic 2-tuple-based approach [20] and the extended hierarchical approach [15] could be interpretable as well. But the computational results are not as natural as those of the proposed approach. The fuzzy set-based approach [18] is hard to interpret.

Besides, some other features of the four approaches are summarized in Table 4. The fuzzy set-based approach [18] brings no limitation to multi-granularity LTSs because it introduces an additional BLTS for unification. The others are based on a linguistic hierarchy and thus the LTS in the bottom level can serve as the role of BLTS. In the unification phase, the linguistic 2-tuple-based approach [20], the extended hierarchical approach [15] and the proposed approach are accurate. The former two are accurate because of the use of linguistic 2-tuple model. The proposed approach uses linguistic hedges to transform linguistic terms based on semantics. The semantics of the unified results coincide with the semantics of the original terms. Thus, the proposed approach is accurate in its unification phase in the sense of semantics.

Table 4: Features of the four comparative approaches.

Approach	BLTS	Unification results	Semantics of unified terms	Multi-granularity LTS
Fuzzy set-based approach [18]	Needed	Fuzzy sets	Changed	Any
Linguistic 2-tuple-based approach [20]	No	Linguistic 2-tuples	Changed	Linguistic hierarchy
Extended hierarchical approach [15]	Needed	Linguistic 2-tuples	Changed	Any
The proposed approach	No	LTWHs	Unchanged	Linguistic hierarchy

Based on the above analysis, we can draw some conclusions with regard to the strengths and weaknesses of using LTWHs in the unification of multi-granularity linguistic information. The strengths are:

(1) The use of linguistic hedges enables to unify linguistic information while maintaining its semantics. This is because the idea of granular transformation is to search for the LTWH whose semantics coincide with a targeted term. The uncertainty included in the targeted term can be modelled by means of a certain hedge and a term with a finer granularity.

(2) The proposed unification algorithm possesses the best interpretability, comparing with other approaches. The results can be interpreted by LTWHs which are close to our linguistic convention. Because of the approximation step in Definition 8, the computational results are natural linguistic expressions in the same range of the original information.

(3) The proposed approach provides a more flexible manner to represent multi-granularity linguistic information. In fact, we do not construct a real linguistic hierarchy. Thus, given a LTS and a WHS, the available linguistic expressions are thus not restricted to the hierarchy. As can be seen in Fig. 5, the expressions “roughly low” is not included in the hierarchy. It cannot be handled in existing multi-granularity models, but can be modelled by a LTWH.

However, the weakness of the algorithms is the limitation of multi-granularity LTSs. Given two adjacent LTSs $S^{(\tau_q)}$ and $S^{(\tau_{q+1})}$, Algorithm 1 requires that τ_{q+1}/τ_q must be a multiple of 2. When constructing the set of multi-granularity LTSs, we have to start with a fine enough LTS which serves as the bottom level of the linguistic hierarchy. This is caused by the definition of the semantics of LTWHs.

In addition, the original information may not take the form of Eq. (24) but Eq. (25). In this case, the experts present their evaluations by means of natural linguistic expressions taking the form of LTWHs. The proposed computational model of LTWH can form a multi-criteria decision making approach in the setting of LTWHs. The aggregation results fall into the same domain of the unified information. To avoid the loss of information caused by high interpretability, an alternative way to obtain accurate and fine aggregation results, by ignoring the approximation step, is also available if the alternatives cannot be ranked by interpretable results.

6. Conclusions

Linguistic hedges are usually considered to strengthen or weaken the power of an adjective. When expressing one's preference in qualitative setting, linguistic terms are preferred. If it is hard to select from a set of linguistic terms, using weakened hedges is a natural way to express the uncertainty. In this study, we have presented a novel CWW model which comprises weakened hedges as a component. A LTWH is a 2-tuple formed by a hedge and an atomic term. The syntax and semantics of LTWHs have been defined to serve as the basis of the CWW model. To enable computing with LTWHs, the negation, order relations and some basic operations have been presented as well. Finally, the connection between LTWHs and some multi-granularity linguistic models has been exploited. When linguistic information is from some specific hierarchical trees, an algorithm, which outputs interpretable results, has been developed to serve as an alternative solution for unification.

The main contributions of this paper lie in the following aspects:

(1) The domain of the values of a linguistic variable has been enlarged. Linguistic hedges, especially weakened hedges, have been introduced in the field of linguistic decision making by defining the representational and computational models of LTWHs. As a natural and frequent way of human thinking, complex linguistic expressions taking the form of LTWHs, such as “more or less *good*”, can be considered by the experts to express preference information in QDM problems.

(2) The application of LTWH in the unification of multi-granularity linguistic information has been proposed and thereby the effectiveness of the proposed model has been clarified. The unification algorithm provides high interpretability with tolerant computational complexity.

Especially, the advantages of the proposed model of LTWHs can be concluded as follows:

(1) Based on a predefined LTS with semantics, the syntax and semantics of a LTWH are clearly defined. The semantics is defined without the necessity of determining any parameters subjectively because the similarity measure is defined objectively according to the distribution of the domain.

(2) The basic operations defined in the computational model make sure that the computational process is interpretable. Following the traditional framework of CWW, the computational results fall into the same domain of the inputs. In addition, to avoid loss of information and derive accurate aggregation results for some special cases, an alternative solution of computation has been provided as well.

There are also some limitations in the proposed model. Firstly, we mainly focus on two frequently used hedges in the case study. In fact, more linguistic hedges should be determined and classified based on the linguistic knowledge and the experts' linguistic convention. Secondly, the defined basic operations may lead to loss of information if high interpretability is required. Thirdly, sometimes, linguistic hedges, such as “more or less”, could be an outlier. This case is not considered in the current study.

Besides, although linguistic hedges have been introduced in the field of linguistic decision making, there are many open problems which are worthy investigating in future:

(1) The semantics of LTWHs could be defined in a more general case. In this paper, the semantics of atomic terms in a LTS is defined by TriFNs, and $\tau_q + 1$ TriFNs share $\tau_q + 1$ points in the domain. Generally, semantics may be represented by trapezoidal fuzzy numbers. In this case, the definition of similarity measures is more complex.

(2) Preference relations and their consistency measures [4, 6] are another hot issue in linguistic decision making. Entries of linguistic preference relations have been extended to ULTs, HFLTSS and EHFLTSS when experts cannot

express their preferences by one linguistic term. It is also natural that experts may express the preferences by means of linguistic expressions like “more or less *good*”. If so, how to get priority from this kind of preference relation is interesting.

(3) In practice, it is not rational to assume that experts can only use ULTs, HFLTS or LTWHs for evaluation. On the contrary, they may use several different kinds of linguistic expression for the sake of convenience. Therefore, it is also interesting to develop integrated or hybrid decision making approach which can deal with multiple kinds of uncertain linguistic information at the same framework.

(4) In the framework of granular computing [36, 37, 38], it is interesting to observe that a weakened hedge can transform knowledge (represented by linguistic terms) of a finer granule into knowledge of a coarser granule, and the transformation is very easy. Thus we speculate that the proposed technique may benefit constructing multi-granularity models for data sciences [50]. In this sense, some existing techniques of granular computing, such as Refs. [9, 22, 39, 58], would be helpful for the developments of new models.

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