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New estimators for galactic ellipticity estimation

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Abstract

Based on the reformulation of the problem of galactic shape measurement, introduced by Tessore and Bridle, we develop and test new estimators which more efficiently solve what is now a statistical problem. Frequently only a single measurement is available from an observation made by telescopes, which may be distorted due to noise introduced in the measuring process. To soften these effects, we investigate a number of approaches, such as scaling, data censoring or mixtures of estimators. We then run simulations for various setups which emulate real-life conditions, to determine practical suitability of all proposed estimation approaches.

Index terms— Ellipticity, Shear, Single sample estimation, Weak lensing

1 Introduction

An important pillar of modern Cosmology is the measurement of *Cosmic Shear*, which describes the subtle change of the observed light distribution of distant galaxies due to weak gravitational lensing by the large-scale structure of the universe (for an introduction, see Bartelmann & Schneider (2001); for a recent review, see Kilbinger (2015)). The measurement depends on the accurate estimation of the shape of millions to billions of individual galaxies, for which generally only a single observation is available. Recently, Tessore and Bridle have shown how a certain method for shape measurement can be reduced to a classical problem of parameter estimation (Tessore & Bridle 2018). The shape of a galaxy is described by the complex-valued *ellipticity* ϵ , which for a perfectly elliptical galaxy with axis ratio $q \in (0, 1]$ between the minor and major axes and angle ϕ between the major and first coordinate axes is defined as

$$\epsilon = \frac{1-q}{1+q} e^{2i\phi}. \quad (1)$$

For a perfectly round galaxy image $q = 1$ and $\epsilon = 0$. For a practically one-dimensional galaxy image with vanishing minor axis $q = 0$ and $\epsilon = 1$. Therefore $0 \leq |\epsilon| < 1$ for the absolute value $|\epsilon|$ of the ellipticity. For the complicated morphology of a real galaxy, it is necessary to define the ellipticity in terms of the moments of the observed brightness distribution. Introducing the so-called Stokes parameters u, v, s , which are related to the central second-order image moments, the ellipticity becomes

$$\epsilon = \frac{u + iv}{s + \sqrt{s^2 - u^2 - v^2}}, \quad (2)$$

where $u^2 + v^2 < s^2$ is guaranteed by the Cauchy-Schwarz inequality. Estimation of the ellipticity (2) is rendered difficult by the noise in the observed images, as well as image distortions due to atmospheric conditions and imperfect optics. Recently, Tessore and Bridle have shown (Tessore & Bridle 2018) that for a certain shape measurement method, the noisy image data can be reduced to three independent and jointly normal random variables:

$$X \sim \mathcal{N}(u, \sigma^2), \quad (3)$$

$$Y \sim \mathcal{N}(v, \sigma^2), \quad (4)$$

$$Z \sim \mathcal{N}(s, 2\sigma^2), \quad (5)$$

so that the parameters u , v , s for the ellipticity (2) are recovered through the means of X , Y , Z as $\mathbb{E}[X] = u$, $\mathbb{E}[Y] = v$ and $\mathbb{E}[Z] = s$. The variance σ^2 is determined by the amount of noise in the observed data and can be assumed known. In this way, the problem of measuring the ellipticity ϵ becomes a classical problem of parameter estimation.

This paper is structured as follows, in the next section we focus on previously proposed estimation techniques to the newly formulated statistical problem. We shed light on the well understood properties the parameters need to fulfil, and explain how they inspired the novel estimation approach listed in the upcoming section. With the completion of the method section, we elaborate on the simulation setup used to investigate the performance of the proposed estimators, and discuss their properties in relation to the composure of the parameter values. In the fourth and final section we give a conclusion on what our observations were, and give an indication on how the approximation methods should be optimally chosen in regard to signal to noise ratio (SNR) values and different parameter combinations. Additionally, we give a motivation for further research and new questions opened up by our results.

2 Methods

The two main estimation methods are on the one hand the previously proposed unbiased estimation technique,

$$\hat{\epsilon}_I(X, Y, Z) = (X + iY)h\left(\sqrt{X^2 + Y^2}, Z\right), \quad (6)$$

$$h(r, z) = \int_0^\infty e^{-\frac{3\sigma^2 k^2}{2}} \frac{I_1(rk)}{rk} e^{-zk} dk, \quad (7)$$

and the rather crude mean estimator

$$\hat{\epsilon}_{II}(X, Y, Z) = \frac{X + iY}{Z + \sqrt{Z^2 - X^2 - Y^2}}, \quad (8)$$

where the first estimator was introduced by Tessore & Bridle (2018), constructed as an unbiased approximation approach.

As previously shown, both estimators are unbiased in the argument $\arg \epsilon$ of the ellipticity, so the main improvement lies in the estimation of the radius of the complex parameter ϵ .

In addition to the random sample values X , Y and Z , we are aware of certain restrictions which have to hold. While we do not know the means of the normal distributions, we do know the standard deviation fairly well, and it will therefore be presumed known in our simulation efforts. As initially mentioned, a second implication for the parameters u , v , s can be derived via the Cauchy-Schwarz inequality for the underlying moments,

$$u^2 + v^2 < s^2. \quad (9)$$

For a significantly larger sample size, we could consider the asymptotic property of the mean estimator to hold, giving us a similar restriction to apply onto the observations made. This would serve as an approximation to the hard mean value rule:

$$\left(\frac{1}{n} \sum_{i=1}^n X_i\right)^2 + \left(\frac{1}{n} \sum_{i=1}^n Y_i\right)^2 < \left(\frac{1}{n} \sum_{i=1}^n Z_i\right)^2. \quad (10)$$

This, in our admittedly simplistic case, would be $X^2 + Y^2 < Z^2$. We do realise this is not a hard criterion, due to the limited sample size, it may however still serve as an indicator of the viability of the sample values. In Table 1 we have given a number of estimators, which seek to incorporate this information into ellipticity estimation.

Secondly, and perhaps more obviously, the ellipticity's absolute value (for example, it's radius in the Euler formulation of the complex value), should not exceed 1. An ellipticity of 1 itself would essentially mean an object with one dimension value vanishing entirely. Hence, we desire the outcome of the estimation to fulfill this condition:

$$|\hat{\epsilon}(X, Y, Z)| = |re^{\phi i}| = r < 1. \quad (11)$$

Again we have tried to incorporate this condition into our estimators. Most of them will be marked as *scaled* as many of the estimators seek to rectify the initial estimation by adjusting the radius. For the crude estimator we can very easily show that both conditions are the same, but not so for the unbiased estimator.

If any of these conditions are violated by the data given, we have some indication of problematic data. We can then proceed with one of the following steps, or a combination thereof:

- **Censoring**

The data in question can be discarded, since it is to be expected that an estimation based on the values would be flawed.

- **Scaling**

Based on the relative values of the sample to each other, it can be concluded that the estimation effort would not accurately capture the correct radius of the ellipticity. The most straightforward case would be an ellipticity which leaves the unit circle, which would have to have the radius cropped to produce a permissible value.

- **Partial scaling**

If we can pinpoint with sufficient confidence which sample values are flawed, we might be able to scale only a single, or multiple values, instead of the entire outcome of the estimation. The scaled sample would then be reintroduced into other estimation techniques.

- **Mixed models**

With several estimators having known properties such as over- or underestimation, we could produce a more reliable estimator by merging individual estimation methods together. This might lead to a dampening effect when investigating samples with values, which would throw otherwise reliable estimators off.

Estimator	Based on	Category
$\hat{\epsilon}_1(X, Y, Z) = (X + iY)h(X, Y, Z)$	I	crude
$\hat{\epsilon}_2(X, Y, Z) = \begin{cases} \hat{\epsilon}_I(X, Y, Z), & \text{if } (X + iY)h(X, Y, Z) < 1, \\ \frac{(X + iY)}{\sqrt{X^2 + Y^2}}, & \text{else} \end{cases}$	I	scaled
$\hat{\epsilon}_3(X, Y, Z) = \begin{cases} \hat{\epsilon}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \frac{(X + iY)}{\sqrt{X^2 + Y^2}}, & \text{else} \end{cases}$	I	scaled
$\hat{\epsilon}_4(X, Y, Z) = \begin{cases} \hat{\epsilon}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ NA, & \text{else} \end{cases}$	I	censored
$\hat{\epsilon}_5(X, Y, Z) = \hat{\epsilon}_{II}(X, Y, Z)$	II	crude
$\hat{\epsilon}_6(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \frac{(X + iY)}{\sqrt{X^2 + Y^2}}, & \text{else} \end{cases}$	II	scaled
$\hat{\epsilon}_7(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ NA, & \text{else} \end{cases}$	II	censored
$\hat{\epsilon}_8(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \frac{X + iY}{Z + \sqrt{X^2 + Y^2}}, & \text{else} \end{cases}$	II	scaled
$\hat{\epsilon}_9(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ NA, & \text{if } X^2 + Y^2 > 2Z^2, \\ \frac{X + iY}{Z + \sqrt{X^2 + Y^2}}, & \text{else} \end{cases}$	II	scaled/censored
$\hat{\epsilon}_{10}(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \frac{X + iY}{2Z}, & \text{else} \end{cases}$	II	scaled/censored

Table 1: Definition of crude estimators.

The crude estimators of Table 1 introduce our two main estimators, the integration based unbiased estimator $\hat{\epsilon}_I$ and the estimator $\hat{\epsilon}_{II}$ as described above with only minor modifications. Those include scaling and censoring of data, if one of the two feasibility conditions is violated, for example, $X^2 + Y^2 < Z^2$ or the radius of the ellipticity exceeding 1. Other variations of $\hat{\epsilon}_{II}$ modify the scaling component, which is in this case the denominator of the estimator in

case of presumed false sample values. For now this is done by either rounding overshoots down to $r = 1$ or simply by avoiding the root altogether.

Estimator	Based on	Category
$\hat{\epsilon}_{11}(X, Y, Z) = \begin{cases} \hat{\epsilon}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_I(\alpha X, \alpha Y, Z), & \text{else} \end{cases}$	I	censored, $\alpha = \frac{1}{\sqrt{1+\sigma}}$
$\hat{\epsilon}_{12}(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_{II}(\alpha X, \alpha Y, Z), & \text{else} \end{cases}$	II	censored, $\alpha = \frac{1}{\sqrt{1+\sigma}}$
$\hat{\epsilon}_{13}(X, Y, Z) = \begin{cases} \hat{\epsilon}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_I(X, Y, \alpha Z), & \text{else} \end{cases}$	I	scaled, $\alpha = \sqrt{1+\sigma}$
$\hat{\epsilon}_{14}(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_{II}(X, Y, \alpha Z), & \text{else} \end{cases}$	II	scaled, $\alpha = \sqrt{1+\sigma}$
$\hat{\epsilon}_{15}(X, Y, Z) = \begin{cases} \hat{\epsilon}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_I\left(X, Y, \sqrt{\frac{X^2+Y^2}{\frac{2}{\pi} \tan^{-1}\left(\frac{X^2+Y^2}{Z^2}\right)}}\right), & \text{else} \end{cases}$	I	scaled, $\alpha = \sqrt{1+\sigma}$
$\hat{\epsilon}_{16}(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_{II}\left(X, Y, \sqrt{\frac{X^2+Y^2}{\frac{2}{\pi} \tan^{-1}\left(\frac{X^2+Y^2}{Z^2}\right)}}\right), & \text{else} \end{cases}$	II	scaled, $\alpha = \sqrt{1+\sigma}$

Table 2: Definition of scaled parameter estimators.

Next up we seek to rectify the sample values by either scaling the directional components X, Y simultaneously down to fulfill the inequality, or scale Z up to achieve the same effect, see Table 2. The whole process is repeated for both base estimators, to enable an accurate comparison between their advantages and disadvantages, and changing behaviour to data scaling and censoring. For the most part these estimators are differing in the scaling method. While the first four try to incorporate the known variance of the samples, the latter two introduce an arcus tangent scaling function, which we rely on for further estimators as well. The idea is to map a potentially infinite radius, or quotient of sample values into the feasible range $[0, 1]$.

In short we can write the transformation of the rescaling procedure using an inverse tangent function in the following way:

$$X^2 + Y^2 > Z^2 \tag{12}$$

$$\iff \frac{X^2 + Y^2}{Z^2} > 1 \tag{13}$$

$$\implies \frac{X^2 + Y^2}{(Z')^2} \stackrel{!}{=} \frac{\pi}{2} \tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right) < 1 \quad (\text{introduce new variable } Z') \tag{14}$$

$$\implies Z' = \sqrt{\frac{X^2 + Y^2}{\frac{\pi}{2} \tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right)}}. \tag{15}$$

Equivalently we can retrieve a scaled version for the X and Y values, depending on their relative contribution to the violation of the inequality. We investigate further how the different combinations of conditions, scaling methods and underlying base estimator behave, see Table 3.

Estimator		Based on	Category
$\hat{e}_{17}(X, Y, Z)$	$= \begin{cases} \hat{e}_I(X, Y, Z), & \text{if } \hat{e}_I(X, Y, Z) < 1, \\ iIm(\hat{e}_I(X, Y, Z)) + \frac{\pi}{2} \tan^{-1} Re(\hat{e}_I(X, Y, Z)), & \text{if } Re(\hat{e}_I(X, Y, Z)) > 1 \wedge Im(\hat{e}_I(X, Y, Z)) < 1, \\ \frac{i\pi}{2} \tan^{-1} Im(\hat{e}_I(X, Y, Z)) + Re(\hat{e}_I(X, Y, Z)), & \text{if } Re(\hat{e}_I(X, Y, Z)) < 1 \wedge Im(\hat{e}_I(X, Y, Z)) > 1, \\ \frac{X+iY}{\sqrt{X^2+Y^2}} \frac{2}{\pi} \tan^{-1}(\hat{e}_I(X, Y, Z)), & \text{else} \end{cases}$	I	scaled
$\hat{e}_{18}(X, Y, Z)$	$= \begin{cases} \hat{e}_{II}(X, Y, Z), & \text{if } \hat{e}_{II}(X, Y, Z) < 1, \\ iIm(\hat{e}_{II}(X, Y, Z)) + \frac{\pi}{2} \tan^{-1} Re(\hat{e}_{II}(X, Y, Z)), & \text{if } Re(\hat{e}_{II}(X, Y, Z)) > 1 \wedge Im(\hat{e}_{II}(X, Y, Z)) < 1, \\ \frac{i\pi}{2} \tan^{-1} Im(\hat{e}_{II}(X, Y, Z)) + Re(\hat{e}_{II}(X, Y, Z)), & \text{if } Re(\hat{e}_{II}(X, Y, Z)) < 1 \wedge Im(\hat{e}_{II}(X, Y, Z)) > 1, \\ \frac{X+iY}{\sqrt{X^2+Y^2}} \frac{2}{\pi} \tan^{-1}(\hat{e}_{II}(X, Y, Z)), & \text{else} \end{cases}$	II	scaled
$\hat{e}_{19}(X, Y, Z)$	$= \begin{cases} \hat{e}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{e}_I\left(\sqrt{\frac{2}{\pi}}(Z^2 - Y^2) \tan^{-1}\left(\frac{X^2}{Z^2 - Y^2}\right), Y, Z\right), & \text{if } X^2 < Z^2, Y^2 > Z^2, \\ \hat{e}_I\left(X, \sqrt{\frac{2}{\pi}}(Z^2 - X^2) \tan^{-1}\left(\frac{Y^2}{Z^2 - X^2}\right), Z\right), & \text{if } X^2 > Z^2, Y^2 < Z^2, \\ \hat{e}_I(X, Y, Z) \sqrt{\frac{2}{\pi} \frac{X^2 + Y^2}{\tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right)}}, & \text{else} \end{cases}$	I	scaled
$\hat{e}_{20}(X, Y, Z)$	$= \begin{cases} \hat{e}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{e}_{II}\left(\sqrt{\frac{2}{\pi}}(Z^2 - Y^2) \tan^{-1}\left(\frac{X^2}{Z^2 - Y^2}\right), Y, Z\right), & \text{if } X^2 < Z^2, Y^2 > Z^2, \\ \hat{e}_{II}\left(X, \sqrt{\frac{2}{\pi}}(Z^2 - X^2) \tan^{-1}\left(\frac{Y^2}{Z^2 - X^2}\right), Z\right), & \text{if } X^2 > Z^2, Y^2 < Z^2, \\ \hat{e}_{II}(X, Y, Z) \sqrt{\frac{2}{\pi} \frac{X^2 + Y^2}{\tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right)}}, & \text{else} \end{cases}$	II	scaled
$\hat{e}_{21}(X, Y, Z)$	$= \begin{cases} \hat{e}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{e}_I\left(Z^2 \tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right) \frac{2}{\pi} - Y^2, Y, Z\right), & \text{if } X^2 < Z^2, Y^2 > Z^2, \\ \hat{e}_I\left(X, Z^2 \tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right) \frac{2}{\pi} - X^2, Z\right), & \text{if } X^2 > Z^2, Y^2 < Z^2, \\ \hat{e}_I(X, Y, Z) \sqrt{\frac{2}{\pi} \frac{X^2 + Y^2}{\tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right)}}, & \text{else} \end{cases}$	I	scaled
$\hat{e}_{22}(X, Y, Z)$	$= \begin{cases} \hat{e}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{e}_{II}\left(Z^2 \tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right) \frac{2}{\pi} - Y^2, Y, Z\right), & \text{if } X^2 < Z^2, Y^2 > Z^2, \\ \hat{e}_{II}\left(X, Z^2 \tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right) \frac{2}{\pi} - X^2, Z\right), & \text{if } X^2 > Z^2, Y^2 < Z^2, \\ \hat{e}_{II}(X, Y, Z) \sqrt{\frac{2}{\pi} \frac{X^2 + Y^2}{\tan^{-1}\left(\frac{X^2 + Y^2}{Z^2}\right)}}, & \text{else} \end{cases}$	II	scaled

Table 3: Definition for result and parameter scaling estimators (I).

Alternatively, we can scale the estimator after evaluation, see Table 4. Again, this can be done either with regard to the entire radius itself, or depending on the exact deviation of the estimate on the real and imaginary parts (depending how grossly the estimate seems to lean one way or the other), this can be done for either part separately. Furthermore, we can apply parameter scaling techniques onto the input values themselves, if the error origin to either one of the input values X or Y alone can be traced reasonably. Failing this, the entire estimate can still be scaled in its entirety to fulfill either the radius condition or the mean value inequality. These selective estimators are described in Table 5

Estimator		Based on	Category
$\hat{e}_{23-29}(X, Y, Z)$	$= \begin{cases} \hat{e}_I(X, Y, Z), & \text{if } \hat{e}_I(X, Y, Z) < 1, \\ \frac{X+iY}{ X+iY } \frac{\pi}{2} \tan^{-1}(\beta \hat{e}_I(X, Y, Z)), & \text{else} \end{cases}$	I	scaled, $\beta = 0.25, 0.5, 1, 1.5, 2, 3, \frac{1}{\sigma}$
$\hat{e}_{30-36}(X, Y, Z)$	$= \begin{cases} \hat{e}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \frac{X+iY}{ X+iY } \frac{\pi}{2} \tan^{-1}(\beta \hat{e}_{II}(X, Y, Z)), & \text{else} \end{cases}$	II	scaled, $\beta = 0.25, 0.5, 1, 1.5, 2, 3, \frac{1}{\sigma}$
$\hat{e}_{37-43}(X, Y, Z)$	$= \begin{cases} \hat{e}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \frac{X+iY}{ X+iY } \frac{\pi}{2} \tan^{-1}(\beta \hat{e}_I(X, Y, Z)), & \text{else} \end{cases}$	I	scaled, $\beta = 0.25, 0.5, 1, 1.5, 2, 3, \frac{1}{\sigma}$

Table 4: A definition for result-only scaling estimators.

Furthermore, we can apply parameter scaling techniques directly onto the input values, if we can reasonably assume

that only one of the input variables is straying far from its true value. Failing this, the entire estimate can still be scaled to fulfill either the radius condition, or the mean value inequality, see Table 5.

Estimator	Based on	Category
$\hat{\epsilon}_{44}(X, Y, Z) = \begin{cases} \hat{\epsilon}_I(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_I\left(\sqrt{\frac{2}{\pi} \tan^{-1}\left(\frac{X^2}{Z^2 - Y^2}\right)}(Z^2 - Y^2), Y, Z\right), & \text{if } X^2 > Z^2 \wedge Y^2 < Z^2, \\ \hat{\epsilon}_I\left(X, \sqrt{\frac{2}{\pi} \tan^{-1}\left(\frac{Y^2}{Z^2 - X^2}\right)}(Z^2 - X^2), Z\right), & \text{if } X^2 < Z^2 \wedge Y^2 > Z^2, \\ \frac{X+iY}{ X+iY } \frac{2}{\pi} \tan^{-1}\left(\frac{1}{\sigma} \hat{\epsilon}_I(X, Y, Z) \right), & \text{else} \end{cases}$	I	scaled, parameters scaled
$\hat{\epsilon}_{45}(X, Y, Z) = \begin{cases} \hat{\epsilon}_{II}(X, Y, Z), & \text{if } X^2 + Y^2 < Z^2, \\ \hat{\epsilon}_{II}\left(\sqrt{\frac{2}{\pi} \tan^{-1}\left(\frac{X^2}{Z^2 - Y^2}\right)}(Z^2 - Y^2), Y, Z\right), & \text{if } X^2 > Z^2 \wedge Y^2 < Z^2, \\ \hat{\epsilon}_{II}\left(X, \sqrt{\frac{2}{\pi} \tan^{-1}\left(\frac{Y^2}{Z^2 - X^2}\right)}(Z^2 - X^2), Z\right), & \text{if } X^2 < Z^2 \wedge Y^2 > Z^2, \\ \frac{X+iY}{ X+iY } \frac{2}{\pi} \tan^{-1}\left(\frac{1}{\sigma} \hat{\epsilon}_{II}(X, Y, Z) \right), & \text{else} \end{cases}$	II	scaled, parameters scaled

Table 5: Definition for result and parameter scaling estimators (II).

In addition, we investigate the averages of a selection of previously introduced estimators, see Table 6. As stated earlier, the idea is to potentially dampen the shortcomings a single estimator might have on its own. We simply use the arithmetic mean on the unbiased and crude version of the more promising estimators. This concludes the list of possible candidates for a more accurate estimation, yet this is by no means an exhaustive compilation as we have only used basic estimation approaches on which to improve upon via varying measures of censoring and scaling.

Estimator	Based on	Category
$\hat{\epsilon}_{46}(X, Y, Z) = (\hat{\epsilon}_1(X, Y, Z) + \hat{\epsilon}_5(X, Y, Z)) / 2$	I+II	crude
$\hat{\epsilon}_{47}(X, Y, Z) = (\hat{\epsilon}_2(X, Y, Z) + \hat{\epsilon}_6(X, Y, Z)) / 2$	I+II	scaled
$\hat{\epsilon}_{48}(X, Y, Z) = (\hat{\epsilon}_3(X, Y, Z) + \hat{\epsilon}_7(X, Y, Z)) / 2$	I+II	censored
$\hat{\epsilon}_{49}(X, Y, Z) = (\hat{\epsilon}_{11}(X, Y, Z) + \hat{\epsilon}_{12}(X, Y, Z)) / 2$	I+II	scaled
$\hat{\epsilon}_{50}(X, Y, Z) = (\hat{\epsilon}_{13}(X, Y, Z) + \hat{\epsilon}_{14}(X, Y, Z)) / 2$	I+II	parameter scaled
$\hat{\epsilon}_{51}(X, Y, Z) = (\hat{\epsilon}_{19}(X, Y, Z) + \hat{\epsilon}_{20}(X, Y, Z)) / 2$	I+II	parameter scaled
$\hat{\epsilon}_{52}(X, Y, Z) = (\hat{\epsilon}_{25}(X, Y, Z) + \hat{\epsilon}_{32}(X, Y, Z)) / 2$	I+II	result scaled
$\hat{\epsilon}_{53}(X, Y, Z) = (\hat{\epsilon}_{44}(X, Y, Z) + \hat{\epsilon}_{45}(X, Y, Z)) / 2$	I+II	mixed

Table 6: Definition for averaged estimators.

We now we wish to determine the performance and suitability of the new suggestions. We will therefore discuss each cluster under the criteria introduced in this section. The accuracy will be tested in different ways, on the one hand via predetermined values which are exemplary for certain regions of parameter sets, and on the other hand via truly random data. This means that the necessary mean values u , v and s which we rely upon for the determination of the ellipticity need to be randomised as well. This approach will give us the chance to investigate potential areas of dominant performance, as well as the overall performance in indiscriminate testing for each candidate.

3 Estimation comparison

In this section, we investigate what the strengths and weaknesses of the proposed estimators are, when and where their properties are of the most benefit. With the actual parameters u , v and s known, we can determine the true ellipticity beforehand. To gain an insight into this, we firstly design an experiment for specific pairings of the mean values u , v and s . This helps determining the effective ranges of each individual estimator. Essentially we begin with the true ϵ value, and then choose the mean parameter values accordingly. This is done by the well-known distributions below, from which we can sample X , Y and Z values:

$$X_i \sim \mathcal{N}(u, \sigma^2), \quad (16)$$

$$Y_i \sim \mathcal{N}(v, \sigma^2), \quad (17)$$

$$Z_i \sim \mathcal{N}(s, 2\sigma^2). \quad (18)$$

The mean values u , v and s , as well as the standard deviation σ are deterministic, but chosen in several combinations to cover a large set of realistic parameter values.

The ellipticities range from close to the origin $0 + i0$ to the border of the unit circle, for example, $\sqrt{\Im(\epsilon)^2 + \Re(\epsilon)^2} =$

$r \sim 1$. We have conducted the experiment for the pairings of parameter values in Table 7. A visual representation of the corresponding ellipticity values can be found in Figure 1.

Combination	u	v	s
1	0	0	5
2	0	1	5
3	0	2	5
4	0	3	5
5	0	4	5
6	1	1	5
7	1	2	5
8	1	3	5
9	1	4	5
10	2	2	5
11	2	3	5
12	2	4	5
13	3	3	5
14	3	4	5
15	-1	-2	5

Table 7: An overview of the tested parameter combinations.

We pair this with the sigma values $\sigma = 0.05, 0.1, 0.25, 0.5, 0.75, 1, 1.5, 2, 3, 5$, using the sigma value for the sampling from the mean values u, v and s . To minimise variance of the experiment, we choose a sufficiently large sample size for each combination of $n = 25000$. Each estimator is tested against the true ellipticity value and an absolute error measure is taken. The mean value of these errors can be found in the Tables 8 to 11.

Figure 1: Overview of investigated ellipticity values.

We discuss a selected pairings of values only, which serve as examples for the various ellipticity locations. The full set of tables can be provided upon inquiry.

		Signal to Noise Ratio										
		100	50	20	10	6.66	5	3.33	2.5	1.66	1	
Estimator Number	Crude	1	0.008852652	0.017818046	0.044400409	0.087290748	0.128403327	0.170999087	0.24270885	0.309275981	0.424400518	0.615987134
		2	0.008852652	0.017818046	0.044400409	0.087213346	0.127273776	0.165127783	0.223619726	0.273861663	0.339899062	0.4217586
		3	0.008852652	0.017818046	0.044400409	0.089393732	0.144198561	0.205435346	0.311630383	0.400994964	0.522467491	0.693821495
		4	0.008852652	0.017818046	0.044400409	0.084749625	0.11268145	0.135024924	0.175464377	0.213566004	0.291407887	0.474481682
		5	0.0088622	0.01788621	0.04550289	0.09935748	0.168975743	0.243897484	0.371717536	0.48772887	0.685952203	0.95750345
		6	0.0088622	0.01788621	0.04550289	0.099126376	0.166073572	0.234254974	0.344370871	0.4436028	0.629102802	0.906686952
		7	0.0088622	0.01788621	0.04550289	0.094556803	0.13579439	0.167859573	0.218671314	0.277780852	0.47886888	0.916771943
		8	0.0088622	0.01788621	0.04550289	0.095509738	0.141793707	0.182336262	0.262459402	0.41348159	0.846976342	1.286531754
		9	0.0088622	0.01788621	0.04550289	0.095484847	0.140584146	0.175882356	0.233005719	0.36355646	0.875666946	1.49467554
		10	0.0088622	0.01788621	0.04550289	0.095691118	0.144854781	0.201247842	0.74207854	1.387255148	2.075621817	5.58432053
	Scaled	11	0.008852652	0.017818046	0.044400409	0.085695991	0.115693797	0.138255402	0.176286814	0.213228844	0.276032195	0.383173709
		12	0.0088622	0.01788621	0.04550289	0.095933102	0.140841971	0.174525124	0.222955921	0.278832652	0.463057869	0.826661677
		13	0.008852652	0.017818046	0.044400409	0.085862743	0.117899759	0.147618716	0.213218094	0.325035371	14.11756533	32285.26732
		14	0.0088622	0.01788621	0.04550289	0.096014629	0.141457293	0.17597463	0.22582594	0.282152765	0.462204655	0.806975887
		15	0.008852652	0.017818046	0.044400409	0.085020616	0.113502181	0.135617381	0.172231233	0.205689825	0.266913867	0.378387215
		16	0.0088622	0.01788621	0.04550289	0.094957773	0.138478138	0.176133017	0.246965533	0.326012955	0.495775036	0.755947099
	Result and Parameter	17	0.008852652	0.017818046	0.044400409	0.087105162	0.125826204	0.159647658	0.210767151	0.252605014	0.308312373	0.376583189
		18	0.0088622	0.01788621	0.04550289	0.095699835	0.143592118	0.186534735	0.258748433	0.330448018	0.483207574	0.712244957
		19	0.008852652	0.017818046	0.044400409	41.10896765	1.249870064	0.536713898	0.373776683	0.371161363	0.40914466	0.510427186
		20	0.0088622	0.01788621	0.04550289	0.094934	0.14492189	0.200633042	0.303674875	0.401699656	0.607522702	0.948036093
		21	0.008852652	0.017818046	0.044400409	0.085357133	0.13086634	0.212589233	7.94764E+19	8.90038E+11	2.40895E+16	1.02029E+72
		22	0.0088622	0.01788621	0.04550289	0.094695091	0.139277916	0.176715852	0.242273886	0.321487562	0.535234109	0.938614352
	Result with dampening Parameter	23	0.008852652	0.017818046	0.044400409	0.08707377	0.125490847	0.158473486	0.208068088	0.248266653	0.300436453	0.362303356
		24	0.008852652	0.017818046	0.044400409	0.087065191	0.125424352	0.158250762	0.208659492	0.250087883	0.304579303	0.371525846
		25	0.008852652	0.017818046	0.044400409	0.087114048	0.126010748	0.160353531	0.213395729	0.25768268	0.316047075	0.388451502
		26	0.008852652	0.017818046	0.044400409	0.087141395	0.126353769	0.161630999	0.216171976	0.262081978	0.322557004	0.397653354
		27	0.008852652	0.017818046	0.044400409	0.087157523	0.126558051	0.162400623	0.21782569	0.264698868	0.326416621	0.403053785
		28	0.008852652	0.017818046	0.044400409	0.087175142	0.126782754	0.163252865	0.219644259	0.267574892	0.330651688	0.408948228
		29	0.008852652	0.017818046	0.044400409	0.087157523	0.126259852	0.160353531	0.210436607	0.250087883	0.301072921	0.361230124
		30	0.26077587	0.260793011	0.260825428	0.262717758	0.272192125	0.288862892	0.320708763	0.356948901	0.425897902	0.534247101
		31	0.212111033	0.212162013	0.212349164	0.215702983	0.230631716	0.254647373	0.298253242	0.342084681	0.418793291	0.534314523
		32	0.118663183	0.118897307	0.120286989	0.130185675	0.157527911	0.195615044	0.260600867	0.318038751	0.408618643	0.536434542
		33	0.033270383	0.034819372	0.04809177	0.08110904	0.120812819	0.16567119	0.240367397	0.305611423	0.404837699	0.541036967
		34	0.043762365	0.044988023	0.056939267	0.089710434	0.127630531	0.16950014	0.241637168	0.307179631	0.408701233	0.548671543
		35	0.167064199	0.167228197	0.169032699	0.176988939	0.194007275	0.219688689	0.27499517	0.332997155	0.430375154	0.569874418
		36	0.588212965	0.491257185	0.258384572	0.089710434	0.127540736	0.195615044	0.284431403	0.342084681	0.423365835	0.534387775
		37	0.260749623	0.260694231	0.260190369	0.262125941	0.285155004	0.325511434	0.409307223	0.487500027	0.58309431	0.67699124
		38	0.212059472	0.211968094	0.211108869	0.212488432	0.238798379	0.285478931	0.381396398	0.469095831	0.577071766	0.685157128
		39	0.118566947	0.118536588	0.118058805	0.126191384	0.167873499	0.230389347	0.348326051	0.450506068	0.574700457	0.698366095
		40	0.033142213	0.034379334	0.046981817	0.085263824	0.146289491	0.219123974	0.347707407	0.455455171	0.585751963	0.714403428
		41	0.043913519	0.045537101	0.059676711	0.102020697	0.164529942	0.236565958	0.364604341	0.472027518	0.602783582	0.731554953
		42	0.167233705	0.16786195	0.172905979	0.194103134	0.2391399	0.297743228	0.41255912	0.513265089	0.639365363	0.763219438
		43	0.588266382	0.491628302	0.262179512	0.102020697	0.147956302	0.230389347	0.366283069	0.469095831	0.580926324	0.674770691
	RP II	44	0.008852652	0.017818046	0.044400409	0.085050779	0.11903741	0.156296551	0.22123391	0.272417601	0.350764418	0.4964604
		45	0.0088622	0.01788621	0.04550289	0.094934	0.14492189	0.200633042	0.303674875	0.401699656	0.607522702	0.948036093
	Mixed	46	0.008857151	0.017850224	0.044923207	0.092930483	0.145762085	0.198330099	0.279723657	0.346084115	0.457461156	0.611487728
		47	0.008857151	0.017850224	0.044923207	0.092947217	0.145968914	0.198144088	0.27983652	0.350869982	0.464790617	0.61564302
		48	0.008857151	0.017850224	0.044923207	0.08943057	0.123528553	0.149877119	0.192846343	0.236097111	0.346941957	0.557336053
		49	0.008857151	0.017850224	0.044923207	0.090378917	0.12612659	0.151760423	0.189882472	0.229269493	0.330653139	0.504137239
		50	0.008857151	0.017850224	0.044923207	0.090466136	0.126827556	0.153437655	0.193582915	0.242983876	0.579269421	25237.0175
		51	0.008857151	0.017850224	0.044923207	0.089713502	0.130230878	0.172970846	0.247945561	0.309450769	0.412805806	0.564757524
		52	0.059439985	0.06006146	0.066378612	0.092968128	0.12880749	0.167893597	0.230927388	0.283940856	0.359983585	0.460737366
		53	0.008857151	0.017850224	0.044923207	0.08976916	0.13123445	0.17673827	0.256890038	0.320105134	0.421475928	0.569468349

Table 8: Error table for the different estimators and sigma values for $u = -2$, $v = -2$ and $s = 5$, for example, $\epsilon = -0.219224 - 0.219224i$.

To investigate the influence of negative values on the estimators, we have included an example with negative mean values $u = -1$, $v = -2$ and $s = 5$. The results are listed in Table 8. In comparison to the positive values, we see that the negative values make very little difference in the performance of the estimators. As u and v are rather directly estimated through the values x and y , the negative sample values directly translate into the estimation. We will therefore consider positive mean values without loss of generality.

		Signal to Noise Ratio										
		100	50	20	10	6.66	5	3.33	2.5	1.66	1	
Estimator Number	Crude	1	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.124015493	0.182464752	0.238481817	0.340184355	0.511853449
		2	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.124005822	0.181113799	0.230883518	0.304161985	0.383030131
		3	0.006280896	0.012464567	0.03142067	0.062427249	0.09358092	0.128214057	0.228560289	0.336670509	0.499438923	0.679989477
		4	0.006280896	0.012464567	0.03142067	0.062427249	0.0933996	0.120899945	0.153380475	0.170962492	0.212855269	0.370504126
		5	0.006283096	0.0124802	0.031662745	0.064463318	0.10162241	0.149602613	0.285358132	0.420328157	0.676871667	0.9617187
		6	0.006283096	0.0124802	0.031662745	0.064463318	0.10162241	0.149602613	0.285358132	0.420328157	0.676871667	0.9617187
		7	0.006283096	0.0124802	0.031662745	0.064463318	0.101442699	0.142467947	0.21571349	0.275518868	0.491872669	0.924696475
		8	0.006283096	0.0124802	0.031662745	0.064463318	0.10153098	0.14636472	0.272690827	0.474513951	1.052963919	1.638313551
		9	0.006283096	0.0124802	0.031662745	0.064463318	0.101511437	0.144497787	0.246260373	0.413544698	1.076933818	1.972636276
		10	0.006283096	0.0124802	0.031662745	0.064463318	0.101542371	0.150802108	0.370029459	1.360994214	1.873728845	2.892488013
	Scaled	11	0.006280896	0.012464567	0.03142067	0.062427249	0.093447169	0.122614604	0.164686393	0.189998768	0.224990329	0.296532808
		12	0.006283096	0.0124802	0.031662745	0.064463318	0.101477357	0.143807427	0.225373969	0.290527162	0.494956399	0.836355747
		13	0.006280896	0.012464567	0.03142067	0.062427249	0.093456781	0.1232536	0.175836597	0.23305254	0.597791449	30.23347778
		14	0.006283096	0.0124802	0.031662745	0.064463318	0.101479167	0.1438909	0.226228841	0.292199816	0.493686574	0.819258806
		15	0.006280896	0.012464567	0.03142067	0.062427249	0.093448091	0.122690343	0.167378258	0.19726687	0.243133865	0.333069053
		16	0.006283096	0.0124802	0.031662745	0.064463318	0.101518212	0.145868386	0.249596255	0.348289342	0.556259682	0.809850556
	Result and Parameter	17	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123979763	0.179697021	0.224479402	0.283857412	0.341440681
		18	0.006283096	0.0124802	0.031662745	0.064463318	0.10152241	0.145442613	0.240958132	0.320388157	0.494831667	0.7158987
		19	0.006280896	0.012464567	0.03142067	0.062427249	0.093454981	0.1235111	0.1748518	0.21597464	0.279069483	0.392510633
		20	0.006283096	0.0124802	0.031662745	0.064463318	0.101505987	0.145091107	0.242165898	0.334797842	0.570547308	0.919268879
		21	0.006280896	0.012464567	0.03142067	0.062427249	0.093439071	0.122832909	0.471557264	0.315477049	8.59915E+12	7.97E+036
		22	0.006283096	0.0124802	0.031662745	0.064463318	0.10148974	0.144466644	0.233003104	0.31611629	0.547536464	0.938827921
	Result with dampening Parameter	23	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123973487	0.179004078	0.221547727	0.274164737	0.321494937
		24	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123979971	0.179473619	0.223644593	0.281236221	0.336633966
		25	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123988558	0.180041996	0.22614979	0.289347944	0.353338788
		26	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123993275	0.180339431	0.227460616	0.293493586	0.361717556
		27	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123996074	0.180513414	0.228228497	0.295901191	0.3665508
		28	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123999144	0.180703037	0.229066349	0.298515781	0.371779541
		29	0.006280896	0.012464567	0.03142067	0.062427249	0.093473299	0.123988558	0.179708326	0.223644593	0.276837416	0.317547199
		30	0.000999634	0.001983785	0.005000556	0.009933942	0.014948306	0.023189823	0.064914805	0.123328048	0.225485587	0.347620354
		31	0.001999259	0.003967495	0.009999992	0.019857926	0.02976781	0.042178926	0.086962661	0.144638448	0.244831157	0.367216293
		32	0.003998442	0.007934394	0.019990319	0.039636849	0.059158436	0.079624078	0.130122089	0.186039742	0.280973685	0.400699827
		33	0.005997475	0.0119001	0.02996177	0.059261175	0.088011153	0.115947077	0.171428432	0.225338597	0.314310091	0.429601546
		34	0.007996282	0.015864019	0.039905025	0.078661588	0.116123761	0.150791119	0.210395456	0.262130979	0.345003853	0.455229826
		35	0.011992919	0.023784134	0.059671361	0.11655589	0.169591751	0.215326951	0.28059535	0.327683212	0.398847868	0.49879977
		36	0.079190934	0.078579102	0.079221916	0.078661588	0.078466981	0.079624078	0.101513359	0.144638448	0.232050438	0.343315083
		37	0.000999984	0.001986273	0.005039079	0.0102578	0.01633291	0.030775199	0.119440003	0.233642757	0.403795333	0.545591012
		38	0.001999959	0.003972471	0.010076937	0.020504494	0.032412681	0.053052194	0.14922476	0.265233558	0.435562903	0.579873901
		39	0.003999843	0.007944343	0.020144126	0.040920956	0.064217694	0.0964914	0.205719467	0.323443248	0.489103025	0.629326038
		40	0.005999576	0.011915018	0.030191915	0.061165534	0.09528147	0.13782738	0.257228239	0.375016566	0.533991009	0.66703305
		41	0.007999083	0.015883899	0.040210841	0.081162005	0.125366263	0.176705022	0.303582135	0.420286635	0.572162334	0.697668522
		42	0.011997118	0.02381391	0.06012571	0.120156332	0.182054294	0.246880624	0.382401575	0.494902752	0.633117452	0.744777899
		43	0.079218141	0.078675756	0.07981985	0.081162005	0.085025826	0.0964914	0.168560887	0.265233558	0.414940584	0.53711567
	RP II	44	0.006280896	0.012464567	0.03142067	0.062427249	0.093426821	0.121742033	0.158549166	0.178994577	0.220264168	0.338996954
		45	0.006283096	0.0124802	0.031662745	0.064463318	0.101505987	0.145091107	0.242165898	0.334797842	0.570547308	0.919268879
	Mixed	46	0.006281996	0.012472383	0.031541707	0.063445284	0.097542623	0.136342335	0.225747431	0.302474944	0.430309131	0.573913184
		47	0.006281996	0.012472383	0.031541707	0.063445284	0.097547855	0.136804218	0.232504001	0.322011147	0.473010572	0.625460672
		48	0.006281996	0.012472383	0.031541707	0.063445284	0.097421149	0.131683946	0.183586348	0.217485994	0.314466476	0.508096468
		49	0.006281996	0.012472383	0.031541707	0.063445284	0.097446088	0.132612992	0.189443188	0.225635453	0.313151723	0.455128439
		50	0.006281996	0.012472383	0.031541707	0.063445284	0.097448484	0.132726899	0.190742057	0.22892868	0.449549864	22.09565019
		51	0.006281996	0.012472383	0.031541707	0.063445284	0.097461263	0.133038743	0.195900389	0.242887254	0.344126637	0.49329472
		52	0.005139669	0.01019948	0.025705494	0.051032049	0.076315868	0.101806318	0.155082043	0.206094766	0.285160815	0.377019307
		53	0.006281996	0.012472383	0.031541707	0.063445284	0.097466404	0.133390677	0.198589424	0.247303487	0.349219282	0.49492551

Table 9: Error table for the different estimators and sigma values for $u = 0$, $v = 0$ and $s = 5$, for example, $\epsilon = 0 + 0i$.

The color coding is designed to help scan the experiment results more easily. Higher error averages are in red, and gradually turning into green background colouring the lower the absolute error gets. The main estimator we compare our newly designed estimators against, is the unaltered $\hat{\epsilon}_I$ or the scaled version, since they are considered to be the strongest contenders for a new estimator against the cruder direct version based on $\hat{\epsilon}_{II}$, which simply plugs in the single sample values into the ellipticity formula.

Around the origin $u = 0$, $v = 0$, $s = 5$ in Table 9, we can see mostly indifference between the estimators, with a bit of preference for estimators 30 – 32 and 37 – 38. Also the differences in performance naturally increase for larger variances. In this range there is a slight preference for 11, 15, 17, 19 as well as 23 – 29 in addition to the overall 30 – 32 performances. This however may be due to the scaling function in certain estimators, which for an ellipticity in the origin will automatically create better results. Therefore this set of results has to be taken with a grain of salt.

		Signal to Noise Ratio										
		100	50	20	10	6.66	5	3.33	2.5	1.66	1	
Estimator Number	Crude	1	0.0092933558	0.0186109789	0.0463971399	0.0919338727	0.135875812	0.1767216285	0.2510954087	0.3212897986	0.4548836706	0.6522562725
		2	0.0092933558	0.0186109789	0.0463971399	0.0918733435	0.1337510683	0.1701997932	0.2276696405	0.2760931359	0.345280003	0.4261770309
		3	0.0092933558	0.0186109789	0.0463971399	0.0949824873	0.1548882147	0.216592283	0.3183276822	0.4095206274	0.5515975302	0.7623392765
		4	0.0092933558	0.0186109789	0.0463971399	0.088517271	0.115341531	0.1376027929	0.1803454674	0.2283254326	0.3501945107	0.6311211712
		5	0.0093032703	0.018688652	0.0477605332	0.1074207365	0.1829833794	0.2562187438	0.3805607239	0.488766262	0.6714684916	0.9672456357
		6	0.0093032703	0.018688652	0.0477605332	0.1070105134	0.1785448814	0.2436885511	0.3475520689	0.4385923236	0.6045918966	0.9077776766
		7	0.0093032703	0.018688652	0.0477605332	0.1006800609	0.1407897089	0.1692440047	0.2198208115	0.2730236124	0.4449084913	0.9352074806
		8	0.0093032703	0.018688652	0.0477605332	0.1017346374	0.1469634292	0.182920432	0.276868261	0.4591967165	0.8101807625	1.3790945074
		9	0.0093032703	0.018688652	0.0477605332	0.101722173	0.1455393877	0.1762303409	0.2510308897	0.4300283734	0.8330540805	1.6684763526
		10	0.0093032703	0.018688652	0.0477605332	0.1019826547	0.151201289	0.2244238109	0.7296378691	1.3661529566	2.826952757	2.9843341939
	Scaled	11	0.0092933558	0.0186109789	0.0463971399	0.0896028514	0.1184245633	0.1392422935	0.1793197357	0.2232264007	0.3123545401	0.4620431114
		12	0.0093032703	0.018688652	0.0477605332	0.1024451759	0.1464275048	0.17552725	0.2224558679	0.2725243802	0.4324909498	0.8403391551
		13	0.0092933558	0.0186109789	0.0463971399	0.0898282808	0.1214207968	0.1486293125	0.2338442632	0.43172537	2.5866160153	123.96091888
		14	0.0093032703	0.018688652	0.0477605332	0.1025597078	0.147207014	0.1772300735	0.2254009473	0.2757029879	0.4312701142	0.8181160176
		15	0.0092933558	0.0186109789	0.0463971399	0.0886331811	0.1152611044	0.1357593276	0.1722302451	0.2121726382	0.2987069785	0.4548355144
		16	0.0093032703	0.018688652	0.0477605332	0.1009334517	0.1427548429	0.1757583218	0.2435198028	0.3157540925	0.4704675211	0.7567712898
	Result and Parameter	17	0.0092933558	0.0186109789	0.0463971399	0.0917228032	0.1307576233	0.1629952183	0.2131343267	0.2547778257	0.3132973003	0.3821375476
		18	0.0093032703	0.018688652	0.0477605332	0.1021552582	0.1502268777	0.1897955352	0.2612366572	0.3275661764	0.4667463268	0.7231750681
		19	0.0092933558	0.0186109789	0.0463971399	7.1033850712	0.295571039	0.2454033443	0.2972566048	0.3508436723	0.4286666819	0.565563362
		20	0.0093032703	0.018688652	0.0477605332	0.1010561959	0.1447593752	0.1817514923	0.2541000006	0.3291660712	0.5177902113	0.906901762
		21	0.0092933558	0.0186109789	0.0463971399	124684.75773	161193.26001	129891.84115	22894164874	2.458818E+38	8.312798E+36	3.979334E+81
		22	0.0093032703	0.018688652	0.0477605332	0.1007045577	0.1412896133	0.1736253282	0.2399481553	0.3138452553	0.5031550968	0.9473756806
	Result with dampening Parameter	23	0.0092933558	0.0186109789	0.0463971399	0.0917469349	0.13079161	0.1627966499	0.2110246819	0.250350849	0.3048909064	0.3678494287
		24	0.0092933558	0.0186109789	0.0463971399	0.0917229124	0.1305145212	0.1622814995	0.2110240945	0.2512398656	0.3082368375	0.3756349836
		25	0.0092933558	0.0186109789	0.0463971399	0.0917701444	0.1314649593	0.1646453711	0.2160496544	0.2588977387	0.3199655643	0.3922674493
		26	0.0092933558	0.0186109789	0.0463971399	0.0917981783	0.1320784868	0.1661416376	0.2191860019	0.2635564541	0.3268581693	0.4015931539
		27	0.0092933558	0.0186109789	0.0463971399	0.0918148815	0.1324474278	0.1670390065	0.221064083	0.2663373671	0.3309542787	0.4070875895
		28	0.0092933558	0.0186109789	0.0463971399	0.0918332501	0.1328553187	0.1680292196	0.2231342221	0.2693974087	0.3354521142	0.4130954287
		29	0.0092933558	0.0186109789	0.0463971399	0.0918148815	0.1319096368	0.1646453711	0.2127713503	0.2512398656	0.3052557459	0.3671395693
		30	0.2804043225	0.2804307968	0.2804306849	0.2824419265	0.2936358162	0.3076015603	0.3395945866	0.3726129599	0.4360081681	0.5062501671
		31	0.2282028398	0.2282720534	0.2283966889	0.2321115886	0.2501402053	0.2722994779	0.3161712349	0.3570298077	0.4282832163	0.5055097363
		32	0.1285571959	0.12882096	0.1300780405	0.1405004365	0.1728035053	0.2102623683	0.2759177807	0.3309764085	0.4166000229	0.5064062417
		33	0.0385279117	0.0399050315	0.0520014552	0.0857644851	0.1296492248	0.1737897684	0.2510038596	0.3149634844	0.4107153963	0.5095728029
		34	0.0414132861	0.0426644562	0.0556013911	0.0906996974	0.1315909129	0.1719615084	0.2472302909	0.3125580914	0.4120135512	0.5155324076
		35	0.1667398492	0.1668083896	0.1684998765	0.1769278694	0.1939463449	0.2164690183	0.2743919497	0.332996571	0.4298723116	0.5338709901
		36	0.571901225	0.4809868748	0.2572310197	0.0906996974	0.13846664	0.2102623683	0.3015985631	0.3570298077	0.4332965089	0.5065879619
		37	0.280374101	0.2803163577	0.2796857329	0.2824324514	0.3107251439	0.3523075161	0.4319112186	0.5050852594	0.6004873057	0.6906699291
		38	0.2281436344	0.2280480056	0.2269462585	0.2290164315	0.2618807843	0.3107762054	0.402524231	0.4852818567	0.5932125242	0.6978180412
		39	0.1284477781	0.1284082141	0.127503214	0.1364400564	0.1856183001	0.2499230837	0.3638732456	0.4622161314	0.5877153656	0.7082910499
		40	0.0383836412	0.0393932639	0.0505641755	0.0910704865	0.1589769715	0.2325011823	0.3572636797	0.4625343445	0.5956390451	0.7219060842
		41	0.0415800071	0.0432692838	0.0585694874	0.1051077781	0.1733832251	0.2453867476	0.3697923698	0.4755239198	0.6103624532	0.7372749639
		42	0.1669220072	0.1674933399	0.1727060989	0.1963843073	0.2445050078	0.301897672	0.4128672354	0.5126055384	0.6439943919	0.7668833986
		43	0.5719546673	0.4813638513	0.261267523	0.1051077781	0.1622670241	0.2499230837	0.3859206726	0.4852818567	0.597888571	0.6886076915
	RP II	44	0.0092933558	0.0186109789	0.0463971399	0.088576152	0.114865669	0.1355715116	0.1763069476	0.222114438	0.3254002405	0.5245940425
		45	0.0093032703	0.018688652	0.0477605332	0.1010561959	0.1447593752	0.1817514923	0.2541000006	0.3291660712	0.5177902113	0.906901762
	Mixed	46	0.009297958	0.0186473298	0.0470422714	0.0991650407	0.1554296254	0.205523743	0.2853568835	0.3501418172	0.4627159114	0.6332193562
		47	0.009297958	0.0186473298	0.0470422714	0.0991665408	0.1552861933	0.2050964873	0.2830363824	0.3486973798	0.4547343562	0.6180218972
		48	0.009297958	0.0186473298	0.0470422714	0.094322534	0.1271937301	0.1515476625	0.1955934432	0.2393537655	0.358895875	0.6427736262
		49	0.009297958	0.0186473298	0.0470422714	0.0954860337	0.1297666661	0.1523247569	0.1906728742	0.2309192409	0.3400971412	0.569284525
		50	0.009297958	0.0186473298	0.0470422714	0.0956078214	0.1306438392	0.1542404656	0.1940114502	0.2582949295	1.2367952176	88.318646897
		51	0.009297958	0.0186473298	0.0470422714	0.0945727484	0.128027543	0.1537977223	0.2021622097	0.2503410429	0.3660942692	0.5790987322
		52	0.0644099787	0.0650482994	0.0711706356	0.098855636	0.1381806088	0.1770240898	0.2401022101	0.2912557916	0.3660543985	0.4476325155
		53	0.009297958	0.0186473298	0.0470422714	0.0945376055	0.1287850263	0.1556499531	0.2042248396	0.2518048895	0.3646821941	0.5735697486

Table 10: Error table for the different estimators and sigma values for $u = 0$, $v = 3$ and $s = 5$, for example, $\epsilon = 0 + 0.333i$.

When we move a little further away from the origin, the picture changes a little in Table 10. While estimators 23 – 29 are still performing as well or better as the original estimators and their modifications, especially for larger sigmas (for small sigmas there is little difference to the unbiased estimator, as the derivatives are based upon $\hat{\epsilon}_I$), we see the addition of estimators 11, 15, 17, 19 and 23 – 29. Furthermore, estimator 44 works fairly well, with decreasing performance for larger variances. Also the average estimators from number 48 onwards perform consistently well.

			Signal to Noise Ratio									
			100	50	20	10	6.66	5	3.33	2.5	1.66	1
Estimator Number	Crude	1	0.008835695	0.017842126	0.044365329	0.087301107	0.130154326	0.168993772	0.243429645	0.317742371	0.41281676	0.618637165
		2	0.008835695	0.017842126	0.044365329	0.087287664	0.129020789	0.164161515	0.223896727	0.273248337	0.340509067	0.422012458
		3	0.008835695	0.017842126	0.044365329	0.089242343	0.146318434	0.204201617	0.310108897	0.398251593	0.527723223	0.724865545
		4	0.008835695	0.017842126	0.044365329	0.085309995	0.113988325	0.134389148	0.174696079	0.215502248	0.292778435	0.539262673
		5	0.00884453	0.01791062	0.045485113	0.099484401	0.172147021	0.242346092	0.370758292	0.475728722	0.684587844	0.963793582
		6	0.00884453	0.01791062	0.045485113	0.099319861	0.168970852	0.232713075	0.343063858	0.432766315	0.627997361	0.912181845
		7	0.00884453	0.01791062	0.045485113	0.095452833	0.137958123	0.166828343	0.218119576	0.266949186	0.47158396	0.926728128
		8	0.00884453	0.01791062	0.045485113	0.096236864	0.144153782	0.18144512	0.260591812	0.381112873	0.735061704	1.418286917
		9	0.00884453	0.01791062	0.045485113	0.096229425	0.142880219	0.17478258	0.230455807	0.322051253	0.713803439	1.711780811
		10	0.00884453	0.01791062	0.045485113	0.096363858	0.147429435	0.209496594	0.502025111	8.759807411	3.35635988	3.584963471
	Scaled	11	0.008835695	0.017842126	0.044365329	0.08603652	0.117144348	0.137559131	0.17605862	0.215929156	0.275911593	0.414185135
		12	0.00884453	0.01791062	0.045485113	0.096633434	0.143385081	0.173519996	0.222497852	0.269855674	0.455133403	0.836234503
		13	0.008835695	0.017842126	0.044365329	0.086166097	0.119449985	0.145884854	0.21548345	0.632091229	1.070305659	311.010474
		14	0.00884453	0.01791062	0.045485113	0.096707696	0.143998611	0.174907139	0.225343327	0.272903204	0.45460903	0.814957009
		15	0.008835695	0.017842126	0.044365329	0.085512261	0.114875951	0.135075385	0.171499445	0.20709598	0.267024018	0.41091029
		16	0.00884453	0.01791062	0.045485113	0.095759991	0.140726254	0.17532851	0.246661567	0.318117114	0.49142709	0.762276073
	Result and Parameter	17	0.008835695	0.017842126	0.044365329	0.087232011	0.127294534	0.159312785	0.210039703	0.251428907	0.308287205	0.376066388
		18	0.00884453	0.01791062	0.045485113	0.096372047	0.146171762	0.185586421	0.258331166	0.321971974	0.478215	0.719590294
		19	0.008835695	0.017842126	0.044365329	19.61335688	1.248350099	0.486566399	0.343741334	0.346703111	0.382669124	0.511173264
		20	0.00884453	0.01791062	0.045485113	0.095555656	0.142174385	0.179799232	0.250552575	0.325399576	0.536382957	0.903492544
		21	0.008835695	0.017842126	0.044365329	0.085590365	0.172548603	22.13266953	2430.465463	164468000000	9.25245E+13	4.62E+065
		22	0.00884453	0.01791062	0.045485113	0.095533236	0.13984624	0.172314253	0.235054861	0.303333491	0.523080171	0.946102502
	Result with dampening Parameter	23	0.008835695	0.017842126	0.044365329	0.087221276	0.12691442	0.158299606	0.207447466	0.247049058	0.300200267	0.36182045
		24	0.008835695	0.017842126	0.044365329	0.087209709	0.126778042	0.158177134	0.208077766	0.248822992	0.304730218	0.370893038
		25	0.008835695	0.017842126	0.044365329	0.087232011	0.127465587	0.160029786	0.213058351	0.256656572	0.31653606	0.388083762
		26	0.008835695	0.017842126	0.044365329	0.087246528	0.127882121	0.161142267	0.215995132	0.261178313	0.32312861	0.397455069
		27	0.008835695	0.017842126	0.044365329	0.08725545	0.128132898	0.161809332	0.21774762	0.263863638	0.327014064	0.402956064
		28	0.008835695	0.017842126	0.044365329	0.087265445	0.128410487	0.16254585	0.21967653	0.266811729	0.331262909	0.408961008
		29	0.008835695	0.017842126	0.044365329	0.08725545	0.12776745	0.160029786	0.209940806	0.248822992	0.300972918	0.360878851
		30	0.260791277	0.260783724	0.260771917	0.2621983	0.27247403	0.2875482	0.322065794	0.36483303	0.413429291	0.504988286
		31	0.212141318	0.21214357	0.212244064	0.214973446	0.230985747	0.253251063	0.299457059	0.349922371	0.406429636	0.504992544
		32	0.118719877	0.118861221	0.120090232	0.129304626	0.158221946	0.193986745	0.261478786	0.32575618	0.39632468	0.506790172
		33	0.033346644	0.034767828	0.047942764	0.080685747	0.12230015	0.163744354	0.24119242	0.312978103	0.392512001	0.511138157
		34	0.043677195	0.045020301	0.057172835	0.089714259	0.129276253	0.167409788	0.242516907	0.31438273	0.396306177	0.518681792
		35	0.166966767	0.167269557	0.16935368	0.177694452	0.195499424	0.218084321	0.275851293	0.340323256	0.417509218	0.539791271
		36	0.588182678	0.491278289	0.258699254	0.089714259	0.128778382	0.193986745	0.285524941	0.349922371	0.410945179	0.505150309
		37	0.260765056	0.260684878	0.260135038	0.261329778	0.285987021	0.325011945	0.408758051	0.482574148	0.587290773	0.67678448
		38	0.21208981	0.211949524	0.211000254	0.211456916	0.239677674	0.284921888	0.380642897	0.463961174	0.581267415	0.684549765
		39	0.118623736	0.118500261	0.117855931	0.125213129	0.169619414	0.229432614	0.347235104	0.44485049	0.578906889	0.697368368
		40	0.033218573	0.034328047	0.046840819	0.084794052	0.148880839	0.217896875	0.346442401	0.449495004	0.58989621	0.713336904
		41	0.043828207	0.045569509	0.059933342	0.101959189	0.167093456	0.2351541	0.363214261	0.466014657	0.606559697	0.730487265
		42	0.167136147	0.16790361	0.173233769	0.194699651	0.241394981	0.296886994	0.411270336	0.508017152	0.64234109	0.762342379
		43	0.588236072	0.49164952	0.262499414	0.101959189	0.150424989	0.229432614	0.36535784	0.463961174	0.585145677	0.674635698
	RP II	44	0.008835695	0.017842126	0.044365329	0.085353612	0.114880603	0.134648607	0.170883151	0.210995154	0.275861144	0.454743341
		45	0.00884453	0.01791062	0.045485113	0.095555656	0.142174385	0.179799232	0.250552575	0.325399576	0.536382957	0.903492544
	Mixed	46	0.00883983	0.017874459	0.044896707	0.093057778	0.148130711	0.196629774	0.27903957	0.345025637	0.448592943	0.613906034
		47	0.00883983	0.017874459	0.044896707	0.093088547	0.148293086	0.196913514	0.279552947	0.345614402	0.464160997	0.61866333
		48	0.00883983	0.017874459	0.044896707	0.090166324	0.125268261	0.149069954	0.192550232	0.232853034	0.341389515	0.588465708
		49	0.00883983	0.017874459	0.044896707	0.090979599	0.128082682	0.151053441	0.189777276	0.227420808	0.324218827	0.527845046
		50	0.00883983	0.017874459	0.044896707	0.091057954	0.128797908	0.152669477	0.193105321	0.415158808	0.45592654	231.1902332
		51	0.00883983	0.017874459	0.044896707	0.090218196	0.126846696	0.152392467	0.197473188	0.243765612	0.346699146	0.53991876
		52	0.059517405	0.060009002	0.066172985	0.092494803	0.129979007	0.166874269	0.231116291	0.287290885	0.354068022	0.44573681
		53	0.00883983	0.017874459	0.044896707	0.090239376	0.127792183	0.15540431	0.203880211	0.250391732	0.34986851	0.537360147

Table 11: Error table for the different estimators and sigma values for $u = 2$, $v = 2$ and $s = 5$, for example, $\epsilon = 0.219 + 0.219\%$.

For the combination $u = 2$, $v = 2$, $s = 5$ the same candidates stand out again in Table 11, delivering sensible alternatives to the original estimators. We do recognise some issues with estimators 19 and 21, who grossly misestimate the true ellipticity value for larger sigma values. For number 19 this might be due to singularities, but it seems there is some intrinsic problem with the estimator 21, which produces arbitrarily large values.

		Signal to Noise Ratio										
		100	50	20	10	6.66	5	3.33	2.5	1.66	1	
Estimator Number	Crude	1	0.0172139934	0.0344334018	0.0835400012	0.1569544484	0.2582418477	0.2921803664	0.3729540922	0.4503571119	0.5746668084	0.7702535092
		2	0.0172139934	0.0344334018	0.082889058	0.1450561682	0.1911304877	0.2276149286	0.2910051765	0.3343266574	0.4094769606	0.5058752237
		3	0.0172139934	0.0344334018	0.089981068	0.1829946111	0.2545107613	0.305726096	0.3839012481	0.4420763634	0.5529695023	0.7832287162
		4	0.0172139934	0.0344334018	0.0738773426	0.1163470895	0.1661187024	0.213579277	0.2953900393	0.3557195995	0.4757150433	0.8059934627
		5	0.0173245306	0.0354105964	0.1031081046	0.2155833824	0.3026207387	0.3700827236	0.4791590173	0.5637510988	0.7386371794	1.0387902845
		6	0.0173245306	0.0354105964	0.1012675313	0.1952370661	0.2567652628	0.2997530866	0.3691966257	0.4284851989	0.5812298168	0.8987942669
		7	0.0173245306	0.0354105964	0.0856729022	0.1316142803	0.1693342739	0.2043193716	0.269963593	0.3303629493	0.5337015717	1.0662762347
		8	0.0173245306	0.0354105964	0.0842816592	0.1180708092	0.1447446605	0.1740624772	0.242913326	0.3553225723	0.7253469846	1.8323320148
		9	0.0173245306	0.0354105964	0.0842816592	0.1183031598	0.1465051207	0.1767513961	0.2403951565	0.3530186141	0.8168788878	2.5848735853
		10	0.0173245306	0.0354105964	0.0840034858	0.1200135045	0.1579657606	0.2191891928	0.6567387063	1.4493026967	5.5287281444	5.0294966559
	Scaled	11	0.0172139934	0.0344334018	0.0744811658	0.1118608372	0.1590583311	0.2045124818	0.2789035982	0.3350497226	0.4291558633	0.6127087883
		12	0.0173245306	0.0354105964	0.089058361	0.1309140709	0.160253973	0.1905921683	0.2526428596	0.3124957289	0.5074058694	0.9588057398
		13	0.0172139934	0.0344334018	0.0747618445	0.1136829809	0.175695382	0.2444418187	0.3597241668	0.862127666	94.121228313	850.41768041
		14	0.0173245306	0.0354105964	0.0893116485	0.1324184645	0.1614671066	0.1907913968	0.2513687914	0.3101620137	0.4993984729	0.932273949
		15	0.0172139934	0.0344334018	0.0764749834	0.1195298037	0.1593465615	0.1945222421	0.2567012352	0.305460823	0.3963807691	0.5878422593
		16	0.0173245306	0.0354105964	0.0875527739	0.1283234398	0.1538394077	0.1804501271	0.2426912405	0.3024879702	0.4568308264	0.768703921
	Result and Parameter	17	0.0172139934	0.0344334018	0.0815583764	0.133486988	0.1720490443	0.2035452956	0.261196713	0.3041234484	0.3746755635	0.4658291885
		18	0.0173245306	0.0354105964	0.0882455143	0.1504242037	0.2047924286	0.2527837007	0.3347627821	0.3995645884	0.5471975669	0.8216140628
		19	0.0172139934	0.0344334018	1.521895E+25	636643.58511	93.921863362	5.6071158972	1.0208668748	0.6501459817	0.5597069566	0.6822520941
		20	0.0173245306	0.0354105964	0.0856729022	0.1345371522	0.1777184558	0.212394646	0.2740768629	0.3452197287	0.5566552941	0.979750524
		21	0.0172139934	0.0344334018	0.0764749834	4.2106343085	1.7372687146	244.14405543	709648.83597	4.075051E+13	3.494182E+13	1.279707E+38
		22	0.0173245306	0.0354105964	0.0856729022	0.1348111712	0.1792452572	0.2157821238	0.2817604557	0.3546273063	0.5768111749	1.0771819548
	Result with dampening Parameter	23	0.0172139934	0.0344334018	0.0826345844	0.1419151918	0.1847966387	0.218037924	0.2766743776	0.3178128698	0.3854496537	0.4700587662
		24	0.0172139934	0.0344334018	0.0821199854	0.1371353442	0.1770489441	0.2086852865	0.2668104836	0.3089894729	0.3783726979	0.4667299097
		25	0.0172139934	0.0344334018	0.0815430064	0.133266437	0.1720915069	0.2041961891	0.2638467517	0.3074589927	0.3798749965	0.472470746
		26	0.0172139934	0.0344334018	0.0817670584	0.1355232365	0.1758391673	0.2089462816	0.2694668334	0.3130971807	0.3863811256	0.4802841441
		27	0.0172139934	0.0344334018	0.0820084066	0.1375623838	0.1790766612	0.2128761369	0.2739537094	0.3174785853	0.3911658947	0.4856531293
		28	0.0172139934	0.0344334018	0.0822812864	0.1398843721	0.1827964695	0.2174109211	0.2791694382	0.3226030007	0.3967323487	0.4918249968
		29	0.0172139934	0.0344334018	0.082427367	0.1375623838	0.1744547684	0.2041961891	0.2639177232	0.3089894729	0.3820876524	0.4718512702
		30	0.4671673681	0.4670786432	0.4617624105	0.4480847438	0.484776657	0.4751204856	0.4971797783	0.5378036029	0.6003519979	0.6507289206
		31	0.3826585887	0.3825345463	0.3828389705	0.3895723269	0.4401300367	0.4389915298	0.4721670059	0.5192699078	0.5891998822	0.6460209594
		32	0.2325583695	0.23259579	0.2430550882	0.2844576684	0.3590037025	0.3729169825	0.4261928055	0.4853198707	0.5696979694	0.6391999685
		33	0.1133362934	0.1138639491	0.1333315902	0.2004897075	0.292892764	0.3183148102	0.3875902834	0.4567882198	0.5537494586	0.6344506557
		34	0.0234210925	0.029938752	0.0666852317	0.1438109477	0.2450122033	0.2773856278	0.3575838773	0.4343918802	0.541481319	0.6314803635
		35	0.1006919354	0.1010858196	0.1069078771	0.139355956	0.2202677749	0.2449970765	0.3253389348	0.4083104566	0.5274927235	0.6301647004
		36	0.3879076322	0.3316118109	0.1761994351	0.1438109477	0.3130617126	0.3729169825	0.4561502672	0.5192699078	0.5965035364	0.6518598523
		37	0.4670282013	0.4665014499	0.4663514191	0.4874900238	0.5188014422	0.5502808308	0.6096372913	0.6573065017	0.7354169761	0.815966739
		38	0.3823953968	0.3814465559	0.3842572053	0.4229497737	0.4671141292	0.5065838336	0.5773064133	0.6323070705	0.7219925079	0.8168585441
		39	0.2321260237	0.2308258873	0.2406388908	0.3105889936	0.3774093841	0.4313079547	0.5224451659	0.5907669827	0.699491665	0.815034378
		40	0.1128369654	0.1118505853	0.1318027696	0.2283165855	0.312432656	0.3774120976	0.4837897409	0.5623943357	0.6850964007	0.8143208255
		41	0.0229862445	0.0290881804	0.0756727565	0.1876116947	0.2784605688	0.3482882699	0.4619117745	0.5467284574	0.6784093393	0.8161833443
		42	0.1011386628	0.1028699923	0.1249758725	0.2019845956	0.2775568981	0.3411667501	0.4519065045	0.5400109512	0.6796659858	0.8251321113
		43	0.3879981104	0.3323170117	0.1930364533	0.1876116947	0.3308134189	0.4313079547	0.5573931945	0.6323070705	0.7308710273	0.81537442
	RP II	44	0.0172139934	0.0344334018	0.0738773426	0.1168958604	0.1638987552	0.2070100694	0.284032977	0.3451078008	0.4575432104	0.6773666324
		45	0.0173245306	0.0354105964	0.0856729022	0.1345371522	0.1777184558	0.212394646	0.2740768629	0.3452197287	0.5566552941	0.979750524
	Mixed	46	0.0172655987	0.0348948018	0.092222544	0.1753849394	0.2553601345	0.2905784441	0.3577125433	0.4171885281	0.5240492015	0.7119196455
		47	0.0172655987	0.0348948018	0.0917009192	0.1680471835	0.2188976854	0.2551357761	0.3152839826	0.3602432844	0.4622721158	0.6446209327
		48	0.0172655987	0.0348948018	0.0793854977	0.1215916474	0.161981262	0.2001592209	0.2719294062	0.3295258295	0.4627897335	0.797285227
		49	0.0172655987	0.0348948018	0.0809327902	0.1153966274	0.1513584548	0.1888114845	0.2605754702	0.3181766533	0.4453764727	0.721214127
		50	0.0172655987	0.0348948018	0.0811470126	0.1160596102	0.1504497936	0.1863324979	0.2562560481	0.3724676884	73.796577488	686.67918974
		51	0.0172655987	0.0348948018	0.0793854977	0.1235330378	0.1641485783	0.1960919709	0.2547851453	0.3117104812	0.4432380939	0.6999220468
		52	0.1164380495	0.1165263271	0.1364691604	0.1971961235	0.2599595968	0.2843969771	0.3413132573	0.3928753669	0.4711262237	0.5521425604
		53	0.0172655987	0.0348948018	0.0793854977	0.1230891508	0.1620548626	0.1930209115	0.2513986528	0.3075304411	0.4374883172	0.6887459361

Table 12: Error table for the different estimators and sigma values for $u = 3$, $v = 4$ and $s = 5$, for example, $\epsilon = 0.6 + 0.8i$.

In the matrix of scatter plots in Figures 2 and 3 (continued), we can see the behaviour of each estimator with a distribution of empirical ellipticities. Each individual estimate for a random experiment is marked by a blue dot, the true value is marked by a single red spot. Furthermore, we have added the average of all estimates of all random experiments, to give a sense of what the bias is.

It becomes evident where some estimators have their largest shortcomings by observing their distribution. The unscaled estimators often overshoot the feasible range of the unit circle, and therefore produce avoidable overestimation. On the other hand some of the scaling approaches, especially the parameter scaling approaches distort the actual value. We can see this through the way the scaled values on the unit circle boundary exhibit a grossly miscalculated angle. On the other hand, we can see some decent performances, where the individual estimates exhibit a tighter grouping around the true value without producing a noticeable bias shift. These candidates seem promising, and we will review

their practical potential by looking into their range-specific behaviour. We see this confirmed at the border of the unit circle with parameter values $u = 3$, $v = 4$, $s = 5$ in Table 12. The problems with estimators 19 and 20 persist for this configuration, yet our favoured estimators exhibit a decent performance when compared to the original estimators. Additionally, the $\hat{\epsilon}_{II}$ based estimators perform rather well for the mid-range variance values around $\sigma = 1$.

Figure 2: Matrix of estimators' 1-28 performance, with true values and bias added.

Figure 3: Matrix of estimator' 29-53 performance, with true values and bias added.

3.1 Non-perfect ellipticity

Sampling from the means u , v and s via a normal distribution implicitly assumes the shape of the galaxies to be perfectly elliptical, which of course is a strong assumption. We have therefore tested a simulation in which the X , Y and Z samples are distributed according to a non-central Student-t distribution (Gosset 1908).

We may assume the previous values u , v and s as the means, as well as the standard deviation σ and retrieve the corresponding characterising parameters ν and μ which are the degrees of freedom and noncentrality parameters, respectively.

If $T \sim S(\nu, \mu)$ is a non-central Student- t random variable with $\nu > 2$,

$$\mathbb{E}[T] = \mu, \quad \text{Var}[T] = \frac{\nu}{\nu - 2}. \quad (19)$$

Assume that $X \sim S(\nu_X, \mu_X)$, $Y \sim S(\nu_Y, \mu_Y)$ and $Z \sim S(\nu_Z, \mu_Z)$. By setting $\text{Var}[X] = \text{Var}[Y] = \sigma^2$, $\text{Var}[Z] = 4\sigma^2$, $\mathbb{E}[X] = u$, $\mathbb{E}[Y] = v$ and $\mathbb{E}[Z] = s$, we can estimate values of ν and μ . We would like to note that we have to assume $\sigma > 1$ due to the nature of the non-central Student- t distribution. This enables us to do the exact same experiment we have done with the assumption of normally distributed measurements.

We have listed the results as before for an example problem with values $u = 2$, $v = 2$ and $s = 5$ across different standard deviations, which translate to the SNR values 4.7619, 4, 3.333, 2.5, 2, 1.75, 1.5, 1.25 and 1. The matrix can be seen in Table 13.

Furthermore we have repeated the graphic representation of the estimator spread and behaviour in Figures 4 and 5. The colour scheme remains the same as previously introduced to help distinguish the classes of estimators.

The most striking change we can observe in the tables is how much more vulnerable the estimators have become to low SNR values. This is especially visible in the $\hat{\epsilon}_I$ based estimators. The scaling approaches with dampening parameter seem to work better across all SNR ranges, even though sometimes only slightly so. This of course depends on the position of the true mean values. The closer the true ellipticity gets to the unit circle boundary, the better the scaling approach works in comparison to the original estimator. We have observed similar behaviour in the normal sampling experiment, but here we restricted ourselves to only certain value pairings, full error tables may be provided upon request.

		Signal to Noise Ratio											
		4.7619048	4	3.3333333	2.8571429	2.5	2	1.7500263	1.5000015	1.25	1		
Estimator Number	Crude	1	0.191930334	12264.52813	663200.8383	16265868.4	18589172.28	7.72235E+14	1.35164E+36	3.47057E+54	5.63407E+44	1.7998E+229	
		2	0.1564019	0.160704369	0.166205097	0.168437421	0.169748981	0.173667234	0.179955524	0.185787632	0.190522788	0.200437073	
		3	0.206402216	12264.56316	663200.8828	16265867.85	18589172.34	7.72235E+14	2.57048E+20	0.43630731	5.63407E+44	1.7998E+229	
		4	0.15415414	13808.86678	761984.4534	18961423.75	21989652.02	9.26697E+14	3.10925E+20	0.364797685	6.95667E+44	2.2315E+229	
		5	0.227200723	0.245893122	0.267857262	0.287325327	0.293259671	0.311734435	0.32993535	0.336205728	0.358507041	0.36294665	
		6	0.220184822	0.236791888	0.256750173	0.274197109	0.279496121	0.296723114	0.313457032	0.319383667	0.341000415	0.345175898	
		7	0.169363372	0.176483352	0.186982674	0.198415899	0.19598358	0.206609342	0.221076973	0.22212436	0.242098804	0.244178294	
		8	0.180870587	0.193223173	0.210717082	0.228627169	0.226851572	0.246009581	0.26955152	0.264420302	0.287158647	1.237892878	
		9	0.175714589	0.186039522	0.200454733	0.216242245	0.211836184	0.228195982	0.250587288	0.241092047	0.264326595	1.302588472	
		10	0.210950403	0.663555232	0.264042228	0.292469341	0.334544684	0.413953541	0.384997258	0.363018579	0.575953597	0.418582325	
	Scaled	11	0.15551673	12264.49922	663200.8072	16265867.77	18589172.25	7.72235E+14	2.57048E+20	1168063.333	5.63407E+44	1.7998E+229	
		12	0.174055454	0.181144443	0.190329926	0.201251319	0.197031493	0.205605494	0.216931659	0.217261319	0.234173289	0.23584365	
		13	0.310382616	12264.53543	663200.8361	16265867.84	18589172.28	7.72235E+14	1.27937E+30	1.56021E+48	5.63407E+44	1.7998E+229	
		14	0.175344062	0.182848111	0.192431833	0.203558791	0.199816821	0.208442295	0.219734573	0.219611396	0.23611156	0.236512647	
		15	0.15236453	12264.49795	663200.8064	16265867.77	18589172.25	7.72235E+14	2.57048E+20	0.328143426	5.63407E+44	1.7998E+229	
		16	0.17504664	0.184115388	0.197284818	0.210583472	0.210274338	0.223864739	0.239424932	0.24326882	0.261093796	0.264301252	
	Result and Parameter	17	0.153257495	0.157440582	0.161941265	0.164279595	0.165624706	0.170156997	0.176822306	0.183125007	0.187295385	0.198077307	
		18	0.183087571	0.193379521	0.207006317	0.220693568	0.220648155	0.233502112	0.248694163	0.251553559	0.269378392	0.272160243	
		19	0.425095494	12264.65362	663200.9006	16265867.83	18589172.43	7.72235E+14	2.57048E+20	0.4197688	5.63407E+44	1.7998E+229	
		20	0.17839212	0.187644463	0.200832111	0.214661032	0.214652528	0.228314424	0.241291183	0.245257425	0.26517431	0.267536446	
		21	229.1403947	13391.18159	8466668.257	3.57303E+11	1.81945E+38	5.5599E+27	6.20434E+52	7543144.347	5.63407E+44	1.7998E+229	
		22	0.173179274	0.181263439	0.192408514	0.204638286	0.203199643	0.215715115	0.22859229	0.230129132	0.250070903	0.25270241	
	Result with dampening Parameter	23	0.152521687	0.156637551	0.161290489	0.163573915	0.165094424	0.169779306	0.176498959	0.182903198	0.187082001	0.198022452	
		24	0.152540532	0.156824145	0.161528286	0.163974014	0.165459488	0.170269405	0.177038444	0.183413398	0.187525823	0.198417945	
		25	0.153739268	0.158092654	0.162987038	0.165391219	0.166816588	0.171381528	0.178014858	0.184224089	0.188493373	0.199091684	
		26	0.154455566	0.158810129	0.163855166	0.166218164	0.167610642	0.172009008	0.178550665	0.184658737	0.189046436	0.199462111	
		27	0.154885172	0.159233448	0.164374905	0.16671067	0.168084671	0.172379487	0.178865407	0.184912604	0.189374838	0.199680164	
		28	0.155359773	0.159696652	0.164948149	0.167252373	0.168606531	0.172784833	0.179208968	0.185188898	0.18973536	0.199918496	
		29	0.153647573	0.157661686	0.162070792	0.164202918	0.165459488	0.170024327	0.176650638	0.18297032	0.187082001	0.197956114	
		30	0.29605063	0.286390979	0.288587516	0.89346738	0.289828418	11650.4301	1.35164E+36	3.47057E+54	14372.34069	4.92838E+18	
		31	0.260528177	0.253803316	0.259027343	0.86611694	0.264394322	11650.40756	1.35164E+36	3.47057E+54	14372.32355	4.92838E+18	
		32	0.199414478	0.197077475	0.206989132	0.817447821	0.218754818	11650.36638	1.35164E+36	3.47057E+54	14372.29152	4.92838E+18	
		33	0.168834806	0.165232123	0.174368252	0.784314814	0.185656226	11650.33392	1.35164E+36	3.47057E+54	14372.26361	4.92838E+18	
		34	0.174241921	0.166291834	0.170876512	0.776999218	0.175000929	11650.31767	1.35164E+36	3.47057E+54	14372.24206	4.92838E+18	
		35	0.228411234	0.21425474	0.211878037	0.811595796	0.203248643	11650.33342	1.35164E+36	3.47057E+54	14372.22992	4.92838E+18	
		36	0.204353326	0.217947883	0.240392784	0.858595569	0.264394322	11650.41645	1.35164E+36	3.47057E+54	14372.34069	4.92838E+18	
		37	0.309526521	0.321427143	0.334069234	0.3435318	0.351047649	0.360847187	0.367619599	0.373218827	0.378966182	0.382029442	
		38	0.267712371	0.28162486	0.296355672	0.307027345	0.315599866	0.326641611	0.334435732	0.340800147	0.346763578	0.350334535	
		39	0.210077319	0.227483276	0.245757515	0.258328843	0.26870673	0.281716003	0.291669871	0.298921453	0.305711793	0.309941999	
		40	0.20017035	0.218488524	0.238109637	0.25157654	0.263059664	0.276762287	0.287864602	0.294853269	0.302522931	0.307164493	
		41	0.220786028	0.2383995	0.257774683	0.271429328	0.282863574	0.296307246	0.307709124	0.314377791	0.322445806	0.326962452	
		42	0.287331976	0.302738494	0.320812197	0.33339027	0.343529153	0.355762545	0.367005728	0.372680626	0.380642086	0.384652473	
		43	0.213446035	0.243578073	0.274722029	0.297611638	0.315599866	0.339787976	0.356094634	0.368346131	0.378966182	0.388783802	
		RP II	44	0.154462692	13093.01596	711558.2988	17430952.63	20066032.21	8.35065E+14	2.78347E+20	0.33993442	6.10779E+44	1.9579E+229
			45	0.17839212	0.187644463	0.200832111	0.214661032	0.214652528	0.228314424	0.241291183	0.245257425	0.26517431	0.267536446
		Mixed	46	0.201997568	6132.375506	331600.5365	8132934.322	9294586.267	3.86117E+14	6.7582E+35	1.73528E+54	2.81703E+44	8.9989E+228
			47	0.186520185	0.195824319	0.206364783	0.214183348	0.215522143	0.22100835	0.226830584	0.22849778	0.236377481	0.235081858
	48		0.159695073	6904.516955	380992.3113	9480711.957	10994826.09	4.63348E+14	1.55462E+20	0.260989574	3.47833E+44	1.1158E+229	
	49		0.160219064	6450.733817	351868.0797	8685691.354	9938607.995	4.14841E+14	1.38454E+20	0.248270604	3.039E+44	9.665E+228	
	50		0.161652478	6450.74737	351868.0892	8685691.375	9938608.005	4.14841E+14	1.38454E+20	0.247499175	3.039E+44	9.665E+228	
	51		0.16177151	6546.593867	355779.237	8715476.402	10033016.19	4.17533E+14	1.39173E+20	0.252674326	3.05389E+44	9.7895E+228	
	52		0.16573028	0.168317692	0.177425166	0.485318974	0.188023067	5825.266023	6.7582E+35	1.73528E+54	7186.239078	2.46419E+18	
	53		0.16380382	6546.596541	355779.2403	8715476.406	10033016.2	4.17533E+14	1.39173E+20	0.256486867	3.05389E+44	9.7895E+228	

Table 13: Error table for the different estimators and sigma values for $u = 2$, $v = 2$ and $s = 5$, for example, $\epsilon = 0.219 + 0.219\%$, assuming Student-t distribution.

Figure 4: Matrix of estimators' 1-28 performance, with true values and bias added for the Student-t distribution.

Figure 5: Matrix of estimators' 29-53 performance, with true values and bias added for the Student-t distribution.

In the figures we see much of the behaviour from previous experiment, yet notice a recurring pattern of the individual estimators, which seem to group along the axes, giving the estimate cloud a diamond-shaped appearance. However,

generally most of the scaling approaches seem to be doing their job, and resize estimates based on measurements exceeding the unit circle to a sensible new approximation.

3.2 Performance by signal to noise ratio

With the investigation of the performance of the estimators for predetermined parameter sets complete, we have settled on a number of promising new candidates. While the original estimator delivers a solid performance, it neglects the potential infeasibility of output values, or the input's violation of the ellipticity formula rules.

To further test the estimator for performance on randomly selected parameter ranges, we now first off select the ellipticity. This can be via a two-dimensional normal distribution as a first approximation:

$$\epsilon \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right). \quad (20)$$

This technically allows for ellipticities outside the feasible range of the unit circle. However, this can be somewhat corrected by choosing the sampling variance accordingly. Alternatively, we can directly proceed to the truncated normal distribution, which essentially crops the normal distribution to our needs. A rather simple modification of the normal distribution from Horrace (2005) could be considered (as the R package `tmvtnorm` is available on CRAN (Wilhelm & Manjunath 2010)):

$$f(x, \mu, \Sigma, a, b) = \frac{\exp\left\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right\}}{\int_a^b \exp\left\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right\} dx} \mathbb{I}_{(a \leq x \leq b)}, \quad (21)$$

which provides a scaled probability function, restrained between a and b , which can be limited to the unit circle. However, in the test we have been conducting, the truncated normal samples seem to place much stress on the boundaries, falsifying the results. It is hard to determine what constitutes a close approximation to the real life underlying processes, therefore we will rely on the untruncated normal distribution (also suggested in Tessore & Bridle (2018)).

Once the ellipticity samples are randomly drawn from the distribution decided upon, we are still left with the task of constructing the respective parameters u , v and s accordingly. We therefore construct an error plot which lists the mean absolute error of each estimator against the SNR (Schroeder 1999; Bushberg 2006):

$$\text{SNR} = \frac{s}{\sigma}. \quad (22)$$

Since this provides a set of s values for every experiment, we can therefore determine u and v simply through the imaginary and real parts of the ellipticity:

$$u(s, \epsilon) = \frac{2\Re(\epsilon)s}{\Re(\epsilon)^2 + \Im(\epsilon)^2 + 1}, \quad v(s, \epsilon) = \frac{2\Im(\epsilon)s}{\Im(\epsilon)^2 + \Re(\epsilon)^2 + 1}. \quad (23)$$

From this we can then in turn sample the actual observations X , Y and Z . To summarise, with the variance σ_e for the initial ellipticity sampling, and σ_o for the observation sample, we draw values as follows:

$$\epsilon \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma_e \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad (24)$$

$$X \sim \mathcal{N}(u(s, \epsilon), \sigma_o), \quad (25)$$

$$Y \sim \mathcal{N}(v(s, \epsilon), \sigma_o), \quad (26)$$

$$Z \sim \mathcal{N}(s, \sigma_o). \quad (27)$$

This procedure of procuring the necessary randomised observations ensures that we do not focus on singular areas and parameter combinations as before. This should provide a more complete view of how the estimators are likely to perform in practical applications.

We have settled on the standard deviations $\sigma_e = 0.2$ and $\sigma_o = 1$ and a SNR range of $[0, 100]$ which should cover most values of interest. Since there are more points to investigate, we have lowered the sample size for each individual SNR to $n = 500$ for a first glance at the estimator performance in relation to the SNR. The results are visualised in Figures 6 and 7, where we split the estimators into two groups for better visibility.

The sample size mainly controls the variance of the error dots in the plots. We split the plot matrices in two groups, 1 – 28 and 29 – 53. We observe the generally expected shape of the curve for most estimators, with higher values of $s = \text{SNR} * \sigma_o$ the absolute error subsides. Since a higher s value diminishes the ellipticity ϵ , as it is only recurring in the denominator of the fraction, we get a smaller spread in initial samples.

Figure 6: The SNR versus mean absolute errors for estimators 1-28, normal distribution.

Furthermore, we can see that some estimators are susceptible to failure, mostly due to the lack of a scaling mechanism. This becomes visible in some of the outliers, which seem to become more frequent as $s \rightarrow 0$, leading to a divergence of the ellipticity value. We therefore measure the success of the other estimators mainly against $\hat{\epsilon}_2$ and $\hat{\epsilon}_5$, since the original unbiased estimator fails for ellipticities around the unit circle (as previously discussed). We see many of the trends from the static parameter tables confirmed. The odd numbered estimators based on the unbiased estimators 15, 17, 19, 21 and 23 for example exhibit a lower mean absolute error, which is especially visible in the lower SNR regions near the coordinate origin. The y -intercept for $\hat{\epsilon}_2$ is between 0.9 and 1, a value which a number of estimators seem to underbid.

Figure 7: The SNR versus mean absolute errors for estimators 29-53, normal distribution.

Figure 8: Average absolute error, $\text{SNR} = [0, 100]$, logarithmic x-scale.

For the parameter and result based scaling techniques, we see good success in estimators such as number 44 or 49, especially for the tail regions there seems to be a fairly quick convergence. Unfortunately, the approach does not seem stable for small SNR values, as the error values scatter a lot more and eventually break due to missing values in the experiment output. We now move on to a direct comparison of all the methods, to give us an impression of how the performance relate to one another, and if there is significant potential for the newly constructed empirical ellipticities.

Figure 9: Average estimator variance, $\text{SNR} = [0, 100]$, logarithmic x-scale.

Figure 10: Average absolute error of selected estimators, $\text{SNR} = [0, 30]$.

Figures 8 and 9 depict all approaches in a single frame, showing the averages of absolute error and variance, respectively. The crude and unbiased estimators ($\hat{\epsilon}_2$ to be precise) are highlighted in red and superimposed over the bulk of the other estimators. The plots primarily give us a first glance at the outcome of the experiment, before we highlight single estimation approaches. As we can see, several monochrome point clouds shape up underneath the red line plots. This is a first indication that stronger estimators for arbitrary values exist.

Figure 11: Average variance of selected estimators, $\text{SNR} = [0, 30]$.

We single out the approaches found to deliver the strongest results, based on the error tables for fixed ellipticity values and the individual and comparative plots. We narrow down the observed estimators to the preferred selection, in order to make the plots more readable. Since the SNR area of interest is below 30, we have omitted the larger tail of the plot, which served to check the estimators for convergence. In the focus plots, we get a much clearer picture of the comparative performance of the estimators, depicted in Figures 10 and 11.

Starting with the largest SNR, we can see that estimator 39 exhibits the lowest average error, converging towards zero the fastest. This effect takes effect roughly from $\text{SNR} = 20$ onwards, whereas the estimator has a seemingly weaker performance for smaller SNR values, for example, higher standard deviations, eventually matching the benchmark of $\hat{\epsilon}_{II}$. The variances in this interval are very tightly grouped, with estimator 37 narrowly beating out the aforementioned approach number 39.

The next interval which is clearly marked by a difference in accuracy covers $\text{SNR} = [10, 20]$, edging closer to the area of practical applicability. Both the scaled estimator number 15 and the averaged estimator 50 show a reduction in absolute error by as much as 25 – 30%. The variance plots show an even clearer gap to the next best competitor, where the faint beige colored error points of estimator 50 seem to surpass the scaled unbiased estimator once again, by roughly 60%. It should be stated at this point that there are some outliers visible in the plot from both approaches, suggesting robustness issues. This becomes even more evident when both estimators fail due to missing values being introduced, thus questioning the practical use of those estimators for $\text{SNR} \leq 10$, and the robustness in general.

		SNR range			
		[0,100]	[0,50]	[0,30]	[0,15]
Estimator number	Crude 2	0.06164	0.10051	0.14769	0.23746
	5	0.20597	0.38931	0.62909	1.19773
	Scaled 15	2.04E+82	4.08E+82	6.80E+82	1.37E+83
	R & P 17	0.05287	0.08320	0.11972	0.18952
	19	4.95E+304	8.23E+155	1.96E+104	1.37E+83
	Dampening	23	0.04935	0.07624	0.10838
		24	0.05194	0.08137	0.11676
		25	0.05496	0.08733	0.12643
		37	0.03898	0.07675	0.12479
		39	0.04900	0.08906	0.13926
		41	0.07119	0.11396	0.16599
	R&P II 44	3.29E+82	6.59E+82	1.10E+83	2.21E+83
	Mixed 48	1.75E+82	3.51E+82	5.86E+82	1.18E+83
		50	1.66E+82	3.33E+82	5.56E+82
					1.12E+83

Table 14: Average absolute error over the respective SNR ranges.

Lastly, and most importantly we focus on $[0, 10]$, representing the interval of the highest practical interest. Once more it becomes clear which approach delivers the best results, as estimator 23 undercuts the other approximations consistently throughout the range in question. The absolute difference is about $0.075 - 0.1$, which is roughly an error reduction of 10%. Perhaps more importantly, we cannot see the same amount of outliers as for estimators 15 and 50, suggesting greater stability of the estimators, less susceptible to variances in the input parameters. Additionally, we see a reduction in variance, as the estimator 23 exhibits the lowest variances for the range $[0, 10]$. The similar estimators 24, 25 and onwards perform similarly, albeit less accurately. Possibly the parameter $\beta = 0.25$ is not yet optimally chosen, and can be altered for even better results in the future.

		SNR range			
		[0,100]	[0,50]	[0,30]	[0,15]
Estimator number	Crude 2	0.15750	0.26457	0.37684	0.56244
	5	0.18141	0.31217	0.45510	0.70801
	Scaled 15	1.37E+38	2.74E+38	4.57E+38	9.17E+38
	R & P 17	0.14649	0.24270	0.34125	0.50089
	19	6.77E+163	1.62E+75	2.36E+49	9.17E+38
	Dampening	23	0.14105	0.23173	0.32296
		24	0.14515	0.24003	0.33692
		25	0.14961	0.24900	0.35170
		37	0.19766	0.26113	0.34431
		39	0.15969	0.25069	0.35927
		41	0.17986	0.29598	0.41850
	R & P II 44	2.21E+38	4.43E+38	7.38E+38	1.48E+39
	Mixed 48	2.35E+38	4.71E+38	7.87E+38	1.58E+39
		50	2.24E+38	4.48E+38	7.47E+38
					1.50E+39

Table 15: Average variance over the respective SNR ranges.

We have analysed and discussed the test results for all newly suggested approaches, and have settled on a recommended estimation technique for a given SNR range. As a last step we have averaged the mean absolute values across different SNR intervals, thus putting our observations from the visualised results in comprehensible numbers. Table 14 lists the mean error results, whereas Table 15 contains the variances summarised in the same way.

Most strikingly, we recognise the issues of stability, which are visible for estimators 15, 17, 44, 48 and 50. The tremendously high errors and variances are due to the inherent flaw to these estimation techniques, that they allow for arbitrarily large estimates, which can be caused by unchecked or unscaled problematic input parameters. Starting with the largest range of SNR in $[0, 100]$, we recognise the lowest average error values from estimators 37, 39, closely followed by number 23, all offering a better option than the benchmark $\hat{\epsilon}_2$. However, this behaviour does not persist for all SNR intervals. We had realised earlier that the estimators 37 and 39 give their strongest performance only from $\text{SNR} = 30$ onwards, which lies outside of the realistically sensible ranges. If we draw the circle a little bit closer, and omit results larger than $\text{SNR} = 50$ estimator 23 delivers the best results, undercutting all other approaches. As we reduce the investigated interval, the effect becomes stronger, with estimator 23 reducing the error of $\hat{\epsilon}_2$ by almost 29% for the narrowest range $[0, 15]$.

For the variance the observations are even clearer, here the scaled estimator 23 exhibits the smallest variance for all listed ranges. Realistically speaking, the smallest range $[0, 15]$ is of the highest significance for practical applications, therefore leaving us with a clear recommendation on which estimator has the largest potential in practice.

In a more categorical overview, we try to investigate how the different types of estimators perform in a given setting. The first category we dubbed ‘crude’ estimators, due to their straightforward nature and simplistic scaling and censoring approach, do not perform remarkably better than any baseline method. In the ‘parameter-only’ scaling section estimator 15 alone performed note-worthily better than the baseline approach. The more sophisticated estimators which scale either parameters, or the entire result based on certain parameter values perform considerably better. This may lead us to the conclusion that result scaling is the general approach, a claim that is verified by the section of exclusively result-scaling approaches. Censoring on the other hand does only seem to slightly increase performance, certainly not to a degree that would allow for routine discarding of data. Another problem with data censoring is the censoring criterion. While $X^2 + Y^2 < Z^2$ effectively works for the Type 2 basic estimator, it does not necessarily omit corrupted data for the Type 1 estimator. The most complex estimators of the second to last category perform best in the very high SNR region, beginning with a ratio of 17 or higher. This puts them outside the most common region of ratio values.

What strikes us is that the result-only scaling introduces only very little bias. As the basic estimator is unbiased, it has to be expected to trade some degree of underestimation of the ellipticity (as we always scale down) for a generally more accurate result overall, observable in Figure 2. However, the slight bias only occurs in the individual ellipticity estimation of a galaxy, and as the scaling is centered towards the origin (for example, only the radius, not the angle is corrected), no bias will appear in an average of galactic ellipticities. The dampening parameters introduced as β could for this purpose be analysed more thoroughly, as the values we have chosen were somewhat arbitrary to give us a more diverse picture. It is imaginable that a dependence on the SNR or standard deviation σ could be derived for an optimal β either empirically or theoretically.

Other scaling functions besides the inverse tangent function may be considered as well, as only the asymptotic values of the scaling function are fixed. A number of different functions with the correct properties and numerous optimisation parameters may be tested, to increase accuracy, while even further minimising variance and bias of the estimation. We may consider this question in further work, as this would have gone beyond the scope of this paper.

At this point we want to stress that these simulation are highly idealised. The assumption of normal distributed samples corresponds to galaxies exhibiting the shape of perfect ellipses, which in nature obviously will not have to be the case. Furthermore the process of deriving the measurement values from the actual galaxy is much more complex than our simulation does justice. Various steps along the actual process of obtaining measurement may introduce bias or distortions which this experimental setup can not reproduce. The performance analysis of this section should be seen as a very simple assessment of theoretical applicability of these approaches, rather than emulating practical effects.

4 Conclusions

In this paper, we have introduced and discussed a number of extensions to the base estimators $\hat{\epsilon}_I$ and $\hat{\epsilon}_{II}$. Due to the limited sample size, strictly speaking we were not able to employ common statistical methods. We employed scaling and censoring techniques based on the conditions the parameters have to fulfill, or are at least desirable to the coherence of the input data.

As we have demonstrated in the simulation experiments, and have explained in the discussion part of Section 3, some of the proposed techniques have definite potential for practical use. Both for fixed underlying means and randomised ellipticities, we see strong performances for a number of our proposed approaches, which retain sufficient numerical stability to be of use in application scenarios.

Other application areas may be considered as well, as the estimation problem is not strictly bound to the original problem anymore. We would have to reconfigure the scaling criterion according to the new application, and might reconsider the scaling function, but the general approach may be much more widely applicable.

We believe there might be even further room for improvement, if the estimators get used in accordance to the SNR or true ellipticity region (for example, $r \approx 0$ or $r \approx 1$). As the variance is known in most cases, it is possible to create confidence areas of the true location of the ellipticity. Hence, an estimation selection algorithm could be developed, which based on the known input factors chooses the optimal approach. As touched upon in the simulation section, scaling functions other than the inverse tangent function may be considered as well. As the dampening parameter β in the most successful category of estimators had tremendous influence over the performance of the estimator, an optimal parameter dependent on σ would be desirable and could help to minimise. We have given first possible starting points with respect to selective estimators.

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References

- Bartelmann, M., Schneider, P., 2001, Physics Reports, 340, 4
Bushberg, J. T., 2006, The Essential Physics of Medical Imaging. Lippincott Williams and Wilkins, Philadelphia
Gosset, W. S., 1908, Biometrika, 6
Horrace, W. C., 2005, Journal of Multivariate Analysis, 94
Kilbinger, M., 2015, Reports on Progress in Physics, 78
Schneider, P., Kochanek, C. S., Wambsganss, J., 2006, Gravitational Lensing: Strong, Weak and Micro. Springer Verlag, New York
Schroeder, D. J., 1999, Astronomical Optics. Academic Press, London
Student (William Sealy Gosset) The Probable Error of a Mean Biometrika, Vol. 6, No. 1 1908.
Tessore, N., Bridle, S., 2018, preprint (arXiv:1807.06116)
Wilhelm, S., Manjunath, B. G., 2010, The R Journal, 2