Partial Myopia, Downside Risk Aversion and Its Application in Option Pricing

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ABSTRACT

The aim of this thesis is to extend the current research on portfolio investment and asset pricing under uncertainty with a special focus on partial myopia and downside risk aversion. To address this goal, the thesis first proposes the use of partial myopia as an alternative approach to dynamic programming for solving a multi-period investment problem with background risks. I provide numerical examples to show that the partial myopia approach could lead to the same optimal investment decision as the dynamic programming method, even in the presence of background risks. Next, the thesis explores the drawbacks of the five existing downside risk aversion measures, and proposes a new local measure. While the proposed measure is limited by its local property, our numerical examples show that it gives the right preference ordering while the other five measures provide inconsistent signals. Finally, the thesis investigates the relationship between downside risk aversion and option pricing by analysing the elasticity of the pricing kernel. We conclude that decreasing absolute risk aversion and increasing downside risk aversion will increase the option price given the forward price. However, the two risk measures are not independent, and they interact to affect option price jointly.
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Chapter 1

Introduction

The aim of this thesis is to extend the current research on portfolio investment and asset pricing under uncertainty covering three related topics concerning partial myopia, downside risk aversion and option pricing.

Chapter 2 provides a detailed review of the literature on partial myopia, downside risk aversion and option pricing. It starts with a brief introduction to expected utility theory along with the most important results on risk aversion, prudence and risk vulnerability. It then proceeds to review previous studies on the multi-period investment problem, especially the seminal work by Mossin (1968) on the myopia and partial myopia investment policies to be studied in Chapter 3. Then, to provide the necessary background for Chapter 4, the definition of and conditions for downside risk aversion are provided, and the existing measures of it are briefly introduced. In addition, applications of downside risk aversion in self-protection and self-insurance are also mentioned. Finally, the incomplete market and its as-
assumptions are discussed, together with previous literature on option pricing and option price bounds, such as Rubinstein (1976)’s discrete time model, Bernardo and Ledoit (2000)’s gain-loss ratio, and Cochrane and Saa-Requejo (2000)’s good deal bounds. A more detailed review is provided of the most important paper for this thesis, viz. Franke, Stapleton and Subrahmanyam (1999), which concerns the relationship between the elasticity of the pricing kernel and the option price. The paper laid the foundation for the extension provided in Chapter 5 regarding how risk preferences, such as downside risk aversion, affect option prices.

Following the overview of previous literature in Chapter 2, the main studies of the thesis follow in Chapters 3, 4 and 5.

Chapter 3 analyses the partial myopia investment decision making process in the presence of background risks. In a multi-period investment problem, an investor behaves **partially myopically** if he makes each period’s decision as if it were a one-period problem and only a risk free investment would be available in all subsequent periods. This differs from the popular dynamic programming method in which each period’s decision must take into account its impact on the subsequent periods. Theoretically, dynamic programming will always give the optimal solution to the multi-period investment problem. In practice, however, it is difficult to apply because it involves very complicated calculations as the number of time periods increases and as the information on a more distant future becomes more uncertain. Therefore, the partial myopia approach to the investment deci-
sion making process is an attractive alternative. Some pioneering work on myopia and partial myopia can be found in Mossin (1968). However, Mossin assumes a complete market with only market risk. In Chapter 3, I introduce additive and multiplicative background risks into the model for the HARA class of investor, and show that the partial myopia method could lead to the same optimal solution as the dynamic programming method, even in the presence of background risks, if certain conditions are met. This result is supported by numerical examples in which the certainty equivalent due to the suboptimal partial myopia investment policy is extremely small. Hence, it can be argued that the simplicity of the partial myopia approach outweighs the marginal efficiency loss and is a superior approach for HARA class investor.

Chapter 4 concentrates on how best to measure downside risk aversion. Downside risk aversion refers to individuals’ tendency to avoid situations that offer the potential for substantial gain but also the possibility of critical level of losses. Unlike risk aversion, downside risk aversion critically depends on the higher order risk preference concerning the third derivative of the individual’s utility function. Since it was first introduced in Menezes, Geiss and Tressler (1980), downside risk aversion has become a popular topic, especially in recent years, as an additional measure of risk preference besides risk aversion. Despite the progress made in previous studies, many issues remain unresolved. The most critical among these is how to determine if one individual is more downside risk averse than another, and how to measure the difference. At least five downside risk aversion measures
have been proposed in the literature but each has its own shortcomings, in terms of conceptual clarity, derivation and application. I therefore propose a new local measure for downside risk aversion by first defining what it means to be more downside risk averse. My new measure differs from the previous five in including the mean of the risky assets in the measurement. The intuition behind this is that, unlike risk aversion, in order to measure downside risk aversion we need to know not only by how much, but also where in the probability distribution the increase in downside risk occurs. The different values of the mean dictate if the probability weight of the distribution has been moved into the critical area to which the investor is particularly averse to and has therefore changed his attitude towards the increase in downside risk. Although our measure is limited by its local property, our numerical examples show that the proposed new measure provides the correct preference ordering while all previous measures provide inconsistent signals.

Chapter 5 considers how risk preferences, such as downside risk aversion, affect the option prices in incomplete markets. This chapter starts by reviewing and extending the results in Franke, Stapleton and Subrahmanyam (1999) concerning the relationship between the elasticity of the pricing kernel and the option price so that the option prices under any two pricing kernels can be compared, and not just those between kernels with declining and constant elasticities. I then proceed to analyse the impacts of different risk measures and how, together, they affect option prices. I also establish the conditions under which the change of elastic-
ity in one pricing kernel is higher than that in another, using the transformation function. In addition, I derive conditions under which the option price is higher for some specific utility functions, and use the HARA case in numerical examples to support the theoretical results.

Finally, Chapter 6 concludes with a summary of the main contributions of this thesis, and a brief discussion of possible extensions for future research.
Chapter 2

Literature Review

This chapter will be divided into three parts. In the first part, expected utility theory will be introduced, along with important results such as risk aversion, prudence and risk vulnerability. After that there will be a review of the multi-period problem and background risks, which together will lead to the concepts of myopia and partial myopia. In the second part, the concept of downside risk aversion will be defined and examined. As an extension of this, the research on self-protection will also be reviewed. In the third part, the research on pricing kernels and option pricing will be reviewed.

2.1 Expected Utility Theory

Although the notion of risk aversion and decision theory under uncertainty can be traced back at least as far as Bernoulli, who in 1738 raised the famous St. Petersburg Paradox, modern expected utility theory was mainly founded in 1944
by John von Neumann and Oskar Morgenstern, in their famous book ‘Theory of Games and Economic Behaviour’. In it, they exhibit that any rational agent satisfying four axioms has a real-valued utility function, and every preference of the agent is characterized by maximizing the expected value of the utility function. The four axioms are completeness, which says that, for any pair of lotteries \((L^a, L^b)\), either \(L^a\) is preferred to \(L^b\), or \(L^b\) is preferred to \(L^a\) (or both); transitivity, which says that, if \(L^a\) is preferred to \(L^b\) and \(L^b\) is preferred to \(L^c\), then \(L^a\) is preferred to \(L^c\); continuity, which says that, if \(L^a\) is preferred to \(L^b\) and \(L^b\) is preferred to \(L^c\), then there exist a scalar \(\alpha \in [0,1]\) such that \(L^b\) is indifferent to \(\alpha L^a + (1-\alpha) L^c\); and independence, which says that, if there is preference ordering for two lotteries \(L^a\) and \(L^b\), and each of them is mixed with a third lottery \(L^c\), then the preference ordering is independent of the third lottery \(L^c\).

Among these axioms, the independence axiom has caused great controversy since its introduction. The oldest and most enduring counterexample to expected utility theory is that proposed by Allais (1953), which is now generally known as the Allais paradox. In Allais’ experiment, there was an urn containing 100 balls numbered from 0 to 99, and four lotteries \(L^a\), \(L^b\), \(M^a\) and \(M^b\) whose monetary outcomes depended on the number of the ball taken out of the urn, as shown in Table 2.1.

In the experiment, many people claimed to prefer \(L^a\) to \(L^b\) and \(M^b\) to \(M^a\). Yet, with \(L^c = (0, 0, 50)\), we have \(L^a = L^c + M^a\) and \(L^b = L^c + M^b\) and according to the independence axiom, if \(L^a\) is preferred to \(L^b\), then \(M^a\) must be preferred to
Table 2.1: Monetary Outcome as a Function of the Number of the Ball

<table>
<thead>
<tr>
<th>Lottery</th>
<th>0</th>
<th>1 – 10</th>
<th>11 – 99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_a$</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>$L_b$</td>
<td>0</td>
<td>250</td>
<td>50</td>
</tr>
<tr>
<td>$M_a$</td>
<td>50</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>$M_b$</td>
<td>0</td>
<td>250</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: There is an urn containing 100 balls numbered from 0 to 99. The monetary outcomes of the lotteries depend on the number on the ball taken out of the urn.

$M_b$, and vice versa. Thus, the experiment shows an inconsistency between actual observed choices and the predictions of expected utility theory. Nevertheless, expected utility theory has proved its power through its prevalent usage in the last seventy years, and the entire theory of risk aversion is built upon it.

### 2.1.1 The Arrow-Pratt Measure of Risk Aversion

When considering the issue of the investment decision under uncertainty, it is traditional to begin with Pratt (1964)’s paper, which involves a wide range of concepts and results that have been widely used in subsequent research. The paper tries to answer the question of how the risk aversion of different agents can be compared. Suppose agent $u_1$ and $u_2$ have the same initial wealth. Then agent $u_1$ is more risk-averse than agent $u_2$ if $u_1$ dislikes all lotteries that $u_2$ dislikes, independent of their initial wealth. This is equivalent to saying that, with premium $\pi$ that satisfies

$$Eu_1 (w_0 + \tilde{x}) = u_1 (w_0 - \pi)$$

Savage (1954) is an early supporter of expected utility theory. He claimed that a more cautious reading of the problem would be sufficient to avoid the Allais Paradox. More recently, Wakker and Tversky (1993) re-examine the issue.
where \( w_0 \) is the initial wealth and \( \tilde{x} \) is the random return of the risky asset, we will have

\[
E u_2 (w_0 + \tilde{x}) > u_2 (w_0 - \pi)
\]

independent of the initial wealth. Pratt finds that the measure of local aversion for the individual, which is the famous Arrow-Pratt approximation\(^2\)

\[
A(w_0) = -\frac{u''(w_0)}{u'(w_0)},
\]

provides a standardized measure of an agent’s risk aversion, and that any two agents’ attitudes towards risk, i.e. their exposure to risky assets, can be compared accordingly. Pratt shows in his paper that the conditions of the following theorem are equivalent.

**Theorem 1.** Denote \( w \) as the initial wealth and \( \tilde{x} \) as the local risk. Let \( A_i(w) \) and \( \pi_i(w, \tilde{x}) \) be the local risk aversion and risk premium corresponding to the utility function \( u_i, i = 1, 2 \).\(^3\) Then the following conditions are equivalent:

1. \( A_1(w) \geq A_2(w) \) for all \( w \) [and \( > \) for at least one \( w \) in every interval].
2. \( \pi_1(w, \tilde{x}) \geq [>] \pi_2(w, \tilde{x}) \) for all \( w \) and \( \tilde{x} \).
3. \( u_1\left(u_2^{-1}(t)\right) \) is a [strictly] concave function of \( t \).

Condition (3) is used to build equivalence between conditions (1) and (2). If one individual \( u \) is more locally risk averse than another individual \( v \), then for any

\(^2\)It is only fair to mention, however, that the measure of absolute risk aversion was independently developed by Arrow (1963) and de Finetti (1952).

\(^3\)The local risk is defined as a small risk whose third and higher moments are of such small order that they can be omitted. The local risk aversion measure, \(-u''/u'\), is derived from the Taylor expansion of the risk premium around the local risk.
wealth \( w \) and local risk \( \tilde{x} \), \( u \) will always be willing to pay more than \( v \) to avoid that risk. One natural extension of Theorem 1 is to transform local risk aversion into global risk aversion\(^4\) so that \( A(w) \) can be used to measure risk aversion in respect to any risk. This is done by Pratt in what is presented here as Theorem 2.

**Theorem 2.** The following conditions are equivalent:

1. The local risk aversion function \( A(w) \) is [strictly] decreasing.
2. The risk premium \( \pi(w, \tilde{x}) \) is a [strictly] decreasing function of \( w \) for all \( \tilde{x} \).

Notice that in Theorem 2 there is no restriction on the size of the risk \( \tilde{x} \), so that, although the local risk aversion \( A(w) \) is derived as a local measure for a small risk, its application is far greater as the difference between two individuals’ risk premiums for any risk, large or small, can readily be assessed through the calculation of their respective local risk aversions \( A(w) \). The issue of individuals’ exposure to risk is turned into a comparison of local risk aversion, which in turn is a matter of comparing the concavity of the agents’ different utility functions. One interesting aspect is that the comparison of risk aversion does not have to be between two agents. It can also be applied to the same person with and without background risk or to the same person in different time periods, as shall be seen in later sections.

Another important concept related to risk aversion is prudence. Here, an agent is defined as prudent if adding an uninsurable zero-mean risk to his future wealth will increase the optimal amount for him to save. Kimball (1990) shows that

\(^4\)In contrast to local risk, global risk is of such order that none of its higher moments can be omitted.
absolute prudence, which is denoted as $P$ and is calculated as

$$P(w) = -\frac{u'''(w)}{u''(w)},$$

measures the strength of the precautionary saving motive. Moreover, decreasing absolute prudence can be interpreted as a precautionary saving motive that decreases in intensity with wealth.

Pratt (1964), together with Arrow (1963) and de Finetti (1952), built the foundation for risk aversion theory. Pratt’s paper can be thought of as a comprehensive and pioneering work, with extremely rich application in analysing problems of investment decisions under uncertainty such as demand in insurance and asset markets, the properties of risk taking in taxation models, and the interaction between risk and life-cycle savings etc. In addition, the Arrow-Pratt measure of risk aversion has spurred a wide range of research on characterizing the full aspects of risk aversion. In order to proceed, we illustrate the remaining issues of the theory using simple equations as follows:

$$Eu(w_0 + \tilde{x}) = u(w_0 - \pi)$$  \hspace{1cm} (2.1)

with the notation having the same meaning as before. As can be seen here, the effect of risk on utility depends on three factors: the risk $\tilde{x}$ itself, the wealth level of the individual $w_0$ and his utility function $u$. Pratt’s dense paper already gives a rather neat picture of risk aversion measurement. However, many issues readily emerge if the assumptions in the original paper are loosened slightly. For instance,
if there is more than one random variable on the left hand side of (2.1), then it turns into an issue of multiple risks. In particular, the individual’s problem can then be turned into determining their optimal exposure to the risky asset under background risk, which is a topic that has been vigorously explored in the last twenty years. In addition, if there are two or more time periods and the individual needs to determine the optimal allocation of wealth in each period, this gives the dynamic portfolio problem.

2.1.2 The Effect of Background Risk

2.1.2.1 Background Risk

In Pratt’s framework, he assumes that the risky asset can be traded in the market and there is only one risky asset. Both assumptions are not necessarily true in reality. In the real world, it is more usual for individuals to face several sources of risk at once. Even if the risks are independent, they interact with each other and the omission of these interactions will lead to errors in estimating the optimal exposure to risk. Here we will relax the assumptions and introduce background risk.

Background risk is a second non-insurable risk besides marketable risky assets. Strictly speaking, background risk can be further divided into additive background risk and multiplicative background risk. The previous literature, such as Pratt and Zeckhauser (1987), Kimball (1993) and Gollier and Pratt (1996), focuses mainly on additive background risk, often simply calling it background risk. A widely
used example of additive background risk that is non-insurable is human capital, due to asymmetric information between employer and employee, and incentive issues.

2.1.2.2 Derived Utility Function

Generally speaking, we want to analyse the effect of background risk on an agent’s attitude towards risky assets. Will the presence of background risk increase or decrease the investor’s demand for risky assets? Consider the portfolio problem

$$
\max_{\alpha} Eu(w_0 + \alpha \tilde{x} + \tilde{y})
$$

where $w_0$ is the initial wealth, and $\tilde{x}$ is the random return on the risky asset, which is independent from the background risk $\tilde{y}$. The agent needs to decide on the optimal amount $\alpha$ to invest in the risky asset. The technique employed here to analyse the effect of background risk, which uses the idea of the derived utility function\(^5\), was first introduced by Kihlstrom, Romer and Williams (1981) and Nachman (1982). The idea behind it is that, if we define a new utility function for all $z$,

$$
 v(z) = Eu(z + \tilde{y}), \quad (2.2)
$$

then the decision problem for an individual with background risk becomes

$$
\max_{\alpha} Ev(w_0 + \alpha \tilde{x})
$$

\(^5\)Some papers alternatively call it the indirect utility function.
which is quite similar to the form given in Section 2.1.1. Now the original problem of analysing the effect of background risk is turned into comparing the risk aversion of the original utility function with that of the derived utility function. This is like comparing the preferences of two individuals in the absence of background risk, and since there is already a sound foundation in this area, the problem becomes much more familiar and easier to handle.

One issue is to find out whether the basic properties of the original utility function are inherited by the derived one. The advantage of inherited properties is that the results concerning investment decisions without background risks can be readily applied to the derived utility function with background risks. Some properties of the original utility function are easily preserved by the new derived utility function. For instance, Nachman (1982) finds that decreasing absolute risk aversion (DARA) is preserved by the introduction of an independent background risk. This is actually a special case of the property in Pratt (1964)’s paper, which shows that the sum of DARA utility functions retains the DARA property.\(^6\) On the other hand, some other properties are not easily preserved. For instance, it is hard to say whether, or under what conditions the convexity of absolute risk tolerance will be preserved. In addition, as is shown by Kihlstrom, Romer and Williams (1981), if \(u_1\) is more risk averse than \(u_2\) in the sense of Arrow-Pratt, this is still not enough to guarantee that \(u_1\) will behave in a more risk averse way than \(u_2\) in the presence of background risk. The authors actually give a counter example in which both \(u_1\) and \(u_2\) exhibit increasing absolute risk aversion yet

\(^6\)For further details, see Gollier (2001), Chapter 8, page 114.
the preservation of comparative risk aversion fails. This result also illustrates the importance of the DARA assumption in analysis. As will be seen later, although DARA is a natural and simple assumption, it is a sufficient condition for many useful results. For instance, $u$ being DARA is equivalent to the prudence measure being uniformly larger than the risk aversion measure. In addition, if $u$ is DARA, then the derived utility function $v$ in (2.2) will inherit DARA from $u$.

2.1.2.3 Background Risk on Risk Aversion

An old question, initially raised by Samuelson (1963), is why people will decline a bet offering a $200 gain against a $100 loss based on the toss of a coin but will accept a large number of the same bet. Samuelson argues that, if a person rejects a bet, and rejects it at all wealth levels, then any number of the same bet should also be rejected. He suggests that people might misinterpret the Law of Large Numbers, since the law applies to averages and not to sums.\footnote{The issue is more complicated than it first appears. Although Samuelson’s own explanation about the fallacy of the Law of Large Numbers is generally accepted by scholars, it also draws a great number of critics and alternative explanations. For instance, Tversky and Bar-Hillel (1983) explain the phenomenon on the basis of prospect theory and framing theory, Benartzi and Thaler (1996) find it to be an undesirable aspect of expected utility theory itself, whereas Ross (1999) shows that accepting sequences of good bets is actually both consistent with expected utility theory and quite usual.}

Samuelson’s question is just a special case of the general question about how the presence of one risk will affect the demand for another independent risk. Intuition suggests that independent risks should behave as substitutes and that the presence of one should have an adverse effect on the others. This idea has led to a wide range of research following Samuelson, most notably Pratt and
Pratt and Zeckhauser’s Notion of ‘Proper Risk Aversion’  Pratt and Zeckhauser (1987) defines the concept of ‘proper risk aversion’ as a significant behavioural condition on utility functions for wealth, namely an undesirable lottery can never be made desirable by the presence of an independent, undesirable lottery, or equivalently that, if two lotteries are individually unattractive, then the compound lottery offering both together will be less attractive than either one alone. Technically speaking, this means that

\[
Eu(\tilde{\omega} + \tilde{x}_1) \leq Eu(\tilde{\omega}) \\
Eu(\tilde{\omega} + \tilde{x}_2) \leq Eu(\tilde{\omega})
\]

where \( \tilde{\omega} \) is the random wealth, and \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are the random returns for the risky assets, will imply that

\[
Eu(\tilde{\omega} + \tilde{x}_1 + \tilde{x}_2) \leq Eu(\tilde{\omega}).
\]

The authors find several separate sufficient and necessary conditions for utility functions to be proper, and prove properness for all mixtures of risk-averse power and logarithmic functions.
Kimball’s Notion of ‘Standard Risk Aversion’ Kimball (1993) extends Pratt and Zeckhauser (1987)’s work and further defines a von Neumann-Morgenstern utility function as having ‘standard risk aversion’ when, in the case of statistical independence, every loss-aggravating risk aggravates every undesirable, independent risk or equivalently when every risk that has a negative interaction with a small reduction in wealth also has a negative interaction with any undesirable, independent risk. Define a risk $\tilde{x}_1$ that aggravates a reduction in wealth of size $\varepsilon$ if and only if

$$\frac{1}{2}u(w-\varepsilon) + \frac{1}{2}Eu(w+\tilde{x}_1) \geq \frac{1}{2}u(w) + \frac{1}{2}Eu(w-\varepsilon+\tilde{x}_1)$$  \hspace{1cm} (2.3)$$

where $w$ is the initial wealth, $\varepsilon$ is some reduction in wealth and $\tilde{x}_1$ is the random return for some risky asset. (2.3) is equivalent to

$$E[u(w+\tilde{x}_1) - u(w-\varepsilon+\tilde{x}_1)] \geq u(w) - u(w-\varepsilon).$$  \hspace{1cm} (2.4)$$

Moreover, when $\varepsilon$ is small, i.e. a small reduction in wealth, (2.4) becomes

$$Eu'(w+\tilde{x}_1) \geq u'(w).$$  \hspace{1cm} (2.5)$$

The paper shows that condition (2.5), together with

$$Eu(w+\tilde{x}_2) \leq u(w)$$

38
i.e. $\tilde{x}_2$ is an undesirable risk, will imply that

$$Eu (w + \tilde{x}_1 + \tilde{x}_2) \leq Eu (w).$$

Other than extending the case from constant initial wealth to random initial wealth, the distinction between proper risk aversion and standard risk aversion is that proper risk aversion requires both risks to be undesirable while standardness only asks for one risk to be undesirable, with the other risk being loss-aggravating, which makes standard risk aversion applicable to more problems in the theory of multiple risk bearing. In addition, the author finds formal relationships between the conditions for standard risk aversion and the conditions in models of the consumption and saving decision under uncertainty, and show that, for $u'(w) > 0$ and $u''(w) < 0$, the sufficient and necessary condition for standard risk aversion is that both absolute risk aversion and absolute prudence should be decreasing in wealth.

**Gollier and Pratt’s Notion of ‘Risk Vulnerability’**  The most powerful result to date is Gollier and Pratt (1996)’s notion of ‘risk vulnerability’. Preferences exhibit ‘risk vulnerability’ if the presence of an exogenous background risk with a non-positive mean, namely an unfair risk, increases the aversion to any other independent risk. According to Theorem 2 in the previous section, this is true if and only if the derived utility function is more concave than the original one. That is,

$$E (\tilde{y}) \leq 0$$
will lead to
\[ \forall z: -\frac{E u'' (z + \hat{y})}{E u' (z + \hat{y})} \geq -\frac{u'' (z)}{u' (z)}, \]
or equivalently
\[ E (\hat{y}) \leq 0 \]
will lead to
\[ \forall z: E [A (z + \hat{y}) u' (z + \hat{y})] \geq [A (z)] E u' (z + \hat{y}). \]

This is the technical condition for risk vulnerability. The sufficient and necessary conditions for risk vulnerability turn out to be rather complex. Instead Gollier and Pratt (1996) provide\(^8\)

**Theorem 3.** Two independent necessary conditions for risk vulnerability are that

1. The agent’s preference exhibits decreasing absolute risk aversion.
2. \[ \forall z: A'' (z) \geq 2 A' (z) A (z). \]

**Theorem 4.** The two independent sufficient conditions for risk vulnerability are that

1. Absolute risk aversion is decreasing and convex.
2. Absolute risk aversion and absolute prudence are decreasing.\(^9\)

Risk vulnerability is closely related to the concepts of proper risk aversion and standard risk aversion. Each of these properties has to do with a certain class of random variables making the agent reject more risks. Proper risk aversion has to do with the effect of two undesirable risks. Standard risk aversion has to do with

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\(^8\)Further details on risk vulnerability can be found in Gollier (2001), Chapter 9.
\(^9\)The proofs of both theorems are rather complicated. Details can be found in Gollier and Pratt (1996).
the effect of expected marginal utility increasing risks. Finally, risk vulnerability has to do with the effect of any undesirable risks. Gollier and Pratt (1996) show that risk vulnerability includes proper risk aversion and standard risk aversion as particular cases, i.e. that all proper and standard utility functions are vulnerable to risk.

2.1.2.4 Multiplicative Background Risk

The above discussions belong to the area of additive background risk, which has attracted a large literature while the multiplicative background risk has been relatively neglected, with some exceptions such as Nachman (1982), Pratt (1988), Finkelshtain, Kella and Scarsini (1999), and Franke, Schlesinger and Stapleton (2006, 2011). The most commonly used example of multiplicative background risk is that associated with the inflation rate, while other examples include the tax legislation uncertainty regarding the pre-tax profits of a firm and exogenous random per-unit profit for farm commodities etc. In the presence of multiplicative background risk, the portfolio problem turns into

$$\max_{\alpha} Eu \left[ \left( w_0 + \alpha \tilde{x} \right) \tilde{q} \right].$$

As before, $w_0$ is the initial wealth, and $\tilde{x}$ is the random return on the risky asset but here $\tilde{q}$ is the multiplicative background risk. The agent needs to determine the optimal amount $\alpha$ to invest in the risky asset. Generally speaking, the effect of additive background risk depends on the properties of the absolute risk aversion,
whereas the effect of multiplicative background risk depends on the properties of the relative risk aversion of the agent. For instance, if the utility function is of the constant relative risk aversion (CRRA) class, then the multiplicative background risk has no effect on the agent’s exposure to risk. On the other hand, the multiplicative background risk can play a significant role in risk taking if the agent’s preference exhibits decreasing or increasing relative risk aversion.

Franke, Schlesinger and Stapleton (2006) establish the concept of ‘multiplicative risk vulnerability’ which is similar to Gollier and Pratt (1996)’s notion of risk vulnerability for additive background risk. They define preferences as being multiplicatively risk vulnerable if an individual behaves more cautiously in the presence of any arbitrary independent multiplicative background risk such that $E(\tilde{q}) \leq 1$.\(^{10}\) Since the sufficient and necessary conditions for multiplicative risk vulnerability turn out to be rather complex, the authors also present some sufficient conditions that are easier to verify. One result particularly important to our research is as follows:

**Theorem 5.** Suppose that $u$ belongs to the hyperbolic absolute risk aversion (HARA) class of utility functions and that absolute risk aversion is decreasing. Then we have the following results:

1. If $u$ satisfies CRRA, then $v$ and $u$ are equivalent.
2. Let $E(\tilde{q}) \leq 1$. If $R_u > 1$ and is decreasing, then $v$ is more risk averse than $u$.
3. Let $E(\tilde{q}) \leq 1$. If $R_u > 1$ and is increasing, then $v$ is less risk averse than $u$.

\(^{10}\)However, the authors admit that there are no empirical studies that suggest that more cautious behaviour is exhibited in the presence of multiplicative background risk.
In the above theorem, $v$ is the derived utility function

$$v(w) \equiv \int u(wq) \, dG(q) = Eu(w\tilde{q})$$

and $R_u$ is the relative risk aversion

$$R_u(w) = -\frac{wu''(w)}{u'(w)}.$$ 

In a recent paper, Franke, Schlesinger, and Stapleton (2011) analyse the simultaneous effect of both additive and multiplicative background risks on optimal portfolio choice. The portfolio choice problem is transformed into

$$\max E (\tilde{w}\tilde{q} + \tilde{y})$$

where $\tilde{w}$ is the random market wealth, $\tilde{q}$ is the multiplicative background risk and $\tilde{y}$ is the non-market wealth. They find that the combined effect depends on the level of non-market wealth, and that the interaction between the background risks needs to be considered when modelling risk-taking decisions.
2.1.3 Multi-Period Investment Problem

2.1.3.1 Horizon Effect on Risk Aversion

Pratt’s framework is static in the sense that there is only one period involved. In reality, the decision making process always involves more than one period. For instance, an individual will normally allocate and reallocate his portfolio many times during his life time, rather than decide on the optimal exposure to risk once and for all. With the introduction of a multi-period framework, one related question is how the length of an investor’s investment horizon will affect the riskiness of his portfolio.

The most common issue in this area is the difference between young people and old people in terms of their exposure to risk. Scholars such as Samuelson (1989) have asked: ‘As you grow older and your investment horizon shortens, should you cut down your exposure to lucrative but risky equities?’ Intuition says ‘yes’, as young people have more time to take and adjust risk. Yet, surprisingly, neither theoretical nor empirical works have provided a solid foundation for such intuition. Barberis (2000) uses a numerical method to estimate the optimal portfolio of a long-horizon Bayesian investor who rebalances annually or not at all and finds that predictability in asset returns leads to strong horizon effects. An investor with a horizon of 10 years allocates significantly more to stocks than someone with a one-year horizon. On the other hand, Guiso, Jappelli and Terlizzese (1996), among others, find that the share of risky assets is lowest for young households and actually increases by 20 percentage points through the life cycle to reach
maximum at age 61.

2.1.3.2 Risk Tolerance and Exposure to Risk

Theoretically, whether young investors will have a higher or lower demand for risky assets depends on the shape of the absolute risk tolerance. The risk tolerance measures the degree of uncertainty that an investor can handle to a negative change in the value of his or her portfolio. It is denoted as $T$ and is calculated as the inverse of absolute risk aversion

$$T(w) = \frac{1}{A(w)} = -\frac{u'(w)}{u''(w)}.$$

Hennessy and Lapan (1998) show that, if absolute risk tolerance is increasing and concave, then risk aversion is standard. Gollier and Zeckhauser (1998) give the more general case as follows:

**Theorem 6.** Consider the two-period investment problem with a zero yield risk-free asset and another risky asset. Then

1. If absolute risk tolerance is linear (HARA case), the time horizon has no effect on the optimal portfolio.
2. If absolute risk tolerance is concave, a longer time horizon induces investors to reduce their demand for stocks.
3. If absolute risk tolerance is convex, more assumptions are needed in order to draw conclusion. In particular, if the portfolio risk is binary or absolute risk aversion is concave, then a longer time horizon increases the demand for stock.
2.1.3.3 Myopia and Partial Myopia

Merton (1969) and Samuelson (1969) were among the first to solve the dynamic investment problem in a continuous time framework with HARA utility functions. A function exhibits hyperbolic absolute risk aversion (HARA) if the absolute risk tolerance is linear in wealth. Due to the special structure of the HARA utility functions, the authors find no relationship between age and exposure to risk.

Around the same period Mossin (1968) introduced the notion of complete myopia and partial myopia. If the investor’s sequence of decisions is obtained as a series of single-period decisions (starting with the first period), where each period is treated as if it were the last one, then he behaves myopically. With myopia, the investor bases each period’s decision on that period’s initial wealth and probability distribution of yields only, with the objective of maximizing the expected utility of final wealth in that period, while disregarding future yields completely. Mathematically, this means that the derived utility function \( v(w) \) is equivalent to the original utility function \( u(w) \) and that the solutions for the optimization problems in both cases are the same\(^{11}\)

\[
v(w_0 + \alpha \tilde{x}_0) \sim u(w_0 + \alpha \tilde{x}_0)
\]

\(^{11}\)Two utility functions \( V(w) \) and \( U(w) \) are said to be equivalent if they have the same preference ordering for any two different portfolios. For instance, with constant \( b \) and \( c \) \((b > 0)\) such that

\[
V(w) = bU(w) + c
\]

\( V(w) \) and \( U(w) \) will always have the same preference ordering and are therefore equivalent

\[
V(w) \sim U(w)
\]
where \( w_0 \) is the initial wealth, \( \tilde{x}_0 \) is the return on the risky asset in the first period, and \( \alpha \) is the amount invested in the risky asset in the first period.

Similarly, an investor behaves partially myopically if he makes his decision in each period as if the entire resulting wealth would have to be invested at the risk free rate(s) for all subsequent periods. Partial myopia differs from complete myopia in that it is necessary to know what the value of the risk free rate(s) will be in the subsequent periods in order to make the optimal decision, while information about the yield distribution for the risky asset in the subsequent periods remains irrelevant and unnecessary for the decision making process. Mathematically, this means that for the two-period investment problem, the derived utility function is equivalent to the original utility function multiplied by \((1 + r)\):

\[
v(w_0 + \alpha \tilde{x}_0) \sim u[(1 + r)(w_0 + \alpha \tilde{x}_0)]
\]

where \( w_0 \) is the initial wealth, \( \tilde{x}_0 \) is the return on the risky asset in the first period, \( r \) is the risk free rate for the second period, and \( \alpha \) is the amount invested in the risky asset. Mossin (1968) proves that the necessary and sufficient conditions for optimality of partial myopia are that the risk tolerance of the utility function is linear in wealth, which means that the utility function will take the exponential form \(-e^{-z/A}\) for \( B = 0 \), the logarithmic form \( \ln(z + A) \) for \( B = 1 \) and the HARA form \( \frac{1}{B-1}(A + Bz)^{1-1/B} \) otherwise. This conclusion also corresponds to the works of Samuelson (1969) and Merton (1969).
2.2 Downside Risk Aversion Measures

2.2.1 Definition of Downside Risk Aversion

It is generally believed in the literature that individuals tend to avoid situations that offer the potential for substantial gains but also leave them the possibility of a critical level of losses. This kind of behaviour is called downside risk aversion. For instance, in an experimental investigation, Mao (1970) asks people to choose between two investment alternatives, \( f(x) \) and \( g(x) \), with return distributions as shown below:

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_r{x = 1} )</td>
<td>3/4</td>
<td>1/4</td>
</tr>
<tr>
<td>( P_r{x = 3} )</td>
<td>1/4</td>
<td>3/4</td>
</tr>
</tbody>
</table>

Mao finds a unanimous preference for \( f(x) \) over \( g(x) \), even though \( f(x) \) and \( g(x) \) have the same mean and variance. This phenomenon is supported by other empirical works as well. The behaviour is attributed to people’s disaster avoidance motive. Menezes, Geiss, and Tressler (1980) formally define the concept of downside risk as a sequence of mean-variance-preserving transformations which are composed of mean-preserving spreads and mean-preserving contractions that shift the probability weight from the right tail to the left tail in a distribution.

\[\footnotesize{12}\] As the downside risk aversion literature is closely related to the new downside risk aversion measure described in Chapter 4, we will only briefly review the related papers here. A more detailed review can be found in Chapter 4.
Menezes, Geiss, and Tressler (1980) further identify the sufficient and necessary condition for an individual to be downside risk averse, namely that the individual’s von Neumann-Morgenstern utility function $u$ has a positive third derivative: $u''' > 0$. The intuition behind this is that downside risk is concerned with the ‘placement’ of risk in a distribution, and for downside risk averse individuals, an increase in downside risk always transfers risk to a range of states to which they are more risk averse, thus making them worse off.

\subsection*{2.2.2 The Intensity of Downside Risk Aversion}

The main issues in this area concern how to define an increase in downside risk aversion and how to measure the intensity of downside risk aversion. To our knowledge, the current literature on downside risk aversion has provided at least five different measures concerning the intensity of downside risk aversion, viz. Kimball (1990), Modica and Scarsini (2005), Keenan and Snow (2002, 2009 and 2010), Huang and Stapleton (2013), and Liu and Meyer (2012), referred to in this thesis as $d_1$, $d_2$, $d_3$, $d_4$ and $d_5$:

\begin{align*}
  d_1 & = -\frac{u'''}{u''} \\
  d_2 & = \frac{u'''}{u'} \\
  d_3 & = \frac{u'''}{u'} - \frac{3}{2} \left( \frac{u''}{u'} \right)^2 \\
  d_4 & = \frac{u'''}{u'} \frac{u'}{(u'')^2} \\
  d_5 & = \frac{u'''}{u'} \left( \frac{u''}{u'} \right)^2 .
\end{align*}
Although defined differently, these measurements all suffer from one or more shortcomings, such as having overly strong assumptions and restrictions, being local measures that cannot lead to global ones, and inconsistency.

### 2.2.3 Self-Protection

Self-protection is defined as the expenditure on reducing the probability of suffering a loss in contrast to insurance which is the expenditure on reducing the severity of a loss. Examples of self-protection include buying a house in a safer area or purchasing better locks and alarm system to reduce the probability of burglary. To use a numerical example, suppose we have the following portfolio:

<table>
<thead>
<tr>
<th></th>
<th>Probability</th>
<th>Net Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good State</td>
<td>0.5</td>
<td>10</td>
</tr>
<tr>
<td>Bad State</td>
<td>0.5</td>
<td>0</td>
</tr>
</tbody>
</table>

By spending 1 unit of money on self-protection, which increases the probability of the good state by 10%, we will have the new portfolio $Y$:

<table>
<thead>
<tr>
<th></th>
<th>Probability</th>
<th>Net Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good State</td>
<td>0.6</td>
<td>9</td>
</tr>
<tr>
<td>Bad State</td>
<td>0.4</td>
<td>−1</td>
</tr>
</tbody>
</table>

This self-protection is fair in the sense that the mean of the original portfolio $X$, which is 5, is the same as that of the new portfolio $Y$. However, it should be noted that the variance has changed from 25 to 24. Indeed, most examples of
fair self-protection cannot guarantee that the variance of the portfolio remains the same.

Despite their similar definitions, self-protection has properties that are very different from those of insurance. For instance, self-protection can be appealing to both risk averse and risk loving individuals, and a more risk averse individual does not necessarily purchase more self-protection. This is because self-protection in general does not reduce the riskiness of individuals’ final wealth: indeed it will make the individual worse off if they do suffer a loss because of the additional cost of the self-protection, and may result in an increase in downside risk.

Considerable effort has been made to clarify the connection between risk aversion and the optimal choice of self-protection, for example by Briys and Schlesinger (1990), Eeckhoudt and Schlesinger (2006), Liu, Rettenmaier, and Saving (2009), and Chiu (2000, 2005). Chiu (2000) suggests that individuals’ aversion to downside risk plays at least an equally important role as risk aversion in determining the optimal choice of self-protection. Chiu (2005) tries to establish the relationships between an agent’s optimal level of self-protection and his downside risk aversion relative to risk aversion. To start with, suppose say there are two individuals whose preferences are thrice differentiable von Neumann-Morgenstein functions $u(x)$ and $v(x)$. Each is endowed with wealth $w$ and there is a probability $p_2$ of them losing an amount $l$, which is smaller than $w$, and a probability $1 - p_2$ of them losing nothing at all. Denote their self-protection expenditure as $k_u$ and $k_v$. 
and the effect of the self-protection as \( p(k_u) \) and \( p(k_v) \). Then their expected utilities are

\[
(p_2 - p(k_u)) u [w - l - k_u] + [1 - p_2 + p(k_u)] u [w - k_u]
\]

and

\[
(p_2 - p(k_v)) u [w - l - k_v] + [1 - p_2 + p(k_v)] u [w - k_v]
\]

Chiu (2005) first shows that with \( p'(k_u) l = 1 \), then

\[
\frac{-v'''(w)}{v''(w)} \leq -\frac{u'''(w)}{u''(w)} \quad (2.6)
\]

for any \( w \) implies that

\[ k^*_v \geq k^*_u. \]

He then shows that, for two cumulative distributions \( F(x) \) and \( G(x) \) defined on \( x \in [a, b] \), with the same mean and

(i) \( \int_a^b \int_a^y [G(s) - F(s)] dsdy < 0 \)

(ii) \( \int_a^x [G(y) - F(y)] dy \) exhibits the single-crossing property

then (2.6) stands if and only if the equality

\[
\int_a^b u(y) dG(x) = \int_a^b u(y) dF(x)
\]

\[13\text{That is, } p(k_u) \text{ and } p(k_v) \text{ are the reductions in the probabilities of the loss state occurring, after self-protection expenditure of } k_u \text{ and } k_v \text{ respectively.} \]
always implies that the following holds:

$$\int_{a}^{b} v(y) \, dG(x) \geq \int_{a}^{b} v(y) \, dF(x).$$

In addition, since there exist a downside risk increase $D(x)$ and a mean-preserving contraction (MPC) $C(x)$ such that

$$G(x) - F(x) = D(x) + C(x)$$

(2.6) stands if and only if the equality

$$- \int_{a}^{b} u(y) \, dD(x) = \int_{a}^{b} u(y) \, dC(x)$$

always implies that the following holds:

$$- \int_{a}^{b} v(y) \, dD(x) \leq \int_{a}^{b} v(y) \, dC(x).$$

Chiu (2005) interprets $- \int_{a}^{b} u(y) \, dD(x)$ and $- \int_{a}^{b} v(y) \, dD(x)$ as the effect of downside risk aversion and $\int_{a}^{b} u(y) \, dC(x)$ and $\int_{a}^{b} v(y) \, dC(x)$ as the effect of risk aversion. Thus, if we treat agent $u$ as a ‘benchmark agent’, then any agent $v$ with a lower index of prudence will buy more self-protection than $u$ and will have weaker downside risk aversion relative to risk aversion than $u$. 
2.3 Option Price Bounds

2.3.1 Incomplete Market

The Black-Scholes option pricing model is based on some strong assumptions. The complete market and continuous trading assumptions enable the investors to perfectly replicate the payoffs of the options through continuous trading in stocks and bonds. An option’s price is then determined by the prices of stocks and bonds and the approach renders the Black-Scholes pricing formula preference-free. In practice, the majority of markets are, by nature, incomplete as there are not sufficient financial instruments to hedge all risks, and one may not be able to trade continuously. In fact, the Black-Scholes formula can only be considered what Merton (1975) calls ‘a valid approximation to the discrete-time solution’. When the complete market assumption breaks down, the purely preference-free approach often breaks down as well. One question in this field concerns the bounds of option pricing when some quite modest assumptions are imposed on the incomplete market.

2.3.2 Discrete Time Models

Earlier work, such as Merton (1973), only requires the no-arbitrage argument to derive the option pricing bounds. Merton (1973) shows that, with no dynamic trading, the upper and lower no-arbitrage bounds for European call option prices are $S$ and $\max(0, S - Ke^{-rt})$ respectively, where $S$ is the initial stock price, $K$ the option exercise price, $r$ the risk free rate and $t$ the option maturity.
Rubinstein (1976) starts with no-arbitrage and non-satiation assumptions. Let \( t = 0, 1, 2, \ldots \) denote dates, \( s(t) \) the state at date \( t \), \( X[s(t)] \) the cash flow received from a security in state \( s(t) \), \( \pi[s(t)] \) the probability assessed at date \( t = 0 \) that state \( s(t) \) will occur and \( P[s(t)] \) the price of the security in state \( s(t) \). Then, from the single-price law, there must exist a set of discount factors \( \{m[s(t)]\} \), which are the same for all securities, such that

\[
P_0 = \sum_{t=1}^{\infty} \sum_{s(t)} m[s(t)] X[s(t)].
\]

A risk free asset that has a certain cash flow of 1 at date \( t \) and a certain cash flow of 0 on other dates should earn the risk free rate. Denote \( R_{Ft}^{-1} \) as the current price of the risk free asset maturing at date \( t \). Then it should satisfy

\[
R_{Ft}^{-1} = \sum_{s(t)} m[s(t)].
\]

Non-satiation means that the investor prefers more wealth to less wealth and is equivalent to

\[
m[s(t)] > 0
\]

for all states. Based on these assumptions, Rubinstein shows that there exists a positive random variable \( Y[s(t)] \) with

\[
Y[s(t)] = \frac{E(Y[s(t)]) R_{Ft} m[s(t)]}{\pi[s(t)]}
\]
which is the same for all securities such that for any security

\[ P_0 = \sum_{t=1}^{\infty} \frac{E(X[s(t)]) + Cov(X[s(t)], Y[s(t)])}{E(Y[s(t)]) / R_{ft}}. \]

This is called Rubinstein’s equation and can be interpreted as the current price of the risk-adjusted expectation on future uncertain cash flows. With additional assumptions of perfect, competitive, and Pareto-efficient financial markets; rational, time-additive tastes, and the existence of a single representative investor, Rubinstein shows that

\[ Y[s(t)] = U'_t(C[s(t)]) \]

for all dates and states, where \( U_t \) denotes the representative investor’s utility given consumption \( C[s(t)] \). To arrive at a more specific formula, Rubinstein makes the further assumption of a bivariate lognormal distribution for prices \( X_t \) and \( Y_t \), and shows that the Black-Scholes formula is robust under lognormality conditions, even with discrete trading opportunities and risk aversion.

### 2.3.3 Perrakis and Ryan Bounds

Perrakis and Ryan (1984) follow Rubinstein (1976)’s approach, but in addition to the five assumptions used by Rubinstein, add the assumptions of no taxes, dividends, or transaction costs, and that the normalized conditional expected marginal utility for consumption is non-increasing in the price change of \( X[s(t)] \). Denoting the price change per period as \( \Delta X \), then mathematically, this is equiv-
alent to $Z(\Delta X)$, defined as

$$Z(\Delta X) = \frac{E[U'(c) | \Delta X]}{E[U'(c)]},$$

being a non-increasing function of $\Delta X$. They then construct three portfolios: (A) one share of stock at price $S$, (B) one call option on the stock at price $C$ and $S - C$ invested in the risk free asset, and (C) $S/C$ call options on the stock at price $C$. The terminal values are: (A) $S + \Delta X$, (B) $h_1(\Delta X) = (S - C)(1 + R_{ft}) + \max\{0, S + \Delta X - K\}$, and (C) $h_2(\Delta X) = (S/C)\max\{0, S + \Delta X - K\}$ respectively. By the single-price law

$$V_0[S + \Delta X] = V_0[h_1(\Delta X)] = V_0[h_2(\Delta X)].$$

Thus, they derive the following bounds on the single-period call option price in equilibrium:

$$\max\left\{0, S + \frac{1}{1 + R_{ft}} \left[ -K + \int_0^K F(w - S) dw \right] \right\} \leq C(S, K, R_{ft}) \leq S + \frac{S}{S + E[\Delta X]} \left[ -K + \int_0^K F(w - S) dw \right]$$

where $F[.]$ is the probability distribution of $\Delta X \in [-S, \infty)$. In addition, they further extend the theory into the multi-period and European put option cases. As can be seen above, in their derivation process they do not consider investors’
preferences to a great extent, and only make the general assumption that investors prefer more wealth to less wealth. In the follow-up paper by Perrakis (1986), the option price bounds are tightened up by a further assumption on the return distribution, and their previous methods for deriving European option price bounds are used to develop bounds on American put option prices.

2.3.4 Good-Deal Bounds

Cochrane and Saa-Requejo (2000) use the basis assets to analyse the characteristics of pricing kernels. They try to solve the problem

$$ C = \min_m \left\{ E(mx^c) \right\} $$

subject to

$$ p = E(mx) $$

$$ m \geq 0 $$

$$ \sigma(m) \leq \frac{h}{Rf} $$

where $m$ is the discount factor, $x$ the payoff of the risky asset and $x^c$ the payoff of the option. The first condition means that they use the prices of basis assets to analyse pricing kernels. The second constraint is equivalent to the no-arbitrage argument that says that, for any asset whose payoff is non-negative in every state of nature ($m \geq 0$), its value must also be non-negative. What they add is the third constraint on the variance of the pricing kernels, which in effect rules out
those prices that, in a certain sense, represent ‘deals which are too good’, giving rise to the name ‘good-deal bounds’. To see this, first note that, according to Hansen and Jagannathan (1991), the constraint on the pricing kernel volatility is equivalent to an upper limit on the Sharpe ratio. That is,

\[ E(mR^c) = 0 \]

if and only if

\[ \frac{|E(R^c)|}{\sigma(R^c)} \leq \frac{\sigma(m)}{E(m)}. \]

Therefore, the limit they impose on the pricing kernel volatility can be transformed into a limit on the Sharpe ratio. They then argue that a high Sharpe ratio, indicating ‘good deals’, is unlikely to survive since investors will grab such investment opportunities quickly. Finally, they extend the results into multi-period and continuous cases.

### 2.3.5 Gain-Loss-Ratio Bounds

Bernardo and Ledoit (2000) add the gain-loss-ratio constraint to the no-arbitrage assumption. Central to their research is the duality result linking the existence of investments with a high gain-loss ratio to pricing kernels exhibiting extreme deviations from the benchmark pricing kernel. They argue that since the existence of a high gain-loss ratio is equivalent to pricing kernels exhibiting extreme
deviations from the benchmark pricing kernel, the following holds:

$$\max_{\tilde{x} \in X, \tilde{x} \neq 0} \frac{E^* [\tilde{x}^+] - E^* [\tilde{x}^-]}{\min_{\tilde{m} \in M} \sup_{j=1,...,S} \left( \frac{m_j}{m^*_j} \right)} = \min_{\tilde{m} \in M} \sup_{j=1,...,S} \left( \frac{m_j}{m^*_j} \right)$$

where $X$ denotes the space of zero-price payoffs and $M$ the set of pricing kernels that correctly price all portfolio payoffs. The left hand side is the maximum gain-loss ratio, found by searching for all possible zero-cost portfolios that can be constructed from bonds, stocks and options. The right hand side is the ratio of the highest to lowest value of the pricing kernels across all states. Thus, by imposing a finite limit on the maximum gain-loss ratio, they restrict the admissible set of pricing kernels. The price bounds converge to Black-Scholes prices as the maximum gain-loss ratio goes to one, while the bounds widen to no-arbitrage bounds as the ratio goes to infinity. Though different in their methods, in essence good-deal bounds and gain-loss-ratio bounds are both analysing the characteristics of the pricing kernel, but from two different perspectives, and therefore deal with the same issue.

### 2.3.6 Elasticity of the Pricing Kernels

Franke, Stapleton and Subrahmanyam (1999) analyse the issue from a different perspective. They do not focus on the shape of the return distribution nor on the investor’s attitude toward risk. Rather, they focus on the elasticity of the pricing
kernel, defined as

$$\nu(S_T) = -\frac{\partial \phi_{t,T} / \partial S_T}{\phi_{t,T}}$$

where $\phi_{t,T}$ is the pricing kernel and $S_T$ is the asset price, and find that it is an important determinant in option pricing. They show that for two pricing kernels $\phi_1$ and $\phi_2$ that yield the same forward price of the risky asset, if $\phi_1$ exhibits constant elasticity of the pricing kernel while $\phi_2$ exhibits declining elasticity of the pricing kernel, then the price of any European-style option will be greater under pricing kernel $\phi_2$ than under $\phi_1$. Furthermore, under the assumption that the underlying information process follows a geometric Brownian motion, they demonstrate the equivalence between constant elasticity of the pricing kernel and the forward price of the underlying asset following a Brownian motion. Thus, with constant elasticity of the pricing kernel, the Black-Scholes formula correctly prices options on the asset, while with declining elasticity of the pricing kernel it will always underprice the options.
Chapter 3

Partial Myopia in the Presence of Background Risks

3.1 Introduction

This chapter investigates the partial myopia approach for multi-period investment in the presence of background risks. According to Mossin (1968), an investor behaves *myopically* if his investment decision is made as a single-period decision where each period is treated as if it were the last. An investor behaves *partially myopically* if the investor makes his decision as if his total wealth at the end of the next period will be invested at the risk free rates for all subsequent periods. Motivated by Tobin (1958), Mossin (1968) investigates the types of utility function for which the optimal risky asset investment is independent of initial wealth. It turns out that the answer to this question leads to the conditions un-
der which partial myopia behaviour is optimal. Specifically, Mossin (1968) shows that partial myopia investment behaviour could lead to the optimal decision if the risk tolerance of the utility function, defined as the inverse of risk aversion, is linear in wealth, i.e. if the utility function takes the HARA (hyperbolic absolute risk aversion) form.\footnote{Actually, Mossin does not use the word HARA. The three examples he uses for linear risk tolerance are exponential, logarithmic and power. His power utility, however, has a HARA form and includes exponential and logarithmic utilities as special cases.} Similar to Mossin (1968), Samuelson (1969) solves the life time consumption-investment decision in a discrete time economy, while Merton (1969) solves it using a continuous time framework. When the utility function exhibits CRRA (constant relative risk aversion), both Samuelson’s and Merton’s optimal investment decisions are independent of the investment horizon. Later, Merton (1971) considers the impact of non-capital gains such as uncertain wage income on risky asset investment. The results in Merton (1969) and Samuelson (1969) are natural outcomes for a HARA utility investor as HARA encompasses CRRA, and all HARA class investors have linear risk tolerance. However, this linear risk tolerance condition of HARA is violated when there is non-insurable background risk, as noted by Ross (1981), Gollier and Pratt (1996), and Franke, Schlesinger and Stapleton (2006). It is therefore interesting to investigate the extent to which the quality of partial myopia investment policy is impaired by additive and multiplicative background risks for a HARA investor.

The motivation for our research is that, while the dynamic programming (DP) method gives the optimal solution for the multi-period investment problem, it involves tedious iterative calculations for long investment horizons. Also, it is
difficult to apply the dynamic programming approach in practice since the return distributions of the risky assets, especially those of the distant future, are extremely difficult to predict. On the other hand, partial myopia (PM) investment decision making process is extremely simple and does not require information on the distributions of risky asset returns for all future periods.\(^2\) We therefore argue for partial myopia as an alternative approach for the multi-period investment problem for the HARA class of investors. We show that, for the HARA class investor, partial myopia could lead to the optimal solution in the presence of non-stochastic background risks. Also, even in the presence of stochastic background risk, under which the partial myopia method is no longer optimal as the risk tolerance of the derived utility function is no longer linear in wealth, we show through numerical examples that the certainty equivalents\(^3\) between partial myopia and dynamic programming methods are extremely small, such that the investment decisions are almost identical. Moreover, such results hold for various HARA utility functions with large additive and multiplicative background risks. Our research will be useful for making investment decisions such as pension plans, and our findings provide a rationale for fund managers’ short-term focus in investment strategies.

The remainder of this chapter is organized as follows. Section 3.2 proves

\(^2\)We assume throughout this chapter that the asset returns are intertemporally independent and risk free rates are non-random. However, as noted by Mossin (1968, p.222), the result for partial myopia would hold in the case of two risky assets and in the presence of intertemporal dependence, and the conclusions would still be the same.

\(^3\)Certainty equivalent is defined as the dollar amount added to the initial wealth of the partial myopia investor such that his utility matches the utility of the investor who optimizes portfolio choice based on dynamic programming.
Mossin’s partial myopia results for an $n$-period investment problem without background risks. Section 3.3 introduces background risks, and considers the partial myopia approach for $n$-period investment problems with background risks. Section 3.4 gives numerical examples and discusses the results. Section 3.5 concludes and suggests some possible extensions for future research.

3.2 Multi-Period Investment Problem

3.2.1 HARA Utility and Linear Risk Tolerance

The real life investment decision making process typically involves many periods, since an individual will adjust his portfolio many times throughout his life time. In this multi-period setting, one question that naturally follows is how life span and risk exposure affect the investment decisions of young and old people. Indeed, Samuelson (1989) asks, ‘As one grows older and one’s investment horizon shortens, should one cut down exposure to lucrative but risky equities?’ Such a conjecture seems intuitive since an older person has less time left to enjoy potentially large positive returns and no time to re-adjust should a risky investment turns sour. Incorporating the parameter uncertainty into a Bayesian model, Barberis (2000) finds that an investor with a 10-year horizon, who rebalances at regular intervals, allocates significantly more to stocks than another investor with a one-year investment horizon. In contrast, Guiso, Jappelli and Terlizzese (1996) find that young households invest the least in risky assets, but gradually increase their risky investment by 20% throughout the life cycle to reach a maximum at age 61. They
suggest that this finding may be due to the fact that financial information on more complex financial assets is acquired more slowly during a person’s life time, as well as the fact that young people are more likely to be liquidity constrained.

Concerning partial myopia, Mossin shows that the sufficient and necessary condition for partial myopia to be optimal is that the absolute risk tolerance of the utility function is linear in wealth. The risk tolerance, \( T(w) \), is calculated as the inverse of absolute risk aversion as shown below:

\[
T(w) = \frac{1}{\alpha(w)} = -\frac{u'(w)}{u''(w)}.
\]

For a portfolio of two assets, Mossin proves that the necessary and sufficient condition for optimality of (partial) myopia is that the risk tolerance of the utility function is linear in wealth as summarised in the table below:

<table>
<thead>
<tr>
<th>Asset Configuration</th>
<th>Myopia</th>
<th>Partial Myopia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk free asset with zero yield</td>
<td>( -\frac{U'(z)}{U''(z)} = \frac{A}{A + Bz} )</td>
<td>( -\frac{U'(z)}{U''(z)} = Bz )</td>
</tr>
<tr>
<td>Risk free asset with non-zero yield</td>
<td>( -\frac{U'(z)}{U''(z)} = Bz )</td>
<td>( -\frac{U'(z)}{U''(z)} = \frac{A}{A + Bz} )</td>
</tr>
<tr>
<td>Both assets with random yields</td>
<td>( -\frac{U'(z)}{U''(z)} = Bz )</td>
<td></td>
</tr>
</tbody>
</table>

Note: Utility functions that satisfy \( -\frac{U'(z)}{U''(z)} = A + Bz \) will allow the partial myopic investment policy to be optimal when the risk free rate is zero, a non-zero constant or a random variable. Meanwhile, for the more demanding complete myopia, the same condition can be applied when the risk free rate is zero. However, only utility functions such that \( -\frac{U'(z)}{U''(z)} = Bz \) allow the completely myopic investment policy to be optimal when the risk free rate is a non-zero constant or a random variable.

Mossin uses HARA as an example of a utility function that has absolute risk
tolerance that is linear in wealth. The HARA utility takes the form

\[ u(z) = \frac{1}{B - 1} (A + Bz)^{1 - \frac{1}{B}} \]  

(3.1)

where \( z \) is the wealth, and \( A \) and \( B \) should be selected such that \( u(z) \) is increasing and concave. The HARA utility in (3.1) includes the exponential utility, \(-e^{-z/A}\), for \( B = 0 \), and logarithmic utility, \( \ln(z + A) \), for \( B = 1 \). We provide numerical results later to show that, for the HARA class of investor, the investment horizon is irrelevant: old and young investors with the same HARA utility function will reach the same investment decision.

3.2.2 \( n \)-Period Partial Myopia without Background Risks

Mossin (1968) provides the sufficient and necessary conditions for (partial) myopia in an \( n \)-period setting. Later, Gollier (2001) provides an alternative proof of partial myopia for sufficiency of HARA class utility with zero risk free rate for a two-period investment problem. Here, we extend Gollier (2001)’s approach to prove the \( n \)-period case with non-zero risk free rate. The myopia result is obtained immediately when the risk free rate is equal to zero.

**Theorem 7 (\( n \)-Period Partial Myopia).** Consider an \( n \)-period two-asset portfolio investment problem for an investor exhibiting HARA preferences. The investor makes his investment decision as if the wealth at the end of that investment period would be reinvested at the risk free rate(s) for all subsequent periods.

**Proof:**
We consider the simple case with no intermediate consumption. Both DP and PM investors seek to determine the optimal amount of risky asset to hold in the first period. The amount of money not invested in the risky asset is to be placed in the risk free asset. Let \( t = 0, 1, 2, \ldots, n \) denote the time horizon. The excess returns of the risky asset at the end of each period are \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \). The amounts invested in the risky asset are \( b_1, b_2, \ldots, b_n \) in each time period, and the accumulated wealth at the end of each period is \( z_1, z_2, \ldots, z_n \). The risk free rates are \( r_1, r_2, \ldots, r_n \) with \( R_k = (1 + r_k) \) for \( k = 1, 2, \ldots, n \). Both DP and PM investors are endowed with initial wealth \( z_0 \) at \( t = 0 \) and have the HARA utility function given in (3.1).

Since DP and PM investors’ decision making criteria are the same in the single period case, we start our analysis at \( t = n - 2 \). At \( t = n - 2 \), DP investor seeks to maximise the objective function:

\[
\max_{b_{n-1}} Ev_{n-1} (z_{n-1}) = \max_{b_{n-1}} Ev_{n-1} (z_{n-2} R_{n-1} + b_{n-1} \tilde{x}_{n-1}) \tag{3.2}
\]

where \( v_{n-1}(z_{n-1}) \) is the derived utility function defined as

\[
v_{n-1} (z_{n-1}) \equiv \max_{b_n} Eu (z_{n-1} R_n + b_n \tilde{x}_n). \tag{3.3}
\]

---

4 Excess returns of the risky assets are assumed to be independent and risk free rates are assumed to be non-random, although such assumptions are not critical. For more details, see Footnote 2 in this chapter.
For the partial myopia investor, the objective function at \( t = n - 2 \) is

\[
\max_{b_{n-1}} E u \left[ (z_{n-2} R_{n-1} + b_{n-1} \tilde{x}_{n-1}) R_n \right] = \max_{b_{n-1}} E u \left[ \tilde{z}_{n-1} R_n \right].
\]

Define the derived utility function for \( z_{n-1} \) for the partial myopia investor as

\[
m_{n-1} (z_{n-1}) \equiv u (z_{n-1} R_n).
\]

Then we have

\[
\max_{b_{n-1}} E u (\tilde{z}_{n-1} R_n) = \max_{b_{n-1}} E m_{n-1} (\tilde{z}_{n-1}). \tag{3.4}
\]

Comparing (3.2) and (3.4), if we can prove that \( v_{n-1} (z_{n-1}) \) is equivalent to \( m_{n-1} (\tilde{z}_{n-1}) \), then DP and PM investors will reach the same optimal investment decision for \( b_{n-1} \) given the same initial fund level \( z_{n-2} \). To prove this equivalence, start from equation (3.3) and denote the optimal risky asset investment for the DP investor as \( b_n^* \). Then,

\[
v_{n-1} (z_{n-1}) = E u [z_{n-1} R_n + b_n^* \tilde{x}_n].
\]

Substituting the HARA utility function given in (3.1) into this, we have

\[
v_{n-1} (z_{n-1}) = E \frac{1}{B - 1} [A + B (z_{n-1} R_n + b_n^* \tilde{x}_n)]^{1 - \frac{1}{\gamma}}. \tag{3.5}
\]
Following Gollier (2001), let

\[ b_n^* = c_{n-1} [A + B z_{n-1} R_n] \] (3.6)

such that each \( b_n^* \) has a one-to-one unique \( c_{n-1} \) associated with it. Substitute (3.6) into (3.5) to give

\[
v_{n-1} [z_{n-1}] = E \left[ \frac{1}{B - 1} \left\{ A + B \left[ z_{n-1} R_n + c_{n-1} \tilde{x}_n (A + B z_{n-1} R_n) \right] \right\}^{1 - \frac{1}{\beta}} \right]
\]
\[
= \frac{1}{B - 1} E \left[ (A + B z_{n-1} R_n) (1 + Bc_{n-1} \tilde{x}_n) \right]^{1 - \frac{1}{\beta}}
\]
\[
= E \left[ 1 + Bc_{n-1} \tilde{x}_n \right]^{1 - \frac{1}{\beta}} \left\{ - \frac{1}{B - 1} [A + B z_{n-1} R_n]^{1 - \frac{1}{\beta}} \right\}
\]
\[
= K_{n-1} u [z_{n-1} R_n]
\]
\[
= K_{n-1} m_{n-1} [z_{n-1}] \tag{3.7}
\]

where \( K_{n-1} \) is some constant that satisfies

\[ K_{n-1} = E \left[ 1 + Bc_{n-1} \tilde{x}_n \right]^{1 - \frac{1}{\beta}}. \]

That is, \( v_{n-1} (z_{n-1}) \) is equivalent to \( m_{n-1} (z_{n-1}) \)

\[ v_{n-1} (z_{n-1}) \sim m_{n-1} (z_{n-1}). \]

Now consider the case at \( t = n - 3 \) where we need to prove that \( v_{n-2} (z_{n-2}) \) is equivalent to \( m_{n-2} (z_{n-2}) \). At \( t = n - 3 \), the DP investor maximizes \( v_{n-2} (z_{n-2}) \),
which can be written as

\[ v_{n-2}(z_{n-2}) = \max_{b_{n-1}} Ev_{n-1}(\tilde{z}_{n-1}). \]

Given the result in (3.7), we can write

\[
v_{n-2}(z_{n-2}) = \max_{b_{n-1}} [K_{n-1}u(z_{n-1}R_n)]
= K_{n-1} \max_{b_{n-1}} [K_{n-1}u[(z_{n-2}R_{n-1} + b_{n-1}\tilde{x}_{n-1}) R_n]].
\]

Denoting the optimal investment for the DP investor as \( b^*_n \), we have, for HARA utility,

\[
v_{n-2}(z_{n-2}) = K_{n-1} E \frac{1}{B-1} \left[ A + B \left( z_{n-2}R_{n-1} + b^*_n\tilde{x}_{n-1} \right) R_n \right]^{1-\frac{1}{\gamma}} (3.8)
\]

As before, let

\[
b^*_{n-1} = c_{n-2} \left[ A + B z_{n-2}R_{n-1}R_n \right] (3.9)
\]

such that each \( b^*_n \) has a one-to-one unique \( c_{n-2} \) associated with it. Substitute
this expression into (3.8) to give

\[ v_{n-2} [z_{n-2}] = K_{n-1} E \left[ \frac{1}{B-1} \left\{ A + B [z_{n-1} R_{n-1} + c_{n-2} \tilde{x}_{n-1} (A + B z_{n-2} R_{n-1} R_n)] R_n \right\}^{1-\frac{1}{\gamma}} \right] \]

\[ = K_{n-1} \frac{1}{B-1} E \left[ (A + B z_{n-2} R_{n-1} R_n) (1 + B c_{n-2} \tilde{x}_{n-1}) R_n \right]^{1-\frac{1}{\gamma}} \]

\[ = K_{n-1} K_{n-2} u (z_{n-2} R_{n-1} R_n) \]

\[ = K_{n-1} K_{n-2} m_{n-2} (z_{n-2}) \]

where \( K_{n-2} \) is some constant that satisfies

\[ K_{n-2} = E \left[ 1 + B c_{n-2} \tilde{x}_{n-1} R_n \right]^{1-\frac{1}{\gamma}} \]

That is, \( v_{n-2} (z_{n-2}) \) is equivalent to \( m_{n-2} (z_{n-2}) \):

\[ v_{n-2} (z_{n-2}) \sim m_{n-2} (z_{n-2}) \]

Finally, by repeating the process for \( t = n - 3, n - 4, \ldots, 1 \), we get

\[ v_1 (z_1) \sim m_1 (z_1) \]

that is, we arrive at the conclusion that \( m_1 [z_1] \) is equivalent to \( v_1 [z_1] \). Q.E.D.

For the HARA investor, if the risk free rates are all equal to zero, then the myopia solution is also optimal for the \( n \)-period case. This is because \( R_1, R_2, \ldots, R_n \) are all equal to 1; only the next period’s risky asset return affects the partial
myopia investment decision.

3.3 n-Period Partial Myopia with Background Risks

Background risk, which may be additive or multiplicative, is uninsurable, unlike the market risk of some tradable risky assets. Pratt and Zeckhauser (1987), Kimball (1993) and Gollier and Pratt (1996) study additive background risk, while Nachman (1982), Pratt (1988), and Franke, Schlesinger and Stapleton (2006, 2011) examine multiplicative background risk. The most commonly used examples of additive background risk are human capital and wages, and the most commonly used example of multiplicative background risk is the investment risk associated with the inflation rate. In this section, we will start by considering non-stochastic background risks, and then use the results to analyse the more general case of stochastic background risks.

3.3.1 The Model

Before we proceed, it is worth pointing out the difference between previous studies on background risk and our current research. Previous research on background risk focuses on how it affects the optimal demand for a risky asset. Here, our research question is not about whether the introduction of background risk will affect the demand for the risky asset, but rather the effectiveness and efficiency of the partial myopia investment policy vis-a-vis the dynamic programming method.
That is, we are interested in the difference in the optimal demand for the risky asset, $b_1$, between

$$\max_{b_1} E v_b [R_1 z_0 + b_1 \tilde{x}_1]$$

and

$$\max_{b_1} E m_b [R_1 z_0 + b_1 \tilde{x}_1]$$

where $v_b$ and $m_b$ are the derived utilities under additive and multiplicative background risks for DP and PM investors respectively.

It is also worth pointing out that when Mossin first introduced the idea of myopia and partial myopia in 1968, little was known about background risks. Thus, with the incorporation of background risks into the model, the partial myopia that is discussed in the subsequent sections does not exactly have its original meaning. In the later sections of this chapter partial myopia is in effect redefined as meaning that an investor is partial myopia if, in the presence of background risks, the investor makes his investment decision for the next period as if the resulting total wealth at the end of the next period would be invested at the risk free rates for all subsequent periods.

### 3.3.2 Non-Stochastic Background Risks

Now we expand the analysis in the previous section by including background risks.\footnote{Unless otherwise stated, all background risks occur in the last period.} The simplest case is when the background risks are non-stochastic which is a necessary step for us to prove the stochastic background risk case.
Theorem 8 (n-Period Non-Stochastic BGR). Consider an n-period portfolio investment problem with an investor exhibiting HARA preferences. When background risks are non-stochastic, partial myopia still holds even in the presence of background risks.

Proof:

It can be seen from the proof in the previous section that, in the absence of background risks, $v_1 [z_1]$ and $m_1 [z_1]$ are equivalent:

$$v_1 [z_1] \sim m_1 [z_1]$$

which means that there must exist a constant $K (K > 0)$ such that

$$v_1 [z_1] = Km_1 [z_1] = Ku [z_1 R_2 \ldots R_n].$$

Then, for any non-stochastic additive background risk $\overline{y}$, using the same notation as before, at the end of period $n - 1$, when $t = n - 1$, the DP investor seeks to maximize the objective function given the realization of $z_{n-1}$:

$$\max_{b_n} Eu (\tilde{z}_n) = \max_{b_n} Eu (z_{n-1} R_n + b_n \tilde{x}_n + \overline{y})$$

$$\equiv v_{n-1} \left( z_{n-1} + \frac{\overline{y}}{R_n} \right).$$

(3.10)

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At time $t = n - 2$, the DP investor has objective function

$$
\max_{b_{n-1}} E v_{n-1} \left( \bar{z}_{n-1} + \frac{\overline{y}}{R_n} \right) = \max_{b_{n-1}} E v_{n-1} \left( z_{n-2} R_{n-1} + b_{n-1} \bar{x}_{n-1} + \frac{\overline{y}}{R_n} \right)
\equiv v_{n-2} \left( z_{n-2} + \frac{\overline{y}}{R_{n-1} R_n} \right).
$$

(3.11)

This process goes on until, at time $t = 1$, the DP investor has the objective function

$$
\max_{b_2} E v_2 \left( \bar{z}_2 + \frac{\overline{y}}{R_3 \ldots R_n} \right) = \max_{b_2} E v_2 \left( z_1 R_2 + b_2 \bar{x}_2 + \frac{\overline{y}}{R_3 \ldots R_n} \right)
\equiv v_1 \left( z_1 + \frac{\overline{y}}{R_2 \ldots R_n} \right).
$$

(3.12)

Meanwhile, for the PM investor, we have

$$
\max_{b_1} E u \left[ (z_0 R_1 + b_1 \bar{x}_1) R_2 \ldots R_n + \overline{y} \right] = \max_{b_1} E u \left( \bar{z}_1 R_2 \ldots R_n + \overline{y} \right)
\equiv \max_{b_1} m_1 \left( \bar{z}_1 + \frac{\overline{y}}{R_2 \ldots R_n} \right).
$$

(3.13)

Let $v_{b_a} [z_1]$ and $m_{b_a} [z_1]$ denote, respectively, the DP and the PM investor’s derived utility functions for additive background risks. Then, as

$$
v_1 (z_1) \sim m_1 (z_1)
$$
for any $z_1$, it is clear that there also exists some constant $K_y (K_y > 0)$ such that

$$v_{ba} [z_1] = v_1 \left[ z_1 + \frac{\bar{y}}{R_2 \cdots R_n} \right]$$

$$= K_y m_1 \left[ z_1 + \frac{\bar{y}}{R_2 \cdots R_n} \right]$$

$$= K_y m_{ba} [z_1].$$

Indeed, the non-stochastic additive background risk $\bar{y}$ here can be thought of as an increase in the accumulated wealth $z_1$ by $\frac{\bar{y}}{R_2 \cdots R_n}$. Thus $v_{ba} [z_1]$ is equivalent to $m_{ba} [z_1]$,

$$v_{ba} [z_1] \sim m_{ba} [z_1].$$

Hence, partial myopia holds in the presence of non-stochastic additive background risk. Similarly, for non-stochastic multiplicative background risk $\bar{q}$, we have

$$v_{bm} [z_1] = v_1 [z_1 \bar{q}]$$

$$= K_q u [z_1 R_2 \cdots R_n \bar{q}]$$

$$= K_q m_{bm} [z_1].$$

Hence, $v_{bm} [z_1]$ is equivalent to $m_{bm} [z_1]$,

$$v_{bm} [z_1] \sim m_{bm} [z_1],$$

and, hence, partial myopia holds in the presence of non-stochastic multiplicative
background risk as well. Q.E.D.

In addition, it is not difficult to see that, when both additive and multiplicative non-stochastic background risks are present, we will still reach the same conclusion.

### 3.3.3 Stochastic Background Risks

Now consider the stochastic additive background risk $\tilde{y}$ in the $n$-period framework when background risk is independent from risky asset returns. Without the loss of generality, suppose that $\tilde{y}$ follows a discrete distribution\(^6\) such that it has $p_i$ ($i = 1, \ldots, I$) probability for $y_i$ with $p_i > 0$ and $\sum_{i=1}^{I} p_i = 1$, and that $\tilde{x}_1$ follows a discrete distribution such that it has $p_j$ ($j = 1, \ldots, J$) probability for $x_{1j}$ with $p_j > 0$ and $\sum_{j=1}^{J} p_j = 1$. We know from the previous analysis that, for each realization of $y_i$, we can have that

$$ v_1 \left[ z_1 + \frac{y_i}{R_2 \ldots R_n} \right] = K_{y_i} m_1 \left[ z_1 + \frac{y_i}{R_2 \ldots R_n} \right] \tag{3.13} $$

where $K_{y_i}$ is some positive constant and $y_i$ and $z_1$ are independent. Therefore, for the DP investor’s derived utility function with additive background risk $u_{ba}[z_1],$

---

\(^6\)The continuous distribution can be similarly proved, although the notations will be more complicated.
we have

\[ v_{ba} [z_1] = E_p v_1 \left[ z_1 + \frac{\tilde{y}}{R_2...R_n} \right] \]
\[ = \sum_{i=1}^{I} p_i v_1 \left[ z_1 + \frac{y_i}{R_2...R_n} \right] \]
\[ = \sum_{i=1}^{I} p_i K_{yi} m_1 \left[ z_1 + \frac{y_i}{R_2...R_n} \right] \]
\[ = \sum_{i=1}^{I} p_i K_{yi} u [z_1 R_2...R_n + y_i] . \]  
(3.14)

For PM’s derived utility function with additive background risk \( m_{ba} [z_1] \), we have

\[ m_{ba} [z_1] = \sum_{i=1}^{I} p_i u [z_1 R_2...R_n + y_i] . \]

Unlike in the case of non-stochastic background risk, with stochastic background risk, \( K_{yi} \) is different for every state \( i \), and \( v_{ba} [z_1] \) and \( m_{ba} [z_1] \) do not share a simple linear relationship such that they can satisfy the condition

\[ v_{ba} [z_1] = C m_{ba} [z_1] + D \]

where \( C \) and \( D \) are constants with \( C > 0 \). Hence, Mossin’s linear risk tolerance condition does not hold, and our concern turns to how much the difference will be between the DP solution, say \( b^*_1 \), and the PM solution, say \( b^#_1 \), and how much
the certainty equivalent will be. For \( v_{ba} \) we have the first order condition that

\[
\frac{\partial [E_x v_{ba} (z_0 R_1 + b_1 \tilde{x}_1)]}{\partial b_1} = E_{x_1} \left[ \frac{\partial v_{ba} (z_0 R_1 + b_1 \tilde{x}_1)}{\partial b_1} \right]
\]

\[
= E_{x_1} \left[ \frac{\partial v_{ba} (z_1)}{\partial z_1} \right] \frac{\partial z_1}{\partial b_1}
\]

\[
= E_{x_1} \left[ v_{ba}' (z_1) x_1 \right]
\]

(3.15)

where

\[ z_{1j} = R_1 z_0 + b_1 x_{1j}. \]

Substituting (3.14) into (3.15), we have

\[
\frac{\partial [E v_{ba} (z_0 R_1 + b_1 \tilde{x}_1)]}{\partial b_1} = E_{x_1} \left[ v_{ba}' (z_1) x_1 \right]
\]

\[
= E_{x_1} \left\{ x_1 \left[ R_2 \ldots R_n \sum_{i=1}^{l} p_i K_{y_i} u' (z_{1j} R_2 \ldots R_n + y_i) \right] \right\}
\]

\[
= E_{x_1} \left\{ x_1 R_2 \ldots R_n \sum_{i=1}^{l} p_i K_{y_i} u' (z_{1j} R_2 \ldots R_n + y_i) \right\}
\]

\[
= \left\{ \sum_{j=1}^{i} p_j x_{1j} R_2 \ldots R_n \sum_{i=1}^{l} p_i K_{y_i} u' [z_{1j} R_2 \ldots R_n + y_i] \right\}
\]

(3.16)

(3.16) is equal to zero when \( b_1 = b_1^* \). Similarly for \( m_{ba} \) we have the first order condition that

\[
\frac{\partial [E_x m_{ba} (z_0 R_1 + b_1 \tilde{x}_1)]}{\partial b_1} = E_{x_1} \left[ \frac{\partial m_{ba} (z_0 R_1 + b_1 \tilde{x}_1)}{\partial b_1} \right]
\]

\[
= E_{x_1} \left[ m_{ba}' (z_1) x_1 \right]
\]

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and we therefore have

\[
\frac{\partial \left[ Em_{b_1}(z_0 R_1 + b_1 \tilde{x}_1) \right]}{\partial b_1} = E_{x_1} \left[ m'_{b_1} (z_1) x_1 \right] \\
= E_{x_1} \left\{ x_1 \left[ R_2 \ldots R_n \sum_{i=1}^{I} p_i u' (z_{ij} R_2 \ldots R_n + y_i) \right] \right\} \\
= E_{x_1} \left\{ x_1 R_2 \ldots R_n \sum_{i=1}^{I} p_i u' (z_{ij} R_2 \ldots R_n + y_i) \right\} \\
= \left\{ \sum_{j=1}^{J} p_j x_{1j} R_2 \ldots R_n \sum_{i=1}^{I} p_i u' [z_{1j} R_2 \ldots R_n + y_i] \right\} 
\] (3.17)

(3.17) is equal to zero when \( b_1 = b_1^\# \).

Therefore, we can see that, generally speaking, the more similar (3.17) is to (3.16), the smaller the difference will be between \( b_1^\# \) and \( b_1^* \). Moreover, as the difference between (3.17) and (3.16) comes from \( K_{y_i} \), we argue that the difference between \( b_1^\# \) and \( b_1^* \) depends on the values of \( K_{y_i} \). If the values of \( K_{y_i} \) are very similar to each other, then it can be predicted from (3.17) and (3.16) that \( b_1^\# \) and \( b_1^* \) will be very close to each other as well.

### 3.4 Numerical Examples

In this section, we use a three-period example to demonstrate and confirm the analyses in the previous sections. Assume that an investor exhibits HARA preferences of the form shown in (3.1). All the input parameter values are listed in the table below. Unless otherwise stated, these values are the default values in the sensitivity analyses. The risky asset return distribution is binomial with 0.5
probability of being in the up state and 0.5 probability of being in the down state. 
We denote the risky asset investment amounts by the vector \( b \) with the first ele-
ment, \( b_1 \), being the amount of risky asset investment in the first period; \( Eu \) is the 
expected utility of the investment policy.

<table>
<thead>
<tr>
<th>Risky asset excess returns</th>
<th>Period 3 background risks</th>
<th>Risk preference parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{x}_1 ): (0.302, -0.242)</td>
<td>Additive ( \tilde{y} ): (10, -30)</td>
<td>( A = 8 )</td>
</tr>
<tr>
<td>( \tilde{x}_2 ): (0.102, -0.092)</td>
<td>Multiplicative ( \tilde{q} ): (0.9, 1.02)</td>
<td>( B = 2 )</td>
</tr>
<tr>
<td>( \tilde{x}_3 ): (0.245, -0.195)</td>
<td>Risk free asset</td>
<td>Initial wealth</td>
</tr>
<tr>
<td>( R_1 = R_2 = R_3 = 1.03 )</td>
<td>( z_0 = 100 )</td>
<td></td>
</tr>
</tbody>
</table>

For the DP investor, the objective function is

\[
\max_{b_1} \left\{ \max_{b_2|b_1} \left\{ \max_{b_3|b_1,b_2} E (u [(R_3 (R_2 (R_1 z_0 + b_1 \tilde{x}_1) + b_2 \tilde{x}_2) + b_3 \tilde{x}_3) \tilde{q} + \tilde{y}] |_{b_1,b_2}) \right\} \right\}
\]

(3.18)

where \( b_2 \) is the decision choice in the second period given a first period investment 
amount of \( b_1 \), and the realization of \( \tilde{x}_1 \), and \( b_3 \) is the decision choice in the third 
period given investment amounts \( b_1 \) and \( b_2 \), and the realization of \( \tilde{x}_1 \) and \( \tilde{x}_2 \).

For the PM investor, the objective function is

\[
\max_{b_1} Eu [R_3 R_2 (R_1 z_0 + b_1 \tilde{x}_1) \tilde{q} + \tilde{y}] .
\]

(3.19)

However, (3.19) is not comparable to (3.18) since the PM investor has yet to 
determine \( b_2 \) for the second period, and \( b_3 \) for the third period. Assuming that 
\( b_1 = b_1^* \) from (3.19), the PM investor solves the second period investment problem
by maximizing

$$max_{b_3|b_1} Eu \left[ R_3 \left( R_2 \left( R_1 z_0 + b_1^{\#} \tilde{x}_1 \right) + b_2 \tilde{x}_2 \right) \tilde{q} + \tilde{y} \right]$$

(3.20)

for each realization of $x_1$. Similarly, given $b_1 = b_1^{\#}$ in (3.19) and $b_2 = b_2^{\#}$ in (3.20), the partial myopia investor solves the following third period investment problem

$$max_{b_3|b_1, b_2} Eu \left[ \left\{ R_3 \left( R_2 \left( R_1 z_0 + b_1^{\#} \tilde{x}_1 \right) + b_2^{\#} \tilde{x}_2 \right) + b_3 \tilde{x}_3 \right\} \tilde{q} + \tilde{y} \right]$$

for each realization of $x_1$ and $x_2$.

Both the DP and the PM cases are solved numerically using 100 grid points for each investment weight. As such, the accuracy of the solution is subject to the size of the grid, which in turns depends on the range of the investment weights. For the three-period cases, there are $100^3 = 1,000,000$ calculations for each sensitivity analysis without background risks. With background risks, there are four million calculations since each terminal payoff is subject to four different background risk scenarios. We start the calculation using investment weights with the maximum range i.e. from 0 to 200%. In the second iteration, the range of the investment weights is narrowed based on the initial solutions so as to reduce the size of the grid and produce more accurate answers. The estimation results are presented in Figure 3.1. For comparison, the CEQ (certainty equivalent) values\(^7\) for cases where the PM investment policies are suboptimal are reported in Table 3.3. Figure 3.1

---

\(^7\)As before, certainty equivalent is defined as the dollar amount added to the initial wealth of the partial myopia investor such that his utility matches the utility of the investor who optimizes portfolio choice based on dynamic programming.
(a) shows that the expected utility for the DP investor increases for four different values of preference parameters, $A$, when there is no background risk. Note that the $A < 0$ and $A > 0$ cases correspond, respectively, to decreasing and increasing relative risk aversion. We note from Figure 3.1 (a) that, as $A$ increases, expected utility and $b_1$ (around 84%) increase, albeit only very moderately. Indeed, Figure 3.1 (a) shows that, for a specific $A$, the $b_1$ values within a range are associated with almost identical expected utility; the rate of change in the expected utility is very small for similar values of $b_1$. For the given value of $A$, both DP and PM approaches lead to identical optimal investment weights and expected utilities, which supports Mossin’s theory for HARA investors.

In Figure 3.1 (b), for each given value of $A$, the expected utility and the optimal value of $b_1$ are smaller than the corresponding values in Figure 3.1 (a). This is expected as all HARA utility functions will satisfy local properness, which in turn leads to risk vulnerability according to Gollier and Pratt (1996). Therefore, background risk with a non-positive mean will raise the investor’s risk aversion and reduce his holding in the risky asset. As before, as $A$ increases, expected utility and $b_1$ (around 71%) increase only moderately. The PM investor’s optimal $b_1$ and risky investment weights are identical to those for DP except when $A = 2$, and even in that case the risky investment weights are only slightly different (subject to the accuracy implied by the grid size). However, the difference in expected utility between the DP and PM investors is extremely small in that case. Table 3.3 shows that the CEQ is 0.9 basis points (or $0.00009) per $100 of initial investment.

Figure 3.1 (c) demonstrates the effect of changing the additive background risk
while the multiplicative background risk is fixed at (0.9, 1.02) and $A = 8$. As the additive risk variation increases and as its mean decreases, the expected utility and risky investment amount $b_1$ decrease. However, the key points to note here are the identical values of $b_1$ and indeed practically identical risky investment weights in all subsequent periods under the DP and PM approaches; the PM solution is identical to that of the DP approach, and this result is not affected by the (changes in) additive background risk. There is only one case where the result is marginally different, and then the CEQ is 0.6 basis points (or $0.00006$) per $\$100$ initial investment.

Figure 3.1 (d) demonstrates the effect of changing multiplicative background risk while holding additive background risk fixed at (10, -30), with $A = 8$. Both $b_1$ and the expected utility are lower due to the introduction of multiplicative background risk. However, in contrast to the previous case, the subsequent increase in the variation and the decrease in the mean value of the multiplicative background risk have very little impact on either $b_1$ or the expected utility. Again, the most important result to note here is the identical values of both $b_1$ and the expected utility for the DP and PM investors in all cases. The PM approach leads to the same optimal decision as the DP approach and this outcome is not affected by the (changes in) multiplicative background risk.

To see why the CEQs between partial myopia and dynamic programming are so small, we take the framework in Figure 3.1 (b) as an example. For the given value of $z_1^8$, and the realizations of additive background risk $y_i^9$ and multiplicative

---

8 which depends on $b_1$ and the realization of $\tilde{x}_1$.
9 $i = 1, 2$ corresponds to two different outcomes of $\tilde{y}$ such that $y_1 = 10$ and $y_2 = -30$. 

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background risk \( q_j ^{10} \), we can calculate \( K_{x_1, y_i, q_j} \) according to the relationship similar to that in (3.13), albeit now with the addition of multiplicative background risk, such that it will meet

\[
v_1 \left[ z_1 q_j + \frac{y_i}{R_2 R_3} \right] = K_{x_1, y_i, q_j} m_1 \left[ z_1 q_j + \frac{y_i}{R_2 R_3} \right].
\]

Since there are two possible outcomes of each of \( \tilde{x}_1 \), \( \tilde{y} \) and \( \tilde{q} \), we should have eight different values of \( K_{x_1, y_i, q_j} \) accordingly for each HARA utility function. Their results are shown in Table 3.4. We can see from the table that most of the \( K \) are very close to each other. Therefore, according to the analysis concerning the relationship between (3.16) and (3.17) in Section 3.3.3, the optimal solutions from the PM and DP methods are very close to each other, as indeed is shown in Figure 3.1 (b).

### 3.5 Conclusion

This chapter examines the partial myopia policy in the presence of additive and multiplicative background risks. Although the partial myopia solutions are only suboptimal compared with the dynamic programming solutions when there is stochastic background risk, the method is much simpler, and has very little reliance on future information. In addition, the numerical examples show that the partial myopia solutions are very close to the dynamic programming solutions. The surprisingly neat results will help provide a rationale for fund managers’

\(^{10} j = 1, 2 \) corresponds to two different outcomes of \( \tilde{q} \) such that \( q_1 = 0.9 \) and \( q_2 = 1.02 \).
Figure 3.1: Dynamic Programming and Partial Myopia 3-Period Investment Outcomes

(a) Effect of Risk Preference Parameter, A (No background Risk)

(b) Effect of Risk Preference Parameter, A (Background Risk)

(c) Effect of Additive Background Risk (A=8)

(d) Effect of Multiplicative Background Risk (A=8)
Table 3.3: Certainty Equivalent in Basis Points per $100 of Initial Investment

<table>
<thead>
<tr>
<th>Panel I:</th>
<th>No Background Risks</th>
<th>With Background Risks</th>
</tr>
</thead>
<tbody>
<tr>
<td>A=-1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>A=2</td>
<td>-</td>
<td>0.00009</td>
</tr>
<tr>
<td>A=5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>A=8</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel II:</th>
<th>Additive BR</th>
<th>Multiplicative BR</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0,0]</td>
<td>-</td>
<td>[1,1]</td>
</tr>
<tr>
<td>[1,-3]</td>
<td>-</td>
<td>[.99,1.005]</td>
</tr>
<tr>
<td>[2,-6]</td>
<td>-</td>
<td>[.98,1.01]</td>
</tr>
<tr>
<td>[3,-9]</td>
<td>-</td>
<td>[.97,1.015]</td>
</tr>
<tr>
<td>[4,-8]</td>
<td>-</td>
<td>[.96,1.02]</td>
</tr>
<tr>
<td>[5,-15]</td>
<td>-</td>
<td>[.95,1.025]</td>
</tr>
<tr>
<td>[6,-18]</td>
<td>-</td>
<td>[.94,1.03]</td>
</tr>
<tr>
<td>[7,-21]</td>
<td>-</td>
<td>[.93,1.035]</td>
</tr>
<tr>
<td>[8,-24]</td>
<td>-</td>
<td>[.92,1.04]</td>
</tr>
<tr>
<td>[9,-27]</td>
<td>0.00006</td>
<td>[.91,1.045]</td>
</tr>
<tr>
<td>[10,-30]</td>
<td>-</td>
<td>[1.05]</td>
</tr>
</tbody>
</table>

Note: This table reports the CEQ (certainty equivalent), or the dollar amount added to the initial wealth of the partial myopia investor such that his utility matches the utility of the investor who optimizes portfolio choice based on dynamic programming. '-' indicates that CEQ=0, and the two types of investor have the same utilities. Both investors have a HARA utility function with A=8, B=2 and initial wealth=100 dollars. In panel I, the case with background risk has additive risk $y=[10-30]$ and multiplicative risk $q=[0.9 1.02]$. Here, we test the effect of the risk preference parameter 'A'. In panel II, we vary the amounts of additive and multiplicative background risks. For 'Additive BR case', A=8 and multiplicative risk $q=[0.9 1.02]$. For 'Multiplicative BR case', A=8 and additive risk $y=[10-30]$.
Table 3.4: The Values of $K_{x_{11}, y_1, q_1}$ Equating Derived Utility of DP Investor with Derived Utility of PM Investor

<table>
<thead>
<tr>
<th>HARA Parameters</th>
<th>$K_{x_{11}, y_1, q_1}$</th>
<th>$K_{x_{12}, y_1, q_1}$</th>
<th>$K_{x_{11}, y_1, q_2}$</th>
<th>$K_{x_{12}, y_1, q_2}$</th>
<th>$K_{x_{11}, y_2, q_1}$</th>
<th>$K_{x_{12}, y_2, q_1}$</th>
<th>$K_{x_{11}, y_2, q_2}$</th>
<th>$K_{x_{12}, y_2, q_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = -1, B = 2$</td>
<td>1.00132787 1.00129204</td>
<td>1.00129967 1.00124160</td>
<td>1.00109857 1.00094472</td>
<td>1.00127906 1.00124103</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A = 2, B = 2$</td>
<td>1.00132796 1.00129539</td>
<td>1.00130037 1.00124708</td>
<td>1.00110829 1.00095262</td>
<td>1.00128113 1.00123947</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A = 5, B = 2$</td>
<td>1.00132867 1.00129511</td>
<td>1.00130282 1.00124762</td>
<td>1.00111075 1.00097624</td>
<td>1.00128031 1.00124484</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A = 8, B = 2$</td>
<td>1.00132873 1.00129819</td>
<td>1.00130343 1.00125272</td>
<td>1.00111958 1.00098265</td>
<td>1.00128226 1.00124330</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The table shows the values of $K_{x_{11}, y_1, q_1}$ for four different HARA utility functions. For instance, $K_{x_{11}, y_1, q_1}$ corresponds to the $K$ value when the realization of risky asset $\tilde{x}_1$ is $x_{11} = 0.302$, the realization of the additive background risk $\tilde{y}$ is $y_1 = 10$ and the realization of the multiplicative background risk $\tilde{q}$ is $q_1 = 0.9$. The four cases above correspond to 'With Background Risks' in Panel I of Table 3.3.
short-term focus on investment strategies, since the partial myopia policy, which disregards all future information on the risky assets, will work perfectly well for HARA investors. However, it should be noted that in this chapter we assume that all future background risks are known.

In the numerical example in the previous section, we assume investors have perfect information about all future outcomes and their associated probabilities. This is not true in real life, especially not for the distant future. Another common assumption is the normal distribution for risky asset returns. This is far from reality, in which fat tailed and skewed distributions are prevalent. It will be useful to study the potential efficiency loss when the wrong assumptions are made. As we noted earlier, a key advantage of the partial myopia policy is that it only requires information about future interest rates. If the unpredictable nature of risky asset returns is considered, it will weaken the appeal of dynamic programming and give more support to the partial myopia approach.
Chapter 4

Downside Risk Aversion

4.1 Introduction

This chapter investigates how best to define and measure downside risk and downside risk aversion. Downside risk aversion refers to the individual’s tendency to avoid any possibility of big losses even if it is associated with the potential for substantial gains. While risk aversion depends on the second derivative of the utility function, downside risk aversion depends on the third derivative of the utility function. Since it was first discussed in Menezes, Geiss and Tressler (1980), downside risk aversion has become a popular risk measure in addition to risk aversion, for focusing on the downside. Despite the progress made to date, there are still many unresolved issues, the most critical of which is the determination and measurement of greater downside risk aversion. At least five downside risk aversion measures have been proposed in the literature, viz. by Kimball (1990), Modica and Scarsini (2005), Keenan and Snow (2002, 2009 and 2010), Huang and Staple-
ton (2013), and Liu and Meyer (2012) which we will refer to in this chapter as \( d_1 \), \( d_2 \), \( d_3 \), \( d_4 \) and \( d_5 \) respectively. Our literature review indicates that none of these five measures is perfect, and that each has its own flaws and shortcomings. In this chapter, we first define the meaning of greater downside risk aversion, and then suggest how it should be measured. In contrast to the previous five measures, we include the mean of the risky assets in the measurement because it is crucial to know not only by how much, but also where in the probability distribution the increase in downside risk occurs. Different values of the mean may cause more or less of the probability mass of the distribution to be moved into the critical area to which the investor is particularly averse and therefore change his attitude toward the increase in downside risk. Admittedly, as our downside risk aversion measure is derived through Taylor’s expansion, it is restricted by its local property or is valid only when a cubic utility function is used. The global property of the downside risk aversion measure is rather complex. As Keenan and Snow (2009) suggest, no measure to date can capture it. Nevertheless, our numerical examples show that only our new measure is able to give the correct preference ordering according to greater downside risk aversion, while the other five cannot; this is clear evidence of the fundamental weakness of the five existing downside risk aversion measures.

The remainder of this chapter is organized as follows. Section 4.2 describes downside risk and Section 4.3 examines the five existing downside risk aversion measures. Section 4.4 proposes stricter definitions for greater downside risk aversion and the downside risk premium, and uses them to derive a new measure of
downside risk aversion. In this section, we use numerical examples to support our new measure and to demonstrate the weakness of the other five measures. Section 4.5 concludes with some suggestions for future research. In addition, Section 4.6 provides more detailed proof of MGT (1980) and Pratt (1964).

4.2 Downside Risk

A downside risk averse individual tends to avoid any possibility of big losses even if it is associated with the potential for substantial gains.\footnote{An individual is risk averse if he dislikes zero-mean risk at all wealth levels. There is certainly some similarity between risk aversion and downside risk aversion as risk aversion captures some aspects of downside risk aversion. However, downside risk aversion concerns the third moment, i.e. skewness, of the risk, whereas risk aversion concerns the second moment. As a result, while risk aversion deals with people’s aversion to the probability mass shifting from the centre to the tails, downside risk aversion deals with their aversion to the probability mass shifting from the right tail to the left tail.} For instance, Mao (1970) asks participants in an experiment to choose between two investment choices, \( f(x) \) and \( g(x) \), as below

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Pr{x = 1} ) = 3/4</td>
<td>( Pr{x = 0} ) = 1/4</td>
<td></td>
</tr>
<tr>
<td>( Pr{x = 3} ) = 1/4</td>
<td>( Pr{x = 2} ) = 3/4</td>
<td></td>
</tr>
</tbody>
</table>

and to find a unanimous preference for \( f(x) \) over \( g(x) \), even though \( f(x) \) and \( g(x) \) have the same mean and variance. This kind of behaviour is called downside risk aversion, and is attributed to an individual’s disaster avoidance tendency. Menezes, Geiss and Tressler (1980) define downside risk as a sequence of mean-variance-preserving transformations consisting of a series of mean-preserving spreads.
and mean-preserving contractions that shift the probability mass from the right tail to the left tail of the distribution.\(^2\)

4.2.1 Mean-Variance-Preserving Transformation

Rothschild and Stiglitz (1970) are the first to define the increase in risk as a mean-preserving spread that shifts the probability mass from the centre to the tails of a distribution. Let \(f(x)\) and \(g(x)\) be probability density functions, normalized and defined on \([0, 1]\).\(^3\) The distribution \(g(x)\) is more risky than \(f(x)\) if

\[
\int_0^y [G(x) - F(x)] \, dx \geq 0. \tag{4.1}
\]

Equality holds in (4.1) when \(y = 1\) as shown below:

\[
\begin{align*}
E_g &= E_f \\
\int_0^1 xg(x) \, dx &= \int_0^1 xf(x) \, dx \\
\int_0^1 [G(x) - F(x)] \, dx &= 0.
\end{align*}
\]

When \(x < 1\), \(\int_0^1 G(x) \, dx > \int_0^1 F(x) \, dx\), and \(g(x)\) has a longer left tail than \(f(x)\).

Menezes, Geiss and Tressler (MGT, 1980) extend this idea, and define the

\(^2\)Indeed this property can also be applied to risk aversion. However, risk aversion concerns the refusal of zero mean risk, i.e. mean-preserving spread, whereas downside risk aversion concerns the refusal of a mean-variance-preserving transformation of risk. Therefore, an increase in the downside risk will make no change to the risk aversion so long as the variance of the risk remains the same.

\(^3\)Following Menezes, Geiss and Tressler (1980), we use distributions defined on \([0, 1]\) since any economically meaningful distribution can be normalized to the unit interval.
increase in downside risk as a mean-variance-preserving transformation (or probability measure change) that unambiguously shifts the risk from the right to the left. An investor whose utility function’s third derivative $u''' > 0$ will be averse to downside risk as he will be decreasing risk averse (or increasing risk taking).

According to MGT, $g(x)$ has more downside risk than $f(x)$ if $g(x)$ can be associated with $f(x)$ through a mean-variance-preserving transformation (MVPT), $t(x)$, as follows:

$$g(x) - f(x) = t(x)$$

$$= s(x) + c(x)$$

where $s(x)$ is a mean-preserving spread (MPS) and $c(x)$ is a mean-preserving contraction (MPC) such that

$$f(x) + s(x) \geq 0, \quad \text{and} \quad f(x) + c(x) \geq 0,$$  \hspace{1cm} (4.3)

$$\int_{0}^{1} s(x) \, dx = 0, \quad \text{and} \quad \int_{0}^{1} c(x) \, dx = 0$$  \hspace{1cm} (4.4)

$$\int_{0}^{1} xs(x) \, dx = 0, \quad \text{and} \quad \int_{0}^{1} xc(x) \, dx = 0$$  \hspace{1cm} (4.5)

$$s(x) \begin{cases} 
\leq 0, & \text{for } x \in [a, b] \text{ where } [a, b] \subset (0, 1) \\
< 0, & \text{for some subinterval of } [a, b] \\
\geq 0, & \text{for } x \notin [a, b]
\end{cases}$$  \hspace{1cm} (4.6)
\[
\begin{align*}
\begin{cases}
\geq 0, & \text{for } x \in [d, e] \text{ where } [d, e] \subset (0, 1) \\
> 0, & \text{for some subinterval of } [d, e] \\
\leq 0, & \text{for } x \notin [d, e].
\end{cases}
\end{align*}
\]

Conditions (4.3) and (4.4) ensure that \(f(x) + s(x)\) (and similarly \(f(x) + c(x)\)) is still a probability function. Condition (4.5) indicates that \(s(x)\) (and similarly \(c(x)\)) does not change the mean of \(f(x)\).\(^4\) Condition (4.6) characterizes the essential property of an MPS, i.e. the transfer of weight from the centre to the tails of a distribution, whereas condition (4.7) characterizes the essential property of an MPC, i.e. the transfer of weight from the tails to the centre of a distribution. So long as the impact of \(s(x)\) on the distribution is to the left of that of \(c(x)\), the net effect is to transfer dispersion from the right to the left for each value of \(x\), making \(g(x)\) more left skewed than \(f(x)\) while sharing the same mean and variance.

Define

\[
S(x) = \int_0^x s(y) \, dy \quad \quad \quad C(x) = \int_0^x c(y) \, dy
\]
\[
S^*(x) = \int_0^x S(y) \, dy = \int_0^x \int_0^y s(z) \, dz \, dy \quad \quad \quad C^*(x) = \int_0^x C(y) \, dy = \int_0^x \int_0^y c(z) \, dz \, dy
\]

\(^4\)From equation (4.5), \(\int_0^1 x [f(x) + s(x)] \, dx = \int_0^1 x f(x) + \int_0^1 x s(x) \, dx = E(x)\) and similarly for \(c(x)\).
with the following properties:

\[ S(0) = S(1) = 0, \quad \text{and} \quad C(0) = C(1) = 0, \quad (4.8) \]

\[ S^*(x) \geq 0, \quad \text{and} \quad C^*(x) \leq 0 \quad (4.9) \]

\[ S^*(0) = S^*(1) = 0, \quad \text{and} \quad C^*(0) = C^*(1) = 0. \quad (4.10) \]

Then, there exists an \( x_s \) in \((a,b)\) such that

\[
S(x) \begin{cases} 
\geq 0, & \text{for all } x \leq x_s \\
\leq 0, & \text{for all } x \geq x_s.
\end{cases} \quad (4.11)
\]

Similarly, there exists an \( x_c \) in \((d,e)\) such that

\[
C(x) \begin{cases} 
\leq 0, & \text{for all } x \leq x_c \\
\geq 0, & \text{for all } x \geq x_c.
\end{cases} \quad (4.12)
\]

With these definitions and properties in place, a function \( t(x) \) is a MVPT if

\[
t(x) = s(x) + c(x) \quad (4.13)
\]

\[
\int_{0}^{1} T^*(x) \, dx = \int_{0}^{1} [S^*(x) + C^*(x)] \, dx = 0 \quad (4.14)
\]

\[
\int_{y}^{0} T^*(x) \, dx = \int_{y}^{0} [S^*(x) + C^*(x)] \, dx \geq 0. \quad (4.15)
\]

For \( g(x) \) and \( f(x) \) as defined in (4.2), condition (4.13) ensures the mean is preserved such that \( E_g = E_f = \mu \). This is because condition (4.5) and footnote 4
show that $E(x)$ is not at all affected by $s(x)$ and $c(x)$.

Given the mean-preserving condition (4.13), condition (4.14) guarantees variance preservation such that $V_g = V_f$ (see Appendix, Section 4.6.1.1 for details). Condition (4.15), when re-expressed as $\int_0^x S^*(y) dy \geq -\int_0^x C^*(y) dy$ indicates that for each $x$, the impact of the spread is at least as great as that of the contraction. In another words, risk is transferred from higher to lower values of $x$, thus increasing downside risk (see Appendix, Section 4.6.1.2, for more details). MGT summarize these properties in the following theorems.

**Theorem 9 (MGT more downside risk)** $g(x)$ has more downside risk than $f(x)$, i.e. $g(x) = f(x) + \sum_i t_i(x)$, if and only if

(i) $E_g = E_f$,

(ii) $\int_0^1 \int_0^z [G(y) - F(y)] dydz = 0$, and

(iii) $\int_0^x \int_0^z [G(y) - F(y)] dydz \geq 0$

for all $x$ in $[0, 1]$ and for some $x$ in $(0, 1)$.

Since $G(y) - F(y) = \int_0^1 [g(y) - f(y)] dy = \int_0^1 t(y) dy = T(y)$, condition (ii) is equivalent to $\int_0^1 T^*(y) dy = 0$. Hence, conditions (i) and (ii) are the mean-variance-preserving conditions. Similarly, condition (iii) is equivalent to condition (4.15) that ensures $g(x)$ has more downside risk than $f(x)$. With Theorem 9 in place, a downside risk averter will prefer $f(x)$ to $g(x)$ according to Theorem 10 below.

**Theorem 10 (MGT downside risk aversion)** $E_f u(x) \geq E_g u(x)$ for all $u(x)$ in $U_3^* = \{u'' > 0\}$ if and only if $g(x)$ has more downside risk than $f(x)$, i.e. $g(x)$ and $f(x)$ satisfy conditions (i), (ii), and (iii) of Theorem 9. (See Appendix,
4.2.2 Important Remarks

1. It is important to note that, according to MGT, \( u'' > 0 \) is the only necessary and sufficient condition for the investor to be a downside risk averter. This means that both the risk averter with \( u'' < 0 \) and the risk lover with \( u'' > 0 \) can be downside risk averters. Furthermore, individuals with increasing or decreasing utility functions can be downside risk averters. As \( u''' > 0 \) is also the sufficient and necessary condition for an individual to be prudent, MGT’s result corresponds to the claim in Crainich, Eeckhoudt, and Trannoy (2012) that even risk lovers can be prudent.

2. Rothschild and Stiglitz (1970)’s increasing risk and MGT’s increasing downside risk are conceptually distinct. Rothschild and Stiglitz (1970)’s definition of increasing risk implies that, if \( g(x) \) is riskier than \( f(x) \) according to (4.1), then, with \( E_g = E_f = \mu \) and \( g(x) = f(x) + s(x) \), an investor with \( u'' < 0 \) will prefer \( f(x) \) to \( g(x) \). By the same token, a risk averter could have an increasing or a decreasing concave utility function. It is clear that neither the conditions nor the outcomes of Rothschild and Stiglitz (1970)’s risk aversion and MGT’s downside risk aversion subsume each other.

3. MGT show that risk aversion and downside risk aversion can be linked to stochastic dominance as stated below. Distribution \( f(x) \) is preferred to \( g(x) \), i.e. \( E_f u(x) \geq E_g u(x) \) if one of the following conditions is true:
**FSD (First order stochastic dominance)** \( F(x) \leq G(x) \) for all \( u(x) \) in \( U_1 = \{u' > 0\} \) and all \( x \in [0, 1] \).

**SSD (Second order stochastic dominance)** \( \int_0^x F(y) dy \leq \int_0^x G(y) dy \) for all \( u(x) \) in \( U_2 = \{u' > 0, u'' < 0\} \) and all \( x \in [0, 1] \).

**TSD (Third order stochastic dominance)** \( \int_0^x \int_0^y F(z) dz dy \leq \int_0^x \int_0^y G(z) dz dy \) for all \( u(x) \) in \( U_3 = \{u' > 0, u'' < 0, u''' > 0\} \) and all \( x \in [0, 1] \).

Higher order SD implies lower order SD, i.e. \( TSD \subset SSD \subset FSD \). If \( f(x) \) dominates \( g(x) \) by TSD, it will also dominates \( g(x) \) by SSD and FSD. However, \( U_3 \) is a much smaller set than \( U_3^* \). TSD is a subset of MGT, and SSD is a subset of Rothschild and Stiglitz (1970). MGT and Rothschild and Stiglitz (1970) overlap but do not subsume each other. As for asset return moment conditions, MGT’s mean-variance preserving condition is a subset of Rothschild and Stiglitz (1970)’s mean-preserving condition. However, there is a wider set that includes all asset returns that are neither mean-nor variance-preserving. This is where a (global) downside risk aversion measure is most needed.
4.3 Existing Downside Risk Aversion Measures

MGT claim that the sufficient and necessary condition for an individual to be downside risk averse is that the individual’s $u'' > 0$. To our knowledge, the current literature on downside risk aversion has provided at least five different downside risk aversion measures, viz. Kimball (1990), Modica and Scarsini (2005), Keenan and Snow (2002, 2009 and 2010), Huang and Stapleton (2013), and Liu and Meyer (2012), which we refer to here as $d_1$, $d_2$, $d_3$, $d_4$ and $d_5$ respectively.

4.3.1 $d_1$: Kimball (1990)

$$d_1 = -\frac{u'''}{u''} \quad (4.16)$$

Chiu (2005) interprets $d_1$ as the strength of downside risk aversion relative to risk aversion. Actually, Kimball (1990) does not refer to downside risk explicitly; he points out that the index of prudence, defined in (4.16), can be used to measure the motive for precautionary saving. Since many papers suggest that prudence is related to downside risk aversion, it is included here as a variant of the downside risk aversion measure.

Consider the case where an agent has an income flow $(w_0, w_1)$, and he selects a consumption plan $(c_0, c_1)$ at $t = 0, 1$. His saving in the first period is $s = w_0 - c_0$, which allows him to consume an additional amount, $(1 + r)s$, in the second period, where $r$ is the interest earned on the amount saved. The agent’s aim is to maximize
utility over the two periods:

$$\max_s \{u_0 (w_0 - s) + u_1 [w_1 + (1 + r) s]\}$$

where $u_0$ and $u_1$ are the agent’s utility functions for the first and second periods which are both increasing and concave. The optimal saving level $s^*$ is obtained when

$$u_0' (w_0 - s^*) = (1 + r) u_1' [w_1 + (1 + r) s^*].$$

Suppose now that there is uncertainty about the income in the second period, i.e. the income at $t = 1$ becomes $w_1 + \tilde{z}$ where $\tilde{z}$ is a pure risk with $E\tilde{z} = 0$. With this uncertainty, the agent’s objective is

$$\max_s H (s) = u_0 (w_0 - s) + Eu_1 [w_1 + (1 + r) s + \tilde{z}].$$

Since $u_0$ and $u_1$ are concave, $H (s)$ is also concave and the agent will increase the amount he saves from $s^*$ if and only if $\frac{\partial H}{\partial s} > 0$, or

$$Eu_1' [w_1 + (1 + r) s^* + \tilde{z}] \geq u_1' [w_1 + (1 + r) s^*]$$

$$Eu_1' (x + \tilde{z}) \geq u_1' (x)$$

where $x = w_1 + (1 + r) s^*$. According to Pratt (1964), (4.17) is true if and only if $u_1'$ is convex as it is then a natural consequence of Jensen’s inequality, i.e. $\frac{\partial^2 u'}{\partial x^2} > 0$, or $u'' > 0$ as pointed out by Leland (1968). In addition, Kimball (1990) defines
the precautionary premium $\psi$ as:

$$Eu'_1 (x + \tilde{z}) = u'_1 (x - \psi (x, u_1, \tilde{z})). \quad (4.18)$$

The more prudent is the investor, the larger is his precautionary premium $\psi$. According to Pratt (1964),

$$\psi (x, u_1, \tilde{z}) \geq \psi (x, v_1, \tilde{z})$$

if and only if the index of prudence bears the following relationship:

$$-\frac{u''''_1 (x)}{u''_1 (x)} \geq -\frac{v''''_1 (x)}{v''_1 (x)} \quad (4.19)$$

for all $x$. Since both downside risk aversion and prudence concern $u''' > 0$, downside risk aversion and prudence are closely related. However, we can see from the above analysis that prudence concerns the amount of precautionary saving in response to future uncertainty, rather than downside risk per se. To put $d_1$ in the context of downside risk, we ask, for a distribution that exhibits more downside risk, if Pratt’s prudent condition in (4.19) is sufficient to guarantee that $\psi (x, u_1, \epsilon) \geq \psi (x, v_1, \epsilon)$. To answer this question, we look to Theorem 11, which comes from Ross (1981):

**Theorem 11** (Ross (1981)) For two agents with utility functions $A$ and $B$, the three conditions below are equivalent
(i) There exist a positive constant such that for all wealth levels

\[
\frac{A''(x)}{A'(x)} \geq \lambda \geq \frac{B''(y)}{B'(y)}
\]

is equivalent to

\[
\inf A''(x) \geq \sup B''(y).
\]

(4.20)

(ii) There exists a transformation function \( \vartheta: \mathbb{R} \to \mathbb{R} \) with \( \vartheta' \geq 0 \) and \( \vartheta'' \leq 0 \), such that \( A(x) = \vartheta(B(x)) \) for any wealth level \( x \in \mathbb{R} \).

(iii) For any random variables \( \tilde{w} \) and \( \tilde{\varepsilon} \), such that \( E\{\tilde{\varepsilon} | \tilde{w}\} = 0 \),

\[
E\{A(\tilde{w} + \tilde{\varepsilon})\} = E\{A(\tilde{w} - \pi_A)\}
\]

\[
E\{B(\tilde{w} + \tilde{\varepsilon})\} = E\{B(\tilde{w} - \pi_B)\}
\]

imply that

\[ \pi_A \geq \pi_B. \]

Now let \( A = u'_1 \) and \( B = v'_1 \); then, the sufficient and necessary condition for

\[ \psi(x, u_1, \tilde{\varepsilon}) \geq \psi(x, v_1, \tilde{\varepsilon}) \]

becomes

\[
\inf \frac{u''_1(x)}{u'_1(x)} \geq \sup \frac{v''_1(y)}{v'_1(y)}.
\]

(4.22)

which is a much stronger condition than Pratt’s condition shown in (4.19). Hence, the index of prudence \( d_1 \) cannot fully measure the intensity of downside risk

\[ ^{5}\text{Note that the wealth in (4.21) is stochastic, whereas the wealth in Kimball’s equation (4.18) is constant.} \]
aversion. Indeed, Ross provides a counter-example to show that (4.19) can lead to a small premium for $u$.

4.3.2 $d_2$: Modica and Scarsini (2005)

$$d_2 = \frac{u'''}{u'}$$  \hfill (4.23)

Modica and Scarsini (2005) propose the use of $d_2$ as a measure of downside risk aversion. Following Ross (1981), they define the ‘stronger measure of downside risk aversion’ in the following proposition:

**Proposition 1** For investors with utility functions $u$ and $v$ that are increasing functions with convex derivatives, the following three conditions are equivalent:

(i) There exists a positive constant such that, for all wealth levels $x, y \in \mathbb{R}$

$$\frac{u'''(x)}{v'''(x)} \geq \lambda \geq \frac{u'(y)}{v'(y)}$$  \hfill (4.24)

which is equivalent to

$$\inf \frac{u'''(x)}{v'''(x)} \geq \sup \frac{u'(y)}{v'(y)}$$

for all wealth levels $x, y \in \mathbb{R}$, which in turn means that for the special case where $y = x$,

$$\inf \frac{u'''(x)}{v'''(x)} \geq \sup \frac{u'(x)}{v'(x)}.$$  \hfill (4.25)

Condition (4.25) requires investor $u$ to be more downside risk averse than $v$ at all wealth levels. This condition is stronger than just demanding investor $u$ to be more downside risk averse than investor $v$ at each wealth level.
(ii) There exists a transformation function \( \vartheta : \mathbb{R} \to \mathbb{R} \) with \( \vartheta' \leq 0 \) and \( \vartheta''' \geq 0 \), and a positive constant \( \lambda \), as defined in (4.24), such that \( u(x) = \lambda v(x) + \vartheta(x) \) for any wealth level \( x \in \mathbb{R} \).

(iii) If \( Y \) has more downside risk than \( X \), \( Eu(Y) = Eu(X - \pi_u) \), and \( Ev(Y) = Ev(X - \pi_v) \), then \( \pi_u \geq \pi_v \).

The index of downside risk aversion in (i) is connected to the downside risk premium in (iii) by condition (ii), so that a greater index of downside risk aversion leads to a greater downside risk premium. To see that (i) implies (ii), first define the transformation function \( \vartheta : \mathbb{R} \to \mathbb{R} \) as

\[ u = \lambda v + \vartheta \]

where \( u \) and \( v \) are scaled to satisfy condition (i), and \( \lambda > 0 \). Differentiating with respect to \( x \), we have

\[ u' = \lambda v' + \vartheta'. \]

Since, from condition (i),

\[ u' - \lambda v' \leq 0 \]

we have

\[ \vartheta' \leq 0. \]
Similarly, for the third order differentiation, we have

\[ u''' = \lambda v''' + \varphi''' \]
\[ \frac{u'''}{v'''} = \lambda + \frac{\varphi'''}{v'''} \geq \lambda. \]

Since \( v''' \geq 0 \), the above expression implies

\[ \varphi''' \geq 0. \]

Thus from condition (i), we prove the existence of a function \( \varphi \) that meets condition (ii).

To see that (ii) implies (iii), we start with

\[ Eu (X - \pi_u) = Eu (Y) = \lambda Ev (Y) + E\varphi (y). \] (4.26)

Since \( Y \) has more downside risk than \( X \), and \( \varphi''' \geq 0 \) we have \( E\varphi (Y) \leq E\varphi (X) \).

Hence, from (4.26)

\[ Eu (X - \pi_u) \leq \lambda Ev (Y) + E\varphi (X). \]

Then, writing \( Ev (Y) = Ev (X - \pi_v) \), we get

\[ Eu (X - \pi_u) \leq \lambda Ev (X - \pi_v) + E\varphi (X). \]
Since \( \vartheta' \leq 0 \), \( \vartheta \) is a decreasing function, and we have \( E\vartheta(X) \leq E\vartheta(X - \pi_v) \).

Thus,

\[
Eu(X - \pi_u) \leq \lambda Ev(X - \pi_v) + E\vartheta(X - \pi_v).
\]

Applying the inverse of the transformation function to the RHS, we get

\[
Eu(X - \pi_u) \leq Eu(X - \pi_v)
\]

and \( \pi_u \geq \pi_v \) must be true.

To prove that (iii) implies (i), the authors only give a special case in which \( X \) and \( Y \) have discrete distributions, which is fine as an illustration but inadequate as a proof. In summary, Modica and Scarsini’s index of downside risk aversion as shown in (4.23) is based on the very strong condition shown in (4.25). Similar to Ross’s condition in (4.22), it may be very difficult to find utility functions that meet such a strong condition, which may limit its use in practice. Furthermore, Modica and Scarsini (2005) specify neither what downside risk is, nor the distinction between it and risk. Hence, it is not possible to conclude that the difference between \( \pi_u \) and \( \pi_v \) is due to downside risk alone rather than to risk.

Crainich and Eeckhoudt (2008) use the Arrow-Pratt approximation approach to reach the same conclusion that \( d_2 \) may be used as the downside risk measure. First, they present the downside risk aversion from Eeckhoudt and Schlesinger (2006) as

\[
u(x - k) + Eu(x + \tilde{\epsilon}) > u(x) + Eu(x - k + \tilde{\epsilon})
\]

(4.27)
where \( x \) is the initial wealth, \( k \) a positive constant and \( \tilde{\varepsilon} \) a zero mean risk. We can re-arrange equation (4.27) as follows:

\[
Eu(x + \tilde{\varepsilon}) - Eu(x - k + \tilde{\varepsilon}) > u(x) + u(x - k)
\]

(4.28)

and we can see that, following Pratt (1964), (4.28) is true if \( u' \) is convex. Then to equalise the two sides of (4.27), we add premium \( m \) to the certain part on the right hand side to give:

\[
u(x - k) + Eu(x + \tilde{\varepsilon}) = u(x + m) + Eu(x - k + \tilde{\varepsilon}).
\]

(4.29)

Using Taylor’s expansion up to the second-order term, we have\(^6\)

\[
u(x - k) + \left[ u(x) + \frac{\sigma^2}{2} u''(x) \right] \approx u(x + m) + \left[ u(x - k) + \frac{\sigma^2}{2} u''(x - k) \right]
\]

\[
\frac{\sigma^2}{2} (u''(x) - u''(x - k)) \approx u(x + m) - u(x)
\]

\[
\frac{\sigma^2}{2} ku'''(x) \approx mu'(x)
\]

\[
m \approx \frac{\sigma^2}{2} k \frac{u''(x)}{u'(x)}.
\]

Since the premium is a function of \( \frac{u''(x)}{u'(x)} \), Crainich and Eeckhoudt (2008) support the use of \( d_2 \) as the index of downside risk aversion. However, if we instead define \( m^* \) as the amount of money received in all states on the right hand side of equation

---

\(^6\)The first-order term, \( E(\tilde{\varepsilon}) u'(x) \), is equal to zero. The second order term is \( \frac{1}{2} E(\tilde{\varepsilon}^2) u''(x) = \frac{1}{2} \sigma^2 u''(x) \).
(4.27), we get

\[
\begin{align*}
    u(x-k) + Eu(x+\hat{e}) &= u(x+m^*) + Eu(x-k+m^*+\hat{e}) \\
    u(x-k) + \left[u(x) + \frac{\sigma^2}{2}u''(x)\right] &\approx u(x+m^*) + \left[u(x-k+m^*) + \frac{\sigma^2}{2}u''(x-k+m^*)\right] \\
    \frac{\sigma^2}{2} [u''(x) - u''(x-k+m^*)] &\approx u(x+m^*) + u(x-k+m^*) - u(x-k) - u(x) \\
    \frac{\sigma^2}{2} (k-m^*) u'''(x) &\approx k u'(x) - k u' (x+m^*) \\
    \frac{\sigma^2}{2k} (k-m^*) u'''(x) &\approx -m^* u''(x) \\
    -\frac{\sigma^2}{2k} (k-m^*) \frac{u''(x)}{u''(x)} &\approx m^*
\end{align*}
\]

which in turn supports the use of the index of prudence, \(d_1\), as the index of downside risk aversion. This serves as an illustration of the arbitrary nature of such reasoning.

### 4.3.3 \(d_3\): Keenan and Snow (2002)

\[
d_3 = \frac{u'''}{u'} - \frac{3}{2} \left(\frac{u''}{u'}\right)^2
\]

Note that it is the square of risk aversion that enters the definition of \(d_3\). In this case, therefore, it is clear that both risk-averse and risk-loving investors could be downside risk averse. The only condition for downside risk aversion is that \(u''' > 0\). Indeed, Keenan and Snow (2010) argue that \(d_1\), or prudence, is not an accurate measure of downside risk aversion, although an increase in prudence relative to risk aversion is sufficient to lead to greater downside risk aversion, so long as the prudence is more than three times the risk aversion (see equation (4.35) and the
related comments).

The core of Keenan and Snow (2002, 2009, 2010)'s work is based on a compensated increase in downside risk, which is an extension of Diamond and Stiglitz (1974)'s notion of a compensated increase in risk. Here, the argument is that, instead of the probability distribution of the random variable, it is the induced distribution of the ex post utility that is important in decision making. In Diamond and Stiglitz (1974), if the expected utility remains the same, \( E_g u = E_f u \), and the utility distribution has a different spread, \( V_g u > V_f u \), then the shift in the distribution represents a compensated increase in risk. Moreover, the compensated riskier distribution must depend on the risk preferences of the decision maker, viz. the greater the risk aversion the greater the risk premium. Keenan and Snow (2002) extend this concept, and define a compensated increase in downside risk as a shift that preserves both the mean and variance of the utility distribution.

Define \( \delta(x) = F(x) - G(x) \) on \([0,1]\), and let \( u_x \) denote the partial derivative of \( u \) with respect to \( x \). Then, for investor \( \rho \) with utility \( u(x, \rho) \), a shift \( \delta(x) \) increases the downside risk if

\[
\int_0^1 u_x(x, \rho) \delta(x) \, dx = 0 \tag{4.30}
\]

\[
\int_0^1 u_x(x, \rho) \left( \int_0^x u_x(y, \rho) \delta(y) \, dy \right) \, dx = 0 \tag{4.31}
\]

\[
\int_0^y u_x(x, \rho) \left( \int_0^x u_x(y, \rho) \delta(y) \, dy \right) \, dx \geq 0 \tag{4.32}
\]

for all \( y \in [0,1] \). Condition (4.30) guarantees that the mean of the utility function
remains the same:

\[
\int_0^1 u_x(x, \rho) \delta(x) \, dx = \int_0^1 u_x(x, \rho) [F(x) - G(x)] \, dx \\
= u(x, \rho) [F(x) - G(x)] \big|_0^1 - \int_0^1 u(x, \rho) [f(x) - g(x)] \, dx \\
= E_g u(x, \rho) - E_f u(x, \rho).
\]  \tag{4.33}

Condition (4.31) guarantees that the variance of the utility function also remains the same:

\[
\int_0^1 u_x(x, \rho) \int_0^x u_x(y, \rho) \delta(y) \, dy \, dx = u(x, \rho) \int_0^x u_x(y, \rho) \delta(y) \, dy \big|_0^1 - \int_0^1 u(x, \rho) u_x(x, \rho) \delta(x) \, dx.
\]

The first term on the RHS is zero. Now, by applying \( du^2 = 2 uu_x \, dx \), we get

\[
- \int_0^1 u(x, \rho) u_x(x, \rho) \delta(x) \, dx = - \int_0^1 1 \cdot \delta(x) \, du^2(x, \rho) \\
= -u^2(x, \rho) \delta(x) \big|_0^1 + \int_0^1 u^2(x, \rho) \frac{d\delta(x)}{du^2} \, du^2 \\
= \int_0^1 u^2(x, \rho) [f(x) - g(x)] \, dx \\
= Var_f[u(x, \rho)] - Var_g[u(x, \rho)].
\]

Condition (4.31) then implies \( Var_f u = Var_g u \). Finally, condition (4.32) ensures that the shift in utility weight is from the right tail to the left tail. The derivation of \( d_3 = \frac{u''}{u'} - \frac{3}{2} \left( \frac{u''}{u'} \right)^2 \) is rather complicated, but \( d_3 \) can be expanded as

\[
d_3 = \left[ \frac{u''}{u'} - \left( \frac{u''}{u'} \right)^2 \right] - \frac{1}{2} \left( \frac{u''}{u'} \right)^2 \quad \tag{4.34}
\]
where the first part, the [ ] term, in (4.34) is the slope of absolute risk aversion. Keenan and Snow (2002) show that, for small increases in downside risk, a larger value of $d_3$ is the sufficient and necessary condition for an individual being more downside risk averse. Since $d_3$ is based on the mean-variance preservation of the induced utility distributions after the shift in downside risk, it is not consistent with the classical expected utility framework that assumes the agent will maximize the expected utility of final wealth, giving no consideration at all to the variance of the utility.

The result in Keenan and Snow (2002) is also restricted by its local property, and it is a sufficient but not a necessary indicator of greater downside risk aversion in the large. Keenan and Snow (2009) extend their 2002 result to produce a global property, and show that an investor $v$ is more downside risk averse than investor $u$ if and only if

$$\vartheta''''(u) > 0$$

where $\vartheta(u)$ is a transformation function defined as $v(x) = \vartheta(u(x))$ for any wealth level $x \in R$. Subsequently, Keenan and Snow (2010) establish restrictions on the preferences under which greater prudence implies greater downside risk aversion. An investor $v$ is more prudent than investor $u$ if and only if

$$-\frac{v'''}{v''} \geq -\frac{u'''}{u''} \quad \frac{(\vartheta'(u))'''}{(\vartheta'(u))''} \geq -\frac{u'''}{u''} \quad -\frac{\vartheta'''(u')^3 + 3\vartheta''u'u'' + \vartheta'u'''}{\vartheta''(u')^2 + \vartheta'u''} \geq -\frac{u'''}{u''}$$

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\[
\vartheta''' (u')^3 + 3 \vartheta'' u' u'' + \vartheta' u''' \geq \left( -\frac{u'''}{u''} \right) \left( -\vartheta'' (u')^2 - \vartheta' u'' \right) \\
\vartheta''' (u')^3 + 3 \vartheta'' u' u'' + \vartheta' u''' \geq \frac{\vartheta''' u'' u^2}{u'} + \frac{\vartheta'' u''' u''}{u''} \\
\vartheta''' (u')^3 + 3 \vartheta'' u' u'' \geq -3 \vartheta'' u' u'' + \frac{\vartheta''' u'' u^2}{u''} \\
\vartheta''' u' \geq \vartheta'' \left( -\frac{3u''}{u'} + \frac{u''' u'}{u''} \right). 
\] (4.35)

Therefore \( \vartheta''' \geq 0 \) if prudence is more than three times the degree of risk aversion, and, according to Keenan and Snow (2009), such a condition indicates greater downside risk aversion.

4.3.4 \( d_4 \): Huang and Stapleton (2013)

\[
d_4 = \frac{u''' u'}{(u'')^2}
\]

\( d_4 \) is actually the measure of cautiousness first defined in Wilson (1968) as the first derivative of risk tolerance\(^7\) and then used in Leland (1980) to study the demand for portfolio insurance:

\[
\frac{u''' u'}{(u'')^2} = \frac{d_1}{A}
\]

where \( d_1 \) represents prudence and \( A \) represents risk aversion. Huang and Stapleton

\(^7\)Excluding here the constant -1 in the expression as

\[
\left( -\frac{u'}{u''} \right)' = \frac{u''' u'}{(u'')^2} - 1.
\]
(2013) show that cautiousness impacts on the demand for options. They show that, in a simple economy with one bond, one stock and one convex derivative, if investor $i$ is more cautious than investor $j$, then investor $j$ will hold a positive position in the derivative only if investor $i$ does so, as follows:

$$\inf_{w_i} C_i(w_i) \geq \sup_{w_j} C_j(w_j)$$

for all wealth levels $w_i$ and $w_j$, where $C$ is the index of cautiousness. This condition can be restated as

$$C_i(w_i) \geq \lambda \geq C_j(w_j)$$

for any positive constant $\lambda$. In addition, they show that agent $i$ is always more likely to buy options than $j$, and demands a higher ratio of options to shares than $j$. If the convex derivative in the model is a put option, then agent $i$’s demand for options minus his demand for shares is greater than $j$’s

### 4.3.5 $d_5$: Liu and Meyer (2012)

$$d_5 = \frac{u''}{u'} - \left( \frac{u''}{u'} \right)^2$$

If $A_u = -\frac{u''}{w}$ is the absolute risk aversion, then $d_5 = -A_u'$ is the slope of the absolute risk aversion. $d_5$ is also closely connected to $d_4$ as follows

$$d_4 = d_5/ \left( \frac{u''}{u'} \right)^2$$

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even though they may not always produce the same preference ordering.\footnote{For instance, suppose we have two agents \( u \) and \( v \) with HARA utility functions
\[
\begin{align*}
u(x) &= \frac{1}{B - 1} \left( A_1 + Bx \right)^{1 - \frac{1}{B}} \\
v(x) &= \frac{1}{B - 1} \left( A_2 + Bx \right)^{1 - \frac{1}{B}}
\end{align*}
\]
where \( A_1 \neq A_2 \). Then, we have
\[
\begin{align*}
d_4(u(x)) &= B, \quad d_4(v(x)) = B \\
d_5(u(x)) &= \frac{B}{(A_1 + Bx)^2}, \quad d_5(v(x)) = \frac{B}{(A_2 + Bx)^2}
\end{align*}
\]
Thus, \( u \) and \( v \) always have the same constant value of \( d_4 \), but not the same value of \( d_5 \).}

We can interpret cautiousness, \( d_4 \), as a measure of the strength of downside risk aversion relative to that of risk aversion. The contribution of Liu and Meyer (2012) is the connection that they establish between decreasing absolute risk aversion (hereafter DARA) and downside risk aversion. First they prove the equivalence of \( u''' > 0 \) and \( A'_u < 0 \).

**Theorem 12** For all \( x \in [a, b] \), any \( H(x) \) such that \( \int_a^b H(x) \, dx = 0 \), \( \int_a^b u'(x) \, H(x) \, dx \geq 0 \) for all \( u'''(x) > 0 \) if and only if \( \int_a^b u'(x) \, H(x) \, dx \geq 0 \) for all \( A'_u(x) < 0 \).

Then, by assigning
\[
H(x) = G(x) - F(x)
\]
with \( G(x) \) having more downside risk than \( F(x) \), the first part of the theorem turns into the condition for downside risk aversion. Based on Theorem 12, Liu and Meyer (2012) propose the use of the slope of the absolute risk aversion, \( A'_u \), as the measure for downside risk aversion.

Furthermore, Liu and Meyer (2012) show that investor \( v \) is more downside risk averse than \( u \) if and only if \( -A'_v > -A'_u \), and if and only if \( d_5(v) > d_5(u) \).
definition is transitive and asymmetric. In relation to prudence, \( d_5(v) > d_5(u) \) is true, if and only if \( P_v > P_u \), where the latter are the prudence measures for \( v \) and \( u \). That is, being universally more prudent leads to being more downside risk averse. In this way, the authors have built the connection between downside risk aversion and prudence.

4.3.6 Shortcomings of the Current Measures

The downside risk measures \( d_1 \) to \( d_5 \) above suffer from various shortcomings. First, some of the restrictions and assumptions are too strong. This is the case for Modica and Scarsini’s condition in (4.25). Second, some of the measures are constrained by their local property and cannot be used in the case of a large amount of risk. This is the case for \( d_1, d_2 \) and \( d_3 \). Keenan and Snow (2009), having realised that \( d_3 \) is a sufficient but not a necessary indicator of greater downside risk aversion, conclude that no measure can fully characterize the degree of downside risk aversion. Third, some of the existing downside risk aversion measures are inconsistent. Given the definition of \( d_3 \) in Keenan and Snow (2002), Keenan and Snow (2009) claim that investor \( v \) is more downside risk averse than investor \( u \) if and only if \( \vartheta''''_1(u) > 0 \). However, if we have investors \( u, v \) and \( w \) with \( v(x) = \vartheta_1(u(x)) \) and \( w(x) = \vartheta_2(v(x)) \), then, even if \( \vartheta''''_1 > 0 \) and \( \vartheta''''_2 > 0 \), i.e. even if \( v \) is more downside risk averse than \( u \), and \( w \) is more downside risk averse
than \( v \), there is no guarantee that \( w \) is more downside risk averse than \( u \) since

\[
(\varphi_2(\varphi_1))''' = (\varphi'_2(\varphi_1) \cdot \varphi''_1)''
\]

\[
= (\varphi'_2(\varphi_1) \cdot \varphi''_1 + \varphi'''_2(\varphi_1) \cdot \varphi''_1)'
\]

\[
= \varphi'_2(\varphi_1)^3 + 3\varphi''_2(\varphi_1)^2 \varphi''_1 + \varphi'''_2 \varphi''_1
\]

does not necessarily lead to \( (\varphi_2(\varphi_1))''' > 0 \). Thus the definition is non-transitive.

In addition, the definition is asymmetric since, if two agents have exponential utility functions \( u(x) = e^x \) and \( v(x) = e^{32x} \), they can each be more downside risk averse than the other because \( \varphi''_1 > 0 \) and \( (\varphi^{-1}_1)''' > 0 \). Due to these limitations and shortcomings, it would be useful to have a better downside risk aversion measure with less restrictive assumptions.

### 4.4 A New Downside Risk Aversion Measure

Despite much research on measures of downside risk aversion, all the previous studies seem to ignore the important difference between risk and downside risk in that, unlike risk, downside risk can only be discussed in a comparative sense in terms of the difference between two risks. That is, we can define ‘pure risk’ as a zero-mean risk, but we cannot define ‘pure downside risk’ in a similar manner. Indeed, what Menezes, Geiss and Tressler (1980) define is not ‘pure’ downside risk, but rather an increase in downside risk. Accordingly, risk aversion is defined as individual’s aversion to pure risk. Indeed, Pratt (1964) defines investor \( u \) as
being more risk averse than \( v \) if \( u \) is willing to pay a higher risk premium \( \pi \) than \( v \) for the same risky asset at every wealth level \( w \). That is, for pure risk \( X \), we have

\[
Eu (w + X) = u (w - \pi_u) \quad (4.36)
\]
\[
Ev (w + X) = v (w - \pi_v) \quad (4.37)
\]

If \( u \) is more risk averse than \( v \), then \( \pi_u > \pi_v \). One may deduce that an appealing approach for addressing the concept of “greater downside risk aversion” is to use the downside risk premium, and define investor \( u \) as being more downside risk averse than investor \( v \) if \( u \) is willing to pay a greater downside risk premium than \( v \) at every wealth level. The difficulty here is that, if there is only one risky asset, the premium paid for the risky asset can be fully explained by Arrow-Pratt’s risk aversion measure alone. It is interesting to note that, to our knowledge, there is no definition of a downside risk premium in the literature to date.

Here, we follow Modica and Scarsini (2005) to define “greater downside risk aversion”. Instead of two investors facing the same risky asset, now there are two investors \( u \) and \( v \) and two risky assets \( X \) and \( Y \), with \( Y \) having more downside risk than \( X \). That is, \( Y \) can be derived from \( X \) by mean-variance-preserving transformations according to MGT (1980). Now we have

\[
Eu (w + Y) = Eu (w + X - \pi_u) \quad (4.38)
\]
\[
Ev (w + Y) = Ev (w + X - \pi_v) \quad (4.39)
\]
and investor $u$ is more downside risk averse than $v$ if $\pi_u > \pi_v$.

### 4.4.1 Deriving the New Measure

First, the RHS of (4.38) can be approximated as

$$
Eu[(w + X) - \pi_u] \approx E[u(w + X) - \pi_u u'(w + X)]
= E[u(w + X)] - \pi_u E[u'(w + X)].
$$

(4.40)

Next, expanding both utility functions on the RHS up to $u'''$ gives

$$
Eu[(w + X) - \pi_u]
\approx E\left[u(w) + Xu'(w) + \frac{1}{2}X^2u''(w) + \frac{1}{6}X^3u'''(w)\right]
- \pi_u E\left[u'(w) + Xu''(w) + \frac{1}{2}X^2u'''(w)\right].
$$

(4.41)

Similarly, the LHS of (4.38) can be expanded to give

$$
Eu(w + Y) \approx E\left[u(w) + Yu'(w) + \frac{1}{2}Y^2u''(w) + \frac{1}{6}Y^3u'''(w)\right].
$$

Note from MGT Theorem 10 that $u''' > 0$ is the only necessary and sufficient condition for $E_fu(x) \geq E_gu(x)$ if $g(x)$ has more downside risk than $f(x)$, and that $g(x)$ is derived from $f(x)$ through mean-variance-preservation transformations. By expanding the utility function up to $u'''$, we may have restricted our downside risk aversion measure to small risks only. We will see later that, when the mean of the risky asset is equal to zero, our new measure is exactly the same as Modica and Scasini (2005)'s downside risk aversion measure $d_2$, which is a global measure under the condition set out in (4.25). In addition, for cubic utility, (4.40) and (4.41) will be strictly equal.
With mean-variance preservation,

\[ E(X) = E(Y) = \mu \]  

(4.42)

and

\[ E\{(X - \mu)^2\} = E\{(Y - \mu)^2\} = \sigma^2. \]  

(4.43)

Then, we have for investor \( u \),

\[
\frac{u'(w) + \mu u''(w) + \frac{1}{2}(\sigma^2 + \mu^2) u'''(w)}{E\left\{\frac{1}{6}(Y^3 - X^3) u'''(w)\right\}} = -\frac{1}{\pi_u}
\]

and for investor \( v \),

\[
\frac{v'(w) + \mu v''(w) + \frac{1}{2}(\sigma^2 + \mu^2) v'''(w)}{E\left\{\frac{1}{6}(Y^3 - X^3) v'''(w)\right\}} = -\frac{1}{\pi_v}
\]

If investor \( u \) is more downside risk averse than investor \( v \), i.e. \( \pi_u > \pi_v \), we have

\[
\frac{u'(w) + \mu u''(w) + \frac{1}{2}(\sigma^2 + \mu^2) u'''(w)}{E\left\{\frac{1}{6}(Y^3 - X^3) u'''(w)\right\}} > \frac{v'(w) + \mu v''(w) + \frac{1}{2}(\sigma^2 + \mu^2) v'''(w)}{E\left\{\frac{1}{6}(Y^3 - X^3) v'''(w)\right\}}.
\]

Now define a new measure \( d_6 \) of downside risk aversion for utility function \( u \) as follows:

\[
d_6 = \frac{u'(w) + \mu u''(w) + \frac{1}{2}(\sigma^2 + \mu^2) u'''(w)}{E\left\{\frac{1}{6}(Y^3 - X^3) u'''(w)\right\}}.
\]

Since \( X \) and \( Y \) have the same means, \( E(X) = E(Y) \), and variances, \( E(X - \mu)^2 = E(Y - \mu)^2 \), but different skewnesses, \( E(X - \mu)^3 > E(Y - \mu)^3 \), the expression
can be rearranged into

\[
d_6 = \frac{6}{E(Y^3 - X^3)} \left[ \frac{u'(w) + \mu u''(w)}{u'''(w)} + \frac{1}{2} \left( \sigma^2 + \mu^2 \right) \right]
\]

\[
= -\frac{u'(w) + \mu u''(w)}{u'''(w)} K_1 + K_2
\]

where

\[
K_1 = \frac{6}{E(X^3 - Y^3)} \quad \text{and} \quad K_2 = \frac{3 \left( \sigma^2 + \mu^2 \right)}{E(X^3 - Y^3)}
\]

are two positive-valued constants that are not dependent on the investor’s risk preferences. Thus the new measure \(d_6\) for downside risk aversion can be simplified to

\[
d_6 = -\frac{u'(w) + \mu u''(w)}{u'''(w)}.
\] (4.44)

The larger is the value of \(d_6\), the more downside risk averse is the investor, and the greater is the downside risk premium. Note that due to the condition of mean-variance preservation in (4.42) and (4.43), \(\mu\) is fixed and identical to the two distributions. So \(d_6\) can only be affected by the difference in risk preference and not due to the different values of \(\mu\).

As we can see from the above, the new measure \(d_6\) differs from the other measures, \(d_1\) to \(d_5\), by explicitly including the mean \(\mu\) in the risk measure, whereas \(d_1\) to \(d_5\) only go so far as assuming that two risky assets have the same mean. It is easy to show that different values of \(\mu\) will result in a preference ordering by \(d_6\) that is different from those produced by \(d_1, d_2, d_3, d_4\) and \(d_5\), as shown in the
The importance of including the mean in the downside risk aversion measure is that, unlike with risk aversion, we need to know not only how great the downside risk is, but also where the increase in downside risk occurs. In particular, the location of the mean dictates whether the change in the probability measure falls into the critical region to which the investor is particularly averse. This in turn will affect his attitude towards the increase in downside risk. In the special case where $\mu = 0$, i.e. $X$ and $Y$ are pure risks, $d_6$ can be further simplified to

$$
\frac{d_6^*}{d_6} = -\frac{1}{u(w)} w''(w) u'(w)
$$

which will then give the same preference ordering as $d_2$.

### 4.4.2 Numerical Examples

As mentioned in the previous section, there are at least five different downside risk aversion measures in the existing literature, and in this chapter, we propose to use the new $d_6$, as defined in (4.44), to measure the intensity of downside risk aversion. The main difference between $d_6$ and the previous five measures is that it includes the mean of the risky assets in the definition. The comparisons of downside risk aversion produced by the previous five measures are not sensitive to the mean value. In the numerical examples below, the investors have initial wealth $w_0 = 100$. As before, we assume two investors, $u$ and $v$, and two risky assets, $X$ and $Y$, with $Y$ having more downside risk than $X$. Although the downside risk
differs, $X$ and $Y$ have the same mean and the same variance. In particular, $Y$ is derived from $X$ through the mean-variance-preserving transformations described in Section 4.2.1. The distribution parameters for $X$ and $Y$, with $\mu = +100$ and $-100$, are reported in Table 4.1, and the corresponding mean-variance-preserving transformations in Table 4.2.

We define utility functions $u$ and $v$ that take the form

$$u(w) = a_1 + b_1 w + c_1 w^2 + d_1 w^3$$
$$v(w) = a_2 + b_2 w + c_2 w^2 + d_2 w^3.$$

The parameter values for the utility functions are shown in Table 4.3.

The downside risk aversion measure $d_6$, and the risk premiums, calculated according to (4.38) and (4.39), are shown in Table 4.4. Table 4.4 clearly shows that investor $v$ is more downside risk averse than $u$ when the mean equals $+100$, but is less downside risk averse than $u$ when the mean equals $-100$. Hence, we have $\pi_u > \pi_v$ when $d_6(u) > d_6(v)$, and $\pi_u < \pi_v$ when $d_6(u) < d_6(v)$. This suggests that the measure $d_6$ is a good indicator of downside risk aversion.

The values of the downside risk aversion measures $d_1$ to $d_5$ for both $u$ and $v$ are shown in Table 4.5. Since $d_1$ to $d_5$ do not take the mean of the risky asset into consideration, the values of downside risk aversion measures are the same whether the mean is equal to $+100$ or $-100$. As can be seen in Table 4.5, since $d_1$ to $d_5$ are not sensitive to the mean, these five downside risk measures may produce incorrect preference ordering. For example, $u$ is always more downside
Table 4.1: Return Distributions of the Risky Assets X and Y

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th></th>
<th>$Y_1$</th>
<th></th>
<th>$X_2$</th>
<th></th>
<th>$Y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Probability</td>
<td>Outcome</td>
<td>Probability</td>
<td>Outcome</td>
<td>Probability</td>
<td>Outcome</td>
<td>Probability</td>
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<tr>
<td>0.2</td>
<td>0.2</td>
<td>87.81</td>
<td>0.2</td>
<td>100.02</td>
<td>0.2</td>
<td>-112.19</td>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>96.00</td>
<td>0.3</td>
<td>88.72</td>
<td>0.3</td>
<td>-104.00</td>
<td>0.3</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>107.28</td>
<td>0.5</td>
<td>106.76</td>
<td>0.5</td>
<td>-92.72</td>
<td>0.5</td>
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</tbody>
</table>

Mean, $\mu$

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th></th>
<th>$Y_1$</th>
<th></th>
<th>$X_2$</th>
<th></th>
<th>$Y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean, $\mu$</td>
<td>+100</td>
<td>+100</td>
<td>-100</td>
<td>-100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance, $\sigma^2$</td>
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<td>61</td>
<td>61</td>
<td>61</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Skewness</td>
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<td>-0.579</td>
<td>-0.397</td>
<td>-0.579</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The return distributions $X_1$ and $X_2$ (and similarly $Y_1$ and $Y_2$) differ by their means. Moments above the third order are not relevant under a cubic utility function.

Table 4.2: Mean-Variance-Preserving Transformation for X and Y

<table>
<thead>
<tr>
<th></th>
<th>$MPS_1$ for $X_1$ and $Y_1$</th>
<th>$MPC_1$ for $X_1$ and $Y_1$</th>
<th>$MPS_2$ for $X_2$ and $Y_2$</th>
<th>$MPC_2$ for $X_2$ and $Y_2$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Outcome</td>
<td>Probability</td>
<td>Outcome</td>
<td>Probability</td>
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<td>88.72</td>
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<td>0.3</td>
</tr>
<tr>
<td>-0.3</td>
<td>96.00</td>
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<td>-0.3</td>
</tr>
<tr>
<td>-0.324</td>
<td>100.02</td>
<td>0.176</td>
<td>106.76</td>
<td>-0.324</td>
</tr>
<tr>
<td>0.324</td>
<td>106.76</td>
<td>-0.5</td>
<td>107.28</td>
<td>0.324</td>
</tr>
</tbody>
</table>

Mean, $\mu$

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th></th>
<th>$Y_1$</th>
<th></th>
<th>$X_2$</th>
<th></th>
<th>$Y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean, $\mu$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: MPS and MPC denote mean-preserving spread and mean-preserving contraction, respectively. We can check that $MPS_1$ for $X_1$ and $Y_1$ and $MPS_2$ for $X_2$ and $Y_2$ meet conditions (4.3), (4.4), (4.5) and (4.6), while $MPC_1$ for $X_1$ and $Y_1$ and $MPC_2$ for $X_2$ and $Y_2$ meet the conditions (4.3), (4.4), (4.5) and (4.7). In addition, the combination of $MPS_1$ and $MPC_1$ and the combination of $MPS_2$ and $MPC_2$ both meet conditions (4.13), (4.14) and (4.15).
Table 4.3: Parameter Values for Utility Functions $u(w)$ and $v(w)$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0</td>
<td>$a_2$</td>
<td>0</td>
</tr>
<tr>
<td>$b_1$</td>
<td>10</td>
<td>$b_2$</td>
<td>15</td>
</tr>
<tr>
<td>$c_1$</td>
<td>-5</td>
<td>$c_2$</td>
<td>-10</td>
</tr>
<tr>
<td>$d_1$</td>
<td>1</td>
<td>$d_2$</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: The utility function $u(w)$ takes the form $u(w) = a_1 + b_1 w + c_1 w^2 + d_1 w^3$, and the utility function $v(w)$ takes the form $v(w) = a_2 + b_2 w + c_2 w^2 + d_2 w^3$. Both utility functions are increasing and concave in the support of $[-112.19, -92.72]$, though they are increasing and convex in the support of $[87.81, 107.28]$. However, as shown in previous sections, both risk lovers and risk averters can be downside risk averse.

Table 4.4: New Downside Risk Aversion Measure, $d_6$, and Associated Risk Premium

<table>
<thead>
<tr>
<th>Mean</th>
<th>$d_6(u)$</th>
<th>$\pi_u$</th>
<th>$d_6(v)$</th>
<th>$\pi_v$</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>+100</td>
<td>-14668</td>
<td>0.0024</td>
<td>-14336</td>
<td>0.0025</td>
<td>As $d_6(v) &gt; d_6(u)$, $\pi_v &gt; \pi_u$</td>
</tr>
<tr>
<td>-100</td>
<td>4998.33</td>
<td>0.4452</td>
<td>4997.5</td>
<td>0.4297</td>
<td>As $d_6(v) &lt; d_6(u)$, $\pi_v &lt; \pi_u$</td>
</tr>
</tbody>
</table>

Note: The table shows the downside risk aversion values for $u$ and $v$ according to the new downside risk aversion measure $d_6$ and their corresponding downside risk premiums. The values will differ when the mean of the risk changes.

Risk averse than $v$ according to $d_3$, but actually $u$ is less downside risk averse than $v$ when the mean is equal to +100. Another interesting observation is that, there is no consensus even among $d_1$ to $d_5$ themselves. For example, according to $d_1$ and $d_2$, $v$ is more downside risk averse than $u$ while according to $d_3$, $d_4$ and $d_5$, $u$ is more downside risk averse than $v$.

Table 4.5: Downside Risk Aversion Measures, $d_1$ to $d_5$

<table>
<thead>
<tr>
<th></th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>For $u$</td>
<td>0.010169</td>
<td>0.000207</td>
<td>-0.000414</td>
<td>0.5000</td>
<td>-0.000207</td>
</tr>
<tr>
<td>For $v$</td>
<td>0.010345</td>
<td>0.000214</td>
<td>-0.000429</td>
<td>0.4997</td>
<td>-0.000214</td>
</tr>
</tbody>
</table>

Note: This table shows the downside risk aversion values according to the downside risk aversion measures $d_1$ to $d_5$. Their values are the same whether the mean is equal to +100 or -100.
4.4.3 On the Search for a Global Measure

After deriving our version of a local downside risk aversion measure, one natural question is whether there is a corresponding global measure $D$ such that, for two investors $u$ and $v$, $u$ will always pay a higher risk premium for the increase in downside risk if and only if $D_u > D_v$ at all wealth levels. One possible way forward is to use a transformation function as in Pratt (1964) to derive the global risk aversion measure. Indeed Keenan and Snow (2002, 2009) show that $u$ is more downside risk averse than $v$ if and only if $v = \phi(u)$ and $\phi''' > 0$. However, as we have shown in Section 4.3.3, it is possible to find some transformation functions that will lead to $u$ and $v$ each having more downside risk averse than the other. The main difference between global risk aversion and global downside risk aversion is that Pratt’s global risk aversion measure deals with the risk premium for the introduction of a pure risk into a zero-risk base case (see Section 4.6.2, equation (4.55)). On the other hand, the downside risk premium for the increase in downside risk involves two scenarios with different levels of uncertainty (see equation (4.38)). Hence, Pratt’s transformation function approach is not a viable way forward. In the case of downside risk, perhaps it is more appropriate to follow Ross’s (1981) analysis of stronger risk aversion in which he considers the risk premium for an increase in risk. Modica and Scarsini (2005) follow this approach but at the cost of a very strong restriction in (4.25). Thus, unfortunately, until now no simple global downside risk measure has existed. The approach adopted in Crainich and Eeckhoudt (2008), and our approach, is to limit the analysis to a local context. Keenan and Snow (2002, 2009, 2010), on the other hand, resort to
constructing a global measure based on the induced or *ex post* utility distributions that are mean and variance preserving.

One element that most previous studies omit to consider is the interaction between risk aversion and downside risk aversion. For instance, Keenan and Snow’s transformation has third derivative,

\[ \vartheta'''(u) = \frac{v'(u)}{(u')^3} [d_2(v) - d_2(u)] - \frac{3r(u)v'(u)}{(u')^3} [r(v) - r(u)] \]

where \( d_2(v) = \frac{v''}{v'} \), \( d_2(u) = \frac{u''}{u'} \), \( r(v) = -\frac{v''}{v'} \), and \( r(u) = -\frac{u''}{u'} \). While we do not agree that \( \vartheta'''(u) > 0 \) is a sufficient condition for \( u \) being more downside risk averse than \( v \), the above expression clearly shows that the comparative downside risk aversion is connected to the level of risk aversion.

### 4.5 Conclusion

This chapter examines the previous literature on downside risk aversion and proposes a new downside risk aversion measure that differs from the previous five measures through the inclusion of the mean of the risky asset. We argue that this is necessary because, for downside risk aversion, we need to know not only how great the downside risk is, but also where the increase in downside risk occurs. To support our new measure, we provide numerical examples in which we compare investors’ local downside risk aversion. One natural extension of the research would be to derive a global downside risk aversion measure. However, as suggested by Keenan and Snow (2009), there might be no global measure that
can fully characterize the degree of downside risk aversion.

4.6 Appendix

4.6.1 Appendix A: Downside Risk & Downside Risk Aversion

4.6.1.1 Variance Preservation

To verify the variance-preserving property, start with the LHS of (4.14) and write $T^*(x) = 1 \cdot T^*(x)$, then

$$
\int_0^1 1 \cdot T^*(x) \, dx = x T^*(x) \big|_0^1 - \int_0^1 x T(x) \, dx. \quad (4.46)
$$

The first term of the RHS of (4.46) is zero because

$$
x T^*(x) \big|_0^1 = \int_0^1 T(x) \, dx - 0
= x T(x) \big|_0^1 - \int_0^1 x \cdot t(x) \, dx
= 0 - 0
$$
due to conditions (4.8) and (4.5).

Since $d \left( \frac{1}{2} x^2 \right) = x \, dx$, we can write the second term on the RHS of (4.46) as

$$
- \int_0^1 1 \cdot T(x) \, d \left( \frac{1}{2} x^2 \right) = - \frac{1}{2} x^2 T(x) \big|_0^1 + \int_0^1 \frac{1}{2} x^2 \cdot t(x) \cdot x \, dx = \frac{1}{2} \int_0^1 x^2 t(x) \, dx. \quad (4.46)
$$
Because, according to (4.14), \( \int_0^1 T^* (x) \, dx = 0 \), this means that

\[
\int_0^1 x^2 t (x) \, dx = 0. \tag{4.47}
\]

Then for \( g (x) = f (x) + t (x) \), and given that \( E_g = E_f = \mu \), we have

\[
V_g = V_f + \int_0^1 (x - \mu)^2 \, t (x) \, dx
\]

\[
= V_f + \int_0^1 x^2 t (x) \, dx - 2 \mu \int_0^1 x t (x) \, dx + \mu^2 \int_0^1 t (x) \, dx
\]

\[
= V_f
\]

according to (4.47), (4.5) and (4.8).

\section{4.6.1.2 Increasing Downside Risk}

For \( \int_0^y T^* (x) \, dx > 0 \), any MVPT, \( t (x) \), will shift the probability mass from the right tail to the left tail. To see this, write, as before, \( T^* (x) = 1 \cdot T^* (x) \), which gives

\[
\int_0^y T^* (x) \, dx = xT^* (x) \big|_0^y - \int_0^y xT (x) \, dx = yT^* (y) - \int_0^y xT (x) \, dx.
\]
Since $d\left(\frac{1}{2}x^2\right) = xdx$,

\[
\int_0^y T^*(x) \, dx = y \int_0^y T(x) \, dx - \left[ \frac{1}{2}x^2T(x) \right]_0^y - \frac{1}{2} \int_0^y x^2t(x) \, dx
\]
\[
= y \left[ xT(x) \right]_0^y - \int_0^y x \, t(x) \, dx - \frac{1}{2}y^2T(y) + \frac{1}{2} \int_0^y x^2t(x) \, dx
\]
\[
= \frac{1}{2}y^2T(y) - y \int_0^y x \, t(x) \, dx + \frac{1}{2} \int_0^y x^2t(x) \, dx
\]
\[
= \frac{1}{2} \left[ \int_0^y y^2t(x) \, dx - 2 \int_0^y yx \, t(x) \, dx + \int_0^y x^2t(x) \, dx \right]
\]
\[
= \frac{1}{2} \int_0^y (y - x)^2 t(x) \, dx
\]
\[
> 0.
\]

We note that from (4.14), it implies that for any $y \in [0, 1]$,

\[
\int_0^y T^*(x) \, dx = -\int_y^1 T^*(x) \, dx.
\]

If $\int_0^y T^*(x) \, dx > 0$, it must mean that $\int_y^1 T^*(x) \, dx < 0$ and

\[
\int_y^1 (y - x)^2 t(x) \, dx < 0.
\]

Thus, adding $t(x)$ to $f(x)$ increases the dispersion below $y$ while decreases the dispersion above $y$ by exactly the same amount, i.e. it shifts the probability weight from the right tail to the left tail.
4.6.1.3 Condition for Being Downside Risk Averse

Here we explain Menezes, Geiss and Tressler (1980)'s proof of Theorem 10, which states that $E_f u(x) \geq E_g u(x)$ for all $u(x)$ in $U^*_3 = \{u''' > 0\}$ if and only if $g(x)$ has more downside risk than $f(x)$, i.e. $g(x)$ and $f(x)$ satisfy the following conditions:

(i) $E_g = E_f$,
(ii) $\int_0^1 \int_0^y [G(y) - F(y)] dydz = 0$, and
(iii) $\int_0^x \int_0^z [G(y) - F(y)] dydz \geq 0$

First note that whether or not an asset distribution $f$ is preferred to $g$, i.e. $E_f u(x) \geq E_g u(x)$, can be assessed by considering $u$, $u'$, $u''$ or $u'''$

$$ E_f u(x) - E_g u(x) = \int_0^1 u(x) [f(x) - g(x)] dx $$

$$ = u(x) [F(x) - G(x)] \big|_0^1 - \int_0^1 u'(x) [F(x) - G(x)] dx $$

$$ = \int_0^1 u'(x) [G(x) - F(x)] dx \quad (4.48) $$

$$ = - \int_0^1 u''(x) \int_0^x [G(y) - F(y)] dydx \quad (4.49) $$

$$ = \int_0^1 u'''(x) \int_0^x \int_0^y [G(z) - F(z)] dzdy dx \quad (4.50) $$

To prove they are equal in mean, MGT use the following pair of utility functions:

$$ u_1(x) = \frac{1}{3} \theta x^3 + x, \quad \text{and} \quad u_2(x) = \frac{1}{3} \theta x^3 - x $$

where $\theta$ is a positive constant. We can see that $u'''_1 = u'''_2 = 2\theta > 0$. Substituting
$u'_1$ and $u'_2$ into (4.48) and taking the limit as $\theta \to 0$, we get

$$\lim_{\theta \to 0} [E_f u_1(x) - E_g u_1(x)] = \lim_{\theta \to 0} \int_0^1 \left( \theta x^2 + 1 \right) [G(x) - F(x)] \, dx$$

$$= \int_0^1 [G(x) - F(x)] \, dx$$  \hspace{1cm} (4.51)$$

$$\lim_{\theta \to 0} [E_f u_2(x) - E_g u_2(x)] = \lim_{\theta \to 0} \int_0^1 \left( \theta x^2 - 1 \right) [G(x) - F(x)] \, dx$$

$$= \int_0^1 - [G(x) - F(x)] \, dx. \hspace{1cm} (4.52)$$

As both (4.51) and (4.52) should be larger than or equal to zero, $\int_0^1 [G(x) - F(x)] \, dx = E_f u - E_g u$ must be equal to zero, which is condition (i).

To prove the variance-preserving property, consider another pair of utility functions:

$$u_3(x) = \frac{1}{3} \theta x^3 + \frac{1}{2} x^2, \quad \text{and} \quad u_4(x) = \frac{1}{3} \theta x^3 - \frac{1}{2} x^2.$$  \hspace{1cm}

Again, $u'''_3 = u'''_4 = 2\theta > 0$. Substituting $u'''_3$ and $u'''_4$ into (4.49) and taking the limit as $\theta \to 0$, we get

$$\lim_{\theta \to 0} [E_f u_3(x) - E_g u_3(x)] = \lim_{\theta \to 0} \int_0^1 (2\theta x + 1) \int_0^x [G(z) - F(z)] \, dz \, dx$$

$$= \int_0^1 \int_0^x [G(z) - F(z)] \, dz \, dx \hspace{1cm} (4.53)$$
\[
\lim_{\theta \to 0} [E_f u_4(x) - E_g u_4(x)] = \lim_{\theta \to 0} \int_0^1 (2\theta x - 1) \int_0^x [G(z) - F(z)] \, dz \, dx
\]
\[
= -\int_0^1 \int_0^x [G(z) - F(z)] \, dz \, dx.
\]
(4.54)

Again, as both (4.53) and (4.54) should be larger than or equal to zero, \( \int_0^1 \int_0^x [G(z) - F(z)] \, dz \, dx = V_g - V_f \) must be equal to zero which is condition (ii).

Finally, the third condition is proved by contradiction. Suppose condition (iii) is false and that \( \int_0^x \int_0^y [G(z) - F(z)] \, dz \, dy < 0 \) for some \( x \in [\alpha, \beta] \). Next consider the utility function

\[
\begin{align*}
u_5(x) &= \begin{cases} 
\frac{1}{6} \theta x^3 & \text{for } x \in [\alpha, \beta] \\
\frac{1}{6} x^3 & \text{otherwise}
\end{cases}
\end{align*}
\]
with \( u'''_5 = \theta > 0 \) for \( x \in [\alpha, \beta] \) and 1 otherwise. Substituting \( u'''_5 \) into (4.50) and taking the limit as \( \theta \to 0 \), we get

\[
\lim_{\theta \to 0} [E_f u_5(x) - E_g u_5(x)] = \lim_{\theta \to 0} \int_0^1 \int_0^x \int_0^y [G(z) - F(z)] \, dz \, dy \, dx < 0
\]

which contradicts the original assumption that \( E_f u(x) \geq E_g u(x) \). Therefore, for \( E_f u(x) \geq E_g u(x) \), condition (iii) must be met.

For the sufficient part, if conditions (i), (ii) and (iii) are met, then for any utility function with \( u''' > 0 \), we have \( E_f u(x) \geq E_g u(x) \) from (4.50).
4.6.2 Appendix B: Pratt’s Risk Aversion in the Large

Pratt (1964)’s Theorem 1 proves, with the use of a transformation function, $u_1(u_2^{-1}(t))$, that the risk aversion measure, $r(x)$, is a global risk measure for pure risk. The proof involves demonstrating that the following conditions are equivalent:\footnote{Let the transformation function $f(t) = u_1(u_2^{-1}(t))$. If $t = u_2(x)$, then $x = u_2^{-1}(t)$ and $f(u_2(x)) = u_1(x)$.}

(i) $r_1(x) \geq r_2(x)$.

(ii) $\pi_1(x, \tilde{z}) \geq \pi_2(x, \tilde{z})$.

(iii) $u_1(u_2^{-1}(t))$ is a concave function of $t$.

To show that (ii) is equivalent to (iii), let us assume that $x$ is a certain amount of wealth, and $\tilde{z}$ is a pure risk such that $E(\tilde{z}) = 0$. Then

$$Eu(x + \tilde{z}) = u(x + E(\tilde{z}) - \pi). \tag{4.55}$$

For the case of investor 1, take $u_1^{-1}$ on both sides of (4.55) to give:

$$\pi_1 = x + E(\tilde{z}) - u_1^{-1}(Eu_1(x + \tilde{z})).$$

Repeat the same process for investor 2 and take the difference of the two risk premia,

$$\pi_1 - \pi_2 = u_2^{-1}(Eu_2(x + \tilde{z})) - u_1^{-1}(Eu_1(x + \tilde{z})).$$
Define $\tilde{t} = u_2(x + \tilde{z})$ and $u_2^{-1}(\tilde{t}) = x + \tilde{z}$. Then we have

$$\pi_1 - \pi_2 = u_2^{-1}(E(\tilde{t})) - u_1^{-1}(Eu_1(u_2^{-1}(\tilde{t}))).$$  \hfill (4.56)

From condition (ii) $\pi_1 \geq \pi_2$, the RHS of (4.56) must be greater than or equal to zero. Re-arranging and taking $u_1$ of both sides gives:

$$Eu_1(u_2^{-1}(\tilde{t})) \leq u_1(u_2^{-1}[E(\tilde{t})]).$$

Hence, by Jensen’s inequality, $u_1(u_2^{-1}(\tilde{t}))$ must be a concave function satisfying condition (iii).

To show that (i) and (iii) are equivalent, first note that

$$\frac{d}{dt}u_1(u_2^{-1}(t)) = u_1'(u_2^{-1}(t)) \frac{d}{dt}u_2^{-1}(t) = \frac{u_1'(u_2^{-1}(t))}{u_2(u_2^{-1}(t))}.$$  \hfill (4.57)

From condition (i),

$$r_2(x) - r_1(x) = \frac{d}{dx}\log \frac{u_1'(x)}{u_2(x)} \leq 0.$$  \hfill (4.58)

Since the logarithmic transformation does not change the sign of the derivative,

\footnote{Let $f(f^{-1}(x)) = x$. Differentiating both sides with respect to $x$ gives $f'(f^{-1}(x))(f^{-1})'(x) = 1$ or $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. Now if we write $f(x)$ as $u_2(t)$, we get $(u_2^{-1})'(t) = \frac{1}{u_2(u_2^{-1}(t))}$.}

\footnote{Note that $d\log \frac{u_1'}{u_2} = \frac{u_1''}{u_1} \left[ u_2 \right]' = \frac{u_1}{u_2} \frac{u_1'' - u_1'u_2'' + u_1u_2''}{u_2} = \frac{u_1}{u_2} \left[ \frac{u_1''}{u_2} - \frac{u_1'u_2''}{u_2^2} \right] = \frac{u_1''}{u_2} - \frac{u_1'u_2''}{u_2^2} = r_2 - r_1$.}
we have

\[ \frac{d}{dx} \frac{u_1'(x)}{u_2'(x)} \leq 0. \]

Substituting \( x = u_2^{-1}(t) \) and making a change of variable, we get

\[ \frac{d}{dt} \frac{u_1'(u_2^{-1}(t))}{u_2'(u_2^{-1}(t))} \leq 0. \]

Substituting the result into (4.57) gives

\[ \frac{d^2}{dt^2} u_1(u_2^{-1}(t)) \leq 0. \]

Hence, the transformation function, \( u_1(u_2^{-1}(t)) \), must be concave.

Since conditions (ii) and (iii) are equivalent, and (i) and (iii) are also equivalent, this must mean that (i) and (ii) are equivalent. Hence, greater risk aversion will attract a greater risk premium.
Chapter 5

Downside Risk Aversion and Option Price Bounds

5.1 Introduction

Traditionally, option pricing under Black and Scholes (1973) assumes a dynamically complete market in which options can be priced and continuously hedged under no arbitrage. In a separate branch of the literature, Merton (1973), Rubinstein (1976) and Brennan (1979) show that in a single period setting, the Black-Scholes model holds if the underlying asset has a lognormal distribution and the representative investor exhibits constant relative risk aversion (CRRA). Since then, attention has focused on the pricing of options when the assumption of CRRA utility does not hold. For example, Theorem 1 in Franke, Stapleton and Subrahmanyam (1999) (hereafter FSS) shows that, if the market portfolio
has a lognormal payoff, the Black-Scholes model holds if and only if the representative investor has a CRRA utility. If she has a utility that exhibits declining relative risk aversion (DRRA), however, she will price options higher than the Black-Scholes formula. This result is proved in Poon and Stapleton (2005). This chapter extends FSS’s Theorem 1 and links the rate of decline in the elasticity of the pricing kernel, $\nu'$, to the representative investor’s risk aversion and downside risk aversion.

Several studies in utility theory define and analyse the property of downside risk aversion. Downside risk was defined in Menezes, Geiss and Tressler (1980) and measures of downside risk aversion were analysed in Kimball (1990), Chiu (2005), Modica and Scarsini (2005), Eeckhoudt and Schlesinger (2006), Keenan and Snow (2002, 2009, 2010) and Liu and Meyer (2012). Downside risk aversion is an important concept which has been shown to explain the demand for portfolio insurance (Leland (1980)), the demand for options (Huang and Stapleton (2013)), and the demand for self protection (Chiu (2005)). According to Menezes, Geiss, and Tressler (1980), a downside risk averse agent has $u'''' > 0$. While it is commonly accepted that the Pratt (1964) and Arrow (1963) absolute risk aversion, $a = -u''/u'$, is a good measure for risk aversion, no less than five downside risk aversion measures have been proposed in the literature. This includes Kimball (1990)’s and Chiu (2005)’s measure for prudence, defined as $p = -u''''/u''$, which explains the demand for precautionary saving and self protection. Modica and Scarsini (2005) show that $d = u''''/u'$ is related to the premium an agent is willing to pay to avoid downside risk. On the other hand, Leland (1980) and Huang
and Stapleton (2013) show that ‘relative prudence’ or cautiousness measured by 
\( c = \frac{(p - a)}{a} \), is related to the demand for portfolio insurance and options re-
respectively.\(^1\)

An agent is risk averse if \( u'' < 0 \) and downside risk averse if \( u''' > 0 \). In this chapter, we show that the price of an option relative to the price of the 
underlying asset depends on \( u'' \) and on \( u''' \). However, since both \( u'' \) and \( u''' \) change simultaneously, it is the relative change in downside risk aversion compared to 
risk aversion which determines the direction and magnitude of the option price 
change. Moreover, as the representative investor becomes more risk averse or more 
downside risk averse, the forward price will also change. Hence, in the numerical 
example, we have made a static comparison between the pricing results of two 
representative investor economies, \( u \) and \( v \), calibrated to give the same forward 
price of the underlying asset before comparing the option prices associated with 
their utility functions.

The remainder of this chapter is organized as follows. Section 5.2 presents 
and extends Franke, Stapleton and Subrahmanyam (1999)’s Theorem 1 on the 
relationship between the rate of decline in the elasticity of the pricing kernel and 
the option price. This is a crucial extension as the comparison must be made very 
general and applicable to any two pricing kernels, and not just to pricing kernels 
with constant and declining elasticity as in the case of the original FSS’s Theorem 
1. Section 5.3 analyses the relations between the elasticity of the pricing kernel

\(^1\)Keenan and Snow (2002, 2009, 2010) use a different framework for measuring the compensated increase in downside risk. Their compensated increase in downside risk is different from Menezes, Geiss and Tressler (1980)’s original definition in that the mean variance preservation is based on the utility distribution instead of the distribution of the terminal wealth.
and measures of risk preferences. In particular, Section 5.3.1 analyses the impacts of different risk measures and how, together, they affect option price. Section 5.3.2 establishes the conditions for $\nu_1' > \nu_2'$ with the use of a transformation function. Section 5.4 gives the conditions for $\nu_1' > \nu_2'$ for some special utility functions, and the HARA case was used specifically in the numerical examples in Section 5.5. Finally, Section 5.6 concludes.

5.2 Elasticity of the Pricing Kernel and Option Price

Franke, Stapleton and Subrahmanyam (1999)’s main focus is on the elasticity of the pricing kernel which is defined as

$$\nu(S_T) = -\frac{\partial \phi_{t,T}}{\phi_{t,T}} \frac{\partial S_T}{S_T}$$

where $\phi_{t,T}$ is the pricing kernel and $S_T$ is the asset price. They show that for two pricing kernels $\phi_1$ and $\phi_2$ that yield the same forward price of the risky asset, if $\phi_1$ exhibits constant elasticity while $\phi_2$ exhibits declining elasticity, then the price of any European-style option is greater under $\phi_2$ than under $\phi_1$. Furthermore, they show that for a pricing kernel with constant elasticity and lognormal price of underlying asset, the Black-Scholes formula correctly prices options. However, if the pricing kernel exhibits declining elasticity, the Black-Scholes formula always underprices options.
We extend FSS’s (1999) Theorem 1 on pricing kernels with constant and declining elasticity to any two pricing kernels in Lemma 1 and Lemma 2 below. Next, by combining the rate of decline of the elasticity of pricing kernels and risk preferences, we analyse the relationship between (downside) risk aversion and the option price in Proposition 2 and Proposition 3. From these, we infer the general relations between utility functions and option prices.

**Lemma 1.** *(Intersections of Pricing Kernels)* Consider two pricing kernels, \( \phi_1 \) and \( \phi_2 \), each of which yields the same forward price \( F^* \).\(^3\) Suppose that the elasticity of \( \phi_2 \) declines at a faster rate than \( \phi_1 \) at all asset price levels, i.e. \( 0 \geq \nu'_1 > \nu'_2, \forall S_T \). Then the pricing kernels \( \phi_1 \) and \( \phi_2 \) intersect twice.

**Proof:**

The proof here first establishes that \( \phi_1 \) and \( \phi_2 \) will intersect, and intersect more than once. We then show that \( \phi_1 \) and \( \phi_2 \) cannot intersect three times or more.\(^4\) First, \( \phi_1 \) and \( \phi_2 \) intersect at least once or it will be impossible for the condition

\[
E (\phi_1) = E (\phi_2) = 1
\]

(5.1)

to hold because if, say, \( \phi_1 \) always lies above \( \phi_2 \), then \( E (\phi_1) > E (\phi_2) \) and this contradicts (5.1). Furthermore, the two pricing kernels \( \phi_1 \) and \( \phi_2 \) must intersect more than once. If they only intersect once, say at \( S_T = \hat{S}_T \), then, if we also

---

\(^2\)Although not necessary in the proof, the pricing kernels are normally assumed to be monotonically decreasing.

\(^3\)Specifically, this refers to the \( T \)-period forward price. For simplicity, the time subscript is omitted.

\(^4\)The original proofs are in FSS (1999) and Poon and Stapleton (2005). The first part of the proof here is the same as in the original papers. Here, we include a new result in the second part of the proof.
suppose $\phi_1 > [<] \phi_2$ for $S_T < [>] \hat{S}_T$, it will lead to

$$
(S_T - \hat{S}_T) (\phi_2 - \phi_1) \geq 0, \ \forall S_T.
$$

For a claim paying $(S_T - \hat{S}_T)$ at time $T$,

$$
E \left[ (S_T - \hat{S}_T) \phi_2 \right] > E \left[ (S_T - \hat{S}_T) \phi_1 \right]
$$

$$
E [S_T \phi_2] - \hat{S}_T > E [S_T \phi_1] - \hat{S}_T
$$

$$
E [S_T \phi_2] > E [S_T \phi_1].
$$

This contradicts the condition that the forward price must be the same under the two pricing kernels. Similarly, a contradiction can be found under the condition $\phi_1 < [>] \phi_2$ for $S_T < [>] \hat{S}_T$. Thus $\phi_1$ and $\phi_2$ intersect more than once.

For the second part of the proof, we have $\phi_1$ and $\phi_2$ intersect three times or more with the first three intersections at $S^A_T$, $S^B_T$ and $S^C_T$ ($S^A_T < S^B_T < S^C_T$). Then

$$
\phi_1 (S^A_T) = \phi_2 (S^A_T)
$$

$$
\phi_1 (S^B_T) = \phi_2 (S^B_T)
$$

$$
\phi_1 (S^C_T) = \phi_2 (S^C_T).
$$

Suppose $\phi_2$ intersects $\phi_1$ from above at $S^A_T$,

$$
- \frac{\partial \phi_1 (S^A_T)}{\partial S_T} < - \frac{\partial \phi_2 (S^A_T)}{\partial S_T}.
$$
Then

\[ \nu_1 (S_T^A) = -\frac{\partial \phi_1 (S_T^A)}{\partial S_T} \cdot \frac{S_T^A}{\phi_1 (S_T)} < -\frac{\partial \phi_2 (S_T^A)}{\partial S_T} \cdot \frac{S_T^A}{\phi_2 (S_T)} = \nu_2 (S_T^A) \]

\[ \nu_1 (S_T^A) - \nu_2 (S_T^A) < 0. \quad (5.2) \]

Similarly \( \phi_2 \) intersects \( \phi_1 \) from below at \( S_T^B \) and \( \phi_2 \) intersects \( \phi_1 \) from above at \( S_T^C \)

\[ \nu_1 (S_T^B) - \nu_2 (S_T^B) > 0 \quad (5.3) \]

\[ \nu_1 (S_T^C) - \nu_2 (S_T^C) < 0. \quad (5.4) \]

According to condition \( 0 \geq \nu_1' > \nu_2', \forall S_T \),

\[ \int_{S_T^B}^{S_T^C} \nu_1' d\eta_1 > \int_{S_T^B}^{S_T^C} \nu_2' d\eta_2 \]

\[ \nu_1 (S_T^C) - \nu_1 (S_T^B) > \nu_2 (S_T^C) - \nu_2 (S_T^B) \]

\[ \nu_1 (S_T^C) - \nu_2 (S_T^C) > \nu_1 (S_T^B) - \nu_2 (S_T^B). \]

This contradicts (5.3) and (5.4).

On the other hand, if \( \phi_2 \) intersects \( \phi_1 \) from below at \( S_T^a \), it will lead to

\[ -\frac{\partial \phi_1 (S_T^A)}{\partial S_T} > -\frac{\partial \phi_2 (S_T^A)}{\partial S_T}. \]

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Then

\[\nu_1(S_T^A) = -\frac{\partial \phi_1(S_T^A)}{\partial S_T} \cdot \frac{S_T^A}{\phi_1(S_T)} > -\frac{\partial \phi_2(S_T^A)}{\partial S_T} \cdot \frac{S_T^A}{\phi_2(S_T)} = \nu_2(S_T^A)\]

\[\nu_1(S_T^A) - \nu_2(S_T^A) > 0 \quad (5.5)\]
\[\nu_1(S_T^B) - \nu_2(S_T^B) < 0 \quad (5.6)\]
\[\nu_1(S_T^C) - \nu_2(S_T^C) > 0. \quad (5.7)\]

According to condition \(0 \geq \nu'_1 > \nu'_2, \forall S_T,\)

\[\int_{S_T^A}^{S_T^B} \nu'_1 d\eta_1 > \int_{S_T^A}^{S_T^B} \nu'_2 d\eta_2\]

\[\nu_1(S_T^B) - \nu_1(S_T^A) > \nu_2(S_T^B) - \nu_2(S_T^A)\]
\[\nu_1(S_T^B) - \nu_2(S_T^B) > \nu_1(S_T^A) - \nu_2(S_T^A)\]

which contradicts (5.5) and (5.6). Therefore, the pricing kernels \(\phi_1\) and \(\phi_2\) cannot intersect three times or more. Since we have already shown that the pricing kernels \(\phi_1\) and \(\phi_2\) must intersect, but they cannot intersect only once, we arrive at the conclusion that they must intersect exactly twice. Q.E.D.

One interesting fact to note is that whether \(\phi_1\) and \(\phi_2\) are increasing, constant or decreasing will not affect the proof. The only necessary assumptions are that they yield the same forward price \(F^*\), and that \(\nu'_1 > \nu'_2\) for all \(S_T\), i.e. the

\[\text{For two pricing kernels that intersect exactly twice, the reverse is true if } \phi_2 \text{ intersects } \phi_1 \text{ from above.}\]
elasticity of $\phi_2$ declines at a faster rate than $\phi_1$ at all asset price levels. When these conditions are met, all different shapes of elasticities of pricing kernels can readily be compared. Lemma 1, together with Lemma 2 below, are the core theories for the initial setup of this chapter.

**Lemma 2.** *(The Pricing of European-Style Options)* Consider two pricing kernels, $\phi_1$ and $\phi_2$, both of which yield the same forward price for the risky asset. Suppose that the elasticity of $\phi_2$ declines at a faster rate than $\phi_1$ at all asset price levels, i.e. $0 \geq \nu'_1 > \nu'_2, \forall S_T$. Then, the price of any European-style call option is greater under pricing kernel $\phi_2$ than $\phi_1$.

**Proof:**

The proof is similar to the proof of Theorem 1 in FSS (1999). Suppose $\phi_2$ intersects $\phi_1$ from above at $S_T^A$, and then from below at $S_T^B$, then

$$\phi_1 < \phi_2 \text{ for } S_T < S_T^A$$

$$\phi_1 > \phi_2 \text{ for } S_T^A < S_T < S_T^B$$

$$\phi_1 < \phi_2 \text{ for } S_T^B < S_T.$$  

For a European call option on $S_T$ with strike price at $k$, its payoff should be

$$(S_T - k)^+.$$  

Then by choosing $a_k$ and $b_k$, there should be a line $L_k(S_T) = a_k + b_k S_T$ that intersects with $(S_T - k)^+$ at $S_T = S_T^A$ and $S_T = S_T^B$. Then the forward price of
the call option under $\phi_1$ should be

$$C_{k,1} = E \left[ (S_T - k)^+ \phi_1 \right]$$

$$= E \left[ ((S_T - k)^+ - L_k (S_T)) \phi_1 \right] + E \left[ L_k (S_T) \phi_1 \right]$$

and similarly

$$C_{k,2} = E \left[ ((S_T - k)^+ - L_k (S_T)) \phi_2 \right] + E \left[ L_k (S_T) \phi_2 \right].$$

Since

$$E \left[ L_k (S_T) \phi_1 \right] = E \left[ (a_k + b_k S_T) \phi_1 \right]$$

$$= E [a_k \phi_1] + E [b_k S_T \phi_1]$$

$$= a_k E [\phi_1] + b_k E [S_T \phi_1]$$

$$= a_k + b_k E [S_T \phi_1]$$

and

$$E \left[ L_k (S_T) \phi_2 \right] = E \left[ (a_k + b_k S_T) \phi_2 \right]$$

$$= E [a_k \phi_2] + E [b_k S_T \phi_2]$$

$$= a_k E [\phi_2] + b_k E [S_T \phi_2]$$

$$= a_k + b_k E [S_T \phi_1]$$

and since $\phi_1$ and $\phi_2$ will yield the same forward price for the risky assets i.e.
\[ E\left[ S_T \phi_1 \right] = E\left[ S_T \phi_2 \right], \]
we have
\[ E\left[ L_k \left( S_T \right) \phi_1 \right] = E\left[ L_k \left( S_T \right) \phi_2 \right], \]
and this will lead to
\[ C_{k,2} - C_{k,1} = E \left[ \left( \left( S_T - k \right)^+ - L_k \left( S_T \right) \right) \left( \phi_2 - \phi_1 \right) \right]. \]

Since
\[ \left( S_T - k \right)^+ - L_k \left( S_T \right) > 0 \quad \text{for } \phi_2 > \phi_1 \]
and
\[ \left( S_T - k \right)^+ - L_k \left( S_T \right) < 0 \quad \text{for } \phi_2 < \phi_1 \]
it must be the case that
\[ C_{k,2} > C_{k,1}. \]
Q.E.D.
5.3 Elasticity of the Pricing Kernel and Preference

Here we analyse the general relationship between the utility functions of representative investors and option prices through the elasticity of the pricing kernel. By definition, the elasticity of the pricing kernel is

$$\nu = -\frac{\partial (\frac{u'}{Eu'})}{\partial w} \frac{w}{u'/Eu'}$$

for pricing kernel $\phi = u'/Eu'$. Differentiating $\nu$ with respect to $w$ gives us the rate of change of the elasticity of pricing kernel as

$$\nu'(w) = -w \left[ \frac{u''u'}{w'u'} - \frac{u''u''}{u'} \right] - \frac{u''}{u'}$$

$$= -w \left[ d - a^2 \right] + a$$

$$= wa^2 + a - wd$$ \hspace{1cm} (5.9)

where $d = \frac{u''}{u'}$ is the downside risk aversion measure (following Modica and Scarsini (2005)) and $a = -\frac{u''}{u'}$ is the risk aversion measure. As the expression for $\nu'$ in (5.8) consists of only $w$, $u'$, $u''$ and $u'''$, according to Lemma 1 and Lemma 2 the option prices are only affected by the representative investor’s non-satiation ($u' > 0$), aversion to risk ($u'' < 0$) and aversion to downside risk ($u''' > 0$).
5.3.1 Changes in Risk Measures

For a representative investor who is downside risk neutral with $u''' = 0$, $\nu'$ in (5.9) becomes

$$\nu' (w) = wa^2 + a.$$  

This means that, everything else being equal, the greater the risk aversion, $a$, the greater is $\nu'$, and the smaller is the option price.

However, if $u''' \neq 0$, the rate of decline of the elasticity of the pricing kernel with respect to wealth level changes is jointly determined by $a$ and $d$ as shown below:

$$\frac{d\nu'}{dw} = \frac{d[wa^2 + a - wd]}{dw} = \left[ 2aw \frac{da}{dw} + a^2 + \frac{da}{dw} - d - w \frac{dd}{dw} \right]$$

$$= (2aw + 1) \frac{da}{dw} - w \frac{dd}{dw} + (a^2 - d).$$

Since

$$(a^2 - d) = \frac{u'' - u'''u'}{u'^2} = \frac{da}{dw},$$

we have

$$\frac{d\nu'}{dw} = (2aw + 2) \frac{da}{dw} - w \frac{dd}{dw}. \quad (5.10)$$

Therefore according to Lemma 2 and (5.10), a sufficient condition for $\frac{d\nu'}{dw} < 0$ and
the representative investor to produce a higher option price is that

\[ [2aw + 2] \frac{da}{dw} < 0 \quad \text{and} \quad w \frac{dd}{dw} > 0. \]

Such conditions will be met when the representative investor has decreasing absolute risk aversion \( \frac{da}{dw} < 0 \), increasing downside risk aversion \( \frac{dd}{dw} > 0 \), and also relative risk aversion \( aw > -1 \).

**Proposition 2.** Given the same forward price, a representative investor will give a higher option price if she has a greater decrease in absolute risk aversion, a greater increase in downside risk aversion and that her relative risk aversion is larger than \(-1\). Such an investor will give an option price greater than the Black-Scholes option price.

We can re-cast proposition 2 in terms of cautiousness, which is defined as the first derivative of risk tolerance

\[ c = \left( -\frac{u'}{u''} \right)' = \frac{u'' u' - u'''^2}{u'^2} = \frac{d}{a^2} - 1. \]
Then from (5.9),

\[
\frac{d\nu'}{dw} = d\left[wa^2 + a - wa^2(1 + c)\right] = d\left[a - cwa^2\right] = \frac{da}{dw} - a^2c - 2awc \frac{da}{dw} - a^2w c \frac{dc}{dw} = (1 - 2awc) \frac{da}{dw} - a^2c - a^2w c \frac{dc}{dw}.
\]

Since

\[
a^2c = \frac{u'^{n2}u''' - u'^{n2}}{u'^2} = \frac{u''u' - u'^{n2}}{u'^2} = -\frac{da}{dw}
\]

we have

\[
\frac{d\nu'}{dw} = (2 - 2awc) \frac{da}{dw} - a^2w c \frac{dc}{dw}
\]

Therefore, according to Lemma 2 and (5.11), the sufficient condition for \( \frac{d\nu'}{dw} < 0 \) and the representative investor to give the option a higher price is that

\[
(2 - 2awc) \frac{da}{dw} < 0 \quad \text{and} \quad a^2w c \frac{dc}{dw} > 0.
\]

Such conditions will be met when the representative investor has decreasing absolute risk aversion \( \frac{da}{dw} < 0 \), increasing cautiousness \( \frac{dc}{dw} > 0 \), and also that \( awc < 1 \).

**Proposition 3.** Given the same forward price, a representative investor will give a higher option price if she has a greater decrease in absolute risk aversion, a
greater increase in cautiousness and that her relative risk aversion and cautiousness meet the condition that \( awc < 1 \). Such an investor will give an option price greater than the Black-Scholes option price.
5.3.2 Transformation Conditions for $\nu'_v > \nu'_u$

Here we compare the option prices for two representative investor economies $u$ and $v$. As we can see from Lemma 2, when they yield the same forward price for the risky asset, investor $v$ will yield a lower option price than investor $u$ if $\nu'_v > \nu'_u$. That is, if

$$-w \left[ \frac{v'''v' - v''v''}{v'v'} \right] - \frac{v''}{v'} > -w \left[ \frac{u'''u' - u''u''}{u'u'} \right] - \frac{u''}{u'}. \quad (5.12)$$

Define a transformation function $s$ for $u$ and $v$ as $v = s(u)$. In the equations below, we write $s(u)$ as $s$ for simplicity:

$$v' = s'u' \quad (5.13)$$

$$v'' = s''u'^2 + s'u'' \quad (5.14)$$

$$v''' = s'''u'^3 + 3s''u'u'' + s'u'''. \quad (5.15)$$

Putting (5.13), (5.14) and (5.15) into (5.12), we then have

$$-w \left[ \frac{v'''}{v'} - \left( \frac{v''}{v'} \right)^2 \right] - \frac{v''}{v'} + w \left[ \frac{u'''}{u'} - \left( \frac{u''}{u'} \right)^2 \right] + \frac{u''}{u'} > 0$$

$$-w \left[ \frac{s'''u'^2}{s'} + 3s''u''}{s'} + \frac{u''}{u'} - \left( \frac{s''u'}{s'} + \frac{u''}{u'} \right)^2 \right]$$

$$- \left( \frac{s''u'}{s'} + \frac{u''}{u'} \right) + w \left[ \frac{u'''}{u'} - \left( \frac{u''}{u'} \right)^2 \right] + \frac{u''}{u'} > 0$$

$$-w \left[ \frac{s'''u'^2 + s''u''}{s'} - \frac{s''u'^2}{s'^2} \right] - \frac{s''u'}{s'} > 0. \quad (5.16)$$
Assuming that $u' > 0$ and $u'' < 0$, the sufficient condition for (5.16) to be true is that $s' > 0$, $s'' < 0$, and

$$s'''u'^2 + s''u'' < 0$$

which is equivalent to

$$[s'(u)]'' < 0.$$

We therefore have the following:

**Proposition 4.** Consider the case with two representative investor economies $u$ and $v$ that yield the same forward price for the risky asset. Define their transformation function $s$ as $v = s(u)$ and suppose $u' > 0$ and $u'' < 0$. Then the sufficient condition for $v$ to yield a lower option price than $u$ is that $s'(u) > 0$, $s''(u) < 0$, and $[s'(u)]'' < 0$.

### 5.4 Special Utility Functions

Some interesting results can be obtained if the representative investor has the special types of utility functions described below.

**Proposition 5.** Assume representative investors $u$ and $v$ with exponential utility
functions

\[ u(w) = -\exp(-aw) \]
\[ v(w) = -\exp(-bw) \]

where \( a, b \geq 0 \). Then, for the same forward price, the price of any European-style option is greater under \( u \) than \( v \) if and only if \( a < b \).

Proof:

According to (5.8),

\[ \nu'_u(w) = -[a^2 - a^2] w + a = a \]
\[ \nu'_v(w) = -[b^2 - b^2] w + b = b. \]

Then, from Lemma 2, the European option price is greater under \( u \) than \( v \) if and only if \( \nu'_u(w) < \nu'_v(w) \), which is the same as \( a < b \). Q.E.D.

Proposition 6. Assume representative investors \( u \) and \( v \) with HARA utility functions

\[ u(w) = \frac{1}{B_1 - 1} (A_1 + B_1w)^{1-\frac{1}{\beta_1}} \]  \hspace{1cm} (5.17)
\[ v(w) = \frac{1}{B_2 - 1} (A_2 + B_2w)^{1-\frac{1}{\beta_2}} \]  \hspace{1cm} (5.18)

where \( B_1, B_2 \neq 0,1 \) and \( A_i + B_iw > 0 \) for \( i = 1,2 \). Then, for the same forward
price, the European option price is greater under \( u \) than \( v \) if and only if

\[
\frac{A_1}{(A_1 + B_1 w)^2} < \frac{A_2}{(A_2 + B_2 w)^2}
\]

(5.19)

at all wealth levels.

Proof:

From (5.8) and (5.17),

\[
\nu_u'(w) = - \left[ (B_1 + 1)(A_1 + B_1 w)^{-2} - (A_1 + B_1 w)^{-2} \right] w - (A_1 + B_1 w)^{-1}
\]

\[
= - \frac{B_1 w}{(A_1 + B_1 w)^2} + \frac{1}{(A_1 + B_1 w)}
\]

\[
= \frac{A_1}{(A_1 + B_1 w)^2}.
\]

Similarly, for investor \( v \),

\[
\nu_v'(w) = \frac{A_2}{(A_2 + B_2 w)^2}.
\]

Then, according to Lemma 2, the European option price is greater under \( u \) than \( v \) if and only if \( \nu_u'(w) < \nu_v'(w) \), or

\[
\frac{A_1}{(A_1 + B_1 w)^2} < \frac{A_2}{(A_2 + B_2 w)^2}.
\]

Q.E.D.
5.5 Numerical Examples

This section presents some numerical examples to illustrate the findings in the previous sections. We assume here that the terminal stock price follows a uniform distribution on $[10, 19]$, and set the strike price for both call and put options are set to be 15. We assume also that representative investors $u$ and $v$ have HARA utilities in (5.17) and (5.18) with $A_1 = -8.975$, $B_1 = 2$, $A_2 = 0$ and $B_2 = \frac{4}{3}$. The forward price for the stock is set at $14.059$. We note that investor $v$ has pricing kernel with constant elasticity while investor $u$ has pricing kernel with declining elasticity. Table 5.1 reports the risk aversion, downside risk aversion, cautiousness, pricing kernel, change in elasticity of pricing kernel and option prices for representative investors $u$ and $v$.

It is clear from Table 5.1 that investor $u$ yields higher call and put option prices than investor $v$. This supports the result in Lemma 2 that, when $\nu'_u < \nu'_v$ on all wealth levels, $u$ will yield the higher option price. It also supports Proposition 6, as condition (5.19) is met at all wealth levels. When comparing the risk aversion, downside risk aversion and cautiousness values of $u$ and $v$, we do not observe a monotonic pattern, which reflects the complex interactions that occur between them in affecting option price; none of the risk measures alone gives a definitive change in option price. On the other hand, the rate of decline of the pricing kernel $\nu'$ dictates whether the option price will be higher or lower.

---

6We first use $A_2$ and $B_2$ to calculate the forward price for investor $v$. Then given $B_1 = 2$, the value of $A_1$ is calibrated such that investor $u$ will produce the same forward price.
Table 5.1: Representative Investor $u$ with Declining Elasticity of the Pricing Kernel

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>$\phi_u$</th>
<th>$d_u$</th>
<th>$a_u$</th>
<th>$c_u$</th>
<th>$\nu'_u$</th>
<th>$\phi_u (S_T - K)^+$</th>
<th>$\phi_u (\overline{S}_T - S_T)^+$</th>
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<td>0.0041</td>
<td>0.0370</td>
<td>2</td>
<td>-0.0122</td>
<td>2.4957</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>0.8027</td>
<td>0.0035</td>
<td>0.0344</td>
<td>2</td>
<td>-0.0106</td>
<td>3.2110</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: Representative investor $u$ has HARA utility function with $A_1 = -8.975$ and $B_1 = 2$. The forward price for the stock is set to be 14.059. We first use $A_1$ and $B_1$ to calculate the forward price for investor $v$. Then, given $B_2$, the value of $A_2$ is calibrated such that investor $u$ will produce the same forward price. The asset price, $S_T$, follows a uniform distribution on $[10, 19]$, and the strike price for both call and put options is set at 15. $\phi$ is the pricing kernel, $d$ the downside risk aversion value, $a$ the risk aversion value, $c$ the cautiousness value, $\nu'$ the change in elasticity of the pricing kernel, $\phi_u (S_T - K)^+$ and $\phi_v (S_T - K)^+$ the call option payoff times the pricing kernel and $\phi_u (K - S_T)^+$ and $\phi_v (K - S_T)^+$ the put option payoff times the pricing kernel. We calculate the call option prices by taking expectations on the $\phi_u (S_T - K)^+$ and $\phi_v (S_T - K)^+$ columns respectively and calculate the put option prices by taking expectations on the $\phi_u (K - S_T)^+$ and $\phi_v (K - S_T)^+$ columns respectively.

Table 5.2: Representative Investor $v$ with Constant Elasticity of the Pricing Kernel

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>$\phi_v$</th>
<th>$d_v$</th>
<th>$a_v$</th>
<th>$c_v$</th>
<th>$\nu'_v$</th>
<th>$\phi_v (S_T - K)^+$</th>
<th>$\phi_v (\overline{S}_T - S_T)^+$</th>
<th>$a_u - a_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.2861</td>
<td>0.0131</td>
<td>0.0750</td>
<td>1.3333</td>
<td>0</td>
<td>6.4305</td>
<td>-0.0011</td>
<td>-0.0157</td>
</tr>
<tr>
<td>11</td>
<td>1.1973</td>
<td>0.0108</td>
<td>0.0681</td>
<td>1.3333</td>
<td>0</td>
<td>4.7895</td>
<td>-0.0068</td>
<td>-0.0086</td>
</tr>
<tr>
<td>12</td>
<td>1.1217</td>
<td>0.0091</td>
<td>0.0625</td>
<td>1.3333</td>
<td>0</td>
<td>3.3652</td>
<td>-0.0041</td>
<td>-0.0040</td>
</tr>
<tr>
<td>13</td>
<td>1.0563</td>
<td>0.0077</td>
<td>0.0576</td>
<td>1.3333</td>
<td>0</td>
<td>2.1127</td>
<td>-0.0025</td>
<td>-0.0010</td>
</tr>
<tr>
<td>14</td>
<td>0.9992</td>
<td>0.0066</td>
<td>0.0535</td>
<td>1.3333</td>
<td>0</td>
<td>0.9992</td>
<td>-0.0015</td>
<td>0.0010</td>
</tr>
<tr>
<td>15</td>
<td>0.9488</td>
<td>0.0058</td>
<td>0.0500</td>
<td>1.3333</td>
<td>0</td>
<td>0</td>
<td>-0.0009</td>
<td>0.0024</td>
</tr>
<tr>
<td>16</td>
<td>0.9040</td>
<td>0.0051</td>
<td>0.0468</td>
<td>1.3333</td>
<td>0</td>
<td>0.9040</td>
<td>0</td>
<td>-0.0005</td>
</tr>
<tr>
<td>17</td>
<td>0.8638</td>
<td>0.0045</td>
<td>0.0411</td>
<td>1.3333</td>
<td>0</td>
<td>1.7277</td>
<td>0</td>
<td>-0.0002</td>
</tr>
<tr>
<td>18</td>
<td>0.8276</td>
<td>0.0040</td>
<td>0.0346</td>
<td>1.3333</td>
<td>0</td>
<td>2.4828</td>
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</tr>
<tr>
<td>19</td>
<td>0.7947</td>
<td>0.0036</td>
<td>0.0394</td>
<td>1.3333</td>
<td>0</td>
<td>3.1788</td>
<td>0</td>
<td>0.0007</td>
</tr>
</tbody>
</table>

Note: Representative investor $v$ has HARA utility function with $A_2 = 0$ and $B_2 = 4/3$. The forward price for the stock is set to be 14.059. We first use $A_1$ and $B_1$ to calculate the forward price for investor $v$. Then, given $B_2$, the value of $A_2$ is calibrated such that investor $u$ will produce the same forward price. The asset price, $S_T$, follows a uniform distribution on $[10, 19]$, and the strike price for both call and put options is set at 15. $\phi$ is the pricing kernel, $d$ the downside risk aversion value, $a$ the risk aversion value, $c$ the cautiousness value, $\nu'$ the change in elasticity of the pricing kernel, $\phi_u (S_T - K)^+$ and $\phi_v (S_T - K)^+$ the call option payoff times the pricing kernel and $\phi_u (K - S_T)^+$ and $\phi_v (K - S_T)^+$ the put option payoff times the pricing kernel. We calculate the call option prices by taking expectations on the $\phi_u (S_T - K)^+$ and $\phi_v (S_T - K)^+$ columns respectively and calculate the put option prices by taking expectations on the $\phi_u (K - S_T)^+$ and $\phi_v (K - S_T)^+$ columns respectively.
5.6 Conclusion

In this chapter, the effect of high order risk preferences on option prices is analysed. First we build the relationship between the rate of decline of the elasticity of the pricing kernel and the option prices, and then we connect this to the relationship between risk preference and the elasticity of the pricing kernel. The results here support the intuition that, everything else being equal, the option price increases as the individual becomes more downside risk averse and more cautious, and lower as he becomes more risk averse. However, as the risk measures are not independent from each other, none of the risk measures alone can give a definitive change in the option price. On the other hand, the rate of decline of the pricing kernel $\nu'$ gives a clear indication of whether the option will be priced higher or lower.
Chapter 6

Conclusion

This thesis begins with an introductory chapter that outlines the main ideas in each chapter and the main contributions of the thesis. In the literature review chapter, I first discuss the original expected utility theory and further developments that have been made to it, with a special focus on background risk and myopia/partial myopia. I then review the downside risk aversion literature and the existing downside risk aversion measures. The chapter also includes a discussion on the pricing kernel and its relation to option pricing and option price bounds.

Chapter 3 examines Mossin’s work on myopia and partial myopia investment policies as alternatives to the dynamic programming solution. My research makes a contribution towards a better understanding of the partial myopia policy in the presence of background risk. Although dynamic programming is theoretically the optimal method, partial myopia is very simple to apply and places little reliance
on future returns information. My numerical examples show that, for a HARA investor, partial myopia could be optimal even in the presence of background risks, provided certain conditions are met. As future returns are often unknown in real life, especially those in the distant future, future research could examine the potential efficiency loss due to making incorrect assumptions about future returns.

Chapter 4 examines the previous literature on downside risk aversion and proposes a new local downside risk aversion measure. The new measure differs from existing measures by including the mean of the risky asset in the definition. While it is based on a previously existing framework and is restricted by its local property, the new measure reflects individuals’ downside risk aversion better than the measures provided in the literature. Future research could investigate its global property. However, to date, the overwhelming view is that there is no global measure that can fully characterize the degree of downside risk aversion.

Chapter 5 analyses option pricing based on the representative investor’s preferences towards risks. It starts by extending the lemmas in Franke, Stapleton and Subrahmanyam (1999) concerning the relationship between the elasticity of the pricing kernel and the option price. My extension has a much wider application in the sense that it allows the option pricing impacts under any two pricing kernels to be readily compared instead of restricting the comparison to constant and declining elasticities of pricing kernels. In addition, although previous research such as Bernado and Ledoit (2000) has already shown the importance of pricing ker-
nels in option pricing, I move a step further by examining the connection with the elasticity of the pricing kernel, and therefore gain a better understanding of how risk preferences, such as downside risk aversion, affect the option price through the pricing kernel.
Appendix

Table 6.1: Index of Theorems, Lemmas and Propositions

<table>
<thead>
<tr>
<th>Theorem 1</th>
<th>on page 30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2</td>
<td>on page 31</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>on page 40</td>
</tr>
<tr>
<td>Theorem 4</td>
<td>on page 40</td>
</tr>
<tr>
<td>Theorem 5</td>
<td>on page 42</td>
</tr>
<tr>
<td>Theorem 6</td>
<td>on page 45</td>
</tr>
<tr>
<td>Theorem 7</td>
<td>on page 68</td>
</tr>
<tr>
<td>Theorem 8</td>
<td>on page 76</td>
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<tr>
<td>Theorem 9</td>
<td>on page 100</td>
</tr>
<tr>
<td>Theorem 10</td>
<td>on page 100</td>
</tr>
<tr>
<td>Theorem 11</td>
<td>on page 105</td>
</tr>
<tr>
<td>Theorem 12</td>
<td>on page 118</td>
</tr>
<tr>
<td>Lemma 1</td>
<td>on page 145</td>
</tr>
<tr>
<td>Lemma 2</td>
<td>on page 149</td>
</tr>
<tr>
<td>Proposition 1</td>
<td>on page 107</td>
</tr>
<tr>
<td>Proposition 2</td>
<td>on page 154</td>
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<tr>
<td>Proposition 3</td>
<td>on page 155</td>
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<tr>
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<td>on page 158</td>
</tr>
<tr>
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<td>on page 158</td>
</tr>
<tr>
<td>Proposition 6</td>
<td>on page 159</td>
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<tr>
<td>Notation</td>
<td>Description</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
</tr>
<tr>
<td>$w$</td>
<td>wealth/initial wealth</td>
</tr>
<tr>
<td>$S$</td>
<td>spot price of the asset</td>
</tr>
<tr>
<td>$z$</td>
<td>wealth/accumulated wealth</td>
</tr>
<tr>
<td>$t/T$</td>
<td>time period/option maturity</td>
</tr>
<tr>
<td>$\dot{w}$</td>
<td>random wealth</td>
</tr>
<tr>
<td>$u,v$</td>
<td>utility functions</td>
</tr>
<tr>
<td>$v$</td>
<td>indirect utility function</td>
</tr>
<tr>
<td>$\pi$</td>
<td>risk premium</td>
</tr>
<tr>
<td>$r^*$</td>
<td>optimal choice</td>
</tr>
<tr>
<td>$\tilde{x}$</td>
<td>excess return of the risky asset</td>
</tr>
<tr>
<td>$\delta$</td>
<td>risk (probability distribution)</td>
</tr>
<tr>
<td>$\tilde{y}/\overline{y}$</td>
<td>additive background risk</td>
</tr>
<tr>
<td>$\tilde{q}/\overline{q}$</td>
<td>multiplicative background risk</td>
</tr>
<tr>
<td>$X,Y$</td>
<td>risky assets/portfolios</td>
</tr>
<tr>
<td>$\alpha/b_1$</td>
<td>wealth invested in the risky asset in the first period</td>
</tr>
<tr>
<td>$\beta/b_2$</td>
<td>wealth invested in the risky asset in the second period</td>
</tr>
<tr>
<td>$s$</td>
<td>saving</td>
</tr>
<tr>
<td>$\mu$</td>
<td>mean</td>
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<td>$\sigma$</td>
<td>standard deviation</td>
</tr>
<tr>
<td>$\varepsilon/\hat{\varepsilon}$</td>
<td>error term</td>
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<tr>
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<td>precautionary premium</td>
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<tr>
<td>$Pr/p$</td>
<td>probability</td>
</tr>
<tr>
<td>$f(x)/g(x)$</td>
<td>probability density function</td>
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<tr>
<td>$C$</td>
<td>call option price</td>
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<tr>
<td>$P$</td>
<td>put option price</td>
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