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Efficient expressions for moments of dependent random sums using copulas

by

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Abstract: Mao and Zhao [IMA Journal of Management Mathematics, 25, 2014, 421-433] derived the mean and variance of a dependent random sum, where the dependence was specified by a copula. Here, we derive expressions for the general moment of the dependent random sum. We also extend Mao and Zhao's results to the case that the components of the sum are not identically distributed. **The practical usefulness of the results (in terms of computational time and computational accuracy) is examined by simulation.**

Keywords: Farlie-Gumbel-Morgenstern copula; Insurance; Inter-arrival time; Poisson process

1 Introduction

Sums of a random number of random variables arise in many areas of the sciences and engineering. A prominent example is the total insurance claim over a fixed period. In this example, the number is the number of insurance claims and the variables are the amounts claimed. A related variable is the inter-arrival time between insurance claims.

There is a vast amount of literature on the distributions and moments of the random sums. Many of the papers suppose that the claims are independently and identically distributed conditioned on the number of claims. They suppose no relationship between the amounts claimed and the inter-arrival times between insurance claims. This may not be a realistic assumption. Recently, researchers have supposed that the amounts claimed and the inter-arrival times are dependent, but are marginally independently and identically distributed conditioned on the number of claims. Some papers based on this assumption are Albrecher and Teugels (2006), Asimit and Jones (2008a), Cossette et al. (2008), Marceau (2008), Ambagaspitiya (2009), Asimit and Badescu (2010) and Chueng et al. (2010). See also Asimit and Jones (2008b), Asimit et al. (2014a, 2014b), Asimit and Chen (2015) and Asimit et al. (2016).

A most recent of these papers is by Mao and Zhao (2014). They derived the first and second moments of the total insurance claim by supposing that the joint distribution of amounts claimed and the inter-arrival times is specified by a copula. They specialized their results to one of the simplest known copulas, the Farlie-Gumbel-Morgenstern (FGM) copula due to Morgenstern (1956), see also Nelsen (2006).

In this note, we extend Mao and Zhao (2014)'s results. We derive the general moment of the total insurance claim. Our results hold for a wide range of copulas not just the FGM copula. We also extend Mao and Zhao (2014)'s results to the case where the identical assumption does not hold.

In management science and related areas, higher order moments are of interest, not just the mean and variance. Examples include: portfolio selection (Harvey et al., 2010); value-at-risk forecasting (Polanski and Stoja, 2010); market risk assessment (Sihem and Slaheddine, 2014).

The contents of this note are organized as follows. Section 2 derives the general moment of the random sums, under the assumptions given in Mao and Zhao (2014). Section 3 extends the results of Section 2 to the case that the inter-arrival times between claims are independent but not identically distributed, the amounts claimed are independent but not identically distributed, and the copulas of the joint distributions are not identical. **Section 4 performs a simulation study to examine the practical values of the results in Section 2.**

2 Main results

In this section, we suppose $N(t)$ is a renewal process with inter arrival times W_i , and (X_i, W_i) are independent and identically distributed with common copula function C , common copula density c , common joint probability density function $f_{X,W}$, common joint cumulative distribution function $F_{X,W}$, common marginal probability density functions f_X , f_W , and common marginal cumulative distribution functions F_X , F_W .

Let $S_t = X_1 + \dots + X_{N(t)}$. Theorem 1 derives the k th moment of S_t . It generalizes Theorems 3.1 and 4.1 in Mao and Zhao (2014). These theorems derived the first and second moments of S_t .

Theorem 1 *The k th moment of S_t can be expressed as*

$$\begin{aligned} E(S_t^k) = & \sum_{n=1}^{\infty} \sum_{k_1+\dots+k_n=k} \binom{k}{k_1 \dots k_n} \left\{ \int_{t-w_{i_1}-\dots-w_{i_m}>0} F^{*(n-m)}(t-w_{i_1}-\dots-w_{i_m}) \right. \\ & \cdot \prod_{j=1}^m [g_{X^{k_{i_j}}}(w_{i_j}) f_W(w_{i_j}) dw_{i_j}] \\ & - \int_0^t \int_{t-y-w_{i_1}-\dots-w_{i_m}>0} F^{*(n-m)}(t-y-w_{i_1}-\dots-w_{i_m}) \\ & \cdot \prod_{j=1}^m [g_{X^{k_{i_j}}}(w_{i_j}) f_W(w_{i_j}) dw_{i_j}] f_W(y) dy \Big\}, \end{aligned}$$

where

$$g_{X^k}(w) = \int_0^{+\infty} x^k f_X(x) c(F_X(x), F_W(w)) dx, \quad (1)$$

and $k_{i_1}, \dots, k_{i_m}$ denote the positive values among $k_1, \dots, k_n$ and $w_{i_1}, \dots, w_{i_m}$ are the corresponding ws.

Proof: Set $B_n = \{N(t) = n\} = \{W_1 + \dots + W_n \leq t\} - \{W_1 + \dots + W_{n+1} \leq t\}$ and $b_n = \{w_1 + \dots + w_n \leq t\} - \{w_1 + \dots + w_{n+1} \leq t\}$. Let I_A denote the indicator function, that is

$I_A = 1$ if A is true and $I_A = 0$ if A is false. By the multinomial theorem,

$$\begin{aligned}
E[S_t^k] &= E[(X_1 + \dots + X_{N(t)})^k] \\
&= E\left\{E[(X_1 + \dots + X_{N(t)})^k \mid N(t)]\right\} \\
&= \sum_{n=1}^{\infty} E[(X_1 + \dots + X_n)^k \mid N(t) = n] \Pr(N(t) = n) \\
&= \sum_{n=1}^{\infty} E\left[\sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \prod_{t=1}^n X_t^{k_t} \mid N(t) = n\right] \Pr(N(t) = n) \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} E\left[\prod_{t=1}^n X_t^{k_t} \mid N(t) = n\right] \Pr(N(t) = n) \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} E\left[I_{B_n} \prod_{t=1}^n X_t^{k_t}\right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int \dots \int_{b_n} \left\{ \prod_{t=1}^m \int x_{i_t}^{k_{i_t}} f_{X,W}(x_{i_t}, w_{i_t}) dx_{i_t} \right\} \\
&\quad \cdot \left[\prod_{j=1, j \neq i_1, \dots, i_m}^{n+1} f_W(w_j) dw_j \right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int \dots \int_{b_n} \left\{ \prod_{t=1}^m \int x_{i_t}^{k_{i_t}} c(F_X(x_{i_t}), F_W(w_{i_t})) f_X(x_{i_t}) dx_{i_t} \right\} \\
&\quad \cdot \left[\prod_{j=1}^{n+1} f_W(w_j) dw_j \right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int \dots \int_{b_n} \left\{ \prod_{t=1}^m g_{X^{k_{i_t}}}(w_{i_t}) \right\} \left[\prod_{j=1}^{n+1} f_W(w_j) dw_j \right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int_{w_1 + \dots + w_n \leq t} \left\{ \prod_{t=1}^m g_{X^{k_{i_t}}}(w_{i_t}) \right\} \left[\prod_{j=1}^{n+1} f_W(w_j) dw_j \right] \\
&\quad - \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int_{w_1 + \dots + w_{n+1} \leq t} \left\{ \prod_{t=1}^m g_{X^{k_{i_t}}}(w_{i_t}) \right\} \left[\prod_{j=1}^{n+1} f_W(w_j) dw_j \right]. \quad (2)
\end{aligned}$$

The two integrals in (2) reduce to

$$\int_{t-w_{i_1}-\dots-w_{i_m}>0} F^{*(n-m)}(t-w_{i_1}-\dots-w_{i_m}) \prod_{j=1}^m [g_{X^{k_{i_j}}}(w_{i_j}) f_W(w_{i_j}) dw_{i_j}]$$

and

$$\int_0^t \int_{t-y-w_{i_1}-\dots-w_{i_m}>0} F^{*(n-m)}(t-y-w_{i_1}-\dots-w_{i_m}) \prod_{j=1}^m [g_{X^{k_{i_j}}}(w_{i_j}) f_W(w_{i_j}) dw_{i_j}] f_W(y) dy \Big\}.$$

Hence, the result.

Corollaries 1 and 2 specialize Theorem 1 for $k = 3, 4$. These corollaries can be used to compute among others the skewness and kurtosis of S_t .

Corollary 1 *The third moment of S_t can be expressed as*

$$\begin{aligned}
E(S_t^3) &= 3 \sum_{n=1}^{\infty} \int_0^t F^{*(n-1)}(t-w) g_{X^3}(w) f_W(w) dw \\
&+ 9 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} F^{*(n-2)}(t-w_1-w_2) g_X(w_1) g_{X^2}(w_2) f_W(w_1) f_W(w_2) dw_2 dw_1 \\
&+ 9 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} F^{*(n-2)}(t-w_1-w_2) g_{X^2}(w_1) g_X(w_2) f_W(w_1) f_W(w_2) dw_2 dw_1 \\
&+ 6 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} \int_0^{t-w_1-w_2} F^{*(n-3)}(t-w_1-w_2) \prod_{j=1}^3 [g_X(w_j) f_W(w_j)] dw_3 dw_2 dw_1 \\
&- 3 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} F^{*(n-1)}(t-y-w) g_{X^3}(w) f_W(w) f_W(y) dw dy \\
&- 9 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} F^{*(n-2)}(t-y-w_1-w_2) g_X(w_1) g_{X^2}(w_2) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(y) dw_2 dw_1 dy \\
&- 9 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} F^{*(n-2)}(t-y-w_1-w_2) g_{X^2}(w_1) g_X(w_2) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(y) dw_2 dw_1 dy \\
&- 6 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} \int_0^{t-y-w_1-w_2} F^{*(n-3)}(t-y-w_1-w_2-w_3) \\
&\quad \cdot \prod_{j=1}^3 [g_X(w_j) f_W(w_j)] f_W(y) dw_3 dw_2 dw_1 dy.
\end{aligned}$$

Corollary 2 *The fourth moment of S_t can be expressed as*

$$\begin{aligned}
E(S_t^4) &= 4 \sum_{n=1}^{\infty} \int_0^t F^{*(n-1)}(t-w) g_{X^4}(w) f_W(w) dw \\
&+ 36 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} F^{*(n-2)}(t-w_1-w_2) g_{X^2}(w_1) g_{X^2}(w_2) f_W(w_1) f_W(w_2) dw_2 dw_1 \\
&+ 24 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} F^{*(n-2)}(t-w_1-w_2) g_X(w_1) g_{X^3}(w_2) f_W(w_1) f_W(w_2) dw_2 dw_1 \\
&+ 24 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} F^{*(n-2)}(t-w_1-w_2) g_{X^3}(w_1) g_X(w_2) f_W(w_1) f_W(w_2) dw_2 dw_1 \\
&+ 48 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} \int_0^{t-w_1-w_2} F^{*(n-3)}(t-w_1-w_2-w_3) g_{X^2}(w_1) g_X(w_2) g_X(w_3) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(w_3) dw_3 dw_2 dw_1 \\
&+ 48 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} \int_0^{t-w_1-w_2} F^{*(n-3)}(t-w_1-w_2-w_3) g_X(w_1) g_{X^2}(w_2) g_X(w_3) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(w_3) dw_3 dw_2 dw_1 \\
&+ 48 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} \int_0^{t-w_1-w_2} F^{*(n-3)}(t-w_1-w_2-w_3) g_X(w_1) g_X(w_2) g_{X^2}(w_3) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(w_3) dw_3 dw_2 dw_1 \\
&+ 24 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-w_1} \int_0^{t-w_1-w_2} \int_0^{t-w_1-w_2-w_3} F^{*(n-4)}(t-w_1-w_2-w_3-w_4) \\
&\quad \cdot \prod_{j=1}^4 [g_X(w_j) f_W(w_j)] dw_4 dw_3 dw_2 dw_1 \\
&- 4 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} F^{*(n-1)}(t-y-w) g_{X^4}(w) f_W(w) f_W(y) dw dy \\
&- 36 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} F^{*(n-2)}(t-y-w_1-w_2) g_{X^2}(w_1) g_{X^2}(w_2) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(y) dw_2 dw_1 dy \\
&- 24 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} F^{*(n-2)}(t-y-w_1-w_2) g_X(w_1) g_{X^3}(w_2) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(y) dw_2 dw_1 dy \\
&- 24 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} F^{*(n-2)}(t-y-w_1-w_2) g_{X^3}(w_1) g_X(w_2) \\
&\quad \cdot f_W(w_1) f_W(w_2) f_W(y) dw_2 dw_1 dy
\end{aligned}$$

$$\begin{aligned}
& -48 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} \int_0^{t-y-w_1-w_2} F^{*(n-3)}(t-y-w_1-w_2-w_3) \\
& \quad \cdot g_{X^2}(w_1) g_X(w_2) g_X(w_3) \\
& \quad \cdot f_W(w_1) f_W(w_2) f_W(w_3) f_W(y) dw_3 dw_2 dw_1 dy \\
& -48 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} \int_0^{t-y-w_1-w_2} F^{*(n-3)}(t-y-w_1-w_2-w_3) \\
& \quad \cdot g_X(w_1) g_{X^2}(w_2) g_X(w_3) \\
& \quad \cdot f_W(w_1) f_W(w_2) f_W(w_3) f_W(y) dw_3 dw_2 dw_1 dy \\
& -48 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} \int_0^{t-y-w_1-w_2} F^{*(n-3)}(t-y-w_1-w_2-w_3) \\
& \quad \cdot g_X(w_1) g_X(w_2) g_{X^2}(w_3) \\
& \quad \cdot f_W(w_1) f_W(w_2) f_W(w_3) f_W(y) dw_3 dw_2 dw_1 dy \\
& -24 \sum_{n=1}^{\infty} \int_0^t \int_0^{t-y} \int_0^{t-y-w_1} \int_0^{t-y-w_1-w_2} \int_0^{t-y-w_1-w_2-w_3} F^{*(n-4)}(t-y-w_1-w_2-w_3-w_4) \\
& \quad \cdot \prod_{j=1}^4 [g_X(w_j) f_W(w_j)] f_W(y) dw_4 dw_3 dw_2 dw_1 dy.
\end{aligned}$$

Corollary 3 specializes Theorem 1 to the case that $N(t)$ is a Poisson process with rate parameter λ .

Corollary 3 Suppose $\{N(t)\}$ is a Poisson process with rate parameter λ . Then the k th moment of S_t can be expressed as

$$\begin{aligned}
E(S_t^k) &= \sum_{n=1}^{\infty} \sum_{k_1+\dots+k_n=k} \binom{k}{k_1 \dots k_n} \frac{\lambda^m}{(n-m)!} \\
& \cdot \left\{ \int_{t-w_{i_1}-\dots-w_{i_m}>0} \gamma(n-m, \lambda(t-w_{i_1}-\dots-w_{i_m})) \right. \\
& \quad \cdot \exp\left(-\lambda \sum_{j=1}^m w_{i_j}\right) \prod_{j=1}^m [g_{X^{k_{i_j}}}(w_{i_j}) dw_{i_j}] \\
& - \lambda \int_0^t \int_{t-y-w_{i_1}-\dots-w_{i_m}>0} \gamma(n-m, \lambda(t-y-w_{i_1}-\dots-w_{i_m})) \\
& \quad \cdot \exp\left(-\lambda y - \lambda \sum_{j=1}^m w_{i_j}\right) \prod_{j=1}^m [g_{X^{k_{i_j}}}(w_{i_j}) dw_{i_j}] dy \Big\},
\end{aligned}$$

where $g_{X^k}(\cdot)$ is given by (1), $\gamma(a, x)$ denotes the incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} \exp(-t) dt,$$

and $k_{i_1}, \dots, k_{i_m}$ denote the positive values among $k_1, \dots, k_n$ and $w_{i_1}, \dots, w_{i_m}$ are the corresponding w s.

Nadarajah (2015) showed that a wide range of bivariate copulas (including the FGM copula) can be expressed as

$$c(u, v) = \sum_{i=1}^n \alpha_i a_i b_i u^{a_i-1} v^{b_i-1} \quad (3)$$

for $n \geq 1$ an integer and $\{(\alpha_i, a_i, b_i) : i \geq 1\}$ some real numbers. Using (3), we can write

$$g_{X^k}(w) = \sum_{i=1}^n \alpha_i a_i b_i F_W^{b_i-1}(w) \int_0^{+\infty} x^k f_X(x) F_X^{a_i-1}(x) dx. \quad (4)$$

If a_i are positive integers and if $X_1, \dots, X_{a_i}$ are independent and identical copies of X then (4) can be further rewritten as

$$g_{X^k}(w) = \sum_{i=1}^n \alpha_i b_i F_W^{b_i-1}(w) E \left[Z_i^k \right],$$

where $Z_i = \max(X_1, \dots, X_{a_i})$.

3 An extension

Theorem 2 extends Theorem 1 to the case that $(X_1, W_1), (X_2, W_2), \dots$ are independent but not identical. We suppose that (X_i, W_i) has copula function $C^{(i)}$, copula density $c^{(i)}$, joint probability density function f_{X_i, W_i} , joint cumulative distribution function F_{X_i, W_i} , marginal probability density functions f_{X_i} , f_{W_i} , and marginal cumulative distribution functions F_{X_i} , F_{W_i} .

Theorem 2 *The k th moment of S_t can be expressed as*

$$\begin{aligned} E(S_t^k) = & \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \left\{ \int_{t-w_{i_1}-\dots-w_{i_m} > 0} F^{*(n-m)}(t-w_{i_1}-\dots-w_{i_m}) \right. \\ & \cdot \prod_{j=1}^m \left[g_{X^{k_{i_j}}}^{(i_j)}(w_{i_j}) f_{W_{i_j}}(w_{i_j}) dw_{i_j} \right] \\ & - \int_0^t \int_{t-y-w_{i_1}-\dots-w_{i_m} > 0} F^{*(n-m)}(t-y-w_{i_1}-\dots-w_{i_m}) \\ & \cdot \prod_{j=1}^m \left[g_{X^{k_{i_j}}}^{(i_j)}(w_{i_j}) f_{W_{i_j}}(w_{i_j}) dw_{i_j} \right] f_{W_{n+1}}(y) dy \Big\}, \end{aligned}$$

where

$$g_{X^k}^{(i)}(w) = \int_0^{+\infty} x^k f_{X_i}(x) c^{(i)}(F_{X_i}(x), F_{W_i}(w)) dx,$$

and $k_{i_1}, \dots, k_{i_m}$ denote the positive values among $k_1, \dots, k_n$ and $w_{i_1}, \dots, w_{i_m}$ are the corresponding w s.

Proof: Define B_n , b_n and I_A as in the proof of Theorem 1. By the multinomial theorem,

$$\begin{aligned}
E[S_t^k] &= E[(X_1 + \dots + X_{N(t)})^k] \\
&= \sum_{n=1}^{\infty} E[(X_1 + \dots + X_n)^k | N(t) = n] \Pr(N(t) = n) \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} E\left[\prod_{t=1}^n X_t^{k_t} \middle| N(t) = n\right] \Pr(N(t) = n) \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} E\left[I_{B_n} \prod_{t=1}^n X_t^{k_t}\right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int \dots \int_{b_n} \left\{ \prod_{t=1}^m \int x_{i_t}^{k_{i_t}} f_{X_{i_t}, W_{i_t}}(x_{i_t}, w_{i_t}) dx_{i_t} \right\} \\
&\quad \cdot \left[\prod_{j=1, j \neq i_1, \dots, i_m}^{n+1} f_{W_j}(w_j) dw_j \right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int \dots \int_{b_n} \left\{ \prod_{t=1}^m \int x_{i_t}^{k_{i_t}} c^{(i_t)}(F_{X_{i_t}}(x_{i_t}), F_{W_{i_t}}(w_{i_t})) f_{X_{i_t}}(x_{i_t}) dx_{i_t} \right\} \\
&\quad \cdot \left[\prod_{j=1}^{n+1} f_{W_j}(w_j) dw_j \right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int \dots \int_{b_n} \left\{ \prod_{t=1}^m g_{X^{k_{i_t}}}^{(i_t)}(w_{i_t}) \right\} \left[\prod_{j=1}^{n+1} f_{W_j}(w_j) dw_j \right] \\
&= \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int_{w_1 + \dots + w_n \leq t} \left\{ \prod_{t=1}^m g_{X^{k_{i_t}}}^{(i_t)}(w_{i_t}) \right\} \left[\prod_{j=1}^{n+1} f_{W_j}(w_j) dw_j \right] \\
&\quad - \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n = k} \binom{k}{k_1 \dots k_n} \int_{w_1 + \dots + w_{n+1} \leq t} \left\{ \prod_{t=1}^m g_{X^{k_{i_t}}}^{(i_t)}(w_{i_t}) \right\} \left[\prod_{j=1}^{n+1} f_{W_j}(w_j) dw_j \right]. \tag{5}
\end{aligned}$$

The two integrals in (5) reduce to

$$\int_{t-w_{i_1}-\dots-w_{i_m}>0} F^{*(n-m)}(t-w_{i_1}-\dots-w_{i_m}) \prod_{j=1}^m \left[g_{X^{k_{i_j}}}^{(i_j)}(w_{i_j}) f_{W_{i_j}}(w_{i_j}) dw_{i_j} \right]$$

and

$$\int_0^t \int_{t-y-w_{i_1}-\dots-w_{i_m}>0} F^{*(n-m)}(t-y-w_{i_1}-\dots-w_{i_m}) \prod_{j=1}^m \left[g_{X^{k_{i_j}}}^{(i_j)}(w_{i_j}) f_{W_{i_j}}(w_{i_j}) dw_{i_j} \right] f_{W_{n+1}}(y) dy \Big\}.$$

Hence, the result.

4 Simulation

In this section, we examine computational efficiency of the expressions derived in Section 2. Computational efficiency is assessed in terms of time and accuracy.

Suppose $N(t)$ is a Poisson process with rate parameter λ , X_i are independent and identical exponential random variables with rate parameter λ and (X_i, W_i) have the common copula function $C(u, v) = uv + \theta uv(1 - u)(1 - v)$, the FGM copula function. We computed $E(S_t^k)$ by simulation and using Theorem 1 for every $k = 1, 2, \dots, 100$. The simulation was performed as follows:

1. simulate N from a Poisson distribution with parameter λ ;
2. simulate $(U_1, V_1), \dots, (U_N, V_N)$ from the FGM copula;
3. set $X_i = -\frac{1}{\lambda} \log(1 - U_i)$ and $W_i = -\frac{1}{\lambda} \log(1 - V_i)$ for $i = 1, \dots, N$;
4. compute $S_t^k = (X_1 + \dots + X_N)^k$;
5. repeat steps 1 to 4 one million times;
6. compute the average of the values of S_t^k .

$E(S_t^k)$ was also computed using Theorem 1 for every $k = 1, 2, \dots, 100$. The Maple software was used, please see the code listed. Maple like other algebraic manipulation packages allows for arbitrary precision. For a given value of Digits, Maple approximates $E(S_t^k)$ by truncated versions of the infinite sums in Theorem 1 and makes sure that the difference between successive truncated sums does not exceed the error specified by Digits. That is, the values computed using Maple can be considered *exact* up to a given decimal place.

```

answer:=evalf(Sum(inn(n,k,lambda,theta,t),n=0..infinity));

inn := proc (n::integer,k::integer,lambda::real,theta::real,t::real)::real;
local integral,m,tt,mm,ss,j,uu,l,p,q,v,w,ww,vv,ws,wm,kk,y,intt,intt0,intt1,intt2;

integral:=0;
for m from 1 to n do
    tt:=combinat(k,m);
    mm:=nops(tt);

    ss:=0;
    for j from 1 to mm do
        uu:=1;
        for l from 1 to m do
            ww[l]:=w[tt[j][1]];
            kk[l]:=tt[j][1];
            uu:=uu*factorial(kk[l]);
        od;
        ss:=ss+factorial(k)/uu;

        if (m>1) then

```

```

    for p from 1 to (m-1) do
        uu:=t;
        for q from 1 to p do
            v[p]:=t-ww[q];
        od;
    od;
end if;

intt0:=1;
wsum:=0;
for p from 1 to m do
    vv:=factorial(kk[p])*(1+theta)/lambda**(kk[p]);
    vv:=vv-factorial(kk[p])*theta/(2*lambda)**(kk[p]);
    vv:=vv-2*factorial(kk[p])*theta*exp(-lambda*ww[p])/lambda**(kk[p]);
    vv:=vv-2*factorial(kk[p])*theta*exp(-lambda*ww[p])/(2*lambda)**(kk[p]);
    intt0:=intt0*vv*lambda*exp(-lambda*ww[p]);
    wsum:=wsum+ww[p];
od;
intt:=intt0*(1-GAMMA(n-m,lambda*(t-wsum))/GAMMA(n-m));

if (m>1) then
    for p from (m-1) by -1 to 1 do
        intt:=Int(intt,ww[p+1]=0..v[p]);
    od;
end if;
intt1:=evalf(Int(intt,ww[1]=0..t));

if (m>1) then
    for p from 1 to (m-1) do
        uu:=t;
        for q from 1 to p do
            v[p]:=t-y-ww[q];
        od;
    od;
end if;

intt:=intt0*(1-GAMMA(n-m,lambda*(t-y-wsum))/GAMMA(n-m));
if (m>1) then
    for p from (m-1) by -1 to 1 do
        intt:=Int(intt,ww[p+1]=0..v[p]);
    od;
end if;

intt2:=evalf(Int(lambda*exp(-lambda*y)*Int(intt,ww[1]=0..t-y),y=0..t));

integral:=integral+ss*(intt1-intt2);
od;
od;

```

end proc;

Figure 1: Central processing unit times in seconds taken to compute $E(S_t^k)$ by simulation (solid line) and by using Theorem 1 (broken line).

Figure 2: The exact minus the simulated values of $E(S_t^k)$ versus k .

The central processing unit times taken in seconds to compute $E(S_t^k)$ are plotted in Figure 1. The differences between the simulated and exact values of $E(S_t^k)$ are plotted in Figure 2. We have taken $\lambda = 1$, $\theta = -0.6, -0.2, 0.2, 0.6$ and $k = 1, \dots, 100$.

We see that our expression in Theorem 1 appears little more efficient for all values of k in terms of the central processing unit times and accuracy. But simulations can be sped up considerably by using a general purpose language such as Python or C++ to give better central processing unit times and accuracy. Hence, it is difficult to say which of the two methods (using Theorem 1 or the simulation approach) is better.

In Figures 1 and 2, we have taken the copula to be the FGM copula. But the conclusion on the choice between using Theorem 1 and the simulation approach was similar for a wide range of copulas of the form (3) and for a wide range of parameter values.

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