ON CERTAIN SUBGROUPS OF $E_8(2)$

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2018

Alexander McGaw
School of Mathematics

Contents

The University of Manchester

Alexander McGaw
Doctor of Philosophy
On Certain Subgroups of $E_8(2)$
September 11, 2018

In this thesis, we study certain subgroups of the exceptional group of Lie-type, $E_8(2)$. In particular, we explore subgroups H of $E_8(2)$ where $F^*(H)$ is isomorphic to one of the following simple groups: $U_4(2)$, $Sp_6(2)$, $\Omega_8^-(2)$, $\Omega_8^+(2)$, $\Omega_8^+(4)$, $Sp_8(2)$, $Sp_4(4)$, or $L_4(4)$. In the case of $F^*(H)$ being isomorphic to $\Omega_8^+(2)$, $\Omega_8^+(4)$, or $Sp_8(2)$, we construct representatives of all classes of subgroups isomorphic to H in $E_8(2)$. In the other five cases, we find representatives of some, but possibly not all, classes of subgroups isomorphic to H in $E_8(2)$. In all cases except where $F^*(H)$ is isomorphic to $\Omega_8^-(2)$, $\Omega_8^+(4)$, or $L_4(4)$, we prove that H is not maximal in $E_8(2)$. All of the main results in this thesis are proved computationally, making use of the computer algebra package, Magma. Furthermore, this thesis is accompanied with files compatible with Magma containing representatives of the classes of subgroups of $E_8(2)$ constructed throughout the thesis, as well as some groups which were used in the construction of said subgroups. Much of this work is a contribution toward classifying the maximal subgroups of $E_8(2)$.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Acknowledgements

I offer my sincerest thanks to my supervisor Professor Peter Rowley for his wisdom, guidance, and patience over the last four years. I am grateful to have worked with someone so passionate about group theory and an excellent mathematics educator and communicator.

I also extend thanks to my whole group. In particular, Jamie Phillips and David Ward for easing my initial PhD doubts and anxieties, John Ballantyne for his knowledge and expertise, and Peter Neuhaus for accompanying me on this challenging journey and always being available for me to pester with my endless questions. In addition, I also offer thanks to the entire School of Mathematics at the University of Manchester. The Alan Turing Building has been home to a welcoming and stimulating community for me these last eight years.

Thank you to my teachers Mr Swarbrick and Mike Moss. Their enthusiasm helped to foster my emerging passion for mathematics.

Next, I would like to thank my parents Jayne Kilgarriff, Jim McGaw, and Kevin Kilgarriff, who have always generously supported me, and my brother Oliver McGaw for his continued encouragement. In addition, I thank my friends, *the group*, for their unending supply of creativity, humour, and loyalty. I also thank my *maths friends* for always being ready for a sanity-restoring weekend of board games.

Finally, my biggest thanks to Natasza Klas for her warmth, support, and for carrying me through the toughest times. It is impossible to overstate how much your companionship means to me.

Chapter 1

Introduction

For almost as long as the notion of a group has been defined, there has been interest in classifying maximal subgroups of finite groups. While this is a subject that has enjoyed much success in recent decades, significant progress was made in the early 20th Century. In 1901, Dickson in [17] classified the maximal subgroups of $L_2(q)$, for q a power of a prime, and so began the classification of maximal subgroups of classical groups of small dimension. In the next two decades, the maximal subgroups of $L_3(q)$ were classified by Mitchell in [37] for q odd, and by Hartley in [21] for q even.

More recent progress was made in 1979 with the O'Nan-Scott Theorem, which classified the maximal subgroups of symmetric groups. This is stated in the appendix of [2] by Aschbacher and Scott. This was later expanded upon in 1986 by Liebeck, Praeger, and Saxl in [28], where they classify the maximal subgroups of finite symmetric and alternating groups, while also listing cases for which subgroups of the finite symmetric and alternating groups are not maximal.

Turning our attention to recent work on the finite simple groups, in 1984 Aschbacher classified the maximal subgroups of finite simple classical groups in [1]. As for the sporadic groups, the maximal subgroups of all the sporadic groups except the Monster group are described in the $\mathbb{ATLAS}$ [14]. However, this was not true at the time of publication. We will provide a brief survey of maximal subgroups of sporadic simple groups here, but a more detailed history can be found in [53].

This is a topic that received much attention in the seventies through to the nineties, but began with Janko categorising the maximal subgroups of the Janko group J_1 in 1966. In 1970, Magliveras classified the maximal subgroups of the Higman-Sims group,

HS [36], while Choi began work on M_{24} in 1972 [12]. The maximal subgroups of M_{24} were later classified by Curtis [15] in 1976. Later that decade, Finkelstein classified the maximal subgroups of the Janko group J_2 [19] in 1973 in collaboration with Rudvalis, and, later that year and the year after, the Conway group Co_3 and McLaughlin group McL in [18] and the Janko group J_3 in [20], the latter again being in collaboration with Rudvalis. Work in the eighties began with Butler, who classified the maximal subgroups of the Held group, He in [11]. The rest of the eighties was dominated by Wilson, who classified the maximal subgroups of the Suzuki group Suz [43] and Conway's groups Co_2 [42] and Co_1 [44] in 1983; the Rudvalis group Ru [45] in 1984; the O'Nan group $O'N$ [47] and the Lyons group Ly in [48]; the Harada-Norton group HN [39] in 1986 in collaboration with Norton; the Fischer group Fi_{22} [46] in 1987 and later the Fischer group Fi_{23} [25] in 1989 in collaboration with Kleidman and Parker; and the Janko group J_4 [26] in 1988 in collaboration with Kleidman. Wilson's first Ph. D. student, Linton, then classified the maximal subgroups of the Thompson group Th [30] in 1989, who later collaborated with Wilson to classify the maximal subgroups of the Fischer group Fi_{24} [31] in 1991. Finally, Wilson completed the classification of the maximal subgroups of the Baby Monster $\mathbb{B}$ in [50]. A turning point for this was, as Wilson describes in [53], when Wilson built a degree-4370 representation of $\mathbb{B}$ over $GF(2)$ in [49], which enabled a new computational approach. This is a theme for much of the work on maximal subgroups; early work was done by hand whereas later work was carried out computationally.

Note that we have not yet discussed the Mathieu groups M_{11}, M_{12}, M_{22}, and M_{23}. As Wilson notes in [53], there do not seem to be any publications detailing the proofs of the classification of the maximal subgroups of these groups. However, these results are well known and can be verified with modern computational techniques. Finally, we have the Monster group $\mathbb{M}$. Again, we recommend [53] for an in-depth examination of the current state of the classification of its maximal subgroups.

A good source for the whole history of finite simple groups and efforts to find their maximal subgroups is Wilson's book *The Finite Simple Groups* [51]. In his section on applications of the classification of finite simple groups, he discusses the problem of finding maximal subgroups of finite simple groups, concluding with the remark that, hopefully, lists of maximal subgroups will one day exist for the exceptional groups of

Lie type.

Such lists exist already for certain exceptional groups of Lie type. Efforts to produce such lists began in 1981 with Cooperstein classifying the maximal subgroups of $G_2(q)$ for q even in [13]. The classification for $G_2(q)$ was extended to q odd by Kleidman [23] in 1988. The same year, Kleidman classified the maximal subgroups of ${}^3D_4(q)$ in [24]. The maximal subgroups of $F_4(2)$ were classified in [40] in 1989, while Magaard, in his 1990 Ph. D. thesis [35], classified the maximal subgroups of $F_4(q)$ where q is not a power of 2 or 3. In the same year, Kleidman and Wilson also published their work in [27], classifying the maximal subgroups of $E_6(2)$. Even more recently the maximal subgroups of $E_7(2)$ were classified in 2015 by Ballantyne, Bates, and Rowley in [6]. As for ${}^2E_6(2)$, the classification of its maximal subgroups has been known for many years, but only earlier this year was a proof published in [52] by Wilson.

This brings us today. An as yet unpublished paper [5] by Aubad, Ballantyne, McGaw, Neuhaus, Rowley, and Ward classifies the maximal subgroups of $E_8(2)$. This thesis details the work undertaken on certain subgroups of $E_8(2)$, finding conjugacy classes of subgroups in $E_8(2)$ and proving that certain subgroups are not maximal in $E_8(2)$. The vast majority of this work is computer-assisted, making use of the computer algebra package MAGMA [7].

The bulk of this thesis is contained within eight chapters, each exploring a different group. Before we get there, we have chapter 2 dedicated to the background material and notation used throughout. Included in Chapter 2 are elementary group theoretic results which are of particular use to us, as well as some more advanced results about groups in general and about $E_8(2)$ specifically. An entire section is devoted to conjugacy class fusion of semisimple elements in $E_8(2)$. In general, if G is any group, then fusion refers to the question of, given $H \leq G$ and $h \in H$, which G-conjugacy class h – and, therefore, all of its H-conjugacy class – belongs to. Through the use of Brauer characters, we are able to deduce that for a certain groups H, if H is isomorphic to a subgroup of G, then there are limited possibilities for how the H-conjugacy classes fuse to the G-conjugacy classes. Sometimes, this information is enough for us to deduce immediately that a subgroup isomorphic to H is not maximal in G.

A crucial part of Chapter 2 is a list of potential maximal subgroups of $E_8(2)$, which serves as a motivation for the subgroups considered in this thesis. This list is an amalgamation of the work of several authors. In our paper [5] on the maximal subgroups of $E_8(2)$, it serves as a checklist for all the groups we must eliminate as being maximal in $E_8(2)$. This thesis focuses on several of the subgroups considered there, with the added focus of finding all conjugacy classes in $E_8(2)$ of these subgroups.

We then move to Chapter 3, another chapter preceding the main eight chapters, which provides an overview of the methodology used throughout the thesis. This includes Section 3.1 on computing in Magma, as, although the reader might be familiar with Magma, there are certain current limitations of Magma that deserve to be explored in full. Section 3.2 is a section devoted to computational methods applied to working with $E_8(2)$ specifically. It should be noted that the methodology discussed here is applicable to any group, although, it will usually only be required when working with large groups. The methods required to prove the results of the eight main chapters are similar, so this section should be seen as a menu of techniques and computational procedures which the later chapters will select from.

The first topic of this section discusses a technique for calculating sets of centralising involutions of subgroups of $E_8(2)$. This is often a challenging task, considering the size of $E_8(2)$, but we have developed routines of several steps allowing these sets to be constructed for certain subgroups of $E_8(2)$. The second topic discussed in Section 3.2 is that of sieving sets of involutions. Again, we note that several of the procedures discussed here can be applied to any set, not just sets of involutions. The proof of the main result of nearly every subsequent chapter involves sieving a set of involutions, so we provide a list of the most common sieves we utilise there. Finally, we provide some results and techniques for how we show two isomorphic subgroups of $E_8(2)$ are conjugate in $E_8(2)$. The primary theme of this thesis can be thought of as finding all subgroups of $E_8(2)$ isomorphic to certain almost simple groups, up to conjugacy in $E_8(2)$. Showing subgroups are conjugate in $E_8(2)$ is, again, often a difficult computational task.

We will now give an outline of the main eight chapters, beginning with Chapter 4. Here, we will explore the existence of $U_4(2)$ subgroups in $E_8(2)$. We note that we often make use of the term *copy* when refering to subgroups of $E_8(2)$. For example, instead of saying that we have a subgroup of $E_8(2)$ isomorphic to $U_4(2)$, we simply say that we have a copy of $U_4(2)$ in $E_8(2)$. We begin by looking at a Sylow 3-subgroup of $E_8(2)$, and finding all subgroups in there isomorphic to a Sylow 3-subgroup of $U_4(2)$. This serves as a starting point for our hunt for $U_4(2)$ subgroups. Using the $U_4(2)$ fusion information given in Proposition 2.2, we are able to further limit the Sylow 3-subgroups we can build up to a $U_4(2)$ subgroup. Once we have obtained $U_4(2)$ subgroups, we then find their automorphism extensions $U_4(2):2$. We then show that no $U_4(2)$ or $U_4(2):2$ subgroup of $E_8(2)$ is a maximal subgroup. All this forms the first main result of this thesis. Before we state it, let us clarify the language used regarding conjugacy class fusion. In Proposition 2.2, we see that there are nine possibilities for how the conjugacy classes of any $U_4(2)$ subgroup fuse into the conjugacy classes of $E_8(2)$. These are numbered (i)–(ix). Throughout this thesis, if we want to assume that a potential $U_4(2)$ subgroup of $E_8(2)$ is such that its conjugacy classes fuse into the $E_8(2)$-conjguacy classes as stated in Proposition 2.2 (i), we use the phrase, "assume $U_4(2)$ follows fusion possibility 2.2 (i)", or the phrase "assume $H \leq E_8(2)$ with $H \cong U_4(2)$ and suppose H follows $U_4(2)$ fusion possibility (i)".

Before we state the first main result, let $G \cong E_8(2)$. Throughout this thesis, whenever $G \cong E_8(2)$, it should be assumed for computational purposes that G is the degree-248 matrix representation of $E_8(2)$ over $GF(2)$, so $G \leq GL_{248}(2)$. This can be constructed in Magma using Procedure B.1. Additionally, throughout the whole thesis V will denote the 248-dimensional $GF(2)$ $E_8(2)$-module. We also note that in the statement of these main results we are using the fact that the automorphism group of all simple groups H which we consider are split extensions, as is shown in [34]. As for the structure of these automorphism groups, see section 1.7.2 of [9].

Theorem 1.1. *Suppose $H \leq G$ with $F^*(H) \cong U_4(2)$. Moreover, suppose $F^*(H)$ does not follow fusion possibility 2.2 (viii) or (ix). Then:*

(i) if $H \cong U_4(2)$, then there are exactly three G-conjugacy classes of subgroups isomorphic to H;

(ii) if $H \cong U_4(2):2$, then there are exactly three G-conjugacy classes of subgroups isomorphic to H.

We limit our focus to those $U_4(2)$ subgroups not following fusion possibility 2.2 (viii) or (ix), as cases (viii) and (ix) are not computationally viable using the techniques developed in this thesis. Moreover, as far as the maximality question is concerned, we will see that any $U_4(2)$ subgroup of $E_8(2)$ following fusion possibility 2.2 (viii) or (ix) can easily be shown to not be maximal.

In Chapter 5, we will turn our focus to $Sp_6(2)$. We will provide the number of conjugacy classes of $Sp_6(2)$ subgroups of $E_8(2)$ following certain fusion possibilites for $Sp_6(2)$ given in Proposition 2.3. The main tactic used in this chapter is the fact that $U_4(2) : 2$ is a maximal subgroup of $Sp_6(2)$. So, after careful consideration of the $Sp_6(2)$ fusion possibilities, we can construct $Sp_6(2)$ by "building up" from the copies of $U_4(2) : 2$ found in the preceding chapter. This forms the basis for the second main result, which we will now state.

Theorem 1.2. *Suppose $H \leq G$ with $H \cong Sp_6(2)$. If H follows fusion possibility 2.3 (ii) or (iii), then there are three G-conjugacy classes of subgroups isomorphic to H.*

Again, we limit our focus to those $Sp_6(2)$ subgroups of $E_8(2)$ not following fusion possibility 2.3 (v) or (vi), as those copies following fusion possibility 2.3 (v) or (vi) contain $U_4(2)$ subgroups which necessarily follow fusion possibility 2.2 (viii) or (ix). We did not construct these $U_4(2)$ subgroups in Chapter 4.

Chapter 6 focuses on compiling certain classes of $\Omega_8^-(2)$ subgroups of $E_8(2)$. Again, the primary tactic involved here is to observe that $Sp_6(2)$ is a maximal subgroup of $\Omega_8^-(2)$, so we construct $\Omega_8^-(2)$ subgroups by building up from the copies of $Sp_6(2)$ found in Chapter 5. This gives rise to the third main result.

Theorem 1.3. *Suppose $H \leq G$ with $F^*(H) \cong \Omega_8^-(2)$. Assume that $F^*(H)$ follows $\Omega_8^-(2)$ fusion possibility 2.4 (ii), then:*

(i) if $H \cong \Omega_8^-(2)$, then there is exactly one G-conjugacy class of subgroups isomorphic to H;

(ii) if $H \cong \Omega_8^-(2) : 2$, then there is exactly one G-conjugacy class of subgroups isomorphic to H.

Again, our focus is restricted to certain fusion possibilities for $\Omega_8^-(2)$ due to the restriction to certain fusion possibilities for $Sp_6(2)$. It is at the end of Chapter 6 where we will actually present a proof of Theorems 1.1, 1.2, and 1.3. This is because the proofs of the earlier theorems rely on results proved in later chapters. For example, part of the proof of Theorem 1.1 regarding $U_4(2)$ relies on results found about $\Omega_8^-(2)$ in Chapter 6.

Next, we shift our attention to $\Omega_8^+(2)$ in Chapter 7, which is the most substantial chapter in this thesis. There, we will no longer build up from subgroups already constructed in previous chapters. Instead, we start with a Sylow 5-subgroup of $E_8(2)$ and construct all possible subgroups isomorphic to a Sylow 5-subgroup of $\Omega_8^+(2)$. Like with the Sylow 3-subgroup of $U_4(2)$, this list of all Sylow 5-subgroups is used as a base for our construction of copies of $\Omega_8^+(2)$. The reason this chapter is comparatively long is because we make no limitation regarding the fusion possibilities of our $\Omega_8^+(2)$ subgroups. While this results in some arduous computations, the reward is arguably the cleanest main result in the thesis, as we wind up constructing, up to conjugacy, all possible subgroups of $E_8(2)$ isomorphic to $\Omega_8^+(2)$ or an automorphism extension of $\Omega_8^+(2)$.

Theorem 1.4. *Suppose $H \leq G$ with $F^*(H) \cong \Omega_8^+(2)$. Then:*

(i) if $H \cong \Omega_8^+(2)$, then there are exactly seven G-conjugacy classes of subgroups isomorphic to H;

(ii) if $H \cong \Omega_8^+(2):2$, then there are exactly seventeen G-conjugacy classes of subgroups isomorphic to H;

(iii) if $H \cong \Omega_8^+(2):3$, then there are exactly six G-conjugacy classes of subgroups isomorphic to H;

(iv) if $H \cong \Omega_8^+(2):\mathrm{Sym}(3)$, then there are exactly ten G-conjugacy classes of subgroups isomorphic to H.

Continuing on from Chapter 7 is Chapter 8. Here, we find all $\Omega_8^+(4)$ subgroups up to conjugacy in $E_8(2)$. This is a continuation as $\Omega_8^+(2) \leq \Omega_8^+(4)$, and so, as with building up from $U_4(2)$ to $Sp_6(2)$, we build up subgroups of $E_8(2)$ isomorphic to $\Omega_8^+(4)$ from the copies of $\Omega_8^+(2)$ found in Chapter 7.

Theorem 1.5. *Suppose $H \leq G$ with $F^*(H) \cong \Omega_8^+(4)$. Then:*

(i) if $H \cong \Omega_8^+(4)$, then there are exactly two G-conjugacy classes of subgroups isomorphic to H;

(ii) if $H \cong \Omega_8^+(4) : 2$, then there are exactly six G-conjugacy classes of subgroups with the same shape as H;

(iii) if $H \cong \Omega_8^+(4) : 3$, then there is exactly one G-conjugacy classes of subgroups isomorphic to H;

(iv) if $H \cong \Omega_8^+(4) : 2^2$, then there are exactly two G-conjugacy classes of subgroups isomorphic to H.

(v) if $H \cong \Omega_8^+(4) : 6$, then there is exactly one G-conjugacy classes of subgroups isomorphic to H;

(vi) if $H \cong \Omega_8^+(4) : \mathrm{Sym}(3)$, then there are exactly two G-conjugacy classes of subgroups isomorphic to H;

(vii) if $H \cong \Omega_8^+(4) : \mathrm{Dih}(12)$, then there is exactly one G-conjugacy classes of subgroups isomorphic to H.

Also following on from Chapter 7 is Chapter 9, in which we seek to find all $Sp_8(2)$ subgroups of $E_8(2)$. Again, we use the fact that $\Omega_8^+(2) \leq Sp_8(2)$ and build up to $Sp_8(2)$ using the copies of $\Omega_8^+(2)$ found in Chapter 7.

Theorem 1.6. *Suppose $H \leq G$ with $F^*(H) \cong Sp_8(2)$. Then there are exactly four G-conjugacy classes of subgroups isomorphic to H.*

From here, we are finished with subgroups containing $\Omega_8^+(2)$. Then we turn our attention to $Sp_4(4)$ in Chapter 10. We observe that a Sylow 5-subgroup of $Sp_4(4)$ is isomorphic to a Sylow 5-subgroup of $\Omega_8^+(2)$. Hence, from our work on $\Omega_8^+(2)$ in Chapter 7, we have a head start on constructing $Sp_4(4)$ subgroups, since we can build up from the same Sylow 5-subgroups as used to construct copies of $\Omega_8^+(2)$. We state the result regarding $Sp_4(4)$ now, remarking that we are back to restricting our attention to considering only those $Sp_4(4)$ following certain fusion possibilities given in Proposition 2.5. We will now state the main result which is to be proved in Chapter 10.

Theorem 1.7. *Suppose $H \leq G$ with $F^*(H) \cong Sp_4(4)$ and that $F^*(H)$ does not follow $Sp_4(4)$ fusion 2.5 (iii) or (iv). Then:*

(i) if $H \cong Sp_4(4)$, then there are exactly five G-conjugacy classes of subgroups of G isomorphic to H;

(ii) *if* $H \cong Sp_4(4) : 2$*, then there are exactly four* G*-conjugacy classes of subgroups of* G *isomorphic to* H*;*

(iii) $H \ncong Sp_4(4) : 4$.

Again, the techniques developed in this thesis are not suitable for constructing all $Sp_4(4)$ subgroups following fusion possibilities 2.5 (iii) and (iv).

Finally, in Chapter 11 we examine subgroups of $E_8(2)$ isomorphic to $L_4(4)$. Here, we use the fact that $Sp_4(4) \leq L_4(4)$ and build up to $L_4(4)$ using the copies of $Sp_4(4)$ constructed in Chapter 10.

Theorem 1.8. *Suppose* $H \leq G$ *with* $F^*(H) \cong L_4(4)$ *and that* $F^*(H)$ *does not follow* $L_4(4)$ *fusion possibility 2.6 (i). Then:*

(i) *if* $H \cong L_4(4)$*, then there is one* G*-conjugacy class of subgroups of* G *isomorphic to* H*;*

(ii) *if* $H \sim L_4(4) : 2$*, then there are exactly three* G*-conjugacy classes of subgroups of* G *with the same shape as* H*;*

(iii) *if* $H \cong L_4(4) : 2^2$*, then there is one* G*-conjugacy class of subgroups of* G *isomorphic to* H*.*

We conclude Chapter 11 with a proof of Theorem 1.8.

Finally, we conclude this thesis with two appendices. Appendix A provides a detailed description of all the files compatible with Magma which accompany this thesis. Appendix B lists some procedures utilised in the computations throughout the thesis, all of which can be applied in Magma.

Chapter 2

Background

2.1 Notation

We begin this chapter by stating all the notation used throughout this thesis. Let G be a group. Then if H is a subgroup of G we write $H \leq G$ and if H is a normal subgroup of G we write $H \trianglelefteq G$. We will use $|G|$ to denote the order of G and $|S|$ for the cardinality of a set S. Furthermore, $[G : H]$ will denote the index of H in G. We will also use 1 to denote both the identity of a group and the trivial group – the context will always be clear. $Z(G)$ will denote the centre of G; $C_G(g)$ denotes the centraliser in G of an element g while $N_G(H)$ denotes the normaliser in G of a subgroup H. If G acts on a vector space V and $g \in G$, then $C_V(g) = \{v \in V : v^g = v\}$ denotes the fixed space in V of g and $\mathrm{Stab}_G(U) = \{g \in G : u^g \in U \text{ for all } u \in U\}$ denotes the stabiliser in G of a subspace U of V. If $N \trianglelefteq G$ then G/N will denote the factor group. If p is a prime dividing $|G|$ then $\mathrm{Syl}_p(G)$ is the set of all Sylow p-subgroups of G. Now, let $g, h \in G$. Then $g^h = h^{-1}gh$ and $[g, h]$ will denote the commutator $g^{-1}h^{-1}gh$. We will also use $o(g)$ to refer to the order of g. If q is a power of a prime, then $GF(q)$ denotes the Galois field of cardinality q. Before we move onto talking about specific groups, we also state that we will use $A \sqcup B$ to denote a union of disjoint sets A and B.

Let $n, k \in \mathbb{N}$. The cyclic group of order n is denoted n; if p is prime then a p-group of unknown structure denoted $[p^k]$; the elementary abelian group of order p^k is denoted p^k. A direct product of groups G and H is denoted $G \times H$ and their central product $G \circ H$. If G is isomorphic to H then we write $G \cong H$, but if G and H only have the same shape then we write $G \sim H$. Split extensions will be denoted with a colon –

$G = N : H$ where here $N \trianglelefteq H$ and $G/N \cong H$. When discussing particular groups, we will adhere to many conventions set down by the Atlas of Finite Groups [14], which we will refer to as the $\mathbb{ATLAS}$, with the following exceptions. We use Sym(n) and Alt(n) to denote, respectively, the symmetric and alternating groups of degree n, while Dih(n) denotes the dihedral group of order n. Additionally, $\Omega_n^+(q)$ and $\Omega_n^-(q)$ denote the simple projective orthogonal groups of plus type or minus type respectively, while $Sp_n(q)$ denotes the symplectic group of degree n over $GF(q)$.

Finally, we will also use $\mathbb{ATLAS}$ notation when referring to conjugacy classes. If G is a group and $n \in \mathbb{N}$ is such that there exists an element of order n in G, then nA refers to the G-conjugacy class of elements of order n with the shortest length, nB for the G-conjugacy class of elements of order n with the second-shortest length, and so on. If two classes have the same length, they will either be ordered according to $\mathbb{ATLAS}$ convention or arbitrarily.

Also, if $H \leq G$ and $g \in H$, then we will use the subscript H to denote the H-conjugacy class of g and a G subscript to refer to its G-conjugacy class. For instance, suppose $g \in H \leq G$ is of order 3. Then it could be the case that g belongs to the H-conjugacy class 3A while belonging to the G-conjugacy class 3B. To clarify this distinction in classes, we write $g \in 3\mathrm{A}_H$ and $g \in 3\mathrm{B}_G$.

2.2 Motivation

We continue this chapter with a short introduction on the motivation behind the selection of subgroups of $E_8(2)$ examined in this thesis. Although this thesis is themed around an exploration of subgroups of $E_8(2)$, exactly which subgroups we have selected to examine is down to their potential for maximality in $E_8(2)$. In our unpubished paper, *The Maximal Subgroups of* $E_8(2)$ [5], we compile a list of all the potential maximal subgroups of $E_8(2)$, then proceed to show that most of them are, in fact, not maximal. Here, we will provide a specific case of the result used to compile this list which is relevant to the groups discussed in this thesis.

Theorem 2.1. *Let $G \cong E_8(2)$ and suppose $H \leq G$ is maximal such that $F^*(H) = H(2^n)$ is simple and in Lie(2). Then $rk(H(2^n)) \leq 4$ and one of the following holds:*

(i) $2^n \leq 9$;

(ii) $H(2^n) \cong A_1(16)$ *or* ${}^2A_2(16)$*;*

(iii) $2^n \leq 1312$ *and* $H(2^n) \cong A_1(2^n)$ *or* ${}^2B_2(2^n)$*.*

Proof. This is a special case of Theorem 2.1 (vi) in [5] and can be found in Theorem 8 VI of [29]. ■

If we combine this result with the classification of finite simple groups, we obtain a list of potential maximal subgroups of $E_8(2)$ which are simple and belong to Lie(2). These are, for each $n \in \{1, 2, 3\}$:

(i) $L_5(2^n)$, $L_4(2^n)$, $L_3(2^n)$, and $L_2(2^m)$ for $m \in \{3, \dots, 10\}$;

(ii) $Sp_8(2^n)$, $Sp_6(2^n)$, $Sp_4(2^3)$, $Sp_4(2^2)$ and $Sp_4(2)'$;

(iii) $U_5(2^n)$, $U_4(2^n)$, $U_3(2^4)$, $U_3(2^3)$, and $U_3(2^2)$;

(iv) $\Omega_8^+(2^n)$ and $\Omega_8^-(2^n)$;

(v) $G_2(2^3)$, and $G_2(2^2)$;

(vi) $Sz(2^l)$ for $l \in \{3, 5, 7, 9\}$;

(vii) $F_4(2^n)$;

(viii) ${}^3D_4(2^n)$;

(ix) ${}^2F_4(2^3)'$ and ${}^2F_4(2)'$;

In particular, the groups we will examine in this thesis – $U_4(2)$, $Sp_6(2)$, $\Omega_8^-(2)$, $\Omega_8^+(2)$, $\Omega_8^+(4)$, $Sp_8(2)$, $Sp_4(4)$, and $L_4(4)$ – appear in the list.

2.3 Conjugacy Class Fusion in $E_8(2)$

Propositions 2.2 to 2.6 will be stated without proof. These results were all obtained by Neuhaus and a full description of how they were obtained will be given in his thesis [38]. Before we proceed, we will explain the notation used throughout these results. In a given statement, the φ_i will denote the irreducible Brauer characters of the given group over $GF(2)$. If the proposition refers to the fusion of a group H into $G \cong E_8(2)$, then notation $n\mathrm{X} \to n\mathrm{Y}$ means that the H-conjugacy class $n\mathrm{X}$ fuses to the G-conjugacy class nY. As an example, let us examine $U_4(2)$ fusion possibility 2.2 (i). It states 3A→3C. So, if $U_4(2) \cong H \leq G \cong E_8(2)$ and we assume that H follows $U_4(2)$ fusion possibility 2.2 (i), then we have that if $h \in H \cap 3\mathrm{A}_H$ then $h \in 3\mathrm{C}_G$.

We also use the term *cohomological dimension* throughout these results, which we will define now. More generally, let S be a group and V be a KS-module.

(i) A 1-cocycle is a map $\varphi : S \to V$ such that $\varphi(gh) = \varphi(g) + g.\varphi(h)$. The additive group of all 1-cocycles is denoted $Z^1(S, V)$;

(ii) A 1-coboundary is a 1-cocycle φ such that $\varphi(g) = g.v - v$ for some $v \in V$. The subgroup of $Z^1(S, V)$ of all 1-coboundaries is denoted $B^1(S, V)$.

(iii) The cohomology group for V is denoted by $H^1(S, V)$ and is defined to be the quotient $Z^1(S, V)/B^1(S, V)$.

The cohomology group is a K-vector space and its dimension is what we term the cohomological dimension of V.

Proposition 2.2. *(Neuhaus) Suppose $U_4(2) \cong H \leq G \cong E_8(2)$. Then the conjugacy classes of H must fuse into the conjugacy classes of G in one of the following nine ways. Moreover, the cohomological dimensions of the φ_i are $\varphi_2 = 0$, $\varphi_3 = 2$, $\varphi_4 = 1$, $\varphi_5 = 0$, $\varphi_6 = 0$.*

(i) $6\varphi_1+4\varphi_2+4\varphi_3+3\varphi_4+2\varphi_5+1\varphi_6$ (3A→3C, B$**$ →3C, 3C→3D, 3D→3C, 5A→5B, 9A→9C, B$**$ →9C);

(ii) $8\varphi_1+8\varphi_2+7\varphi_3+4\varphi_4+2\varphi_5+0\varphi_6$ (3A→3C, B$**$ →3C, 3C→3D, 3D→3B, 5A→5B, 9A→9D, B$**$ →9D);

(iii) $12\varphi_1 + 0\varphi_2 + 5\varphi_3 + 2\varphi_4 + 1\varphi_5 + 2\varphi_6$ (3A→3C, B$**$ →3C, 3C→3D, 3D→3B, 5A→5B, 9A→9B, B$**$ →9B);

(iv) $2\varphi_1+5\varphi_2+2\varphi_3+4\varphi_4+2\varphi_5+1\varphi_6$ (3A→3C, B$**$ →3C, 3C→3C, 3D→3D, 5A→5B, 9A→9D, B$**$ →9D);

(v) $10\varphi_1 + 5\varphi_2 + 6\varphi_3 + 4\varphi_4 + 1\varphi_5 + 1\varphi_6$ (3A→3C, B$**$ →3C, 3C→3C, 3D→3B, 5A→5B, 9A→9C, B$**$ →9C);

(vi) $4\varphi_1+2\varphi_2+1\varphi_3+4\varphi_4+1\varphi_5+2\varphi_6$ (3A→3C, B∗∗ →3C, 3C→3B, 3D→3D, 5A→5B, 9A→9C, B∗∗ →9C);

(vii) $6\varphi_1+6\varphi_2+4\varphi_3+5\varphi_4+1\varphi_5+1\varphi_6$ (3A→3C, B∗∗ →3C, 3C→3B, 3D→3C, 5A→5B, 9A→9D, B∗∗ →9D);

(viii) $46\varphi_1 + 10\varphi_2 + 16\varphi_3 + 1\varphi_4 + 0\varphi_5 + 0\varphi_6$ (3A→3C, B∗∗ →3C, 3C→3B, 3D→3A, 5A→5A, 9A→9A, B∗∗ →9A);

(ix) $18\varphi_1 + 17\varphi_2 + 2\varphi_3 + 8\varphi_4 + 0\varphi_5 + 0\varphi_6$ (3A→3C, B∗∗ →3C, 3C→3A, 3D→3B, 5A→5A, 9A→9B, B∗∗ →9B).

Proposition 2.3. *(Neuhaus) Suppose $Sp_6(2) \cong H \leq G \cong E_8(2)$. Then the conjugacy classes of H must fuse into the conjugacy classes of G in one of the following six ways. Moreover, the cohomological dimensions of the φ_i are $\varphi_2 = 1$, $\varphi_3 = 0$, $\varphi_4 = 0$, $\varphi_5 = 1$, $\varphi_6 = 0$, $\varphi_7 = 0$, $\varphi_8 = 0$.*

(i) $6\varphi_1 + 4\varphi_2 + 2\varphi_3 + 3\varphi_4 + 2\varphi_5 + 1\varphi_6 + 0\varphi_7 + 0\varphi_8$ (3A→3D, 3B→3C, 3C→3C, 5A→5B, 7A→7B, 9A→9C, 15A→15G);

(ii) $8\varphi_1 + 8\varphi_2 + 5\varphi_3 + 4\varphi_4 + 2\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8$ (3A→3D, 3B→3C, 3C→3B, 5A→5B, 7A→7B, 9A→9D, 15A→15G);

(iii) $4\varphi_1 + 2\varphi_2 + 0\varphi_3 + 4\varphi_4 + 1\varphi_5 + 2\varphi_6 + 0\varphi_7 + 0\varphi_8$ (3A→3B, 3B→3C, 3C→3D, 5A→5B, 7A→7B, 9A→9C, 15A→15F);

(iv) $6\varphi_1 + 6\varphi_2 + 3\varphi_3 + 5\varphi_4 + 1\varphi_5 + 1\varphi_6 + 0\varphi_7 + 0\varphi_8$ (3A→3B, 3B→3C, 3C→3C, 5A→5B, 7A→7B, 9A→9D, 15A→15F);

(v) $46\varphi_1 + 10\varphi_2 + 16\varphi_3 + 1\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8$ (3A→3B, 3B→3C, 3C→3A, 5A→5A, 7A→7A, 9A→9A, 15A→15A);

(vi) $18\varphi_1 + 17\varphi_2 + 2\varphi_3 + 8\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8$ (3A→3A, 3B→3C, 3C→3B, 5A→5A, 7A→7B, 9A→9B, 15A→15B).

Proposition 2.4. *(Neuhaus) Suppose $\Omega_8^-(2) \cong H \leq G \cong E_8(2)$. Then the conjugacy classes of H must fuse into the conjugacy classes of G in one of the following five ways. Moreover, the cohomological dimensions of the φ_i are $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 2$, $\varphi_5 = 1$, $\varphi_6 = 0$, $\varphi_7 = 0$, $\varphi_8 = 1$, $\varphi_9 = 2$, $\varphi_{10} = 0$, $\varphi_{11} = 0$, $\varphi_{12} = 0$.*

(i) $2\varphi_1+0\varphi_2+0\varphi_3+0\varphi_4+0\varphi_5+0\varphi_6+0\varphi_7+1\varphi_8+0\varphi_9+0\varphi_{10}+0\varphi_{11}+0\varphi_{12}$ (3A→3B, 3B→3D, 3C→3C, 5A→5B, 7A→7B, 9A→9C, 15A→15G, B∗ →15G, 15C→15F, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D;

(ii) $4\varphi_1+2\varphi_2+2\varphi_3+2\varphi_4+1\varphi_5+1\varphi_6+0\varphi_7+0\varphi_8+0\varphi_9+0\varphi_{10}+0\varphi_{11}+0\varphi_{12}$ (3A→3D, 3B→3B, 3C→3C, 5A→5B, 7A→7B, 9A→9D, 15A→15F, B∗ →15F, 15C→15G, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D;

(iii) $8\varphi_1+5\varphi_2+0\varphi_3+4\varphi_4+2\varphi_5+0\varphi_6+0\varphi_7+0\varphi_8+0\varphi_9+0\varphi_{10}+0\varphi_{11}+0\varphi_{12}$ (3A→3A, 3B→3B, 3C→3C, 5A→5A, 7A→7B, 9A→9B, 15A→15E, B∗ →15E, 15C→15B, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D;

(iv) $16\varphi_1 + 1\varphi_2 + 1\varphi_3 + 8\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12}$ (3A→3A, 3B→3B, 3C→3C, 5A→5A, 7A→7B, 9A→9B, 15A→15A, B∗ →15A, 15C→15B, 17A→17A/B, B∗2→17A/B, C∗3→17A/B, D∗6→17A/B;

(v) $30\varphi_1 + 8\varphi_2 + 8\varphi_3 + 1\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12}$ (3A→3B, 3B→3A, 3C→3C, 5A→5A, 7A→7A, 9A→9A, 15A→15B, B∗ →15B, 15C→15A, 17A→17A/B, B∗2→17A/B, C∗3→17A/B, D∗6→17A/B.

Proposition 2.5. *(Neuhaus) Suppose $Sp_4(4) \cong H \leq G \cong E_8(2)$. Then the conjugacy classes of H must fuse into the conjugacy classes of G in one of the following eight ways. Moreover, the cohomological dimensions of the φ_i are $\varphi_2 = 2$, $\varphi_3 = 2$, $\varphi_4 = 0$, $\varphi_5 = 0$, $\varphi_6 = 0$, $\varphi_7 = 0$, $\varphi_8 = 0$, $\varphi_9 = 0$, $\varphi_{10} = 0$.*

(i) $8\varphi_1 + 4\varphi_2 + 2\varphi_3 + 2\varphi_4 + 0\varphi_5 + 1\varphi_6 + 0\varphi_7 + 0\varphi_8 + 1\varphi_9 + 0\varphi_{10}$ (3A→3D, 3B→3B, 5A→5B, B∗ →5B, 5C→5B, D∗ →5B, 5E→5B, 15A→15G, B∗ →15G, 15C→15F, D∗ →15F, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D);

(ii) $8\varphi_1 + 2\varphi_2 + 4\varphi_3 + 0\varphi_4 + 2\varphi_5 + 0\varphi_6 + 1\varphi_7 + 1\varphi_8 + 0\varphi_9 + 0\varphi_{10}$ (3A→3B, 3B→3D, 5A→5B, B∗ →5B, 5C→5B, D∗ →5B, 5E→5B, 15A→15F, B∗ →15F, 15C→15G, D∗ →15G, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D);

(iii) $32\varphi_1 + 8\varphi_2 + 1\varphi_3 + 1\varphi_4 + 8\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10}$ (3A→3B, 3B→3A, 5A→5A, B∗ →5A, 5C→5A, D∗ →5A, 5E→5A, 15A→15A, B∗ →15A, 15C→15B, D∗ →15B, 17A→17A/B, B∗2→17A/B, C∗3→17A/B, D∗6→17A/B);

(iv) $32\varphi_1 + 1\varphi_2 + 8\varphi_3 + 8\varphi_4 + 1\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10}$ (3A→3A, 3B→3B, 5A→5A, B∗ →5A, 5C→5A, D∗ →5A, 5E→5A, 15A→15B, B∗ →15B, 15C→15A, D∗ →15A, 17A→17A/B, B∗2→17A/B, C∗3→17A/B, D∗6→17A/B);

(v) $16\varphi_1 + 4\varphi_2 + 9\varphi_3 + 0\varphi_4 + 4\varphi_5 + 0\varphi_6 + 2\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10}$ (3A→3B, 3B→3D, 5A→5A, B∗ →5A, 5C→5B, D∗ →5B, 5E→5B, 15A→15E, B∗ →15E, 15C→15G, D∗ →15G, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D);

(vi) $16\varphi_1 + 9\varphi_2 + 4\varphi_3 + 4\varphi_4 + 0\varphi_5 + 2\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10}$ (3A→3D, 3B→3B, 5A→5B, B∗ →5B, 5C→5A, D∗ →5A, 5E→5B, 15A→15G, B∗ →15G, 15C→15E, D∗ →15E, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D);

(vii) $8\varphi_1 + 4\varphi_2 + 2\varphi_3 + 0\varphi_4 + 0\varphi_5 + 2\varphi_6 + 0\varphi_7 + 1\varphi_8 + 0\varphi_9 + 0\varphi_{10}$ (3A→3D, 3B→3D, 5A→5B, B∗ →5B, 5C→5B, D∗ →5B, 5E→5B, 15A→15G, B∗ →15G, 15C→15G, D∗ →15G, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D);

(viii) $8\varphi_1 + 2\varphi_2 + 4\varphi_3 + 0\varphi_4 + 0\varphi_5 + 0\varphi_6 + 2\varphi_7 + 0\varphi_8 + 1\varphi_9 + 0\varphi_{10}$ (3A→3D, 3B→3D, 5A→5B, B∗ →5B, 5C→5B, D∗ →5B, 5E→5B, 15A→15G, B∗ →15G, 15C→15G, D∗ →15G, 17A→17C/D, B∗2→17C/D, C∗3→17C/D, D∗6→17C/D).

Proposition 2.6. *(Neuhaus) Suppose $L_4(4) \cong H \leq G \cong E_8(2)$. Then the conjugacy classes of H must fuse into the conjugacy classes of G in one of the following two ways. Moreover, the cohomological dimensions of the φ_i are $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 0$, $\varphi_5 = 0$, $\varphi_6 = 0$, $\varphi_7 = 2$, $\varphi_8 = 0$, $\varphi_9 = 0$, $\varphi_{10} = 0$, $\varphi_{11} = 0$, $\varphi_{12} = 2$, $\varphi_{13} = 2$, $\varphi_{14} = 0$, $\varphi_{15} = 0$, $\varphi_{16} = 0$, $\varphi_{17} = 0$, $\varphi_{18} = 0$, $\varphi_{19} = 0$, $\varphi_{20} = 0$, $\varphi_{21} = 2$, $\varphi_{22} = 0$, $\varphi_{23} = 0$, $\varphi_{24} = 0$.*

(i) $8\varphi_1 + 0\varphi_2 + 0\varphi_3 + 4\varphi_4 + 2\varphi_5 + 2\varphi_6 + 1\varphi_7 + 2\varphi_8 + 1\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14} + 0\varphi_{15} + 0\varphi_{16} + 0\varphi_{17} + 0\varphi_{18} + 0\varphi_{19} + 0\varphi_{20} + 0\varphi_{21} + 0\varphi_{22} + 0\varphi_{23} + 0\varphi_{24}$ (3AB→ 3C, 3C→ 3A, 3D→ 3B, 5AB→ 5A, 5CD→ 5A, 5E→ 5A, 7A→ 7B, 7B→ 7B, 9AB→ 9B, 15AB→ 15B, 15CD→ 15B, 15EF→ 15B, 15G→ 15B, 15H→ 15B, 15IJ→ 15D, 15KL→ 15D, 15MN→ 15A, 17AB→ 17AB, 17CD→ 17AB, 21AB→ 21F, 21CD→ 21F);

(ii) $4\varphi_1 + 2\varphi_2 + 2\varphi_3 + 2\varphi_4 + 1\varphi_5 + 1\varphi_6 + 1\varphi_7 + 1\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 1\varphi_{12} + 1\varphi_{13} + 0\varphi_{14} + 0\varphi_{15} + 0\varphi_{16} + 0\varphi_{17} + 0\varphi_{18} + 0\varphi_{19} + 0\varphi_{20} + 0\varphi_{21} + 0\varphi_{22} + 0\varphi_{23} + 0\varphi_{24}$ (3AB→

3C, 3C→ 3D, 3D→ 3B, 5AB→ 5B, 5CD→ 5A, 5E→ 5B, 7A→ 7B, 7B→ 7B, 9AB→ 9D, 15AB→ 15C, 15CD→ 15C, 15EF→ 15G, 15G→ 15G, 15H→ 15G, 15IJ→ 15D, 15KL→ 15D, 15MN→ 15E, 17AB→ 17CD, 17CD→ 17CD, 21AB→ 21F, 21CD→ 21F).

We conclude this section with a result providing us with a means of establishing whether a given subgroup of G following a specific fusion possibility fixes a non-zero vector in V.

Proposition 2.7. *Suppose $H \leq G$ where H follows the fusion possibility provided by $\lambda_1\varphi_1 + \ldots + \lambda_n\varphi_n$, where φ_i for each $i \in \{1, \ldots, n\}$ is an irreducible Brauer character of H over $GF(2)$ and φ_1 is the trivial character. Let d_i be the cohomological dimension of φ_i. If $\lambda_1 > \lambda_2 d_2 + \ldots + \lambda_n d_n$, then $\dim C_V(H) > 0$.*

Proof. See Proposition 3.6 (i) in the Ph. D. thesis of Litterick [33]. ■

2.4 Useful Results

Here we provide some general group theoretic results which we make extensive use of throughout this thesis, as well as some established facts about $E_8(2)$.

Proposition 2.8. *If G is a group with $S \subset G$, then for all $g \in G$ we have*

$$C_G(S^g) = C_G(S)^g \text{ and } N_G(S^g) = N_G(S)^g.$$

Proposition 2.9. *(Burnside's Theorem) Suppose G is a group with $|G| = p^n q^m$, where p and q are prime and $n, m \in \mathbb{N} \cup \{0\}$. Then G is soluble.*

Proof. See [10]. ■

Proposition 2.10. *Suppose G is a finite group and let $P \in \mathrm{Syl}_p(G)$. Then P is a characteristic subgroup of $N_G(P)$.*

Proof. We will show that P is the unique Sylow p-subgroup of $N_G(P)$. First note that $P \leq N_G(P)$ so it is true that $P \in \mathrm{Syl}_p(N_G(P))$. Now suppose that $P_0 \in \mathrm{Syl}_p(N_G(P))$. By Sylow's Theorems, P and P_0 are conjugate in $N_G(P)$, so there is some element $n \in N_G(P)$ with $P^n = P_0$. But n normalises P, so we have $P = P^n = P_0$. Hence P is the unique Sylow p-subgroup of $N_G(P)$, so any automorphism of $N_G(P)$ must fix P. ■

Much of this thesis deals with the issue of computing centralisers of elements inside large groups. The next result provides us with a method for calculating centralisers of involutions.

Proposition 2.11. *(This is the basis of a method devised by Bray in 2.2 of* [8].*) Suppose G is a finite group and let $t \in G$ be an involution. Now let $h \in G$. Let n be the order of the commutator $[t,h]$. Now set*

$$g = \begin{cases} [t,h]^{\frac{n}{2}}, & \text{if } n \text{ even,} \\ h[t,h]^{\frac{n-1}{2}}, & \text{otherwise.} \end{cases}$$

Then $g \in C_G(t)$. Upon repeating this process we can find multiple elements in $C_G(t)$, which, in some cases, may eventually form a generating set for the whole of $C_G(t)$.

Proposition 2.12. *Let G be a finite group with $K \leq H \leq G$. Suppose also that we have a complete list $K_1, \ldots, K_m \leq G$ such that for all $K_0 \leq G$ with $K_0 \cong K$, K_0 is G-conjugate to K_i for some $i \in \{1, \ldots, m\}$. Then given any $H_0 \leq G$ with $H_0 \cong H$, there exists $g \in G$ such that $K_i \leq H_0^g$, for some $i \in \{1, \ldots, m\}$. In other words, H_0 is G-conjugate to an overgroup of some K_i.*

Proof. If $H_0 \leq G$ and $H_0 \cong H$, then we must have $K_0 \leq H_0$ with $K_0 \cong K$. But K_0 must be G-conjugate to K_i for some $i \in \{1, \ldots, m\}$. Let $g \in G$ be such that $K_0^g = K_i$ and now

$$K_i = K_0^g \leq H_0^g$$

as required. ■

It may have been a straightforward result, but it is essential in our work in $E_8(2)$. Often, we are in a situation where we have already calculated representatives $K_1, \ldots, K_m$ of all the subgroups of G isomorphic to K up to G-conjugacy. We then want to find representatives of all the subgroups of G isomorphic to H up to G-conjugacy. But since $K \leq H$ we may assume, without loss of generality, that every H subgroup we desire exists as an overgroup to one of our K_i.

Another problem we frequently encounter is the following. Once we have found $H \leq G$ where H is simple, we want to know whether $H \leq K \leq G$ where K is isomorphic to a subgroup of $\mathrm{Aut}(H)$. The next result, known as Frattini's Argument, gives us a starting point to look for potential generators of $\mathrm{Aut}(H)$.

Proposition 2.13. *(Frattini's Argument) If G is a finite group with $H \trianglelefteq G$ and $P \in \mathrm{Syl}_p(H)$ for some prime p, then*

$$G = N_G(P)H.$$

Proof. Showing $G \supset N_G(P)H$ is trivial as both H and $N_G(P)$ are subgroups of G. So let $g \in G$. Then $P^g \leq H^g = H$ as H is normal in G. Since $|P| = |P^g|$ we have that $P^g \in \mathrm{Syl}_p(H)$ and so, by Sylow's Theorems, there is some $h \in H$ for which $P^g = P^h$. Hence $P^{gh^{-1}} = P$ which gives $gh^{-1} \in N_G(P)$ and so $g \in N_G(P)h$. Thus, $G = N_G(P)H$ as the choice of g was arbitrary. ■

Now we state a specific case of Frattini's Argument that relates directly to automorphism groups.

Corollary 2.14. *Suppose $H \leq G$ with G isomorphic to a subgroup of $\mathrm{Aut}(H)$ for some finite simple group H. Let $P \in \mathrm{Syl}_p(H)$. Then*

$$G = N_G(P)H.$$

Proof. As G is isomorphic to a subgroups of $\mathrm{Aut}(H)$ we clearly have that for all $g \in G$, $H^g = H$ and so $H \trianglelefteq G$. Now we apply Frattini's Argument and the result follows. ■

We use the corollary as a starting point for how we might construct subgroups of $\mathrm{Aut}(H)$ inside a larger group. The next few results explore this further. Beyond this background chapter, G will denote $E_8(2)$, as this is the group we are trying to construct subgroups of. The next few results can be thought of in that specific context, but the results will be presented in general.

Proposition 2.15. *Let G be a finite group. Let $H \leq G$ be simple with $P \in \mathrm{Syl}_p(H)$. Suppose $g \in G$ such that $\langle H, g\rangle$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$. Then*

(i) $g \in N_{N_G(P)}(N_H(P))$;

(ii) $g \in N_G(H)$.

Proof. (i) By Corollary 2.14 we know we can choose $g \in N_G(P)$. Hence $P^g = P$ and so $N_H(P^g) = N_H(P)$. This gives us $N_H(P)^g = N_H(P)$ by Proposition 2.8. Thus, $g \in N_{N_G(P)}(N_H(P))$.

(ii) Since $g \in \langle H, g\rangle$ where $\langle H, g\rangle$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$, we must have $H^g = H$ and hence $g \in N_G(H)$.

■

To clear up some of the cumbersome notation here, we will commonly set

$$\mathcal{E}_0(H) = N_{N_G(P)}(N_H(P)) \cap N_G(H)$$

as our initial set of potential generators of a subgroup of $\mathrm{Aut}(H)$.

Proposition 2.16. *Suppose $x, y \in \mathcal{E}_0(H)$ such that x and y are conjugate in $\mathcal{E}_0(H)$. Then*

(i) $x \notin H$ if and only if $y \notin H$;

(ii) for all $n \in \mathbb{N}$ we have that $x^n \in H$ if and only if $y^n \in H$;

(iii) $\langle H, x\rangle = \langle H, y\rangle$ are conjugate groups.

Proof. Let $g \in \mathcal{E}_0(H)$ such that $x^g = y$. Recall that, by definition of $\mathcal{E}_0(H)$, we have that $H^g = H$.

(i) Assume $y \notin H$ and $x \in H$. Then

$$y = x^g \in H^g = H,$$

a contradiction, so we must also have that $x \notin H$. A similar argument holds for showing that $x \notin H$ implies $y \notin H$.

(ii) Now let $n \in \mathbb{N}$ and assume that $x^n \in H$. Then

$$y^n = (x^g)^n = (x^n)^g \in H^g = H$$

and similarly to show that $y^n \in H$ implies $x^n \in H$.

(iii) Here, we simply observe that $\langle H, x\rangle^g = \langle H^g, x^g\rangle = \langle H, y\rangle$.

■

Proposition 2.17. *Suppose $H \trianglelefteq H_0 \leq G$ with $H_0 \cong H : q$ for some $q \in \mathbb{N}$. If $g \in G$ such that $\langle H, g\rangle = H_0$, then $g^q \in H$.*

Proof. This is clear from the fact that the qth power of any element in the cyclic group q is the identity. ■

We complete our discussion on automorphism extensions of groups with the following result, which often eliminates many cases without the need for computation.

Proposition 2.18. *Suppose G is a finite group acting on a vector space V over $GF(2)$. Now suppose we have $H \leq G$ such that $\dim C_V(H) > 0$. Then if $H \leq H_0 \leq G$ where $H_0 \cong H : [2^k]$ for some $k \in \mathbb{N}$, then $\dim C_V(H_0) > 0$.*

Proof. Suppose $\dim C_V(H) = n > 0$. Then there are $2^n - 1$ non-trivial vectors in $C_V(H)$ which split into orbits under the action of H_0. Each orbit has length dividing $|H_0/H| = 2^k$, so each orbit either has length 1 or has even length. However, the sum of all these orbit lengths is $2^n - 1$, which is odd, so there must be at least one orbit which has length 1. This orbit contains a non-zero vector which is fixed by H_0 and so belongs to $C_V(H_0)$. ■

We will now shift our focus and explore some useful results about $E_8(2)$ specifically. For the rest of this chapter, let $G \cong E_8(2)$. The adjoint representation of $E_8(2)$ is the representation of smallest degree. This has degree 248, and so, usually, G can be thought of as a subgroup of $GL_{248}(2)$. The first result states some useful facts about the four conjugacy classes of involutions in G.

Proposition 2.19. *Let V be the 248-dimensional $GF(2)$ G-module. Suppose t is an involution of G, and set $U = O_2(C_G(t))$, the maximal normal unipotent subgroup of $C_G(t)$.*

(i) If $t \in 2A$, then $\dim(C_V(t)) = 190$ and $C_G(t) = UL$ with $U \sim 2^{1+56}$ and $L \cong E_7(2)$.

(ii) If $t \in 2B$, then $\dim(C_V(t)) = 156$ and $C_G(t) = UL$ where $U \sim [2^{78}]$ and $L \cong Sp_{12}(2)$.

(iii) If $t \in 2C$, then $\dim(C_V(t)) = 138$ and $C_G(t) = UL$ with $U \sim [2^{81}]$ and $L \cong \mathrm{Sym}(3) \times F_4(2)$.

(iv) If $t \in 2D$, then $\dim(C_V(t)) = 128$ and $C_G(t) = UL$ with $U \sim [2^{84}]$ and $L \cong Sp_8(2)$.

Proof. See (17.15) of Aschbacher, Seitz [3] for the shape of $C_G(t)$. In Magma we use the commands `Dimension` and `Eigenspace` to determine $\dim C_V(t)$. ■

Theorem 2.20. (Aubad, Ballantyne, McGaw, Neuhaus, Phillips, Rowley, Ward)*: Let $G \cong E_8(2)$ and let V be the 248-dimensional $GF(2)$ G-module. The semisimple conjugacy classes of G, their centraliser structures, dimensions of their fixed spaces of V, and power maps are displayed in Table 2.1.*

Proof. See [4]. ■

Conjugacy Class	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$1A$	$E_8(2)$	$\lvert E_8(2)\rvert$	248	-
$3A$	$3 \times E_7(2)$	$2^{63} \cdot 3^{12} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 73 \cdot 127$	134	-
$3B$	$3 \times \Omega_{14}^-(2)$	$2^{42} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 43$	92	-
$3C$	$3.({}^2E_6(2) \times U_3(2)).3$	$2^{39} \cdot 3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	86	-
$3D$	$3 \times U_9(2)$	$2^{36} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 43$	80	-
$5A$	$5 \times \Omega_{12}^-(2)$	$2^{30} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	68	-
$5B$	$SU_5(4)$	$2^{20} \cdot 3^2 \cdot 5^5 \cdot 13 \cdot 17 \cdot 41$	48	-
$7A$	$7 \times E_6(2)$	$2^{36} \cdot 3^6 \cdot 5^2 \cdot 7^4 \cdot 13 \cdot 17 \cdot 31 \cdot 73$	80	-
$7B$	$7 \times L_3(2) \times {}^3D_4(2)$	$2^{15} \cdot 3^5 \cdot 7^4 \cdot 13$	38	-
$9A$	$9 \times \Omega_{10}^-(2)$	$2^{20} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	48	3C
$9B$	$9 \times \mathrm{Sym}(3) \times {}^3D_4(2)$	$2^{13} \cdot 3^7 \cdot 7^2 \cdot 13$	34	3C
$9C$	$9 \times \mathrm{Sym}(3) \times U_5(2)$	$2^{11} \cdot 3^8 \cdot 5 \cdot 11$	30	3C
$9D$	$9 \times \mathrm{Sym}(3) \times U_3(8)$	$2^{10} \cdot 3^7 \cdot 7 \cdot 19$	28	3C
$11A$	$11 \times U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11^2$	28	-
$13A$	$13 \times {}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13^2$	32	-
$13B$	$13 \times U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13^2$	20	-
$15A$	$15 \times \Omega_{10}^+(2)$	$2^{20} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 17 \cdot 31$	48	3B,5A
$15B$	$5 \times 3^2 : 2 \times \Omega_8^-(2)$	$2^{13} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 17$	34	3A,5A
$15C$	$15 \times U_5(2)$	$2^{10} \cdot 3^6 \cdot 5^2 \cdot 11$	28	3D,5A
$15D$	$5 \times GU_3(2) \times L_4(2)$	$2^9 \cdot 3^6 \cdot 5^2 \cdot 7$	26	3C,5A
$15E$	$15 \times L_2(4) \times U_4(2)$	$2^8 \cdot 3^6 \cdot 5^3$	24	3B,5A
$15F$	$15 \times U_3(4)$	$2^6 \cdot 3^2 \cdot 5^3 \cdot 13$	20	3B,5B
$15G$	$15 \times L_2(16)$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 17$	16	3D,5B
$17AB$	$17 \times \Omega_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17^2$	32	-
$17CD$	$17 \times L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17^2$	16	-
$19A$	$19 \times 3^{\cdot}PGU_3(2)$	$2^3 \cdot 3^4 \cdot 19$	14	-

Table 2.1: Conjugacy classes of semisimple elements of $E_8(2)$

Conjugacy Class	$C_G(x)$	$\|C_G(x)\|$	$\dim(C_V(x))$	Powers
$21A$	$21 \times L_6(2)$	$2^{15} \cdot 3^5 \cdot 5 \cdot 7^3 \cdot 31$	38	3A,7A
$21B$	$21 \times {}^3D_4(2)$	$2^{12} \cdot 3^5 \cdot 7^3 \cdot 13$	32	3A,7B
$21C$	$21 \times \Omega_8^-(2)$	$2^{12} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 17$	32	3B,7A
$21D$	$7 \times 3.(3^2 : Q_8 \times L_3(4)) : 3$	$2^9 \cdot 3^6 \cdot 5 \cdot 7^2$	26	3C,7A
$21E$	$21 \times L_3(2) \times L_2(8)$	$2^6 \cdot 3^4 \cdot 7^3$	20	3A,7B
$21F$	$7 \times L_3(2) \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^6 \cdot 3^5 \cdot 7^2$	20	3C,7B
$21G$	$21 \times 3 \times L_2(8)$	$2^3 \cdot 3^4 \cdot 7^2$	14	3B,7B
$21H$	$21 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^5 \cdot 7$	14	3D,7B
$31ABC$	$31 \times L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31^2$	28	-
$31D$	31^2	31^2	8	-
$33AB$	$33 \times U_4(2)$	$2^6 \cdot 3^5 \cdot 5 \cdot 11$	20	3D,11A
$33CD$	$11 \times \mathrm{Sym}(3) \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^4 \cdot 3^5 \cdot 11$	16	3C,11A
$33E$	$33 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^5 \cdot 11$	14	3A,11A
$33F$	$33 \times 3 \times \mathrm{Sym}(3)^2$	$2^2 \cdot 3^4 \cdot 11$	12	3B,11A
$35A$	$35 \times U_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	5A,7A
$39A$	$13 \times \mathrm{Sym}(3) \times L_2(8)$	$2^4 \cdot 3^4 \cdot 7 \cdot 13$	14	3A,13A
$39B$	$13 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^4 \cdot 13$	14	3C,13A
$39C$	195	$3 \cdot 5 \cdot 13$	8	3B,13B
$41AB$	205	$5 \cdot 41$	8	-
$43ABC$	$129 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 43$	10	-
$45A$	$45 \times L_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	3C,9A,5A,15D
$45B$	$45 \times 3 \times \mathrm{Alt}(5)$	$2^2 \cdot 3^4 \cdot 5^2$	12	3C,9A,5A,15D
$45C$	$45 \times 3 \times \mathrm{Sym}(3)$	$2 \cdot 3^4 \cdot 5$	10	3C,9C,5A,15D
$51AB$	$51 \times L_4(2)$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$	20	3B,17AB
$51CD$	$51 \times \mathrm{Sym}(3) \times \mathrm{Alt}(5)$	$2^3 \cdot 3^3 \cdot 5 \cdot 17$	14	3A,17AB
$51EF$	$17 \times GU_3(2)$	$2^3 \cdot 3^4 \cdot 17$	14	3C,17AB
$51GH$	255	$3 \cdot 5 \cdot 17$	8	3D,17CD

Table 2.2: Conjugacy classes of semisimple elements of $E_8(2)$

Conjugacy Class	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$55A$	165	$3 \cdot 5 \cdot 11$	8	5A,11A
$57AB$	$19 \times 3^{\cdot} PGU_3(2)$	$2^3 \cdot 3^4 \cdot 19$	14	3C,19A
$57C$	$3 \times 19 \times 9$	$3^3 \cdot 19$	8	3A,19A
$57DE$	57×3	$3^2 \cdot 19$	8	3D,19A
$63ABC$	$63 \times \mathrm{Sym}(3) \times L_3(2)$	$2^4 \cdot 3^4 \cdot 7^2$	16	3C,9B,7B,21F
$63D$	$63 \times \mathrm{Alt}(5)$	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	12	3C,9A,7A,21D
$63E$	$63 \times 7 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 7^2$	10	3C,9B,7A,21D
$63FGH$	$63 \times 3 \times \mathrm{Sym}(3)$	$2 \cdot 3^4 \cdot 7$	10	3C,9D,7B,21F
$65ABCD$	$65 \times \mathrm{Alt}(5)$	$2^2 \cdot 3 \cdot 5^2 \cdot 13$	12	5B,13B
$65EF$	13×5^2	$5^2 \cdot 13$	8	5A,13B
$73ABCD$	$73 \times L_3(2)$	$2^3 \cdot 3 \cdot 7 \cdot 73$	14	-
$85AB$	$85 \times \mathrm{Sym}(3)^2$	$2^2 \cdot 3^2 \cdot 5 \cdot 17$	12	5A,17AB
$85CDEF$	255	$3 \cdot 5 \cdot 17$	8	5B,17CD
$91ABC$	$91 \times L_3(2)$	$2^3 \cdot 3 \cdot 7^2 \cdot 13$	14	7B,13A
$91D$	91×7	$7^2 \cdot 13$	8	7A,13A
$93ABC$	$93 \times L_3(2)$	$2^3 \cdot 3^2 \cdot 7 \cdot 31$	14	3A,31ABC
$93DEF$	$93 \times \mathrm{Alt}(5)$	$2^2 \cdot 3^2 \cdot 5 \cdot 31$	12	3B,31ABC
$99AB$	$99 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 11$	10	3C,9C,11A,33CD
$99CD$	99×3	$3^3 \cdot 11$	8	3C,9A,11A,33CD
$105AB$	$35 \times GU_3(2)$	$2^3 \cdot 3^4 \cdot 5 \cdot 7$	14	3C,5A,7A,15D,21D,35A
$105C$	$105 \times \mathrm{Sym}(3)^2$	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	12	3B,5A,7A,15A,21C,35A
$105D$	$105 \times 3 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 5 \cdot 7$	10	3A,5A,7A,15B,21A,35A
$117ABC$	$117 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 13$	10	3C,9B,13A,39B
$119AB$	357	$3 \cdot 7 \cdot 17$	8	7A,17AB
$127ABCDEFGHI$	$127 \times \mathrm{Sym}(3)$	$2 \cdot 3 \cdot 127$	10	-

Table 2.3: Conjugacy classes of semisimple elements of $E_8(2)$

Conjugacy Class	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_V(x))$	Powers
$129ABCDEF$	$129 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 43$	10	3D,43ABC
$129GHI$	129×3	$3^2 \cdot 43$	8	3A,43ABC
$129JKLMNO$	129×3	$3^2 \cdot 43$	8	3B, 43ABC
$151ABCDE$	151	151	8	-
$153AB$	153	$3^2 \cdot 17$	8	3C,9A,17AB,51EF
$155ABC$	465	$3 \cdot 5 \cdot 31$	8	5A,31ABC
$165AB$	165	$3 \cdot 5 \cdot 11$	8	3D,5A,11A,15C,33AB,55A
$171ABCDEF$	$171 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 19$	10	3C,9D,19A,57AB
$195ABCD$	195	$3 \cdot 5 \cdot 13$	8	3B,5B,13B,15F,39C,65ABCD
$205ABCDEFGH$	205	$5 \cdot 41$	8	5B,41AB
$217ABCDEF$	$217 \times \mathrm{Sym}(3)$	$2 \cdot 3 \cdot 7 \cdot 31$	10	7A,31ABC
$219ABCD$	219	$3 \cdot 73$	8	3A,73ABCD
$241ABCDEFGHIJ$	241	241	8	-
$255ABCD$	$255 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 5 \cdot 17$	10	3A,5A,15B,17AB,51CD,85AB
$255EF$	255×3	$3^2 \cdot 5 \cdot 17$	8	3B,5A,15A,17AB,51AB,85AB
$255GHIJKLMN$	255	$3 \cdot 5 \cdot 17$	8	3D,5B,15G,17CD,51GH,85CDEF
$273ABC$	273	$3 \cdot 7 \cdot 13$	8	3A,7B,13A,21B,39A,91ABC
$315AB$	315	$3^2 \cdot 5 \cdot 7$	8	3C,5A,7A,9A,15D,21D,35A,45A,63D,105AB
$331ABCDEFGHIJK$	331	331	8	-
$357ABCD$	357	$3 \cdot 7 \cdot 17$	8	3B,7A,17AB,21C,51AB,119AB
$381ABCDEFGHI$	381	$3 \cdot 127$	8	3A,127ABCDEFGHI
$465ABCDEF$	465	$3 \cdot 5 \cdot 31$	8	3B,5A,15A,31ABC,93DEF,155ABC
$511ABCDEFGH$	511	$7 \cdot 73$	8	7A,73ABCD
$651ABCDEF$	651	$3 \cdot 7 \cdot 31$	8	3A,7A,21A,31ABC,93ABC,217ABCDEF

Table 2.4: Conjugacy classes of semisimple elements of $E_8(2)$

Proposition 2.21. *Let $g \in G$ have order 2^k for some $k \in \mathbb{N}$. Then g is unipotent. Moreover, if $U \leq G$ such that $|U| = 2^n$ for some $n \in \mathbb{N}$, then U is a unipotent group.*

Proposition 2.22. *Suppose T is a maximal torus of G. Then:*

(i) T is cyclic of order 85, 105, 127, 195, 205, 217, 219, 241, 255, 273, 315, 331, 357, 381, 465, 511, or 651; or

(ii) T is isomorphic to one of: 3×51, 3×63, 3×93, 3×105, 5×35, 5×65, 7×91, 13^2, 15^2, 17^2, 21^2, 31^2, 3×5^2, $3^2 \times 21$, $3^2 \times 15$, $5^2 \times 15$, $7^2 \times 21$, 3^4, 5^4, *or* 7^4.

Moreover, if $H \leq G$ where $F^*(H)$ is isomorphic to $U_4(2)$, $Sp_6(2)$, $\Omega_8^-(2)$, or $Sp_4(4)$, then H does not contain a maximal torus of G.

Proof. For the isomorphism types of maximal tori in G, see Table IV of [16]. The result regarding subgroups H of G with $F^*(H)$ isomorphic to $U_4(2)$, $Sp_6(2)$, $\Omega_8^-(2)$, or $Sp_4(4)$ can easily be verified by Lagrange's theorem and Magma. ■

This next result is key to showing that many subgroups we examine in this thesis are not maximal. It is an adaptation of Lemma 1.3 found in [41] by Sietz.

Proposition 2.23. *If $H \leq G$ and $\dim C_V(H) > 0$, and H is neither a maximal parabolic subgroup of G nor contains a maximal torus of G, then H is not maximal in G.*

Proof. See [41]. ■

As a result of Proposition 2.22, we can apply Proposition 2.23 to subgroups $H \leq G$ with $F^*(H)$ isomorphic to $U_4(2)$, $Sp_6(2)$, $\Omega_8^-(2)$, or $Sp_4(4)$. Since these groups are not maximal parabolic groups of G, if they fix a non-zero vector of V, then they are not maximal. Note that, as stated in [29], the maximal parabolic subgroups of G correspond to those obtained by removing a node of the E_8 Dynkin diagram and therefore do not correspond to any of the above subgroups.

Chapter 3

Methodology

All the results in this thesis are proved computationally using the computer algebra package Magma. Similar methods and variations of the same algorithms are employed in order to prove each of the main results. In this chapter, we will provide a general breakdown of these methods and algorithms.

3.1 Computing in Magma

This section will outline some basic techniques on how to use Magma as well as some common tactics we employ when the computational power of Magma falls short. We begin with a discussion on how groups can be presented in Magma. Cayley's theorem tells us that every finite group G is isomorphic to a subgroup of $\mathrm{Sym}(n)$, for some $n \in \mathbb{N}$. A consequence of this is that G can be faithfully representated as a subgroup of $GL(V)$, the general linear group over some vector space V. In Magma, any finite group can be presented as a group of permutations or as a group of matrices. Both have their advantages, and often it is beneficial to switch between them.

The faithful matrix representation of $E_8(2)$ with smallest degree has degree 248. Magma can handle some operations on 248×248 matrices without issue. For example, most linear algebra operations, such as finding the eigenspace or determinant of a given matrix, are carried out quickly. But there are two major drawbacks with working with such large matrices. The first is that group theoretic operations are often very slow. These are operations like finding the order of a group generated by a selection of such matrices. The second is that such matrices take a lot of memory to store – a single element of $GL_{248}(2)$ takes up approximately 181KB of memory, so storing lots of such matrices quickly becomes impractical or, indeed, impossible. We combat both of these issues by shifting group presentations, but it should be noted that this is not always possible.

Before we proceed, for the rest of this chapter we will discuss various Magma functions. These will be written in the standard `verbatim` typeface. It should always be assumed that, if we are discussing an object represented symbolically by a single letter, that same letter will be used to represent that object in Magma. For instance, if we are discussing a group G and a matrix x, then these will be represented in Magma using `G` and `x` respectively. Furthermore, if subscripts are used, then the numbers will be appended to the Magma objects – x_1 and x_2 will be called `x1` and `x2`, for example. If the context requires objects with more complex names, they will always be explained as and when they are introduced.

Given a finite group G, the command

```
p,P:=PermutationRepresentation(G);
```

returns P, a subgroup of $\mathrm{Sym}(n)$ for some $n \in \mathbb{N}$ which is isomorphic to G. It also gives an isomorphism $p : G \to P$. Several group theoretic operations work much faster in the permutation group setting. Note that some randomness is associated with the algorithm Magma uses to determine P. As a result, it is possible to obtain a permutation representation of high degree. We often wish to find a permutation group isomorphic to G of smaller degree, where computations may run even faster. Here, we might use

```
R,r:=DegreeReduction(P);
```

to find $R \cong P$ where the degree of R is at most that of P, and an isomorphism $r : P \to R$.

A common tactic is to run the necessary computations in P, then map the results back to the matrix setting, G. For example, let's say we would like to find all the subgroups of G. This is often time consuming (or impossible) to run in the matrix setting, so we run the following.

```
p,P:=PermutationRepresentation(G);
Subs:=Subgroups(P);
```

```
S:={Subs[i]‘subgroup@@p : i in [1..#subs]};
```

This finds S, a set of all subgroups of G, even though the `Subgroups` command was run on P. Here is the same example after finding a permutation representation of G of smaller degree.

```
p,P:=PermutationRepresentation(G);
R,r:=DegreeReduction(P);
Subs:=Subgroups(R);
S:={Subs[i]‘subgroup@@r@@p : i in [1..#subs]};
```

This is such a common tactic for us that it would be burdensome to have to detail the same set of procedures whenever it is employed. Instead, we will use the phrase "turn G into a permutation group" whenever we are finding a permutation group isomorphic to G. We realise that this is not strictly true, but, for our purposes, this language is simply shorthand. Through an abuse of notation, we will also commonly refer to the permutation group isomorphic to G as G itself. With this language, the above process of finding the subgroups of G would be described as such: We turn G into a permutation group and employ the command `Subgroups`, then pull the subgroups back into the matrix setting.

Permutation groups also alleviate the second issue – the storage problem – described earlier. If we want to save a multitude of matrices, we can turn the group to which they belong into a permutation group and save the elements as permutations. So, if G is a matrix group and we want to save some element $g \in G$ we could carry out

```
p,P:=PermutationRepresentation(G);
gp:=p(g);
PrintFileMagma("gp",gp);
```

to save g as a permutation. This causes a new problem, however, for when we open a new session and wish to load the permutation g, there is no guarantee that running `PermutationRepresentation` will yield the same isomorphism or even a permutation group of the same degree. Thus, we have no way of recovering the original matrix g. To solve this, we must always save the original isomorphism and re-load it in a new session. We save maps by running

```
Gg:=SetToSequence(Generators(G));
Pg:=[p(g) : g in Gg];
PrintFileMagma("Gg",Gg);
PrintFileMagma("Pg",Pg);
```

which saves the ordered generating set $\{g_1, \ldots, g_k\}$ of G and the ordered generating set $\{p(g_1), \ldots, p(g_k)\}$. Since any homomorphism is defined by its action on the group generators, we can now setup the same map in a new session by carrying out

```
Gg:=eval Read("Gg");
Pg:=eval Read("Pg");
P:=Universe(Pg);
p:=Homomorphism(G,P,Gg,Pg);
gp:=eval Read("gp");
g:=gp@@p;
```

and now g has been recovered as the original matrix.

Another form of group presentation we frequently take advantage of are power-conjugate presentations. Let G be a finite soluble group. A power-conjugate presentation of G is a presentation of the form

$$\langle a_1, \ldots, a_n : a_j^{p_j} = w_{jj},\ 1 \leq j \leq n,\ a_j^{a_i} = w_{ij},\ 1 \leq i < j \leq n \rangle$$

where

(i) p_j is the least prime such that $a_j^{p_j} \in \langle a_{j+1}, \ldots, a_n \rangle$ for $j < n$ and $a_j^{p_j} = 1$ for $j = n$;

(ii) w_{ij} is a word in the generators $a_{i+1}, \ldots, a_n$.

We refer to groups presented this way as pc-presentations of pc-groups. Like permutation groups, pc-groups lend themselves to certain computations which are slow in a matrix setting. The phrase "turn G into a pc-group" will be used in the same context as with permutation groups.

Note that only soluble groups have pc-presentations. If G is a matrix group, the command

```
S,P,p:=LMGSolubleRadical(G);
```

outputs S, the soluble radical of G, $P \cong S$ where P is a pc-group, and an isomorphism $p : S \to P$. Note that the soluble radical of G is the largest soluble normal subgroup of G. If G itself is soluble, then its soluble radical is G. In this case, $S = G$ and we have an isomorphism $p : G \to P$, effectively turning G into a pc-group.

We remark here that `LMGSolubleRadical` is an LMG command, which stands for *large matrix group*. These commands can only be called on matrix groups and use algorithms which are typically faster than their non-LMG counterpart when called on a matrix group of sufficiently large degree. For example, calling `LMGOrder` often works faster than simply calling `Order`. Since we are working with $E_8(2)$ – in particular, its 248-degree matrix representation – we make extensive use of these LMG commands throughout the thesis. As stated in the Magma handbook, there is a small probability of failure or returning incorrect results. We use the command `LMGInitialise` to combat this.

3.2 Working with $E_8(2)$

We now examine some of the specific methods we use when working with $E_8(2)$, and some algorithms used throughout this thesis. On top of the fact that the smallest matrices forming the elements of $E_8(2)$ are 248×248, there is another, more glaring computational hurdle to overcome when working with $E_8(2)$. Having over 10^{74} elements, $E_8(2)$ is a truly gargantuan group with more elements than there are atoms in the Milky Way galaxy. Let $G \cong E_8(2)$ and $H \leq G$. A frequent challenge for us is to find $C_G(H)$. Simply running through the elements of G and collecting the ones centralising H is not a sensible idea. Given the size of G, it is likely that such a procedure would continue until long after the death of the sun. Fortunately, there are ways in which we can drastically reduce the group through which we must sift for centralising elements. It should be noted that, throughout, G will denote $E_8(2)$ as a subgroup of $GL_{248}(2)$, and that many of these methods could be generalised to working with other matrix groups.

To give some context, the next few results should be read with the following problem in mind. Given $H \leq G \cong E_8(2)$, construct $\mathcal{I}(C_G(H))$. (Here, given a group or a set of group elements S, $\mathcal{I}(S)$ denotes the set of involutions in S.) We will explore the process of whittling down where the centralising involutions can be found in G. Many of these results will be drawn upon throughout the following chapters.

A frequent place to start is by choosing an element $h \in H$ and constructing $C_G(h)$. If h is an involution, we can use the intrinsic MAGMA command

```
CentraliserOfInvolution(G,h)
```

which implements the Bray method (see 2.11) and outputs $C_G(h)$. (Note that, contrary to most intrinsic MAGMA commands, this only works with the U.K. spelling "centraliser" and not the U.S. spelling "centralizer".) If h is semisimple (of odd order), we can often use the `FindCent` procedure (see B.3) developed by Ballantyne and Rowley to find $C_G(h)$. For computational reasons, it is often beneficial to choose $h \in Z(H)$, as the next result shows.

Proposition 3.1. *Let $z \in Z(H)$. Then $H \leq C_G(z)$ and $C_G(H) \leq C_G(z)$.*

Proof. Let $h \in H$. Then as z is central in H, we must have $zh = hz$ and so $h \in C_G(z)$. Now, if $g \in C_G(H)$, then g commutes with everything in H, including z, so $g \in C_G(z)$. ■

While these facts are fairly clear, it gives a crucial starting point in cutting the group we must sieve down from G to $C_G(z)$. Moreover, the fact that $H \leq C_G(z)$ enables us to utilise further computational techniques, as we will employ in the next result and its following procedure.

Proposition 3.2. *Let $C = C_G(z)$ where $z \in Z(H)$. Let $\overline{C} = C/N$ where $N \trianglelefteq C$. Then $C_G(H)$ is contained in the full inverse image of $C_{\overline{C}}(\overline{H})$. (That is, B such that $N \leq B \leq C$ and for all $b \in B$, $\overline{b} \in C_{\overline{C}}(\overline{H})$.)*

Proof. Let $g \in C_G(H)$. We must show that $\overline{g} \in C_{\overline{C}}(\overline{H})$. Since $C_G(H) \leq C$, we know $g \in C$. And now for all $h \in H$ we have that $gh = hg$ which implies $\overline{gh} = \overline{hg}$ and so $\overline{g}\overline{h} = \overline{h}\overline{g}$. Thus, $\overline{g} \in C_{\overline{C}}(\overline{H})$, as required. ■

It is worth noting that the inverse image of $C_{\overline{C}}(\overline{H})$ and $C_C(H)$ are not necessarily the same group – the former contains, but may contain much more than, the latter. Regardless, we have again trimmed down the group through which we must sieve. We will commonly refer to the inverse image of $C_{\overline{C}}(\overline{H})$ as $\mathcal{C}_1(H)$. We will now discuss a procedure we can use in order to construct $\mathcal{C}_1(H)$ in Magma, in the case where $N = O_2(C)$. Further to our earlier discussion on how we name objects in Magma, we will name any object expressed as a letter with a bar with that letter and append the letter "b". For example, to refer to the object $\overline{C}$ in Magma we will use the name `Cb`.

Procedure 3.3. Starting with C, we wish to construct $C/O_2(C)$. The command

```
Cb,p,W:=LMGRadicalQuotient(C);
```

outputs $\overline{C} \cong C/\mathrm{Rad}(C)$, where $\mathrm{Rad}(C)$ is the soluble radical of C. (In all cases in this thesis where this procedure is used, we will have $\mathrm{Rad}(C) = O_2(C)$.) Note that $\overline{C}$ is given as a permutation group. We also have an epimorphism $p : C \to \overline{C}$ with kernel W, which in our case is $O_2(C)$.

Since $H \leq C = C_G(z)$ by Proposition 3.1, H is in the domain of p so we can apply p to H and obtain $\overline{H}$. We can then use the command `Centraliser` to compute $C_{\overline{C}}(\overline{H})$ and find the inverse image by applying the inverse of p. If we use the name `C1H` to refer to $\mathcal{C}_1(H)$, this whole process is carried out using

```
C1H:=Centraliser(Cb,p(H)) @@ p;
```

and hence we have $\mathcal{C}_1(H)$.

Now we seek to trim $\mathcal{C}_1(H)$ to an even smaller group which still contains $\mathcal{I}(C_G(H))$. Before we get to the next procedure, however, we require two lemmas.

Lemma 3.4. *Let $S \in \mathrm{Syl}_2(\mathcal{C}_1(H))$ and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a right transversal for S in $\mathcal{C}_1(H)$. Then given any involution $t \in \mathcal{C}_1(H)$, there exists $\gamma_i \in \Gamma$ such that $t \in S^{\gamma_i}$.*

Proof. By Sylow's Theorems, we have that t belongs to some Sylow 2-subgroup of $\mathcal{C}_1(H)$, and that this Sylow 2-subgroup is conjugate to S. Say $t \in S^g$ for some $g \in \mathcal{C}_1(H)$. Now, since $\mathcal{C}_1(H) = S\gamma_1 \sqcup \ldots \sqcup S\gamma_n$, we have that $g \in S\gamma_i$ for some $\gamma_i \in \Gamma$. Then $g = s\gamma_i$ for some $s \in S$. And now

$$t \in S^g = S^{s\gamma_i} = S^{\gamma_i}$$

as required. ∎

Recall that V is the natural 248-dimensional $GF(2)$ G-module, and that if U is a subspace of V and A is some subgroup of G, then the stabiliser of U in A is defined to be

$$\mathrm{Stab}_A(U) = \{g \in A : u^g \in U, \text{ for all } u \in U\}.$$

Recall, also, that we are looking for involutions centralising H. We know they must all be contained in $\mathcal{C}_1(H)$, so we are looking for $C_{\mathcal{C}_1(H)}(H)$ or something containing this group.

Lemma 3.5. *Let $F = C_V(H)$. Then $C_{\mathcal{C}_1(H)}(H) \leq \mathrm{Stab}_{\mathcal{C}_1(H)}(F)$.*

Proof. Let $g \in C_{\mathcal{C}_1(H)}(H)$. We must show that given any $v \in F$, we have $v^g \in F$. So, let $v \in F$ and $h \in H$. Then, as v is in the fixed space of H, we have $v^h = v$. And, as g centralises H, we have $gh = hg$. And so

$$v^g = (v^h)^g = (v^g)^h$$

and hence $v^g \in F$.

■

Now we get to the crux of the last two lemmas.

Proposition 3.6. *Using the notation established in the last two results, we have*

$$\mathcal{I}(C_G(H)) \subseteq \langle \mathrm{Stab}_{S^{\gamma_i}}(F) : \gamma_i \in \Gamma \rangle.$$

Proof. Let $t \in \mathcal{I}(C_G(H))$. Then by Proposition 3.2 we know $t \in \mathcal{C}_1(H)$. By Lemma 3.4 we know that for some $\gamma_i \in \Gamma$, we have $t \in S^{\gamma_i}$. And now, by Lemma 3.5 we have $C_{S^{\gamma_i}}(H) \leq \mathrm{Stab}_{S^{\gamma_i}}(F)$. Hence, every element in $\mathcal{I}(C_G(H))$ is contained in $\mathrm{Stab}_{S^{\gamma_i}}(F)$ for some $i \in \{1, \ldots, n\}$ and so the result follows. ■

We will set $\mathcal{C}_2(H) = \langle \mathrm{Stab}_{S^{\gamma_i}}(F) : \gamma_i \in \Gamma \rangle$, as this is the next subgroup in our ever-decreasing chain of subgroups containing $\mathcal{I}(C_G(H))$. We will now look at calculating $\mathcal{C}_2(H)$ in practice. Note that there are several different procedures here depending on the size of $\mathcal{C}_1(H)$, and which one we use is situational. For each procedure, we will give the approximate circumstances under which it ought to be used, however, there isn't a complete set of hard and fast rules to follow, as certain situations call for slight variations of the presented algorithms. In later chapters when we actually implement these procedures, we will always state exactly which procedure we will use.

For the following procedures, let $|\mathcal{C}_1(H)| = 2^m k$ where $m, k \in \mathbb{N}$ and such that $2^{m+1} \nmid |\mathcal{C}_1(H)|$. In practice, m will typically be large (around 90) and the procedure used will depend on the size of k. Before we begin, we must first discuss how to compute fixed space stabilisers in Magma. First, note that to calculate $F = C_V(H)$ we can carry out

```
F:=Fix(GModule(H));
```

Now, given a 2-group S, note that S is a unipotent group by Proposition 2.21, so we may use

```
UnipotentStabiliser(S,F);
```

to calculate the stabiliser in S of F. With that out of the way, let us list the procedures we can use to generate $\mathcal{C}_2(H)$.

Procedure 3.7. As stated previously, we have $|\mathcal{C}_1(H)| = 2^m k$ with $S \in \mathrm{Syl}_2(\mathcal{C}_1(H))$ and Γ a right transversal for S in $\mathcal{C}_1(H)$. The procedure used to find $\mathcal{C}_2(H)$ depends on k.

(i) $k = 1$: This is the simplest case, for here, $\mathcal{C}_1(H)$ is a 2-group, so $S = \mathcal{C}_1(H)$ and $\Gamma = \{1\}$, so we can simply calculate $\mathcal{C}_2(H) = \mathrm{Stab}_{\mathcal{C}_1(H)}(F)$ using the `UnipotentStabiliser` command.

(ii) $k > 1$ and k is "small": The word "small" here is situation-dependent, but we will clarify what it means shortly. Here, $\mathcal{C}_1(H)$ is not a 2-group so we obtain S using

```
S:=LMGSylow(C1H,2);
```

where here, recall that `C1H` is a stand-in for $\mathcal{C}_1(H)$. Now we find Γ using

```
Gamma:=Transversal(C1H,S);
```

which outputs a right transversal of S in $\mathcal{C}_1(H)$. We say that k is "small" when Magma can actually execute the `Transversal` command. This is dependent on various factors such as the available memory and processing power of the machine used, and the order of $\mathcal{C}_1(H)$. Now we run through each element $\gamma \in \Gamma$

and calculate $\mathrm{Stab}_{S^\gamma}(F)$, building the group generated by all of them. Here is a sample code we can use to execute this procedure, where the object `Q` denotes $GL_{248}(2)$.

```
Gamma:=Transversal(C1H,S);
U:=sub<Q|Id(Q)>;
for g in Gamma do;
 Sg:=S^g;
 UU:=UnipotentStabiliser(Sg,F);
 U:=sub<Q|U,UU>;
end for;
LMGInitialise(U:Al:="CompositionTree");
LMGFactoredOrder(U);
```

Note that at the end of this code we request the order of U using the command `LMGFactoredOrder`. However, as discussed, there is a small possibility of receiving an erroneous output here, unless we first initialise U, hence the command `LMGInitialise`.

Now we will discuss a different procedure we can use in the case where k is still small enough that we can use `Transversal`, but so large that the above algorithm becomes impractical. This happens because on each pass, we are adding more generators to the generating set of U. If k is sufficiently large, there are so many generators that the commands `LMGInitialise` and `LMGFactoredOrder` take a long time – or, in some cases, fail – to execute.

This is a procedure suggested by Derek Holt, and it essentially involves building U from a subset of Γ. Then we run through the remaining $\gamma \in \Gamma$ and check first whether $\mathrm{Stab}_{S^\gamma}(F)$ is a subgroup of U. Only when it's not do we add it to the generators of U. This accomplishes the result, however, the generators of U only increase when U itself increases. This algorithm is, therefore, more efficient, and is used in cases where $k > 150$. Here is some sample code carrying out this algorithm.

```
/*
```

```
First we build a subset of at most 50 elements of Gamma.
(Sometimes, due to the Random command, we build a set of fewer
than 50 elements, but this doesn't matter for our purposes.)
*/
Gamma:=Transversal(C1H,S);
SubGamma:={@@};
for i in [1..50] do;
 Include(~SubGamma,Random(Gamma));
end for;
/*
Now we build an initial U from the elements of SubGamma.
*/
U:=sub<Q|Id(Q)>;
for g in SubGamma do;
 Sg:=S^g;
 UU:=UnipotentStabiliser(Sg,F);
 U:=sub<Q|U,UU>;
end for;
LMGInitialise(U:Al:="CompositionTree");
LMGFactoredOrder(U);
/*
Now we run through the remaining elements of Gamma, only including
UU into U if UU is not already a subgroup of U.
*/
for g in Gamma do;
 if g notin SubGamma then;
  Sg:=S^g;
  UU:=UnipotentStabiliser(Sg,F);
  if LMGIsSubgroup(U,UU) eq false then;
   U:=sub<Q|U,UU>;
  end if;
 end if;
```

```
end for;
LMGInitialise(U:Al:="CompositionTree");
LMGFactoredOrder(U);
```

In either case, it is helpful to reduce the generators of U before continuing. This is because, at this stage, U could have a generating set of potentially thousands of elements. To reduce generators, we simply take a random selection of 50 (or similar) elements of U and find the order of the group U_0 they generate. If it is smaller than U, we take a random element U which is not in U_0 and add it to the generators of U_0. We repeat this process until $|U| = |U_0|$, after which we redefine U as U_0. Hence U is the group $\mathcal{C}_2(H)$.

(iii) $k > 1$ and k "large": Here we mean that k is so large that we cannot execute the `Transversal` command. These cases are infrequent in this thesis, and will be discussed in greater detail as and when they occur.

From here, $\mathcal{C}_2(H)$ is usually sufficiently small that we can calculate $\mathcal{C}_3(H) := C_{\mathcal{C}_2(H)}(H)$ directly. Note that $\mathcal{C}_2(H)$ is usually soluble so we can turn it into a pc-group using `LMGSolubleRadical` and running `Centraliser` in the pc-group setting. That concludes our discussion on how to construct a group small enough that we can start sieving for involutions, which is the subject of the next subsection.

Sieving for Involutions

This subsection will discuss several sieving procedures. There are many reasons why sieves might need to be implemented, but the primary reason we require them is to solve the following problem. Let G and H be a finite groups with $S \leq G$ and $X \subset G$, a set of involutions. We want to sieve X for involutions x such that $\langle S, x\rangle \cong H$. Again, this thesis is concerned with constructing subgroups of $E_8(2)$, but the sieves will be discussed in a general setting.

A sieve is any procedure that removes unwanted involutions from X, hopefully by performing a few simple checks to rule out $x \in X$ as a potential generator of H. Sieves are utilised because running through all the elements of X and checking which are such that $\langle S, x\rangle \cong H$ is often impractical for two main reasons. Firstly, in practice, X can be a very large set (multiple billions) and, secondly, the MAGMA command `IsIsomorphic` can often take a while to execute.

The first sieve we will discuss is used in every subsequent chapter of this thesis and will be referred to as the "order of random elements" sieve.

Procedure 3.8. Let $L = \{o(h) : h \in H\}$, the set of all possible element orders of elements in H. This sieves takes $x \in X$ and builds $Y = \langle S, x\rangle$. Now, it takes a random element $r_1 \in Y$ and checks if $o(r_1) \in L$. If $o(r_1) \notin L$, then we can eliminate x as a potential generator of H, as the subgroup it generates cannot possibly be isomorphic to H, seeing as it contains an element of invalid order. If $o(r_1) \in L$, then we choose another random element $r_2 \in Y$ and make the same enquiry. We repeat this process n times (where n is usually 100). If $o(r_i) \in L$ for every $i \in \{1, \ldots, n\}$ then we say x "survives" the sieve and we store it as a potential generator.

There are many advantages of this sieve which make it a fast one to implement. Determining element orders, even when those elements are large matrices, is a quick process. Moreover, as soon as we encounter some r_i for which $o(r_i) \notin L$, we can eliminate x immediately and move onto the next element of X. Finally, since we are usually working in $E_8(2)$, if $Y \ncong H$ then it is quite likely that we will quickly encounter a random element of invalid order, given the plethora of possible element orders in $E_8(2)$.

Below is a sample code of this procedure where $n = 100$. It makes use of the `break` command, which forces an early exit of a `for..do` loop. We use this to stop checking element orders as soon as a random element is found of invalid order; we only keep x in the set named `KeepX` when all 100 random elements pass the test.

```
KeepX:={@@};
for x in X do;
 Y:=sub<Q|S,x>;
 for j in [1..100] do;
  r:=Random(Y);
  if Order(r) notin L then break j;
  end if;
  if j eq 100 then Include(~KeepX,x);
```

```
  end if;
 end for;
end for;
```

Generally, this is the fastest and most efficient sieve, so it is usually the first one we run when faced with a large set of involutions.

We will now discuss how we deal with X when it is too large. As usual, "large" is a vague term which depends on available computational power but here it simply means so large that the order of random elements sieves is impractical to implement directly. We will now suppose $X \leq G$ and discuss the task of constructing and sieving $\mathcal{I}(X)$. There are a few methods we can implement here.

The first is used when X is a 2-group. Let $E \leq Z(X)$ where E is elementary abelian. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a right transversal for E in X. Then the following results, while straightforward, provide us with a clear way of accessing the involutions of X.

Lemma 3.9. *Let $x \in X$ and $\gamma_i \in \Gamma$ such that $x \in E\gamma_i$. Then $x^2 = 1$ if and only if $\gamma_i^2 = 1$.*

Proof. Let $\varepsilon \in E$ such that $x = \varepsilon\gamma_i$. Then recall first that, since E is an elementary abelian 2-group, we have $\varepsilon^2 = 1$ and, secondly, that $E \leq Z(X)$ so ε commutes with everything in X. Now assume that $x^2 = 1$. Then $1 = x^2 = (\varepsilon\gamma_i)^2 = \varepsilon^2\gamma_i^2 = \gamma_i^2$, as required. Conversely, suppose γ_i^2. Then $x^2 = \varepsilon^2\gamma_i^2 = 1$. ∎

We remark here that we have used $\gamma_i^2 = 1$ instead of stating "γ_i is an involution" as the latter would rule out the possibility that $x \in E$.

Proposition 3.10. *Without loss of generality, let $\{\gamma_1, \ldots, \gamma_k\}$ be such that $\gamma_i^2 = 1$ for each $i \in \{1, \ldots, k\}$. Then*

$$\mathcal{I}(X) \subseteq E\gamma_1 \sqcup \ldots \sqcup E\gamma_k.$$

Proof. Let $x \in \mathcal{I}(X)$ and let $\gamma_i \in \Gamma$ such that $x \in E\gamma_i$. By Lemma 3.9 we know that we must have $\gamma_i^2 = 1$ and hence $i \in \{1, \ldots, k\}$. ∎

Again, we remark that the only reason we don't have an equality of sets is that the identity appears in $E\gamma_1 \sqcup \ldots \sqcup E\gamma_k$.

The advantage of this construction is that if $1 \neq \gamma \in \Gamma$ is *not* an involution, then we know that $E\gamma$ contains no involutions. In effect, we are sieving for involutions in Γ and throwing away entire cosets of unwanted elements, instead of sieving for involutions in the entirety of X. In practice, we are often fortunate enough that E is usually large enough that the amount of elements we must sieve is reduced by several orders of magnitude.

The following code utilises this process instead of building $\mathcal{I}(X)$ directly. Note that `Sieve(x)` is a stand-in for whatever sieve we wish to run, and that `Gamma` will be used to represent Γ. This code simply runs through each element of $\gamma \in \Gamma$, then, if $\gamma^2 = 1$, runs through each element of $\varepsilon \in E$, and runs the sieving procedure on $\varepsilon\gamma$ – essentially, this sieves each coset $E\gamma$ in turn.

```
for g in Gamma do;
 if g^2 eq Id(X) then;
  for e in E do;
   x:=e*g;
   Sieve(x);
  end for;
 end if;
end for;
```

We will now discuss how we handle some of these situations in practice because, despite the above process being an efficient way of buliding $\mathcal{I}(X)$, it doesn't actually sieve the elements themselves any faster. The reader should keep in mind that, while we are sieving the cosets $E\gamma_1, \ldots, E\gamma_k$, in practice k is often very large (hundreds of thousands or even millions). As a result, there are often too many elements to construct and sieve. To solve this problem, we utilise parallel processing – the practice of partitioning $\mathcal{I}(X)$ into many disjoint subsets and sieving each of them simultaneously across multiple Magma sessions. The next procedure will discuss an effective way of partitioning $\mathcal{I}(X)$.

We keep with $E \leq Z(X)$ where E is elementary abelian, and now let $E \leq S \leq X$ such that $[X : S]$ is the desired number of sets in our partition of X. We will refer to this as the number of "screens", since the `screen` command enables us to open multiple MAGMA sessions which we may detach from and leave running without being present.

Now, we redefine $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ to be a right transversal for E in S, and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a transversal for S in X.

Proposition 3.11. *The set*

$$\{\gamma_i\delta_j : \gamma_i \in \Gamma,\ \delta_j \in \Delta\}$$

is a right transversal for E in X.

Proof. We must show that these elements form a set of distinct right coset representatives for E in X – that is, that

$$X = \bigsqcup_{j=1}^{n}(E\gamma_1\delta_j \sqcup \ldots \sqcup E\gamma_m\delta_j).$$

We start by showing that these cosets are disjoint. Clearly, $S\delta_1, \ldots, S\delta_n$ are disjoint cosets, as Δ is a transversal. Hence, we fix $k \in \{1, \ldots, n\}$ and if we show that $E\gamma_1\delta_k, \ldots, E\gamma_m\delta_k$ are disjoint, we are done. Suppose we have $i, j \in \{1, \ldots, m\}$ such that $i \neq j$ and $E\gamma_i\delta_k \cap E\gamma_j\delta_k \neq \varnothing$. Then we can choose $x \in E\gamma_i\delta_k \cap E\gamma_j\delta_k$. Hence we have $\varepsilon_i, \varepsilon_j \in E$ with $x = \varepsilon_i\gamma_i\delta_k = \varepsilon_j\gamma_j\delta_k$. But now, $x\delta_k^{-1} = \varepsilon_i\gamma_i = \varepsilon_j\gamma_j$ and so $x\delta_k^{-1} \in E\gamma_i \cap E\gamma_j$, which is a contradiction, as Γ is a transversal for E in S.

We will now prove equality of the sets. A counting argument is sufficient here but we will give more detail, as it will make the essence of the procedures later in this section clearer. Showing $\supseteq$ is trivial, so now let $x \in X$. Since $X = S\delta_1 \sqcup \ldots \sqcup S\delta_n$, we have $x \in S\delta_j$ for some $\delta_j \in \Delta$. Hence there exists $s \in S$ such that $x = s\delta_j$. Now, recall that $S = E\gamma_1 \sqcup \ldots \sqcup E\gamma_n$, so we have $s \in E\gamma_i$ for some $\gamma_i \in \Gamma$. But if $s \in E\gamma_i$, then $x = s\delta_j \in E\gamma_i\delta_j$, as required. ∎

In practice, we typically take $|\Delta| = n$ to be 8, 16, or 32 and save it as an ordered set. We then open one screen per element of Δ; δ_1 is loaded in screen 1, δ_2 is loaded in screen 2, and so on. Then screen i sieves the coset $S\delta_i$, meaning that so long as we save S and Δ, we can partition X across n screens having only saved a handful of elements. We remark that there is no need to save Γ, because the last result holds for *any* right transversal of E in S, so we can calulate Γ afresh in each screen. The following procedure discusses how to employ this technique in practice.

Procedure 3.12. First, a note on constructing S. Recall that $E \leq S \leq X$ such that $[X : S] = n$. To build S, we simply take a random element $r_1 \in X$ and calculate the order of $\langle E, r_1 \rangle$. If this is too small, we take another random element $r_2 \in X$ and calculate the order of $\langle E, r_1, r_2 \rangle$. We repeat this process, adding more elements to the generating set until we have a group of the desired order. (If we generate a group that's too large, we can remove generating elements.) Since we always include E in the generating set, E will be a subgroup of the generated group. Once the order is correct, this becomes S. Now, we find Δ using

```
Delta:=Transversal(X,S);
```

which outputs an ordered right transversal of X in S. We now save Δ and S before proceeding.

We will now fix $i \in \{1, \ldots, n\}$ and focus on what happens in screen i of n parallel screens. The following code sieves the coset $S\delta_i$ for involutions. Recall that we will compute Γ to be a transversal of E in S and that $x \in S\delta_i$ is such that $x^2 = 1$ if and only if $(\gamma\delta_i)^2 = 1$, where $\gamma \in \Gamma$ is such that $x \in E\gamma\delta_i$ by Lemma 3.9. Again, the command `Sieve` is a stand-in for whatever sieve we wish to implement.

```
d:=Delta[i];
Gamma:=Transversal(S,E);
for g in Gamma do;
 if (g*d)^2 eq Id(X) then;
  for e in E do;
   x:=e*g*d;
   Sieve(x);
  end for;
 end if;
end for;
```

The final procedure we will discuss in this section on sieving can be used when X is not a 2-group. The essence of this procedure will be to sieve for involutions which are in the 2-core of X, then for those not in the 2-core, while implementing the procedures we have discussed already. All the groups encountered where this method will be used are split extensions of the form $X = [2^n] : L$, where $O_2(X) = [2^n]$ and $\overline{X} = X/O_2(X) \cong L$.

Let $t \in X$ be an involution. Then, clearly, either $t \in O_2(X)$ or $t \notin O_2(X)$. Since $O_2(X)$ is a 2-group we can sieve it for involutions using Procedure 3.12. Hence the rest of this section will be devoted to creating a set containing $\mathcal{I}(X) \setminus O_2(X)$. Once this and $O_2(X)$ have been sieved, we will have sieved every involution in X.

Suppose $t \notin O_2(X)$. Then we observe that $\bar{t}$ is an involution in $\overline{X}$. Indeed, if $t^2 = 1$, then $\bar{t}^2 = \bar{1}$ which implies $\bar{t} = \bar{1}$ or $\bar{t}$ is an involution. But $\bar{t} = \bar{1}$ if only if $t \in O_2(X)$, which contradicts our assumption. Now let $\bar{t}_1, \ldots, \bar{t}_k$ be a complete list of $\overline{X}$-conjugacy class representatives for its classes of involutions. Finally, let T_i be the full inverse image of $\langle \bar{t}_i \rangle$ in X and let $\overline{R}_i$ be a right transversal for $C_{\overline{X}}(\bar{t}_i)$ in $\overline{X}$, for each $i \in \{1, \ldots, k\}$.

Lemma 3.13. *We have $\bar{t} = \bar{t}_i^{\bar{r}}$ for some $i \in \{1, \ldots, k\}$ and some $\bar{r} \in \overline{R}_i$.*

Proof. We have already established that $\bar{t}$ is an involution in $\overline{X}$, so it must be $\overline{X}$-conjugate to $\bar{t}_i$ for some $i \in \{1, \ldots, k\}$. Say $\bar{t} = \bar{t}_i^{\bar{x}}$ for some $\bar{x} \in \overline{X}$. But $\overline{R}_i$ is a transversal for $C_{\overline{X}}(\bar{t}_i)$ in $\overline{X}$. Hence there is some $\bar{r} \in \overline{R}_i$ for which $\bar{x} \in C_{\overline{X}}(\bar{t}_i)\bar{r}$ and so $\bar{x} = \overline{cr}$ for some $\bar{c} \in C_{\overline{X}}(\bar{t}_i)$. And now we have that

$$\bar{t} = \bar{t}_i^{\overline{cr}} = \bar{t}_i^{\bar{r}}.$$

as required. ■

We now define a set called R_i. This will be a collection of elements in X such that

$$\{\bar{r} : r \in R_i\} = \overline{R}_i,$$

and given any $\bar{r} \in \overline{R}_i$, there is some unique $s \in R_i$ such that $\bar{s} = \bar{r}$. Essentially, it is a set of representatives taken from each coset $\bar{r} \in \overline{R}_i$. Note that it is not the inverse image of $\overline{R}_i$, as we have only taken one representative from each coset. Now we can state a second result about this setup.

Lemma 3.14. *We have $t \in T_i^r$ for some $r \in R_i$.*

Proof. Choose $r \in R_i$ such that $\bar{t} = \bar{t}_i^{\bar{r}}$, which must exist due to Lemma 3.13. This implies that $\bar{t}^{\overline{r^{-1}}} = \bar{t}_i$ and so $\overline{t^{r^{-1}}} \in \langle \bar{t}_i \rangle$. Now, since T_i is the full inverse image of $\langle \bar{t}_i \rangle$, we have that $t^{r^{-1}} \in T_i$. Thus, $t \in T_i^r$, as required. ■

Thus we conclude that given any $t \in \mathcal{I}(X) \setminus O_2(X)$, we have $t \in T_i^r$ for some $i \in \{1, \ldots, k\}$ and some $r \in R_i$. But we can actually do better than this. Let

$$N_i = T_i \setminus O_2(X)$$

and we claim that

$$t \in T_i^r \setminus O_2(X) \text{ if and only if } t \in N_i^r.$$

Indeed, suppose $t \in T_i^r \setminus O_2(X)$ with $t = t_i^r$ for some $t_i \in T_i$. Then we have $t_i \notin O_2(X)$, otherwise $t_i \in O_2(X)$ which implies $t = t_i^r \in O_2(X)$ as $O_2(X)$ is normal in X. Hence $t_i \in N_i$ by the definition of N_i and hence $t \in N_i^r$. Now suppose $t \in N_i^r$, so $t = t_i^r$ for some $t_i \in N_i$. Clearly, $t \in T_i^r$ but assume $t \in O_2(X)$. But now $t_i = t^{r^{-1}} \in O_2(X)$ as, again, $O_2(X)$ is normal in X. This contradicts the supposition that $t_i \notin O_2(X)$. Hence the claim holds and so we have

$$\mathcal{I}(X) \setminus O_2(X) = \bigcup_{i=1}^{k} \bigcup_{r \in R_i} \mathcal{I}(N_i)^r.$$

The advantages of this method are numerous. Firstly, we have a natural way of partitioning $X \setminus O_2(X)$ – we can split the various N_i across multiple screens. Secondly, all of these sets are easy to construct in Magma and we don't have to store large sets in order to reconstruct the partitions of $X \setminus O_2(X)$; we only have to store N_i and R_i for each $i \in \{1, \ldots, k\}$. Finally, we are often in a situation when we are sieving that we only require involutions belonging to a certain conjugacy class of G. Say we only require $t \in \mathcal{I}(X) \setminus O_2(X)$ such that $t \in 2\mathrm{D}_G$. Then we can change N_i to the following set:

$$N_i = (T_i \setminus O_2(X)) \cap 2\mathrm{D}_G.$$

Then all of the involutions in N_i^r will also lie in $2\mathrm{D}_G$.

We will not provide a detailed breakdown of how to implement this prodcedure in practice here. Instead, we will provide such detail as and when this procedure is used. This is simply because this procedure is very case-specific.

Showing Subgroups are Conjugate in $E_8(2)$

Another common challenge we encounter is the following. Suppose $G \cong E_8(2)$ with $H_1, H_2 \leq G$ such that $H_1 \cong H_2$. How do we determine whether H_1 and H_2 are conjugate in G as groups? There is an awfully large number of elements $g \in G$ to run through and test if $H_1^g = H_2$. Fortunately, we can limit the group through which we must search for such conjugating elements, as explored in the next result. It requires many assumptions about H_1 and H_2, but these will often be satisfied due to how we construct subgroups in G in this thesis. Again, we will present this result for an arbitrary finite group G but in this thesis it will only ever be applied to $E_8(2)$.

Proposition 3.15. *Let G be a finite group with $H_1, H_2 \leq G$ such that $H_1 \cong H_2$. Let $P \in \mathrm{Syl}_p(H_1) \cap \mathrm{Syl}_p(H_2)$ and suppose $N := N_{H_1}(P) = N_{H_2}(P)$ (so N is a subgroup common to H_1 and H_2). Then*

H_1 and H_2 are conjugate in G if and only if H_1 and H_2 are conjugate in $N_{N_G(P)}(N)$.

Proof. Clearly, if H_1 and H_2 are conjugate in $N_{N_G(P)}(N)$, then they are conjugate in G, as $N_{N_G(P)}(N) \leq G$. So now let $g \in G$ such that $H_1^g = H_2$. Since $P \leq H_1$ we have that $P^g \leq H_1^g = H_2$. As $|P| = |P^g|$ we have that $P, P^g \in \mathrm{Syl}_p(H_2)$. By Sylow's Theorems, there is $h \in H_2$ such that $P^h = P^g$. Therefore, $N_{H_2}(P^h) = N_{H_2}(P^g)$ and so $N_{H_2}(P)^h = N_{H_2}(P)^g$ by Proposition 2.8. Recalling that $N = N_{H_2}(P)$, we now see that $N^h = N^g$ and therefore $N = N^{gh^{-1}}$. Thus, we conclude that $gh^{-1} \in N_G(N)$.

Now let $g = nh$ for some $n \in N_G(N)$. Now we have $H_2 = H_1^g = H_1^{nh}$ which implies that $H_2^{h^{-1}} = H_1^n$ but since $h \in H_2$ we know $H_2^{h^{-1}} = H_2$. Therefore, $H_1^n = H_2$ where $n \in N_G(N)$. Since n normalises the normaliser of a Sylow p-subgroup, it must also normalise that Sylow p-subgroup by Proposition 2.10. Hence $n \in N_G(P)$ and so $n \in N_{N_G(P)}(N)$, as required. ∎

We now provide a straightforward result which is useful for demonstrating that two subgroups of G are not conjugate.

Proposition 3.16. *Suppose $K_1, K_2 \leq G$ with $K_1 \cong K_2$. Now suppose that there is some group $H_1 \leq G$ such that $K_1 \leq H_1$, and no subgroup H_2 with $K_2 \leq H_2$ and $H_1 \cong H_2$. Then K_1 and K_2 are not conjugate in G.*

Proof. Assume that there is some $g \in G$ such that $K_1^g = K_2$. But now we have $K_2 = K_1^g \leq H_1^g \cong H_1$, a contradiction. ∎

We conclude this chapter with a simple yet useful technique which addresses the following problem. Suppose we have a set of elements $S \subseteq G$ and we generate $H =$

$\langle S \rangle \leq G$. Now suppose we wish to find $|H|$ or the structure of H. However, it could very well be the case that $H = G$. Calling `LMGFactoredOrder` on H is then asking Magma to calculate the order of $E_8(2)$. This is a lengthy calculation. Therefore, it is helpful to rule out the case that $H = G$ before calling `LMGFactoredOrder`. We do this by noting that if the output of

```
#CompositionFactors(GModule(H))
```

is greater than 1, then $H < G$.

Chapter 4

$U_4(2)$ and Its Extensions

For the rest of this thesis, G will denote $E_8(2)$. In this chapter, we will prove that if $H \leq G$ with $F^*(H) \cong U_4(2)$ and $F^*(H)$ not following $U_4(2)$ fusion possibility (viii) or (ix) as shown in Proposition 2.2, then, up to conjugacy in G, there are at most three subgroups of G isomorphic to $U_4(2)$ and at most three subgroups of G isomorphic to $U_4(2) : 2$. We will also show that any $H \leq G$ with $F^*(H) \cong U_4(2)$ is not maximal in G. Let us eliminate fusion possibilities (viii) and (ix) from our discussion in our first result.

Lemma 4.1. *Suppose that $U_4(2) \cong H \leq G$ such that H follows $U_4(2)$ fusion possibility (viii) or (ix). Then H is not maximal in G.*

Proof. By Proposition 2.7, H fixes a non-zero vector in V. Hence, by Proposition 2.23, H is not maximal in G. ■

We will now proceed to construct all $U_4(2)$ subgroups of G which do not follow fusion possibilities (viii) or (ix).

4.1 Constructing $U_4(2)$ Subgroups of G

For the rest of this section, we will suppose we have $H \leq G$ with $H \cong U_4(2)$ following neither fusion possibility (viii) nor (ix). Then $|H| = 2^6.3^4.5$ and the following result will outline our approach for constructing such subgroups in G.

Proposition 4.2. *Let $H \cong U_4(2)$ and $R \in \mathrm{Syl}_3(H)$. Then:*

(i) *There exists a unique subgroup $E \leq R$ such that $E \cong 3^3$;*

(ii) *$N_H(E) = ES$ where $S \cong \mathrm{Sym}(4)$ and any element of order 3 in S lies in $3D_{U_4(2)}$;*

(iii) *$H = \langle ES, x\rangle$ where x is the involution such that $\langle x\rangle = C_H(S)$. Note also that x is H-conjugate to the single class of involutions in $S \setminus O_2(S)$, and that $C_H(E) \cap S = 1$.*

Proof. All of these facts can be easily verified using Magma. ■

We will now outline how to implement these results in the setting of G. This list also provides the structure of this section.

1. We will find all the subgroups in G isomorphic to R, up to G conjugacy, then extract their elementary abelian subgroups of order 3^3. We will call these groups $E_1, \ldots, E_m$.
2. For each $i \in \{1, \ldots, m\}$, we will find, up to G-conjugacy, all subgroups $S_1^{(i)}, \ldots, S_{n_i}^{(i)}$ isomorphic to Sym(4) which normalise E_i.
3. For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n_i\}$, we will construct and sieve the set of involutions centralising $S_j^{(i)}$ for involutions x such that $\langle E_i S_j^{(i)}, x\rangle \cong U_4(2)$.

To proceed with the first step – finding all subgroups in G isomorphic to $R \in \mathrm{Syl}_3(U_4(2))$ up to G-conjugacy – we must first examine how the elements of a potential R will fuse in G.

Lemma 4.3. *Let $H \cong U_4(2)$ and $R \in \mathrm{Syl}_3(H)$. Suppose now that $H \leq G$. Then R has 44 elements of order 3 and 36 elements of order 9, and the possibilities for how many belong to each G-class is given in Table 4.1.*

Proof. This can be seen by noting that R has four elements in $3\mathrm{A}_{U_4(2)}$, four elements in $3\mathrm{B}_{U_4(2)}$, six elements in $3\mathrm{C}_{U_4(2)}$, and thirty elements in $3\mathrm{D}_{U_4(2)}$. R also has eighteen elements in $9\mathrm{A}_{U_4(2)}$ and eighteen elements in $9\mathrm{B}_{U_4(2)}$. This all obtainable through direct calculation in Magma. By examining the fusion possibilities in Proposition 2.2 we arrive at the information in the table. ■

With these possibilities in mind, we can begin our hunt for subgroups of G which are isomorphic to R.

Fusion Combination	3A	3B	3C	3D	9A	9B	9C	9D
(i)	0	0	38	6	0	0	36	0
(ii)	0	30	8	6	0	0	0	36
(iii)	0	30	8	6	0	36	0	0
(iv)	0	0	14	30	0	0	0	36
(v)	0	30	14	0	0	0	36	0
(vi)	0	6	8	30	0	0	36	0
(vii)	0	6	38	0	0	0	0	36
(viii)	30	6	8	0	36	0	0	0
(ix)	6	30	8	0	0	36	0	0

Table 4.1: Fusion of elements of order 3 and order 9 of a Sylow 3-subgroup of $U_4(2)$ in $E_8(2)$

Lemma 4.4. *Up to G-conjugacy, there are at most fourteen subgroups of G which are isomorphic to R.*

Proof. Recall that we are only interested in these subgroups up to G-conjugacy. As the desired subgroups are 3-groups, they must all lie inside Sylow 3-subgroups. Thus, we may restrict our search to a single Sylow 3-subgroup of G, which we will call K. This is saved (see Appendix A) and for details on how a Sylow 3-subgroup of G was constructed, see 4.1 in [5], case $p = 3$. Now, we turn K into a permutation group and use

```
Subgroups(K : OrderEqual:=81)
```

to find all the subgroups of K with order 3^4, up to conjugacy in K. We find 719,558 of them, but only 24,435 are isomorphic to R.

We bring these subgroups back into the matrix setting and, for each, we determine how many of their elements of order 3 belong to each G-class using the `Eigenspace` command. If one of our subgroups has elements of order 3 which do *not* match one of the rows in Table 4.1, we discard it. This leaves us with 449 subgroups, which we sort into five sets depending on which fusion combination they correspond to, as shown in Table 4.2.

Fusion Combination	Number of subgroups isomorphic to R
(ii)	102
(iii)	26
(vi)	304
(viii)	4
(ix)	13

Table 4.2: Number of subgroups isomorphic to R corresponding to each fusion combination

Note that all of these subgroups represent distinct classes of subgroups in K. However, some of these subgroups may be conjugate under the action of a larger group. We introduce $L \leq G$ such that $L \sim 3^8.(2.\Omega_8^+(2).2)$ and $K \leq L$. See [5] for details on how L was constructed. Therefore, all of our potential subgroups lie inside L and we may see if any are L-conjugate. Note that subgroups with different fusion combinations cannot possibly be conjugate in G. We run Procedure B.4 to obtain L-classes of subgroups for each fusion combination, and our results are displayed in Table 4.3.

Fusion Combination	Number of L-classes of subgroups isomorphic to R
(ii)	6
(iii)	2
(vi)	3
(viii)	1
(ix)	2

Table 4.3: Number of L-classes of subgroups isomorphic to R corresponding to each fusion combination

Here are our fourteen subgroups, completing the proof. ■

We will now introduce some notation. For each subgroup isomorphic to R we will take its elementary abelian subgroup of order 3^3 and denote it by $E_j^{(i)}$. The superscript will denote which fusion possibility it follows, but i will be given as the number the fusion Roman numeral represents. So, we have $E_1^{(2)}, \ldots, E_6^{(2)}$, the six copies of 3^3 corresponding to fusion possibility (ii); $E_1^{(3)}, E_2^{(3)}$ corresponding to fusion possibility (iii); and $E_1^{(6)}, E_2^{(6)}, E_3^{(6)}$ corresponding to fusion possibility (vi). We proceed by showing that some of these groups are actually G-conjugate.

Lemma 4.5. *For all $i, j \in \{1, \ldots, 6\}$, we have that $E_i^{(2)}$ and $E_j^{(2)}$ are G-conjugate. Also, for each $i, j \in \{1, \ldots, 3\}$, we have that $E_i^{(6)}$ and $E_j^{(6)}$ are G-conjugate.*

Proof. We will use the same method on each pair of copies of 3^3 we wish to show are conjugate. To simplify the notation, suppose E_1 and E_2 are two subgroups following the same fusion possibility. We begin by taking random elements $h \in L$ until we find one such that $E_1^h \cap E_2 \neq 1$. Now we seek to show that E_1^h is G-conjugate to E_2. Note that any element conjugating E_1^h into E_2 must fix their intersection. Hence, a good place to look for such elements is in the centraliser of a non-trivial element in $E_1^h \cap E_2$. So, let $r \in E_1^h \cap E_2$. In all cases, we have that $r \in 3\mathrm{B}_{E_8(2)}$ or $r \in 3\mathrm{D}_{E_8(2)}$, which means we can calculate $C_G(r)$ using the `FindCent` procedure (see B.3).

Let $C = C_G(r)$. We now use the command `LMGRadicalQuotient` to find $\overline{C} = C/O_2(C)$ and the natural homomorphism $\varphi : C \to \overline{C}$. Mapping E_1^h and E_2 into $\overline{C}$, we use the command `IsConjugate` to find an element $\overline{x} \in \overline{C}$ such that $\overline{E_1^h}^{\overline{x}} = \overline{E_2}$. Now, taking an inverse image of $\overline{x}$, we have $E_1^{hx} = E_2$ and we take $g = hx \in G$ to be our conjugating element. ■

Hence, removing conjugate cases, we now proceed with the group $E_1^{(2)}$ following fusion possibility (ii); $E_1^{(3)}, E_2^{(3)}$ following fusion possibility (iii); and $E_1^{(6)}$ following fusion possibility (vi). We remark that in Lemma 4.4 we found three subgroups isomorphic to R which follow $U_4(2)$ fusion (viii) or (ix). Since we are attempting to construct $U_4(2)$ subgroups of G which do not follow fusion possibility (viii) or (ix), we abandon these subgroups here and proceed with those following (ii), (iii), or (vi). We must now find copies of $\mathrm{Sym}(4)$ which normalise each of these copies of 3^3.

Lemma 4.6. *Let $H \cong U_4(2)$, $R \in \mathrm{Syl}_3(H)$, and $E \leq R$ where E is the unique elementary abelian subgroup of R with order 3^3. Now take $x, y \in E \cap 3C_{U_4(2)}$ with $\langle x \rangle \neq \langle y \rangle$. Then:*

$$N_H(E) = \langle N_{C_H(x)}(E), N_{C_H(y)}(E) \rangle.$$

Proof. This can be directly verified using Magma. ■

Now, if $H \leq G$, it is clear that:

$$N_H(E) = \langle N_{C_H(x)}(E), N_{C_H(y)}(E)\rangle \leq \langle N_{C_G(x)}(E), N_{C_G(y)}(E)\rangle.$$

Hence we will begin by calculating $\langle N_{C_G(x)}(E), N_{C_G(y)}(E)\rangle$ for each of our cases and hunting in there for subgroups isomorphic to Sym(4).

Lemma 4.7. *Let $(i, j) \in \{(2, 1), (3, 1), (3, 2), (6, 1)\}$ and let $E = E_j^{(i)}$. Now set*

$$D_j^{(i)} := \langle N_{C_G(x)}(E), N_{C_G(y)}(E)\rangle$$

where $x, y \in E$ are such that $\langle x\rangle \neq \langle y\rangle$ and belong to $3C_{U_4(2)}$. Note that $3C_{U_4(2)}$ fuses to a different G-conjugacy class of elements of order 3 depending on i. Then $|D_j^{(i)}|$ for each case is given in Table 4.4

i	j	$\lvert D_j^{(i)}\rvert$
2	1	$2^8.3^9$
3	1	$2^5.3^{10}$
3	2	$2^5.3^{10}$
6	1	$2^6.3^9$

Table 4.4: Orders of $D_j^{(i)}$

Proof. First, we will focus on the cases where $i = 2, 3$. In these cases, $3C_{U_4(2)}$ fuses to $3D_{E_8(2)}$. Hence we choose $x, y \in E \cap 3D_{E_8(2)}$ with $\langle x\rangle \neq \langle y\rangle$. By using the `FindCent` procedure (see B.3) we can calculate $C_G(x)$ and $C_G(y)$, both of which are isomorphic to $3 \times U_9(2)$ by Theorem 2.20.

Now, we use the command `LMGRadicalQuotient` to obtain $\overline{C_G(x)} := C_G(x)/3 \cong U_9(2)$ as a permutation group, as well as the natural homomorphism $\varphi_x : C_G(x) \to \overline{C_G(x)}$. Note that as E is abelian, we have $E \leq C_G(x)$, and so we may apply φ_x to E and obtain $\overline{E}$. In this permutation setting, we are able to directly calculate $N_{\overline{C_G(x)}}(\overline{E})$. Finally, we take the full inverse image – a group which necessarily contains $N_{C_G(x)}(E)$ – and find the normaliser in here of E. Hence we have found $N_{C_G(x)}(E)$.

We repeat these steps for $C_G(y)$, finding $N_{C_G(y)}(E)$. Now, finding $D_j^{(i)}$ is simply a matter of calculating $\langle N_{C_G(x)}(E), N_{C_G(y)}(E)\rangle$.

The case where $i = 6$ has $3C_{U_4(2)}$ fuse to $3B_{E_8(2)}$, so we choose $x, y \in E \cap 3B_{E_8(2)}$ with $\langle x\rangle \neq \langle y\rangle$. We are, again, able to use the `FindCent` procedure to calculate $C_G(x)$

and $C_G(y)$. This time, these groups are isomorphic to $3 \times \Omega_{14}^-(2)$. Still, we are able to follow the exact steps as in the $i = 1, 2$ cases in order to obtain $N_{C_G(x)}(E)$ and $N_{C_G(y)}(E)$, thereby obtaining $D_j^{(i)}$. ■

Each of the groups $D_j^{(i)}$ are relatively small, so we can begin hunting in there for viable subgroups isomorphic to Sym(4).

Proposition 4.8. *Let $(i, j) \in \{(2, 1), (3, 1), (3, 2), (6, 1)\}$. Then the orders of $C_{D_j^{(i)}}(E_j^{(i)})$ and number of viable* Sym(4) *subgroups normalising each $E_j^{(i)}$ is given in Table 4.5.*

i	j	$\lvert D_j^{(i)}\rvert$	$\lvert C_{D_j^{(i)}}(E_j^{(i)})\rvert$	Number of Sym(4) subgroups
2	1	$2^8.3^9$	$2^4.3^8$	4
3	1	$2^5.3^{10}$	$2^4.3^9$	0
3	2	$2^5.3^{10}$	$2^4.3^9$	0
6	1	$2^6.3^9$	$2^2.3^8$	2

Table 4.5: Orders of $C_{D_j^{(i)}}(E_j^{(i)})$ and Number of Viable Sym(4) Subgroups in $D_j^{(i)}$

Proof. Recall that by Lemma 4.6 and the remark following the proof that every Sym(4) normalising $E_j^{(i)}$ is to be found in $D_j^{(i)}$. Before we begin the search, we make it clear which Sym(4) subgroups in $D_j^{(i)}$ are deemed viable. Suppose $\mathrm{Sym}(4) \cong S \leq D_j^{(i)}$. We only wish to keep the subgroups S which could potentially generate a copy of $U_4(2)$. These must satisfy the following conditions:

- By Lemma 4.2 (iii), $S \cap C_{D_j^{(i)}}(E_j^{(i)}) = 1$ and hence, since $E_j^{(i)}$ is abelian, we know $S \cap E_j^{(i)} = 1$;
- By Lemma 4.2 (ii), for all $s \in S$ where $o(s) = 3$, $s \in 3\mathrm{D}_{U_4(2)}$.

We begin by looking at the case where $i = 3$. Given that $|D_j^{(3)}| = 2^5.3^{10}$ and $|C_{D_j^{(3)}}(E_j^{(3)})| = 2^4.3^9$, we see that $D_j^{(3)}$ cannot possibly contain any subgroups S of order $24 = 2^3.3$ such that $S \cap C_{D_j^{(3)}}(E_j^{(3)}) = 1$. Indeed, if such an S did exist then for all $s_1, s_2 \in S$ where $s_1 \neq s_2$, the cosets $s_1 C_{D_j^{(3)}}(E_j^{(3)})$ and $s_2 C_{D_j^{(3)}}(E_j^{(3)})$ are disjoint. But now:

$$D_j^{(3)} = \bigsqcup_{s \in S} s C_{D_j^{(3)}}(E_j^{(3)})$$

which implies

$$2^5.3^{10} = |D_j^{(3)}| = \sum_{s\in S} |sC_{D_j^{(3)}}(E_j^{(3)})| = 24(2^4.3^9) = 2^7.3^{10},$$

which is a clear contradiction. Hence the case where $i = 3$ has been eliminated entirely from our potential set of generators of a $U_4(2)$ subgroup of G.

Now, we move on to examine the cases where $i = 2, 6$. We first convert $D_j^{(i)}$ into a permutation group and use the `Subgroups` command to find, up to conjugacy in $D_j^{(i)}$, all the subgroups of $D_j^{(i)}$ of order 24. In the case where $i = 2$, we find 3,920 of them; for $i = 6$ we find 954. Now we sieve these for subgroups which are actually isomorphic to Sym(4) and have trivial intersection with $C_{D_j^{(i)}}(E_j^{(i)})$. For those that survive, we also sieve by the following condition, recalling that all viable Sym(4) subgroups are such that all their elements of order 3 belong to $3\mathrm{D}_{U_4(2)}$.

- For $i = 2$, $3\mathrm{D}_{U_4(2)}$ fuses to $3\mathrm{B}_{E_8(2)}$, so we only keep the Sym(4) subgroups whose elements of order 3 belong to $3\mathrm{B}_{E_8(2)}$.
- For $i = 6$, $3\mathrm{D}_{U_4(2)}$ fuses to $3\mathrm{D}_{E_8(2)}$, so we only keep the Sym(4) subgroups whose elements of order 3 belong to $3\mathrm{D}_{E_8(2)}$.

Note that we determine which G-class the elements of order 3 belong to using the `Eigenspace` command. After checking these conditions, we find four Sym(4) subgroups in the $i = 2$ case, and two in the $i = 6$ case. We name these $S_k^{(i,j)}$ where: if $i = 2$ and $j = 1$, then $k \in \{1, 2, 3, 4\}$; and if $i = 6$ and $j = 1$ then $k = 1, 2$. ■

We now seek to find all the involutions centralising our copies of Sym(4), then sieve them for elements forming a generating set for $U_4(2)$. The process we follow will be the same for each Sym(4) subgroup, so, to simplify notation, let $S = S_k^{(i,j)}$. We will start by taking $\mathrm{Dih}(8) \cong Y \in \mathrm{Syl}_2(S)$ and $z \in Z(Y)$ to be the unique involution. Then we calculate $C_G(z)$ using `CentraliserOfInvolution`. By Proposition 3.1 we have that $C_G(S) \leq C_G(z)$. We now find $\mathcal{C}_1(S)$ as defined in Proposition 3.2 and the remarks that follow, which, by the same proposition, contains all the involutions of interest. Next, we find $\mathcal{C}_2(S)$ as defined in Proposition 3.6 and the remarks that follow – that proposition also tells us that $\mathcal{C}_2(S)$ also contains the involutions of interest. The results of this process are given in the next proposition.

Proposition 4.9. *The orders of $\mathcal{C}_m(S_k^{(i,j)})$ for viable choices of i, j, k and $m \in \{1, 2\}$ are given in Table 4.6.*

i	j	k	$\mathcal{C}_1(S_k^{(i,j)})$	$\mathcal{C}_2(S_k^{(i,j)})$
2	1	1	$2^{98}.3^2.5$	$2^{29}.3^2.5$
2	1	2	$2^{94}.3$	$2^{19}.3$
2	1	3	$2^{94}.3$	$2^{19}.3$
2	1	4	2^{94}	2^{18}
6	1	1	$2^{95}.3$	2^{11}
6	1	2	$2^{95}.3$	2^{11}

Table 4.6: Orders of $\mathcal{C}_m(S_k^{(i,j)})$ for $m \in \{1, 2\}$

Proof. We follow Procedure 3.3 to find $\mathcal{C}_1(S_k^{(i,j)})$ and the procedures detailed in Procedure 3.7 to find $\mathcal{C}_2(S_k^{(i,j)})$. Specifically, we use Procedure 3.7 (i) when $\mathcal{C}_1(S_k^{(i,j)})$ is a 2-group, and Procedure 3.7 (ii) otherwise. ∎

Taking stock of our current situation, we have $\mathcal{C}_2(S_k^{(i,j)})$, in which we must locate all the involutions centralising $S_k^{(i,j)}$. This turns out to be a straightforward process for most cases of $S_k^{(i,j)}$, as we will see in the next result.

Proposition 4.10. *Let $S_k^{(i,j)}$ be such that $(i, j, k) \neq (2, 1, 1)$, so $|\mathcal{C}_2(S_k^{(i,j)})| \neq 2^{29}.3^2.5$. Let $\mathcal{C}_3(S_k^{(i,j)}) = C_{\mathcal{C}_2(S_k^{(i,j)})}(S_k^{(i,j)})$. Then there is exactly one involution t such that $\langle S_1^{(6,1)}, t\rangle \cong U_4(2)$, and no involutions such that $\langle S_k^{(i,j)}, t\rangle \cong U_4(2)$ for $(i, j, k) \neq (6, 1, 1)$.*

Proof. First, we calculate $\mathcal{C}_3(S_k^{(i,j)})$ directly by turning $\mathcal{C}_2(S_k^{(i,j)})$ into a permutation group and running `Centraliser` in the permutation group setting. Now, we sieve $\mathcal{C}_3(S_k^{(i,j)})$ using sieves described in Chapter 3. Again, for simplicity of notation, let $S = S_k^{(i,j)}$.

Let $\mathcal{I}_0(S)$ be the set of all involutions in $\mathcal{C}_3(S)$, which we can find easily by running through every element of $\mathcal{C}_3(S)$ and collecting the involutions. Now, we run an order of random elements sieve on each element of $\mathcal{I}_0(S)$, storing the elements that survive in a set called $\mathcal{I}_1(S)$. For full details on this sieve, see Procedure 3.8. Note that to use this sieve, we require list of all $U_4(2)$ element orders, which is

$$\{1, 2, 3, 4, 5, 6, 9, 12\}$$

as seen in the $\mathbb{ATLAS}$ [14]. Finally, let

$$\mathcal{I}_2(S) = \{x \in \mathcal{I}_1(S) : \langle S, x\rangle \cong U_4(2)\}.$$

The results of this process are displayed in Table 4.7.

i	j	k	$\lvert\mathcal{C}_3(S_k^{(i,j)})\rvert$	$\lvert\mathcal{I}_0(S_k^{(i,j)})\rvert$	$\lvert\mathcal{I}_1(S_k^{(i,j)})\rvert$	$\lvert\mathcal{I}_2(S_k^{(i,j)})\rvert$
2	1	2	$2^{14}.3$	2,815	4	0
2	1	3	$2^{14}.3$	2,815	6	0
2	1	4	2^{14}	2,303	2	0
6	1	1	2^7	127	2	1
6	1	2	2^7	127	1	0

Table 4.7: Involutions generating $U_4(2)$ for $(i, j, k) \neq (2, 1, 1)$

The result follows, as is evident from the table. ■

We now turn our attention back to $(i, j, k) = (2, 1, 1)$. In this case, $|\mathcal{C}_2(S_k^{(i,j)})| = 2^{29}.3^2.5$. We also have that this group has 2-core of order 2^{25}. We sieve for involutions in $\mathcal{C}_2(S_k^{(i,j)})$ by first sieving for those which lie in its 2-core.

Lemma 4.11. *Let $(i, j, k) = (2, 1, 1)$. To simplify notation, let $E = E_1^{(2)}$, $S = S_1^{(2,1)}$, and $C = \mathcal{C}_2(S_1^{(2,1)})$. Then there are ten involutions $t \in O_2(C)$ such that $\langle ES, t\rangle \cong U_4(2)$.*

Proof. First, note that $O_2(C) \cong [2^{25}]$ which we will proceed to sieve in the same way as in Proposition 4.10. The only difference, this time, is that make use of the fact that by Proposition 4.2 (iii) we have that the involutions we're hunting for must be chosen to be G-conjugate to the single class of involutions in $S \setminus O_2(S)$. In this case, that class is $2\mathrm{C}_G$. Hence we construct the set:

$$\mathcal{I}_0(S) = O_2(C) \cap 2\mathrm{C}_G$$

which we then sieve using an order of random elements sieve (see Procedure 3.8 for full details) and store the surviving involutions in $\mathcal{I}_1(S)$. Finally,

$$\mathcal{I}_2(S) = \{x \in \mathcal{I}_1(S) : \langle ES, x\rangle \cong U_4(2)\}.$$

We find that $|\mathcal{I}_0(S)| = 13,110$, $|\mathcal{I}_1(S)| = 12$, and $\mathcal{I}_2(S) = 10$, as required. ■

Now, we will sieve for the involutions in $\mathcal{C}_2(S_1^{(2,1)})$ which do not lie in its 2-core. To do this, we refer the reader to the discussion at the end of Chapter 3 for further details.

Lemma 4.12. *Let $(i, j, k) = (2, 1, 1)$. To simplify notation, again let $E = E_1^{(2)}$, $S = S_1^{(2,1)}$ and $C = \mathcal{C}_2(S_1^{(2,1)})$. Then there are no involutions $t \in C \setminus O_2(C)$ for which $\langle ES, t\rangle \cong U_4(2)$.*

Proof. We have $\overline{C} = C/O_2(C) \cong \mathrm{Sym}(6)$ which we obtain by running the command `LMGRadicalQuotient`. If $t \in C \setminus O_2(C)$ is an involution, then $\bar{t} \in \overline{C}$ is an involution. Hence we let $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_3$ be representatives of the conjugacy classes of involutions in Sym(6). We also let C_i be the full inverse image of $\langle \bar{c}_i \rangle$ in C and $\overline{R}_i$ a right transversal for $C_{\overline{C}}(\bar{c}_i)$ in $\overline{C}$, for each $i \in \{1, 2, 3\}$. Now let R_i be a set of representatives of each coset in $\overline{R}_i$. By Lemma 3.14 we know that $t \in C_i^r$ for some $i \in \{1, 2, 3\}$ and $r \in R_i$. However, since we know that $t \notin O_2(C)$, we also know that $t \in (C_i \setminus O_2(C))^r$ (see remarks following the proof of Lemma 3.14). Finally, recall that by Proposition 4.2 we know that t must be G-conjugate to the single class of involutions in $S \setminus O_2(S)$. In this case, this requires that $t \in 2\mathrm{C}_G$. Hence we construct

$$N_i = (C_i \setminus O_2(C)) \cap 2\mathrm{C}_G$$

and know that $t \in N_i^r$.

In practice, we construct N_i then for all $x \in N_i$, we run an order of random elements sieve on x^r for each $r \in R_i$. For full details on the order of random elements sieve, see Procedure 3.8. In all cases, no involutions survive this process, so we can be certain that none of them generate a $U_4(2)$ subgroup of G, as required. ■

In summary, having exhausted all possibilities, we have found eleven $U_4(2)$ subgroups. However, we will show that this number can be reduced in this next result, which will conclude the section.

Proposition 4.13. *An upper bound for the number of $U_4(2)$ subgroups up to G-conjugacy containing $E_j^{(i)} S_k^{(i,j)}$, for $(i, j, k) \in \{(2, 1, 1), (6, 1, 1)\}$, is shown in Table 4.8.*

Proof. In the case where $(i, j, k) = (6, 1, 1)$, we only have one $U_4(2)$ subgroup, so no additional argument is required to reduce the number of cases. Now fix $(i, j, k) =$

i	j	k	Number of $U_4(2)$ subgroups containing $E_j^{(i)} S_k^{(i,j)}$
2	1	1	2
6	1	1	1

Table 4.8: Number of $U_4(2)$ subgroups up to G-conjugacy

$(2, 1, 1)$ and for simpler notation let $E = E_1^{(2)}$, $S = S_1^{(2,1)}$, and $H_1, \dots, H_{10}$ be the ten $U_4(2)$ subgroups containing ES. First, we find R such that for all $l \in \{1, \dots, 10\}$, $R \in \mathrm{Syl}_3(H_l)$. This is straightforward to find – we choose any Sylow 3-subgroup of H_1 then conjugate it by random elements of H_1 until it is also a subgroup of $H_2, \dots, H_{10}$.

Now recall that $N_{H_l}(E) = ES$ and note that E is unique in R, so anything normalising R must also normalise E. Therefore, $N_{H_l}(R) \leq N_{H_l}(E) = ES$. This implies that $N_{H_l}(R) = N_{H_{l_0}}(R)$ for any $l, l_0 \in \{1, \dots, 10\}$. Hence Proposition 3.15 applies and we have that H_l and H_{l_0} are conjugate in G if only if they are conjugate in $N_{N_G(R)}(N_{H_l}(R))$. And now observe that

$$N_{N_G(R)}(N_{H_l}(R)) \leq N_{N_G(E)}(N_{H_l}(R)).$$

Hence we will find conjugating elements in $D_1^{(2)}$ (see Lemma 4.7). We find that H_1 is not conjugate to H_2 in $D_1^{(2)}$, and H_2 is conjugate in $D_1^{(2)}$ to H_l for $l \in \{3, \dots, 10\}$. ■

Hence we have, in total, at most three $U_4(2)$ subgroups of G, following fusion possibility (i) to (vii), up to conjugacy in G. We will now attempt to find overgroups of these groups isomorphic to $\mathrm{Aut}(U_4(2))$.

4.2 Extending $U_4(2)$ to $U_4(2) : 2$

Of all the automorphism extensions we consider in this thesis, extending copies of $U_4(2)$ to $\mathrm{Aut}(U_4(2)) \cong U_4(2) : 2$ is the most straightforward. We obtain a result analogous to Proposition 4.2 which will allow us to quickly find copies of $U_4(2) : 2$.

Proposition 4.14. *Let $H \cong U_4(2) : 2$ and $R \in \mathrm{Syl}_3(H)$. Then:*

(i) There exists a unique subgroup $E \leq R$ where $E \cong 3^3$;

(ii) $N_H(E) = ES$ where $S \cong \mathrm{Sym}(4) \times 2$ and any element of order 3 in S lies in $3C_{U_4(2):2}$;

(iii) $H = \langle ES, x\rangle$ *where* x *is the involution such that* $\langle x\rangle = C_H(S)$*. Note also that* x *is* H*-conjugate to the single class of involutions in* $S\setminus O_2(S)$*, and that* $C_H(E)\cap S = 1$.

Proof. All these results can be shown through direct calculation in MAGMA. ■

Note that the only difference between this result and Proposition 4.2 is that we have $S \cong \mathrm{Sym}(4) \times 2$ instead of $\mathrm{Sym}(4)$. We also have the following result analogous to Lemma 4.6.

Lemma 4.15. *Let* $H \cong U_4(2) : 2$*,* $R \in \mathrm{Syl}_3(H)$*, and* $E \leq R$ *where* E *is the unique elementary abelian subgroup of* R *with order* 3^3*. Now take* $x, y \in E \cap 3C_{U_4(2):2}$ *with* $\langle x\rangle \neq \langle y\rangle$*. Then:*

$$N_H(E) = \langle N_{C_H(x)}(E), N_{C_H(y)}(E)\rangle.$$

Hence, our strategy is clear. Given $U_4(2) : 2 \cong H_0$ such that $H \leq H_0 \leq G$ and $H \cong U_4(2)$ follows $U_4(2)$ fusion possibility 2.2 (i) to (vii), then H is G-conjugate to one of the three copies of $U_4(2)$ discovered in Proposition 4.13. Thus, we may assume, by Proposition 2.12, that H_0 is an overgroup of one of these groups. Recall that each of these groups are generated by $\langle ES, x\rangle$ where $E \cong 3^3$ and $S \cong \mathrm{Sym}(4)$. If H_0 exists as an overgroup of one of these copies of $U_4(2)$, then there must be a copy of $S_0 \cong \mathrm{Sym}(4) \times 2$ normalising E. If such an S_0 exists, it must be conjugate to an overgroup of S. We will find, by Lemma 4.15, all such $\mathrm{Sym}(4) \times 2$ in $D_j^{(i)}$ as defined in Lemma 4.7 as overgroups of S, and our copies of $U_4(2) : 2$ will be given by $\langle ES_0, x\rangle$.

Before we find all viable copies of $\mathrm{Sym}(4) \times 2$, we will state a lemma which will cut down the number of cases we must consider.

Lemma 4.16. *Suppose* $U_4(2) \cong H \leq H_0 \cong U_4(2) : 2$*. Then the classes of elements of order 3 in* H *fuse to the classes in* H_0 *as follows.*

$$3A_H \to 3A_{H_0},\ 3B_H \to 3A_{H_0},\ 3C_H \to 3B_{H_0},\ 3D_H \to 3C_{H_0}.$$

Proof. This can be verified in MAGMA. ■

Proposition 4.17. *Up to* G*-conjugacy, there are at most three subgroups* $H \leq G$ *such that* $H \cong U_4(2) : 2$ *and such that* H *follows one of* $U_4(2)$ *fusion possibilities 2.2 (i) to (vii).*

Proof. We will execute the strategy described in the remarks following Lemma 4.15. Let H be one of the three $U_4(2)$ subgroups found in Proposition 4.13. Let $E = E_j^{(i)}$ be the elementary abelian 3^3 subgroup that H is built up from (so $E_1^{(2)}$, or $E_1^{(6)}$ depending on the case), and let $D = D_j^{(i)}$ as defined in Lemma 4.7. Then we turn D into a permutation group and find all subgroups of D of order 48, storing them in a set called $\mathcal{S}_0(D)$; then we see which of these subgroups are isomorphic to $\mathrm{Sym}(4) \times 2$, storing them in a set called $\mathcal{S}_1(D)$; then we keep only the ones which intersect trivially with $C_D(E)$ in a set called $\mathcal{S}_2(D)$.

Now recall that by Proposition 4.14, all of the elements of order 3 in our copies of $\mathrm{Sym}(4) \times 2$ must lie inside $3\mathrm{C}_{U_4(2):2}$. By Lemma 4.16, we know that $3\mathrm{D}_{U_4(2)} \rightarrow 3\mathrm{C}_{U_4(2):2}$. Supposing first that H follows $U_4(2)$ fusion possibility (ii), we know that, by Proposition 2.2, $3\mathrm{D}_H \rightarrow 3\mathrm{B}_G$ and hence $3\mathrm{C}_{H_0} \rightarrow 3\mathrm{B}_G$. Thus we may keep only those $\mathrm{Sym}(4) \times 2$ such that all of their elements of order 3 lie inside $3\mathrm{B}_G$. Secondly, if H follows $U_4(2)$ fusion possibility (vi), then we have, again by Proposition 2.2, that $3\mathrm{D}_H \rightarrow 3\mathrm{D}_G$ and hence $3\mathrm{C}_{H_0} \rightarrow 3\mathrm{D}_G$. So, in this case, we may keep only the $\mathrm{Sym}(4)\times 2$ subgroups such that all of their elements of order 3 lie inside $3\mathrm{D}_G$. In either case, we store the desired subgroups in a set called $\mathcal{S}_3(D)$.

Finally, recall that we need only take copies of $\mathrm{Sym}(4) \times 2$ which are conjugate to an overgroup of one of the copies of $\mathrm{Sym}(4)$ used in the construction of $U_4(2)$ – recall that these were named $S_1^{(i,j)}$. To sieve for this criterion, we take S_0, our candidate copy of $\mathrm{Sym}(4) \times 2$, then find $S_1, S_2 \leq S_0$ such that $S_1 \cong \mathrm{Sym}(4) \cong S_2$. Then we check if either S_1 or S_2 are D-conjugate to $S_1^{(i,j)}$. Note that we do this in the permutation group setting using the command `IsConjugate`. If they are conjugate, then this command also returns a conjugating element $g \in D$ such that $S_1^{(i,j)} = S_k^g$, where $k = 1, 2$. Then we have that S_0^g is an overgroup of $S_1^{(i,j)}$. We store the copies of $\mathrm{Sym}(4) \times 2$ which can be conjugated to an overgroup of $S_1^{(i,j)}$ in a set called $\mathcal{S}_4(D)$.

The results of this process are summarised in Table 4.9

i	j	$\lvert\mathcal{S}_0(D_j^{(i)})\rvert$	$\lvert\mathcal{S}_1(D_j^{(i)})\rvert$	$\lvert\mathcal{S}_2(D_j^{(i)})\rvert$	$\lvert\mathcal{S}_3(D_j^{(i)})\rvert$	$\lvert\mathcal{S}_4(D_j^{(i)})\rvert$
2	1	2,764	49	5	4	2
6	1	273	5	3	1	1

Table 4.9: Viable $\mathrm{Sym}(4) \times 2$ subgroups of $D_j^{(i)}$

To complete the proof, we note that every copy of $\mathrm{Sym}(4) \times 2$ found this way can be used in the generation of a $U_4(2) : 2$ subgroup. Recall that when $i = 2$, we have two copies of $U_4(2)$. Call these $H_1^{(2)}$ and $H_2^{(2)}$. We find that $H_1^{(2)}$ can be extended to two distinct groups isomorphic to $U_4(2) : 2$ using either copy of $\mathrm{Sym}(4) \times 2$, and that $H_2^{(2)}$ cannot be extended. This yields two copies of $U_4(2) : 2$. Now fix $i = 6$ and $j = 1$. Recall here that we only have one copy of $U_4(2)$, and we found that this can be extended to a group isomorphic to $U_4(2) : 2$ using the sole viable copy of $\mathrm{Sym}(4) \times 2$ found in $D_1^{(6)}$. This yields the third copy of $U_4(2) : 2$. ■

So far, we have that there are at most three classes of subgroups isomorphic to $U_4(2)$ and at most three classes of subgroups isomorphic to $U_4(2)$, which do not follow $U_4(2)$ fusion possibilities 2.2 (viii) or (ix). However, Theorem 1.1 states that there are exactly three classes of each subgroup following fusion possibility (viii) or (ix). We will prove this later, as the lower bound presents itself naturally following results proved later in the thesis. Now, we will conclude this chapter by demonstrating that any $U_4(2)$ or $U_4(2) : 2$ subgroup of G is not maximal.

Proposition 4.18. *Let $H_0 \leq G$ such that $F^*(H_0) \cong U_4(2)$. Then H_0 is not a maximal subgroup of G.*

Proof. Let $H_0 \leq G$ such that $F^*(H_0) \cong U_4(2)$. Let $H = F^*(H_0)$ and by Proposition 2.2, H follows one of nine fusion possibilities. First, suppose that H follows $U_4(2)$ fusion possibility (viii) or (ix). Then, by Proposition 2.7 we have that H fixes a non-zero vector of V, so, by Proposition 2.18, H_0 fixes a non-zero vector of V. Hence, by Proposition 2.23 we know H_0 is not maximal in G.

Secondly, suppose H neither follows $U_4(2)$ fusion possibility (viii) nor (ix). Then, by construction, H is G-conjugate to one of the three $U_4(2)$ subgroups found in Proposition 4.13. If H is G-conjugate to the copy of $U_4(2)$ which does not extend to $U_4(2) : 2$, then we find that $\dim C_V(H) = 1$ and hence is not maximal by Proposition 2.23. If H is G-conjugate to one of the two copies of $U_4(2)$ which do extend to $U_4(2) : 2$, then we either have $H_0 = H$ – in which case, H_0 is not maximal as it is contained in a $U_4(2) : 2$ subgroup – or H_0 is G-conjugate to one of the three copies of $U_4(2) : 2$ found in Proposition 4.17. We find that, if H follows $U_4(2)$ fusion possibility (ii), then both cases for H_0 are such that $\dim C_V(H_0) = 3$. If H follows $U_4(2)$ fusion possibility (vi), then we have that $\dim C_V(H_0) = 1$. In either case, we have that H_0 is not maximal by Proposition 2.23. ■

Chapter 5

$Sp_6(2)$

As always, G will denote $E_8(2)$. In this chapter, we will show that there are three conjugacy classes in G of subgroups isomorphic to $Sp_6(2)$ which do not follow $Sp_6(2)$ fusion possibility (v) or (vi) as given in Proposition 2.3. We will also show that any $Sp_6(2)$ subgroup of G is not a maximal subgroup. The key to this result is noting that $Sp_6(2)$ contains $U_4(2):2$ as a subgroup. Therefore, our strategy will be to build $Sp_6(2)$ subgroups from the copies of $U_4(2):2$ we found in Proposition 4.17. However, we must take care here, as we did not find *every* $U_4(2):2$ subgroup of G up to conjugacy; we found all $U_4(2):2$ subgroups containing $U_4(2)$ following fusion possibility 2.2 (ii) or (vi). But we never ruled out the possibility of $U_4(2)$ subgroups existing in G which follow fusion possibility 2.2 (viii) or (ix). The following result will show that, in order to show $Sp_6(2)$ is not maximal in G, we need not concern ourselves with such cases.

Lemma 5.1. *Suppose $K \leq H \leq G$ where $H \cong Sp_6(2)$ and $K \cong U_4(2)$. Then*

(i) H cannot follow $Sp_6(2)$ fusion possibility (i) or (iv);

(ii) if H follows $Sp_6(2)$ fusion possibility (ii) or (iii), then K follows $U_4(2)$ fusion possibility (ii) or (vi) respectively;

(iii) if H follows $Sp_6(2)$ fusion possibility (v) or (vi), then K follows $U_4(2)$ fusion possibility (viii) or (ix) respectively and, moreover, H is not maximal in G.

Proof. Proving these results is, for the most part, a simple matter of comparing the $U_4(2)$ fusion information given in Proposition 2.2 with the $Sp_6(2)$ fusion information given in Proposition 2.3. First, we recall that in Chapter 4 we proved that any $U_4(2)$

subgroup of G can only follow fusion possibility 2.2 (ii), (vi), (viii), or (ix). Indeed, $U_4(2)$ contains a Sylow 3-subgroup R of order 3^4, but the only subgroups of G isomorphic to R following *any* valid $U_4(2)$ fusion possibility followed $U_4(2)$ fusion possibility (ii), (iii), (vi), (viii), or (ix), as seen in Lemma 4.4, and the cases following fusion possibility (iii) were ruled out in Proposition 4.8. Now we will prove each statement in turn.

(i) Suppose H follows $Sp_6(2)$ fusion possibility (i). Then every element of order 9 in H fuses to $9C_G$, hence K must follow $U_4(2)$ fusion possibility (i), (v), (vi). Now, every element of order 3 in H fuses to either $3C_G$ or $3D_G$. If K follows $U_4(2)$ fusion possbility (v) or (vi), then there is an element in K fusing to $3B_G$, contradicting our assumption of K and hence K follows $U_4(2)$ fusion possibility (i). But this is not possible, as discussed at the beginning of this proof. Thus, H cannot follow $Sp_6(2)$ fusion possibility (i).

   Now, suppose H follows $Sp_6(2)$ fusion possibility (iv). Then every element of order 9 in H fuses to $9D_G$, hence K must follow $U_4(2)$ fusion possibility (ii), (iv), or (vii). Now, every element of order 3 in H fuses to either $3B_G$ or $3C_G$, so K cannot follow $U_4(2)$ fusion possibility (ii) or (iv), else it would contain an element of order 3 fusing to $3D_G$. Hence K follows $U_4(2)$ fusion possibility (vii), which is impossible. Therefore, H cannot follow $Sp_6(2)$ fusion possibility (iv), as required.

(ii) Assume first that H follows $Sp_6(2)$ fusion possibility (ii). Then every element of order 9 in H fuses to $9D_G$, and hence K must follow $U_4(2)$ fusion possibility (ii), (iv), or (vii). Now we note that, of these options for K, we must have K following (ii), as the others do not exist in G.

   Secondly, assume H follows $Sp_6(2)$ fusion possibility (iii). Then every element of order 9 in H fuses to $9C_G$, so K must follow $U_4(2)$ fusion possibility (i), (v), or (vi). But of these, only fusion possibility (vi) is possible for K.

(iii) Finally, assume H follows $Sp_6(2)$ fusion possibility (v) or (vi). In both of these cases, every element of order 5 in H fuses to $5A_G$, so K must follow $U_4(2)$ fusion possibility (viii) or (ix). Now, if H follows $Sp_6(2)$ fusion possibility (v), then every element of order 9 in H fuses to $9A_G$ and hence K must now follow $U_4(2)$

fusion possibility (viii). Similarly, if we assume H follows $Sp_6(2)$ fusion possibility (vi), then every element of order 9 in H fuses to $9\mathrm{B}_G$, and hence K must follow $U_4(2)$ fusion possibility (ix).

Now, we note that if H follows $Sp_6(2)$ fusion possibility (v) or (vi), then H fixes a non-zero vector of V by Proposition 2.7. Hence, by Proposition 2.23, we have that H is not maximal in G.

■

We will now state some facts about $Sp_6(2)$ which will allude to how we intend to build $Sp_6(2)$ subgroups from our $U_4(2):2$ subgroups.

Proposition 5.2. *Suppose* $H \cong Sp_6(2)$ *and* $U_4(2):2 \cong K \leq H$. *Let* $S \in \mathrm{Syl}_2(K)$. *Then*

(i) there is a unique elementary abelian subgroup $W \leq S$ *of order* 2^4 *and such that* $N_K(W) \sim 2^4.\mathrm{Sym}(5)$*;*

(ii) $C_H(N_K(W)) = \langle t \rangle$ *where* t *is an involution;*

(iii) $H = \langle K, t \rangle$.

Proof. All of these facts can be verified directly in Magma. ■

We will now focus on building $Sp_6(2)$ subgroups from copies of $U_4(2):2$ found in Chapter 4.

5.1 Constructing $Sp_6(2)$ subgroups of G

For the rest of this chapter, we construct $H \leq G$ such that $H \cong Sp_6(2)$ and H follows $Sp_6(2)$ fusion possibility (ii) or (iii). Hence, by Lemma 5.1, $K \leq H$ with $K \cong U_4(2):2$ contains $U_4(2)$ following $U_4(2)$ fusion possibility (ii) or (vi). Therefore, K is G-conjugate to one of the three $U_4(2):2$ subgroups constructed in Proposition 4.17. Without loss of generality, we will actually assume that K is equal to one of these three subgroups. We will name these subgroups $K_j^{(i)}$. Here, $i \in \{2,6\}$, and $K_j^{(2)}$ contains $U_4(2)$ following fusion possibility (ii) for $j \in \{1,2\}$, whereas $K_1^{(6)}$ contains $U_4(2)$ following fusion possibility (vi).

Now we implement Proposition 5.2 by following these steps, which form an outline of this chapter. Fix i and j where $i \in \{2, 6\}$, $j \in \{1, 2\}$ if $i = 2$, and $j = 1$ if $i = 6$.

1. We will locate $2^4 \cong W_j^{(i)} \leq K_j^{(i)}$ such that $N_j^{(i)} := N_{K_j^{(i)}}(W_j^{(i)}) \sim 2^4.\mathrm{Sym}(5)$.
2. We will construct $\mathcal{I}(C_G(N_j^{(i)}))$.
3. We will sieve $\mathcal{I}(C_G(N_j^{(i)}))$ for involutions t such that $\langle K_j^{(i)}, t\rangle \cong Sp_6(2)$.

The first step is straightforward. Using the MAGMA command `LMGSylow` we can find $S_j^{(i)} \in \mathrm{Syl}_2(K_j^{(i)})$, then use `ElementaryAbelianSubgroups` to find representatives of the three classes of subgroups of $S_j^{(i)}$ isomorphic to 2^4. Using `Normaliser` we calculate each of their normalisers in $K_j^{(i)}$. We identify the 2^4 subgroup with normaliser in $K_j^{(i)}$ with shape $2^4.\mathrm{Sym}(5)$ and name it $W_j^{(i)}$. Now we let $N_j^{(i)} = N_{K_j^{(i)}}(W_j^{(i)})$ and our task is now to construct $\mathcal{I}(C_G(N_j^{(i)}))$.

Our strategy here is the same regardless of our choice of i and j, so, to simplify notation, let $K = K_j^{(i)}$, $W = W_j^{(i)}$, and $N = N_j^{(i)}$. We start by taking $R \leq N$ such that $R \cong \mathrm{Sym}(5)$ and $D \in \mathrm{Syl}_2(R)$, so $D \cong \mathrm{Dih}(8)$. Now let $z \in Z(D)$ be the unique involution. Note that, in all cases, $z \in 2\mathrm{D}_G$ so $C_G(z) \sim [2^{84}].Sp_8(2)$ by Proposition 2.19. We calculate $C_G(z)$ using

`CentraliserOfInvolution`

which, by Proposition 3.1, contains D and $C_G(N)$.

Now we find $\mathcal{C}_1(D)$ as defined in Proposition 3.2. Recall that if we let $C = C_G(z)$ and $\overline{C} = C/O_2(C) \cong Sp_8(2)$, then $\mathcal{C}_1(D)$ is defined to be the inverse image of $C_{\overline{C}}(\overline{D})$ in C. Again, by Proposition 3.2, we have $\mathcal{I}(C_G(N)) \subseteq \mathcal{C}_1(D)$.

Next, we find

$$\mathcal{C}_2(D) = \langle \mathrm{Stab}_X(F) : X \in \mathrm{Syl}_2(\mathcal{C}_1(D))\rangle$$

where $F = C_V(N)$. By Proposition 3.6 we have that $\mathcal{I}(C_G(N)) \subseteq \mathcal{C}_2(D)$.

Finally, let

$$\mathcal{C}_3(D) = C_{\mathcal{C}_2(D)}(N)$$

which we find directly. The next result shows the outcome of this process.

Proposition 5.3. *The order of $\mathcal{C}_k(D_j^{(i)})$ where $k \in \{1, 2, 3\}$ is given in Table 5.1.*

i	j	$\lvert\mathcal{C}_1(D_j^{(i)})\rvert$	$\lvert\mathcal{C}_2(D_j^{(i)})\rvert$	$\lvert\mathcal{C}_3(D_j^{(i)})\rvert$
2	1	$2^{98}.3^2.5$	$2^{12}.3$	$2^4.3$
2	2	$2^{98}.3^2.5$	2^{10}	2^4
6	1	$2^{98}.3^2.5$	$2^{31}.3$	2.3

Table 5.1: Order of $\mathcal{C}_k(D_j^{(i)})$ for $k \in \{1, 2, 3\}$

Proof. We find $\mathcal{C}_1(D_j^{(i)})$ using Procedure 3.3; $\mathcal{C}_2(D_j^{(i)})$ using Procedure 3.7 (ii); and $\mathcal{C}_3(D_j^{(i)})$ directly by using `Centraliser`. ∎

At this stage, we start sieving $\mathcal{C}_3(D_j^{(i)})$ for involutions t such that $\langle K_j^{(i)}, t\rangle \cong Sp_6(2)$. We observe that, as far as groups we must sieve go, each $\mathcal{C}_3(D_j^{(i)})$ is tiny compared with groups we've faced in other cases, making the sieving process quick and straightforward.

Proposition 5.4. *For each $K_j^{(i)}$, there exists $H_j^{(i)} \leq G$ with $H_j^{(i)} \cong Sp_6(2)$ such that $K_j^{(i)} \leq H_j^{(i)}$ and $H_j^{(i)}$ is unique up to G-conjugacy.*

Proof. Since $\mathcal{C}_3(D_j^{(i)})$ is so small for each choice of i and j, we need not apply any complicated sieves. We simply construct the following chain of subgroups directly in Magma.

$$\mathcal{I}_0(N_j^{(i)}) = \{t \in \mathcal{C}_3(D_j^{(i)}) : o(t) = 2\},$$
$$\mathcal{I}_1(N_j^{(i)}) = \{t \in \mathcal{I}_0(N_j^{(i)}) : |\langle K_j^{(i)}, t\rangle| = |Sp_6(2)|\},$$
$$\mathcal{I}_2(N_j^{(i)}) = \{t \in \mathcal{I}_1(N_j^{(i)}) : \langle K_j^{(i)}, t\rangle \cong Sp_6(2)\}.$$

The sizes of these sets are displayed in Table 5.2 We see that $|\mathcal{I}_2(N_j^{(i)})| = 1$ for each

i	j	$\lvert\mathcal{I}_0(N_j^{(i)})\rvert$	$\lvert\mathcal{I}_1(N_j^{(i)})\rvert$	$\lvert\mathcal{I}_2(N_j^{(i)})\rvert$
2	1	19	1	1
2	2	11	1	1
6	1	1	1	1

Table 5.2: $|\mathcal{I}_k(N_j^{(i)})|$ for $k \in \{0, 1, 2\}$

choice of i and j, proving that there is exactly one involution t in each case for which $\langle K_j^{(i)}, t\rangle \cong Sp_6(2)$. ∎

We will call the three subgroups isomorphic to $Sp_6(2)$ found in Proposition 5.4 $H_j^{(i)}$, where $H_j^{(i)}$ contains $K_j^{(i)}$ for each $(i, j) \in \{(2, 1), (2, 2), (6, 1)\}$. We note here that none of these three subgroups are G-conjugate, though, we will not give a full proof here, as it depends on results established later in the thesis. We will conclude this chapter by showing that none of these $Sp_6(2)$ subgroups are maximal in G.

Proposition 5.5. *Let $H \leq G$ such that $H \cong Sp_6(2)$. Then H is not maximal in G.*

Proof. Suppose $H \leq G$ with $H \cong Sp_6(2)$. By Proposition 2.3, there are six fusion possibilities for H. By Lemma 5.1, H cannot follow fusion possibilities (i) and (iv), and if H follows fusion possibility (v) or (vi), then H is not maximal. Now suppose H follows fusion possibility (ii) or (iii). Then, by construction, H is G-conjugate to one of the three subgroups found in Proposition 5.4, which we named $H_1^{(2)}$, $H_2^{(2)}$, and $H_1^{(6)}$. Now, we simply observe that

$$\dim C_V(H_j^{(i)}) = \begin{cases} 3, & \text{if } i = 2 \\ 1, & \text{if } i = 6 \end{cases}$$

and hence each H fixes a non-zero vector in V. Therefore, by Proposition 2.23, we have that H is not maximal in G. ■

Chapter 6

$\Omega_8^-(2)$ and Its Extensions

In this chapter, we will prove Theorem 1.3, which is arguably the weakest result in this thesis, seeing as there are unaddressed cases of $\Omega_8^-(2)$ which could exist in G and indeed be maximal. For this reason, this chapter should be viewed as the beginnings of a proof of such a result, as well as the start of categorising all the $\Omega_8^-(2)$ subgroups of G up to conjugacy.

We will start with the observation that $Sp_6(2) \leq \Omega_8^-(2)$, so our strategy will be to build up $\Omega_8^-(2)$ subgroups from the three copies of $Sp_6(2)$ found in Chapter 5. Let us begin with a result exploring the fusion possibilities for $\Omega_8^-(2)$ in G.

Lemma 6.1. *Suppose $K \leq H \leq G$ with $K \cong Sp_6(2)$ and $H \cong \Omega_8^-(2)$. Then*

(i) if H follows $\Omega_8^-(2)$ fusion possibility (i), then K follows $Sp_6(2)$ fusion possibility (iii);

(ii) if H follows $\Omega_8^-(2)$ fusion possibility (ii), then K follows $Sp_6(2)$ fusion possibility (ii);

(iii) if H follows $\Omega_8^-(2)$ fusion possibility (iii) or (iv), then K follows $Sp_6(2)$ fusion possibility (vi);

(iv) if H follows $\Omega_8^-(2)$ fusion possibility (v), then K follows $Sp_6(2)$ fusion possibility (v). Moreover, H is not maximal in G.

Proof. We prove these results by recalling some results about $Sp_6(2)$ subgroups of G from Chapter 5, and making comparisons between the fusion information given in

Propositions 2.3 and 2.4. Recall that if $Sp_6(2) \cong K \leq G$, then K cannot follow fusion possibility (i) or (iv). This was proved in Lemma 5.1. With this in mind, we will prove each statement in turn.

(i) Suppose H follows $\Omega_8^-(2)$ fusion possibility (i). Then every element in H of order 9 fuses to 9C_G, hence K must follow $Sp_6(2)$ fusion possibility (i) or (iii). However, as stated earlier, no $Sp_6(2)$ subgroups of G exist with fusion possibility (i), hence K follows fusion possibility (iii).

(ii) Suppose H follows $\Omega_8^-(2)$ fusion possibility (ii). Then every element in H of order 9 fuses to 9D_G, hence K must follow $Sp_6(2)$ fusion possibility (ii) or (iv). However, no $Sp_6(2)$ subgroups of G exist which follow fusion possibility (iv), so K must follow fusion possibility (ii).

(iii) If H follows $\Omega_8^-(2)$ fusion possibility (iii) or (iv), then every element in H of order 9 fuses to 9B_G. Hence, K follows fusion possibility (vi).

(iv) Finally, assume H follows $\Omega_8^-(2)$ fusion possibility (v). Then every element in H of order 9 fuses to 9A_G. Therefore, K must follow $Sp_6(2)$ fusion possibility (v). In this case, we also see that, by Proposition 2.7, H fixes a non-zero vector of V. Thus, H is not maximal by Proposition 2.23.

■

Here we see the problem, and why Theorem 1.3 is a weaker result. The fusion information does not rule out the possibility that an $\Omega_8^-(2)$ subgroup of G contains a copy of $Sp_6(2)$ which follows fusion possibility 2.3 (vi). Moreover, this case of $\Omega_8^-(2)$ may not necessarily fix a non-zero vector of V, as we cannot invoke Proposition 2.7 here. We would require all $Sp_6(2)$ subgroups of G up to G-conjugacy which follow $Sp_6(2)$ fusion possibility (vi) to construct these $\Omega_8^-(2)$ subgroups, which we have yet been unable to accomplish. We will continue under the assumption, then, that $H \cong \Omega_8^-(2)$ follows fusion possibility (i) or (ii).

6.1 Constructing $\Omega_8^-(2)$ subgroups of G

Our strategy here is to build up from our $Sp_6(2)$ subgroups found in Proposition 5.4. Recall that there are three such copies of $Sp_6(2)$. The next two results will explore some information about $\Omega_8^-(2)$.

Lemma 6.2. *Suppose $K \cong Sp_6(2)$. Let $R \in \mathrm{Syl}_2(K)$. Then R has a unique class of subgroups represented by S of order 2^7 meeting the three following criteria.*

(i) S is normal in R;

(ii) $|N_K(S)| = 2^9.3$;

(iii) S is isomorphic to the group given by the intrinsic `SmallGroup(128,1755)`.

Proof. S is directly obtainable – and its uniqueness shown – by direct calculation in Magma. First, we use `Subgroups` to obtain all 75 subgroups of R of order 2^7. Sieving by class length, we find that there are 35 subgroups which have class length 1 (i.e. are normal in R); three of these have normaliser in K of order $2^9.3$; only one of these is isomorphic to `SmallGroup(128,1755)`. ■

Note that `SmallGroup` is an intrinsic function of Magma which uses the `SmallGroup` library – a catalogue of all groups of certain small orders. For our purposes, we need not know the exact structure of this group, we simply require a way of uniquely defining it within R.

Proposition 6.3. *Let $Sp_6(2) \cong K \leq H \cong \Omega_8^-(2)$ and let $S \leq K$ be the group defined in Lemma 6.2. Then there are four involutions $x \in C_H(S)$ for which $\langle K, x\rangle = H$ and $x \in 2A_H$.*

Proof. This is easily verifiable in Magma. ■

As usual, we will detail our strategy for employing this result in the context of G, which also doubles as an outline of the remainder of this chapter. Recall that we have three $Sp_6(2)$ subgroups of G up to G-conjugacy which follow fusion possibility 2.3 (ii) or (iii) (and hence each of these potentially exist as a subgroup of $\Omega_8^-(2)$ following fusion possibility 2.4 (i) or (ii) by Lemma 6.1). We will name these subgroups $K_j^{(2)}$ for $j \in \{1, 2\}$ which follow fusion possibility (ii), and $K_1^{(3)}$ which follows fusion possibility (iii). Now fix i and j, where $(i, j) \in \{(2, 1), (2, 2), (3, 1)\}$.

1. We identify $S_j^{(i)} \leq K_j^{(i)}$ where $S_j^{(i)}$ is as defined in Lemma 6.2.

2. We will calculate $\mathcal{I}(C_G(S_j^{(i)}))$.

3. Finally, we will sieve $\mathcal{I}(C_G(S_j^{(i)}))$ for involutions t such that $\langle K_j^{(i)}, t\rangle \cong \Omega_8^-(2)$.

The first step is a simple one – we obtain $S_j^{(i)}$ in Magma by following the same steps as in the proof of Lemma 6.2. We will follow the same strategy in every case of i and j, so, to simplify notation, let $K = K_j^{(i)}$ and $S = S_j^{(i)}$. In every case, we can choose $x \in Z(S) \cap 2\mathrm{D}_G$, and hence we can find $C_G(z) \sim [2^{84}] : Sp_8(2)$ using the command `CentraliserOfInvolution`. By Proposition 3.1 we know that $\mathcal{I}(C_G(S)) \subseteq C_G(z)$ and $S \leq C_G(z)$.

Now, we define $\mathcal{C}_1(S)$ as in Proposition 3.2: Let $C = C_G(z)$ and $\overline{C} = C/O_2(C) \cong Sp_8(2)$, then $\mathcal{C}_1(S)$ is the inverse image of $C_{\overline{C}}(\overline{S})$. By Proposition 3.2, we know that $\mathcal{I}(C_G(S)) \subseteq \mathcal{C}_1(S)$.

Next, we define $\mathcal{C}_2(S) = \langle \mathrm{Stab}_X(F) : X \in \mathrm{Syl}_2(\mathcal{C}_1(S))\rangle$, where $F = C_V(S)$, and by Proposition 3.6 we have that $\mathcal{I}(C_G(S)) \subseteq \mathcal{C}_2(S)$. Finally, let

$$\mathcal{C}_3(S) = C_{\mathcal{C}_2(S)}(S),$$

which is also clearly such that $\mathcal{I}(C_G(S)) \subseteq \mathcal{C}_3(S)$. The outcome of this process is explored in the following result.

Proposition 6.4. *The orders of $\mathcal{C}_k(S_j^{(i)})$ for $k \in \{1, 2, 3\}$ are given in Table 6.1.*

i	j	$\lvert\mathcal{C}_1(S_j^{(i)})\rvert$	$\lvert\mathcal{C}_2(S_j^{(i)})\rvert$	$\lvert\mathcal{C}_3(S_j^{(i)})\rvert$
2	1	$2^{96}.3.5$	$2^{35}.3$	$2^{25}.3$
2	2	2^{96}	2^{31}	2^{25}
3	1	$2^{98}.3^2.5$	$2^{42}.3^2$	$2^8.3$

Table 6.1: Orders of $\mathcal{C}_k(S_j^{(i)})$ for $k \in \{1, 2, 3\}$

Proof. We find $\mathcal{C}_1(S_j^{(i)})$ using Procedure 3.3; $\mathcal{C}_2(S_j^{(i)})$ using Procedure 3.7 (ii); and $\mathcal{C}_3(S_j^{(i)})$ directly by using `Centraliser`. ■

We now sieve $\mathcal{C}_3(S)$ for involutions t such that $\langle K, t\rangle \cong \Omega_8^-(2)$. We will employ the usual sieving techniques. The results are given in the following proposition.

Proposition 6.5. *If* $(i,j) = (2,2)$*, there is a unique (up to* G*-conjugacy) subgroup* $H_j^{(i)} \cong \Omega_8^-(2)$ *such that* $K_j^{(i)} \leq H_j^{(i)}$*. Otherwise, there does not exist an overgroup of* $K_j^{(i)}$ *isomorphic to* $\Omega_8^-(2)$*.*

Proof. For any i, j, if an overgroup of $K_j^{(i)}$ isomorphic to $\Omega_8^-(2)$ exists in G, then there is some $t \in \mathcal{C}_3(S_j^{(i)})$ such that $\langle K_j^{(i)}, t\rangle \cong \Omega_8^-(2)$. Hence we must sieve $\mathcal{C}_3(S_j^{(i)})$ for such involutions. Again, we will simplify notation by letting $K = K_j^{(i)}$ and $S = S_j^{(i)}$.

We start by letting $\mathcal{I}_0(K) = \mathcal{I}(\mathcal{C}_3(S))$, the set of all involutions in $\mathcal{C}_3(S)$. We find this in Magma by first turning $\mathcal{C}_3(S)$ into a pc-group using `LMGSolubleRadical`. We can do this as each $\mathcal{C}_3(S)$ is soluble by Burnside's Theorem (see Proposition 2.9).

Next, we run an order of random elements sieve on $\mathcal{I}_0(K)$. For full details on this sieve, see Procedure 3.8. Note that using this sieve requires a set of all possible element orders appearing in $\Omega_8^-(2)$, which is

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 17, 21, 30\}$$

as can be seen in the $\mathbb{ATLAS}$ [14]. We gather the surviving involutions in a set called $\mathcal{I}_1(K)$. Now we define

$$\mathcal{I}_2(K) = \{t \in \mathcal{I}_1(S) : |\langle K, t\rangle| = |\Omega_8^-(2)|\}$$

and finally $\mathcal{I}_3(K)$, a set of elements in $\mathcal{I}_2(K)$ which generate distinct $\Omega_8^-(2)$ subgroups. The results of this process are displayed in Table 6.2.

i	j	$\lvert\mathcal{I}_0(K_j^{(i)})\rvert$	$\lvert\mathcal{I}_1(K_j^{(i)})\rvert$	$\lvert\mathcal{I}_2(K_j^{(i)})\rvert$	$\lvert\mathcal{I}_3(K_j^{(i)})\rvert$
2	1	851,967	1	0	0
2	2	425,983	5	4	1
3	1	255	1	0	0

Table 6.2: Sizes of $|\mathcal{I}_k(K_j^{(i)})|$, for $k \in \{0, 1, 2, 3\}$

Now we simply observe that $\mathcal{I}_3(K_j^{(i)})$ contains one involution for $(i,j) = (2,2)$ and is empty otherwise. Hence the result follows. ■

We give this $\Omega_8^-(2)$ subgroup of G the name H and proceed to attempt to find overgroups in G isomorphic to $\mathrm{Aut}(\Omega_8^-(2)) \cong \Omega_8^-(2) : 2$.

6.2 Extending $\Omega_8^-(2)$ to $\Omega_8^-(2):2$

We begin by stating some facts about $\Omega_8^-(2):2$.

Proposition 6.6. *Suppose* $H \cong \Omega_8^-(2):2$ *and* $R \in \mathrm{Syl}_3(H)$ *with* $E \leq R$ *the unique elementary abelian subgroup of* R *of order* 3^3*. Let* $x, y \in E \cap 3A_H$ *with* $\langle x\rangle \neq \langle y\rangle$*. Now let* $\Omega_8^-(2) \cong K \leq H$ *and define*

$$D = \langle N_{C_H(x)}(E), N_{C_H(y)}(E)\rangle.$$

Then there is some involution $t \in D$ *such that* $\langle K, t\rangle = H$*.*

Proof. All of these facts can be verified in Magma. ∎

Because we have adjusted and simplified our notation as we have progressed through this thesis, let us summarise which groups we have in play. Recall that H is our sole copy of $\Omega_8^-(2)$ which were constructed from our $Sp_6(2)$ subgroups, which were constructed from $U_4(2)$ subgroups, which were constructed from elementary abelian subgroups of order 3^3 which were found and named in Lemma 4.4 in the remarks that followed. These were named $E_1^{(2)}$, which follows $U_4(2)$ fusion possibility (ii), and $E_1^{(6)}$, which follows $U_4(2)$ fusion possibility (vi). Note now that we have $E_1^{(2)} \leq H$.

In employing Proposition 6.6, we may take, without loss of generality, E to be $E_1^{(2)}$. Now, we must choose $x, y \in E \cap 3\mathrm{A}_{\Omega_8^-(2):2}$ with $\langle x\rangle \neq \langle y\rangle$ and construct

$$D = \langle N_{C_G(x)}(E), N_{C_G(y)}(E)\rangle.$$

But now note that H follows $\Omega_8^-(2)$ fusion possibility (ii), and hence we will be taking $x, y \in 3\mathrm{D}_G$. Hence D is the same group as constructed in Lemma 4.7. Now, as implied by Proposition 6.6, we must sieve D for involutions t such that $\langle H, t\rangle \cong \Omega_8^-(2):2$. We do this by first constructing $\mathcal{I}_0(D) = \mathcal{I}(D)$, the set of all involutions in D. Then, we note that if $t \in D$ such that $\langle H, t\rangle \cong \Omega_8^-(2):2$, then $t \notin H$ and $H^t = H$. Hence t can be found in

$$\mathcal{I}_1(D) := (\mathcal{I}_0(D) \setminus H) \cap N_G(H).$$

The next proposition states the results of this process.

Proposition 6.7. *We have* $|\mathcal{I}_0(D)| = 40,215$ *and* $|\mathcal{I}_1(D)| = 136$*.*

Proof. We calcuate $\mathcal{I}_0(D)$ by first turning D into a permutation group and sifting through its elements, keeping only the involutions. Then we run through the elements $t \in \mathcal{I}_0(D)$ and keep them only if they satisfy $t \notin H$ and $H^t = H$; we keep them in $\mathcal{I}_1(D)$. ■

Now we use the following simple result to drastically reduce the number of involutions we need to concern ourselves with.

Lemma 6.8. *Let $A = \langle \mathcal{I}_1(D) \rangle$. Then*

(i) $A \leq N_G(H)$;

(ii) If $t, s \in \mathcal{I}_1(D)$ such that t and s are conjugate in A, then $\langle H, t\rangle$ and $\langle H, s\rangle$ are conjugate groups.

Proof. (i) Recall that, by construction, for all $t \in \mathcal{I}_1(D)$, we have $H^t = H$. Hence any element $t \in A = \langle \mathcal{I}_1(D) \rangle$ will also be such that $H^t = H$.

(ii) Let $a \in A$ such that $t = s^a$. Then by (i) we have $H^a = H$ and therefore

$$\langle H, s\rangle^a = \langle H^a, s^a\rangle = \langle H, t\rangle,$$

as required.

■

We can now complete the process of constructing subgroups of G isomorphic to $\Omega_8^-(2) : 2$.

Proposition 6.9. *There is a unique (up to G-conjugacy) subgroup of G isomorphic to $\Omega_8^-(2) : 2$ containing H.*

Proof. Here, we pick up from where Proposition 6.7 left off, implementing Lemma 6.8 by constructing $A := \langle \mathcal{I}_1(D) \rangle$. Now, we let A act on $\mathcal{I}_1(D)$ by conjugation and let $\mathcal{I}_2(D)$ be a set of orbit representatives. We find $|\mathcal{I}_2(D)| = 6$ and, moreover, each $t \in \mathcal{I}_2(D)$ is such that $\langle H, t\rangle \cong \Omega_8^-(2) : 2$. However, all six involutions in $\mathcal{I}_2(D)$ produce the same copy of $\Omega_8^-(2) : 2$, so this subgroup is unique up to G-conjugacy. ■

In previous chapters, we have concluded with a proof that the subgroup in question – in this case, $\Omega_8^-(2)$ – is not maximal in G. However, due to the aforementioned limitations, we are unable to provide this for $\Omega_8^-(2)$. Still, we can rule out some cases of $\Omega_8^-(2)$ subgroups of G as being maximal.

Proposition 6.10. *Suppose $H_0 \leq G$ such that $F^*(H_0) \cong \Omega_8^-(2)$, and that $F^*(H_0)$ follows $\Omega_8^-(2)$ fusion possibility 2.4 (i), (ii), or (v). Then H_0 is not maximal in G.*

Proof. Let $\Omega_8^-(2) \cong H = F^*(H_0)$ and assume first that H follows $\Omega_8^-(2)$ fusion possibility (v). Then, by Proposition 2.7 we have that H fixes a non-zero vector of V. By Proposition 2.18, H_0 fixes a non-zero vector of V. Therefore H_0 is not maximal in G by Proposition 2.23.

Now assume that H follows $\Omega_8^-(2)$ fusion possibility (i) or (ii). Then, by construction, H is G-conjugate to the sole copy of $\Omega_8^-(2)$ found in Proposition 6.5. Therefore, H_0 is G-conjugate to the copy of $\Omega_8^-(2) : 2$ found in Proposition 6.9. Calculation in MAGMA reveals that $\dim C_V(H_0) = 2$. Thus, by Proposition 2.23, $H_0 : 2$ is not maximal. ■

We will now finish this chapter by proving Theorems 1.1, 1.2, and 1.3.

6.3 Proof of Theorems 1.1, 1.2, and 1.3

All the results we require to prove these three main results are provided in Chapters 4, 5, and 6. This short section exists to collate this information. We will proceed by first proving Theorem 1.1 (i), then by proving Theorem 1.1 (ii) and Theorem 1.2 at the same time. We will then conclude by proving Theorem 1.3.

Recall that Theorem 1.1 states that if $H_0 \leq G$ with $F^*(H_0) \cong U_4(2)$ and $F^*(H_0)$ not following $U_4(2)$ fusion possibility 2.2 (viii) or (ix), then there are three G-classes of subgroups $H_0 \cong U_4(2)$ and three classes of subgroups $H_0 \cong U_4(2) : 2$. Assume first that $H_0 \cong U_4(2)$. Then H_0 is G-conjugate to one of the three $U_4(2)$ subgroups constructed in Proposition 4.13. We shall call these $A_1^{(2)}$, $A_2^{(2)}$, and $A_1^{(6)}$, where $A_1^{(2)}$ and $A_2^{(2)}$ follow $U_4(2)$ fusion possibility 2.2 (ii) and $A_1^{(6)}$ follows $U_4(2)$ fusion possibility 2.2 (vi). Clearly, $A_1^{(6)}$ is G-conjugate to neither $A_1^{(2)}$ nor $A_2^{(2)}$, as it follows a different fusion pattern. To see that $A_1^{(2)}$ and $A_2^{(2)}$ are not G-conjugate, simply observe that, by Proposition 4.17, $A_1^{(2)}$ is contained in some copy of $U_4(2) : 2$ while $A_2^{(2)}$ is not. Hence, by Proposition 3.16, $A_1^{(2)}$ and $A_2^{(2)}$ are not G-conjugate. This proves Theorem 1.1 (i).

Now suppose that $H_0 \cong U_4(2):2$. Then, by Proposition 4.17, we have that H_0 is G-conjugate to one of three $U_4(2):2$ subgroups. Let us name these $B_1^{(2)}$, $B_2^{(2)}$, and $B_1^{(6)}$, where $F^*(B_1^{(2)})$ and $F^*(B_2^{(2)})$ follow $U_4(2)$ fusion possibility 2.2 (ii) and $F^*(B_1^{(6)})$ follows $U_4(2)$ fusion possibility 2.2 (vi).

Let us now remind ourselves of the statement of Theorem 1.2. It states that there are three classes of subgroups of G isomorphic to $Sp_6(2)$ which follow $Sp_6(2)$ fusion possibilities 2.3 (ii) or (iii). As shown in Proposition 5.4, we have at most three such classes of $Sp_6(2)$ subgroups. Let us call these $C_1^{(2)}$, $C_2^{(2)}$, and $C_1^{(6)}$ where $B_j^{(i)} \leq C_j^{(i)}$ for each $(i,j) \in \{(2,1),(2,2),(6,1)\}$. Again, it is clear due to fusion patterns that $B_1^{(6)}$ is G-conjugate to neither $B_1^{(2)}$ nor $B_2^{(2)}$, and that $C_1^{(6)}$ is G-conjugate to neither $C_1^{(2)}$ nor $C_2^{(2)}$. We will now show that $B_1^{(2)}$ is not G-conjugate to $B_2^{(2)}$, and that $C_1^{(2)}$ is not G-conjugate to $C_2^{(2)}$. Here, we simply observe that by Proposition 6.5 we have $B_2^{(2)}$ and $C_2^{(2)}$ contained in some $\Omega_8^-(2)$ subgroup of G, while there is no $\Omega_8^-(2)$ subgroup containing $C_1^{(2)}$ or $B_1^{(2)}$. Thus, by Proposition 3.16, $B_1^{(2)}$ and $B_2^{(2)}$ are not G-conjugate, and $C_1^{(2)}$ and $C_2^{(2)}$ are not G-conjugate. This proves Theorem 1.1 (ii) and Theorem 1.2.

Finally, we must prove Theorem 1.3. This states that if $H_0 \leq G$ such that $F^*(H_0) \cong \Omega_8^-(2)$ and $F^*(H_0)$ follows $\Omega_8^-(2)$ fusion possibility 2.4 (i) or (ii), then there is one G-class of subgroups $H_0 \cong \Omega_8^-(2)$ and one class of subgroups $H_0 \cong \Omega_8^-(2):2$. This follows directly from Propositions 6.5 and 6.9, where we construct all such $\Omega_8^-(2)$ and $\Omega_8^-(2):2$ subgroups. ∎

Chapter 7

$\Omega_8^+(2)$ and Its Extensions

In this chapter, we will prove Theorem 1.4, showing that there are seven conjugacy classes of subgroups of $E_8(2)$ isomorphic to $\Omega_8^+(2)$. Moreover, we will show that if $\Omega_8^+(2) \cong F^*(H) \leq E_8(2)$, then H is not maximal. As always let $G \cong E_8(2)$. This case contains the longest and most complicated set of results in this thesis. This is because, usually, we eliminate the cases where our chosen group fixes a vector at the start of the chapter and focus on constructing all subgroups up to G-conjugacy which follow the other fusion possibilities. We do not do this with $\Omega_8^+(2)$. Instead, we proceed to find *all* $\Omega_8^+(2)$ subgroups in G up to conjugacy, yielding a stronger result.

Suppose now that $\Omega_8^+(2) \cong H \leq G$. From the $\mathbb{ATLAS}$ [14], we know that $|H| = 2^{12}.3^5.5^2.7$. Our starting point for constructing H in G is with its Sylow 5-subgroup. As we know, a Sylow 5-subgroup of G has order 5^5. We also have that a Sylow 5-subgroup of H is isomorphic to 5^2. Hence our first point of attack is to find all the 5^2 subgroups in G up to conjugacy.

7.1 Constructing $\Omega_8^+(2)$ subgroups of G

Let us begin by compiling some facts about $\Omega_8^+(2)$. As usual, we choose a specific way of generating of H in G and exhaust all possible cases for how H can be generated, thus finding all possible $\Omega_8^+(2)$ subgroups up to G-conjugacy.

Proposition 7.1. *Let $H \cong \Omega_8^+(2)$, $P \in \mathrm{Syl}_5(H)$, and $T \in \mathrm{Syl}_2(N_H(P))$. Then the following hold:*

(i) P is elementary abelian of order 5^2;

(ii) $N_H(P) = PT$ *where* $T \cong \mathrm{Dih}(8) \circ \mathbb{Z}_4$, $T \cap C_H(P) = 1$, *and all elements of order 4 in* T *are* H*-conjugate;*

(iii) there exists $x \in C_H(T) \setminus T$ *such that* x *is an involution with* $H = \langle PT, x \rangle$, *and* x *is* H*-conjugate to the unique involution in* $Z(T)$.

Proof. These facts are easily verified using Magma and the intrinsic copy of $\Omega_8^+(2)$ given by the command `POmegaPlus(8,2)`. ■

Now, we must employ this result in the setting of G. To do this, we follow the procedure below, which also outlines the structure of this chapter:

1. We will find $P_1, \ldots, P_m$ such that $5^2 \cong P_i \leq G$, a complete list of G-class representatives of subgroups of G isomorphic to 5^2.
2. For each $i \in \{1, \ldots, m\}$, we will calculate $N_G(P_i)$ and $T_1^{(i)}, \ldots, T_{n_i}^{(i)}$, a complete list of $N_G(P_i)$-class representatives of subgroups isomorphic to $\mathrm{Dih}(8) \circ \mathbb{Z}_4$.
3. For each $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n_i\}$, we will construct and sieve $\mathcal{I}(C_G(T_j^{(i)}))$ for involutions x such that $\langle PT, x \rangle \cong \Omega_8^+(2)$.

As stated at the beginning of this chapter, we will start by finding all 5^2 subgroups of G up to G-conjugacy. This will be completed in the following four results.

Lemma 7.2. *There are at most six subgroups isomorphic to* 5^2 *in* G *up to* G*-conjugacy.*

Proof. Since we are looking for 5^2 subgroups up to G-conjugacy, by Sylow's Theorems, every 5^2 can be found in a Sylow 5-subgroup of G and, moreover, that all Sylow 5-subgroups of G are conjugate. It follows that we can choose any $S \in \mathrm{Syl}_5(G)$ and find all 5^2 subgroups of S up to G-conjugacy. Hence we choose $S \in \mathrm{Syl}_5(G)$ (see Appendix A for a copy of this Sylow 5-subgroup). It can be verified in Magma that S contains a unique subgroup $E \cong 5^4$. Let $N = N_G(E)$, and since $S \leq N$ (as E is normal in S) we will actually find all the 5^2 subgroups of N up to N-conjugacy. We note here that our unpublished paper [5] contains more details regarding the construction of S and $N_G(E)$.

To see why we do this, we should note that when finding subgroups, we use the `Subgroups` command which finds all the subgroups of a given group up to conjugacy within that group. This usually means we would like to execute `Subgroups` on the largest group we are able to – a larger group tends to have a fewer number of subgroups of a given order up to conjugacy, since there are more elements with which to conjugate the subgroups together. However, the larger the group, the longer it takes MAGMA to execute `Subgroups`. Too large, and the command will not work at all. Hence we must find a balance. Finding subgroups of S will result in too many 5^2 subgroups to manage, so we work in N.

Since we are looking for elementary abelian subgroups of order 5^2, we execute

```
ElementaryAbelianSubgroups(N : OrderEqual:=25)
```

which yields six subgroups isomorphic to 5^2 up to conjugacy in N, as required. We remark that some of these groups may yet be conjugate under the action of G. ■

For the rest of this thesis, let us name these groups P_i for $i \in \{1, \ldots, 6\}$. As alluded to at the end of the last proof, some of these groups might, in fact, be conjugate. The next lemma provides us with a clue as to which may be conjugate, as well as allowing us to eliminate one of our subgroups entirely from our list of potential Sylow 5-subgroups of an $\Omega_8^+(2)$ subgroup.

Lemma 7.3. *The numbers of elements in P_i belonging to $5\mathrm{A}_G$ and $5\mathrm{B}_G$, for each $i \in \{1, \ldots, 6\}$, are given in Table 7.1. Moreover, P_6 cannot exist as a Sylow 5-subgroup of an $\Omega_8^+(2)$ subgroup of G.*

i	$\lvert P_i \cap 5\mathrm{A}_G \rvert$	$\lvert P_i \cap 5\mathrm{B}_G \rvert$
1	24	0
2	0	24
3	8	16
4	0	24
5	0	24
6	12	12

Table 7.1: Element structure of $P_1, \ldots, P_6$

Proof. By Theorem 2.20 we know that for all $g \in G$,

$$g \in 5\mathrm{A}_G \text{ if and only if } o(g) = 5 \text{ and } \dim C_V(g) = 68,$$

$$g \in 5\mathrm{B}_G \text{ if and only if } o(g) = 5 \text{ and } \dim C_V(g) = 48,$$

where V is the 248-dimensional $GF(2)$ G-module. Hence, in MAGMA, for each $i \in \{1, \ldots, 6\}$, we can run through all elements $p \in P_i$ of order 5 and carry out the command:

```
Dimension(Eigenspace(p,1)).
```

This provides us with $\dim C_V(p)$ for each $p \in P_i$ of order 5, thereby enabling us to identify which G-class of elements of order 5 that p belongs to.

Now, if $P \in \mathrm{Syl}_5(H)$ where $H \cong \Omega_8^+(2)$, we have that P contains 8 elements in $5\mathrm{A}_H$, 8 elements in $5\mathrm{B}_H$, and 8 elements in $5\mathrm{C}_H$. Regardless of how the H-classes of elements of order 5 fuse in G, it is not possible for them to fall into the pattern seen in P_6. Thus we may eliminate P_6 as a potential Sylow 5-subgroup of an $\Omega_8^+(2)$ subgroup of G. ■

Since three of these groups, P_2, P_4, and P_5, have the same arrangement of elements of order 5, it is natural to wonder whether they are conjugate in G. This is explored in the next two results, which will pin down the exact number of classes of 5^2 subgroups in G. First, we calculate $C_G(P_i)$ for $i \in \{1, \ldots, 5\}$.

Proposition 7.4. *The order of $C_G(P_i)$, for each $i \in \{1, \ldots, 5\}$, is displayed in Table 7.2. Moreover, $C_G(P_i) \cong 5^4$ for $i \in \{2, 4, 5\}$.*

i	$\lvert C_G(P_i)\rvert$
1	$2^{12}.3^5.5^4.7$
2	5^4
3	$2^4.3^2.5^4$
4	5^4
5	5^4

Table 7.2: Orders of $C_G(P_i)$, $i \in \{1, \ldots, 5\}$

Proof. Let $i \in \{1, \ldots, 5\}$. Due to the size of G, it is not practical to use MAGMA to calculate $C_G(P_i)$ directly. Instead, we choose $x_i, y_i \in P_i$ such that $\langle x_i, y_i \rangle = P_i$. Our strategy is to use the fact that $C_G(P_i) = C_{C_G(x_i)}(y_i)$. We break this proof up into two cases, since calculating $C_G(P_i)$ depends on which G-class our generators can be chosen from.

(i) $i = 1$: In this case, $x_1, y_1 \in 5\mathrm{A}_G$. We use the MAGMA procedure `FindCent` (see B.3) to construct $C_G(x_1)$. We know from Theorem 2.20 that $C_G(x_1) \cong$

$5\times\Omega_{12}^-(2)$. Now we use the command `LMGRadicalQuotient` to obtain the soluble radical $R \cong \Omega_{12}^-(2)$ as a permutation group, the natural homomorphism $\varphi : C_G(x_1) \to R$, and $K = \ker\varphi \cong 5$. Then $C_G(P_i) = \langle \varphi^{-1}(C_R(\varphi(y_1))), k\rangle$ where $k \in K^{\#}$. As R is a permutation group, MAGMA can easily calculate this.

(ii) $i \in \{2,3,4,5\}$: In these cases, we can select x_i, y_i such that $x_i, y_i \in 5\mathrm{B}_G$. Again, we use the `FindCent` procedure to find $C_G(x_i)$, which we know from Theorem 2.20 is isomorphic to $SU_5(4)$. This group is sufficiently small that we may calculate $C_{C_G(x_i)}(y_i)$ directly in MAGMA using the command `LMGCentraliser`.

Finally, let $i \in \{2,4,5\}$. We see that $C_G(P_i) \cong 5^4$ by using the MAGMA command `IsElementaryAbelian` on these groups. ■

Proposition 7.5. *There are exactly four subgroups isomorphic to 5^2 in G up to G-conjugacy.*

Proof. From Lemma 7.2 we know there are at most six distinct G-classes of subgroups isomorphic to 5^2 in G, and by Lemma 7.3 we see that there are at least four distinct classes. Indeed, P_1, P_2, P_3, and P_6 represent distinct classes – they contain distinct combinations of elements from $5\mathrm{A}_G$ and $5\mathrm{B}_G$, so they cannot possibly be conjugate. To prove these are, in fact, the four classes, we will show that P_2, P_4, and P_5 are actually G-conjugate.

First we observe by direct calculation that $P_2 \leq E$, where E is the unique elementary abelian subgroup of order 5^4 in S, our fixed Sylow-5 subgroup of G. Now let $i \in \{4,5\}$. We will show that $C_G(P_2)$ is G-conjugate to $C_G(P_i)$. From Proposition 7.4 we know that $C_G(P_2) \cong C_G(P_i) \cong 5^4$. Also, there exists $S_i \in \mathrm{Syl}_5(G)$ such that $C_G(P_i) \leq S_i$. Clearly, as $P_2 \leq E$, we have $C_G(P_2) = E \leq S$. Recall that from the proof of Lemma 7.2, we found that any Sylow-5 subgroup of G contains a unique elementary abelian subgroup of order 5^4. Therefore, $C_G(P_2)$ and $C_G(P_i)$ are the unique elementary abelian subgroups of order 5^4 inside S and S_i respectively.

Now, by Sylow's theorems, we can find $g \in G$ such that $S_i^g = S$ and because of their uniqueness, we must have $C_G(P_i)^g = C_G(P_2)$ which implies $C_G(P_i^g) = C_G(P_2) = E$ using Proposition 2.8, and this implies that $P_i^g \leq E$. Now, using MAGMA we find that E has 806 elementary abelian subgroups of order 5^2, where 96 of them have all of their elements of order 5 inside $5\mathrm{B}_G$. However, we find that all 96 of these are actually

N-conjugate. Since $P_i^g, P_2 \leq E$, both of which contain only elements of order 5 inside $5\mathrm{B}_G$, they are N-conjugate. Thus, there is some $n \in N$ such that $P_i^{gn} = P_2$, so P_i and P_2 are G-conjugate, as required. ∎

Hence we will proceed with only P_1, P_2, and P_3. We now carry out the next step: calculating $N_G(P_i)$ for each $i \in \{1, 2, 3\}$.

Lemma 7.6. *Let $i \in \{1, 2, 3\}$. Then the following holds:*

$$N_G(P_i) = N_N(P_i)C_G(P_i).$$

Proof. We will prove this equality by showing that both groups are contained in each other.

$\supseteq$: Suppose $g \in N_N(P_i)C_G(P_i)$. Then $g = nc$ for some $n \in N_N(P_i)$ and $c \in C_G(P_i)$. Now:

$$P_i^g = P_i^{nc} = (P_i^n)^c = P_i^c = P_i.$$

Hence $g \in N_G(P_i)$.

$\subseteq$: Now suppose $g \in N_G(P_i)$. First, we show that $E, E^g \leq C_G(P_i)$. To see this, recall that E is abelian and note that by direct calculation we know that $P_i \leq E$. Therefore for all $u \in E$ and for all $p \in P_i$, we have that $up = pu$ and hence $E \leq C_G(P_i)$. Now if we take some $u^g \in E^g$ we have:

$$\begin{aligned}
u^g P_i &= g^{-1}ugP_i \\
&= g^{-1}ugg^{-1}P_ig && \text{as } P_i^g = P_i \\
&= g^{-1}uP_ig \\
&= g^{-1}P_iug && \text{as } u \in C_G(P_i) \\
&= g^{-1}P_igg^{-1}ug \\
&= P_iu^g && \text{again, as } P_i^g = P_i.
\end{aligned}$$

Hence $u^g \in C_G(P_i)$ and so $E^g \leq C_G(P_i)$.

Now, since $E, E^g \in \mathrm{Syl}_5(C_G(P_i))$ (which can be verified by considering $|C_G(P_i)|$ given in Lemma 7.4), we can find $c \in C_G(P_i)$ such that $E^c = E^g$. Then $E^{gc^{-1}} = E$ which implies $gc^{-1} \in N_G(E) = N$. Take $gc^{-1} = n$ for some $n \in N$. Finally, observe that $P_i^n = P_i^{gc^{-1}} = P_i^{c^{-1}} = P_i$ and so $n \in N_N(P_i)$. Therefore, $g = nc$ for some $n \in N_N(P_i)$ and $c \in C_G(P_i)$, as required.

Both are subsets of one another, so the result holds. ■

Proposition 7.7. *The orders of $N_N(P_i)$ and $N_G(P_i)$ for each $i \in \{1,2,3\}$ are displayed in Table 7.3.*

i	$\lvert N_N(P_i)\rvert$	$\lvert N_G(P_i)\rvert$
1	$2^9.3.5^4$	$2^{17}.3^6.5^4.7$
2	$2^5.3.5^5$	$2^5.3.5^5$
3	$2^7.5^4$	$2^9.3^2.5^4$

Table 7.3: Orders of $N_N(P_i)$ and $N_G(P_i)$ for $i \in \{1,2,3\}$

Proof. Since N is a fairly small group, we can turn it into a permutation group and calculate $N_N(P_i)$ directly. Now, $C_G(P_i)$ was calculated in Proposition 7.4. Hence by Lemma 7.6, we can calculate $N_G(P_i) = N_N(P_i)C_G(P_i)$. ■

Our task is now to find all $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups of $N_G(P_i)$ for $i \in \{1,2,3\}$. We will show, first, that we need only find such subgroups up to conjugacy in $N_G(P_i)$.

Lemma 7.8. *Let $i \in \{1,2,3\}$. Suppose we have $T_1, T_2 \leq N_G(P_i)$ such that $T_1 \cong \mathrm{Dih}(8) \circ \mathbb{Z}_4 \cong T_2$ and T_1 and T_2 are $N_G(P_i)$-conjugate. Then for any $x \in C_G(T_1)$ such that $\langle PT_1, x\rangle \cong \Omega_8^+(2)$, there exists $y \in C_G(T_2)$ such that $\langle PT_2, y\rangle \cong \Omega_8^+(2)$ and $\langle PT_1, x\rangle$ and $\langle PT_2, y\rangle$ are conjugate groups. Conversely, for all $y \in C_G(T_2)$ for which $\langle PT_2, y\rangle \cong \Omega_8^+(2)$, there exists $x \in C_G(T_1)$ such that $\langle PT_1, x\rangle \cong \Omega_8^+(2)$ and $\langle PT_2, y\rangle$ and $\langle PT_1, x\rangle$ are conjugate groups.*

Proof. We have that T_1 and T_2 are $N_G(P_i)$-conjugate, so let $n \in N_G(P_i)$ such that $T_1^n = T_2$. Assume we have $x \in C_G(T_1)$ such that $\langle PT_1, x\rangle \cong \Omega_8^+(2)$. Then take $y = x^n$. Then $y = x^n \in C_G(T_1)^n = C_G(T_1^n) = C_G(T_2)$ so $y \in C_G(T_2)$. Moreover, we have

$$\langle P_iT_1, x\rangle^n = \langle P_i^nT_1^n, x^n\rangle = \langle P_iT_2, y\rangle$$

so $\langle PT_1, x\rangle$ and $\langle PT_2, y\rangle \cong \Omega_8^+(2)$ are conjugate groups. A similar argument holds for the converse statement. ■

We will now collect some more facts about the $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups we need. Suppose $\mathrm{Dih}(8) \circ \mathbb{Z}_4 \cong T \leq N_G(P_i)$ is a valid subgroup. Then, from Proposition 7.1 (ii), we know that all the elements of order 4 must be G-conjugate. Furthermore, we have that $T \cap C_G(P_i) = 1$. We will now find *all* subgroups of $N_G(P_i)$ isomorphic to

$\mathrm{Dih}(8) \circ \mathbb{Z}_4$ up to conjugacy in $N_G(P_i)$, and then eliminate the ones not satisfying those criteria.

Proposition 7.9. *Let* $i \in \{1,2,3\}$. *Then the number of classes of viable* $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ *subgroups of* $N_G(P_i)$ *is shown in Table 7.4.*

i	$\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups in $N_G(P_i)$
1	351
2	1
3	4

Table 7.4: Number of classes of viable $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups of $N_G(P_i)$ for $i \in \{1,2,3\}$

Proof. Fix $i \in \{1,2,3\}$. Since we are only interested in subgroups of $N_G(P_i)$ up to $N_G(P_i)$-conjugacy, we start by taking $R_i \in \mathrm{Syl}_2(N_G(P_i))$. Firstly, we turn R_i into a permutation group and run the `Subgroups` command to find, up to R_i-conjugacy, all the subgroups of $N_G(P_i)$ of order 16. We store these subgroups in a set called $\mathcal{T}_0(P_i)$. Secondly, we build $\mathcal{T}_1(P_i) = \{T \in \mathcal{T}_0(P_i) : T \cong \mathrm{Dih}(8) \circ \mathbb{Z}_4\}$. Note that the intrinsic MAGMA command `SmallGroup(16,13)` is a group isomorphic to $\mathrm{Dih}(8) \circ \mathbb{Z}_4$, so to build $\mathcal{T}_1(P_i)$ we test whether the groups in $\mathcal{T}_0(P_i)$ are isomorphic to `SmallGroup(16,13)` using `IsIsomorphic`. Thirdly, we construct

$$\mathcal{T}_2(P_i) = \{T \in \mathcal{T}_1(P_i) : T \cap C_G(P_i) = 1\},$$

which must contain the groups we desire by the remarks preceding this result. Similarly, take only the groups $T \in \mathcal{T}_2(P_i)$ for which all the elements of order 4 in T belong to the same G-conjugacy class. We store these in a set called $\mathcal{T}_3(P_i)$. Finally, we let $N_G(P_i)$ act on $\mathcal{T}_3(P_i)$ by conjugation and collect a set of orbit representatives in our last set, $\mathcal{T}_4(P_i)$. Note that we do this using Procedure B.4. This last step ensures we are only collecting subgroups up to conjugacy in the whole of $N_G(P_i)$, not just conjugacy in R_i. The results of this process are displayed in Table 7.5. Looking at the rightmost column, we obtain the numbers stated in the result.

■

Here is where things become tricky, as there are 351 subgroups of $N_G(P_1)$ to deal with. Most of these subgroups will undergo the same process as with the subgroups in $N_G(P_2)$ and $N_G(P_3)$. Before we reach that stage, let us organise our various cases.

i	$\lvert\mathcal{T}_0(P_i)\rvert$	$\lvert\mathcal{T}_1(P_i)\rvert$	$\lvert\mathcal{T}_2(P_i)\rvert$	$\lvert\mathcal{T}_3(P_i)\rvert$	$\lvert\mathcal{T}_4(P_i)\rvert$
1	1,081,838	336,548	67,828	32,084	351
2	3	1	1	1	1
3	431	4	4	4	4

Table 7.5: Size of $\mathcal{T}_k(P_i)$ for $k \in \{0, 1, 2, 3, 4\}$

For $i \in \{1, 2, 3\}$, let $\mathcal{T}^{(i)}$ be the set of viable subgroups of $N_G(P_i)$ isomorphic to $\mathrm{Dih}(8) \circ \mathbb{Z}_4$. (So, using the notation from the latest result, $\mathcal{T}^{(i)} = \mathcal{T}_4(P_i)$.) Then $|T^{(1)}| = 351$, $|T^{(2)}| = 4$, and $|T^{(3)}| = 1$. Now recall that for each $T \in T^{(i)}$, we must find and sieve $\mathcal{I}(C_G(T))$. The starting point for this process will be to find $C_G(z)$, where z is the unique involution in $Z(T)$. Since many of the groups in $\mathcal{T}^{(1)}$ share a common central involution, we will break up $\mathcal{T}^{(1)}$ into subsets corresponding to groups with a central involution in common.

Formally speaking, we will define the relation $\sim$ on $\mathcal{T}^{(1)}$ as follows. Given $T_1, T_2 \in \mathcal{T}^{(1)}$, we have

$$T_1 \sim T_2 \text{ if and only if } Z(T_1) \cap Z(T_2) \neq 1. \tag{7.1}$$

First, let us prove this is an equivalence relation.

- Reflexivity: If $T \in \mathcal{T}^{(1)}$, then clearly we have $Z(T) \cap Z(T) \neq 1$, so $T \sim T$ and the relation is reflexive.

- Symmetry: If $T_1, T_2 \in \mathcal{T}^{(1)}$ and $T_1 \sim T_2$, then $Z(T_1) \cap Z(T_2) \neq 1$ and so $Z(T_2) \cap Z(T_1) \neq 1$, hence $T_2 \sim T_1$ and $\sim$ is symmetric.

- Transitivity: Suppose $T_1, T_2, T_3 \in \mathcal{T}^{(1)}$ with $T_1 \sim T_2$ and $T_2 \sim T_3$. Here we must use the fact that $Z(\mathrm{Dih}(8) \circ \mathbb{Z}_4) \cong \mathbb{Z}_4$. Because of this, we know that the condition $Z(T_1) \cap Z(T_2) \neq 1$ implies that there is a common central *involution* to T_1 and T_2. Indeed, suppose $1 \neq t \in Z(T_1) \cap Z(T_2)$. Then $o(t) = 2$ or $o(t) = 4$. If $o(t) = 2$, we are done, so instead suppose that $o(t) = 4$. But as $Z(T_1) \cap Z(T_2)$ is a group, $t^2 \in Z(T_1) \cap Z(T_2)$ and $o(t^2) = 2$. Now let t be the unique central involution of T_2. Since $Z(T_1) \cap Z(T_2) \neq 1$, we must have $t \in T_1$. Likewise, as $Z(T_2) \cap Z(T_3) \neq 1$, we must also have $t \in T_3$. Thus, $t \in Z(T_1) \cap Z(T_3)$ and so $T_1 \sim T_3$ and $\sim$ is transitive.

Hence $\sim$ induces a partition on $\mathcal{T}^{(1)}$. We explore the consequences of this in the next result.

Lemma 7.10. *When we partition $\mathcal{T}^{(1)}$ according to the relation described in (7.1), we form five subsets $\mathcal{T}^{(1,1)}, \dots, \mathcal{T}^{(1,5)}$. That is, for each $i \in \{1,2,3,4,5\}$, $\mathcal{T}^{(1,i)}$ is a set of groups that share a common central involution. Let $z^{(1,i)}$ be the central involution common to all groups belonging to $\mathcal{T}^{(1,i)}$. Information about these subsets is given in Table 7.6.*

i	$\lvert\mathcal{T}^{(1,i)}\rvert$	G-class of $z^{(1,i)}$
1	4	2D
2	10	2D
3	10	2D
4	200	2D
5	127	2C

Table 7.6: Sizes of $\mathcal{T}^{(1,i)}$ and G-class of $z^{(1,i)}$, for $i \in \{1,2,3,4,5\}$

Proof. These sets are easily constructed in Magma. The G-class that $z^{(1,i)}$ belongs to can be ascertained using the `Eigenspace` command and using the results in Proposition 2.19. ■

To begin, let $T \in \mathcal{T}^{(1,1)} \cup \mathcal{T}^{(1,2)} \cup \mathcal{T}^{(1,3)} \cup \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$ and let $z \in Z(T)$ be the unique central involution in T. Then $z \in 2\mathrm{D}_G$, so we can use the command

`CentraliserOfInvolution`

to calculate $C_G(z)$ in Magma which, by Proposition 2.19, has shape $[2^{84}] : Sp_8(2)$. Now define $\mathcal{C}_1(T)$ to be the group defined in Proposition 3.2, which, by the same result, contains $\mathcal{I}(C_G(T))$. That is, if we let $C = C_G(z)$ and $\overline{C} = C/O_2(C) \cong Sp_8(2)$, then $\mathcal{C}_1(T)$ is the inverse image of $C_{\overline{C}}(\overline{T})$ in C.

Next, let $\mathcal{C}_2(T) = \langle \mathrm{Stab}_X(F) : X \in \mathrm{Syl}_2(\mathcal{C}_1(T)) \rangle$ where $F = C_V(T)$. By Proposition 3.6 we have that $\mathcal{I}(C_G(T)) \subseteq \mathcal{C}_2(T)$. Next, we let $\mathcal{C}_3(T) = C_{\langle \mathcal{C}_2(T), T \rangle}(T)$, which will clearly contain $\mathcal{I}(C_G(T))$. The results of this process are explored in the next result.

Let us introduce some more notation. Let $T_j^{(2)}$ for $j \in \{1,2,3,4\}$ be the four groups belonging to the set $\mathcal{T}^{(2)}$; let $T_1^{(3)}$ be the group belonging to $\mathcal{T}^{(3)}$; and let $T_j^{(1,i)}$ be a group belonging to $\mathcal{T}^{(1,i)}$, where $i \in \{1,\dots,5\}$ and $j \in \{1,\dots,|\mathcal{T}^{(1,i)}|\}$. For now, we will now focus on the $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups belonging to $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$, and $\mathcal{T}^{(1,i)}$, where $i \in \{1,2,3\}$. We will leave the $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups belonging to $\mathcal{T}^{(1,4)}$ and $\mathcal{T}^{(1,5)}$ for later.

Proposition 7.11. *The sizes of the sets $\mathcal{C}_k(T_j^{(i)})$ for every subgroup $T_j^{(i)} \in \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$, $k \in \{1,2,3\}$, are shown in Table 7.7. The sizes of $\mathcal{C}_k(T_j^{(i)})$ for each $T_j^{(1,i)} \in \mathcal{T}^{(1,i)}$ where $i \in \{1,2,3\}$ and $k \in \{1,2,3\}$, are shown in Table 7.8.*

i	j	$\mathcal{C}_1(T_j^{(i)})$	$\mathcal{C}_2(T_j^{(i)})$	$\mathcal{C}_3(T_j^{(i)})$
2	1	2^{92}	2^{23}	2^{17}
3	1	$2^{91}.3$	$2^{25}.3$	$2^{20}.3$
3	2	2^{93}	2^{24}	2^{19}
3	3	2^{91}	2^{23}	2^{14}
3	4	2^{91}	2^{23}	2^{14}

Table 7.7: Sizes of $\mathcal{C}_k(T)$ for $k \in \{1,2,3\}$ and $T \in \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$

Proof. We use Procedure 3.3 to calculate $\mathcal{C}_1(T_j^{(i)})$, Procedures 3.7 (i) and (ii) to find $\mathcal{C}_2(T_j^{(i)})$ (note we use (i) when $\mathcal{C}_1(T_j^{(i)})$ is a 2-group and (ii) otherwise), and we calculate $\mathcal{C}_3(T_j^{(i)})$ directly by turning $\langle \mathcal{C}_2(T_j^{(i)}), T_j^{(i)} \rangle$ into a pc-group using `LMGSolubleRadical` (which we can do since, in all cases, $\langle \mathcal{C}_2(T_j^{(i)}), T_j^{(i)} \rangle$ is soluble by Burnside's Theorem – see Proposition 2.9). ■

Before sieving $\mathcal{C}_3(T)$ for $T \in \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)} \cup \mathcal{T}^{(1,i)}$, $i \in \{1,2,3\}$, we look to construct $\mathcal{C}_3(T)$ for $T \in \mathcal{T}^{(1,4)}$. We do this now because the sieving process will be the same for both sets of groups. However, we encounter a significant hurdle with these cases. Suppose z is the central involution common to every $T \in \mathcal{T}^{(1,4)}$ and let $C = C_G(z)$ with $\overline{C} = C/O_2(C) \cong Sp_8(2)$. Now, for every $T \in \mathcal{T}^{(1,4)}$, we have that $T \leq O_2(C)$. This is a problem because when we find $\mathcal{C}_1(T)$, the full inverse image of $C_{\overline{C}}(\overline{T})$, we will obtain $\mathcal{C}_1(T) = C$. This is because, as $T \leq O_2(C)$, then we have that $\overline{T} = \overline{1}$ and hence $C_{\overline{C}}(\overline{T}) = \overline{C}$. In other words, the first step in our usual routine of whittling down C to a manageable group fails. Moreover, our usual procedures provided in Procedure 3.7 fail us in trying to find $\mathcal{C}_2(T) = \langle \mathrm{Stab}_S(F) : S \in \mathrm{Syl}_2(\mathcal{C}_1(T)) \rangle$, where $F = C_V(T)$. This is because, usually, we find $S \in \mathrm{Syl}_2(\mathcal{C}_1(T))$ and a transversal R of S in $\mathcal{C}_1(T)$. Then we have $\mathcal{C}_2(T) = \langle \mathrm{Stab}_{S^r}(F) : r \in R \rangle$. However, in this case, since $\mathcal{C}_1(T) = C$, we have $|C| = 2^{100}.3^5.5^2.7.17$ and therefore the transversal R would have

i	j	$\lvert\mathcal{C}_1(T_j^{(1,i)})\rvert$	$\lvert\mathcal{C}_2(T_j^{(1,i)})\rvert$	$\lvert\mathcal{C}_3(T_j^{(1,i)})\rvert$
1	1	2^{93}	2^{24}	2^{19}
1	2	2^{93}	2^{24}	2^{19}
1	3	2^{93}	2^{24}	2^{19}
1	4	2^{93}	2^{24}	2^{19}
2	1	2^{89}	2^{20}	2^{17}
2	2	2^{89}	2^{20}	2^{17}
2	3	2^{89}	2^{20}	2^{17}
2	4	2^{89}	2^{20}	2^{17}
2	5	2^{89}	2^{21}	2^{16}
2	6	2^{89}	2^{20}	2^{17}
2	7	2^{89}	2^{21}	2^{17}
2	8	2^{89}	2^{20}	2^{17}
2	9	$2^{93}.3$	2^{25}	2^{22}
2	10	$2^{93}.3$	2^{28}	2^{20}
3	1	2^{93}	2^{24}	2^{18}
3	2	$2^{93}.3$	2^{27}	2^{20}
3	3	2^{92}	2^{23}	2^{18}
3	4	2^{92}	2^{22}	2^{18}
3	5	2^{93}	2^{23}	2^{18}
3	6	2^{93}	2^{23}	2^{18}
3	7	2^{93}	2^{23}	2^{18}
3	8	$2^{93}.3$	2^{27}	2^{20}
3	9	$2^{93}.3$	$2^{28}.3$	$2^{25}.3$
3	10	$2^{93}.3$	2^{28}	2^{25}

Table 7.8: Sizes of $\mathcal{C}_k(T)$ for $k \in \{1,2,3\}$ and $T \in \mathcal{T}^{(1,i)}$, for $i \in \{1,2,3\}$

size $3^5.5^2.7.17 = 722,925$, which is too large for our usual method of calculating $\mathcal{C}_2(T)$ to be feasible. To combat this, we develop a new method using the next result.

Lemma 7.12. *Let $\overline{S} \in \mathrm{Syl}_2(\overline{C})$ and $R = \{\overline{r}_1, \ldots, \overline{r}_m\}$ such that $\overline{S}^{\overline{r}_1} \cup \ldots \cup \overline{S}^{\overline{r}_m}$ is an involution cover of $\overline{C} \cong Sp_8(2)$. That is,*

$$\mathcal{I}(\overline{C}) \subseteq \overline{S}^{\overline{r}_1} \cup \ldots \cup \overline{S}^{\overline{r}_m}.$$

Now let r_i be any representative of the coset $\overline{r}_i$ (so any element in $\overline{r}_i \setminus O_2(C)$), and let S be the full inverse image of $\overline{S}$ in C. Then

$$S^{r_1} \cup \ldots \cup S^{r_m}$$

is an involution cover of C.

Proof. Let $t \in C$ be an involution. Then either $\bar{t} = \bar{1}$ or $\bar{t}$ is an involution in $\overline{C}$. Either way, we have that

$$\bar{t} \in \overline{S}^{\overline{r}_1} \cup \ldots \cup \overline{S}^{\overline{r}_m}$$

and hence there is some $k \in \{1, \ldots, m\}$ for which $\bar{t} \in \overline{S}^{\overline{r}_k}$. Now, we have $\bar{t} = \overline{r}_k^{-1}\overline{s}\overline{r}_k$ for some $\overline{s} \in \overline{S}$. Then, without loss of generality, $t = r_k^{-1}sr_k$ for some $s \in S$ and so $t \in S^{r_k}$. Since this holds for any involution $t \in C$, the result follows. ■

This is an extremely useful result – instead of having to conjugate a Sylow 2-subgroup S of C by every element in a transversal for S in C, we only need to conjugate it by every element in some involution cover for the involutions in $Sp_8(2)$. To build an involution cover of $Sp_8(2)$, we start with the intrinsic copy of $Sp_8(2)$ in Magma given by `Sp(8,2)`. Call this K, and now let $X \in \mathrm{Syl}_2(K)$ found using `Sylow`. Using `Classes`, we can obtain conjugacy class representatives for the classes of involutions in K, and, using `Class`, we can build each class of involutions and take their union using `join` to form $\mathcal{I}(K)$. Now, beginning with $R := \varnothing$ and $Y := \mathcal{I}(X)$, we take random elements $r \in K$ and redefine $Y := Y \cup \mathcal{I}(X^r)$. If Y is bigger than it was before taking its union with $\mathcal{I}(X^r)$, we store r in R. We repeat this until $Y = \mathcal{I}(K)$. By construction, R is such that for all involutions $t \in K$, there is some $r \in R$ such that $t \in X^r$. When we carry this out, we find R such that $|R| = 2,036$. We note that this might not be optimal – a smaller R could exist. However, for our purposes, having $|R| = 2,036$ is adequate. We now save K, X, and R for future use.

To use this in practice, we load K, X, and R in the screen containing C, $\overline{C}$, and T. Since $K \cong \overline{C}$, we can use the command `IsIsomorphic` to obtain an isomorphism $\sigma : K \to \overline{C}$. Now let $\overline{S} = \sigma(X)$ and $\overline{R} = \{\sigma(r) : r \in R\}$, so that now we have $\overline{S}$, a Sylow 2-subgroup of $\overline{C}$ and $\overline{R}$ as described in Lemma 7.12. Now, let $\varphi : C \to \overline{C}$ given by the command `LMGRadicalQuotient`. Using this we can obtain representatives $\{r_1, ..., r_{2036}\}$ of the cosets in $\overline{R}$ as well as the full inverse image S of $\overline{S}$. Let $F = C_V(T)$ and now define

$$\mathcal{C}_2(T) = \langle \mathrm{Stab}_{S^{r_i}}(F) : i \in \{1, \ldots, 2036\} \rangle.$$

Using the same argument as is in the proof of Proposition 3.6 and combined with Lemma 7.12, we see that $\mathcal{I}(C_G(T)) \subseteq \mathcal{C}_2(T)$. We also use the same procedure described in Procedure 3.7 (ii) to calculate $\mathcal{C}_2(T)$, but with $\{r_1, ..., r_{2036}\}$ in place of the right transversal. Finally, as usual, we let

$$\mathcal{C}_3(T) = C_{\langle \mathcal{C}_2(T), T \rangle}(T)$$

which clearly also contains $\mathcal{I}(C_G(T))$. The results of this process are given in the next result.

Proposition 7.13. *The sizes of $\mathcal{C}_2(T)$ and $\mathcal{C}_3(T)$ for every $T \in \mathcal{T}^{(1,4)}$ are given in Table 7.9.*

Proof. We follow the process described before the statement of the proposition. ■

j	$\lvert\mathcal{C}_2(T_j^{(1,4)})\rvert$	$\lvert\mathcal{C}_3(T_j^{(1,4)})\rvert$
1	2^{30}	2^{22}
2	2^{30}	2^{22}
3	2^{30}	2^{22}
4	2^{29}	2^{23}
5	2^{30}	2^{22}
6	2^{29}	2^{23}
7	2^{30}	2^{22}
8	2^{30}	2^{22}
9	2^{30}	2^{22}
10	2^{29}	2^{23}
11	2^{30}	2^{22}
12	2^{29}	2^{23}
13	2^{33}	2^{27}
14	2^{29}	2^{23}
15	2^{31}	2^{24}

16	2^{31}	2^{25}
17	2^{29}	2^{23}
18	2^{29}	2^{23}
19	2^{29}	2^{24}
20	2^{29}	2^{23}
21	2^{29}	2^{23}
22	2^{30}	2^{22}
23	2^{29}	2^{23}
24	2^{30}	2^{22}
25	2^{29}	2^{23}
26	2^{31}	2^{24}
27	2^{30}	2^{22}
28	2^{30}	2^{22}
29	2^{29}	2^{23}
30	2^{30}	2^{22}
31	2^{29}	2^{23}
32	2^{29}	2^{23}
33	2^{30}	2^{22}
34	2^{30}	2^{22}
35	2^{30}	2^{22}
36	2^{29}	2^{23}
37	2^{29}	2^{23}
38	2^{31}	2^{24}
39	2^{31}	2^{24}
40	2^{29}	2^{23}
41	2^{31}	2^{25}
42	2^{31}	2^{25}
43	2^{31}	2^{25}
44	2^{31}	2^{25}
45	2^{29}	2^{24}
46	2^{29}	2^{24}
47	2^{29}	2^{24}
48	2^{29}	2^{24}
49	2^{30}	2^{22}
50	2^{29}	2^{23}
51	2^{29}	2^{23}
52	2^{30}	2^{22}
53	2^{29}	2^{23}
54	2^{31}	2^{25}
55	2^{31}	2^{26}
56	2^{29}	2^{23}
57	2^{31}	2^{26}
58	2^{31}	2^{26}
59	2^{30}	2^{22}
60	2^{31}	2^{26}

61	2^{31}	2^{25}
62	2^{31}	2^{25}
63	2^{31}	2^{25}
64	2^{29}	2^{23}
65	2^{29}	2^{23}
66	2^{31}	2^{25}
67	2^{31}	2^{25}
68	2^{29}	2^{23}
69	2^{29}	2^{23}
70	2^{29}	2^{23}
71	2^{29}	2^{23}
72	2^{29}	2^{23}
73	2^{31}	2^{24}
74	2^{31}	2^{24}
75	2^{29}	2^{23}
76	2^{29}	2^{23}
77	2^{31}	2^{24}
78	2^{29}	2^{23}
79	2^{29}	2^{23}
80	2^{29}	2^{23}
81	2^{29}	2^{23}
82	2^{29}	2^{24}
83	2^{30}	2^{22}
84	2^{30}	2^{22}
85	2^{29}	2^{23}
86	2^{29}	2^{23}
87	2^{29}	2^{24}
88	2^{29}	2^{24}
89	2^{31}	2^{25}
90	2^{29}	2^{23}
91	2^{29}	2^{23}
92	2^{31}	2^{25}
93	2^{29}	2^{23}
94	2^{31}	2^{24}
95	2^{29}	2^{23}
96	2^{29}	2^{23}
97	2^{30}	2^{22}
98	2^{30}	2^{22}
99	2^{29}	2^{23}
100	2^{30}	2^{22}
101	2^{30}	2^{22}
102	2^{29}	2^{23}
103	2^{31}	2^{25}
104	2^{29}	2^{23}
105	2^{29}	2^{23}

106	2^{29}	2^{23}
107	2^{29}	2^{23}
108	2^{29}	2^{23}
109	2^{29}	2^{23}
110	2^{30}	2^{23}
111	2^{30}	2^{23}
112	2^{30}	2^{23}
113	2^{30}	2^{23}
114	2^{31}	2^{24}
115	2^{31}	2^{24}
116	2^{31}	2^{24}
117	2^{29}	2^{23}
118	2^{34}	2^{27}
119	2^{34}	2^{27}
120	$2^{34}.3$	2^{27}
121	2^{34}	2^{27}
122	2^{30}	2^{22}
123	2^{29}	2^{23}
124	2^{29}	2^{23}
125	2^{30}	2^{22}
126	2^{30}	2^{23}
127	2^{30}	2^{23}
128	2^{30}	2^{23}
129	2^{30}	2^{23}
130	2^{34}	2^{27}
131	2^{33}	2^{27}
132	2^{34}	2^{27}
133	$2^{34}.3$	2^{27}
134	2^{30}	2^{22}
135	2^{29}	2^{23}
136	2^{29}	2^{23}
137	2^{30}	2^{22}
138	2^{30}	2^{22}
139	2^{30}	2^{22}
140	2^{30}	2^{22}
141	2^{29}	2^{23}
142	2^{29}	2^{23}
143	2^{31}	2^{25}
144	2^{29}	2^{23}
145	2^{29}	2^{23}
146	2^{29}	2^{23}
147	2^{30}	2^{22}
148	2^{30}	2^{22}
149	2^{30}	2^{22}
150	2^{29}	2^{23}

151	2^{29}	2^{23}
152	2^{30}	2^{22}
153	2^{31}	2^{24}
154	2^{31}	2^{24}
155	2^{29}	2^{23}
156	2^{30}	2^{22}
157	2^{30}	2^{22}
158	2^{29}	2^{23}
159	2^{31}	2^{24}
160	2^{31}	2^{24}
161	2^{29}	2^{23}
162	2^{29}	2^{23}
163	2^{29}	2^{23}
164	2^{29}	2^{23}
165	2^{31}	2^{24}
166	2^{31}	2^{24}
167	2^{29}	2^{23}
168	2^{30}	2^{22}
169	2^{30}	2^{22}
170	2^{30}	2^{22}
171	2^{29}	2^{23}
172	2^{31}	2^{25}
173	2^{29}	2^{23}
174	2^{29}	2^{23}
175	2^{30}	2^{22}
176	2^{30}	2^{23}
177	2^{30}	2^{23}
178	2^{31}	2^{24}
179	2^{31}	2^{24}
180	2^{29}	2^{23}
181	2^{29}	2^{23}
182	2^{30}	2^{22}
183	2^{30}	2^{22}
184	2^{29}	2^{23}
185	2^{30}	2^{22}
186	2^{30}	2^{23}
187	2^{30}	2^{23}
188	2^{30}	2^{22}
189	2^{30}	2^{22}
190	2^{30}	2^{23}
191	2^{30}	2^{23}
192	2^{30}	2^{22}
193	2^{31}	2^{24}
194	2^{30}	2^{22}
195	2^{33}	2^{27}

196	2^{29}	2^{23}
197	2^{29}	2^{23}
198	2^{30}	2^{22}
199	2^{30}	2^{22}
200	2^{30}	2^{22}

Table 7.9: Size of $\mathcal{C}_2(T)$ and $\mathcal{C}_3(T)$ for $T \in \mathcal{T}^{(1,4)}$

Now we will move on to the cases $T \in \mathcal{T}^{(1,5)}$. These cases are more difficult, for the simple reason that $z^{(1,5)} \in 2\mathrm{C}_G$. To see why this presents a new set of challenges, observe that, by Proposition 2.19, we know $C_G(z) \sim [2^{81}] : \mathrm{Sym}(3) \times F_4(2)$. Now recall that the first steps in finding $\mathcal{I}(C_G(T))$ have been to first find $C = C_G(z)$ then $\overline{C} = C/O_2(C) \cong \mathrm{Sym}(3) \times F_4(2)$ by using the command `LMGRadicalQuotient`. While MAGMA is able to calculate $C_G(z)$ using the usual `CentraliserOfInvolution`, $F_4(2)$ is too large a group for it to execute `LMGRadicalQuotient` on C. Hence, our usual process is disrupted. Over the next few results, we will describe a way to circumvent this issue.

The key theme of these results is that we are trying to work with $\overline{C}$ without explicitly calculating it. Recall that usually, after finding $\overline{C}$, we find the inverse image of $C_{\overline{C}}(\overline{T})$. We are not able to calculate this full inverse image, for reasons we will discuss. However, we will devise a way of calculating the inverse image of $C_{\overline{C}}(\overline{a})$ where $a \in T$ is a non-central involution. This is a larger group than the inverse image of $C_{\overline{C}}(\overline{T})$, but it is still a smaller group than the whole of $C_G(z)$ and provides a starting point for trimming down the group we must sieve for involutions.

The first step in this process is to find $O_2(C)$ without using the usual intrinsic commands. This result provides us with a means of finding random elements of $O_2(C)$.

Lemma 7.14. *Let $y \in C$ be of order 56. Then $y^{28} \in O_2(C)$.*

Proof. First note that elements of order 56 do exist in C – these are easily found using the `Element` command (see B.2). Now note that there are no elements of order 56 in the group $F_4(2)$. This can be seen by consulting the $\mathbb{ATLAS}$ [14]. Hence, there are no elements of order 56 in the direct product $\mathrm{Sym}(3) \times F_4(2)$.

Take $y \in C$ of order 56. As C is a split extension, $y = wk$ where $w \in O_2(C)$ and $k \in \mathrm{Sym}(3) \times F_4(2)$. Moreover, by the above, $w \neq 1$. Since $O_2(C)$ is normal in C, we know that $k^{-1}wk = w_0$ for some $w_0 \in O_2(C)$, therefore $wk = kw_0$. This is useful, since we may rewrite $(wk)^n = k^n w_0$ for any $n \in \mathbb{N}$ and some $w_0 \in O_2(C)$. In particular, we have

$$1 = y^{56} = (wk)^{56} = k^{56}w_0$$

for some $w_0 \in O_2(C)$ and we claim that $k^{56} = 1$. Indeed, assume otherwise. Then $k^{56} = w_0^{-1} \in O_2(C)$, a contradiction, as $O_2(C) \cap \mathrm{Sym}(3) \times F_4(2) = 1$. Hence the order of k divides 56 – i.e. $o(k) = \{1, 2, 4, 7, 8, 14, 28\}$.

Now, consider $y^{28} = (wk)^{28} = k^{28}w_0$ for some $w_0 \in O_2(C)$. Note that if $o(k) \neq 8$, then $o(k)$ divides 28, and so $k^{28} = 1$ and we are done. So, suppose $o(k) = 8$. But now, $y^8 = k^8 w_1 = w_1$ for some $w_1 \in O_2(C)$ and so

$$1 = y^{56} = (y^8)^7 = w_1^7.$$

But the order of w_1 must be even or 1, so the only way $w_1^7 = 1$ is if $w_1 = 1$. This is a contradiction, as now, $y^8 = 1$, contradicting the order of y. ■

We use this result to repeatedly find elements $y_1, \ldots, y_n \in C$ of order 56, then take the subgroup of C given by $\langle y_1^{28}, \ldots, y_n^{28} \rangle$. Since for every $i \in \{1, \ldots, n\}$ we have $y_i^{28} \in O_2(C)$, then $\langle y_1^{28}, \ldots, y_n^{28} \rangle \leq O_2(C)$. We keep increasing n until $|\langle y_1^{28}, \ldots, y_n^{28} \rangle| = 2^{81}$, which means this must be $O_2(C)$. Now that we have $O_2(C)$, the next result gives us a way of finding random elements belonging to the inverse image of $C_{\overline{C}}(\overline{a})$. It is a modification of the Bray method (see Proposition 2.11) which we will call the *relative Bray method.*

Proposition 7.15. *Suppose H is a finite group with $N \trianglelefteq H$ and let $\overline{H} = H/N$. Let $g \in H$ be an involution. Let $h \in H$ and suppose $r \in \mathbb{N}$ is minimal such that $[g, h]^r \in N$. Then let*

$$x = \begin{cases} [g, h]^{\frac{r}{2}}, & \text{if } r \text{ even,} \\ h[g, h]^{\frac{r-1}{2}}, & \text{otherwise.} \end{cases} \tag{7.2}$$

Then $\overline{gx} = \overline{xg}$ and hence x is in the inverse image of $C_{\overline{H}}(\overline{g})$.

Proof. Note that $g^{-1} = g$ as g is an involution. First we observe that $\overline{gx} = \overline{xg}$ if only if $(gx)N = (xg)N$, which is equivalent to $[g, x] = gx^{-1}gx \in N$. Hence, to demonstrate that $\overline{gx} = \overline{xg}$ we will demonstrate that $[g, x] \in N$ in both cases. First, suppose r is even. Then:

$$\begin{aligned}[g,x] &= gx^{-1}gx \\ &= g([g,h]^{\frac{r}{2}})^{-1}g[g,h]^{\frac{r}{2}} \\ &= g(gh^{-1}gh\dots gh^{-1}gh)^{-1}g[g,h]^{\frac{r}{2}} \\ &= g(h^{-1}ghg\dots h^{-1}ghg)g[g,h]^{\frac{r}{2}} \\ &= (gh^{-1}gh\dots gh^{-1}gh)[g,h]^{\frac{r}{2}} \\ &= [g,h]^{\frac{r}{2}}[g,h]^{\frac{r}{2}} \\ &= [g,h]^r \in N.\end{aligned}$$

Now, suppose r is odd. Then:

$$\begin{aligned}[g,x] &= gx^{-1}gx \\ &= g(h[g,h]^{\frac{r-1}{2}})^{-1}gh[g,h]^{\frac{r-1}{2}} \\ &= g(h(gh^{-1}gh\dots gh^{-1}gh))^{-1}gh[g,h]^{\frac{r-1}{2}} \\ &= g((h^{-1}ghg\dots h^{-1}ghg)h^{-1})gh[g,h]^{\frac{r-1}{2}} \\ &= (gh^{-1}gh\dots gh^{-1}gh)gh^{-1}gh[g,h]^{\frac{r-1}{2}} \\ &= [g,h]^{\frac{r-1}{2}}[g,h][g,h]^{\frac{r-1}{2}} \\ &= [g,h]^r \in N.\end{aligned}$$

Hence we have $[g,x] \in N$ and so, by our earlier observations, $\overline{gx} = \overline{xg}$ as required. ■

We use this result to repeatedly find elements $x_1, \dots, x_n$ in the inverse image of $C_{\overline{C}}(\overline{a})$. To do this, we use the procedure `ReBray` (see B.5). Let $X = \langle x_1, \dots, x_n \rangle$. We would like to keep increasing n until X is the whole inverse image of $C_{\overline{C}}(\overline{a})$. However, caution must be taken here. Earlier, we took random elements until we generated the whole of $O_2(C)$. This was fine, because when we knew we had generated the whole of $O_2(C)$. However, we do not yet know when we have generated the whole inverse image of $C_{\overline{C}}(\overline{a})$. The next two results will deliver all the potential orders of this inverse image, which is the first step in knowing when to stop adding generators to X.

Lemma 7.16. *Let $L = \mathrm{Sym}(3) \times F_4(2)$ and let $t \in L$ be an involution. Then there are seven possible orders of $C_L(t)$:*

(i) $2^{25}.3^5.5.7$, *(iv)* $2^{25}.3^4.5.7$, *(vii)* $2.|F_4(2)|$.

(ii) $2^{25}.3^3.5$, *(v)* $2^{25}.3^2.5$,

(iii) $2^{21}.3^3$, *(vi)* $2^{21}.3^2$,

Proof. Since L is a direct product, we may consider the elements of L as ordered pairs (g_S, g_F) where $g_S \in \mathrm{Sym}(3)$ and $g_F \in F_4(2)$. Now, observe that there are three possible structures for t. These are $(1_S, t_F)$, (t_S, t_F), and $(t_S, 1_F)$, for some involutions $t_S \in \mathrm{Sym}(3)$, $t_F \in F_4(2)$, and where 1_S and 1_F are the identity elements of $\mathrm{Sym}(3)$ and $F_4(2)$ respectively. Now note that there are three possible centraliser sizes for $C_{F_4(2)}(t_F)$, which are $2^{24}.3^4.5.7$, $2^{24}.3^2.5$, and $2^{20}.3^2$ – this can be verified using the 𝔸𝕋𝕃𝔸𝕊 [14]. And, as $\mathrm{Sym}(3)$ only has one conjugacy class of involutions, we know that $C_{\mathrm{Sym}(3)}(t_S) = 2$.

Firstly, if t is of the form $(1_S, t_F)$, then $C_L(t)$ is of the form $C_{\mathrm{Sym}(3)}(1_S) \times C_{F_4(2)}(t_F) = \mathrm{Sym}(3) \times C_{F_4(2)}(t_F)$. This has an order of six multiplied by the possibilities for $C_{F_4(2)}(t_F)$ given above, which accounts for cases (i), (ii), and (iii).

Secondly, if t is of the form (t_S, t_F), then $C_L(t)$ is of the form $C_{\mathrm{Sym}(3)}(t_S) \times C_{F_4(2)}(t_F) = 2 \times C_{F_4(2)}(t_F)$. This has an order of two multiplied by the possibilities for $C_{F_4(2)}(t_F)$ given above, which accounts for cases (iv), (v), and (vi).

Finally, if t is of the form $(t_S, 1_F)$, then $C_L(t)$ is of the form $C_{\mathrm{Sym}(3)}(t_S) \times F_4(2) = 2 \times F_4(2)$. This has order $2.|F_4(2)|$, accounting for case (vii) and completing the proof. ■

With this, we can now examine a complete list of the possible orders of the inverse image of $C_{\overline{C}}(\overline{a})$.

Proposition 7.17. *Let B be the full inverse image of $C_{\overline{C}}(\overline{a})$ in C. Then there are eight possibilities for $|B|$, which are:*

(i) $2^{106}.3^5.5.7$, *(iv)* $2^{106}.3^4.5.7$, *(vii)* $2^{82}.|F_4(2)|$,

(ii) $2^{106}.3^3.5$, *(v)* $2^{106}.3^2.5$, *(viii)* $|C|$.

(iii) $2^{102}.3^3$, *(vi)* $2^{102}.3^2$,

Proof. By definition, $B/O_2(C) = C_{\overline{C}}(\overline{a})$. Therefore $|B| = |C_{\overline{C}}(\overline{a})||O_2(C)|$. Suppose first that $a \notin O_2(C)$. Then $\overline{a}$ is an involution in $\overline{C}$. Therefore, $C_{\overline{C}}(\overline{a})$ must have one of the orders given in Lemma 7.16, and as $|O_2(C)| = 2^{81}$, this accounts for cases (i)–(vii).

Now suppose $a \in O_2(C)$. Then $\overline{a} = \overline{1}$, so $C_{\overline{C}}(\overline{a}) = C_{\overline{C}}(\overline{1}) = \overline{C}$. Therefore, $B = C$, which accounts for case (viii). ■

Unfortunately, we are not done. For even if we have $|X|$ matching one of the eight possibilities given in Proposition 7.17, we still cannot be sure that $X = B$. For example, if we generate X such that $|X| = 2^{106}.3^3.5$, matching Proposition 7.17 (ii), we can safely rule out the possibility that $|B| = 2^{102}.3^3$, since $X \leq B$. However, it could be the case that no matter how many generators we add to X, it never grows, and yet, we could be in a situation where, for example, $|B| = 2^{106}.3^5.5.7$. For this reason, we need an effective way to know when we can stop growing X and be sure we have the entirety of B. The next result describes a method which achieves this. It involves choosing random elements of C satisfying certain properties which provide a near-instant check in Magma.

Proposition 7.18. *(i) Suppose $|X| = |C|$. Then $X = B$.*

(ii) Suppose $|X| = 2^{106}.3^5.5.7$ and there exists $x \in C$ of order 17 such that $[x, a] \notin O_2(C)$. Then $X = B$.

(iii) Suppose $|X| = 2^{106}.3^3.5$. Suppose also that there exists some $x \in C$ of order 17 such that $[x, a] \notin O_2(C)$ and some $g \in C$ such that, given any $n \in \{1, 2, 3, 4, 6, 12\}$, we have $[a, g]^n \notin O_2(C)$. Then $X = B$.

(iv) Suppose $|X| = 2^{102}.3^3$. Suppose also that there exists $x \in C$ of order 17 such that $[x, a] \notin O_2(C)$ and some $g \in C$ such that, given any $n \in \{1, 2, 3, 4, 5, 6, 12, 15\}$, we have $[g, a]^n \notin O_2(C)$. Then $X = B$.

Proof. We will work through each case in turn.

(i) This case is trivial, for we know that $X \leq B \leq C$ and so $|X| = |C|$ yields $X = B$.

(ii) Suppose $|X| = 2^{106}.3^5.5.7$ and assume that $X \neq B$. Then, as $|X| < |B|$ and by Proposition 7.17, there are only two possibilities for $|B|$, namely $2^{82}.|F_4(2)|$ and $|C|$. But now consider $C_{\overline{C}}(\overline{a}) = B/O_2(C)$. In these cases, $B/O_2(C) \cong 2 \times F_4(2)$ or $B/O_2(C) \cong \mathrm{Sym}(3) \times F_4(2)$ respectively. Therefore, $F_4(2) \leq B/O_2(C) =$

$C_{\overline{C}}(\overline{a})$. Crucially, for all $\overline{x} \in F_4(2)$ of order 17, we have $\overline{xa} = \overline{ax}$. Therefore $(xa)O_2(C) = (ax)O_2(C)$ and so $[x, a] \in O_2(C)$.

Now, let $x \in C$ of order 17 such that $[x, a] \notin O_2(C)$. Then since $x^{17} = 1$, we must have $x \notin O_2(C)$, as $O_2(C)$ is a 2-group and therefore only contains elements of order a power of 2. Hence, $\overline{x}^{17} = \overline{1}$. But note that $\overline{x} \in \mathrm{Sym}(3) \times F_4(2)$ and the only elements of order 17 in $\mathrm{Sym}(3) \times F_4(2)$ are in $F_4(2)$. Hence $\overline{x} \in F_4(2)$. But since $[x, a] \notin O_2(C)$, this is a contradiction.

(iii) Suppose $|X| = 2^{106}.3^3.5$ and again suppose that this is not the whole of B. Then, by Proposition 7.17 (i), (iv), (vii), and (viii) there are four possibilities for $|B|$, namely $2^{106}.3^5.5.7$, $2^{106}.3^4.5.7$, $2^{82}.|F_4(2)|$, and $|C|$.

First, note that the existence of $x \in C$ of order 17 such that $[x, a] \notin O_2(C)$ rules out the possibility that $|B|$ is $2^{82}.|F_4(2)|$ or $|C|$, using the proof of part (ii). So, suppose now that $|B|$ is $2^{106}.3^5.5.7$ or $2^{106}.3^4.5.7$. Then that means that $\overline{a} \in \mathrm{Sym}(3) \times F_4(2)$ is of the form $(1_S, t_F)$ or (t_S, t_F) where t_S is an involution in $\mathrm{Sym}(3)$ and t_F is an involution in $F_4(2)$ such that $|C_{F_4(2)}(t_F)| = 2^{24}.3^4.5.7$. This corresponds to t_F being in $2\mathrm{A}_{F_4(2)}$ or $2\mathrm{B}_{F_4(2)}$. We know, from direct calculation in MAGMA using the intrinsic copy of $F_4(2)$ given by the command `ChevalleyGroup("F",4,2)`, for all $t \in 2\mathrm{A}_{F_4(2)} \cup 2\mathrm{B}_{F_4(2)}$, we have $o(tt^g) \in \{1, 2, 3, 4\}$ where $g \in F_4(2)$. Hence, we claim that for all $\overline{g} \in \mathrm{Sym}(3) \times F_4(2)$, $o(\overline{a}\overline{a}^{\overline{g}}) \in \{1, 2, 3, 4, 6, 12\}$. Indeed, let $\overline{g} = (g_S, g_F)$ for some $g_S \in \mathrm{Sym}(3)$ and $g_F \in F_4(2)$. If $\overline{a} = (1_S, t_F)$ then

$$o(\overline{a}\overline{a}^{\overline{g}}) = o((1_S, t_F)(1_S, t_F)^{(g_S, g_F)}) = o((1_S, t_F t_F^{g_F})) \in \{1, 2, 3, 4\},$$

and if $\overline{a} = (t_S, t_F)$, then we have

$$o(\overline{a}\overline{a}^{\overline{g}}) = o((t_S, t_F)(t_S, t_F)^{(g_S, g_F)}) = o((t_S t_S^{g_S}, t_F t_F^{g_F})) \in \{1, 2, 3, 4, 6, 12\}.$$

Now we observe that for all $\overline{g} \in \overline{C}$, we have $o(\overline{a}\overline{a}^{\overline{g}}) \in \{1, 2, 3, 4, 6, 12\}$ if and only if, $(\overline{a}\overline{a}^{\overline{g}})^n = \overline{1}$ for some $n \in \{1, 2, 3, 4, 6, 12\}$. This holds if and only if $(aa^g)^n \in O_2(C)$ for some $n \in \{1, 2, 3, 4, 6, 12\}$. This holds for every $g \in C$. However, this contradicts the fact that we have an element $g \in C$ such that for all $n \in \{1, 2, 3, 4, 6, 12\}$, $[a, g]^n \notin O_2(C)$.

(iv) Suppose $|X| = 2^{102}.3^3$ and again suppose that this is not the whole of B. By Proposition 7.17 (i), (ii), (iv), (v), (vii), and (viii) there are six possibilities for $|B|$. These are: $2^{106}.3^5.5.7$, $2^{106}.3^3.5$, $2^{106}.3^4.5.7$, $2^{106}.3^2.5$, $2^{82}.|F_4(2)|$, and $|C|$.

First, note that the existence of this particular x rules out possibilities $2^{82}.|F_4(2)|$ and $|C|$ using the argument from part (ii). And the existence of this particular g rules out possibilities $2^{106}.3^5.5.7$ and $2^{106}.3^4.5.7$ using the argument from part (iii).

So, suppose $|B| = 2^{106}.3^3.5$ or $2^{106}.3^2.5$. This proof will continue on the same lines as the proof of part (iii). We see that $\overline{a} \in \mathrm{Sym}(3) \times F_4(2)$ is of the form $(1_S, t_F)$ or (t_S, t_F) where t_S is an involution in Sym(3) and t_F is an involution in $F_4(2)$ such that $|C_{F_4(2)}(t_F)| = 2^{24}.3^2.5$. In other words, $\overline{a} \in 2\mathrm{C}_{F_4(2)}$. From direct calculation in MAGMA, we know that for all $t \in 2\mathrm{C}_{F_4(2)}$, we have $o(tt^g) \in \{1, 2, 3, 4, 5, 6\}$ where $g \in F_4(2)$. Therefore, we claim that for all $\overline{g} \in \mathrm{Sym}(3) \times F_4(2)$, $o(\overline{a}\overline{a}^{\overline{g}}) \in \{1, 2, 3, 4, 5, 6, 12, 15\}$.

Again, by considering $\overline{g}$ as an ordered pair $(g_S, g_F) \in \mathrm{Sym}(3) \times F_4(2)$, we have the following possibilities:

$$\text{if } \overline{a} = (1_S, t_F) \text{ then } o(\overline{a}\overline{a}^{\overline{g}}) = o((1_S, t_F)(1_S, t_F)^{(g_S, g_F)}) = o((1_S, t_F t_F^{g_F}));$$
$$\text{if } \overline{a} = (t_S, t_F) \text{ then } o(\overline{a}\overline{a}^{\overline{g}}) = o((t_S, t_F)(t_S, t_F)^{(g_S, g_F)}) = o((t_S t_S^{g_S}, t_F t_F^{g_F})).$$

We know that $o(t_F t_F^{g_F}) \in \{1, 2, 3, 4, 5, 6\}$, therefore $o((1_S, t_F t_F^{g_F})) \in \{1, 2, 3, 4, 5, 6\}$. Also, as $o(t_S t_S^{g_S}) \in \{1, 3\}$, we have $o((t_S t_S^{g_S}, t_F t_F^{g_F})) \in \{1, 2, 3, 4, 5, 6, 12, 15\}$. This proves the claim. But now we know that $o(\overline{a}\overline{a}^{\overline{g}}) \in \{1, 2, 3, 4, 5, 6, 12, 15\}$ if and only if, $(\overline{a}\overline{a}^{\overline{g}})^n = \overline{1}$ for some $n \in \{1, 2, 3, 4, 5, 6, 12, 15\}$. And this is true if and only if $(aa^g)^n \in O_2(C)$ for some $n \in \{1, 2, 3, 4, 5, 6, 12, 15\}$. Thus, for all $g \in C$, there is some $n \in \{1, 2, 3, 4, 5, 6, 12, 15\}$ for which $(aa^g)^n \in O_2(C)$. However, this contradicts the fact that we have an element $g \in C$ such that for all $n \in \{1, 2, 3, 4, 5, 6, 12, 15\}$, $[a, g]^n \notin O_2(C)$.

■

In practice, we continue adding generators to X until its order matches one of the possibilities given in Proposition 7.17. Then, we utilise simple `repeat..until` loops to acquire elements satisfying the properties given in Proposition 7.18. Once these elements are found, we can be certain that X is the entire inverse image of $C_{\overline{C}}(\overline{a})$. (See Procedure B.6 for procedures which find these elements.)

We also remark here that we have not provided ways of finding these "checking" elements for every possible inverse image order. Indeed, we listed eight possibilities in Proposition 7.17, but only discussed four of them in Proposition 7.18. However, in practice, in all 127 cases of $T \in \mathcal{T}^{(1,5)}$, we only ever encounter an inverse image size matching one of the four discussed in Proposition 7.18. For this reason, we have no need of a method finding elements which confirm we have the other possible four sizes.

So that our notation is consistent with prior sections, we will now refer to the full inverse image of $C_{\overline{C}}(\overline{a})$ as $\mathcal{C}_1(T)$. We make very clear that isn't exactly how $\mathcal{C}_1(T)$ was defined in previous sections – usually, $\mathcal{C}_1(T)$ represents the full inverse image of $C_{\overline{C}}(\overline{T})$. However, we simply use the notation $\mathcal{C}_k(T)$ for $k \geq 0$ as shorthand for groups which contain $\mathcal{I}(C_G(T))$ which are successively smaller as k increases.

To execute these methods in practice, we note that we are actually able to choose non-central involutions common to multiple groups in $\mathcal{T}^{(1,5)}$. Hence we are able to split $\mathcal{T}^{(1,5)}$ into 46 disjoint subsets $\mathcal{T}^{(1,5,i)}$ for $i \in \{1, \dots, 46\}$ such that there is an involution $a \in T$ for all subgroups $T \in \mathcal{T}^{(1,5,i)}$ with $a \neq z$ (where $z \in Z(T)$ is the involution for every $T \in \mathcal{T}^{(1,5)}$) and $a \notin O_2(C)$. There is one exception: there is one $T \in \mathcal{T}^{(1,5)}$ such that $T \leq O_2(C)$ and hence such an a cannot be chosen for this T. We store this on its own in $\mathcal{T}^{(1,5,1)}$. We will use the notation $a^{(i)}$ to refer to the non-central involution common to each $T \in \mathcal{T}^{(1,5,i)}$ for each $i \in \{1, \dots, 46\}$. The advantage of breaking up $\mathcal{T}^{(1,5)}$ this way is that, if $T \in \mathcal{T}^{(1,5,i)}$ we can define $\mathcal{C}_1(T)$ to be the inverse image of $C_{\overline{C}}(\overline{a^{(i)}})$, which will be same for every $T \in \mathcal{T}^{(1,5,i)}$. Hence we must only calculate such an inverse image 46 times instead of 127 (the total number of subgroups in $\mathcal{T}^{(1,5)}$) times.

Once we have obtained $\mathcal{C}_1(T)$ for each $T \in \mathcal{T}^{(1,5)}$, we define $\mathcal{C}_2(T) = \langle \mathrm{Stab}_S(F) : S \in \mathrm{Syl}_2(\mathcal{C}_1(T))\rangle$, which, by Proposition 3.6, contains $\mathcal{I}(C_G(T))$. Finally, let $\mathcal{C}_3(T) = C_{\mathcal{C}_2(T)}(T)$ which clearly also contains $\mathcal{I}(C_G(T))$. These groups are calculated in the next result. Note that we leave the sole case in $\mathcal{T}^{(1,5,1)}$ out of this result.

Proposition 7.19. *The sizes of $\mathcal{C}_1(T)$, $\mathcal{C}_2(T)$, and $\mathcal{C}_3(T)$ for every $T \in \mathcal{T}^{(1,5,i)}$, $i \in \{2, \dots, 46\}$ are given in Table 7.10.*

i	$\lvert\mathcal{C}_1(T_j^{(1,5,i)})\rvert$	j	$\lvert\mathcal{C}_2(T_j^{(1,5,i)})\rvert$	$\lvert\mathcal{C}_3(T_j^{(1,i)})\rvert$
2	$2^{106}.3^5.5.7$	1	$2^{37}.3$	$2^{32}.3$
		2	2^{36}	2^{27}
		3	2^{37}	2^{31}
		4	2^{36}	2^{30}
		5	2^{36}	2^{30}
		6	2^{34}	2^{36}
		7	2^{34}	2^{29}
		8	2^{34}	2^{29}
		9	2^{34}	2^{26}
		10	$2^{34}.3$	$2^{26}.3$
		11	$2^{35}.3^2$	$2^{26}.3$
		12	2^{32}	2^{26}
		13	2^{30}	2^{25}
		14	$2^{32}.3$	$2^{27}.3$
3	$2^{106}.3^3.5$	1	$2^{32}.3^2$	$2^{25}.3$
		2	$2^{35}.3$	$2^{29}.3$
		3	2^{33}	2^{28}
		4	2^{32}	2^{26}
		5	$2^{32}.3$	$2^{27}.3$
		6	2^{32}	2^{25}
		7	$2^{32}.3$	$2^{25}.3$
		8	$2^{32}.3$	$2^{25}.3$
		9	$2^{3}9.3$	$2^{34}.3$
		10	2^{37}	2^{31}
		11	$2^{34}.3$	2^{27}
		12	2^{36}	2^{30}
		13	2^{32}	2^{26}
		14	2^{36}	2^{30}
4	$2^{106}.3^5.5.7$	1	2^{33}	2^{28}
		2	$2^{32}.3$	$2^{27}.3$
		3	$2^{32}.3$	$2^{25}.3$
		4	$2^{33}.3$	$2^{27}.3$
		5	2^{33}	2^{26}
		6	2^{33}	2^{26}
		7	2^{35}	2^{30}
		8	$2^{37}.3$	$2^{31}.3$
5	$2^{106}.3^5.5.7$	1	$2^{35}.3$	$2^{29}.3$
		2	2^{36}	2^{30}
		3	2^{33}	2^{26}
		4	2^{35}	2^{30}
		5	2^{33}	2^{28}
		6	$2^{32}.3$	$2^{25}.3$
6	$2^{106}.3^5.5.7$	1	$2^{37}.3^2$	$2^{32}.3^2$
7	$2^{106}.3^5.5.7$	1	2^{31}	2^{24}

		2	2^{33}	2^{28}
		3	2^{35}	2^{30}
		4	2^{33}	2^{28}
		5	2^{31}	2^{24}
8	$2^{102}.3^3$	1	2^{30}	2^{23}
		2	$2^{29}.3$	$2^{24}.3$
		3	2^{30}	2^{23}
		4	2^{28}	2^{22}
		5	$2^{29}.3$	$2^{24}.3$
9	$2^{106}.3^3.5$	1	2^{31}	2^{24}
		2	2^{36}	2^{30}
		3	$2^{34}.3$	$2^{26}.3$
		4	$2^{32}.3$	$2^{27}.3$
10	$2^{102}.3^3$	1	2^{28}	2^{22}
		2	2^{28}	2^{22}
		3	2^{28}	2^{22}
		4	2^{28}	2^{22}
11	$2^{102}.3^3$	1	$2^{29}.3$	$2^{23}.3$
		2	$2^{31}.3$	$2^{25}.3$
		3	2^{31}	2^{26}
		4	2^{31}	2^{25}
12	$2^{106}.3^3.5$	1	$2^{32}.3$	$2^{25}.3$
		2	$2^{29}.3$	$2^{24}.3$
		3	2^{30}	2^{24}
		4	$2^{33}.3$	$2^{27}.3$
		5	$2^{37}.3$	$2^{32}.3$
13	$2^{106}.3^3.5$	1	2^{29}	2^{23}
		2	2^{29}	2^{23}
		3	2^{29}	2^{23}
14	$2^{106}.3^3.5$	1	$2^{27}.3$	$2^{22}.3$
		2	2^{33}	2^{28}
		3	$2^{35}.3^2$	$2^{29}.3^2$
15	$2^{106}.3^3.5$	1	2^{31}	2^{26}
		2	$2^{34}.3$	$2^{29}.3$
		3	2^{33}	2^{28}
		4	$2^{34}.3$	$2^{31}.3$
16	$2^{102}.3^3$	1	2^{28}	2^{22}
		2	2^{28}	2^{24}
17	$2^{102}.3^3$	1	$2^{28}.3$	$2^{22}.3$
		2	$2^{32}.3$	$2^{26}.3$
18	$2^{106}.3^3.5$	1	2^{27}	2^{22}
		2	2^{27}	2^{22}
19	$2^{102}.3^3$	1	2^{30}	2^{23}
		2	2^{30}	2^{24}
20	$2^{106}.3^3.5$	1	2^{30}	2^{24}

		2	2^{33}	2^{28}
21	$2^{106}.3^3.5$	1	$2^{34}.3$	$2^{26}.3$
		2	$2^{32}.3$	$2^{27}.3$
22	$2^{106}.3^3.5$	1	$2^{32}.3$	$2^{25}.3$
		2	$2^{32}.3$	$2^{27}.3$
23	$2^{102}.3^3$	1	2^{30}	2^{24}
		2	2^{27}	2^{22}
		3	2^{33}	2^{27}
24	$2^{102}.3^3$	1	$2^{31}.3$	$2^{25}.3$
		2	2^{30}	2^{24}
25	$2^{102}.3^3$	1	$2^{32}.3$	$2^{26}.3$
		2	$2^{32}.3$	$2^{24}.3$
26	$2^{102}.3^3$	1	2^{28}	2^{22}
		2	2^{28}	2^{22}
27	$2^{106}.3^3.5$	1	2^{29}	2^{22}
		2	2^{29}	2^{22}
28	$2^{106}.3^3.5$	1	$2^{32}.3$	$2^{27}.3$
29	$2^{106}.3^3.5$	1	$2^{32}.3$	$2^{26}.3$
30	$2^{106}.3^5.5.7$	1	$2^{32}.3$	$2^{27}.3$
31	$2^{106}.3^3.5$	1	$2^{29}.3$	$2^{24}.3$
		2	$2^{33}.3$	$2^{27}.3$
		3	$2^{35}.3$	$2^{29}.3$
32	$2^{106}.3^5.5.7$	1	$2^{31}.3$	$2^{26}.3$
33	$2^{102}.3^3$	1	2^{27}	2^{22}
34	$2^{106}.3^3.5$	1	$2^{31}.3$	$2^{26}.3$
35	$2^{106}.3^3.5$	1	$2^{35}.3$	$2^{29}.3$
36	$2^{102}.3^3$	1	2^{27}	2^{22}
37	$2^{106}.3^3.5$	1	$2^{31}.3$	$2^{25}.3$
38	$2^{102}.3^3$	1	2^{33}	2^{26}
39	$2^{102}.3^3$	1	2^{33}	2^{26}
40	$2^{106}.3^3.5$	1	$2^{36}.3$	$2^{29}.3$
41	$2^{106}.3^5.5.7$	1	2^{40}	2^{35}
42	$2^{106}.3^3.5$	1	$2^{33}.3$	$2^{26}.3$
43	$2^{106}.3^3.5$	1	$2^{35}.3.5$	$2^{27}.3.5$
44	$2^{106}.3^5.5.7$	1	$2^{35}.3^2$	$2^{30}.3^2$
45	$2^{106}.3^5.5.7$	1	$2^{37}.3^2$	$2^{32}.3^2$
46	$2^{106}.3^3.5$	1	$2^{35}.3.5$	$2^{30}.3.5$

Table 7.10: Sizes of $\mathcal{C}_k(T)$ for $k \in \{1,2,3\}$ and $T \in \mathcal{T}^{(1,5)} \setminus \mathcal{T}^{(1,5,1)}$

Proof. We calculate $\mathcal{C}_1(T)$ using Procedure 3.2 and $\mathcal{C}_2(T)$ using Procedure 3.7. When $\mathcal{C}_2(T)$ is soluble, we turn it into a pc-group and calculate $\mathcal{C}_3(T)$ directly. Otherwise, we calculate it using `LMGCentraliser`. ■

Taking stock of where we are currently, we have $\mathcal{C}_3(T)$ for every $T \in \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$. We broke $\mathcal{T}^{(1)}$ into five sets $\mathcal{T}^{(1,1)}, \ldots, \mathcal{T}^{(1,5)}$ and we have $\mathcal{C}_3(T)$ for every $T \in \mathcal{T}^{(1,1)} \cup \ldots \cup \mathcal{T}^{(1,4)}$. Finally, we broke $\mathcal{T}^{(1,5)}$ into 46 sets $\mathcal{T}^{(1,5,1)}, \ldots, \mathcal{T}^{(1,5,46)}$ and we have $\mathcal{C}_3(T)$ for every $T \in \mathcal{T}^{(1,5,2)} \cup \ldots \cup \mathcal{T}^{(1,5,46)}$. Therefore, the *only* group T for which we have not obtained $\mathcal{C}_3(T)$ is the sole T belonging to the set $\mathcal{T}^{(1,5,1)}$. Let us discuss this group now.

Fix $T \in \mathcal{T}^{(1,5,1)}$ and $z \in Z(T)$ the unique central involution. Recall that we have $C = C_G(z) \sim [2^{81}] : \mathrm{Sym}(3) \times F_4(2)$. This case is a nuisance, as $T \leq O_2(C)$. This happened before with the subgroups belonging to $\mathcal{T}^{(1,4)}$, and we dealt with these cases by building an involution cover of $Sp_8(2)$. However, their central involution belonged to $2\mathrm{D}_G$, where here we have $z \in 2\mathrm{C}_G$. If we wanted to tackle this case in the same way, we would need to construct an involution cover of $\mathrm{Sym}(3) \times F_4(2)$. However, $F_4(2)$ is a very large group, and such a construction is not feasible. In fact, this case is so tricky that we will proceed to sieve $\mathcal{C}_3(T)$ for all other cases of T, then return to $T \in \mathcal{T}^{(1,5,1)}$ later.

Now, let $T \in \mathcal{T}^{(i)}$, $i \in \{1,2,3\}$, such that $T \notin \mathcal{T}^{(1,5,1)}$. We must is sieve $\mathcal{C}_3(T)$ for involutions t such that $\langle P_iT, t\rangle \cong \Omega_8^+(2)$. To do this, we must first develop a new sieve. Usually, an order of random elements sieve (see Procedure 3.8) is sufficient for our needs but, in many of these cases, we have so many involutions to sieve that this sieve does not sufficiently cut down our set.

Lemma 7.20. *Suppose $H \cong \Omega_8^+(2)$ is generated as in Proposition 7.1 – $P \in \mathrm{Syl}_5(H)$, $T \in \mathrm{Syl}_2(N_H(P))$ and $x \in C_H(T) \setminus T$ such that $H = \langle PT, x\rangle$. Now, let $O_i(x) = |\{n \in PT : o(nx) = i\}|$, that is, the number of elements of PT with order i product with x. Then, the following hold: $O_2(x) = 8$, $O_4(x) = 24$, $O_5(x) = 8$, $O_6(x) = 64$, $O_8(x) = 32$, $O_9(x) = 32$, $O_{10}(x) = 104$, $O_{12}(x) = 96$, $O_{15}(x) = 32$, and $O_i(x) = 0$ for all $i \notin \{2,4,5,6,8,9,10,12,15\}$.*

Proof. This is easily verified using Magma. ■

We can use this information in our hunt for involutions. If $H \leq G$, and $P_iT_j^{(i)} \leq H$ then there must exist some involution $t \in \mathcal{C}_3(T_j^{(i)})$ such that, if $O_i(t) = |\{n \in$

$P_iT_j^{(i)} : o(nt) = i\}|$, then for each $i \in \{2, 4, 5, 6, 8, 9, 10, 12, 15\}$, $O_i(t)$ is as stated in Lemma 7.20. Hence we will incorporate this into a sieve.

Procedure 7.21. This procedure will be referred to as the *action on PT sieve*. Let X be the set of involutions we seek to sieve, and let $PT = P_iT_j^{(i)}$. For each $g \in X$, we carry out the following. Let $O_i(g) := 0$ for each $i \in \{2, 4, 5, 6, 8, 9, 10, 12, 15\}$. Now, for each $n \in PT$, if $o(ng) = i$ then we set $O_i(x) := O_i(x) + 1$. If, once we have run through every element of PT, we have that $O_i(x)$ is as described in Lemma 7.20 for *every* $i \in \{2, 4, 5, 6, 8, 9, 10, 12, 15\}$, then we keep g in a set called Y. We do this for every $g \in X$. If $O_i(x)$ is incorrect for any i, then we discard g and move on.

Here is a sample code which executes this procedure.

```
Y:={};
for x in X do;
 O2 :=0;
 O4 :=0;
 O5 :=0;
 O6 :=0;
 O8 :=0;
 O9 :=0;
 O10:=0;
 O12:=0;
 O15:=0;
 for n in PT do;
  o:=Order(n*x);
  if o eq 2 then O2:=O2+1; end if;
  if o eq 4 then O4:=O4+1; end if;
  if o eq 5 then O5:=O5+1; end if;
  if o eq 6 then O6:=O6+1; end if;
  if o eq 8 then O8:=O8+1; end if;
  if o eq 9 then O9:=O9+1; end if;
  if o eq 10 then O10:=O10+1; end if;
  if o eq 12 then O12:=O12+1; end if;
```

```
  if o eq 15 then O15:=O15+1; end if;
 end for;
 if O2 eq 8 then;
  if O4 eq 24 then;
   if O5 eq 8 then;
    if O6 eq 64 then;
     if O8 eq 32 then;
      if O9 eq 32 then;
       if O10 eq 104 then;
        if O12 eq 96 then;
         if O15 eq 32 then;
          Include(~Y,x);
         end if;
        end if;
       end if;
      end if;
     end if;
    end if;
   end if;
  end if;
 end if;
end for;
```

We are now ready to sieve each $\mathcal{C}_3(T)$ and find if there are any involutions generating $\Omega_8^+(2)$ containing P_iT for $i \in \{2,3\}$. The following result explores this process for $T \in \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$.

Proposition 7.22. *Let $(i,j) \in \{(2,1),(3,1),(3,2),(3,3),(3,4)\}$. The number of $\Omega_8^+(2)$ subgroups of G up to G-conjugacy for which $P_iT_j^{(i)} \leq \Omega_8^+(2)$ is given in Table 7.11.*

Proof. Let $T = T_j^{(i)}$. First, note that $\mathcal{C}_3(T)$ is soluble in every case, so we can convert it into a pc-group and gather all of its involutions in a set called $\mathcal{I}_0(T)$. Then, we run an order of random elements sieve on each involution in $\mathcal{I}_0(T)$ (see Procedure 3.8 for

i	j	Number of $\Omega_8^+(2)$ subgroups containing $P_iT_j^{(i)}$
2	1	1
3	1	1
3	2	1
3	3	0
3	4	0

Table 7.11: $\Omega_8^+(2)$ subgroups containing $P_iT_j^{(i)}$

full details). Note that we require a full set of all possible orders of elements appearing in $\Omega_8^+(2)$ to employ this sieve. This is {1,2,3,4,5,6,7,8,9,12,15} which can be seen in the 𝔸𝕋𝕃𝔸𝕊 [14]. We store the involutions that survive in a set called $\mathcal{I}_1(T)$. Now, for each involution $t \in \mathcal{I}_1(T)$, we check that $O_k(t)$ (as defined in Lemma 7.20) has the correct size for each $k \in \{2, 4, 5, 6, 8, 9, 10, 12, 15\}$ by running Procedure 7.21. If t survives this sieve, we store it in a set called $\mathcal{I}_2(T)$. Finally, for each $t \in \mathcal{I}_2(T)$, if $\langle P_iT, t\rangle \cong \Omega_8^+(2)$ and t generates a distinct $\Omega_8^+(2)$ subgroup, then we store t in $\mathcal{I}_3(T)$. The results of this process are shown in Table 7.12. The final column of Table 7.12 provides us with the number of $\Omega_8^+(2)$ subgroups, as stated. ■

i	j	$\lvert\mathcal{I}_0(T_j^{(i)})\rvert$	$\lvert\mathcal{I}_1(T_j^{(i)})\rvert$	$\lvert\mathcal{I}_2(T_j^{(i)})\rvert$	$\lvert\mathcal{I}_3(T_j^{(i)})\rvert$
2	1	12,287	28	3	1
3	1	96,255	410	9	1
3	2	43,007	148	3	1
3	3	5,119	148	1	0
3	4	3,071	132	1	0

Table 7.12: $|\mathcal{I}_l(T_j^{(i)})|$ for $l \in \{0, 1, 2, 3\}$

We have found three $\Omega_8^+(2)$ subgroups so far. Let us now look at $T \in \mathcal{T}^{(1,i)}$ for $i \in \{1, 2, 3, 4\}$.

Proposition 7.23. *Let $i \in \{1, 2, 3, 4\}$. The number of $\Omega_8^+(2)$ subgroups of G up to G-conjugacy for which $P_1T_j^{(1,i)} \leq \Omega_8^+(2)$ is given in Table 7.13.*

i	j	Number of $\Omega_8^+(2)$ subgroups containing $P_1T_j^{(1,i)}$
1	$1, 2, 3, 4$	0
2	$1, \dots, 10$	0
3	$1, \dots, 5, 7, \dots, 10$	0
3	6	1
4	$1, \dots, 119, 121, \dots, 200$	0
4	120	1

Table 7.13: $\Omega_8^+(2)$ subgroups containing $P_1T_j^{(1,i)}$, $i \in \{1, 2, 3, 4\}$

Proof. We follow the steps of the proof of Proposition 7.22 identically, constructing sets of involutions $\mathcal{I}_l(T_j^{(1,i)})$ for $l \in \{0,1,2,3\}$. The results for $(i,j) \in \{(3,6),(4,120)\}$ are displayed in Table 7.14. There are no $\Omega_8^+(2)$ subgroups of G containing $P_1T_j^{(1,i)}$ for $(i,j) \notin \{(3,6),(4,120)\}$, so we do not list the results here.

i	j	$\lvert\mathcal{I}_0(T_j^{(1,i)})\rvert$	$\lvert\mathcal{I}_1(T_j^{(1,i)})\rvert$	$\lvert\mathcal{I}_2(T_j^{(1,i)})\rvert$	$\lvert\mathcal{I}_3(T_j^{(1,i)})\rvert$
3	6	720,895	6,095	24	2
4	120	1,245,183	19,455	24	2

Table 7.14: $|\mathcal{I}_l(T_j^{(1,i)})|$ for $l \in \{0,1,2,3\}$

Now let $(i,j) \in \{(3,6),(4,120)\}$. There are two $\Omega_8^+(2)$ subgroups containing $P_1T_j^{(1,i)}$. Let us call them $H_1^{(1,i,j)}$ and $H_2^{(1,i,j)}$. In actuality, these are G-conjugate. By Proposition 3.15, we know that $H_1^{(1,i,j)}$ and $H_2^{(1,i,j)}$ are conjugate in G if and only if they are conjugate in the group $N_{N_G(P_1)}(P_1T_j^{(1,i)})$. Recall that we have $N_G(P_1)$ in MAGMA as a permutation group. In this setting, we can easily calculate $N_{N_G(P_1)}(P_1T_j^{(1,i)})$ and readily find an element that conjugates $H_1^{(1,i,j)}$ to $H_2^{(1,i,j)}$. ■

We have found an additional two $\Omega_8^+(2)$ subgroups. Now, let us see how many $\Omega_8^+(2)$ we can find contining P_1T where $T \in \mathcal{T}^{(1,5)}$, with the exception of $T \in \mathcal{T}^{(1,5,1)}$.

Proposition 7.24. *Recall that we break $\mathcal{T}^{(1,5)}$ into 46 sets: $\mathcal{T}^{(1,5,1)}, \ldots, \mathcal{T}^{(1,5,46)}$. Let $n_i = |\mathcal{T}^{(1,5,i)}|$ for $i \in \{1,\ldots,46\}$. Let $T_1^{(1,5,i)}, \ldots, T_{n_i}^{(1,5,i)}$ be the subgroups contained in the set $\mathcal{T}^{(1,5,i)}$. Then:*

(i) there is a unique (up to G-conjugacy) subgroup of G isomorphic to $\Omega_8^+(2)$ up to G-conjugacy containing $P_1T_4^{(1,5,15)}$;

(ii) there are no $\Omega_8^+(2)$ subgroups containing $P_1T_j^{(1,5,i)}$ for all choices of $i \in \{2,\ldots,46\}$ and $j \in \{1,\ldots,n_i\}$ except $(i,j) \in \{(5,15)\}$.

Proof. For the most part, we can again follow the steps of the proof of Proposition 7.22 identically, constructing sets of involutions $\mathcal{I}_l(T_j^{(1,5,i)})$ for $l \in \{0,1,2,3\}$. However, this method cannot feasibly be applied to six cases of $T_j^{(1,5,i)}$. It should be noted that, given enough computing power and enough time, then *all* of these cases can be sieved for involutions using the techniques described in the proof of Proposition 7.22. However, we will describe the faster methods for how we dealt with these cases. These cases are $T_1^{(1,5,i)}$ where $i \in \{6,41,43,44,45,46\}$ and $T_9^{(1,5,2)}$.

(i) First, we will sieve $\mathcal{C}_3(T_1^{(1,5,41)})$, which is of order 2^{35} by Proposition 7.19. Here, we use Procedure 3.12. The full details of this procedure are in Procedure 3.12, but we will go over some of the details here. Let $X = \mathcal{C}_3(T_1^{(1,5,41)})$ and we construct

$$E \leq Z(X) \leq S \leq X$$

where $E \cong 2^6$ and $|S| = 2^{30}$. Now, let Γ be a right transversal for E in S and $\Delta = \{\delta_1, \ldots, \delta_{32}\}$ be a right transversal for S in X. We open a new screen for each element of Δ; 32 screens in total. Screen i sieves the set $S\delta_i$. Then, combining Propositions 3.10 and 3.11 we know that we must sieve $E\gamma\delta_i$ for each $i \in \{1, \ldots, 32\}$ and for $\gamma \in \Gamma$ such that $(\gamma\delta_i)^2 = 1$. A sample code for the sieving process is given in Procedure 3.12. For each element in $x \in E\gamma\delta_i$, we first check that $x \in 2\mathrm{C}$, then run x through an order of random elements sieve (see Procedure 3.8). No involutions survive this process, so we know there are no $\Omega_8^+(2)$ subgroups containing $T_1^{(1,5,41)}$.

(ii) Next, we will look at sieving $\mathcal{C}_3(T_j^{(1,5,i)})$ for $(i,j) \in \{(2,9),(6,1),(44,1),(45,1)\}$. Here, we have $|\mathcal{C}_3(T_j^{(1,5,i)})| \in \{2^{34}.3, 2^{30}.3^2, 2^{32}.3^2\}$, but we will use the same method for each case. First, let X be a Sylow 2-subgroup of $\mathcal{C}_3(T_j^{(1,5,i)})$ and let R be a right transversal for X in $\mathcal{C}_3(T_j^{(1,5,i)})$. (We calculate R using `Transversal` but in the case $\mathcal{C}_3(T_2^{(1,5,9)})$ we have that $|\mathcal{C}_3(T_2^{(1,5,9)})| = 2^{34}.3$ so we can choose any element $r \in \mathcal{C}_3(T_2^{(1,5,9)})$ of order 3 and $R = \{1, r, r^2\}$ will be a right transversal for X in $\mathcal{C}_3(T_2^{(1,5,9)})$.) Then if $t \in \mathcal{C}_3(T_j^{(1,5,i)})$ is an involution then $t \in X^r$ for some $r \in R$. The method we will employ, therefore, is to find the involutions of $x \in X$ and run x^r through our sieves for every $r \in R$. This will ensure that we will have interrogated each involution in $\mathcal{C}_3(T_j^{(1,5,i)})$. However, we do not actually need to sieve X^r for each $r \in R$ entirely separately. A more efficient way would be to first construct $\mathcal{I}(X)$ then sieve $\mathcal{I}(X)^r$ for each $r \in R$, as then, we only need to construct one set of involutions of a Sylow 2-subgroup then conjugate by r to cover all the involutions in $\mathcal{C}_3(T_j^{(1,5,i)})$. Even better still would be to construct the set $\mathcal{I}(X) \cap 2\mathrm{C}_G$ then sieve $(\mathcal{I}(X) \cap 2\mathrm{C}_G)^r$ for each $r \in R$. Of course, this means we only have to sieve one $\mathcal{I}(X)$ for involutions in $2\mathrm{C}_G$, then any conjugate of these involutions will automatically lie in $2\mathrm{C}_G$.

We will use the method in part (i) to obtain $E \leq Z(X) \leq S \leq X$ where E is elementary abelian, with Γ a right tranversal for E in S and Δ a right transversal for S in X. (Note that for $(i,j) = (2,9)$ we have $E \cong 2^6$ and choose S such that $[X : S] = 32$ and for $(i,j) \in \{(6,1),(44,1),(45,1)\}$ we have $E \cong 2^8$ and choose S such that $[X : S] = 8$.) We open a new screen for each element $\delta \in \Delta$ and each screen then sieves $S\delta$ by first building a set $\mathcal{I}_0(X) = \{\varepsilon\gamma\delta : (\gamma\delta)^2 = 1,\ \varepsilon \in E\}$. Now let $\mathcal{I}_1(X) = \{x \in \mathcal{I}_0(X) : \dim C_V(x) = 138\}$, that is, the elements of $\mathcal{I}_0(X)$ which lie in $2\mathrm{C}_G$. Now, for every $x \in \mathcal{I}_1(X)$ we run an order of random elements sieve on x^r for every $r \in R$. No involutions survive this process, so we know there are no $\Omega_8^+(2)$ subgroups containing $T_j^{(1,5,i)}$ for $(i,j) \in \{(6,1),(44,1),(45,1),(2,9)\}$.

(iii) Finally, we will look at the cases where $(i,j) \in \{(43,1),(46,1)\}$. Here we have $|\mathcal{C}_3(T_j^{(1,5,i)})| \in \{2^{27}.3.5, 2^{30}.3.5\}$, but we will proceed the same way in both cases. Let $M = \mathcal{C}_3(T_j^{(1,5,i)})$ and let $W = O_2(M)$ with $\overline{M} = M/O_2(M) \cong \mathrm{Alt}(5)$. We calculate W and $\overline{M}$ using the command `LMGRadicalQuotient`. Since $|W| = 2^{25}$ or $|W| = 2^{28}$ we can sieve this directly, following the steps of the proof of Proposition 7.22. This accounts for all the involutions $t \in W$, but now we must consider the involutions $t \notin W$.

In this case, we must have that $\overline{t} \in \overline{M}$ is an involution. Let $\overline{m} \in \overline{M}$ be a conjugacy class representative for the unique $\overline{M}$-class of involutions. Then $\overline{t}$ and $\overline{m}$ are conjugate in $\overline{M}$, so there is some $\overline{g} \in \overline{M}$ such that $\overline{t} = \overline{m}^{\overline{g}}$. Now let $\overline{R}$ be a right transversal for $C_{\overline{M}}(\overline{m})$ in $\overline{M}$. As $\overline{g} \in \overline{M}$ we have that $\overline{g} \in C_{\overline{M}}(\overline{m})\overline{r}$ for some $\overline{r} \in \overline{R}$. Therefore, there exists $\overline{c} \in C_{\overline{M}}(\overline{m})$ such that $\overline{g} = \overline{cr}$. But now

$$\overline{t} = \overline{m}^{\overline{g}} = \overline{m}^{\overline{cr}} = (\overline{m}^{\overline{c}})^{\overline{r}} = \overline{m}^{\overline{r}}.$$

Hence, for all $t \in M \setminus W$, there exists $\overline{r} \in \overline{R}$ such that $\overline{t} = \overline{m}^{\overline{r}}$.

Now let A be the inverse image of $\overline{m}$ and let $R = \{r_1, \dots, r_{15}\} \subset M$ be a set of coset representatives of the cosets $\overline{r} \in \overline{R}$. Supposing that $\overline{t} = \overline{m}^{\overline{r_i}}$, for some $i \in \{1, \dots, 15\}$, we claim that $t \in A^{r_i}$. Indeed, we have $\overline{t^{r_i^{-1}}} = \overline{m}$ and hence $t^{r_i^{-1}} \in A$ which gives us that $t \in A^{r_i}$. Since this holds for every involution in

$M \setminus W$ we have that

$$\mathcal{I}(M \setminus W) = \bigcup_{r \in R} (A \setminus W)^r.$$

Moreover, recall that we only require the involutions in $M \setminus W$ which belong to $2\mathrm{C}_G$. Therefore, the involutions we desire will be found in $((A \setminus W) \cap 2\mathrm{C}_G)^r$ for each $r \in R$.

In practice, A has order 2^{26} or 2^{29}. We turn this into a pc-group using the command `LMGSolubleRadical` and run through every element $x \in A$, collecting all the involutions not in W and pulling them back to the matrix setting to keep them if $\dim C_V(x) = 138$. Then, we run an order of random elements sieve (see Procedure 3.8) on x^r for each $r \in R$. No involutions survive this process, so we know there are no $\Omega_8^+(2)$ subgroups containing $T_j^{(1,5,i)}$ for $(i, j) \in \{(43, 1), (46, 1)\}$.

The only case in which we find involutions t such that $\langle P_1 T_j^{(1,5,i)}, t\rangle \cong \Omega_8^+(2)$ is the case when $(i, j) = (15, 4)$, where we find involutions which generate four distinct $\Omega_8^+(2)$ subgroups. However, we are able to readily find elements in $N_G(P_1)$ which conjugate these copies of $\Omega_8^+(2)$ to each other. Hence, there is only one $\Omega_8^+(2)$ containing $P_1 T_4^{(1,5,15)}$ up to G-conjugacy. ■

We will conclude this section by finding, up to G-conjugacy, all $\Omega_8^+(2)$ subgroups containing $P_1 T$ where T is the unique member of the set $\mathcal{T}^{(1,5,1)}$.

Proposition 7.25. *There is exactly one $\Omega_8^+(2)$ subgroup of G up to G-conjugacy containing $P_1 T$ where T is the unique member of the set $\mathcal{T}^{(1,5,1)}$.*

Proof. Let $C = C_G(z)$, where z is the unique central involution in T. Then we know, by Proposition 2.19, that $C \sim [2^{81}] : \mathrm{Sym}(3) \times F_4(2)$. Let $W = O_2(C) \sim [2^{81}]$. In this case, we see that $T \leq W$. This presents us with a difficult situation; usually, we aim to find the inverse image of $C_{\overline{C}}(\overline{T})$ in C, where $\overline{C} = C/W$. However, as $T \leq W$, we have that $\overline{T} = \overline{1}$ and therefore $C_{\overline{C}}(\overline{T})$ is the whole of $\overline{C}$ and thus its inverse image is the whole of C. So, we have failed to cut down where we must look for involutions centralising T.

Instead, let $\Phi(W)$ be the Frattini subgroup of W (the intersection of all maximal subgroups of W). We find that $\Phi(W)$ is of order 2^{29} and so $W/\Phi(W)$ is elementary abelian of order 2^{52}. Moreover, we can calculate $C_W(T)$ directly and we find that

$|C_W(T)| = 2^{28}$ with $|\Phi(W) \cap C_W(T)| = 2^{27}$. Now consider the action of C on $W/\Phi(W)$ by conjugation and suppose $c \in C$ normalises $C_W(T)$. Given that $\Phi(W) \cap C_W(T)$ is normal in $C_W(T)$, for all $w \in C_W(T) \setminus \Phi(W)$ we have $w^c \in C_W(T) \setminus \Phi(W)$. Therefore, the element $\overline{w} \in W/\Phi(W)$ is stabilised by any $c \in N_C(C_W(T))$. Using the command

```
GModule(C,W,FW)
```

where the object `FW` is the Frattini subgroup $\Phi(W)$, we are able to obtain the 52-dimensional vector space over $GF(2)$ isomorphic to the elementary abelian group $W/\Phi(W) \cong 2^{52}$ under the action of C. By the above argument, $N_C(C_W(T))$ will be found in the inverse image of the stabiliser in C of $\overline{w}$, where we consider $\overline{w}$ as a 52-dimensional vector.

Now we claim that $C_G(T)$ lies in $N_C(C_W(T))$. Indeed, $C_G(T) \leq C_C(T)$, so if c centralises T then $c \in C$ and $C_W(T)^c = C_W(T^c) = C_W(T)$, so $c \in N_C(C_W(T))$ as required. Now, let $\langle a, b, c \rangle \leq T$ where a, b, and c are involutions. Furthermore, we find that a, b, c can be chosen such that $a \notin \Phi(W)$, $b \notin \Phi(W)$, and $c \notin \Phi(W)$. Now, any $x \in C_C(T)$ must clearly fix a, b, and c by conjugation, and so must stabilise each of $\overline{a}$, $\overline{b}$, and $\overline{c}$, where here we are considering $\overline{a}$, $\overline{b}$, and $\overline{c}$ as 52-dimensional vectors. Therefore, $C_G(T)$ lies inside the inverse image of the intersection of the stabilisers in C of $\overline{w}$, $\overline{a}$, $\overline{b}$, and $\overline{c}$. Hence we form a chain of nested stabilisers in the vector space setting, where X is the matrix group representing the action of C on $W/\Phi(W)$:

$$S_w = \mathrm{Stab}_X(\overline{w}),\ S_a = \mathrm{Stab}_{S_w}(\overline{a}),\ S_b = \mathrm{Stab}_{S_a}(\overline{b}),\ S_c = \mathrm{Stab}_{S_b}(\overline{c}).$$

So now we know that $C_G(T)$ must lie in the inverse image in C of S_c. We are unable to calculate this inverse image directly using Magma. However, we see that $S_c \cong \Omega_8^+(2)$, and hence we know $C_G(T) \leq [2^{81}] : \Omega_8^+(2) \leq C$. Let $M = [2^{81}] : \Omega_8^+(2)$ – the group which we know must contain $C_G(T)$ but cannot calculate directly. Were we able to calculate M, we would then proceed to calculate $\mathrm{Stab}_M(F)$ where $F = C_V(T)$, which we know also must contain $C_G(T)$. However, we can calculate $\mathrm{Stab}_S(F)$ where $S \in \mathrm{Syl}_2(C)$, and if we were able to run through every Sylow 2-subgroup S of C and take the subgroup generated by all of these stabilisers, we would obtain a group guaranteed to contain every involution centralising T. However, we need not run through *every* Sylow 2-subgroup. After running through 200 Sylow 2-subgroups, we calculate $M_0 = \langle \mathrm{Stab}_{S^{r_i}}(F) : r_1, \ldots, r_{200} \in C \rangle$ and find that $M_0 \sim [2^{33}] : \Omega_8^+(2)$. We also find that $C_{M_0}(T) \sim [2^{28}] : \Omega_8^+(2)$. Hence we conclude that $M = WC_{M_0}(T)$ and so

$$C_M(T) = C_W(T)C_{M_0}(T).$$

However, we now see that $C_W(T) \leq C_{M_0}(T)$ and so $C_M(T) = C_{M_0}(T)$. Finally, we have arrived at a group $C_{M_0}(T)$ we can now proceed to sieve for involutions.

Still, sieving $C_{M_0}(T)$ is a challenge in and of itself. This is the largest subgroup we must sieve for involutions in this entire thesis. From here on, we will use $\mathcal{C}_0(T)$ to refer to $C_{M_0}(T)$ and recall that we have $\mathcal{C}_0(T) \sim [2^{28}] : \Omega_8^+(2)$. We will sieve $\mathcal{C}_0(T)$ in a similar way to other cases, sieving for involutions which lie in $O_2(\mathcal{C}_0(T))$ and for involutions which do not. Suppose $x \in \mathcal{C}_0(T)$ such that $\langle P_1T, x \rangle \cong \Omega_8^+(2)$. Then either $x \notin O_2(\mathcal{C}_0(T))$ or $x \in O_2(\mathcal{C}_0(T))$.

Let us first assume that $x \notin O_2(\mathcal{C}_0(T))$. Let $\overline{\mathcal{C}_0(T)} = \mathcal{C}_0(T)/O_2(\mathcal{C}_0(T)) \cong \Omega_8^+(2)$ and let $\overline{x}_1, \ldots, \overline{x}_5$ be representatives for the classes of involutions in $\overline{\mathcal{C}_0(T)}$. Then $\overline{x}$ is an involution and is conjugate to $\overline{x}_i$ for some $i \in \{1, 2, 3, 4, 5\}$. Let $\overline{R}_i$ be a right transversal in $\overline{\mathcal{C}_0(T)}$ for $C_{\overline{\mathcal{C}_0(T)}}(\overline{x}_i)$ and let R_i be a complete set of representatives for the cosets in $\overline{R}_i$. Finally, let N_i be the inverse image in $\mathcal{C}_0(T)$ of $\langle \overline{x}_i \rangle$. Then, by the same argument seen in the end of the proof of Proposition 7.24 (ii) when dealing with $(i, j) \in \{(43, 1), (46, 1)\}$, we have that

$$x \in N_i^r$$

for some $r \in R_i$. Moreover, since we know that $x \notin O_2(\mathcal{C}_0(T))$ and we can take $x \in 2\mathrm{C}_G$, we know that if

$$X_i = (N_i \setminus O_2(\mathcal{C}_0(T))) \cap 2\mathrm{C}_G.$$

then $x \in X_i^r$ for some $r \in R_i$. Hence, for each $i \in \{1, 2, 3, 4, 5\}$, our task is now to construct R_i and X_i and sieve X_i^r for each $r \in R_i$. First, we will focus on constructing X_i.

Fix $i \in \{1, 2, 3, 4, 5\}$. Using the command `LMGRadicalQuotient` and the homomorphism provided, we can obtain N_i and R_i. We now use last procedure described in Procedure 3.12, obtaining a chain of subgroups

$$E_i \leq Z(N_i) \leq S_i \leq O_2(\mathcal{C}_0(T)) \leq N_i$$

where E_i is elementary abelian. Then, we find a right transversal Γ_i for E_i in S_i, and a right transversal Δ_i for S_i in N_i. Then the involutions we require are of the form $\varepsilon\gamma\delta$ where $\varepsilon \in E_i$, $\gamma \in \Gamma$, and $\delta \in \Delta$ such that $(\gamma\delta)^2 = 1$ and $\delta \notin O_2(\mathcal{C}_0(T))$ (see Procedure 3.12 for full details). Additionally, if $\varepsilon\gamma\delta \in 2\mathrm{C}_G$ then we store $\varepsilon\gamma\delta$ in X_i. Hence we have constructed X_i.

We must now sieve X_i^r for every $r \in R_i$. So, for each $x \in X_i$, we run through every element $r \in R_i$ and let $y = x^r$. First, we run an order of random elements sieve on y (see Procedure 3.8). If y survives this sieve, we run an "action on P_1T" sieve on y (see Procedure 7.21). No elements survive both sieves for any $i \in \{1, 2, 3, 4, 5\}$, so there are no involutions $x \in \mathcal{C}_0(T) \setminus O_2(\mathcal{C}_0(T))$ such that $\langle P_1T, x\rangle \cong \Omega_8^+(2)$.

Finally, we must sieve $O_2(\mathcal{C}_0(T))$ for generating involutions. Here, we simply take $x \in O_2(\mathcal{C}_0(T))$ and run x through a random elements sieve followed by an action on P_1T sieve (see Procedures 3.8 and 7.21) and keep those that survive in a set called X. We find that $|X| = 43,201$. Let x_j be the elements of X for $j \in \{1, \dots, 43201\}$. We find that $\langle P_1T, x_1\rangle \cong \Omega_8^+(2)$. Moreover, we let $N = N_G(P_1)$ and observe that $(P_1T)^n = P_1T$ for all $n \in N$. Now, for every $j \in \{2, \dots, 43201\}$, we can find some n_j such that $x_j^{n_i} \in \langle P_1T, x_1\rangle$. This implies that $\langle P_1T, x_j\rangle$ is a redundant subgroup for all $j \in \{2, \dots, 43201\}$. Indeed, we have

$$\langle P_1T, x_j\rangle^{n_j} = \langle (P_1T)^{n_j}, x_j^{n_j}\rangle = \langle P_1T, x_j^{n_j}\rangle \leq \langle P_1T, x_1\rangle$$

and so if we have equality then $\langle P_1T, x_j\rangle$ is conjugate to $\langle P_1T, x_1\rangle$, but if we don't have equality then $\langle P_1T, x_j\rangle \not\cong \Omega_8^+(2)$. Thus, we have only one $\Omega_8^+(2)$ subgroup containing P_1T, as stated. ∎

Hence we have at most seven copies of $\Omega_8^+(2)$ in G up to G-conjugacy. We will now seek to extend these.

7.2 Extending $\Omega_8^+(2)$ to $\Omega_8^+(2) : \mathrm{Sym}(3)$

We will begin this subsection by simplifying our notation. Our three subgroups isomorphic to 5^2 will remain P_1, P_2, and P_3. We will now renumber the $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups, extracting the ones which actually form part of a generating set for an $\Omega_8^+(2)$ subgroup of G. Let $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$, and $T_4^{(1)}$ be the subgroups $T_9^{(1,3)}$, $T_{152}^{(1,4)}$,

$T_4^{(1,5,15)}$, and $T_1^{(1,5,1)}$ respectively. Let $T_1^{(2)}$ be the sole $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroup normalising P_2 and let $T_1^{(3)}$ and $T_2^{(3)}$ remain as they were in the previous section. Finally, for each $(i,j) \in \{(1,1),(1,2),(1,3),(1,4),(2,1),(3,1),(3,2)\}$, we will let $H_j^{(i)}$ be the unique (up to G-conjugacy) $\Omega_8^+(2)$ subgroup of G containing $P_iT_j^{(i)}$.

We start by looking for $g \in G$ such that $\langle H_j^{(i)}, g\rangle \cong \Omega_8^+(2) : k$, where $k \in \{2,3\}$. By Proposition 2.15 we know that we will find such elements in $N_{N_G(P_i)}(P_iT_j^{(i)}) \cap N_G(H_j^{(i)})$. We have $N_G(P_i)$ as a permutation group, as found in Proposition 7.7. Hence we can easily calculate $N_j^{(i)} := N_{N_G(P_i)}(P_iT_j^{(i)})$ directly in the permutation group setting. The challenge is now to find which elements in $N_j^{(i)}$ normalise $H_j^{(i)}$. This can be difficult, as $H_j^{(i)}$ is not a subgroup of $N_j^{(i)}$, so we cannot apply the command `Normaliser`. Instead, we apply the following result.

Lemma 7.26. *Let $F_j^{(i)} = C_V(H_j^{(i)})$. Then*

$$N_{N_j^{(i)}}(H_j^{(i)}) \leq \mathrm{Stab}_{N_j^{(i)}}(F_j^{(i)}).$$

Proof. Let $g \in N_{N_j^{(i)}}(H_j^{(i)})$. We must show that for all $v \in F_j^{(i)}$, we have $v^g \in F_j^{(i)}$. Note that $v^g \in F_j^{(i)}$ if and only if, $(v^g)^h = v^g$ for all $h \in H_j^{(i)}$. Also note that, since g normalises $H_j^{(i)}$, then for all $h \in H_j^{(i)}$, we have $ghg^{-1} = h_0 \in H$ and therefore $gh = h_0g$. So, let $v \in F_j^{(i)}$ and $h \in H_j^{(i)}$, and now:

$$(v^g)^h = v^{gh} = v^{h_0g} = (v^{h_0})^g = v^g$$

and hence $v^g \in F_j^{(i)}$. ∎

Let $S_j^{(i)} = \mathrm{Stab}_{N_j^{(i)}}(F_j^{(i)})$. We will now calculate $S_j^{(i)}$ and $N_{S_j^{(i)}}(H_j^{(i)})$ for each case.

Proposition 7.27. *For each $(i,j) \in \{(1,1),(1,2),(1,3),(1,4),(2,1),(3,1),(3,2)\}$, the orders of $N_j^{(i)}$, $S_j^{(i)}$, and $N_{S_j^{(i)}}(H_j^{(i)})$ are given in Table 7.15.*

Proof. As previously stated, we can calculate $N_j^{(i)}$ directly in the permutation setting of $N_G(P_i)$. Back in the matrix setting, we calculate $S_j^{(i)}$ using the command `Stabiliser`. Now, in every case except $(i,j) = (1,2)$, we observe that every generator g of $S_j^{(i)}$ normalises $H_j^{(i)}$ and therefore $S_j^{(i)}$ is in fact $N_{S_j^{(i)}}(H_j^{(i)})$. In the case where $(i,j) = (1,2)$, there exists an element $g \in S_2^{(1)}$ which does not normalise $H_2^{(1)}$. Therefore, we have that $N_{S_2^{(1)}}(H_2^{(1)}) \lneqq S_2^{(1)}$. Now, we take random elements of $S_2^{(1)}$ which *do* normalise $H_2^{(1)}$, and take the subgroup they generate. This will be a subgroup of $N_{S_2^{(1)}}(H_2^{(1)})$.

i	j	$\|N_j^{(i)}\|$	$\|S_j^{(i)}\|$	$\|N_{S_j^{(i)}}(H_j^{(i)})\|$
1	1	$2^9.3.5^2$	$2^8.3.5^2$	$2^8.3.5^2$
1	2	$2^{11}.3.5^2$	$2^{11}.3.5^2$	$2^{10}.3.5^2$
1	3	$2^{13}.3.5^2$	$2^{11}.3.5^2$	$2^{11}.3.5^2$
1	4	$2^{17}.3^6.5^4.7$	$2^{11}.3^4.5^2.7$	$2^{11}.3^4.5^2.7$
2	1	$2^5.3.5^2$	$2^5.3.5^2$	$2^5.3.5^2$
3	1	$2^6.3.5^2$	$2^6.3.5^2$	$2^6.3.5^2$
3	2	$2^6.5^2$	$2^6.5^2$	$2^6.5^2$

Table 7.15: Orders of $N_j^{(i)}$, $S_j^{(i)}$, and $N_{S_j^{(i)}}(H_j^{(i)})$

However, we can find a sufficient number of such elements so as we can generate a subgroup of $S_2^{(1)}$ of index 2. This must be the whole of $N_{S_2^{(1)}}(H_2^{(1)})$ – it cannot be any larger, since it must be a proper subgroup of $S_2^{(1)}$. This completes the proof. ∎

We must now sieve each $N_{S_j^{(i)}}(H_j^{(i)})$ for elements g such that $\langle H_j^{(i)}, g\rangle \cong \Omega_8^+(2) : k$, for $k \in \{2, 3\}$. Here, we appeal to Proposition 2.17 to see that if $g \in N_{S_j^{(i)}}(H_j^{(i)})$ is such that $\langle H_j^{(i)}, g\rangle \cong \Omega_8^+(2) : k$, then $g \notin H_j^{(i)}$ and $g^k \in H_j^{(i)}$. Furthermore, by Proposition 2.16 we know that if g and g_0 are conjugate in $N_{S_j^{(i)}}(H_j^{(i)})$, then:

- $g \notin H_j^{(i)}$ if and only if $g_0 \notin H_j^{(i)}$;
- $g^k \in H_j^{(i)}$ if and only if $g_0^k \in H_j^{(i)}$;
- $\langle H_j^{(i)}, g\rangle$ and $\langle H_j^{(i)}, g_0\rangle$ are conjugate groups.

Hence we need only find elements up to conjugacy in $N_{S_j^{(i)}}(H_j^{(i)})$. From now on, we will use $\mathcal{E}_0(H_j^{(i)})$ to denote $N_{S_j^{(i)}}(H_j^{(i)})$ and we will let $\mathcal{E}_1(H_j^{(i)})$ be a set of $\mathcal{E}_0(H_j^{(i)})$-conjugacy class representatives. Now, we define

$$\mathcal{E}_2^{(k)}(H_j^{(i)}) = \{x \in \mathcal{E}_1(H_j^{(i)}) : x \notin H_j^{(i)} \text{ and } x^k \in H_j^{(i)}\}$$

for $k \in \{2, 3\}$. Finally, we construct $\mathcal{E}_3^{(k)}(H_j^{(i)})$, a set of elements from $\mathcal{E}_2^{(k)}(H_j^{(i)})$ which generate distinct $\Omega_8^+(2) : k$ subgroups containing $H_j^{(i)}$. The results of this process are explored now.

Proposition 7.28. *The sizes of $\mathcal{E}_1(H_j^{(i)})$ and $\mathcal{E}_l^{(k)}(H_j^{(i)})$ for $l, k \in \{2, 3\}$ are given in Table 7.16.*

Proof. We construct $\mathcal{E}_1(H_j^{(i)})$ using `Classes` and $\mathcal{E}_l^{(k)}(H_j^{(i)})$ directly. ∎

i	j	$\lvert\mathcal{E}_1(H_j^{(i)})\rvert$	$\lvert\mathcal{E}_2^{(2)}(H_j^{(i)})\rvert$	$\lvert\mathcal{E}_2^{(3)}(H_j^{(i)})\rvert$	$\lvert\mathcal{E}_3^{(2)}(H_j^{(i)})\rvert$	$\lvert\mathcal{E}_3^{(3)}(H_j^{(i)})\rvert$
1	1	80	48	4	2	1
1	2	264	72	4	3	1
1	3	280	108	12	10	0
1	4	320	40	28	3	3
2	1	20	8	4	2	1
3	1	60	28	12	4	0
3	2	40	28	0	2	0

Table 7.16: Sizes of $\mathcal{E}_1(H_j^{(i)})$ and $\mathcal{E}_l^{(k)}(H_j^{(i)})$ for $l, k \in \{2, 3\}$

We will now show that some of the $\Omega_8^+(2) : k$ subgroups we have found are actually conjugate.

Proposition 7.29. *For each $k \in \{2, 3\}$, the number of $\Omega_8^+(2) : k$ subgroups up to G-conjugacy containing each $H_j^{(i)}$ is given in Table 7.17.*

i	j	Number of $\Omega_8^+(2) : 2$ subgroups	Number of $\Omega_8^+(2) : 3$ subgroups
1	1	2	1
1	2	1	1
1	3	6	0
1	4	3	3
2	1	1	1
3	1	2	0
3	2	2	0

Table 7.17: Number of $\Omega_8^+(2) : k$ subgroups of G contining $H_j^{(i)}$

Proof. We run through all the elements in $N_{S_j^{(i)}}(H_j^{(i)})$ looking for elements which conjugate the $\Omega_8^+(2) : k$ subgroups containing $H_j^{(i)}$ to each other. Note that if $L_1 \cong L_2 \cong \Omega_8^+(2) : k$ are two subgroups such that $L_1 \neq L_2$, $H_j^{(i)} \leq L_1$, and $H_j^{(i)} \leq L_2$, then if there is no element $n \in N_{S_j^{(i)}}(H_j^{(i)})$ such that $L_1^n = L_2$, then L_1 and L_2 are not conjugate in G. To see why, for this proof let $H = H_k^{(i,j)}$ and $P = P_i$. Since H is the unique $\Omega_8^+(2)$ subgroup of L_1 and L_2, any $n \in G$ such that $L_1^n = L_2$ must normalise H. Then $P^n \in \mathrm{Syl}_5(H)$ and so there is some $h \in H$ such that $P^{nh} = P$. Now $H^{nh} = H$ where $nh \in N_H(P)$. Moreover, $N_H(P)^{nh} = N_H(P^{nh}) = N_H(P)$. Hence $L_1^{nh} = L_2$ where $nh \in N_{N_G(P)}(N_H(P)) = N_j^{(i)}$, as required. The results of this search are given in the table. ∎

Now, clearly, we have that $\Omega_8^+(2) : \mathrm{Sym}(3) \geq \Omega_8^+(2) : 3$. Therefore, overgroups of $H_j^{(i)}$ isomorphic to $\Omega_8^+(2) : \mathrm{Sym}(3)$ only exist if there exist overgroups of $H_j^{(i)}$ isomorphic to $\Omega_8^+(2) : 3$. Hence, we need only consider extending $H_j^{(i)}$ to $H_j^{(i)} : \mathrm{Sym}(3)$

where $(i,j) \in \{(1,1),(1,2),(1,4),(2,1)\}$. To begin, let us consider the following fact regarding $\Omega_8^+(2) : \mathrm{Sym}(3)$.

Lemma 7.30. *Suppose* $\Omega_8^+(2) \cong H \leq H_0 \cong \Omega_8^+(2) : \mathrm{Sym}(3)$ *and let* $x, y \in H_0$ *such that* $\langle H, y\rangle \cong \Omega_8^+(2) : 3$ *and* $\langle H, x, y\rangle = H_0$. *Then* $x^2 \in H$ *and* $(xy)^2 \in H$. *Moreover, we have* $(xy_0)^2 \in H$ *for every* $y_0 \in \langle H, y\rangle \setminus H$.

Proof. As $H_0 \cong \Omega_8^+(2) : \mathrm{Sym}(3)$ is a split extension, we know that $H_0/H \cong \mathrm{Sym}(3)$. Given $h \in H_0$, denote by $\overline{h}$ the coset hH. As $\langle H, x, y\rangle = H_0$, we have $\langle \overline{x}, \overline{y}\rangle \cong \mathrm{Sym}(3)$. Note that $\overline{y}$ has order 3, as $y^3 \in H$ and $y, y^2 \notin H$. In order for $\overline{x}$ to generate Sym(3) along with an element of order 3, it must have order 2. Thus $\overline{x}^2 = \overline{1}$ which implies that $x^2 \in H$. Finally, we know that in Sym(3) the product of any involution with an element of order 3 is an involution. Therefore, $(\overline{xy})^2 = \overline{1}$, which is to say that $(xy)^2 \in H$.

Now, let $y_0 \in \langle H, y\rangle \setminus H$. As H is maximal in $\langle H, y\rangle$, we have that $\langle H, y\rangle = \langle H, y_0\rangle$ and hence the above argument holds for y_0 in place of y. ∎

For a fixed $H_j^{(i)}$, let $k = 1$ if $(i,j) \in \{(1,1),(1,2),(2,1)\}$ and $k \in \{1,2,3\}$ if $(i,j) = (1,4)$. Now, let y_k be such that $\langle H_j^{(i)}, y_k\rangle$ are the distinct copies of $\Omega_8^+(2) : 3$ found in Proposition 7.28. We now seek to find x such that $\langle H_j^{(i)}, x, y_k\rangle \cong \Omega_8^+(2) : \mathrm{Sym}(3)$. Since x must normalise $H_j^{(i)}$, we know we can restrict our search to $N_{S_j^{(i)}}(H_j^{(i)})$. By Lemma 7.30 we know that $x^2 \in H_j^{(i)}$ and $(xy_k)^2 \in H_j^{(i)}$. Moreover, x must normalise $\langle H_j^{(i)}, y_k\rangle \cong \Omega_8^+(2) : 3$, as $\Omega_8^+(2) : 3$ is a normal subgroup of $\Omega_8^+(2) : \mathrm{Sym}(3)$ (as it has index 2). Let $L_k^{(i,j)} = \langle H_j^{(i)}, y_k\rangle$. Hence we start by constructing

$$\mathcal{X}_0(L_k^{(i,j)}) = \{x \in N_{S_j^{(i)}}(H_j^{(i)}) : x^2 \in H_j^{(i)},\ (xy_k)^2 \in H_j^{(i)},\ \langle H_j^{(i)}, y_k\rangle^x = \langle H_j^{(i)}, y_k\rangle\}.$$

Now, suppose $x, x_0 \in \mathcal{X}_0(L_k^{(i,j)})$ such that x and x_0 are conjugate in $\langle \mathcal{X}_0(L_k^{(i,j)})\rangle$. Then $\langle H_j^{(i)}, x, y_k\rangle$ and $\langle H_j^{(i)}, x_0, y_k\rangle$ are conjugate groups. Indeed, let $g \in \langle \mathcal{X}_0(H_j^{(i)})\rangle$ such that $x^g = x_0$. Note that, as $g \in \langle \mathcal{X}_0(H_j^{(i)})\rangle$, we have $\langle H_j^{(i)}, y_k\rangle^g = \langle H_j^{(i)}, y_k\rangle$. And so we have

$$\begin{aligned}\langle H_j^{(i)}, x, y_k\rangle^g &= \langle \langle H_j^{(i)}, y_k\rangle, x\rangle^g \\ &= \langle \langle H_j^{(i)}, y_k\rangle^g, x^g\rangle \\ &= \langle \langle H_j^{(i)}, y_k\rangle, x_0\rangle \\ &= \langle H_j^{(i)}, x_0, y_k\rangle\end{aligned}$$

as required. Note that $\mathcal{X}_0(L_k^{(i,j)})$ is not a group, which is why we demand conjugacy in $\langle \mathcal{X}_0(L_k^{(i,j)})\rangle$. Hence, we need only take elements from $\mathcal{X}_0(L_k^{(i,j)})$ up to conjugacy in $\langle \mathcal{X}_0(L_k^{(i,j)})\rangle$. In fact, we may actually take an arbitrary set of $\langle \mathcal{X}_0(L_k^{(i,j)})\rangle$-conjugacy class representatives, as the next result will show.

Lemma 7.31. *Let $L = L_k^{(i,j)}$ and $H = H_k^{(i,j)}$ and let $\mathcal{X}_1(L)$ be a set of conjugacy class representatives of $\langle \mathcal{X}_0(L)\rangle$. Let $x \in \mathcal{X}_1(L)$ such that x is $\langle \mathcal{X}_0(L)\rangle$-conjugate to some element $x_0 \in \mathcal{X}_0(L)$. Then*

(i) $x^2 \in H$, x normalises H, and for all $y \in L$ such that $L = \langle H, y\rangle$, we have $(xy)^2 \in H$;

(ii) $\langle H, x, y\rangle$ and $\langle H, x_0, y\rangle$ are conjugate groups.

Proof. Let $g \in \langle \mathcal{X}_0(L)\rangle$ such that $x = x_0^g$. Note that $H^g = H$ and $L^g = L$.

(i) Note that since $x_0 \in \mathcal{X}_0(L)$, we have $x_0^2 \in H$. Hence

$$x^2 = (x_0^g)^2 = (x_0^2)^g \in H^g = H$$

as required. Now let $y \in L$ such that $\langle H, y\rangle = L$. Then

$$L^x = L^{g^{-1}x_0 g} = L$$

as required. Finally, since $L^g = L$ we know that there must exist some $y_0 \in L$ for which $y_0^g = y$. And since $x_0 \in \mathcal{X}_0(L)$ we have that $(x_0 y)^2 \in H$ for all $y \in L \setminus H$.

$$(xy)^2 = (x_0^g y_0^g)^2 = ((x_0 y_0)^g)^2 = ((x_0 y_0)^2)^g \in H^g = H$$

as required. This completes part (i).

(ii) Here, we simply have again that $L = \langle H, y\rangle$ which is normalised by g, and so

$$\langle H, x, y\rangle^g = \langle L, x\rangle^g = \langle L^g, x^g\rangle = \langle L, x_0\rangle = \langle H, x_0, y\rangle$$

as required, which completes part (ii).

■

Hence, we let $\mathcal{X}_1(L_k^{(i,j)})$ be a set of $\langle \mathcal{X}_0(L_k^{(i,j)})\rangle$-conjugacy class representatives of $\mathcal{X}_0(L_k^{(i,j)})$. Finally, we let $\mathcal{X}_2(L_k^{(i,j)})$ be a set of elements $x \in \mathcal{X}_1(L_k^{(i,j)})$ such that $\langle H_j^{(i)}, x, y_k\rangle$ is a distinct copy of $\Omega_8^+(2) : \mathrm{Sym}(3)$. We will give the results of this process before determining whether any of the copies of $\Omega_8^+(2) : \mathrm{Sym}(3)$ are conjugate.

Proposition 7.32. *The sizes of* $\mathcal{X}_0(L_k^{(i,j)})$, $\langle \mathcal{X}_0(L_k^{(i,j)}) \rangle$, $\mathcal{X}_1(L_k^{(i,j)})$, *and* $\mathcal{X}_2(L_k^{(i,j)})$ *are given in Table 7.18.*

i	j	k	$\lvert\mathcal{X}_0(L_k^{(i,j)})\rvert$	$\lvert\langle\mathcal{X}_0(L_k^{(i,j)})\rangle\rvert$	$\lvert\mathcal{X}_1(L_k^{(i,j)})\rvert$	$\lvert\mathcal{X}_2(L_k^{(i,j)})\rvert$
1	1	1	2,400	4,800	16	2
1	2	1	4,800	9,600	16	4
1	4	1	379,200	29,030,400	24	3
1	4	2	14,400	43,200	16	10
1	4	3	43,200	259,200	16	2
2	1	1	1,200	2,400	8	1

Table 7.18: Sizes of $\mathcal{X}_0(L_k^{(i,j)})$, $\langle \mathcal{X}_0(L_k^{(i,j)}) \rangle$, $\mathcal{X}_1(L_k^{(i,j)})$, and $\mathcal{X}_2(L_k^{(i,j)})$

Proof. We construct these sets in Magma directly. ■

The final part of the construction of $\Omega_8^+(2)$: Sym(3) subgroups is to show that many of the $\Omega_8^+(2)$: Sym(3) subgroups we have found are actually conjugate.

Proposition 7.33. *The number of* $\Omega_8^+(2)$: Sym(3) *subgroups containing each* $L_k^{(i,j)}$ *up to* G*-conjugacy is given in Table 7.19.*

i	j	k	Number of $\Omega_8^+(2)$: Sym(3) subgroups
1	1	1	2
1	2	1	1
1	4	1	3
1	4	2	2
1	4	3	1
2	1	1	1

Table 7.19: Number of $\Omega_8^+(2)$: Sym(3) subgroups containing each $L_k^{(i,j)}$

Proof. We follow the same steps as in the proof of Proposition 7.29. ■

Before proving Theorem 1.4, we will provide a proof that none of the subgroups constructed in this chapter are maximal.

Proposition 7.34. *Suppose* $H \leq G$ *with* $F^*(H) \cong \Omega_8^+(2)$*. Then* H *is not a maximal subgroup of* G*.*

Proof. If $F^*(H) \cong \Omega_8^+(2)$, then $F^*(H)$ is G-conjugate to one of the seven $\Omega_8^+(2)$ subgroups constructed in this chapter. Recall that these are named $H_j^{(i)}$ where $(i, j) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (3, 1), (3, 2)\}$. We will work through each case in turn.

(i) Case $H_1^{(1)}$: Here we have L_1 and L_2, two non-conjugate copies of $\Omega_8^+(2):2$ containing $H_1^{(1)}$. Also, we have K_1 and K_2, two non-conjugate copies of $\Omega_8^+(2):\mathrm{Sym}(3)$ containing $H_1^{(1)}$. Then, without loss of generality, we may assume that $L_k \leq K_k$ for $k \in \{1,2\}$. Indeed, K_1 contains one copy of $\Omega_8^+(2):2$ up to conjugacy, and K_2 contains one copy of $\Omega_8^+(2):2$ up to conjugacy. Since K_1 and K_2 are not G-conjugate, it follows that their respective $\Omega_8^+(2):2$ subgroups are not G-conjugate. Since there are only two $\Omega_8^+(2):2$ subgroups containing $H_1^{(1)}$ – namely L_1 and L_2 – they must correspond to the $\Omega_8^+(2):2$ subgroups found in K_1 and K_2.

Define $X_1^{(1)} = \langle K_1, K_2\rangle$. By the argument above, we have that every $H \leq G$ such that $F^*(H) \cong \Omega_8^+(2)$ and $H_1^{(1)} \leq H$ is such that $H \leq X_1^{(1)}$. We construct $X_1^{(1)}$ and find that it is a proper subgroup of G, thereby proving that H is not maximal.

(ii) Case $H_2^{(1)}$: There is $L \cong \Omega_8^+(2):2$ and $K \cong \Omega_8^+(2):\mathrm{Sym}(3)$ such that $H_2^{(1)} \leq L$ and $H_2^{(1)} \leq K$. By the same argument as in part (i), we may assume without loss of generality $L \leq K$. Therefore, every $H \leq G$ such that $F^*(H) \cong \Omega_8^+(2)$ and $H_2^{(1)} \leq H$ is such that $H \leq K$. Now we simply find an element $m \in N_2^{(1)}$ such that $K < \langle K, m\rangle < G$, so K is not maximal.

(iii) Case $H_3^{(1)}$: Here we have $L_1, \ldots, L_6 \cong \Omega_8^+(2):2$ such that $H_3^{(1)} \leq L_k$ for each $k \in \{1,\ldots,6\}$. Now define $X_3^{(1)} = \langle L_k : k \in \{1,\ldots,6\}\rangle$ and we see that $X_3^{(1)}$ is a proper subgroup of G. Hence and $H \leq G$ such that $F^*(H) \cong \Omega_8^+(2)$ and $H_3^{(1)} \leq H$ is not maximal.

(iv) Case $H_4^{(1)}$: In this case, we have three $\Omega_8^+(2):2$ subgroups L_1, L_2, L_3 such that $H_4^{(1)} \leq L_k$ for $k \in \{1,2,3\}$. Moreover, we have six $\Omega_8^+(2):\mathrm{Sym}(3)$ subgroups $K_1,\ldots,K_6$ such that $H_4^{(1)} \leq K_k$ for $k \in \{1,\ldots,6\}$. Finally, we have three $\Omega_8^+(2):3$ subgroups R_1, R_2, R_3 such that $H_4^{(1)} \leq R_k$ for $k \in \{1,2,3\}$. Recall that, by construction, we also have $R_1 \leq K_k$ for $k \in \{1,2,3\}$, $R_2 \leq K_k$ for $k \in \{4,5\}$, and $R_3 \leq K_6$. By the same argument in part (i), we may assume without loss of generality that $L_k \leq K_k$ for each $k \in \{1,2,3\}$.

Similarly to part (i), we define $X_4^{(1)} = \langle K_1,\ldots,K_6\rangle$. By the above, we have that every $H \leq G$ such that $F^*(H) \cong \Omega_8^+(2)$ and $H_4^{(1)} \leq H$ is such that $H \leq X_4^{(1)}$.

Upon constructing $X_4^{(1)}$ we find that it is a proper subgroup of G, so H is not maximal.

(v) Case $H_1^{(2)}$: Let K be the sole $\Omega_8^+(2) : \mathrm{Sym}(3)$ subgroup containing $H_1^{(2)}$. We find an element $m \in N_G(P_2)$ such that $K < \langle K, m\rangle < G$, thereby proving that K is not maximal.

(vi) Case $H_1^{(3)}$: Here, we have two $\Omega_8^+(2) : 2$ subgroups L_1 and L_2 containing $H_1^{(3)}$. Let $X_1^{(3)} = \langle L_1, L_2\rangle$ and we find that $X_1^{(3)}$ is a proper subgroup of G. Therefore any $H \leq G$ with $F^*(H)$ conjugate to $H_1^{(3)}$ is such that $H < X_1^{(3)} < G$ and so H is not maximal.

(vii) Case $H_2^{(3)}$: We follow the same argument as in part (vi).

By exhaustion of cases, any $H \leq G$ with $F^*(H) \cong \Omega_8^+(2)$ is not maximal in G.

■

We will conclude this chapter with a proof of Theorem 1.4. As with other proofs of the main results, this is mostly a matter of compiling results together from the rest of the chapter. As always, let P_1, P_2, and P_3 be the elementary abelian 5^2 subgroups of G used to construct our copies of $\Omega_8^+(2)$ and let $H_j^{(i)} \cong \Omega_8^+(2)$ where $(i, j) \in \{(1, 1),(1, 2),(1, 3),(1, 4),(2, 1),(3, 1),(3, 2)\}$ and $P_i \in \mathrm{Syl}_5(H_j^{(i)})$. Furthermore, recall that $P_iT_j^{(i)} = N_{H_j^{(i)}}(P_i)$ where the subgroups $T_j^{(i)}$ represent distinct classes of $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups of $N_G(P_i)$. Note that $H_j^{(i)}$ were constructed as follows. We found $H_1^{(2)}$, $H_1^{(3)}$, and $H_2^{(3)}$ in Proposition 7.22; $H_1^{(1)}$ and $H_2^{(1)}$ in Proposition 7.23; $H_3^{(1)}$ in Proposition 7.24; and $H_4^{(1)}$ in Proposition 7.25.

(i) This part states that there are exactly seven G-conjugacy classes of subgroups isomorphic to $\Omega_8^+(2)$. We will now prove that each $H_j^{(i)}$ represents a distinct G-conjugacy class of $\Omega_8^+(2)$ subgroups of G. That is, for all $(i, j),(i_0, j_0) \in \{(1, 1),(1, 2),(1, 3),(1, 4),(2, 1),(3, 1),(3, 2)\}$ with $(i, j) \neq (i_0, j_0)$, $H_j^{(i)}$ and $H_{j_0}^{(i_0)}$ are not G-conjugate. Clearly, if $i \neq i_0$, then P_i and P_{i_0} follow different fusion patterns (see Lemma 7.3), and by extension, so do $H_j^{(i)}$ and $H_{j_0}^{(i_0)}$. Hence $H_j^{(i)}$ and $H_{j_0}^{(i_0)}$ cannot be conjugate.

Now assume $i = i_0$ and let $H = H_j^{(i)}$ and $H_0 = H_{j_0}^{(i)}$. Also let $P = P_i$, $T = T_j^{(i)}$, and $T_0 = T_{j_0}^{(i)}$. Let $g \in G$ such that $H^g = H_0$. Then $P, P^g \in \mathrm{Syl}_5(H_0)$ and so there is some $h \in H_0$ such that $P = P^{gh}$. Let $n = gh$ and hence $H^n = H_0$ where $n \in N_G(P)$. Now note that T normalises P and so T^n normalises P as well. Hence $T^n \leq N_{H_0}(P)$ and so $T^n, T_0 \in \mathrm{Syl}_2(N_{H_0}(P))$ which means there is some $n_0 \in N_{H_0}(P)$ such that $T^{nn_0} = T_0$. But now, since $nn_0 \in N_G(P)$, we have a contradiction, as T, T_0 represent distinct classes of $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups in $N_G(P)$. Thus, all seven $H_j^{(i)}$ are not conjugate in G.

(ii) Here, the theorem states that there are exactly seventeen G-conjugacy classes of subgroups isomorphic to $\Omega_8^+(2) : 2$. Let $K, K_0 \leq G$ such that $K \cong \Omega_8^+(2) : 2 \cong K_0$ and suppose $\Omega_8^+(2) \cong L \leq K$ and $\Omega_8^+(2) \cong L_0 \leq K_0$. Note that L is unique inside K and L_0 is unique inside K_0. Then it follows that if L and L_0 are not conjugate in G, then K and K_0 are not conjugate in G. Otherwise, $K^g = K_0$ for some $g \in G$ implies that $L^g = L_0$ by the uniqueness of L and L_0. Therefore, by part (i), if we have $(i, j), (i_0, j_0) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (3, 1), (3, 2)\}$ with $(i, j) \neq (i_0, j_0)$, any $\Omega_8^+(2) : 2$ subgroup containing $H_j^{(i)}$ is not conjugate to any $\Omega_8^+(2) : 2$ subgroup containing $H_{j_0}^{(i_0)}$. By Proposition 7.29, any two $\Omega_8^+(2) : 2$ subgroups containing $H_j^{(i)}$ are not conjugate. Therefore, all seventeen $\Omega_8^+(2) : 2$ subgroups from Proposition 7.29 are not conjugate in G, yielding the result.

(iii) This part states that there are exactly six G-conjugacy classes of subgroups isomorphic to $\Omega_8^+(2) : 3$. By the same argument as in the proof of part (ii), all six $\Omega_8^+(2) : 3$ subgroups constructed in Proposition 7.29 are not conjugate in G.

(iv) Finally, the theorem states here that there are 10 G-conjugacy classes of subgroups isomorphic to $H \cong \Omega_8^+(2) : \mathrm{Sym}(3)$. By the same argument as in part (ii), all ten $\Omega_8^+(2) : \mathrm{Sym}(3)$ subgroups constructed in Proposition 7.33 are not conjugate in G.

This concludes the proof of Theorem 1.4. ∎

Chapter 8

$\Omega_8^+(4)$ and Its Extensions

In this chapter, we will find all classes of subgroups H of G such that $F^*(H) \cong \Omega_8^+(4)$. In contrast to most other chapters, finding all classes of $\Omega_8^+(4)$ subgroups in G is relatively straightforward. Most of the work in this chapter is in aid of extending these copies of $\Omega_8^+(4)$ to various automorphism extension groups. Given that $\mathrm{Aut}(\Omega_8^+(4)) \cong \Omega_8^+(4) : \mathrm{Dih}(12)$, and that there are five isomorphism types of proper subgroups of $\mathrm{Dih}(12)$, there are six possible overgroups of a given $\Omega_8^+(4)$ isomorphic to some automorphism extension of $\Omega_8^+(4)$. We will calculate the possibilities for each of these in turn. First, we must construct $\Omega_8^+(4)$ subgroups of G.

8.1 Constructing $\Omega_8^+(4)$ subgroups of G

We will proceed in a manner similar to how we constructed $Sp_6(2)$ subgroups by building up from copies of $U_4(2)$; since we know that $\Omega_8^+(2) \leq \Omega_8^+(4)$, we can apply Proposition 2.12 and say that, up to G-conjugacy, any $\Omega_8^+(4)$ subgroup of G contains one of the seven $\Omega_8^+(2)$ subgroups found in Chapter 7. This first result will outline our process for building $\Omega_8^+(4)$ subgroups from our $\Omega_8^+(2)$ subgroups.

Proposition 8.1. *Let $\Omega_8^+(2) \cong K \leq H \cong \Omega_8^+(4)$ and let $P \in \mathrm{Syl}_5(K)$. Then $5^4 \cong C_H(P) \not\leq K$ and, since K is maximal in H, we have that $H = \langle K, C_H(P) \rangle$.*

Proof. This is directly verifiable in Magma. ∎

Now, given $\Omega_8^+(2) \cong K \leq G$, if some $H \cong \Omega_8^+(4)$ exists with $K \leq H \leq G$ then we must have $C_H(P) \leq C_G(P)$ and therefore there exists some $S \leq C_G(P)$ with $S \cong 5^4$

such that

$$H = \langle K, S \rangle.$$

Let us now establish some notation and summarise our results on $\Omega_8^+(2)$ from Chapter 7. Recall that there are three elementary abelian subgroups P_1, P_2, and P_3 of G of order 5^2 which are contained in $\Omega_8^+(2)$ subgroups of G. We will name our $\Omega_8^+(2)$ subgroups $K_j^{(i)}$ where $P_i \leq K_j^{(i)}$ and $j \in \{1, 2, 3, 4\}$ when $i = 1$; $j = 1$ when $i = 2$; and $j \in \{1, 2\}$ when $i = 3$. Now recall that in Proposition 7.4 we calculated $C_G(P_i)$ for each $i \in \{1, 2, 3\}$. Hence we can proceed straight to finding 5^4 subgroups of each $C_G(P_i)$ and seeing which ones form a generating set for $\Omega_8^+(4)$.

Proposition 8.2. *There are exactly two classes of $\Omega_8^+(4)$ subgroups in G.*

Proof. Let us first remind ourselves of $|C_G(P_i)|$ for each $i \in \{1, 2, 3\}$. We have $|C_G(P_1)| = 2^{12}.3^5.5^4.7$; $|C_G(P_2)| = 5^4$; and $|C_G(P_3)| = 2^4.3^2.5^4$. Now we note that, for each $i \in \{1, 2, 3\}$, a Sylow 5-subgroup of $C_G(P_i)$ is isomorphic to 5^4. Hence, to sift through all the elementary abelian subgroups of order 5^4 in $C_G(P_i)$, it is sufficient to sift through all Sylow 5-subgroups of $C_G(P_i)$.

Fix $(i, j) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (3, 1), (3, 2)\}$. Let $S_i \in \mathrm{Syl}_5(C_G(P_i))$ and R_i be a right transversal for $N_{C_G(P_i)}(S_i)$, so that $\{S_i^r : r \in R_i\}$ is a complete set of Sylow 5-subgroups of $C_G(P_i)$. Now, for each $r \in R_i$ we build $Y = \langle K_j^{(i)}, S_i^r \rangle$. We now run an order of random elements sieve on Y (see Procedure 3.8 for full details). Note that to use this sieve, we require a list of all possible element orders appearing in $\Omega_8^+(4)$. This is

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$$

as can be seen in the $\mathbb{ATLAS}$ [14].

Let $(i, j) = (1, 1)$. After running this sieve, we find four elements y_1, y_2, y_3, and y_4 such that $Y_k = \langle K_1^{(1)}, S_1^{y_k} \rangle \cong \Omega_8^+(4)$ for $k \in \{1, 2, 3, 4\}$. Note that we confirm that these groups are isomorphic to $\Omega_8^+(4)$ using `LMGChiefFactors`. However, we will now show that all four of these subgroups are G-conjugate. Recall that in Proposition 7.27 we calulated a group normalising $K_1^{(1)}$. Call this $A_1^{(1)}$ and we have that $|A_1^{(1)}| = 2^8.3.5^2$. For each $k \in \{2, 3, 4\}$ we can find $c_k \in A_1^{(1)}$ such that $(S_1^{y_k})^{c_k} = S_1^{y_1}$. But since these elements c_k normalise $K_1^{(1)}$, we have that

$$Y_k^{c_k} = \langle K_1^{(1)}, S_1^{y_k} \rangle^{c_k} = \langle (K_1^{(1)})^{c_k}, (S_1^{y_k})^{c_k} \rangle = \langle K_1^{(1)}, S_1^{y_1} \rangle = Y_1.$$

Hence all four of these groups are conjugate in G.

Now, let $(i, j) = (3, 2)$. After running the order of random elements sieve, we have a unique Sylow 5-subgroup $S_3 \leq C_G(P_3)$ such that $\langle K_2^{(3)}, S_3 \rangle \cong \Omega_8^+(4)$.

Finally, let $(i, j) \in \{(1, 2), (1, 3), (1, 4), (2, 1), (3, 1)\}$. No Sylow 5-subgroups survive the sieve in any of these cases. Hence there are no $\Omega_8^+(4)$ subgroups of G containing $K_j^{(i)}$.

In the end, we have two distinct $\Omega_8^+(4)$ subgroups. Moreover, we know that these groups are not conjugate as they follow different fusion patterns. ■

We will now find automorphism extensions in G of these two copies of $\Omega_8^+(4)$.

8.2 Extending $\Omega_8^+(4)$ to $\Omega_8^+(4) : \mathrm{Dih}(12)$

Let us first identify the groups which we are trying to construct. We know that $\mathrm{Aut}(\Omega_8^+(4)) = \Omega_8^+(4) : \mathrm{Dih}(12)$, and note that there are five non-trivial isomorphism types of proper subgroups of Dih(12). These are the groups 2, 3, 2^2, 6, and Sym(3). Hence we will find all subgroups of G, up to G-conjugacy, isomorphic to each of:

$$\Omega_8^+(4) : 3,\ \Omega_8^+(4) : 2^2,\ \Omega_8^+(4) : 6,\ \Omega_8^+(4) : \mathrm{Sym}(3),\ \text{and}\ \Omega_8^+(4) : \mathrm{Dih}(12),$$

and of shape $\Omega_8^+(4) : 2$. Note that there are three distinct conjugacy classes of groups isomorphic to 2 in Dih(12), so there are three distinct isomorphism types of subgroups of shape $\Omega_8^+(4) : 2$ in $\Omega_8^+(4) : \mathrm{Dih}(12)$.

We will now establish the notation used throughout the remainder of this chapter. As usual, we will use P_1 and P_3 to refer to the elementary abelian 5^2 subgroups of G contained in some $\Omega_8^+(4)$. As we discovered in the previous section, there are two $\Omega_8^+(4)$ subgroups in G up to G-conjugacy. We will call these K_1 and K_3, where $P_i \leq K_i$ for $i \in \{1, 3\}$. Now let $S_i \in \mathrm{Syl}_5(K_i)$ such that $P_i \leq S_i$, as constructed in the previous section.

Fix $i \in \{1, 3\}$. Now apply Corollary 2.14, which implies that if $\langle K_i, h \rangle$ is some automorphism extension of K_i, then g can be chosen such that $g \in N_G(S_i)$ and $\langle K_i, g \rangle = \langle K_i, h \rangle$.

Proposition 8.3. *We have that* $N_G(S_i) \sim 5^4.((4 \circ 2^{1+4}).\mathrm{Alt}(6).2)$ *and we have obtained* $N_G(S_i)$ *for each* $i \in \{1, 3\}$.

Proof. Since we have that $S_i \cong 5^4$, we know that S_i is a maximal torus by Proposition 2.22 and hence $N_G(S_i)$ has the stated shape and is maximal by 4.5 of [5]. Since $P_i \leq S_i \leq N_G(P_i)$, we start by calculating $N_{N_G(P_i)}(S_i)$. Recall that we already have $N_G(P_i)$ from Proposition 7.7, so we can calculate $N_{N_G(P_i)}(S_i)$ directly. We find

$$|N_{N_G(P_i)}(S_i)| = \begin{cases} 2^9.3.5^4, & \text{if } i = 1, \\ 2^7.5^4, & \text{if } i = 3. \end{cases}$$

However, we know that $|N_G(S_i)| = 2^{10}.3^2.5^5$, so we know we have not yet found the whole of $N_G(S_i)$. To find the rest, we observe that $N_G(S_i) \leq \mathrm{Stab}_G(C_V(S_i)) \leq G$. But the maximality of $N_G(S_i)$ yields $N_G(S_i) = \mathrm{Stab}_G(C_V(S_i))$. To generate the rest of $N_G(S_i)$, we take $t \in N_{N_G(P_i)}(S_i) \cap 2\mathrm{D}_G$ and construct $C_G(t)$ using the command `CentraliserOfInvolution`. Let $X \in \mathrm{Syl}_2(C_G(t))$ – which we find using `LMGSylow` – and calculate $\mathrm{Stab}_X(C_V(S_i))$ using `UnipotentStabiliser`. We repeat this process, choosing a new t each time, until we find $\mathrm{Stab}_X(C_V(S_i)) \nleq N_{N_G(P_i)}(S_i)$ and until $|\langle \mathrm{Stab}_X(C_V(S_i)), N_{N_G(P_i)}(S_i)\rangle| = 2^{10}.3^2.5^5$. This is $N_G(S_i)$, as required. ■

Now, clearly, if $g \in N_G(S_i)$ acts as an automorphism on K_i, then $g \in N_{N_G(S_i)}(K_i)$, which we will calculate in the next result.

Proposition 8.4. *We have*

$$|N_{N_G(S_i)}(K_i)| = \begin{cases} 2^8.3^2.5^4, & \text{if } i = 1, \\ 2^8.3.5^4, & \text{if } i = 3. \end{cases}$$

Proof. We calculate this by observing that $N_{N_G(S_i)}(K_i) \leq \mathrm{Stab}_{N_G(S_i)}(C_V(K_i))$, which can be shown the same way as in the proof of Lemma 7.26. Now we calculate $\mathrm{Stab}_{N_G(S_i)}(C_V(K_i))$ using the command `Stabiliser`. But then we find that every generator $g \in \mathrm{Stab}_{N_G(S_i)}(C_V(K_i))$ is such that $K_i^g = K_i$, and so we actually have that $N_{N_G(S_i)}(K_i) = \mathrm{Stab}_{N_G(S_i)}(C_V(K_i))$. Hence we have found $N_{N_G(S_i)}(K_i)$. Their orders are as given in the result. ■

From here on, we will let $\mathcal{E}_0(K_i) = N_{N_G(S_i)}(K_i)$. Now we will begin the process of constructing automorphism extensions of each K_i, beginning with the construction of subgroups of shape $\Omega_8^+(4):2$ and $\Omega_8^+(4):3$. Let $k \in \{2,3\}$. Then by Proposition 2.17 we know that if $g \in \mathcal{E}_0(K_i)$ such that $\langle K_i, g\rangle \sim \Omega_8^+(4):k$, then $g \notin K_i$ and $g^k \in K_i$.

Furthermore, by Proposition 2.16 every $g_0 \in \mathcal{E}_0(K_i)$ which is $\mathcal{E}_0(K_i)$-conjugate to g will be such that $g_0 \notin K_i$ and $g_0^k \in K_i$ and will generate a conjugate copy of $\Omega_8^+(4) : k$. Hence we let $\mathcal{E}_1(K_i)$ be a set of $\mathcal{E}_0(K_i)$-conjugacy class representatives and it is in $\mathcal{E}_1(K_i)$ we look for our elements.

Now, for each $k \in \{2, 3\}$, we construct

$$\mathcal{E}_2^{(k)}(K_i) = \{x \in \mathcal{E}_1(K_i) : x \notin K_i,\ x^k \in K_i\}.$$

Finally, let $\mathcal{E}_3^{(k)}(K_i)$ be a set of elements from $\mathcal{E}_2^{(k)}(K_i)$ which generate distinct subgroups of shape $\Omega_8^+(4) : k$. The next result describes the outcome of this process.

Proposition 8.5. *For each $i \in \{1, 3\}$, the sizes of $\mathcal{E}_1(K_i)$ and $\mathcal{E}_l^{(k)}(K_i)$ for $k, l \in \{2, 3\}$ are given in Table 8.1. Moreover, there are at most six subgroups of shape $\Omega_8^+(4) : 2$ up to conjugacy in G. Three of these contain K_1, and the other three contain K_3. Finally, there is exactly one copy of $\Omega_8^+(4) : 3$ up to conjugacy in G.*

i	$\lvert\mathcal{E}_1(K_i)\rvert$	$\lvert\mathcal{E}_2^{(2)}(K_i)\rvert$	$\lvert\mathcal{E}_3^{(2)}(K_i)\rvert$	$\lvert\mathcal{E}_2^{(3)}(K_i)\rvert$	$\lvert\mathcal{E}_3^{(3)}(K_i)\rvert$
1	92	48	5	11	1
3	93	52	3	0	0

Table 8.1: Sizes of $\mathcal{E}_1(K_i)$ and $\mathcal{E}_l^{(k)}(K_i)$ for $k, l \in \{2, 3\}$

Proof. We calculate $\mathcal{E}_1(K_i)$ using `Classes` on $\mathcal{E}_0(K_i)$ as a permutation group. Then we calculate $\mathcal{E}_2^{(k)}(K_i)$ for $k \in \{2, 3\}$ directly. We are able to find $y_n \in \mathcal{E}_2^{(2)}(K_1)$ for $n \in \{1, \ldots, 5\}$ with $Y_n = \langle K_1, y_n\rangle \cong \Omega_8^+(4) : 2$ such that for all $y \in \mathcal{E}_2^{(2)}(K_i)$, $y \in Y_n$ for some $n \in \{1, \ldots, 5\}$. This implies that all $y \in \mathcal{E}_2^{(2)}(K_i) \setminus \{y_1, \ldots, y_5\}$ generate redundant subgroups. Indeed, let $n \in \{1, \ldots, 5\}$ such that $y \in Y_n$. Then $\langle K_1, y\rangle \leq Y_n$. If we have equality, then $\langle K_1, y\rangle$ is a duplicate of Y_n; else $\langle K_1, y\rangle$ is a proper subgroup of Y_n and so clearly is not of shape $\Omega_8^+(4) : 2$. Since we also have that if $m, n \in \{1, \ldots, 5\}$ are distinct, then $y_n \notin Y_m$, each Y_n is distinct and therefore $\mathcal{E}_3^{(3)}(K_1) = \{y_1, \ldots, y_5\}$. We can now easily find elements $c_4, c_5 \in \mathcal{E}_0(K_1)$ such that $y_4^{c_4} \in Y_1$ and $y_5^{c_5} \in Y_3$. Hence Y_1 is conjugate to Y_4, and, Y_3 is conjugate to Y_5. This brings the total number of subgroups of shape $\Omega_8^+(4) : 2$ up to conjugacy containing K_1 from five down to three, as stated. A similar situation occurs with $\mathcal{E}_2^{(2)}(K_3)$ where we find three elements instead of five.

Now, we find a unique element $z_1 \in \mathcal{E}_2^{(3)}(K_1)$ such that $Z_1 = \langle K_1, z_1\rangle \cong \Omega_8^+(4) : 3$, and for all $z \in \mathcal{E}_2^{(3)}(K_1)$ we have $z \in Z_1$. Hence by the same argument above, we have that $\mathcal{E}_3^{(3)}(K_1) = \{z_1\}$. ■

Before we proceed, a remark on the proof of Proposition 8.5. Due to the sizes of the groups we are considering, it is not practical to ask MAGMA directly whether a given group is isomorphic to $\Omega_8^+(4) : k$ for $k \in \{2, 3\}$. The `IsIsomorphic` command executes within seconds for smaller groups, but for groups of this size it can often run for hours or days without completing. Instead, for example, say we wish to know whether a given group Y is isomorphic to $\Omega_8^+(4)$: 2. First, we check that $|Y| = 2|\Omega_8^+(4)|$. This can easily be checked using `LMGFactoredOrder` or `LMGOrder`. Secondly, we use `LMGChiefFactors` to obtain the composition factors for Y. We know that, in the case of $\Omega_8^+(4)$: 2, these factors will be 2 followed by $\Omega_8^+(4)$. Finally, we must rule out the possibility that $Y \cong \Omega_8^+(4) \times 2$. We do this by running `LMGNormalSubgroups` and seeing that if Y only has three normal subgroups – 1, $\Omega_8^+(4)$, and Y itself – then $Y \cong \Omega_8^+(4) : 2$. We note that if $Y \cong \Omega_8^+(4) \times 2$, then it would also have 2 as a normal subgroup. Finally, we note that due to the sizes of these groups, we cannot, at this stage, demonstrate whether the subgroups of shape $\Omega_8^+(4)$: 2 we have generated are pairwise non-isomorphic.

We will now move on to constructing overgroups of K_i isomorphic to $\Omega_8^+(4) : 2^2$. We will use $H_j^{(i)}$ to denote one of the copies of subgroups of shape $\Omega_8^+(4) : 2$ containing K_i, so $j \in \{1, 2, 3\}$. Now, if an $\Omega_8^+(4) : 2^2$ subgroup containing K_i exists, then it must contain a subgroup of shape $\Omega_8^+(4)$: 2. Hence, without loss, we may assume that an $\Omega_8^+(4) : 2^2$ subgroup containing K_i contains some $H_j^{(i)}$. Now observe that if $g \in G$ is such that $\langle H_j^{(i)}, g\rangle \cong \Omega_8^+(4) : 2^2$, then g must normalise both K_i and $H_j^{(i)}$, as $H_j^{(i)}$ is a normal subgroup of $\langle H_j^{(i)}, g\rangle$ since it has index 2. Since we already know that such g must exist in $\mathcal{E}_0(K_j)$, we now let

$$\mathcal{E}_0(H_j^{(i)}) = \{x \in \mathcal{E}_0(K_i) : (H_j^{(i)})^x = (H_j^{(i)})\}.$$

As usual, by Propositions 2.16 and 2.17, it is sufficient to look for elements in $\mathcal{E}_0(H_j^{(i)})$ up to conjugacy. Hence, we let $\mathcal{E}_1(H_j^{(i)})$ be a set of $\mathcal{E}_0(H_j^{(i)})$-conjugacy class representatives. Now let

$$\mathcal{E}_2(H_j^{(i)}) = \{x \in \mathcal{E}_1(H_j^{(i)}) : x \notin H_j^{(i)} \text{ and } x^2 \in H_j^{(i)}\}$$

and finally $\mathcal{E}_3(H_j^{(i)})$ be a set of elements $g \in \mathcal{E}_2(H_j^{(i)})$ such that $\langle H_j^{(i)}, g\rangle$ is a distinct copy of $\Omega_8^+(4) : 2^2$. The next results explores the outcome of this process.

Proposition 8.6. *Let* $(i, j) \in \{(1, 1), (1, 2), (1, 3), (3, 1), (3, 2), (3, 3)\}$*. Then the sizes of* $\mathcal{E}_k(H_j^{(i)})$ *for* $k \in \{0, 1, 2, 3\}$ *are given in Table 8.2. Moreover, there are two* $\Omega_8^+(4) : 2^2$ *in* G *subgroups up to conjugacy; one containing* K_1 *and one containing* K_3*.*

i	j	$\lvert\mathcal{E}_0(H_j^{(i)})\rvert$	$\lvert\mathcal{E}_1(H_j^{(i)})\rvert$	$\lvert\mathcal{E}_2(H_j^{(i)})\rvert$	$\lvert\mathcal{E}_3(H_j^{(i)})\rvert$
1	1	$2^8.3.5^4$	93	27	1
1	2	$2^8.3^2.5^4$	92	37	2
1	3	$2^8.3.5^4$	93	40	1
3	1	$2^8.3.5^4$	93	27	1
3	2	$2^8.3.5^4$	93	37	1
3	3	$2^8.3.5^4$	93	40	1

Table 8.2: Sizes of $\mathcal{E}_k(H_j^{(i)})$ for $k \in \{0, 1, 2, 3\}$

Proof. We will first describe how we construct $\mathcal{E}_0(H_j^{(i)})$. Recall that $\mathcal{E}_0(H_j^{(i)}) = N_{\mathcal{E}_0(K_i)}(H_j^{(i)})$. Hence we know that $\mathcal{E}_0(H_j^{(i)}) \leq \mathrm{Stab}_{\mathcal{E}_0(K_i)}(C_V(H_j^{(i)}))$. Also, in Proposition 8.5 we found elements $h_j^{(i)}$ such that $H_j^{(i)} = \langle K_i, h_i^{(i)}\rangle$. Hence $C_V(H_j^{(i)}) = C_V(K_i) \cap C_V(h_j^{(i)})$ which we can calculate directly. Upon calculating $\mathrm{Stab}_{\mathcal{E}_0(K_i)}(C_V(H_j^{(i)}))$ we see that all of its generators normalise $H_j^{(i)}$, hence we have found $\mathcal{E}_0(H_j^{(i)})$. Now, to calculate $\mathcal{E}_k(H_j^{(i)})$ for $k \in \{1, 2, 3\}$, we follow the same steps as in the proof of Proposition 8.5.

We can now find elements in $\mathcal{E}_0(H_j^{(1)})$ which conjugate the copies of $\Omega_8^+(4) : 2^2$ containing K_1 to each other, thus proving that there is only one $\Omega_8^+(4) : 2^2$ containing K_1 up to conjugacy. Finally, we see that all three copies of $\Omega_8^+(4) : 2^2$ containing K_3 are actually identical, so there is only one such group. ■

Let us take stock of what we have achieved so far. We have found all copies up to conjugacy of subgroups of shape $\Omega_8^+(4) : 2$, $\Omega_8^+(4) : 3$, and $\Omega_8^+(4) : 2^2$. However, the subgroups $\Omega_8^+(4) : 6$, $\Omega_8^+(4) : \mathrm{Sym}(3)$, and $\Omega_8^+(4) : \mathrm{Dih}(12)$ remain to be constructed. Note that these three remaining subgroups contain $\Omega_8^+(4) : 3$ as a subgroup. This reduces the work we must do, because by Proposition 8.5 only K_1 has an overgroup isomorphic to $\Omega_8^+(4) : 3$, so therefore there are no subgroups of G isomorphic to $\Omega_8^+(4) : 6$, $\Omega_8^+(4) : \mathrm{Sym}(3)$, or $\Omega_8^+(4) : \mathrm{Dih}(12)$ which contain K_3. Hence, from now on, let $K = K_1$.

First, we will construct copies of $\Omega_8^+(4):6$ containing K. To do this, we could build up from the sole copy of $\Omega_8^+(4):3$ containing K. However, we will opt to construct $\Omega_8^+(4):6$ subgroups directly. As usual, we observe that if $g \in G$ is such that $\langle K, g\rangle \cong \Omega_8^+(4):6$, then we know that $g \in \mathcal{E}_0(K)$, where $\mathcal{E}_0(K)$ is as constructed in Proposition 8.4. Now, by Propositions 2.16 and 2.17, we know that it is sufficient to search through a set of conjugacy class representatives of $\mathcal{E}_0(K)$ and that $g^n \notin K$ for $n \in \{1,2,3,4,5\}$ and $g^6 \in K$. Hence we will use the set of $\mathcal{E}_0(K)$-conjugacy class representatives $\mathcal{E}_1(K)$ as constructed in Proposition 8.5 and now let

$$\mathcal{E}_2^{(6)}(K) = \{x \in \mathcal{E}_1(K) : x^6 \in K,\ x^n \notin K \text{ for } n \in \{1,\dots,5\}\}.$$

and finally let $\mathcal{E}_3^{(6)}$ be a set of elements $g \in \mathcal{E}_2^{(6)}(K)$ such that $\langle K, g\rangle$ is a distinct copy of $\Omega_8^+(4):6$. The following result shows the outcome of this process.

Proposition 8.7. *We have* $|\mathcal{E}_1(K)| = 92$, $|\mathcal{E}_2(K)| = 7$, *and* $|\mathcal{E}_3(K)| = 1$. *This implies that there is a unique copy of* $\Omega_8^+(4):6$ *containing* K.

Proof. We follow the same steps as in the proof of Proposition 8.5. ■

Let us now find all the $\Omega_8^+(4):\mathrm{Sym}(3)$ subgroups containing K. As previously discussed, we know that such a subgroup must contain a copy of $\Omega_8^+(4):3$. There is a unique $\Omega_8^+(4):3$ containing K up to conjugacy, as discovered in Proposition 8.5. Let $H \cong \Omega_8^+(4):3$ be this unique subgroup and let h be the element we found such that $\langle K, h\rangle = H$. Now, since $\Omega_8^+(4):3$ is an index-2 subgroup of $\Omega_8^+(4):\mathrm{Sym}(3)$, we know that it is normal. Hence, any $g \in G$ such that $\langle H, g\rangle \cong \Omega_8^+(4):\mathrm{Sym}(3)$ must normalise H. Note that, as always, we already know that $g \in \mathcal{E}_0(K)$, so now we construct $\mathcal{E}_0(H) = N_{\mathcal{E}_0(K)}(H)$. However, we actually find that every generator of $\mathcal{E}_0(K)$ normalises H, so, in fact, $\mathcal{E}_0(H) = \mathcal{E}_0(K)$.

Now we use Lemma 7.30, which deals with finding elements extending $\Omega_8^+(2):3$ to $\Omega_8^+(2):\mathrm{Sym}(3)$ but can easily be adapted to finding elements extending $\Omega_8^+(4):3$ to $\Omega_8^+(4):\mathrm{Sym}(3)$. It tells us that if $g \in \mathcal{E}_0(H)$ such that $\langle H, g\rangle = \langle K, h, g\rangle \cong \Omega_8^+(4):\mathrm{Sym}(3)$, then $g \notin K$, $g^2 \in K$, and $(gh)^2 \in K$. Moreover, any $\mathcal{E}_0(H)$-conjugate g satisfies the same properties (which can be seen by adapting Lemma 7.31) and generates a conjugate copy of $\Omega_8^+(4):\mathrm{Sym}(3)$. Hence we let $\mathcal{E}_1(H)$ be a set of $\mathcal{E}_0(H)$-conjugacy class representatives, let

$$\mathcal{E}_2(H) = \{x \in \mathcal{E}_1(H) : x \notin K,\ x^2 \in K,\ (xh)^2 \in K\},$$

and, finally, let $\mathcal{E}_3(H)$ be a set of elements $g \in \mathcal{E}_2(H)$ such that $\langle H, g\rangle$ generates a distinct copy of $\Omega_8^+(4) : \mathrm{Sym}(3)$. The next result explores the outcome of this process.

Proposition 8.8. *We have $|\mathcal{E}_1(H)| = 92$, $|\mathcal{E}_2(H)| = 37$, and $|\mathcal{E}_3(H)| = 2$. This implies that there are at most two copies of $\Omega_8^+(4) : \mathrm{Sym}(3)$ containing K up to conjugacy.*

Proof. We follow the same steps as in the proof of Proposition 8.5. ■

We will conclude this section by by finding all the $\Omega_8^+(4) : \mathrm{Dih}(12)$ subgroups containing K. Note that $\Omega_8^+(4) : 6$ is a normal subgroup of $\Omega_8^+(4) : \mathrm{Dih}(12)$, and, by Proposition 8.7, there is only one $\Omega_8^+(4) : 6$ subgroup of G containing K up to conjugacy. Hence, without loss, to find all $\Omega_8^+(4) : \mathrm{Dih}(12)$ subgroups up to conjugacy containing K, we may assume that each such $\Omega_8^+(4) : \mathrm{Dih}(12)$ contains the sole copy of $\Omega_8^+(4) : 6$ found earlier. Let this unique (up to conjugacy) $\Omega_8^+(4) : 6$ subgroup containing K be called L, and let λ be the element such that $\langle K, \lambda\rangle = L$. Since L is normal in any potential $\Omega_8^+(4) : \mathrm{Dih}(12)$ subgroup, we start by finding $\mathcal{E}_0(L) = N_{\mathcal{E}_0(K)}(L)$. However, we once again see that every generator of $\mathcal{E}_0(K)$ normalises L, and so $\mathcal{E}_0(L) = \mathcal{E}_0(K)$.

As usual, we need only work with $\mathcal{E}_1(L)$, a set of $\mathcal{E}_0(L)$-conjugacy class representatives. Now, we will state a lemma which will provide us with conditions on the required elements of $\mathcal{E}_1(L)$.

Lemma 8.9. *Suppose $g \in \mathcal{E}_1(L)$ such that $\langle K, \lambda, g\rangle \cong \Omega_8^+(4) : \mathrm{Dih}(12)$. Then $g \notin K$, $g^2 \in K$, and $(g\lambda)^2 \in K$.*

Proof. Suppose $\langle K, \lambda, g\rangle \cong \Omega_8^+(4) : \mathrm{Dih}(12)$. Then, as this is a split extension, we have $\langle K, \lambda, g\rangle / K = \langle \overline{\lambda}, \overline{g}\rangle \cong \mathrm{Dih}(12)$. Since we have that $\lambda^n \notin K$ for any $n \in \{1, 2, 3, 4, 5\}$ and $\lambda^6 \in K$, we have that $\overline{\lambda}$ has order 6 in the quotient. Now, $\overline{g}$ generates $\mathrm{Dih}(12)$ along with λ, and the only elements with this property are involutions with order 2 product with λ. Hence $\overline{g}^2 = \overline{1}$ and $(\overline{g}\overline{\lambda})^2 = \overline{1}$, which is to say that $g^2 \in K$ and $(g\lambda)^2 \in K$. ■

Applying Lemma 8.9, we let

$$\mathcal{E}_2(L) = \{x \in \mathcal{E}_1(L) : x \notin K,\ x^2 \in K,\ (x\lambda)^2 \in K\}.$$

Finally, let $\mathcal{E}_3(L)$ be a set of elements $g \in \mathcal{E}_2(L)$ such that $\langle L, g\rangle$ is a distinct copy of $\Omega_8^+(4) : \mathrm{Dih}(12)$. We will examine the outcome of this process in the next result.

Proposition 8.10. *We have $|\mathcal{E}_1(L)| = 92$, $|\mathcal{E}_2(L)| = 37$, and $|\mathcal{E}_3(L)| = 1$. This implies that there is a unique copy of $\Omega_8^+(4) : \mathrm{Dih}(12)$ containing K.*

Proof. We follow the same steps as in Proposition 8.5. ■

We remark here that all three copies of subgroups of shape $\Omega_8^+(4) : 2$ containing K are contained in this sole copy of $\Omega_8^+(4) : \mathrm{Dih}(12)$.

Now we faced with the question of whether the groups we have found are maximal in G. Let us return to our previous notation established at the beginning of this section, where $K_1, K_3 \cong \Omega_8^+(4)$ where $5^2 \cong P_i \leq K_i$ for $i \in \{1, 3\}$. Recall that containing K_1 we find up to G-conjugacy: three copies of groups of shape $\Omega_8^+(4) : 2$, one copy of $\Omega_8^+(4) : 3$, one copy of $\Omega_8^+(4) : 2^2$, and one copy of $\Omega_8^+(4) : \mathrm{Sym}(3)$; all of these are containined in one copy of $\Omega_8^+(4) : \mathrm{Dih}(12)$. Recall also that containing K_3 we find up to G-conjugacy three copies of groups of shape $\Omega_8^+(4) : 2$, all three of which are contained in a single copy of $\Omega_8^+(4) : 2^2$. Moreover, the sole copy of $\Omega_8^+(4) : 2^2$ containing K_3 is not contained in an $\Omega_8^+(4) : \mathrm{Dih}(12)$ subgroup. Hence there are two subgroups we must test for maximality: the copy of $\Omega_8^+(4) : \mathrm{Dih}(12)$ containing K_1 and the copy of $\Omega_8^+(4) : 2^2$ containing K_3.

Unfortunately, $\Omega_8^+(4)$ contains 5^4, which is a maximal torus in $E_8(2)$ by Proposition 2.22. Hence Proposition 2.23 does not apply, so we cannot show that these groups are not maximal by showing that they fix a non-zero vector in V. If we suppose these groups are not maximal in G, then they must be contained in some other maximal subgroup of G. Here, we appeal to our unpublished paper [5] where we have a complete list of known maximal subgroups of G and a list of potential maximal subgroups of G. By Lagrange's theorem, the only possible maximal subgroups of G which could possibly contain $\Omega_8^+(4) : 2^2$ or $\Omega_8^+(4) : \mathrm{Dih}(12)$ are $\Omega_{16}^+(2)$ and $\Omega_8^+(4) : \mathrm{Dih}(12)$. Over the course of the next two results, we will demonstrate that the copy of $\Omega_8^+(4) : 2^2$ containing K_3 is contained in $\Omega_{16}^+(2)$ and is therefore not maximal, and that the copy of $\Omega_8^+(4) : \mathrm{Dih}(12)$ containing K_1 is in fact maximal.

Lemma 8.11. *There are two classes of $\Omega_8^+(4) : 2^2$ subgroups and no $\Omega_8^+(4) : \mathrm{Dih}(12)$ subgroups in $\Omega_{16}^+(2)$. Moreover, let $\Omega_{16}^+(2) \leq G$. Then these $\Omega_8^+(4) : 2^2$ subgroups are not conjugate in G.*

Proof. We will construct $\Omega_8^+(4):2^2$ subgroups inside $\Omega_{16}^+(2)$ directly. Let $K \cong \Omega_8^+(4)$ and $S \in \mathrm{Syl}_5(K)$. Then $N_K(S) = ST$ where T is isomorphic to the group given by the intrinsic `SmallGroup(192,1493)`. Furthermore, there are two involutions $t \in C_K(T)$ such that $\langle ST, t\rangle = K$. All of the above is directly verifiable in MAGMA.

Now let $H \cong \Omega_{16}^+(2)$. We can construct H as a subgroup of G using the root system, then turn it into a permutation group of degree 32,895 using Procedure B.7 written by Ballantyne. In this permutation setting, we can easily find $S \in \mathrm{Syl}_5(H)$ and $N_H(S)$. By running `Subgroups` on $N_H(S)$ and sieving for subgroups isomorphic to the group given by the intrinsic `SmallGroup(192,1493)`, we find T_1 and T_2 of the correct isomorphism type. Fix $i \in \{1,2\}$ and we find two involutions in $t_1, t_2 \in C_H(T_i)$ such that $\langle ST_i, t_j\rangle \cong \Omega_8^+(4)$ for $j \in \{1,2\}$. However, by running `IsConjugate` we find that $\langle ST_i, t_1\rangle$ and $\langle ST_i, t_2\rangle$ are H-conjugate. Hence, up to H-conjugacy, there is one $\Omega_8^+(4)$ subgroup of H containing ST_1 and one $\Omega_8^+(4)$ subgroup of H containing ST_2. Call these L_1 and L_2 respectively and note that L_1 and L_2 are not conjugate in H (verified using `IsConjugate`). Now we simply take $N_i = N_H(L_i)$ and find that $N_i \cong \Omega_8^+(4):2^2$ for each $i \in \{1,2\}$. Since L_1 and L_2 are not H-conjugate, it follows that N_1 and N_2 are not H-conjugate. Thus there are two $\Omega_8^+(4):2^2$ subgroups up to conjugacy in H, neither of which can be extended to $\Omega_8^+(4):\mathrm{Dih}(12)$ in H.

Now, using the maps provided in Procedure B.7, we can map N_1 and N_2 into G. There, we take $R_i \in \mathrm{Syl}_3(N_i)$ and observe that the elements in R_i follow different fusion patterns in G. Specifically, we see that there is some element in R_1 belonging to $3\mathrm{D}_G$, while $R_2 \cap 3\mathrm{D}_G = \varnothing$. Hence N_1 and N_2 are not conjugate in G, as required. ■

Proposition 8.12. *The $\Omega_8^+(4):2^2$ subgroup of G containing K_3 is not maximal, while the $\Omega_8^+(4):\mathrm{Dih}(12)$ subgroup of G containing K_1 is maximal.*

Proof. Let H_1 be the $\Omega_8^+(4):2^2$ subgroup of G containing K_1 and H_3 be the $\Omega_8^+(4):2^2$ subgroup of G containing K_3. Since $\Omega_{16}^+(2)$ is a subgroup of G, by Lemma 8.11 there are two $\Omega_8^+(4):2^2$ subgroups in $\Omega_{16}^+(2)$ up to G-conjugacy. However, by Proposition 8.6, H_1 and H_3 are the only two $\Omega_8^+(4):2^2$ subgroups in G up to conjugacy. Hence H_1 and H_3 must both sit inside an $\Omega_{16}^+(2)$ subgroup of G. Therefore, H_3 is not maximal.

Now, we know H_1 extends to $\Omega_8^+(4):\mathrm{Dih}(12)$, which is unique up to conjugacy. Since $\Omega_8^+(4):\mathrm{Dih}(12)$ is a known maximal subgroup of G, which can be seen in

Theorem 1.1 of [5], and since $\Omega_8^+(4) : \mathrm{Dih}(12)$ is unique in G up to conjugacy, it must be the case that the copy of $\Omega_8^+(4) : \mathrm{Dih}(12)$ containing K_1 is maximal in G. ■

We will now conclude this chapter with a proof of Theorem 1.5. This is a matter of coalescing the results from this chapter with some extra steps in some of the cases. We will prove each part of the theorem in turn.

(i) There are two classes of subgroups of G isomorphic to $\Omega_8^+(4)$. This was shown in Proposition 8.2.

(ii) There are six classes of subgroups of G of shape $\Omega_8^+(4) : 2$. In Proposition 8.8, it was shown that there are *at most* six subgroups of shape $\Omega_8^+(4) : 2$ up to conjugacy in G. There are three containing K_1 and three containing K_3. Recall that all three containing K_1 are contained in a single copy of $\Omega_8^+(4) : 2^2$, and all three containing K_3 are contained in a single copy of $\Omega_8^+(4) : 2^2$. Since there are three distinct classes of involutions in 2^2, it follows that there are three distinct isomorphism types of subgroups of shape $\Omega_8^+(4) : 2$ inside $\Omega_8^+(4) : 2^2$. Since, by construction, we found all possible subgroups of shape $\Omega_8^+(4) : 2$ containing K_i for $i \in \{1, 3\}$, we conclude that these three distinct groups of shape $\Omega_8^+(4) : 2$ containing K_i must be pairwise non-isomorphic. Hence they cannot by conjugate, and so each represents a distinct G-conjugacy class of subgroups of shape $\Omega_8^+(4) : 2$, as required.

(iii) There is one class of subgroups of G isomorphic to $\Omega_8^+(4) : 3$; this was shown in Proposition 8.5.

(iv) There are two classes of subgroups of G isomorphic to $\Omega_8^+(4) : 2^2$; this was shown in Proposition 8.6.

(v) There is one class of subgroups of G isomorphic to $\Omega_8^+(4) : 6$; this was shown in Proposition 8.7.

(vi) There are two classes of subgroups of G isomorphic to $\Omega_8^+(4) : \mathrm{Sym}(3)$. In Proposition 8.8, it was shown that there are *at most* two $\Omega_8^+(4) : \mathrm{Sym}(3)$ subgroups up to conjugacy in G. Let K_1 and K_2 be these subgroups and L be the $\Omega_8^+(4)$ they contain. However, both K_1 and K_2 are contained in the unique $\Omega_8^+(4) : \mathrm{Dih}(12)$

subgroup of G constructed in Proposition 8.10, which was later shown to be maximal in Proposition 8.12. Since K_1 and K_2 are not conjugate in H, they are not conjugate in G by the same argument as in part (ii)

(vii) There is one class of subgroups of G isomorphic to $\Omega_8^+(4)$: Dih(12); this was shown in Proposition 8.10.

This concludes the proof of Theorem 1.5. ■

Chapter 9

$Sp_8(2)$

Here, we will find all $Sp_8(2)$ subgroups of $E_8(2)$ up to conjugacy. As always, G will denote $E_8(2)$ for the entire chapter. Since we know that $Sp_8(2)$ contains $\Omega_8^+(2):2$ as a maximal subgroup – see the $\mathbb{ATLAS}$ [14] – we know that, by Proposition 2.12, to find all $Sp_8(2)$ subgroups up to conjugacy in G, we can build up from the seventeen copies of $\Omega_8^+(2):2$ found in Chapter 7.

9.1 Constructing $Sp_8(2)$ Subgroups of G

Let us begin with a result which will provide us with a means of buliding $Sp_8(2)$ from our copies of $\Omega_8^+(2):2$.

Proposition 9.1. *Let* $\Omega_8^+(2):2 \cong K \leq H \cong Sp_8(2)$. *Let* $P \in \mathrm{Syl}_5(K)$ *and* $R \in \mathrm{Syl}_2(N_K(P))$. *Then*

(i) $R \cong 4 \wr 2$*;*

(ii) there are four involutions $t_1, t_2, t_3, t_4 \in C_H(R)$ *such that* $\langle K, t_i \rangle = H$ *for* $i \in \{1,2,3,4\}$*;*

(iii) without loss of generality, $t_1 \in 2A_H$, $t_2 \in 2C_H$, *and* $t_3, t_4 \in 2D_H$.

Proof. All of these facts are verifiable in Magma using the intrinsic copy of $Sp_8(2)$ given by `Sp(8,2)`. To obtain $\Omega_8^+(2):2$ as a subgroup of $Sp_8(2)$, we use the command `MaximalSubgroups`. ■

Now, we observe that all six of the classes of involutions in both $K \cong \Omega_8^+(2):2$ and $H \cong Sp_8(2)$ have distinct lengths. Thus, the classes $2\mathrm{A}_K, \dots, 2\mathrm{F}_K$ and $2\mathrm{A}_H, \dots, 2\mathrm{F}_H$ are ordered uniquely. Moreover, we have that $2\mathrm{A}_K \to 2\mathrm{A}_H, 2\mathrm{B}_K \to 2\mathrm{B}_H, \dots, 2\mathrm{F}_K \to 2\mathrm{F}_H$. Again, this is verifiable in Magma by calling `Classes` on both groups to obtain representatives of each class, then calling `IsConjugate` to see that the relevant representatives are indeed conjugate in H.

Here, we will give an outline of how we use Proposition 9.1 to build $Sp_8(2)$ subgroups of G. Let $L \leq G$ be one of the seven $\Omega_8^+(2)$ subgroups found in Chapter 7, and let $K \geq L$ be one of the seventeen $\Omega_8^+(2):2$ subgroups. Then recall that we have $5^2 \cong P \in \mathrm{Syl}_5(L)$ and $T \cong \mathrm{Dih}(8) \circ \mathbb{Z}_4$ such that $PT = N_L(P)$. Now, clearly, we have $PT = N_L(P) \leq N_K(P) = PR$ where $R \cong 4 \wr 2$. Hence we will choose $R \in \mathrm{Syl}_2(N_K(P))$ such that $T \leq R$. We make this choice because, looking at Proposition 9.1, we will build $\mathcal{I}(C_G(R))$ and look for involutions $t \in \mathcal{I}(C_G(R))$ such that $\langle K, t\rangle \cong Sp_8(2)$. But clearly if $T \leq R$ then $\mathcal{I}(C_G(R)) \subseteq \mathcal{I}(C_G(T))$. This is significant to us because in Chapter 7 we found a group, $\mathcal{C}_3(T)$, containing $\mathcal{I}(C_G(T))$ for every possible T. Therefore, we have $\mathcal{I}(C_G(R)) \subseteq C_{\mathcal{C}_3(T)}(R)$, which we will calculate and sieve in Magma.

Let us now establish the notation we will use for the remainder of this chapter. Recall that in Proposition 7.5 we found exactly four subgroups of G isomorphic to 5^2, and we called them P_i for $i \in \{1,2,3,4\}$. Now, only P_i for $i \in \{1,2,3\}$ is contained in some $\Omega_8^+(2)$ subgroup of G. Then we found $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups $T_j^{(i)}$ normalising P_i where: if $i = 1$ then $j \in \{1,2,3,4\}$; if $i = 2$ then $j = 2$; if $i = 3$ then $j \in \{1,2\}$. Now let the seven $\Omega_8^+(2)$ subgroups be denoted by $L_j^{(i)}$, where $P_i T_j^{(i)} \leq L_j^{(i)}$. Next, by Proposition 7.29, we have seventeen $\Omega_8^+(2):2$ subgroups which we will denote by $K_k^{(i,j)}$ where $L_j^{(i)} \leq K_k^{(i,j)}$ and:

- if $(i,j) \in \{(1,2),(2,1)\}$, then $k = 1$;
- if $(i,j) \in \{(1,1),(3,1),(3,2)\}$, then $k \in \{1,2\}$;
- if $(i,j) = (1,4)$, then $k \in \{1,\dots,3\}$;
- if $(i,j) = (1,3)$, then $k \in \{1,\dots,6\}$.

Next, we choose $R_k^{(i,j)} \in \mathrm{Syl}_2(K_k^{(i,j)})$ such that $T_j^{(i)} \leq R_k^{(i,j)}$ and let $r_k^{(i,j)} \in R_k^{(i,j)} \setminus T_j^{(i)}$

so that $\langle T_j^{(i)}, r_k^{(i,j)}\rangle = R_k^{(i,j)}$. Hence we have

$$C_{\mathcal{C}_3(T_j^{(i)})}(R_k^{(i,j)}) = C_{\mathcal{C}_3(T_j^{(i)})}(r_k^{(i,j)}),$$

a group we know contains $\mathcal{I}(C_G(R_k^{(i,j)}))$. From now on, we will refer to $C_{\mathcal{C}_3(T_j^{(i)})}(R_k^{(i,j)})$ simply as $\mathcal{C}(R_k^{(i,j)})$. Once we have obtained $\mathcal{C}(R_k^{(i,j)})$, we sieve it for involutions with the usual methods.

Proposition 9.2. *The order of* $\mathcal{C}(R_k^{(i,j)})$ *and the number of* $Sp_8(2)$ *subgroups containing* $K_k^{(i,j)}$*, up to conjugacy in* G*, are shown in Table 9.1.*

i	j	k	$\lvert\mathcal{C}_3(T_j^{(i)})\rvert$	$\lvert\mathcal{C}(R_k^{(i,j)})\rvert$	Number of $Sp_8(2)$ subgroups containing $K_k^{(i,j)}$
1	1	1	$2^{25}.3$	$2^{16}.3$	0
1	1	2		2^{15}	0
1	2	1	2^{25}	2^{15}	0
1	3	1	$2^{31}.3$	2^{18}	0
1	3	2		$2^{22}.3$	1
1	3	3		2^{21}	0
1	3	4		2^{19}	0
1	3	5		2^{18}	0
1	3	6		$2^{18}.3$	0
1	4	1	$2^{40}.3^5.5^2.7$	$2^{27}.3^4.5.7$	1
1	4	2		$2^{25}.3^2$	0
1	4	3		$2^{25}.3^2$	0
2	1	1	2^{17}	2^9	0
3	1	1	$2^{20}.3$	$2^{15}.3$	2
3	1	2		2^{11}	0
3	2	1	2^{19}	2^{13}	0
3	2	2		2^{12}	0

Table 9.1: Number of $Sp_8(2)$ subgroups containing $K_k^{(i,j)}$

Proof. Let $T = T_j^{(i)}$, $R = R_k^{(i,j)}$, and $r = r_k^{(i,j)}$. Now, choose involutions $a, b, c \in T$ such that $\langle a, b, c\rangle = T$ and hence $R = \langle a, b, c, r\rangle$. When calling the command `LMGCentraliser(X,x)` on a group X and an element x, we obtain $C_X(x)$ even if $x \notin X$. It only requires that x and X exist in the same universe (in our case, $E_8(2)$). Hence, in most cases, we can simply calculate $\mathcal{C}(R) = C_{\mathcal{C}_3(T)}(r)$ directly using this command. However, sometimes this command will not execute (or, at least, not in a realistic timeframe) when $x \notin X$, but will when $x \in X$. For this reason, we sometimes first calculate $\mathcal{C}_0(R) = C_{\langle\mathcal{C}_3(T),r\rangle}(r)$ first, as, clearly, $r \in \langle\mathcal{C}_3(T), r\rangle$. This always executes within a matter of minutes at most. Then we test every generator $g \in \mathcal{C}_0(R)$ to see whether g commutes with each of a, b, and c. If each generator does, then $\mathcal{C}_0(R) = \mathcal{C}(R)$. If some generator fails to commute with any of a, b, and c, then we continue as follows. Suppose $y \in \{a, b, c\}$ such that there exists some generator $g \in \mathcal{C}_0(R)$ for which $gy \neq yg$. Then we calculate $\mathcal{C}_1(R) = C_{\mathcal{C}_0(R)}(y)$. This is a group guaranteed to commute with r and y. In all cases where this step is required, we now observe that every generator $g \in \mathcal{C}_1(R)$ commutes with a, b, and c, and, by construction, r. Hence $\mathcal{C}(R) = \mathcal{C}_1(R)$ and we have successfully found $\mathcal{C}(R)$.

Now, we will sieve $\mathcal{C}(R_k^{(i,j)})$ for involutions $t_k^{(i,j)}$ such that $\langle K_k^{(i,j)}, t_k^{(i,j)} \rangle \cong Sp_8(2)$. We use the same method for every case of i, j, and k except for the case when $(i, j, k) = (1, 4, 1)$. Hence, for now, suppose $(i, j, k) \neq (1, 4, 1)$. Firstly, note that in all cases, we have that $2\mathrm{A}_{K_k^{(i,j)}} \to 2\mathrm{B}_G$. Hence, by Proposition 9.1, it is sufficient to look for involutions in $\mathcal{C}(R_k^{(i,j)})$ which belong to $2\mathrm{B}_G$. Now, for each such involution $t \in \mathcal{C}(R_k^{(i,j)})$, we build $Y = \langle K_k^{(i,j)}, t \rangle$ and run Y through an order of random elements sieve (see Procedure 3.8). Note that to use this sieve, we need a set of all potential element orders appearing in $Sp_8(2)$. This is

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 17, 18, 20, 21, 24, 30\}$$

as can be seen in the $\mathbb{ATLAS}$ [14]. If Y survives this sieve, then we check the order of Y. If $|Y| = |Sp_8(2)|$, then we check that $Y \cong Sp_8(2)$ by using `LMGChiefFactors`. When $(i, j, k) = (1, 3, 2)$ we find that only one involution survives this whole process, resulting in just one $Sp_8(2)$ subgroup containing $K_2^{(1,3)}$.

When $(i, j, k) = (3, 1, 1)$ we find that there are four involutions $t_1, \ldots, t_4$ such that $Y_n = \langle K_1^{(3,1)}, t_n \rangle \cong Sp_8(2)$ for $n \in \{1, 2, 3, 4\}$. Moreover, we find that if $m, n \in \{1, 2, 3, 4\}$ with $m \neq n$, then $Y_m \neq Y_n$. However, we can readily find elements c_2, c_3 inside the group

$$N_{N_G(P_3)}(P_3 R_1^{(3,1)})$$

with $Y_1 = Y_2^{c_2} = Y_3^{c_3}$. Also, for all $c \in N_{N_G(P_3)}(P_3 R_1^{(3,1)})$ we have that $Y_1 \neq Y_4^c$ which implies, by Proposition 3.15, that Y_1 and Y_4 are not conjugate in G. Therefore, there are two $Sp_8(2)$ subgroups of G up to G-conjugacy containing $K_1^{(3,1)}$. For all cases with $(i, j, k) \notin \{(1, 3, 2), (1, 4, 1), (3, 1, 1)\}$, no involutions survive this sieve.

Now we will consider the case when $(i, j, k) = (1, 4, 1)$. Let $K = K_1^{(1,4)}$, $R = R_1^{(1,4)}$, and $C = \mathcal{C}(R)$. In this case, we have $C \sim [2^{18}] : Sp_6(2)$. Let $W = O_2(C)$ and $\overline{C} = C/W \cong Sp_6(2)$, which we can obtain using `LMGRadicalQuotient`. We sieve W exactly as we do above, finding three involutions t_1, t_2, t_3 such that $Y_n = \langle K, t_n \rangle \cong Sp_8(2)$ for each $n \in \{1, 2, 3\}$. However, we find that $t_n \in Y_1$ for $n \in \{2, 3\}$, which implies that $Y_1 = Y_2 = Y_3$. This is the sole copy of $Sp_8(2)$ containing K. We will now proceed to prove that there are no involutions $t \in C \setminus W$ such that $\langle K, t \rangle \cong Sp_8(2)$.

Suppose $t \in C \setminus W$ is an involution. Then $\bar{t} \in \overline{C}$ is an involution. Let $\bar{t}_1, \ldots, \bar{t}_4$ be representatives of the four classes of involutions in $\overline{C}$. We obtain these using `Classes`. Now, let $\overline{S}_n$ be a right transversal for $C_{\overline{C}}(\bar{t}_n)$ in $\overline{C}$ and let N_n be the inverse image in C of $\langle \bar{t}_n \rangle$, for each $n \in \{1, 2, 3, 4\}$. Finally, let S_n be a set of representatives for all the cosets in $\overline{S}_n$. By Lemma 3.14, we have that $t \in N_n^s$ for some $n \in \{1, 2, 3, 4\}$ and $s \in S_n$. Moreover, since we can demand that $t \in 2\mathrm{B}_G$ and $t \notin W$, we have that

$$t \in ((N_n \setminus W) \cap 2\mathrm{B}_G)^s,$$

as shown in the remarks following Lemma 3.14. Let $X_n = (N_n \setminus W) \cap 2\mathrm{B}_G$ which we can construct in Magma directly. The sizes of S_n and X_n are given in Table 9.2. Now, for each $n \in \{1, 2, 3, 4\}$, we simply run through every $x \in X_n$ and every $s \in S_n$

i	$\lvert S_n \rvert$	$\lvert X_n \rvert$
1	63	64
2	315	880
3	945	64
4	3780	0

Table 9.2: Sizes of S_n and X_n

and run the same sieves on x^s that we ran in the other cases. We find that no elements survive the order of random elements sieve, which proves that there are no involutions $t \in C \setminus W$ for which $\langle K, t \rangle \cong Sp_8(2)$, as required. ∎

We remark here that Theorem 1.6 follows immediately from Proposition 9.2. Indeed, the two $Sp_8(2)$ subgroups containing $K_1^{(3,1)}$ are not conjugate, as shown in the proof of Proposition 9.2. Now note that $\Omega_8^+(2) : 2$ is unique in $Sp_8(2)$ up to conjugacy. Therefore, if we have two $Sp_8(2)$ subgroups built up from different $\Omega_8^+(2) : 2$ subgroups, since these $\Omega_8^+(2) : 2$ subgroups are not conjugate (see the proof of Theorem 1.4 at the end of Chapter 7), it must follow that these $Sp_8(2)$ subgroups are not conjugate. Thus we have exactly four classes of $Sp_8(2)$ subgroups of G, as required.

We will now conclude this chapter by proving that no $Sp_8(2)$ subgroup of G is maximal.

Proposition 9.3. *Suppose $H \leq G$ such that $H \cong Sp_8(2)$ or $F^*(H) \cong \Omega_8^+(2)$. Then H is not maximal in G.*

Proof. By Proposition 9.2, there are four $Sp_8(2)$ subgroups in G up to conjugacy. Let $H_k^{(i,j)} \cong Sp_8(2)$ such that $\Omega_8^+(2):2 \cong K_k^{(i,j)} \leq H_k^{(i,j)}$. Recall that in Proposition 7.34, we constructed subgroups $X_j^{(i)}$ such that $K_k^{(i,j)} < X_j^{(i)}$. Now we simply observe that $H_k^{(i,j)} < \langle H_k^{(i,j)}, X_j^{(i)} \rangle < G$, thus each $Sp_8(2)$ subgroup of G is not maximal. ■

Chapter 10

$Sp_4(4)$ and Its Extensions

In this chapter, we will prove Theorem 1.7, showing that $Sp_4(4)$ and its automorphism extensions are not maximal subgroups of $E_8(2)$. In addition, we will determine certain $Sp_4(4)$ subgroups up to conjugacy in $E_8(2)$.

For this chapter, G will continue to denote $E_8(2)$. Let $H \cong Sp_4(4)$. Then $|H| = 2^8.3^2.5^2.17$ and we observe that if $P \in \mathrm{Syl}_5(H)$, then $P \cong 5^2$. To construct all copies of H in G, we will begin by finding all G-classes of subgroups of G isomorphic to 5^2. However, we have a head start here, having already completed this step in Chapter 7 when working with $\Omega_8^+(2)$. In a situation similar to $\Omega_8^+(2)$, we will find specific subgroups normalising these copies of 5^2, then construct their centralisers. The first result of this chapter compiles some facts about H and alludes to how we intend to generate subgroups of G isomorphic to H.

Proposition 10.1. *Let $H \cong Sp_4(4)$, $P \in \mathrm{Syl}_5(H)$, and $D \in \mathrm{Syl}_2(N_H(P))$. Then:*

(i) $P \cong 5^2$*;*

(ii) $N_H(P) = PD$ *where* $D \cong \mathrm{Dih}(8)$ *and* $D \cap C_H(P) = 1$*;*

(iii) there are exactly four involutions $x \in C_H(D)$ *such that* $\langle PD, x\rangle = H$*, and* x *is conjugate in* H *to the unique central involution in* D*.*

We must now employ this result in the setting of G. To do this, we will proceed through the following steps, which also provide an outline of this chapter.

1. Recall that we already have $P_1, \ldots, P_4 \leq G$, a complete list of subgroups of G isomorphic to 5^2 up to G-conjugacy, by Proposition 7.5. We first determine

which ones could exist as a subgroup of H if $H \leq G$.

2. For each viable P_i, we find all subgroups $D_1^{(i)}, \ldots, D_{n_i}^{(i)}$ isomorphic to Dih(8) which normalise P_i, up to conjugacy in $N_G(P_i)$.

3. For each viable P_i and each $j \in \{1, \ldots, n_i\}$, we find which involutions x centralising $D_j^{(i)}$ are such that $\langle P_i D_j^{(i)}, x\rangle \cong Sp_4(4)$.

4. For each copy of $Sp_4(4)$ we find, we attempt to extend it to $Sp_4(4):2$ and $Sp_4(4):4$.

10.1 Constructing $Sp_4(4)$ Subgroups of G

In this section, we will prove Theorem 1.7 part (i) – there are at most five G-classes of subgroups of G which are isomorphic to $Sp_4(4)$ which do not follow $Sp_4(4)$ fusion possibility (iii) or (iv). The first step will be to determine which subgroups of G isomorphic to 5^2 could exist as a Sylow 5-subgroup of an $Sp_4(4)$ subgroup. We will start with a quick result limiting which $Sp_4(4)$ fusion possibilities we will consider.

Lemma 10.2. *Suppose $H \leq G$ with $H \cong Sp_4(4)$ and H following $Sp_4(4)$ fusion possibility (iii) or (iv) from Proposition 2.5. Then H is not maximal in G.*

Proof. We can see immediately that H fixes a nonzero vector in V using Proposition 2.7. Now, by Proposition 2.23, we have that H is not maximal in G ■

For the remainder of this section, suppose $Sp_4(4) \cong H \leq G$ such that H does not follow $Sp_4(4)$ fusion (iii) or (iv). Recall from Chapter 7 that there are exactly four G-classes of subgroups isomorphic to 5^2 and these classes are represented by the 5^2 subgroups P_1, P_2, P_3, and P_4. Recall, also, that their elements fall into G-conjugacy classes as in Table 10.1.

Group	No. of Elements in $5\mathrm{A}_G$	No. of Elements in $5\mathrm{B}_G$
P_1	24	0
P_2	0	24
P_3	8	16
P_4	12	12

Table 10.1: Element structure of $P_1, \ldots, P_4$

However, we actually know right away that some of these cases cannot exist as a Sylow 5-subgroup of H. We do this by examining the $Sp_4(4)$-conjugacy classes of elements of order 5.

Lemma 10.3. *Let $P \in \mathrm{Syl}_5(H)$. Then the elements of order 5 in P fall into the five H-conjugacy classes as shown in Table 10.2.*

H-class	5A	5B	5C	5D	5E
Number of elements of P in that class	4	4	4	4	8

Table 10.2: H-fusion of elements of order 5 in P

Proof. This can be directly verified in MAGMA. ■

Let us now examine the six fusion possibilities for H.

Lemma 10.4. *In $Sp_4(4)$ fusion possibilities (i), (ii), (v), (vi), (vii), and (viii), the elements of order 5 in H fuse to the two G-classes of elements of order 5 as follows*

(i) 5ABCDE → 5B;

(ii) 5ABCDE → 5B;

(v) 5AB → 5A, 5CDE → 5B;

(vi) 5ABE → 5B, 5CD → 5A;

(vii) 5ABCDE → 5B;

(viii) 5ABCDE → 5B.

Proof. See Proposition 2.5. ■

Proposition 10.5. *The elements of order 5 in P fuse to the G-classes of elements of order 5, depending on which fusion possibility H follows, according to Table 10.3. Moreover, H cannot contain a Sylow 5-subgroup which is G-conjugate to P_1 or P_4.*

Fusion Possibility	Number of Elements in $5A_G$	Number of Elements in $5B_G$
(i), (ii), (vii), (viii)	0	24
(v), (vi)	8	16

Table 10.3: G-fusion of elements of order 5 in P

Proof. This is simply a combination of Lemmas 10.3 and 10.4. For the last part of the result, we observe that the element structures for P_1 or P_4 seen in Table 10.1 do not appear in Table 10.3, so they (or any G-conjugate group) cannot possibly appear in H. ■

Hence we proceed with P_2 and P_3 as our viable cases. We must now move onto the next step – to find all Dih(8) subgroups of G which normalise P_i for $i \in \{2, 3\}$. Recall from Chapter 7 we had to calculate $N_G(P_i)$ and in Proposition 7.7 we found the following.

$$|N_G(P_2)| = 2^5.3.5^3,$$
$$|N_G(P_3)| = 2^9.3^2.5^4.$$

Fix $i \in \{2, 3\}$. By the same argument applied in Proposition 7.8, we only need to find subgroups of $N_G(P_i)$ up to $N_G(P_i)$-conjugacy. Recall also that, by Proposition 10.1 part (ii), any potential Dih(8) we desire must have trivial intersection with $C_G(P_i)$.

Proposition 10.6. *Up to $N_G(P_2)$-conjugacy, there is one subgroup of $N_G(P_2)$ isomorphic to* Dih(8) *intersecting $C_G(P_2)$ trivially. Up to $N_G(P_3)$-conjugacy, there are seven subgroups of $N_G(P_3)$ isomorphic to* Dih(8) *intersecting $C_G(P_3)$ trivially.*

Proof. We start by converting $N_G(P_i)$ into a permutation group. In the permutation setting, we now run the command

```
Subgroups(NGPi :  OrderEqual:=8);
```

where the object `NGPi` is a stand-in for $N_G(P_i)$, to find all the subgroups of $N_G(P_i)$ (up to conjugacy in $N_G(P_i)$) of order 8. We now sieve these by demanding that they be isomorphic to Dih(8), then by demanding that they intersect trivially with $C_G(P_i)$.

We find five classes of subgroups of $N_G(P_2)$ of order 8; one of which is a class of subgroups isomorphic to Dih(8), which has trivial intersection with $C_G(P_2)$. Also, we find 159 classes of subgroups of $N_G(P_3)$ of order 8; eleven of which are classes of subgroups isomorphic to Dih(8); seven of which have trivial intersection with $C_G(P_3)$. ■

We name these Dih(8) subgroups as follows. $D_1^{(2)}$ is the unique (up to $N_G(P_2)$-conjugacy) viable subgroup normalising P_2. $D_j^{(3)}$ for $j \in \{1, \ldots, 7\}$ are the seven viable subgroups normalising P_3. We must now work with each of these Dih(8) subgroups in turn, finding all involutions centralising them.

Fix $(i, j) \in \{(2, 1), (3, 1), (3, 2), \ldots, (3, 7)\}$ and let $z_j^{(i)} \in Z(D_j^{(i)})$ be the unique involution. We note that, by direct calculation, in all cases $z_j^{(i)} \in 2\mathrm{D}_G$. To construct all the involutions centralising $D_j^{(i)}$, we will follow the same procedure for each $D_j^{(i)}$ so, for now, we will set $D = D_j^{(i)}$ and $z = z_j^{(i)}$ so the notation is less cumbersome. This procedure is laid out in full detail in Section 3.2, but we will provide a general outline here as we proceed.

We start by computing $C_G(z)$ using the command

`CentraliserOfInvolution`

as, by Proposition 3.1 we have that $\mathcal{I}(C_G(D)) \subseteq C_G(z)$ and $D \leq C_G(z)$. Set $C = C_G(z)$ and by Proposition 2.19 we have $C \sim [2^{84}] : Sp_8(2)$. Let $\overline{C} = C/O_2(C) \cong Sp_8(2)$ and let $\overline{D}$ be the image of D in $\overline{C}$. We now build the group $\mathcal{C}_1(D)$ as defined in the remarks following Proposition 3.2, using the process given in Procedure 3.3. Also by Proposition 3.2, we know that $\mathcal{I}(C_G(D) \subseteq \mathcal{C}_1(D)$.

Now let $F = C_V(D)$ and define

$$\mathcal{C}_2(D) = \langle \mathrm{Stab}_S(F) : S \in \mathrm{Syl}_2(\mathcal{C}_1(D)) \rangle$$

and by Proposition 3.6 we know $\mathcal{I}(C_G(D)) \subseteq \mathcal{C}_2(D)$. We construct $\mathcal{C}_2(D)$ using the procedures given in Procedure 3.7 (specifically, to find $\mathcal{C}_2(D_j^{(i)})$ we use Procedure 3.7 (ii) when $(i, j) = \{(3, 1), (3, 3), (3, 5)\}$, and Procedure 3.7 (i) otherwise).

Finally, we define

$$\mathcal{C}_3(D) = C_{\mathcal{C}_2(D)}(D)$$

which also, clearly, is such that $\mathcal{I}(C_G(D)) \subseteq \mathcal{C}_3(D)$. We can calculate this in Magma by choosing involutions d_1 and d_2 such that $D = \langle d_1, d_2 \rangle$. Then, we use the command `LMGCentraliser` twice to find $C_{\mathcal{C}_2(D)}(d_1)$ then $C_{C_{\mathcal{C}_2(D)}(d_1)}(d_2)$. Alternatively, in the cases where $\mathcal{C}_2(D_j^{(i)})$ is soluble (every case except when $(i, j) = (3, 3)$), we can convert it into a pc-group and run the centraliser computations in the pc-group setting. The results of this process are collated in the next result.

Proposition 10.7. *Let $(i, j) \in \{(2, 1), (3, 1), (3, 2), \ldots, (3, 7)\}$. Then the orders of $\mathcal{C}_1(D_j^{(i)})$, $\mathcal{C}_2(D_j^{(i)})$, and $\mathcal{C}_3(D_j^{(i)})$ are presented in Table 10.4.*

i	j	$\|\mathcal{C}_1(D_j^{(i)})\|$	$\|\mathcal{C}_2(D_j^{(i)})\|$	$\|\mathcal{C}_3(D_j^{(i)})\|$
2	1	2^{90}	2^{26}	2^{26}
3	1	$2^{94}.3$	$2^{30}.3$	$2^{28}.3$
3	2	2^{94}	2^{30}	2^{28}
3	3	$2^{98}.3^2.5$	$2^{42}.3^2.5$	$2^{40}.3^2.5$
3	4	2^{94}	2^{30}	2^{28}
3	5	$2^{95}.3$	$2^{34}.3$	$2^{32}.3$
3	6	2^{94}	2^{30}	2^{28}
3	7	2^{93}	2^{28}	2^{27}

Table 10.4: $|\mathcal{C}_1(D_j^{(i)})|$, $|\mathcal{C}_2(D_j^{(i)})|$, and $|\mathcal{C}_3(D_j^{(i)})|$

Proof. We follow the procedures outlined in the discussion preceding this result. ∎

We must now sieve each $\mathcal{C}_3(D_j^{(i)})$ for involutions x such that $\langle P_i D_j^{(i)}, x\rangle \cong Sp_4(4)$. The results of this process for each case except $(i,j) = (3,3)$ are given in the following result.

Proposition 10.8. *There are two involutions $x \in \mathcal{C}_3(D_1^{(2)})$ such that $\langle P_2 D_1^{(2)}, x\rangle \cong Sp_4(4)$, four involutions $x \in \mathcal{C}_3(D_5^{(3)})$ such that $\langle P_3 D_5^{(3)}, x\rangle \cong Sp_4(4)$, and four involutions $x \in \mathcal{C}_3(D_7^{(3)})$ such that $\langle P_3 D_7^{(3)}, x\rangle \cong Sp_4(4)$.*

Proof. First, we turn $\mathcal{C}_3(D_j^{(i)})$ into a pc-group and define $\mathcal{I}_0(D_j^{(i)}) = \mathcal{I}(\mathcal{C}_3(D_j^{(i)}))$, the set of all involutions in $\mathcal{C}_3(D_j^{(i)})$. We construct this set by simply running through all of the elements of $\mathcal{C}_3(D_j^{(i)})$ and storing the involutions in another set. Now, recall that, by Proposition 10.1 (iii), the involutions we desire can be taken to be H-conjugate to $z_j^{(i)}$. Therefore, we can take only involutions belonging to $2\mathrm{D}_G$. We now run through the elements of $\mathcal{I}_0(D_j^{(i)})$. If the chosen involution is not in $2\mathrm{D}_G$, we skip it and move onto the next one. If it is in $2\mathrm{D}_G$, we put it through an order of random elements sieve – for full details on this, see Procedure 3.8. Note that to use this sieve, we require a set all of possible element orders of elements in $Sp_4(4)$. This is

$$\{1, 2, 3, 4, 5, 6, 10, 15, 17\}$$

which can be seen from the $\mathbb{ATLAS}$ [14]. We store the elements that survive this sieve in sets called $\mathcal{I}_1(D_j^{(i)})$.

Now let

$$\mathcal{I}_2(D_j^{(i)}) = \{x \in \mathcal{I}_1(D_j^{(i)}) : |\langle P_i D_j^{(i)}, x\rangle| = |Sp_4(4)|\}.$$

Finally, we collect a set of involutions in $\mathcal{I}_2(D_j^{(i)})$ which generate a distinct $Sp_4(4)$ subgroup. For example, if $x, y \in \mathcal{I}_2(D_j^{(i)})$ are such that $\langle P_i D_j^{(i)}, x\rangle = \langle P_i D_j^{(i)}, y\rangle \cong$

$Sp_4(4)$, then without loss we can keep only one of x and y. These involutions are gathered in $\mathcal{I}_3(D_j^{(i)})$.

The results of this process are collected in Table 10.5.

i	j	$\lvert\mathcal{I}_0(D_j^{(i)})\rvert$	$\lvert\mathcal{I}_1(D_j^{(i)})\rvert$	$\lvert\mathcal{I}_2(D_j^{(i)})\rvert$	$\lvert\mathcal{I}_3(D_j^{(i)})\rvert$
2	1	950,271	13	12	2
3	1	1,769,471	11	0	0
3	2	1,638,399	13	0	0
3	4	1,441,791	9	0	0
3	5	29,622,271	54	28	4
3	6	1,638,399	17	0	0
3	7	2,523,135	57	40	4

Table 10.5: $|\mathcal{I}_0(D_j^{(i)})|$, $|\mathcal{I}_1(D_j^{(i)})|$, $|\mathcal{I}_2(D_j^{(i)})|$, and $|\mathcal{I}_3(D_j^{(i)})|$ for $(i,j) \neq (3,3)$

An examination of $|\mathcal{I}_3(D_j^{(i)})|$ yields the results stated. ∎

We name these $Sp_4(4)$ subgroups $H_k^{(i,j)}$ where $D_j^{(i)} \leq H_k^{(i,j)}$ and: if $(i,j) = (2,1)$ then $k \in \{1,2\}$; if $(i,j) \in \{(3,5),(3,7)\}$ then $k \in \{1,2,3,4\}$.

Now we turn our attention back to the case where $(i,j) = (3,3)$. This case is more difficult due to the relatively large order of $\mathcal{C}_3(D_3^{(3)})$. For simplicity, let $D = D_3^{(3)}$ and $C = \mathcal{C}_3(D_3^{(3)})$. By running the command `LMGRadicalQuotient` on C we obtain $O_2(C) \cong [2^{36}]$ and $\overline{C} = C/O_2(C) \cong \mathrm{Sym}(6)$ as a permutation group. Then if $x \in \mathcal{C}_3(D_3^{(3)})$ is an involution, we either have $x \in O_2(C)$ or $x \notin O_2(C)$. We will first sieve for viable involutions in $O_2(C)$.

Proposition 10.9. *There are at most 64 $Sp_4(4)$ subgroups up to conjugacy in G which are generated by $\langle P_3 D_3^{(3)}, x\rangle$ where $x \in O_2(C)$.*

Proof. To carry out our computations, we use parallel processing using the method described in full in Procedure 3.12. In this case, we have $Z := Z(O_2(C)) \cong 2^{11}$ and we construct a group $Z \leq S \leq O_2(C)$ such that $|S| = 2^{33}$. We let $\Delta = \{\delta_1,\ldots,\delta_8\}$ be a right transversal for S in $O_2(C)$. Now, we open eight parallel screens, where screen i sieves the coset $S\delta_i$. We define and construct similar sets as before, where $i \in \{1,\ldots,8\}$:

$\mathcal{I}_0(S\delta_i) = \mathcal{I}(S\delta_i)$;

$\mathcal{I}_1(S\delta_i) = \{x \in \mathcal{I}_0(S\delta_i) : x \in 2\mathrm{D}_G$ and x survives an order of random elements sieve$\}$;

$\mathcal{I}_2(S\delta_i) = \{x \in \mathcal{I}_1(S\delta_i) : |\langle P_3 D_3^{(3)}, x\rangle| = |Sp_4(4)|\}$;

j	$\lvert\mathcal{I}_0(S\delta_i)\rvert$	$\lvert\mathcal{I}_1(S\delta_i)\rvert$	$\lvert\mathcal{I}_2(S\delta_i)\rvert$	$\lvert\mathcal{I}_3(S\delta_i)\rvert$
1	18,751	15	11	7
2	18,816	15	13	8
3	18,752	20	14	9
4	18,816	21	15	8
5	18,752	18	14	8
6	18,816	21	15	7
7	18,880	18	14	8
8	18,688	15	13	9

Table 10.6: $|\mathcal{I}_0(S\delta_i)|$, $|\mathcal{I}_1(S\delta_i)|$, $|\mathcal{I}_2(S\delta_i)|$, and $|\mathcal{I}_3(S\delta_i)|$

and $\mathcal{I}_3(S\delta_i)$ is a collection of involutions from $\mathcal{I}_2(S\delta_i)$ which generate distinct $Sp_4(4)$ subgroups. The results of this process are presented in Table 10.6.

Now, we see that $\sum_{i=1}^{8} |\mathcal{I}_3(S\delta_i)| = 64$, which is the required result. ∎

We will now sieve for involutions in $C \setminus O_2(C)$. The method we will employ is described in the remarks following Lemma 3.14.

Proposition 10.10. *There are no $Sp_4(4)$ subgroups in G which are generated by $\langle P_3D_3^{(3)}, x\rangle$ where $x \notin O_2(C)$.*

Proof. Recall that $\overline{C} = C/O_2(C) \cong \mathrm{Sym}(6)$. Let $\overline{c}_1$, $\overline{c}_2$, and $\overline{c}_3$ be representatives of the conjugacy classes of involutions in Sym(6). Furthermore, let C_i be the full inverse image of $\langle\overline{c}_i\rangle$ in C and $\overline{R}_i$ a right transversal for $C_{\overline{C}}(\overline{c}_i)$ in $\overline{C}$, for each $i \in \{1, 2, 3\}$. Finally, let R_i be a set of representatives of each coset in $\overline{R}_i$. And now we form

$$N_i = (C_i \setminus O_2(C)) \cap 2\mathrm{D}_G$$

so that

$$\bigcup_{i=1}^{3} \left(\bigcup_{r \in R_i} N_i^r \right) \supseteq \mathcal{I}(C \setminus O_2(C)) \cap 2\mathrm{D}_G$$

by Lemma 3.14 and the remarks that follow.

In practice, we form these in one "master screen". We obtain $\overline{c}_1$, $\overline{c}_2$, and $\overline{c}_3$ by using `Classes` on $\overline{C}$, and $\overline{R}_i$ by using the command `Transversal`. We find R_i with:

```
Ri:={r@@f : r in Rib};
```

where the objects `Ri` and `Rib` represent R_i and $\overline{R}_i$ respectively, and `f` denotes the homomorphism $f : C \to \overline{C}$ provided when we ran `LMGRadicalQuotient` on C to find $O_2(C)$ and $\overline{C}$. And we find C_i using:

```
Ci:=sub<Cb|cib>@@f;
```

where the objects `Ci`, `Cb`, and `cib` represent C_i, $\overline{C}$, and $\overline{c}_i$ respectively. Note that $|R_i| = 15$ for $i = 1, 2$, and $|R_3| = 45$, so we are able to save these as sets of matrices. Also, note that $Z(C_i) \cong 2^7$ for $i \in \{1, 3\}$ and $Z(C_2) \cong 2^8$.

Now we will fix $i \in \{1, 2, 3\}$ and discuss how we sieve each C_i. We construct $Z(C_i) \leq S_i \leq O_2(C) \leq C_i$ such that $|S_i| = 2^{31}$ and therefore $[C_i : S_i] = 64$, and let Δ_0 be a right transversal for S_i in C_i. Then for all $t \in C_i$, we have $t \in S_i\delta$ for some $\delta \in \Delta_0$. But recall that we are only interested in t when it is an involution which is not in $O_2(C)$. We claim that

$$t \notin O_2(C) \text{ if and only if } \delta \notin O_2(C).$$

Indeed, suppose $\delta \in O_2(C)$. Then we know $t = s\delta$ for some $s \in S_i$. Since $S_i \leq O_2(C)$ we have $s \in O_2(C)$ and thus $t \in O_2(C)$. Now suppose $t \in O_2(C)$. This time, we see that $s^{-1}t = \delta$ and since $t, s \in O_2(C)$ we have $\delta \in O_2(C)$. Therefore, the claim holds, as we have proved both contrapositive results. Hence we need only sieve the cosets $S_i\delta$ such that $\delta \notin O_2(C)$, so let $\Delta = \{\delta \in \Delta_0 : \delta \notin O_2(C)\} = \{\delta_1, \ldots, \delta_{32}\}$.

We now break the process up into 32 screens and run them in parallel. Fix $j \in \{1, \ldots, 32\}$. Screen j will sieve the coset $S_i\delta_j$, so in screen j we load Δ, S_i, R_i, and C_i. Then, we let Γ be a right transversal for $Z(C_i)$ in S_i (we do this by running `Transversal` in the pc-group setting). By Proposition 3.11, we have that every involution in $S_i\delta_j$ is given in the form $z\gamma\delta_j$ for some $z \in Z(C_i)$ and $\gamma \in \Gamma$. We also know, by Lemma 3.9, that $(z\gamma\delta_j)^2 = 1$ if and only if $(\gamma\delta_j)^2 = 1$.

Let $\mathcal{I}_0(S_i\delta_j) = \{\gamma\delta_j : (\gamma\delta_j)^2 = 1\}$. We now build a set called $\mathcal{I}_1(S_i\delta_j)$ by carrying out the following. For each $x_0 \in \mathcal{I}_0(S_i\delta_j)$, we run through the elements $z \in Z(C_i)$ and build $x = zx_0$ (as we are now essentially sieving the cosets $Z(C_i)x_0$). Now, if $x \in 2\mathrm{D}_G$, we run through each $r \in R_i$ and consider x^r. (Recall we are sieving N_i^r for all $r \in R_i$.) If x^r survives an order of random elements sieve (see Procedure 3.8 for full details) then we keep x^r in the set $\mathcal{I}_1(S_i\delta_j)$. Having carried out this process for each $i \in \{1, 2, 3\}$ and each $j \in \{1, \ldots, 32\}$, we now construct

$$\mathcal{I}_1(C) = \bigcup_{i=1}^{3} \left(\bigcup_{j=1}^{32} \mathcal{I}_1(S_i\delta_j) \right),$$

the union of all $\mathcal{I}_1(S_i\delta_j)$ across all screens. Then $|\mathcal{I}_1(C)| = 50$ and now let

$$\mathcal{I}_2(C) = \{x \in \mathcal{I}_1(C) : |\langle P_3 D_3^{(3)}, x\rangle| = |Sp_4(4)|\}$$

but we find that $\mathcal{I}_2(C) = \varnothing$. Hence there are no involutions in $C \setminus O_2(C)$ which generate $Sp_4(4)$ along with $P_3D_3^{(3)}$. ■

We will now work to prove that many of the copies of $Sp_4(4)$ we have found are actually conjugate in G. Let $(i,j) \in \{(2,1),(3,3),(3,5),(3,7)\}$ and recall we have $P_iD_j^{(i)} \leq H_k^{(i,j)} \leq G$ for $H_k^{(i,j)} \cong Sp_4(4)$, where $k \in \{1,\ldots,n_{ij}\}$ and n_{ij} is the number of $Sp_4(4)$ subgroups containing $P_iD_j^{(i)}$ (so $n_{21} = 2$, $n_{33} = 64$, $n_{35} = 4$, and $n_{37} = 4$). Recall that $P_iD_j^{(i)} = N_{H_k^{(i,j)}}(P_i)$, so Proposition 3.15 applies and we have, for $k, k_0 \in \{1,\ldots,n_{ij}\}$, that $H_k^{(i,j)}$ and $H_{k_0}^{(i,j)}$ are G-conjugate if and only if $H_k^{(i,j)}$ and $H_{k_0}^{(i,j)}$ are $N_{N_G(P_i)}(P_iD_j^{(i)})$-conjugate.

Hence the first step in showing some of our found subgroups are conjugate is to construct $N_{N_G(P_i)}(P_iD_j^{(i)})$.

Lemma 10.11. *Let $(i,j) \in \{(2,1),(3,3),(3,5),(3,7)\}$. The orders of $N_{N_G(P_i)}(P_iD_j^{(i)})$ are given in Table 10.7.*

i	j	$\lvert N_{N_G(P_i)}(P_iD_j^{(i)})\rvert$
2	1	$2^5.3^5$
3	3	$2^6.3.5^3$
3	5	$2^6.5^2$
3	7	$2^5.5^2$

Table 10.7: $\lvert N_{N_G(P_i)}(P_iD_j^{(i)})\rvert$ for $(i,j) \in \{(2,1),(3,3),(3,5),(3,7)\}$

Proof. We already have $N_G(P_i)$ for $i \in \{2,3\}$. To find $N_{N_G(P_i)}(P_iD_j^{(i)})$ we first turn $N_G(P_i)$ into a permutation group. Since $P_iD_j^{(i)} \leq N_G(P_i)$ we can map $P_iD_j^{(i)}$ into the permutation group setting and calculate the normaliser there directly. ■

With these groups constructed, we can now proceed to show which of our $Sp_4(4)$ subgroups are conjugate in G.

Proposition 10.12. *The number of $Sp_4(4)$ subgroups up to G-conjugacy containing $P_iD_j^{(i)}$, for $(i,j) \in \{(2,1),(3,3),(3,5),(3,7)\}$, is shown in Table 10.8.*

Proof. First, let $(i,j) \neq (3,3)$. Choose any $k, k_0 \in \{1,\ldots,n_{ij}\}$. Then, using the group $N_{N_G(P_i)}(P_iD_j^{(i)})$ we can readily find an element $n \in N_{N_G(P_i)}(P_iD_j^{(i)})$ such that $(H_k^{(i,j)})^n = H_{k_0}^{(i,j)}$. We find these elements with a simple `repeat..until` loop. For the case where $(i,j) = (3,3)$, we use the same method, except the subgroups fall into two conjugacy classes. Hence the result follows by Proposition 3.15. ■

i	j	Number of $Sp_4(4)$ subgroups containing $P_iD_j^{(i)}$
2	1	1
3	3	2
3	5	1
3	7	1

Table 10.8: Number of $Sp_4(4)$ subgroups up to G-conjugacy

Now that we have constructed all possible subgroups of G isomorphic to $Sp_4(4)$ up to conjugacy in G and which follow $Sp_4(4)$ fusion (i), (ii), (v), (vi), (vii), or (viii), we now seek to extend them to groups isomorphic to subgroups of $\mathrm{Aut}(H)$.

10.2 Extending $Sp_4(4)$ to $Sp_4(4):4$

Before we begin the hunt for subgroups isomorphic to subgroups of $\mathrm{Aut}(H)$, let us first redefine our notation. For the rest of this chapter, we will denote our copies of $Sp_4(4)$ by $H_1^{(2,1)}$, $H_1^{(3,3)}$, $H_2^{(3,3)}$, $H_1^{(3,5)}$, and $H_1^{(3,7)}$, where $P_iD_j^{(i)} \leq H_k^{(i,j)}$. First, we state that $\mathrm{Aut}(Sp_4(4)) \cong Sp_4(4):4$, which can be seen in the $\mathbb{ATLAS}$ [14]. To find elements extending these copies of $Sp_4(4)$, we will appeal to some results in Section 2.4, which we will gather together in the following lemma.

Lemma 10.13. *Suppose $g, g_0 \in G$ such that $\langle H_k^{(i,j)}, g\rangle \cong Sp_4(4):2$ and $\langle H_k^{(i,j)}, g_0\rangle \cong Sp_4(4):4$. Then*

(i) $g, g_0 \in \mathcal{E}_0(H_k^{(i,j)}) := N_{N_G(P_i)}(P_iD_j^{(i)}) \cap N_G(H_k^{(i,j)})$;

(ii) $g, g_0 \notin H_k^{(i,j)}$ and $g^2, g_0^4 \in H_k^{(i,j)}$;

(iii) all the $\mathcal{E}_0(H_k^{(i,j)})$-conjugates of g, respectively g_0, generate conjugate copies of $Sp_4(4):2$, respectively $Sp_4(4):4$.

Proof. Part (i) follows from Proposition 2.15, part (ii) from Proposition 2.17, and part (iii) from Proposition 2.16. ■

This lemma provides us with an outline on how we will proceed to find elements extending our copies of $Sp_4(4)$. After constructing $\mathcal{E}_0(H_k^{(i,j)})$, we will build $\mathcal{E}_1(H_k^{(i,j)})$, a set of $\mathcal{E}_0(H_k^{(i,j)})$-conjugacy class representatives. Then we build

$$\mathcal{E}_2^{(2)}(H_k^{(i,j)}) = \{x \in \mathcal{E}_1(H_k^{(i,j)}) : x \notin H_k^{(i,j)} \text{ and } x^2 \in H_k^{(i,j)}\};$$

$$\mathcal{E}_2^{(4)}(H_k^{(i,j)}) = \{x \in \mathcal{E}_1(H_k^{(i,j)}) : x \notin H_k^{(i,j)} \text{ and } x^4 \in H_k^{(i,j)}\};$$

and finally

$$\mathcal{E}_3^{(2)}(H_k^{(i,j)}) = \{x \in \mathcal{E}_2^{(2)}(H_k^{(i,j)}) : \langle H_k^{(i,j)}, x\rangle \cong Sp_4(4) : 2\};$$

$$\mathcal{E}_3^{(4)}(H_k^{(i,j)}) = \{x \in \mathcal{E}_2^{(4)}(H_k^{(i,j)}) : \langle H_k^{(i,j)}, x\rangle \cong Sp_4(4) : 4\}.$$

The results of this process are given in the next result.

Proposition 10.14. *Let $(i, j, k) \in \{(2, 1, 1), (3, 3, 1), (3, 3, 2), (3, 5, 1), (3, 7, 1)\}$. Then the sizes of the sets $\mathcal{E}_m(H_k^{(i,j)})$ where $m \in \{0, 1\}$ and $\mathcal{E}_m^{(n)}(H_k^{(i,j)})$ where $m \in \{2, 3\}$ and $n \in \{2, 4\}$ are given in Table 10.9. Moreover, the two elements $g, g_0 \in \mathcal{E}_3^{(2)}(H_1^{(3,3)})$ are such that $\langle H_1^{(3,3)}, g\rangle$ and $\langle H_1^{(3,3)}, g_0\rangle$ are conjugate in G as groups.*

i	j	k	$\lvert\mathcal{E}_0(H_k^{(i,j)})\rvert$	$\lvert\mathcal{E}_1(H_k^{(i,j)})\rvert$	$\lvert\mathcal{E}_2^{(2)}(H_k^{(i,j)})\rvert$	$\lvert\mathcal{E}_3^{(2)}(H_k^{(i,j)})\rvert$	$\lvert\mathcal{E}_2^{(4)}(H_k^{(i,j)})\rvert$	$\lvert\mathcal{E}_3^{(4)}(H_k^{(i,j)})\rvert$
2	1	1	400	16	6	1	6	0
3	3	1	24,000	62	16	2	22	0
3	3	2	1,600	46	30	1	36	0
3	5	1	400	16	6	1	6	0
3	7	1	200	14	0	0	0	0

Table 10.9: Sizes of the sets $\mathcal{E}_0(H_k^{(i,j)})$, $\mathcal{E}_1(H_k^{(i,j)})$, and $\mathcal{E}_m^{(n)}(H_k^{(i,j)})$ for $m \in \{2, 3\}$ and $n \in \{2, 4\}$

Proof. We first construct $\mathcal{E}_0(H_k^{(i,j)})$. Note that we already constructed $N_{N_G(P_i)}(P_iD_j^{(i)})$ in Lemma 10.11; now we run through every element in that group and store the ones that normalise $H_k^{(i,j)}$. This is $\mathcal{E}_0(H_k^{(i,j)})$. Using `Classes` we obtain $\mathcal{E}_1(H_k^{(i,j)})$, and now some simple `for..do` loops build the rest of the sets.

Now let $g, g_0 \in \mathcal{E}_3^{(2)}(H_1^{(3,3)})$ with $g \neq g_0$. We can then, using a `repeat..until` loop, easily find an element $n \in N_{N_G(P_i)}(P_iD_j^{(i)})$ such that $\langle H_1^{(3,3)}, g\rangle^n = \langle H_1^{(3,3)}, g_0\rangle$, thus these groups are conjugate in G as required. ■

Hence we have, up to G-conjugacy, at most four subgroups isomorphic to $Sp_4(4) : 2$ and no subgroups isomorphic to $Sp_4(4) : 4$ containing an $Sp_4(4)$ following fusion possibility 2.5 (i), (ii), (v), (vi), (vii), or (viii). Note that Theorem 1.7 states that there are exactly five classes of subgroups isomorphic to $Sp_4(4)$ following these fusion possibilities, and exactly four classes of subgroups isomorphic to $Sp_4(4) : 2$ containing an $Sp_4(4)$ following these fusion possibilities. So far, we have only obtained upper bounds. We will prove Theorem 1.7 at the end of this chapter, as we will use results established later. We will now prove that no $Sp_4(4)$, $Sp_4(4):2$, or $Sp_4(4):4$ subgroups are maximal in G.

Proposition 10.15. *Suppose $H_0 \leq G$ with $H = F^*(H_0) \cong Sp_4(4)$. Then H_0 is not maximal in G.*

Proof. By Proposition 2.5 we know that H must follow one of eight fusion possibilities. If H follows $Sp_4(4)$ fusion possibility (iii) or (iv), then H is not maximal by Lemma 10.2. Moreover, H_0 must also fix a non-zero vector by Proposition 2.18 and so is not maximal in G by Proposition 2.23.

So, assume H follows fusion possibility (i), (ii), (v), (vi), (vii), or (viii). Then H is G-conjugate to one of the five copies of $Sp_4(4)$ constructed in Proposition 10.12 and the preceding results. Hence, without loss of generality, if we show that these five copies of $Sp_4(4)$ – and their automorphism extensions, if they exist – are not maximal in G, we are done.

We start with $H_1^{(3,7)}$, which has no automorphism extensions in G. We also find that $\dim C_V(H_1^{(3,7)}) = 0$ but we can use a `repeat..until` loop to find an element $m \in N_{N_G(P_3)}(P_3 D_7^{(3)})$ with $H_1^{(3,7)} \leq \langle H_1^{(3,7)}, m\rangle < G$. And so, $H_1^{(3,7)}$ is not maximal in G.

Now let $(i,j,k) \in \{(2,1,1),(3,3,1),(3,3,2),(3,5,1)\}$. In all of these cases, we have that $H_k^{(i,j)} \leq H_k^{(i,j)}:2$, so none of $H_k^{(i,j)}$ are maximal in G. As for the groups $H_k^{(i,j)}:2$, we find that

$$\dim C_V(H_k^{(i,j)}:2) = \begin{cases} 1, & \text{if } (i,j,k) \in \{(2,1,1),(3,3,2),(3,5,1)\}, \\ 4, & \text{if } (i,j,k) = (3,3,1). \end{cases}$$

In all cases, $\dim C_V(H_k^{(i,j)}:2) > 0$ so $H_k^{(i,j)}:2$ fixes a non-zero vector of V and therefore is not maximal in G by Proposition 2.23. This exhausts all possibilities for H_0. ∎

We will conclude this chapter by proving Theorem 1.7. To prove part (i), we must show that if $H_0 \leq G$ such that $H = F^*(H_0) \cong Sp_4(4)$ and H does not follow $Sp_4(4)$ fusion possibility 2.5 (iii) or (iv), then there are five classes of subgroups $H_0 \cong Sp_4(4)$, four classes of subgroups $H_0 \cong Sp_4(4):2$, and no subgroups $H_0 \cong Sp_4(4):4$.

We begin by recalling that by Proposition 10.12 and the preceding results we have five subgroups $H_k^{(i,j)}$ isomorphic to $Sp_4(4)$, where $(i,j,k) \in \{(2,1,1),(3,3,1),(3,3,2),$

$(3,5,1),(3,7,1)\}$. Recall also that $P_i \in \mathrm{Syl}_5(H_k^{(i,j)})$. Since P_2 and P_3 follow different fusion patterns, it follows that $H_1^{(2,1)}$ is not G-conjugate to any $H_k^{(i,j)}$ such that $(i,j,k) \in \{(3,3,1),(3,3,2),(3,5,1),(3,7,1)\}$. Moreover, $H_1^{(3,3)}$ and $H_2^{(3,3)}$ are not conjugate by Proposition 10.12. We must now show that if $(j,k),(j_0,k_0) \in \{(3,1),(3,2),(5,1),(7,1)\}$ such that $j \neq j_0$, then $H_k^{(3,j)}$ and $H_{k_0}^{(3,j_0)}$ are not G-conjugate. We claim that $H_k^{(3,j)}$ and $H_{k_0}^{(3,j_0)}$ are G-conjugate if and only if they are $N_G(P_3)$-conjugate. Indeed, suppose $g \in G$ with $(H_k^{(3,j)})^g = H_{k_0}^{(3,j_0)}$. Then $P_3^g \in \mathrm{Syl}_5(H_{k_0}^{(3,j_0)})$ and so there is some $h \in H_{k_0}^{(3,j_0)}$ such that $P_3^{gh} = P_3$ by Sylow's Theorems. Then $gh \in N_G(P_3)$. Now we have $(H_k^{(3,j)})^{gh} = (H_{k_0}^{(3,j_0)})^h = H_{k_0}^{(3,j_0)}$. Hence $H_k^{(3,j)}$ and $H_{k_0}^{(3,j_0)}$ are conjugate in $N_G(P_3)$. Now, assume there exists $n \in N_G(P_3)$ such that $(H_k^{(3,j)})^n = H_{k_0}^{(3,j_0)}$. Then $(D_j^{(3)})^n \leq H_{k_0}^{(3,j_0)}$ but since $(D_j^{(3)})^n$ must also normalise P_3 we have that $(D_j^{(3)})^n \leq N_{H_{k_0}^{(3,j_0)}}(P_3)$. However, $D_{j_0}^{(3)}$ is unique in $N_{H_{k_0}^{(3,j_0)}}(P_3)$ up to conjugacy, so let $h \in N_{H_{k_0}^{(3,j_0)}}(P_3)$ such that $(D_j^{(3)})^{nh} = D_{j_0}^{(3)}$. But now we have $nh \in N_G(P_3)$. This is a contradiction, because – by Proposition 10.6 – $D_j^{(3)}$ and $D_{j_0}^{(3)}$ represent distinct classes of $\mathrm{Dih}(8)$ subgroups in $N_G(P_3)$. Hence $H_k^{(3,j)}$ and $H_{k_0}^{(3,j_0)}$ are not $N_G(P_3)$-conjugate and thus are not G-conjugate, as claimed.

To prove Theorem 1.7 (ii), we have at most four classes of $Sp_4(4):2$ subgroups constructed in Proposition 10.14. We will name the representatives of these classes $K_k^{(i,j)}$ such that $(i,j,k) \in \{(2,1,1),(3,3,1),(3,3,2),(3,5,1)\}$ and $H_k^{(i,j)} \leq K_k^{(i,j)}$. Let $(i,j,k),(i_0,j_0,k_0) \in \{(2,1,1),(3,3,1),(3,3,2),(3,5,1)\}$ such that $(i,j,k) \neq (i_0,j_0,k_0)$ and let $H = H_k^{(i,j)}$, $K = K_k^{(i,j)}$, $H_0 = H_{k_0}^{(i_0,j_0)}$, and $K_0 = K_{k_0}^{(i_0,j_0)}$. We claim that if K is G-conjugate to K_0, then H is G-conjugate to H_0. Indeed, if $g \in G$ such that $K^g = K_0$, then $H^g \leq K_0$. Since H_0 is unique up to conjugacy in K_0, then we have some $h \in K_0$ for which $H^{gh} = H_0$. This proves the claim. We showed above that H is not conjugate to H_0, hence K is not conjugate to K_0. Since this holds for all choices of K, K_0, we conclude that we have exactly four classes of subgroups isomorphic to $Sp_4(4):2$ which contain $Sp_4(4)$ not following fusion possibilities 2.5 (iii) or (iv).

Finally, we see that by Proposition 10.14 there are no $Sp_4(4):4$ subgroups containing an $Sp_4(4)$ subgroup not following fusion possibilities 2.5 (iii) or (iv). This concludes the proof of Theorem 1.7. ■

Chapter 11

$L_4(4)$ and Its Extensions

In this chapter, we will prove Theorem 1.8, showing that there is only one $L_4(4)$ subgroup of G up to G-conjugacy which follows $L_4(4)$ fusion possibility (ii). Compared with other cases, $L_4(4)$ is relatively straightforward and mostly a matter of building up from our copies of $Sp_4(4)$ constructed in Chapter 10.

11.1 Constructing $L_4(4)$ Subgroups of G

For the rest of this section, we will suppose $H \cong L_4(4)$ and follows $L_4(4)$ fusion possibility (ii).

Lemma 11.1. *Let $K \leq H \leq G$ with $K \cong Sp_4(4)$. If H follows $L_4(4)$ fusion possibility (ii), then K follows $Sp_4(4)$ fusion possibility (i), (ii), (v), (vi), (vii), or (viii). Moreover, H is G-conjugate to a copy of $L_4(4)$ subgroup of G containing one of the copies of $Sp_4(4)$ found in Proposition 10.12.*

Proof. If we compare the $Sp_4(4)$ fusion possibilities with $L_4(4)$ fusion possibility (ii), we see that K cannot follow $Sp_4(4)$ fusion possibility (iii) or (iv). This can be seen by observing that, if K follows fusion (iii) or (iv), then there is some $k \in K$ such that $k \in 3\text{A}_G$, but there is no $h \in H$ such that $h \in 3\text{A}_G$. This is a clear contradiction. All the fusion information in $Sp_4(4)$ fusion possibilities (i), (ii), (v), (vi), (vii), or (viii) is consistent with $L_4(4)$ fusion (ii) – this completes the first result in the lemma. Now, we simply observe that in Proposition 10.12, we found all copies of $Sp_4(4)$ which follow these fusion possibilities, up to G-conjugacy. Therefore, Proposition 2.12 yields the second result of the lemma. ■

The rest of this chapter will be devoted to building $L_4(4)$ subgroups of G as overgroups of the five $Sp_4(4)$ subgroups found in Proposition 10.12. The next result lists some properties of $L_4(4)$ and provides us with an efficient means of building $L_4(4)$ as an overgroup of $Sp_4(4)$.

Proposition 11.2. *Let $K \leq H \leq G$ with $K \cong Sp_4(4)$. Let $P \in \mathrm{Syl}_5(K)$. Then*

(i) K is maximal in H;

(ii) $N_H(P) \nleq K$;

(iii) for all $x \in N_H(P)$ where $o(x) = 30$, we have $\langle K, x \rangle = H$.

Proof. (i) See the $\mathbb{ATLAS}$ [14].

(ii) This is directly verifiable in Magma.

(iii) First, note that there no elements of order 30 in K (again, see the $\mathbb{ATLAS}$ [14]). There are elements x of order 30 in $N_H(P)$. Hence we must have that $K < \langle K, x \rangle \leq H$ and so the result follows by the maximality of K in H.

∎

Recall that P_2 and P_3 are subgroups of G isomorphic to 5^2, and that in the remark following Proposition 10.12, after finding the five subgroups isomorphic to $Sp_4(4)$, we named them $H_k^{(i,j)}$ where $(i, j, k) \in \{(2, 1, 1), (3, 3, 1), (3, 3, 2), (3, 5, 1), (3, 7, 1)\}$. In this chapter, we will use the letter K instead, so our five copies of $Sp_4(4)$ are now denoted $K_k^{(i,j)}$. Here, we have $P_i \in \mathrm{Syl}_5(K_k^{(i,j)})$.

Proposition 11.2 tells us that if $K_k^{(i,j)} \leq H \leq G$, then there is an element x of order 30 normalising P_i such that $\langle K_k^{(i,j)}, x \rangle = H$. Recall that we have copies of $N_G(P_i)$ for $i \in \{2, 3\}$, which we found in Proposition 7.7. Hence, for each $i \in \{2, 3\}$, we construct

$$\mathcal{L}_0(K_k^{(i,j)}) = \{x \in N_G(P_i) : o(x) = 30\},$$

a set containing all possible elements forming a required generating set of $L_4(4)$. Now, for each element $x \in \mathcal{L}_0(K_k^{(i,j)})$, we apply a 100 order of random elements sieve to $\langle K_k^{(i,j)}, x \rangle$, which is described in full detail in Procedure 3.8. For the elements x which survive the sieve, we store them in a set called $\mathcal{L}_1(K_k^{(i,j)})$. Finally, we construct a set $\mathcal{L}_2(K_k^{(i,j)}) \subseteq \mathcal{L}_1(K_k^{(i,j)})$ such that

(i) for all $y \in \mathcal{L}_2(K_k^{(i,j)})$, $\langle K_k^{(i,j)}, y\rangle \cong L_4(4)$;

(ii) for all $y, y_0 \in \mathcal{L}_2(K_k^{(i,j)})$ with $y \neq y_0$, $\langle K_k^{(i,j)}, y\rangle \neq \langle K_k^{(i,j)}, y_0\rangle$;

(iii) for all $y_0 \in \mathcal{L}_1(K_k^{(i,j)})$, there exists $y \in \mathcal{L}_2(K_k^{(i,j)})$ such that $y_0 \in \langle K_k^{(i,j)}, y\rangle$.

Essentially, we trim $\mathcal{L}_1(K_k^{(i,j)})$ down to $\mathcal{L}_2(K_k^{(i,j)})$ by choosing only elements that generate all of the distinct copies of $L_4(4)$. We will now state the results of this process.

Proposition 11.3. *Table 11.1 shows $|\mathcal{L}_m(K_k^{(i,j)})|$ for $m \in \{0, 1, 2\}$.*

i	j	k	$\lvert\mathcal{L}_0(K_k^{(i,j)})\rvert$	$\lvert\mathcal{L}_1(K_k^{(i,j)})\rvert$	$\lvert\mathcal{L}_2(K_k^{(i,j)})\rvert$
2	1	1	0	0	0
3	3	1	224,000	800	10
3	3	2	224,000	0	0
3	5	1	224,000	0	0
3	7	1	224,000	0	0

Table 11.1: $|\mathcal{L}_m(K_k^{(i,j)})|$ for $m \in \{0, 1, 2\}$

Proof. To build $\mathcal{L}_0(K_k^{(i,j)})$, we first turn $N_G(P_i)$ into a permutation group. Then, using `Classes`, we obtain a collection of conjugacy class representatives of $N_G(P_i)$ of elements of order 30. Then we use `Class` to construct all conjugacy classes of elements of order 30, and use `join` to form their union. This is $\mathcal{L}_0(K_k^{(i,j)})$. We apply Procedure 3.8 to construct $\mathcal{L}_1(K_k^{(i,j)})$. Note that to apply this sieve, we use the set of possible element orders of $L_4(4)$ given by

$$\{1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 15, 17, 21, 30, 63, 85\}$$

which can be seen by looking at the $\mathbb{ATLAS}$ [14]. To build $\mathcal{L}_2(K_k^{(i,j)})$, we use the following algorithm. Start with $\mathcal{L}_2(K_k^{(i,j)}) := \mathcal{L}_1(K_k^{(i,j)})$ as an ordered set. Start with the first element $y_1 \in \mathcal{L}_2(K_k^{(i,j)})$ such that $\langle K_k^{(i,j)}, y_1\rangle \cong L_4(4)$ then delete from $\mathcal{L}_2(K_k^{(i,j)})$ all x such that $x \in \langle K_k^{(i,j)}, y_1\rangle$. Now take the second element y_2 such that $\langle K_k^{(i,j)}, y_1\rangle \cong L_4(4)$ and delete from $\mathcal{L}_2(K_k^{(i,j)})$ all x such that $x \in \langle K_k^{(i,j)}, y_2\rangle$. We repeat this process until we choose y_n, where $n = |\mathcal{L}_2(K_k^{(i,j)})|$ (so that y_n is the last element in $\mathcal{L}_2(K_k^{(i,j)})$), at which point there is nothing more to delete. ■

Hence we only have one copy of $Sp_4(4)$ for which an overgroup of $L_4(4)$ exists as a subgroup of G, namely $K_1^{(3,3)}$. For the remainder of this section, let $K = K_1^{(3,3)}$, as we have no interest in the other copies of $Sp_4(4)$. Proposition 11.3 tells us that there are at most ten $L_4(4)$ subgroups of G containing K, up to G-conjugacy. The next result reduces this number.

Proposition 11.4. *Up to conjugacy in G, there is exactly one copy of $L_4(4)$ containing K.*

Proof. From Proposition 11.3, we have $y_1, \ldots, y_{10}$ such that $\langle K, y_i \rangle \cong L_4(4)$, where $i \in \{1, \ldots, 10\}$. And, for $i, j \in \{1, \ldots, 10\}$, we have that if $i \neq j$ then $\langle K, y_i \rangle \neq \langle K, y_j \rangle$. Now, for each $i \in \{2, \ldots, 10\}$, we take random elements $n \in N_G(P_3)$ until $y_i^n \in \langle K, y_1 \rangle$. When we find such an n, we then check if $\langle K, y_i \rangle^n = \langle K, y_1 \rangle$. These elements can be readily found for each $i \in \{2, \ldots, 10\}$, so we have that all 10 of these copies of $L_4(4)$ are conjugate in G. ■

Thus, we have only one $L_4(4)$ subgroup of G following fusion possibility (ii), up to conjugacy in G. We call this subgroup H, and now we endeavour to find $H_0 \leq G$ such that H_0 is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

11.2 Extending $L_4(4)$ to $L_4 : 2^2$

Let H be the sole copy of $L_4(4)$ in G up to conjugacy. We start with some facts about $\mathrm{Aut}(L_4(4))$.

Proposition 11.5. *Let $H_0 \cong \mathrm{Aut}(H)$. Then $H_0 \cong L_4(4) : 2^2$ and H_0 contains three classes of maximal subgroups with the shape $L_4(4) : 2$ which are pairwise non-isomorphic. Moreover, if we let Y_1, Y_2, and Y_3 be representatives of these classes, then if $i \neq j$ then $H_0 = \langle Y_i, Y_j \rangle$, for $i, j \in \{1, 2, 3\}$.*

Proof. The existence of such classes of maximal subgroups can be seen in the $\mathbb{ATLAS}$ [14]. The final statement is a direct result of the maximality of these subgroups. ■

Hence our strategy will first be to find all overgroups of H in G with the shape $L_4(4) : 2$. For if $L_4(4) : 2^2$ exists as an overgroup of H in G, it will contain three non-isomorphic copies of $L_4(4) : 2$.

Recall that $P_3 \in \mathrm{Syl}_5(H)$ where $P_3 \cong 5^2$. Define

$$\mathcal{E}_0(H) = N_{N_G(P_3)}(N_H(P_3)) \cap N_G(H),$$

and by Proposition 2.15, any $g \in G$ for which $\langle H, g\rangle \sim L_4(4) : 2$ must be such that $g \in \mathcal{E}_0(H)$. Now we let $\mathcal{E}_1(H)$ be a set of conjugacy class representatives of the classes in $\mathcal{E}_0(H)$, since from Proposition 2.16 we have that if $g, h \in \mathcal{E}_0(H)$ which are conjugate in $\mathcal{E}_0(H)$, then $\langle H, g\rangle$ and $\langle H, h\rangle$ are conjugate groups. Next, let

$$\mathcal{E}_2(H) = \{x \in \mathcal{E}_1(H) : x \notin H \text{ and } x^2 \in H\},$$

as by Proposition 2.17 we have that all $g \in G$ such that $\langle H, g\rangle \sim L_4(4) : 2$ must be such that $g \in \mathcal{E}_2(H)$. Finally, we build $\mathcal{E}_3(H) \subseteq \mathcal{E}_2(H)$, a set containing elements of $\mathcal{E}_2(H)$ which generate distinct copies of $L_4(2) : 2$. The next result details the outcome of this process.

Proposition 11.6. *The sizes of the sets $\mathcal{E}_m(H)$ for $m \in \{0, 1, 2, 3\}$ are shown in Table 11.2.*

$\lvert\mathcal{E}_0(H)\rvert$	$\lvert\mathcal{E}_1(H)\rvert$	$\lvert\mathcal{E}_2(H)\rvert$	$\lvert\mathcal{E}_3(H)\rvert$
7,200	78	31	5

Table 11.2: $|\mathcal{E}_m(H)|$ for $m \in \{0, 1, 2, 3\}$

Proof. In MAGMA, we construct $\mathcal{E}_0(H)$ by calculating $N_H(P_3)$ directly, then turning $N_G(P_3)$ into a permutation group and finding $N_{N_G(P_3)}(N_H(P_3))$ in the permutation setting. Then, we run through the elements of $N_{N_G(P_3)}(N_H(P_3))$ and keep the ones which normalise H. We find $\mathcal{E}_1(H)$ using `Classes`, then $\mathcal{E}_2(H)$ with simple `for..do` procedures. To construct $\mathcal{E}_3(H)$ we use an algorithm like that described at the end of the proof of Proposition 11.3. ∎

Hence we have five distinct overgroups of H in G with shape $L_4(4) : 2$. Now we can begin constructing overgroups of H isomorphic to $L_4(4) : 2^2$.

Proposition 11.7. *Suppose $H \leq H_0 \leq G$ such that $H_0 \cong L_4(4) : 2^2$. Then H_0 is unique up to G-conjugacy.*

Proof. First, we will introduce some notation. From Proposition 11.5 we know that there are three isomorphism types of maximal subgroups of H_0 with shape $L_4(4) : 2$. We will call these $L_4(4) : 2_i$ for $i \in \{1, 2, 3\}$. Now, let $y_1, \ldots, y_5$ be the distinct elements of $\mathcal{E}_3(H)$, and $Y_j = \langle H, y_j\rangle$, for each $j \in \{1, 2, 3, 4, 5\}$. Then we have $Y_1 \cong Y_3 \cong L_4(4) : 2_1$, $Y_2 \cong L_4(4) : 2_2$, and $Y_4 \cong Y_5 \cong L_4(4) : 2_3$. Hence all possible

$L_4(4) : 2^2$ overgroups of H will be given by $\langle Y_i, Y_j \rangle$ such that $i, j \in \{1, 2, 3, 4, 5\}$, $Y_i \ncong Y_j$, and $\langle Y_i, Y_j \rangle \cong L_4(4) : 2^2$. We find that $\langle Y_i, Y_j \rangle \cong L_4(4) : 2^2$ when $(i, j) \in \{(1, 2), (1, 4), (2, 3), (2, 4), (2, 5), (3, 5)\}$. Moreover, let $Y_{ij} = \langle Y_i, Y_j \rangle$ and we find that $Y_{12} = Y_{14} = Y_{24}$ and $Y_{23} = Y_{25} = Y_{35}$ with $Y_{12} \neq Y_{23}$. Finally, we can readily find elements $n \in N_G(P_3)$ such that $Y_{12}^n = Y_{23}$. Hence, up to G-conjugacy, there is only one $L_4(4) : 2^2$ subgroup containing H. ■

We remark that the proof of Proposition 11.7 demonstrates that there are exactly three subgroups of G containing H with shape $L_4(4) : 2$. Indeed, since $L_4(4) : 2^2$ contains a unique copy of $L_4(4) : 2_i$, for each $i \in \{1, 2, 3\}$, it follows that Y_1 and Y_3 are G-conjugate and Y_4 and Y_5 are G-conjugate. Now we will show that none of the groups we have constructed in this chapter are maximal in G.

Proposition 11.8. *Let $H \leq G$ such that $F^*(H) \cong L_4(4)$ and $F^*(H)$ follows $L_4(4)$ fusion possibility 2.6 (ii). Then H is not maximal in G.*

Proof. By Proposition 11.4, $F^*(H)$ is G-conjugate to the sole copy of $L_4(4)$ following fusion possibility 2.6 (ii). Call this group L. By Propositions 11.6 and 11.7, L can be extended into three groups of shape $L_4(4) : 2$, all of which are contained in an $L_4(4) : 2^2$ subgroup, which we will call K. To see that K is not maximal, we find some $x \in N_G(P_3)$ such that $K < \langle K, x \rangle < G$. ■

Let us now conclude this chapter – and thesis – by proving Theorem 1.8. We must show that if $H_0 \leq G$ such that $H = F^*(H_0) \cong L_4(4)$ and H follows $L_4(4)$ fusion possibility 2.6 (ii), then there is one class of subgroups $H_0 \cong L_4(4)$, three classes of subgroups H_0 with shape $L_4(4) : 2$, and one class of subgroups $H_0 \cong L_4(4) : 2^2$. The single class of $L_4(4)$ subgroups is constructed in Proposition 11.4, while the latter two results can be seen from Proposition 11.7 and the remarks following its proof. This concludes the proof of Theorem 1.8. ■

Bibliography

[1] M. Aschbacher: *On the maximal subgroups of the finite classical groups*, Invent. Math. 76 (1984), no. 3, 469–514.

[2] M. Aschbacher, L. L. Scott: *Maximal subgroups of finite groups*, J. 92 (1985), 1–22.

[3] M. Aschbacher, G. M. Seitz: *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. 63 (1976) 1–91.

[4] A. Aubad, J. Ballantyne, A. McGaw, P. Neuhaus, J. Phillips, P. Rowley, D.Ward: *The Semisimple Elements of $E_8(2)$*, `http://eprints.ma.man.ac.uk/2457/`.

[5] A. Aubad, J. Ballantyne, A. McGaw, P. Neuhaus, P. Rowley, D. Ward: *The maximal subgroups of $E_8(2)$*, unpublished manuscript.

[6] J. Ballantyne, C. Bates, P. Rowley: *The Maximal Subgroups of $E_7(2)$*, LMS J. Comput. Math. 18 (2015), 323–371.

[7] W. Bosma, J. Cannon, C. Playoust: *The Magma algebra system. I. The user language*, Symbolic Comput., 24 (1997), 235265

[8] J. N. Bray: *An improved method for generating the centraliser of an involution*, Archiv der Mathematik, 74, (2000), 241–245.

[9] J. N. Bray, D. F. Holt, C. M. Roney-Dougal: *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups*, London Mathematical Society Lecture Note Series 407, Cambridge University Press, 2013.

[10] W. Burnside: *On Groups of Order $p^\alpha q^\beta$*, Proc. London Math. Soc. (s2-1 (1)) (1904), 388–392.

[11] G. Bulter: *The maximal subgroups of the sporadic simple group of Held*, J. Algebra 69 (1981), 67-81.

[12] C. Choi: *On subgroups of M_{24}. II. The maximal subgroups of M_{24}*, Trans. Amer. Math. Soc. 167 (1972), 29-47.

[13] B. N. Cooperstein: *Maximal subgroups of $G_2(2^n)$*, J. Algebra 70 (1981) 23–36.

[14] J. H. Conway, R. A. Parker, R. A. Wilson, S. P. Norton: *Atlas of Finite Groups*, Clarendon press. Oxford, (1985).

[15] R. T. Curtis: *A new combinatorial approach to M_{24}*, Math. Proc. Cambridge Philos. Soc. 79 (1976), 25-42.

[16] D. I. Deriziotis, A. P. Fakiolas: *The Maximal Tori in The Finite Chevalley Groups of E_6E_7 And E_8*, Comm. Algebra 19 (1991), no. 3, 889-903.

[17] L. E. Dickson: *Linear groups, with an exposition of the Galois field theory*, (Dover, 1958) Reprint of the 1901 original.

[18] L. Finkelstein: *The maximal subgroups of Conways group C3 and McLaughlins group*, J. Algebra 25 (1973), 58-89.

[19] L. Finkelstein, A. Rudvalis: *The maximal subgroups of the HallJankoWales group*, J. Algebra 24 (1973), 486-493.

[20] L. Finkelstein, A. Rudvalis: *The maximal subgroups of Jankos simple group of order 50,232,960*, J. Algebra 30 (1974), 122-143.

[21] R. W. Hartley: *Determination of the ternary collineation groups whose coefficients lie in the $GF(2^n)$*, Ann. of Math. (2) 27 (1925), 140–158.

[22] Z. Janko: *A new finite simple group with abelian Sylow 2-subgroups, and its characterization*, J. Algebra 3 (1966), 147-186.

[23] P. B. Kleidman: *The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups $^2G_2(q)$, and their automorphism groups*, J. Algebra 117 (1988), 30–71

[24] P. B. Kleidman: *The maximal subgroups of the Steinberg triality groups* $^3D_4(q)$ *and of their automorphism groups*, J. Algebra 115 (1988), 182–199

[25] P. B. Kleidman, R. A. Parker, R. A. Wilson: *The maximal subgroups of the Fischer group* Fi_{23}, J. London Math. Soc. 39 (1989), 89-101.

[26] P. B. Kleidman, R. A. Wilson: *The maximal subgroups of* $J4$, Proc. London Math. Soc. 56 (1988), 484-510.

[27] P. B. Kleidman, R. A. Wilson: *The maximal subgroups of* $E_6(2)$ *and* $\mathrm{Aut}(E_6(2))$, Proc. Lond. Math. Soc. (3) 60 (1990), no. 2, 266–294.

[28] M. W. Liebeck, C. E. Praeger, J. Saxl: *A classification of the maximal subgroups of the finite alternating and symmetric groups*, J. Algebra, 111 (1987), 365–383.

[29] M. W. Liebeck, G. M. Seitz: *A survey of maximal subgroups of exceptional groups of Lie type*, Groups, Combinatorics and Geometry: Durham, 2001, World Scientific (2003)

[30] S. A. Linton: *The maximal subgroups of the Thompson group*, J. London Math. Soc. 39 (1989), 79-88. Corrections, ibid. 43 (1991), 253-254.

[31] S. A. Linton, R. A. Wilson: *The maximal subgroups of the Fischer groups* Fi_{24} *and* Fi'_{24}, Proc. London Math. Soc. 63 (1991), 113-164.

[32] R. J. List: *On the maximal subgroups of the Mathieu groups*, I. M24, Atti Accad. Naz. Lincei Renc. Cl. Sci. Fis. Mat. Natur. 62 (1977), 432-438.

[33] A. Litterick: *Finite Simple Subgroups of Exceptional Algebraic Groups*, Ph. D. thesis, Imperial College London (2013).

[34] A. Lucchini, F. Menegazzo, and M. Morigi: *On the existence of a complement for a finite simple group in its automorphism group*, Illinois Journal of Mathematics, Volume 47, Number 1–2 (2003), 395–418.

[35] K. Magaard: *The maximal subgroups of the Chevalley groups* $F_4(F)$ *where* F *is a finite or algebraically closed field of characteristic* $\neq 2, 3$, Ph. D. thesis, Calif. Inst. Tech. (1990).

[36] S. S. Magliveras: *The maximal subgroups of the Higman-Sims group*, Bull. Amer. Math. Soc. 77 (1971), no. 4, 535–539.

[37] H. H. Mitchell: *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc. 12 (1911), 207–242.

[38] P. Neuhaus: Ph. D. thesis in preparation, University of Manchester (2018).

[39] S. P. Norton, R. A. Wilson: *Maximal subgroups of the HaradaNorton group*, J. Algebra 103 (1986), 362-376.

[40] S. P. Norton, R. A. Wilson: *The maximal subgroups of* $F_4(2)$ *and its automorphism group*, Comm. Algebra 17 (1989), 2809–2824.

[41] Gary M. Seitz: *Maximal Subgroups of Exceptional Algebraic Groups*, Memoirs of the American Mathematical Society 90 (441), (1991).

[42] R. A. Wilson: *The maximal subgroups of Conways group* $\cdot 2$, J. Algebra 84 (1983), 107-114.

[43] R. A. Wilson: *The complex Leech lattice and maximal subgroups of the Suzuki group*, J. Algebra 84 (1983), 151-188.

[44] R. A. Wilson: *The maximal subgroups of Conways group* Co_1, J. Algebra 85 (1983), 144-165.

[45] R. A. Wilson: *The geometry and maximal subgroups of the simple groups of A. Rudvalis and J. Tits*, Proc. London Math. Soc. 48 (1984), 533-563.

[46] R. A. Wilson: *On maximal subgroups of the Fischer groups* Fi_{22}, Math. Proc. Cambridge Philos. Soc. 95 (1984) 197-222.

[47] R. A. Wilson: *The maximal subgroups of the ONan group*, J. Algebra 97 (1985), 467-473.

[48] R. A. Wilson: *The maximal subgroups of the Lyons group*, Math. Proc. Cambridge Philos. Soc. 97 (1985), 433-436

[49] R. A. Wilson: *A new construction of the Baby Monster and its applications* Bull. London Math. Soc. 25 (1993), 431-437.

[50] R. A. Wilson: *The maximal subgroups of the Baby Monster*, I, J. Algebra 211 (1999), 1-14.

[51] R. A. Wilson: *The Finite Simple Groups*, Graduate Texts in Mathematics, 251. Springer-Verlag London, Ltd., London, 2009.

[52] R. A. Wilson: *Maximal subgroups of* $^2E_6(2)$ *and its automorphism groups*, `https://arxiv.org/abs/1801.08374`.

[53] R. A. Wilson: *Maximal subgroups of sporadic groups*, `https://arxiv.org/pdf/1701.02095.pdf`.

Appendix A

Guide to Provided Files

Here we will provide a breakdown of all the files compatible with MAGMA which accompany this thesis. We provide hard copies of all the subgroups constructed throughout this thesis, as well as some groups used in the construction of these subgroups. While we will provide a detailed description of the files in this chapter, we have endeavoured to name the files as clearly as possible – for example, a file named `O8+2:2` is an $\Omega_8^+(2):2$ subgroup of G. Where there are several copies of $\Omega_8^+(2):2$ constructed in that particular case, they are named `O8+2:2Copy1`, `O8+2:2Copy2`, and so on. All the files are organised into four folders, named: `GeneralE8(2)Files`, `U4(2)`, `O8+2`, and `Sp4(4)`.

The contents of folder `GeneralE8(2)Files` is given in Table A.1. It contains some files related to $E_8(2)$.

File	Description
`CG3B`	$C_G(g)$ where $g \in G \cap 3\mathrm{B}$
`CG3D`	$C_G(g)$ where $g \in G \cap 3\mathrm{D}$
`CG5A`	$C_G(g)$ where $g \in G \cap 5\mathrm{A}$
`CG5B`	$C_G(g)$ where $g \in G \cap 5\mathrm{B}$
`E8Sylow3`	$H \in \mathrm{Syl}_3(G)$
`E8Sylow5`	$H \in \mathrm{Syl}_5(G)$
`NormOfSylow3`	$N_G(H)$ where $H \in \mathrm{Syl}_3(G)$
`NormOfElAb5^4SubgroupSylow5`	$N_G(K)$ where $5^4 \cong K \leq H \in \mathrm{Syl}_5(G)$

Table A.1: Contents of folder `GeneralE8(2)Files`

Folder `U4(2)`

This folder contains files related to Chapter 4 on $U_4(2)$, Chapter 5 on $Sp_6(2)$, and Chapter 6 on $\Omega_8^-(2)$. The files are organised into five folders: `FusionCombination2`, FusionCombination3, FusionCombination6, FusionCombination8, and finally FusionCombination9. Each folder contains files associated to particular $U_4(2)$ fusion possibilities. These files are described in Table A.2.

File	Description
3tothe4s	The full set of $[3^4]$ subgroups described in Table 4.2
Rs	The groups in 3tothe4s reduced by conjugacy, as in Table 4.3
Es	The set of 3^3 subgroups of the groups in Rs

Table A.2: Contents of FusionCombination folders

The folders FusionCombination8 and FusionCombination9 contain nothing more. Folders FusionCombination2 and FusionCombination6 contain elements conjugating the subgroups in Es. For example, the file ConjElt1and2 contains an element c such that $E_1^c = E_2$ where E_1 and E_2 are the first and second subgroups contained in Es respectively. These elements were found in Lemma 4.5.

Now we will go on to describe the additional folders inside FusionCombination2, FusionCombination3 and FusionCombination6. Let i be the number on the folder (so $i \in \{2, 3, 6\}$). Folders FusionCombination2 and FusionCombination6 contain a folder called E1. Folder FusionCombination6 contains folders E1 and E2. These correspond to the 3^3 subgroups named $E_j^{(i)}$ after the proof of Lemma 4.4, where $(i, j) \in \{(2, 1), (3, 1), (3, 2), (6, 1)\}$. These folders contain E, which is $E_j^{(i)}$, R the $[3^4]$ containing $E_j^{(i)}$, D which is $D_j^{(i)}$ as defined in Lemma 4.7, and x, y, CGx, and CGy which are x, y, $C_G(x)$, and $C_G(y)$ used in the construction of $D_j^{(i)}$. Folder FusionCombination3 contains nothing more.

Folder FusionCombination2/E1 contains four more folders, S1, S2, S3, and S4, while folder FusionCombination6/E1 contains two more folders, S1 and S2. These are folders relating to the copies of Sym(4) constructed and named $S_k^{(i,j)}$ in Proposition 4.8. The contents of these folders are explained in Table A.3.

File	Description
S	The Sym(4) subgroup $S_k^{(i,j)}$
D8	The Dih(8) subgroup of $S_k^{(i,j)}$
B	$\mathcal{C}_1(S_k^{(i,j)})$ as defined in Proposition 4.9
U	$\mathcal{C}_2(S_k^{(i,j)})$ as defined in Proposition 4.9
M	$\mathcal{C}_3(S_k^{(i,j)})$ as defined in Proposition 4.10

Table A.3: Contents of the folders related to $S_k^{(i,j)}$

Note that there is one exception. The folder FusionCombination2/E1/S1 does not contain a file named `M`. This is because the case with $S_1^{(2,1)}$ was resolved differently – see Lemma 4.12. Furthermore, this folder also contains files `Sx2-1` and `Sx2-2`. Similarly, folder `FusionCombination6/E1/S1` also contains a file called `Sx2`. These three files are copies of $\mathrm{Sym}(3) \times 2$ used in the generation of $U_4(2) : 2$ in Proposition 4.17. Finally, there are two additional folders inside `FusionCombination2/E1/S1`. These are named `U4(2)Copy1` and `U4(2)Copy2`. The contents of these folders, as well as the additional contents of folder `FusionCombination6/E1/S1`, are named following the conventions stated in Table A.4.

File	Description
`U4(2)`	$U_4(2)$
`InvolutionGeneratingU4(2)`	t such that $\langle E_j^{(i)} S_k^{(i,j)}, t\rangle \cong U_4(2)$
`U4(2):2`	$U_4(2) : 2$
`Sp6(2)`	$Sp_6(2)$
`InvolutionGeneratingSp6(2)`	t such that $\langle U_4(2) : 2, t\rangle \cong Sp_6(2)$
`O8-(2)`	$\Omega_8^-(2)$
`InvolutionGeneratingO8-(2)`	t such that $\langle Sp_6(2), t\rangle \cong \Omega_8^-(2)$
`O8-(2):2`	$\Omega_8^-(2) : 2$
`InvolutionGeneratingO8-(2):2`	t such that $\langle \Omega_8^-(2), t\rangle \cong \Omega_8^-(2) : 2$

Table A.4: Contents of the folders related to $S_k^{(i,j)}$

Where multiple copies of groups are found, we append `Copy1`, `Copy2` etc. to the name of the group. For example, in the folder `FusionCombination2/E1/S1` there are two copies of $U_4(2) : 2$. These are named `U4(2):2Copy1` and `U4(2):2Copy2`. Whenever this occurs, we also label the number the generating elements accordingly so that `Involution1Generating...` is used to generate the group labelled `...Copy1`, and so on. We also append `With...` to the end of a file name where it is unclear which subgroup we are using to generate the next group. For example, in the folder `FusionCombination2/E1/S1` we have $H \cong Sp_6(2)$ where H is labelled `Sp6(2)Copy2`. There exists t such that $\langle H, t\rangle \cong \Omega_8^-(2)$. This element t is labelled `InvolutionGeneratingO8-(2)WithSp6(2)Copy2`. These naming conventions will be used in the description of files associated with other groups.

Folder `O8+2`

These files are organised into folders `P1`, `P2`, and `P3`. These correspond to folders containing files associated with the elementary abelian subgroups P_1, P_2, and P_3 of order 5^2 used in the generation of $\Omega_8^+(2)$. Note that in Proposition 7.5 we conclude that there are four such subgroups up to conjugacy in G. However, in Lemma 7.3 we show that P_4 cannot be built into an $\Omega_8^+(2)$ subgroup of G. For completeness, we provide P_4 in a file called `P4` by itself.

Each folder `P1`, `P2`, and `P3` contains a file of the same name – these are copies of P_1, P_2, and P_3 respectively. Folder `P1` contains files `CGP1` and `NGP1`, which are $C_G(P_1)$ and $N_G(P_1)$ respectively. Folders `P2` and `P3` contain similarly named files for $C_G(P_2)$, $N_G(P_2)$, $C_G(P_3)$, and $N_G(P_3)$. Additionally, folder `P1` contains the following:

- `NGP1gen` – an ordered generating set for $N_G(P_1)$ as a subgroup of G;
- `NGP1permgen` – an ordered generating set for $N_G(P_1)$ as a permutation group of degree 40,563;
- `NGP1permgenreduced` – an ordered generating set for $N_G(P_1)$ as a permutation group of degree 4,050.

Calculations inside the permutation group of degree 4,050 are much faster. We use the following the following procedure to construct `map`, an isomorphism from $N_G(P_1)$ as a matrix group to $N_G(P_1)$ as a degree-40,563 permutation group, and `mapp`, an isomorphism from $N_G(P_1)$ as a degree-40,563 permutation group to $N_G(P_1)$ as a degree-4,050 permutation group.

```
NGP1:=sub<Q|NGP1gen>;
Np:=Universe(NGP1permgen);
Npp:=Universe(NGP1permgenreduced);
map:=Homomorphism(NGP1,Np,NGP1gen,NGP1permgen);
mapp:=Homomorphism(Np,Npp,NGP1permgen,NGP1permgenreduced);
```

Now recall that we have $T_j^{(i)}$ normalising P_i such that $P_iT_j^{(i)}$ is contained in some $\Omega_8^+(2)$ subgroup of G, where $(i,j) \in \{(1,1),(1,2),(1,3),(1,4),(2,1),(3,1),(3,2)\}$. Accordingly, we have a folder for each $T_j^{(i)}$. Thus we have folders `P1/T1`, `P1/T2`, `P1/T3`, `P1/T4`, `P2/T1`, `P3/T1`, and `P3/T2`. Inside each folder is a file of the same name containing a copy of $T_j^{(i)} \cong \mathrm{Dih}(8) \circ \mathbb{Z}_4$. Further contents of these folders are given in Table A.5.

File	Description
`C3T`	$\mathcal{C}_3(T_j^{(i)})$ as defined before Proposition 7.13
`O8+2`	$\Omega_8^+(2)$
`InvolutionGeneratingO8+2`	t such that $\langle P_i T_j^{(i)}, t\rangle \cong \Omega_8^+(2)$
`O8+2:2`	$\Omega_8^+(2) : 2$
`ElementGeneratingO8+2:2`	t such that $\langle \Omega_8^+(2), t\rangle \cong \Omega_8^+(2) : 2$
`O8+2:3`	$\Omega_8^+(2) : 3$
`ElementGeneratingO8+2:3`	t such that $\langle \Omega_8^+(2), t\rangle \cong \Omega_8^+(2) : 3$
`O8+2:Sym(3)`	$\Omega_8^+(2) : \mathrm{Sym}(3)$
`ElementGeneratingO8+2:Sym(3)`	t such that $\langle \Omega_8^+(2) : 3, t\rangle \cong \Omega_8^+(2) : \mathrm{Sym}(3)$

Table A.5: Contents of the folders related to $T_j^{(i)}$

Also in folders `P1/T2` and `P2/T1` are the files `EltToProveO8+2:Sym(3)NotMaximal`. These are elements x such that $H < \langle H, x\rangle < G$ where H is the relevant $\Omega_8^+(2)$: Sym(3) subgroup, which are used to show that H is not maximal in G.

Additionally, in folder `P1` is another folder called `ConstructionFiles`. This is a folder related to the 351 subgroups isomorphic to $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ normalising P_1 (see Proposition 7.9). There are five folders within: `Ts1`, `Ts2`, `Ts3`, `Ts4`, and `Ts5`. these correspond to the sets of $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups labelled $\mathcal{T}^{(1,1)}, \ldots, \mathcal{T}^{(1,5)}$ and constructed in Lemma 7.10. In each folder is a file of the same name containing an ordered set of the $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups in $\mathcal{T}^{(1,k)}$ for $k \in \{1, 2, 3, 4, 5\}$. There is also a file called `zCentralInvolution` which is the central involution common to all $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups in $\mathcal{T}^{(1,k)}$.

In `Ts5` specifically, let z be the element in `zCentralInvolution` there are files named `CGz`, which contains $C_G(z)$, and `Core2CGz` which is $O_2(C_G(z))$ constructed in the remarks following Lemma 7.14. Recall now that in the passage preceding Proposition 7.19 we split $\mathcal{T}^{(1,5)}$ into 46 sets $\mathcal{T}^{(1,5,k)}$ for $k \in \{1, \ldots, 46\}$ where each $\mathcal{T}^{(1,5,k)}$ is a set of $\mathrm{Dih}(8) \circ \mathbb{Z}_4$ subgroups such that there is a non-central involution common to each $T \in \mathcal{T}^{(1,5,k)}$. Associated with these are 46 folders inside `Ts5` called `Tset1`,. . .,`Tset46`. The contents of these 46 folders are given in Table A.6.

File	Description
`Ts`	The ordered set $\mathcal{T}^{(1,5,k)}$
`a`	the non-central involution common to all subgroups in $\mathcal{T}^{(1,5,k)}$
`B`	$\mathcal{C}_1(T)$ as defined before Proposition 7.19
`U1`, `U2`, ...	$\mathcal{C}_2(T)$ for each $T \in \mathcal{T}^{(1,5,k)}$ as defined before Proposition 7.19
`M1`, `M2`, ...	$\mathcal{C}_3(T)$ for each $T \in \mathcal{T}^{(1,5,k)}$ as defined before Proposition 7.19

Table A.6: Contents of the folders related to $S_k^{(i,j)}$

Now we move onto the files related to $\Omega_8^+(4)$ subgroups of G. Recall that in Proposition 8.2 we found a copy of $\Omega_8^+(4)$ containing $P_1T^{(1)}$ and one containing $P_3T_2^{(3)}$. Accordingly, we have folders named `O8+4` in the folders `P1/T1` and `P3/T2` containing files associated with $\Omega_8^+(4)$ subgroups. In these folders we have `Sylow5ForO8+4`, which is $R \cong 5^4$ and `TransvEltForO8+4`, which is t such that $R^t \in \mathrm{Syl}(H)$ where H is the copy of $\Omega_8^+(4)$ in question. These were found in Proposition 8.2 and were used in the construction of $\Omega_8^+(4)$ subgroups. We also have `NormOfSyl5OfO8+4` which is $N_G(R^t)$ which were found in Proposition 8.3 and was used in the construction of automorphism extensions of $\Omega_8^+(4)$ subgroups. The remaining contents of these folders are explained in Table A.7.

File	Description
`O8+4`	$\Omega_8^+(4)$
`ElementGeneratingO8+4`	t such that $\langle \Omega_8^+(2), t\rangle \cong \Omega_8^+(4)$
`O8+4:2`	$\Omega_8^+(4):2$
`ElementGeneratingO8+4:2`	t such that $\langle \Omega_8^+(4), t\rangle \cong \Omega_8^+(4):2$
`O8+4:3`	$\Omega_8^+(4):3$
`ElementGeneratingO8+4:3`	t such that $\langle \Omega_8^+(4), t\rangle \cong \Omega_8^+(4):3$
`O8+4:2^2`	$\Omega_8^+(4):2^2$
`ElementGeneratingO8+4:2^2`	t such that $\langle \Omega_8^+(4):2, t\rangle \cong \Omega_8^+(4):2^2$
`O8+4:6`	$\Omega_8^+(4):6$
`ElementGeneratingO8+4:6`	t such that $\langle \Omega_8^+(4), t\rangle \cong \Omega_8^+(4):6$
`O8+2:Sym(3)`	$\Omega_8^+(4):\mathrm{Sym}(3)$
`ElementGeneratingO8+2:Sym(3)`	t such that $\langle \Omega_8^+(4):3, t\rangle \cong \Omega_8^+(4):\mathrm{Sym}(3)$
`O8+2:Dih(12)`	$\Omega_8^+(4):\mathrm{Dih}(12)$
`ElementGeneratingO8+2:Dih(12)`	t such that $\langle \Omega_8^+(4):\mathrm{Sym}(3), t\rangle \cong \Omega_8^+(4):\mathrm{Dih}(12)$

Table A.7: Contents of the folders related to $\Omega_8^+(4)$

Now we will discuss files associated with $Sp_8(2)$ subgroups. By Proposition 9.2 we have four $Sp_8(2)$ subgroups: one containing $P_1T_3^{(1)}$, one containing $P_1T_4^{(1)}$, and two containing $P_3T_1^{(3)}$. Files associated with these $Sp_8(2)$ subgroups are, therefore, contained in the folders `P1/T3`, `P1/T4`, and `P3/T1`. There, we have files named `Sp8(2)` or `Sp8(2)FromO8+2:2Copy...` in folders where there are multiple copies of $\Omega_8^+(2):2$ (see the naming conventions established after Table A.4). Also, we have elements t such that $\langle \Omega_8^+(2):2, t\rangle \cong Sp_8(2)$, which are saved as `InvolutionGeneratingSp8(2)`. In `P1/T4` we also have the group $\mathcal{C}(R_1^{(1,4)})$ as defined before Proposition 9.2, saved in the file `CR`.

Folder `Sp4(4)`

In this final folder, we have all the files associated with $Sp_4(4)$ and $L_4(4)$ subgroups. Like the `O8+2` folder, folder `Sp4(4)` is organised into folders `P2` and `P3` which correspond to the 5^2 subgroups P_2 and P_3. Folder `P3` contains seven folders `D1`,...,`D7` corresponding to the seven Dih(8) subgroups $D_i^{(3)}$ for $i \in \{1, \ldots, 7\}$ constructed in Proposition 10.6. Each of these folders contains a file `D` which contains the subgroup $D_i^{(3)}$. Similarly, there is a file called `D` inside the folder `P2` containing the subgroup $D_1^{(2)}$. The folders `P2`, `P3/D3`, `P3/D5`, and `P3/D7` also contain files which are described in Table A.8.

File	Description
`Sp4(4)`	$Sp_4(4)$
`InvolutionGeneratingSp4(4)`	t such that $\langle P_i D_j^{(i)}, t\rangle \cong Sp_4(4)$
`Sp4(4):2`	$Sp_4(4) : 2$
`ElementGeneratingSp4(4):2`	t such that $\langle Sp_4(4), t\rangle \cong Sp_4(4) : 2$

Table A.8: Contents of the folders related to $D_j^{(i)}$

Also in folder `P3/D7` is a file `EltToProveSp4(4)NotMaximal` which is an element x such that $H < \langle H, x\rangle < G$ where $P_3 D_7^{(3)} \leq H \cong Sp_4(4)$ used to prove that H is not maximal.

Finally, in folder `P3/D3` containing files associated with $L_4(4)$. The contents of this folder are described in Table A.9.

File	Description
`L4(4)`	$Sp_4(4)$ used to generate $L_4(4)$
`L4(4)`	$L_4(4)$
`ElementGeneratingSp4(4)`	t such that $\langle Sp_4(4), t\rangle \cong L_4(4)$
`L4(4):2`	$L_4(4) : 2$
`L4(4):2^2`	$L_4(4) : 2^2$

Table A.9: Contents of the folders related to $D_j^{(i)}$

We also have a file `EltToProveL4(4):2^2NotMaximal` which contains an element x such that $H < \langle H, x\rangle < G$ where $H \cong L_4(4) : 2^2$. This is used to prove that H is not maximal in G.

Appendix B

Procedures

Here we provide some of the procedures used throughout this thesis. We will start with the code used to construct $G \cong E_8(2)$ as a subgroup of $GL_{248}(2)$. This is included as all of our computations take place in this matrix representation of $E_8(2)$.

Procedure B.1. The following procedure constructs $E_8(2) \cong G \leq GL_{248}(2)$.

```
H:=GroupOfLieType("E8",GF(2));
f:=AdjointRepresentation(H);
Q:=Codomain(f);
Hgens:=[];
for i:=1 to 8 do;
 Append(~Hgens,elt<H|<i,1>>);
end for;
for i:=1 to 8 do;
 Append(~Hgens,elt<H|<120+i,1>>);
end for;
Ggens:=[];
for h in Hgens do;
 Append(~Ggens,f(h));
end for;
G:=sub<Q|Ggens>;
```

Procedure B.2. This procedure is used to find a random element of specified order in a given group. Here, `G` is the group we wish to search in, and `l` is the order we want our element to be. For example, `g:=Element(G,3)` will find an element $g \in G$ of order 3. It works by searching for random elements in G until it finds h such that $o(h)$ divides l. Then it sets $g := h^{\frac{o(h)}{l}}$ so that $o(g) = l$.

```
function Element(G,l);
x:=Id(G);
repeat
r:=Random(G);
o:=Order(r);
if o mod l eq 0 then
 k:=IntegerRing()!(o/l);
 x:=r^k;
end if;
until Order(x) eq l;
return x;
end function;
```

Procedure B.3. This procedure was developed by Ballantyne and Rowley and is used to find the centraliser of an element in $E_8(2)$. Here, the parameters are as follows:

- `G` is the group we wish to find the centraliser in (so in our case $E_8(2)$);
- `g` is the element we wish to find the centraliser of;
- `k` is an integer at least as large as the dimension of the smallest non-trivial irreducible $\langle g \rangle$-module over $\mathbb{F}_2$;
- `H` is a subgroup of G which is isomorphic to $C_G(g)$.

Note that the `Element` command found in Procedure B.2 must be loaded before `FindCent` is used.

```
procedure FindCent(G,g,k,H)
Q:=GL(248,2);
V:=GModule(H);
CompsV:=CompositionFactors(V);
dimsV:={};
```

```
for c in CompsV do
 Include(~dimsV,Dimension(c));
end for;
Cg:=sub<Q|Id(Q)>;
count:=0;
repeat
repeat
 t:=Element(G,2);
 Y:=sub<Q|t,g>;
 U:=GModule(Y);
 CF:=CompositionFactors(U);
until #CF ge 10;
for i:=1 to 5 do
thing:=0;
counter:=0;
repeat
 a:=Element(Y,2);
 L:=sub<Q|a,g>;
 W:=GModule(L);
 CFW:=CompositionFactors(W);
 dims:={};
 for c in CFW do
  Include(~dims,Dimension(c));
 end for;
 counter:=counter+1;
 if Max(dims) le k then
  thing:=1;
 end if;
until Max(dims) le k or counter eq 20;
if thing eq 1 then
 l:=LMGOrder(L); Factorisation(l);
  if l le 2^20 then
```

```
  CL:=Centraliser(L,g);
  Cg:=sub<Q|Cg,CL>;
 end if;
end if;
end for;
VCg:=GModule(Cg);
count:=count+1;
count;
CFVCg:=CompositionFactors(VCg);
dimsVCg:={};
for c in CFVCg do
 Include(~dimsVCg,Dimension(c));
end for;
if #CFVCg le 20 then
 CFVCg;
end if;
until dimsV eq dimsVCg;
PrintFileMagma("Cg",Cg);
end procedure;
```

Procedure B.4. For some group G, this procedure takes a group $H \leq G$ and a set S of subgroups of H and finds orbit representatives in S under the action of H by conjugation. Here, `Calc` is the name given to the set S (and note that it is an *ordered* set) and `ConjGrp` is the name given to the group H. Essentially, the code works by first setting `Set` to be an idential copy of S. Then it takes the first element X of `Set` and deletes all the H-conjugates of X from `Set` (including X itself). It stores X in a new set called `reps`. Then it moves on, choosing the first element of `Set` and repeating the above steps until `Set` is empty. Upon completion, `reps` will be the desired set of representatives. Note that the procedure can be easily be modified to find orbit representatives of a set of group elements rather than subgroups.

This procedure is employed when we have $L \leq H$ and L is sufficiently small that we can run `Subgroups` on L, thereby obtaining all subgroups of the desired order in L up to L-conjugacy. If we want to reduce the number of subgroups by considering conjugacy in the whole of H, we employ this procedure. This usually occurs in situations when H is too large to employ `Subgroups` on H directly.

```
Set:=Calc;
reps:={@@};
for i in [1..(#Calc-1)] do;
 if Calc[i] in Set then;
  Include(~reps,Calc[i]);
  X:=subs[Calc[i]]`subgroup;
  for j in [i..#Calc] do;
   if Calc[j] in Set then;
    Y:=subs[Calc[j]]`subgroup;
    if IsConjugate(ConjGrp,X,Y) then;
     Set:=Set diff {Calc[j]};
    end if;
   end if;
  end for;
 end if;
end for;
```

Procedure B.5. Here is the code for the `ReBray` function which takes a group G with $N \trianglelefteq G$ and an involution $g \in G$, then outputs an element $x \in G$ such that $\overline{gx} = \overline{xg}$, where $\overline{g} = gN$. It works by employing Lemma 7.15.

```
function ReBray(G,N,g)
h:=Random(G);
o:=Order((g,h));
D:=Divisors(o);
M:={d:d in D| LMGIsIn(N,(g,h)^d)};
r:=Min(M);
if r mod 2 eq 0 then;
 m:=(r/2);
 k:=IntegerRing()!m;
 x:=(g,h)^k;
```

```
 else;
 m:=(r-1)/2;
 k:=IntegerRing()!m;
 x:=h*(g,h)^k;
end if;
return(x);
end function;
```

Procedure B.6. These procedures accompany Proposition 7.18 and are used to find the desired elements x and g. Note that, in both procedures, the object `W` represents $O_2(C)$.

(i) This procedure finds $x \in C$ of order 17 such that $[x, a] \notin O_2(C)$. Note that the function `Element` from Procedure B.2 needs to be loaded first.

```
repeat
x:=Element(C,17);
until LMGIsIn(W,x*a*(x^-1)*a) eq false;
```

(ii) This procedure finds $g \in C$ such that, given any $n \in L$, $[a, g]^n \notin O_2(C)$. Depending on the case, L should be taken to either be the set $\{1, 2, 3, 4, 6, 12\}$ or $\{1, 2, 3, 4, 5, 6, 10, 12, 15\}$.

```
repeat;
g:=Random(C);
TF:={};
x:=a*(g^-1)*a*g;
for n in L do;
 y:=x^n;
 tf:=LMGIsIn(W,y);
 Include(~TF,tf);
end for;
until TF eq {false};
```

Procedure B.7. Let $E_8 \cong G \leq GL_{248}(2)$. This procedure, written by Ballantyne, constructs $\Omega^+_{16}(2) \cong K \leq G$ as well as P such that $P \cong K$ and P is a permutation group of degree 32,895, as well as providing an isomorphism $\varphi : P \to K$. Note that the $E_8(2)$ setup procedure found in Procedure B.1 should be run first. Given $p \in P$, by running `k:=Evaluate(w(h),K)` we obtain $k = \varphi(p) \in K$.

```
Q:=GL(248,2);
n:=[];
for i:=1 to 8 do
 Append(~n,f(elt<H|<i,1>>));
end for;
for i:=121 to 128 do
 Append(~n,f(elt<H|<i,1>>));
end for;
Append(~n,f(elt<H|<120,1>>));
Append(~n,f(elt<H|<240,1>>));
K:=sub<Q|n[2],n[3],n[4],n[5],n[6],n[7],n[8],n[10],n[11],n[12],
n[13],n[14],n[15],n[16],n[17],n[18]>;
truth,m1,m2:=ClassicalConstructiveRecognition(K,"Omega+",16,2);
n16:=[];
for i:=1 to 16 do
Append(~n16,m1(K.i));
end for;
Q16:=GL(16,2);
K16:=sub<Q16|n16>;
p,P:=PermutationRepresentation(K16);

sW:=WordGroup(P);
w:=InverseWordMap(P);
WK16:=WordGroup(K16);
wK16:=InverseWordMap(K16);
```