LOCAL TIME-SPACE CALCULUS WITH APPLICATIONS

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This thesis deals with extensions of Itô's formula using the local time, namely the local time on curves and surfaces formulae and the local time-space integral, and existence and uniqueness results for stochastic differential equations involving the local time.

In Chapter 2, I weaken the required conditions for the local time on curves and surfaces formula to hold, and also allow semimartingales with jumps. In addition, under vanishing of a sectional derivative (the so-called 'smooth fit' condition), I remove technical barriers and show that the classical Itô formula still holds. This has applications in optimal stopping and control.

In Chapter 3, I undertake the challenge of Ghomrasni and Peskir by establishing rigorously some manipulations of the local time-space integral introduced by Eisenbaum, which takes the form

$$\Lambda(H, X) = \frac{1}{2} \int_{\mathbb{R}} \int_0^t H(s, a) \, \mathrm{d}\ell_t^a(X). \tag{1}$$

I provide a unifying local time-space calculus which unifies and extends the results of other authors in this direction. This includes new representations for the correction term in Itô's formula which were presently unknown or only derived in special cases.

In Chapter 4 I demonstrate applications of these formulae in the area of stochastic differential equations involving the local time. In particular, I deal with equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}B_s + \int_{\mathbb{R}} \int_0^t h(s, a) \, \mathrm{d}_s \ell_s^a(X) \, \mathrm{d}\nu(a), \tag{2}$$

where I extend the seminal existence and uniqueness results of Le Gall to this time-inhomogeneous case.

Finally in Chapter 5, I demonstrate the difficulties in extending the previous result to higher dimensions. I derive a fully higher-dimensional extension of the local time on curves formula, and outline recent approaches by other authors. Finally I provide some partial results and ideas which form a basis for further work in this area.

Chapters 2, 3 and 4 form the basis of two preprints which have been submitted for publication.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

Itô's formula is the stochastic analogue of the change of variables formula from classical calculus, and is just as fundamental in the theory of stochastic calculus. Given a function $F \in C^2$ and a continuous semimartingale X, Itô's formula provides an integral representation of $F(X)$ as

$$F(X_t) - F(X_0) = \int_0^t F_x(X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t F_{xx}(X_s)\,\mathrm{d}\langle X, X\rangle_s\,. \tag{1.1}$$

Just as life is not always easy, F is not always C^2. As the study of continuous time stochastic processes has evolved, many attempts have been made to establish a formula in the same spirit as Itô's, but in different settings or with greater generality. The first such attempt was undertaken by Itô's student, Tanaka, who connected stochastic calculus and the so-called 'local time'.

The study of local time began with Lévy [44], initially arising as the occupation density of a one-dimensional Brownian path with respect to Lebesgue measure. Fix some time T and consider the measure μ_T on $\mathcal{B}(\mathbb{R})$ whose value $\mu_T(A)$ is defined as 'time spent by the Brownian path in the set A up to time T'. The local time field of B is the (pathwise) Radon-Nikodym derivative of μ_T with respect to Lebesgue measure. Essentially the defining property is the equality

$$\int_0^t f(B_s)\,\mathrm{d}s = \int_{\mathbb{R}} f(a)\,L_t^a\,\mathrm{d}a, \tag{1.2}$$

which holds for all bounded Borel-measurable functions f. The existence of a continuous modification of this field of local times provides both a useful technical tool, and some intuitive connection with our ideas of 'time spent at a point'. In analogy with a real-valued random variable, where all values may have zero probability of occurrence even though some values seem intuitively 'more likely' than others, a typical Brownian path spends Lebesgue-measure zero time at any point but naturally spends longer in the vicinity of certain points, which have a correspondingly larger local time. Therefore the local time at a point represents 'total time spent' by the process at this point, hence its name.

Thereafter, the theory of local time has developed in two major directions. The first involves representation of continuous additive functionals. We do not pursue this direction, but it is interesting that the local time is a nice example of a such a functional, and it is now known that all such continuous additive functionals have a representation in terms of the local time (see [38, p. 190]).

The second direction, and the one we will be concerned with, develops the connection with stochastic calculus. Essentially the local time can be defined as the correction term in a stochastic change-of-variables formula for the absolute value function, in lieu of the usual quadratic variation integral. Formally,

$$|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s)\, \mathrm{d}B_s + L_t^0. \tag{1.3}$$

Thus the local time (at zero) is the process of bounded variation in the Doob-Meyer decomposition of $|B|$, the absolute value of Brownian motion. Using Doob's characterisation of Brownian motion, we immediately see that

$$\int_0^t \operatorname{sgn}(B_s)\, \mathrm{d}B_s \tag{1.4}$$

is another Brownian motion. So intuitively the local time is exactly the required 'boost' necessary to account for the effect of reflection of the Brownian motion from a 'mirror' at a fixed point which disturbs the martingale property. The above formula is called the Tanaka equation, whilst the generalisation to differences of convex functions and general semimartingales is named the Itô-Tanaka formula, and takes the form

$$F(X_t) - F(X_0) = \int_0^t F'_-(X_s)\, \mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a\, \mathrm{d}F'_-(a), \tag{1.5}$$

where F'_- is the left derivative of F, and $\mathrm{d}F'_-$ is its associated Lebesgue-Stieltjes measure. The reader is invited to read the Appendix A.1 to see the justification of these results. We note that in the one-dimensional time-independent case, the class of functions which are the difference of two convex functions is the most general which preserves the semimartingale property. Indeed, as a special case of the deep result of Çinlar, Jacod, Protter and Sharpe [14], given a standard Brownian motion B, we know $F(B)$ is a semimartingale if and only if F is the difference of two convex functions.

Let us now imagine that the reflective 'mirror' in the Tanaka formula changes position in time. The Itô formula for a time-dependent function $F \in C^{1,2}$ of a continuous semimartingale $(X_t)_{t\geq 0}$ reads

$$\begin{aligned} F(t, X_t) = F(0, X_0) + \int_0^t F_t(s, X_s)\,\mathrm{d}s &+ \int_0^t F_x(s, X_s)\,\mathrm{d}X_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s, X_s)\,\mathrm{d}\langle X, X\rangle_s\,. \end{aligned} \tag{1.6}$$

We seek a counterpart of the Tanaka formula for the time-dependent case, which involves replacing the fixed level of the local time by a curve, formally yielding the expression L_t^b for some curve $b : [0, t] \to \mathbb{R}$. This expression is called the local time of Brownian motion on a curve, and the corresponding change-of-variables formula (time-inhomogeneous Itô-Tanaka formula) is called the local time on curves formula. This was derived by Peskir [46], and takes the form

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \frac{1}{2}\big(F_t(s, X_s+) + F_t(s, X_s-)\big)\,\mathrm{d}s \\ &+ \int_0^t \frac{1}{2}\big(F_x(s, X_s+) + F_x(s, X_s-)\big)\,\mathrm{d}X_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s, X_s)\ \mathbb{1}_{\{X_s \neq b(s)\}}\,\mathrm{d}[X, X]_s \\ &+ \int_0^t \frac{1}{2}\big(F_x(s, X_s+) - F_x(s, X_s-)\big)\ \mathbb{1}_{\{X_s = b(s)\}}\,\mathrm{d}_s L_s^b(X). \end{aligned} \tag{1.7}$$

Our considerations must naturally make assumptions on the geometry of the curve b and of the discontinuity of the first derivative, captured by $F_x(s, X_s+) - F_x(s, X_s-)$. Further, if X is an Itô process the other integral terms combine to form the infinitesimal generator of X applied to F, which is often more tractable than the individual components. We will examine the local time on curves formula, its generalisations to surfaces, and prove its validity under weaker conditions including the 'smooth-fit' hypothesis in Chapter 2.

The local time on curves formula allows F to admit a discontinuity in its first derivative over a time-dependent curve. If we seek to admit more general F, we must deal with general measures which admit a Cantor (singular but not discrete) part. Further, the one dimensional decomposition of such measures does not extend naturally to the time-space case (see Appendix A.2).

Other authors have introduced correction terms which aim to overcome these difficulties which we will examine in Chapter 3. To unify these ideas, Eisenbaum [18] and Ghomrasni and Peskir [36] constructed a correction term of the form

$$\int_0^t \int_{\mathbb{R}} F_x(s,a)\,\mathrm{d}\ell_s^a. \tag{1.8}$$

One may obtain many different versions of this term by formal manipulations, which led Ghomrasni and Peskir to invite a rigorous proof of these different forms in [36]. This challenge is undertaken starting in Section 3.3, which leads on to further results and applications.

A natural area of application of these generalised Itô formulae including the local time is in stochastic differential equations. In Chapter 4 we will prove existence and pathwise uniqueness results for equations of the form

$$X_t = X_0 + \int_0^t b(s,X_s)\,\mathrm{d}s + \int_0^t \sigma(s,X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} \int_0^t h(s,a)\,\mathrm{d}_s\ell_s^a(X)\,\mathrm{d}\nu(a). \tag{1.9}$$

Such equations are a natural generalisation of the work of Le Gall, who treated such equations in the seminal paper [42], which are further generalisations of the Skew Brownian motion first introduced in SDE form by [37]. Recent work by Étoré and Martinez [27] has led us to consider this more general case, and has shown modern interest in this problem.

In Chapter 5, we examine some areas for further work. The generalisation of work involving local time to higher dimensions is fundamentally hampered by the lack of existence of a higher dimensional local time. It is natural to determine what a suitable higher dimensional analogue of the one dimensional results might look like, if it exists, and the local time on curves formula is a natural candidate. We can show a version in higher dimensions, and recent work by Flandoli and Bevilacqua [8] also indicates the potential for further developments. We also examine how the construction via the Tanaka formula fails in higher dimensions, and hence what tools might be available to avoid these difficulties or form an alternative line of attack.

Chapter 2

Local Time on Curves and Surfaces

2.1 Introduction

The prototype non-smooth function which necessitates a generalisation of the classical Itô formula for C^2 functions is the absolute value function. Namely, we have Tanaka's formula

$$|X_t| = |X_0| + \int_0^t \operatorname{sgn}(X_s)\, \mathrm{d}X_s + \ell_t^0. \tag{2.1}$$

The absolute value function is a simple linear function away from the origin, but at the origin has an irregularity despite remaining continuous. This irregularity is the motivation for the introduction of the local time of a semimartingale. If we allow this irregular point to move along a time-dependent curve $b : \mathbb{R}_+ \to \mathbb{R}$, our problem becomes to find an expansion for the process

$$|X_t - b(t)|\,, \tag{2.2}$$

whereupon we immediately arrive at the need to develop a time-dependent Itô-Tanaka formula. More generally, take a curve $b : \mathbb{R}_+ \to \mathbb{R}$ and a function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ which is smooth above and below this curve but only continuous over the curve. If the function F can be extended to a $C^{1,2}$ function from both sides of the curve, we can obtain the local time on curves formula

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t F_t(s, X_s-)\, \mathrm{d}s + \int_0^t F_x(s, X_s-)\, \mathrm{d}X_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s, X_s)\, \mathbb{1}_{\{X_s \neq b(s)\}}\, \mathrm{d}[X, X]_s \\ &+ \int_0^t \frac{1}{2}\big(F_x(s, X_s+) - F_x(s, X_s-)\big)\, \mathbb{1}_{\{X_s = b(s)\}}\, \mathrm{d}_s\ell_s^b(X), \end{aligned} \tag{2.3}$$

where the final integral is with respect to the time variable of the local time $\ell^b_s(X)$, defined by $\ell^0_s(X-b)$. Note that for F to extend to a $C^{1,2}$ function from both sides of curve it must obey quite stringent conditions, despite being significantly more general than the global $C^{1,2}$ condition of Itô's formula. However the formula contains only one-dimensional limits parallel to the space axis, which point the way to a broad generalisation which is also very useful in applications, and which will be the subject of this Chapter. Further, it is natural to consider the higher-dimensional analogues of this problem over hyper-surfaces and with multidimensional processes.

The key motivation for the local time on curves formula appeared in [47], where it was used to characterise the boundary of the optimal stopping problem associated to the American put option. The optimal exercise time was shown to be the first hitting time of the underlying stock price to a time-dependent curve, which can be characterised as the solution of a nonlinear Volterra integral equation. The standard method of backward induction used to derive the integral equation in such problems uses the local time on curves formula, due to the irregularity of the value function over the optimal exercise curve. The formula has since been applied in many other problems, for example [33, 39, 51], in the case of jump-diffusion processes [12], and very recently [15]. A formula of this type was also recently used by Étoré and Martinez in [27] to prove existence and uniqueness for a type of stochastic differential equation involving the local time (SDELTs). This will be discussed in detail in Chapter 4.

Firstly in Section 2.2 we will introduce the local time on curves formula, its extension to surfaces, and discuss previous developments. In Section 2.3 we present two special cases, Theorems 2.3.1 and 2.3.2, which provide clear and applicable results for common settings in optimal stopping. It is hoped this will equip the reader to approach the main Theorems 2.4.1 and 2.5.1, in Sections 2.4 and 2.5 respectively. The main predecessor of this work is the thesis of Du Toit [16]; Theorem 2.4.1 is a version of this Du Toit's result extended to include jump processes under certain conditions. Theorem 2.5.1 deals with the smooth-fit case, described below around equation (2.21), which allows us to remove some technical assumptions when the local time term is no longer present. In this case the result is a strict generalisation of Itô's formula. Also present are some Remarks 2.4.2 – 2.4.7 and 2.5.4, which further detail how technical barriers may be removed to help apply the formulae.

2.2 Review of previous work

The local time on curves formula was first derived by Peskir in [46], and later extended to surfaces in [48]. Let us set the scene in the time-space case. First, let $b : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function of bounded variation. Define two sets:

$$C = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid x < b(t)\}, \tag{2.4}$$

$$D = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > b(t)\}. \tag{2.5}$$

The graph of b is a non-smooth but continuous curve. The sets C and D are the (strict) hypograph and epigraph of this curve respectively. A function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the strong smoothness conditions if it is globally continuous, and:

$$F \text{ is } C^{1,2} \text{ on } \bar{C}, \tag{2.6}$$

$$F \text{ is } C^{1,2} \text{ on } \bar{D}. \tag{2.7}$$

This means that the restriction of F to C has a $C^{1,2}$ global extension, and the same for the restriction of F to D. This is a very strong condition, as even uniform convergence of all derivatives from each side of the curve does not guarantee that the function can be extended to a $C^{1,2}$ function everywhere without further assumptions on the curve b. In general terms the geometry of the time-space domain allows us to approach the curve from many directions, forcing us to adopt strong assumptions. In preparation for weakening these assumptions, we say that F obeys the weak smoothness conditions if it is globally continuous and:

$$F \text{ is } C^{1,2} \text{ on } C, \tag{2.8}$$

$$F \text{ is } C^{1,2} \text{ on } D. \tag{2.9}$$

Under the strong smoothness conditions for continuous X, we have the local time on curves formula from [46], which reads

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{1}{2}\big(F_t(s,X_s+) + F_t(s,X_s-)\big)\,\mathrm{d}s \\ &+ \int_0^t \frac{1}{2}\big(F_x(s,X_s+) + F_x(s,X_s-)\big)\,\mathrm{d}X_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\ \mathbb{1}_{\{X_s \neq b(s)\}}\,\mathrm{d}\langle X,X\rangle_s \\ &+ \int_0^t \frac{1}{2}\big(F_x(s,X_s+) - F_x(s,X_s-)\big)\ \mathbb{1}_{\{X_s = b(s)\}}\,\mathrm{d}_s L_s^b(X), \end{aligned} \tag{2.10}$$

where the final integral is with respect to the time variable of $L^b_s(X)$, the symmetric local time of the semimartingale $(X - b)$ at 0.

One feature of this formula is that we approach the curve only along sections in the space direction, meaning along vertical lines in the time-space domain. This is critically important; we avoid dealing with the geometry of the space by restricting to limits along one axis only, and in applications one may not have control over the limits in arbitrary directions or may have to go to great effort to determine them.

The first formula under weaker conditions was given by Peskir [46]. Assume that X solves the SDE

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t, \tag{2.11}$$

for μ_X and $\sigma > 0$ continuous and locally bounded, where $B = (B_t)_{t\geq 0}$ is standard Brownian motion. Let F obey the weak smoothness conditions (2.8) and (2.9) above. Then

$$\begin{aligned} F(t, X_t) = F(0, X_0) \;&+ \int_0^t \big(F_t + \mu F_x + (\sigma^2/2) F_{xx}\big)(s, X_s)\, \mathbb{1}_{\{X_s \neq\, b(s)\}}\, \mathrm{d}s \\ &+ \int_0^t \big(\sigma F_x\big)(s, X_s)\, \mathbb{1}_{\{X_s \neq\, b(s)\}}\, \mathrm{d}B_s \\ &+ \frac{1}{2} \int_0^t \big(F_x(s, X_s+) - F_x(s, X_s-)\big)\, \mathbb{1}_{\{X_s =\, b(s)\}}\, \mathrm{d}_s L^b_s(X), \end{aligned} \tag{2.12}$$

if the following three conditions are satisfied:

$$F_t + \mu F_x + (\sigma^2/2) F_{xx} \text{ is locally bounded on } C \cup D; \tag{2.13}$$

$$F_x(\cdot, b(\cdot) \pm \varepsilon) \to F_x(\cdot, b(\cdot)\pm) \text{ uniformly on } [0, t] \text{ as } \varepsilon \downarrow 0; \tag{2.14}$$

$$\sup_{0 < \varepsilon < \delta} \mathrm{TV}_t\big(F(\cdot, b(\cdot) \pm \varepsilon)\big) < \infty \text{ for some } \delta > 0. \tag{2.15}$$

Here, $\mathrm{TV}_t(G)$ is the variation of G on the interval $[0, t]$.

Note that the expression in (2.13) is the infinitesimal generator of X applied to F. This is a natural object, and usually appears in place of the individual derivatives of F. Its local boundedness ensures the existence of the time integral in (2.12) and weakens the previous conditions by allowing us to cancel oscillation or divergence of individual derivatives. Also, note that condition (2.14) ensures that the limiting jump in F_x is continuous.

Let us generalise the setting in the introduction by moving to the discontinuous case, and introducing a process of bounded variation alongside the full semimartingale

X. Let $(A_t)_{t\geq 0}$ be an adapted process of locally-bounded variation, and let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous, such that $(b(t, A_t))_{t\geq 0}$ is a semimartingale. Note now that A and X may have jumps. Define the sets:

$$C = \{(t, a, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \mid x < b(t, a)\}, \tag{2.16}$$

$$D = \{(t, a, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \mid x > b(t, a)\}. \tag{2.17}$$

We have again constrained the geometry of the problem to focus on surfaces which expressed as a graph of the former coordinates in the final coordinate. This is a fundamental first step, but generalising this idea is a main theme of Chapter 5. The condition that $b_t = b(t, A_t)$ should be a semimartingale is necessary in order to write the local time of X at b as $L_t^0(X - b)$.

The weak and strong smoothness conditions are defined analogously. We say F satisfies the strong smoothness conditions if it is globally continuous, and:

$$F \text{ is } C^{1,1,2} \text{ on } \bar{C}, \tag{2.18}$$

$$F \text{ is } C^{1,1,2} \text{ on } \bar{D}. \tag{2.19}$$

If F is globally continuous and instead we have $C^{1,1,2}$ regularity only on the open sets C and D then we say F obeys the weak smoothness conditions.

In [48], under the strong smoothness conditions, the formula (2.10) is extended to higher dimensions, where (in our setting) it now reads

$$\begin{aligned}
F(t,&A_t, X_t) = F(0, A_0, X_0) \\
&+ \int_0^t \frac{1}{2}\big(F_t(s-, A_{s-}, X_{s-}+) + F_t(s-, A_{s-}, X_{s-}-)\big)\,\mathrm{d}s \\
&+ \int_0^t \frac{1}{2}\big(F_a(s-, A_{s-}, X_{s-}+) + F_a(s-, A_{s-}, X_{s-}-)\big)\,\mathrm{d}A_s \\
&+ \int_0^t \frac{1}{2}\big(F_x(s-, A_{s-}, X_{s-}+) + F_x(s-, A_{s-}, X_{s-}-)\big)\,\mathrm{d}X_s \\
&+ \frac{1}{2}\int_0^t F_{xx}(s-, A_{s-}, X_{s-})\,\mathbb{1}_{\{X_{s-}\neq b_{s-}\}}\,\mathrm{d}[X, X]_s^c \\
&+ \int_0^t \frac{1}{2}\big(F_x(s-, A_{s-}, X_{s-}+) - F_x(s-, A_{s-}, X_{s-}-)\big)\,\mathbb{1}_{\{X_{s-}=b_{s-}\}}\,\mathrm{d}_s L_s^b(X) \\
&+ \sum_{0<s\leq t}\Big(F(s, A_s, X_s) - F(s-, A_{s-}, X_{s-}) \\
&\qquad\qquad - \frac{1}{2}\big(F_a(s-, A_{s-}, X_{s-}+) + F_a(s-, A_{s-}, X_{s-}-)\big)\,\Delta A_s \\
&\qquad\qquad - \frac{1}{2}\big(F_x(s-, A_{s-}, X_{s-}+) + F_x(s-, A_{s-}, X_{s-}-)\big)\,\Delta X_s\Big),
\end{aligned} \tag{2.20}$$

where the final integral is with respect to the time variable of $L_s^b(X)$, the symmetric local time of the semimartingale $(X - b)$ at 0.

Note that the same crucial features as in the time-space case are still present, however, the presence of jumps no longer guarantees that the local time of X possesses a nice modification (see Appendix A.1). In [48], weaker conditions are also given in the special case where X is a continuous diffusion, and both F and the surface b also depend upon the maximum process of X.

In certain problems in optimal stopping, it is possible to directly verify the 'smooth fit' condition, meaning that

$$(t, a) \mapsto F_x(t, a, b(t, a)+) - F_x(t, a, b(t, a)-) \tag{2.21}$$

is identically zero. This indicates that we no longer require the local time component above. It should be noted that the function F may still fail to be $C^{1,1,2}$ at the surface b, so the conditions of the classical Itô formula may not hold.

Our task is to find minimal conditions on the function F, the surface b, and the processes A and X to retain a suitable analogue of formula (2.10) above. The thesis of Du Toit [16], in the continuous case, is the basis for our results in the discontinuous case. The form of his result is the same as our (2.44) without jump terms. Notably Du Toit introduced the condition (2.43) which is a general analogue of local boundedness of the infinitesimal generator. The method of proof also relies on the techniques introduced by Du Toit.

2.3 Results for diffusion and jump processes

The main Theorems 2.4.1 and 2.5.1 are general but difficult to appreciate immediately as they deal with general jump processes, and the conditions may appear arcane. As our motivation arises from diffusion and jump-diffusion processes, we examine the results in this setting where the conditions are very natural.

The reader should note that we will exclusively use the right local time (see A.1), but results can be translated to the symmetric local time case by replacing left limits with symmetric limits in the space variables of the integrands, see Remark 2.4.6. When X satisfies (2.11), then formula (2.12) holds under conditions (2.13), (2.14) and (2.15). Du Toit [16] has removed the requirement (2.15) by using a different method of proof, and further weakened condition (2.14) to continuity of the map $t \mapsto F_x(t, b(t)+) - F_x(t, b(t)-)$. In fact, he shows this in the more general setting when b also depends on a continuous bounded variation process A. We now demonstrate an example which shows that the analogous result holds when A and X are discontinuous, under certain conditions on the jumps.

Theorem 2.3.1. *Assume X solves the following SDE*

$$\mathrm{d}X_t = \mu_X(t-, A_{t-}, X_{t-})\,\mathrm{d}t + \sigma(t-, A_{t-}, X_{t-})\,\mathrm{d}B_t + \lambda_X(t-, A_{t-}, X_{t-})\,\mathrm{d}Y_t, \tag{2.22}$$

where Y is a pure-jump Lévy process with bounded variation. Assume A satisfies the SDE

$$\mathrm{d}A_t = \mu_A(t-, A_{t-}, X_{t-})\,\mathrm{d}t + \lambda_A(t-, A_{t-}, X_{t-})\,\mathrm{d}Y_t. \tag{2.23}$$

Let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, with b Lipschitz. Further assume $F = F(t, a, x)$ obeys the weak smoothness conditions (2.8), (2.9). If the following conditions are satisfied:

$$(F_t + \mu_X F_x + \mu_A F_a + (\sigma^2/2) F_{xx})(t, a, x) \text{ is locally bounded;} \tag{2.24}$$

$$t \mapsto F_x(t, A_t, b_t+) - F_x(t, A_t, b_t-) \text{ is almost surely continuous;} \tag{2.25}$$

$$\mathbb{P}\left[X_{s-} = b_{s-}\right] = 0 \text{ for all } t \in \mathbb{R}_+. \tag{2.26}$$

Then the following change of variable formula holds,

$$\begin{aligned} F(t, A_t, X_t) &= F(0, A_0, X_0) \\ &\quad + \int_0^t (\sigma F_x)(s-, A_{s-}, X_{s-})\, \mathbb{1}_{\{X_{s-} \neq b_{s-}\}}\,\mathrm{d}B_s \\ &\quad + \int_0^t (F_t + \mu_X F_x + \mu_A F_a + (\sigma^2/2) F_{xx})(s-, A_{s-}, X_{s-})\, \mathbb{1}_{\{X_{s-} \neq b_{s-}\}}\,\mathrm{d}s \\ &\quad + \frac{1}{2} \int_0^t (F_x(s-, A_{s-}, b_{s-}+) - F_x(s-, A_{s-}, b_{s-}-))\, \mathbb{1}_{\{X_{s-} = b_{s-}\}}\,\mathrm{d}_s \ell_s^b(X) \\ &\quad + \sum_{0<s\leq t} F(s, A_s, X_s) - F(s-, A_{s-}, X_{s-}), \end{aligned} \tag{2.27}$$

for all $t \geq 0$, where the final integral is with respect to the time variable of $\ell_s^b(X)$, the right-local time of the semimartingale $(X - b)$ at 0.

The requirement that Y has bounded variation ensures that X has bounded variation of jumps, meaning that

$$\sum_{0<s\leq t} |\Delta X_s| < \infty, \tag{2.28}$$

for all $t \in \mathbb{R}_+$, which implies that the local time can be taken right-continuous in space (see A.1.3). Further, if b is Lipschitz, then the process $b_t = b(t, A_t)$ is of locally-bounded variation whenever A is (see [40]). Finally, the stipulation that $\mathbb{P}[X_{s-} = b_{s-}] = 0$ for each $0 < s \leq t$ implies that the presence of the indicator functions in (2.27) do not change the value of the integrals in which they appear. Their introduction allows us to avoid discussion of the limiting values of the derivatives of F as we approach the surface b, which is particularly useful in applications.

We now deal with the smooth fit case, recalling (2.21) above.

Theorem 2.3.2. *Assume X satisfies the following SDE*

$$\mathrm{d}X_t = \mu_X(t-, A_{t-}, X_{t-})\,\mathrm{d}t + \sigma(t-, A_{t-}, X_{t-})\,\mathrm{d}B_t + \lambda_X(t-, A_{t-}, X_{t-})\,\mathrm{d}Y_t, \tag{2.29}$$

where Y_t is a pure-jump Lévy process. Let Z be a pure-jump Lévy process of bounded variation, and assume A satisfies the SDE

$$\mathrm{d}A_t = \mu_A(t-, A_{t-}, X_{t-})\,\mathrm{d}t + \lambda_A(t-, A_{t-}, X_{t-})\,\mathrm{d}Z_t. \tag{2.30}$$

Let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and obey the weak smoothness conditions (2.8), (2.9). Assume that:

$$(F_t + \mu_X F_x + \mu_A F_a + (\sigma^2/2)F_{xx})(t, a, x) \text{ is locally bounded;} \tag{2.31}$$

$$t \mapsto F_x(t, A_t, b_t+) - F_x(t, A_t, b_t-) \text{ is almost surely identically zero;} \tag{2.32}$$

$$\text{We have } \mathbb{P}[X_{s-} = b_{s-}] = 0 \text{ for all } 0 < s \leq t. \tag{2.33}$$

Then the following change of variable formula holds,

$$\begin{aligned} F(t, A_t, X_t) &= F(0, A_0, X_0) + \int_0^t (\sigma F_x)(s-, A_{s-}, X_{s-})\, \mathbb{1}_{\{X_{s-} \neq b_{s-}\}}\,\mathrm{d}B_s \\ &+ \int_0^t (\lambda_X F_x)(s-, A_{s-}, X_{s-}-)\,\mathrm{d}Y_s \\ &+ \int_0^t (F_t + \mu_X F_x + \mu_A F_a + (\sigma^2/2)F_{xx})(s-, A_{s-}, X_{s-})\, \mathbb{1}_{\{X_{s-} \neq b_{s-}\}}\,\mathrm{d}s \\ &+ \sum_{0<s\leq t} \Big(F(s, A_s, X_s) - F(s-, A_{s-}, X_{s-}) - F_x(s-, A_{s-}, X_{s-}-)\Delta X_s \Big), \end{aligned} \tag{2.34}$$

for all $t \geq 0$.

Note that the jumps of X are the product of λ and the jumps of Y, and so the final term above may also be written as

$$\sum_{0<s\leq t} \Big(F(s, A_s, X_s) - F(s-, A_{s-}, X_{s-}) - (\lambda_X F_x)(s-, A_{s-}, X_{s-}-)\Delta Y_s \Big). \tag{2.35}$$

2.4 Local time on curves for jump processes

Throughout we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ obeying the usual conditions. Let $(X_t)_{t\geq 0}$ be an $\mathcal{F}_t$-semimartingale, where $(A_t)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted process of locally-bounded variation. We note that X admits at least one decomposition

$$X = X_0 + K + M, \tag{2.36}$$

into an $\mathcal{F}_t$-local martingale M, an $\mathcal{F}_t$-adapted process K of locally-bounded variation, and an $\mathcal{F}_0$-measurable random variable X_0, such that $M_0 = K_0 = 0$. Let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous such that $(b(t, A_t))_{t\geq 0}$ is a semimartingale, with sets C and D as given in (2.16) and (2.17).

Let $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that:

$$F \text{ is } C^{1,1,2} \text{ on } C, \tag{2.37}$$

$$F \text{ is } C^{1,1,2} \text{ on } D. \tag{2.38}$$

We write $(b_t)_{t\geq 0}$ to mean the process given by $b_t = b(t, A_t)$ for $t \geq 0$.

Theorem 2.4.1. *In the setting given above, assume that the function F obeys the following two criteria:*

$$\textit{The limits } F_x(t, a, b(t, a)\pm) \textit{ exist for all } (t, a) \in \mathbb{R}_+ \times \mathbb{R}; \tag{2.39}$$

$$(t, a) \mapsto F_x(t, a, b(t, a)+) - F_x(t, a, b(t, a)-) \textit{ is jointly continuous on } \mathbb{R}_+ \times \mathbb{R}. \tag{2.40}$$

Further assume that the surface b, process A and semimartingale X satisfy:

$$\textit{The process } (b_t)_{t\geq 0} \textit{ is of locally-bounded variation almost surely;} \tag{2.41}$$

$$\textit{We have } \sum_{0<s\leq t} |\Delta X_s| < \infty \textit{ for all } t \geq 0, \textit{ almost surely.} \tag{2.42}$$

If there exists a signed Radon measure measure λ on $\mathbb{R}_+$ and a locally-bounded function $H : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ such that $H(t, a, x-)$ exists for all $(t, a, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, which satisfy

$$\begin{aligned} &\int_0^t F_t(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\, ,\, b_{s-} + \varepsilon]\}}\, \mathrm{d}s \\ &\quad + \int_0^t F_a(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\, ,\, b_{s-} + \varepsilon]\}}\, \mathrm{d}A^c_s \\ &\quad + \int_0^t F_x(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\, ,\, b_{s-} + \varepsilon]\}}\, \mathrm{d}K^c_s \\ &\quad + \frac{1}{2} \int_0^t F_{xx}(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\, ,\, b_{s-} + \varepsilon]\}}\, \mathrm{d}[X, X]^c_s \\ &= \int_0^t H(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\, ,\, b_{s-} + \varepsilon]\}}\, \mathrm{d}\lambda(s), \end{aligned} \tag{2.43}$$

for all $\varepsilon > 0$ and all $0 < c \leq \delta$ for some fixed $\delta > 0$, then we have

$$\begin{aligned} F(t, A_t, X_t) =\ & F(0, A_0, X_0) \\ &+ \int_0^t H(s-, A_{s-}, X_{s-}-)\, \mathrm{d}\lambda(s) + \int_0^t F_x(s-, A_{s-}, X_{s-}-)\, \mathrm{d}M_s \\ &+ \frac{1}{2} \int_0^t \big(F_x(s-, A_{s-}, b_{s-}+) - F_x(s-, A_{s-}, b_{s-}-)\big)\, \mathrm{d}_s \ell^b_s(X) \\ &+ \sum_{0 < s \leq t} \Big(F(s, A_s, X_s) - F(s-, A_{s-}, X_{s-}) - F_x(s-, A_{s-}, X_{s-}-)\Delta M_s\Big), \end{aligned} \tag{2.44}$$

for all $t \geq 0$.

Proof of this theorem is given in the next section. Note that condition (2.43) is the general equivalent of local boundedness of the infinitesimal generator, and hence can be reduced to (2.13) when X and A solve appropriate SDEs - see Theorems 2.3.1 and 2.3.2 for clarification in the most common cases. More generally, if we have that $\mathrm{d}A^c_s$, $\mathrm{d}K^c_s$ and $\mathrm{d}\,[X, X]^c_s$ all absolutely continuous with respect to Lebesgue measure, then instead it suffices to check that

$$\begin{aligned} &\left(F_t + F_a \frac{\mathrm{d}A^c_s}{\mathrm{d}s} + F_x \frac{\mathrm{d}K^c_s}{\mathrm{d}s} + \frac{1}{2} F_{xx} \frac{\mathrm{d}[X, X]^c_s}{\mathrm{d}s}\right) \mathbb{1}_{\{x - c \notin (b(s,a) - \varepsilon\, ,\, b(s,a) + \varepsilon]\}} \\ &\qquad = H\, \mathbb{1}_{\{x - c \notin (b(s,a) - \varepsilon\, ,\, b(s,a) + \varepsilon]\}}, \end{aligned} \tag{2.45}$$

for H, c, ε as in the Theorem. This is clearly implied by local boundedness with left limits of

$$F_t + F_a \frac{\mathrm{d}A^c_s}{\mathrm{d}s} + F_x \frac{\mathrm{d}K^c_s}{\mathrm{d}s} + \frac{1}{2} F_{xx} \frac{\mathrm{d}[X, X]^c_s}{\mathrm{d}s}, \tag{2.46}$$

the generalised infinitesimal generator. In this case that it is not possible to directly combine these quantities as above, we have the following Remark which stipulates that the equivalent condition on each quantity alone suffices. Further, the other Remarks give further generalisations and remove technical barriers.

Remark 2.4.2. Considering condition (2.43) above, we may generalize to the case where we have finitely many locally-bounded functions $H_1, \dots, H_n : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ and signed Radon measures $\lambda_1, \dots, \lambda_n$ on $\mathbb{R}_+$ such that $H_i(t, a, x-)$ exists for each $(t, a, x) \in \mathbb{R}_+ \times \mathbb{R}^2$ and $i = 1, \dots, n$, which satisfy

$$\begin{aligned} &\int_0^t F_t(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\,,\, b_{s-} + \varepsilon]\,\}}\, \mathrm{d}s \\ &+ \int_0^t F_a(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\,,\, b_{s-} + \varepsilon]\,\}}\, \mathrm{d}A_s^c \\ &+ \int_0^t F_x(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\,,\, b_{s-} + \varepsilon]\,\}}\, \mathrm{d}K_s^c \\ &+ \frac{1}{2} \int_0^t F_{xx}(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\,,\, b_{s-} + \varepsilon]\,\}}\, \mathrm{d}[X, X]_s^c \\ = &\sum_{i=1}^n \int_0^t H_i(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-} - c \notin (b_{s-} - \varepsilon\,,\, b_{s-} + \varepsilon]\,\}}\, \mathrm{d}\lambda_i(s), \end{aligned} \tag{2.47}$$

for all $\varepsilon > 0$ and all $0 < c \leq \delta$ for some fixed $\delta > 0$. The resultant formula (2.44) consequently changes in the obvious way.

Remark 2.4.3. We may also relax condition (2.43), requiring only that the left-limits of H exist for all t outside of a λ-null set almost surely (see (2.26) and (2.33)).

Remark 2.4.4. We may further allow dependence of H, F, b and λ on the underlying probability space, provided they are respectively adapted, predictably measurable processes and a random Radon measure, which obey the conditions of the theorem almost surely.

Remark 2.4.5. Theorem 2.4.1 extends in a straightforward manner when we allow dependence on finitely many one-dimensional processes of locally-bounded variation. That is, if we allow one-dimensional processes of locally-bounded variation $A^1, \dots, A^k$, and a general semimartingale X, the surface b is then defined on $\mathbb{R}_+ \times \mathbb{R}^k$ and takes values in $\mathbb{R}$. We may also allow finitely many non-intersecting surfaces, under the obvious extension of the weak smoothness conditions. Note that (2.43) also changes correspondingly.

Remark 2.4.6. We may replace the use of the right local time by the left or symmetric local time, provided that we replace the left limits in the space variable by the right or symmetric limits respectively. Note especially that the range of the variable c in (2.43) must also be changed correspondingly. If we are also using the weaker conditions on X given by Remark 2.4.7 below, then these conditions must be replaced by the obvious right or two-sided versions.

Remark 2.4.7. The condition (2.42) is connected to the problem of continuity of the local time process of X. We may generalise by replacing (2.42) by the following two conditions:

$$X - b \text{ admits a local time } \ell_t^a \text{ such that } \lim_{a \downarrow 0} \ell_s^a = \ell_s^0 \text{ for all } 0 \leq s \leq t; \tag{2.48}$$

$$\text{We have } \sum_{0<s\leq t} |\Delta X_s| \, \mathbb{1}_{\{X_{s-}=b_{s-},\, X_s > b_s\}} < \infty \text{ for all } t \geq 0, \text{ almost surely.} \tag{2.49}$$

If X is a Lévy process, conditions for existence of a version of its local time satisfying Remark 2.48 is well studied. In particular, see [5] for necessary and sufficient conditions. It is already noted implicitly in the conditions of Theorem 2.4.1 that processes with so-called 'bounded variation of jumps', namely those satisfying (2.42), admit such a local time. A further special case consists of the α-stable Lévy processes, for $1 < \alpha < 2$, which also admit such a local time, a result given in [11].

2.5 An extension of Itô's formula

The local time on curves formula of Section 2.4 provides an alternative to Itô's formula when the function of interest contains a discontinuity in the space derivative over a surface. Formally one can see that in the smooth-fit case when (2.21) vanishes, the local time integral term from (2.44) also vanishes, meaning we may relax the condition that $(b_t)_{t\geq 0}$ be of locally-bounded variation. Unless, for example, b is Lipschitz, meaning $(b_t)_{t\geq 0}$ is automatically of bounded variation whenever A is, proving that $(b_t)_{t\geq 0}$ is of locally-bounded variation can be a difficult task. Relaxing this condition in a more general way shortens many results which employ techniques based on specific cases. In [39] for example, the authors go to some effort to prove that the process corresponding to $(b_t)_{t\geq 0}$ is of bounded variation, whereas Theorem 2.5.1 does not require this.

We are lead to the following theorem, which is now an extension of the classical Itô formula, not including the local time.

Theorem 2.5.1. *In the setting of Section 2.4, assume that:*

$$\text{The limits } F_x(t,a,b(t,a)\pm) \text{ exist for all } (t,a) \in \mathbb{R}_+ \times \mathbb{R}; \tag{2.50}$$

$$(t,a) \mapsto F_x(t,a,b(t,a)+) - F_x(t,a,b(t,a)-) \text{ is identically zero on } \mathbb{R}_+ \times \mathbb{R}. \tag{2.51}$$

If there exists a signed Radon measure λ on $\mathbb{R}_+$ and a locally-bounded function $H : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ such that $H(t,a,x-)$ exists for all $(t,a,x) \in \mathbb{R}_+ \times \mathbb{R}^2$, which satisfy

$$\begin{aligned} &\int_0^t F_t(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-}-c \notin (b_{s-}-\varepsilon\,, b_{s-}+\varepsilon]\}}\, \mathrm{d}s \\ &+ \int_0^t F_a(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-}-c \notin (b_{s-}-\varepsilon\,, b_{s-}+\varepsilon]\}}\, \mathrm{d}A_s^c \\ &+ \int_0^t F_x(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-}-c \notin (b_{s-}-\varepsilon\,, b_{s-}+\varepsilon]\}}\, \mathrm{d}K_s^c \\ &+ \frac{1}{2}\int_0^t F_{xx}(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-}-c \notin (b_{s-}-\varepsilon\,, b_{s-}+\varepsilon]\}}\, \mathrm{d}[X,X]_s^c \\ = &\int_0^t H(s-, A_{s-}, X_{s-} - c)\, \mathbb{1}_{\{X_{s-}-c \notin (b_{s-}-\varepsilon\,, b_{s-}+\varepsilon]\}}\, \mathrm{d}\lambda(s), \end{aligned} \tag{2.52}$$

for all $\varepsilon > 0$ and all $0 < c \leq \delta$ for some fixed $\delta > 0$, then we have

$$\begin{aligned} F(t, A_t, X_t) = \; & F(0, A_0, X_0) + \int_0^t H(s-, A_{s-}, X_{s-}-)\, \mathrm{d}\lambda(s) \\ &+ \int_0^t F_x(s-, A_{s-}, X_{s-}-)\, \mathrm{d}M_s \\ &+ \sum_{0<s\leq t} \Big(F(s, A_s, X_s) - F(s-, A_{s-}, X_{s-}) - F_x(s-, A_{s-}, X_{s-}-)\Delta M_s \Big) \end{aligned} \tag{2.53}$$

for all $t \geq 0$.

Finally, let us note that Remarks 2.4.2 − 2.4.5 still hold in this new setting, with obvious modifications to the conditions required.

Proof of Theorem 2.5.1. The proof follows the method of Du Toit [16], making adjustments for the existence of jumps. The method of proof is to approximate the process $t \mapsto b_t = b(t, A_t)$ pathwise, from above and below, by a process of locally-bounded variation. After truncating and smoothing the function F, we then take limits to return to the original problem. Smoothing and truncation allows us to apply the

Lebesgue-Stieltjes chain rule, and standard Itô formula, whereupon we can combine derivatives in the form of (2.52). Boundary terms appear from the truncation, which either vanish or converge to the local-time correction term in (2.44).

We assume, through localisation, that the semimartingale $(t, A_t, X_t)_{t\geq 0}$ is bounded and therefore takes values in a compact set. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function which is supported on $[0,1]$ with $\int_{\mathbb{R}} \rho(y)\,\mathrm{d}y = 1$.

If $t \mapsto b_t$ is of locally-bounded variation, define $\tilde{b}^m = b$ for each $m \in \mathbb{N}$. If not, we define the Moreau envelope of b for each $m \in \mathbb{N}$ by

$$\tilde{b}^m(t,a) = \inf_{(s,y)\,\in\,\mathbb{R}_+\times\mathbb{R}} \left\{ b(s,y) + \frac{1}{2m}\|(t,a)-(s,y)\|^2 \right\}, \tag{2.54}$$

where $\|\cdot\|$ is the usual Euclidean norm. Intuitively, this approximation is the vector sum of the epigraphs of the surface b and $\frac{1}{2m}\|x-y\|^2$.

It is known (for example, [53]) that the Moreau envelope of a continuous function is Lipschitz and converges pointwise monotonically to b from below as $m \to \infty$. By Dini's theorem, this implies uniform convergence on compact sets.

It has also been shown in the one-dimensional case by Josephy [40] that the composition of a Lipschitz function with a process of locally-bounded variation is again of locally-bounded variation. We observe that this generalises to higher dimensions by considering increments in each variable separately and using continuity.

Fix $\varepsilon > 0$, and $n \in \mathbb{N}$. By uniform convergence, and recalling the localisation above, we can choose m large enough such that $\|\tilde{b}^m - b\| < \varepsilon$ on the compact set containing (t, A_t). We define two truncated convolution approximations to F, on either side of the surface $\tilde{b}^m$, by

$$\begin{aligned} G^{n,m,\varepsilon}(s,a,x) &= \int_{-\infty}^{\infty} F(s,a,x-z/n+\tilde{b}^m(s,a))\ \mathbb{1}_{\{x-z/n>2\varepsilon\}}\,\rho(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} n\,F(s,a,k+\tilde{b}^m(s,a))\,\mathbb{1}_{\{k>2\varepsilon\}}\,\rho(n(x-k))\,\mathrm{d}k, \end{aligned} \tag{2.55}$$

$$\begin{aligned} H^{n,m,\varepsilon}(s,a,x) &= \int_{-\infty}^{\infty} F(s,a,x-z/n+\tilde{b}^m(s,a))\ \mathbb{1}_{\{x-z/n\leq-\varepsilon\}}\,\rho(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} n\,F(s,a,k+\tilde{b}^m(s,a))\,\mathbb{1}_{\{k\leq-\varepsilon\}}\,\rho(n(x-k))\,\mathrm{d}k. \end{aligned} \tag{2.56}$$

In anticipation of letting $\varepsilon \to 0$, we also define

$$\begin{aligned} G^{n,m}(s,a,x) &= \int_{-\infty}^{\infty} F(s,a,x-z/n+\tilde{b}^m(s,a))\ \mathbb{1}_{\{x-z/n>0\}}\,\rho(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} n\,F(s,a,k+\tilde{b}^m(s,a))\,\mathbb{1}_{\{k>0\}}\,\rho(n(x-k))\,\mathrm{d}k, \end{aligned} \tag{2.57}$$

$$\begin{aligned}H^{n,m}(s,a,x) &= \int_{-\infty}^{\infty} F(s,a,x-z/n+\tilde{b}^m(s,a))\, \mathbb{1}_{\{x-z/n\le 0\}}\, \rho(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} n\, F(s,a,k+\tilde{b}^m(s,a))\, \mathbb{1}_{\{k\le 0\}}\, \rho(n(x-k))\,\mathrm{d}k.\end{aligned} \tag{2.58}$$

We deal only with the G approximations - note that the same arguments apply to the H approximations. It is easily verified that these approximations have x derivatives of all orders, and that:

$$\lim_{\varepsilon\to 0} G^{n,m,\varepsilon}(s,a,x) = G^{n,m}(s,a,x); \tag{2.59}$$

$$\lim_{n\to\infty}\lim_{m\to\infty} G^{n,m}(s,a,x) + H^{n,m}(s,a,x) = F(s,a,x+\tilde{b}^m(s,a)). \tag{2.60}$$

Fix $t > 0$. Define an arbitrary sequence of refining partitions of the interval $[0,t]$ whose mesh tends to zero. Specifically, for each $\tilde{n} \in \mathbb{N}$, define $T^{\tilde{n}} = \{t_0^{\tilde{n}} < \cdots < t_{\tilde{n}}^{\tilde{n}}\}$, such that $T^{\tilde{n}} \subset T^{\tilde{n}+1}$, and $\max_{1\le i\le \tilde{n}} |t_i^{\tilde{n}} - t_{i-1}^{\tilde{n}}| \to 0$ as $\tilde{n}\to\infty$.

We deal first with an arbitrary semimartingale Y, which will allow us to make an appropriate substitution later. Let us approximate $G^{n,m,\varepsilon}$ applied to (t,A,Y) by splitting it into purely stochastic and purely bounded variation increments across the partition $T^{\tilde{n}}$. Consider a single increment of $G^{n,m,\varepsilon}$, temporarily fix $\tilde{n}\in\mathbb{N}$, and write $T = T^{\tilde{n}} = \{0 = t_0 < t_1 \le \cdots < t_{\tilde{n}} = t\}$. Lack of smoothness prevents us from directly applying Itô's formula. However, we can write this increment as

$$\begin{aligned}G^{n,m,\varepsilon}(t_i,A_{t_i},Y_{t_i}) - G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{t_{i-1}}) = {}& G^{n,m,\varepsilon}(t_i,A_{t_i},Y_{t_i}) \\ - G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{t_i}) + G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{t_i}) &- G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{t_{i-1}}).\end{aligned} \tag{2.61}$$

The observation that the latter (stochastic) increment is adapted allows us to make use of Itô's formula. We can then apply the usual deterministic calculus pathwise to the non-adapted bounded variation increment. More precisely, apply the extended Itô formula from [49, Thm. 18, p. 278], yielding

$$\begin{aligned}&G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{t_i}) - G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{t_{i-1}}) = \\ &\int_{t_{i-1}}^{t_i} G_x^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{s-})\,\mathrm{d}Y_s + \frac{1}{2}\int_{t_{i-1}}^{t_i} G_{xx}^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{s-})\,\mathrm{d}[Y,Y]_s^c \\ &+ \sum_{t_{i-1}<s\le t_i}\Big(G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_s) - G^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{s-}) \\ &\qquad\qquad - G_x^{n,m,\varepsilon}(t_{i-1},A_{t_{i-1}},Y_{s-})\Delta Y_s\Big).\end{aligned} \tag{2.62}$$

To deal with the deterministic increment, we apply the Lebesgue-Stieltjes change of variables pathwise, to obtain

$$
\begin{aligned}
&G^{n,m,\varepsilon}(t_i, A_{t_i}, Y_{t_i}) - G^{n,m,\varepsilon}(t_{i-1}, A_{t_{i-1}}, Y_{t_i}) = \\
&\quad \int_{-\infty}^{\infty} \Bigg\{ \int_{t_{i-1}}^{t_i} F_s(s-, A_{s-}, Y_{t_i} - z/n + \tilde{b}^m_{s-})\, \mathrm{d}s \\
&\qquad + \int_{t_{i-1}}^{t_i} F_a(s-, A_{s-}, Y_{t_i} - z/n + \tilde{b}^m_{s-})\, \mathrm{d}A^c_s \\
&\qquad + \int_{t_{i-1}}^{t_i} F_x(s-, A_{s-}, Y_{t_i} - z/n + \tilde{b}^m_{s-})\, \mathrm{d}\tilde{b}^{m,c}_s \\
&\qquad + \sum_{t_{i-1} < s \le t_i} F(s, A_s, Y_{t_i} - z/n + \tilde{b}^m_s) \\
&\qquad\qquad - F(s-, A_{s-}, Y_{t_i} - z/n + \tilde{b}^m_{s-}) \Bigg\}\, \mathbb{1}_{\{Y_{t_i} - z/n > 2\varepsilon\}}\, \rho(z)\, \mathrm{d}z,
\end{aligned}
\tag{2.63}
$$

where $\mathrm{d}\tilde{b}^{m,c}_s$ is the continuous part of the measure $\mathrm{d}\tilde{b}^m_s$. Note that the presence of the indicator function ensures each non-zero increment is in the area where F is $C^{1,1,2}$.

With the aim of combining the representations (2.62) and (2.63), we calculate the x derivatives in (2.62) directly. Write $G^{n,m,\varepsilon}$ as

$$
G^{n,m,\varepsilon}(s, a, x) = \int_{2\varepsilon}^{\infty} n\, F(s, a, k + \tilde{b}^m(s, a))\, \rho(n(x - k))\, \mathrm{d}k. \tag{2.64}
$$

Now we can differentiate under the integral sign in x. Integrating by parts in k, noting that F is $C^{1,1,2}$ where required, then changing variables, we obtain

$$
\begin{aligned}
G^{n,m,\varepsilon}_x(s, a, x) &= n\, F(s, a, 2\varepsilon + \tilde{b}^m(s, a))\, \rho(n(x - 2\varepsilon)) \\
&\quad + \int_{-\infty}^{\infty} F_x(s, a, x - z/n + \tilde{b}^m(s, a))\, \mathbb{1}_{\{x - \frac{z}{n} > 2\varepsilon\}}\, \rho(z)\, \mathrm{d}z.
\end{aligned}
\tag{2.65}
$$

Repeating this procedure gives

$$
\begin{aligned}
G^{n,m,\varepsilon}_{xx}(s, a, x) &= n^2\, F(s, a, 2\varepsilon + \tilde{b}^m(s, a))\, \rho'(n(x - 2\varepsilon)) \\
&\quad + n\, F_x(s, a, 2\varepsilon + \tilde{b}^m(s, a))\, \rho(n(x - 2\varepsilon)) \\
&\quad + \int_{-\infty}^{\infty} F_{xx}(s, a, x - z/n + \tilde{b}^m(s, a))\, \mathbb{1}_{\{x - \frac{z}{n} > 2\varepsilon\}}\, \rho(z)\, \mathrm{d}z.
\end{aligned}
\tag{2.66}
$$

We may now substitute into the original expression (2.61), making use of the deterministic and stochastic Fubini theorems (see [49, Thm. 64–65, pp. 210–213]), yielding

$$
\begin{aligned}
&G^{n,m,\varepsilon}(t_i, A_{t_i}, Y_{t_i}) - G^{n,m,\varepsilon}(t_{i-1}, A_{t_{i-1}}, Y_{t_{i-1}}) = \\
&\int_{t_{i-1}}^{t_i} n\, F(t_{i-1}, A_{t_{i-1}}, 2\varepsilon + \tilde{b}^m_{t_{i-1}})\, \rho(n(Y_{s-} - 2\varepsilon))\, \mathrm{d}Y_s \\
&+ \frac{1}{2} \int_{t_{i-1}}^{t_i} n^2\, F(t_{i-1}, A_{t_{i-1}}, 2\varepsilon + \tilde{b}^m_{t_{i-1}})\, \rho'(n(Y_{s-} - 2\varepsilon)) \qquad (2.67) \\
&\qquad + n\, F_x(t_{i-1}, A_{t_{i-1}}, 2\varepsilon + \tilde{b}^m_{t_{i-1}})\, \rho(n(Y_{s-} - 2\varepsilon))\, \mathrm{d}[Y,Y]^c_s \\
&+ \int_{-\infty}^{\infty} \Bigg\{ \int_{t_{i-1}}^{t_i} F_s(s-, A_{s-}, Y_{t_i} - z/n + \tilde{b}^m_{s-})\, \mathbb{1}_{\{Y_{t_i} - \frac{z}{n} > 2\varepsilon\}}\, \mathrm{d}s \\
&\qquad + \int_{t_{i-1}}^{t_i} F_a(s-, A_{s-}, Y_{t_i} - z/n + \tilde{b}^m_{s-})\, \mathbb{1}_{\{Y_{t_i} - \frac{z}{n} > 2\varepsilon\}}\, \mathrm{d}A^c_s \\
&\qquad + \int_{t_{i-1}}^{t_i} F_x(s-, A_{s-}, Y_{t_i} - z/n + \tilde{b}^m_{s-})\, \mathbb{1}_{\{Y_{t_i} - \frac{z}{n} > 2\varepsilon\}}\, \mathrm{d}\tilde{b}^{m,c}_s \\
&\qquad + \frac{1}{2} \int_{t_{i-1}}^{t_i} F_{xx}(t_{i-1}, A_{t_{i-1}}, Y_{s-} - z/n + \tilde{b}^m_{t_{i-1}}))\, \mathbb{1}_{\{Y_{s-} - \frac{z}{n} > 2\varepsilon\}}\, \mathrm{d}[Y,Y]^c_s \\
&\qquad + \int_{t_{i-1}}^{t_i} F_x(t_{i-1}, A_{t_{i-1}}, Y_{s-} - z/n + \tilde{b}^m_{t_{i-1}})\, \mathbb{1}_{\{Y_{s-} - \frac{z}{n} > 2\varepsilon\}}\, \mathrm{d}Y_s \Bigg\}\, \rho(z)\, \mathrm{d}z \\
&+ \sum_{t_{i-1} < s \le t_i} G^{n,m,\varepsilon}(t_{i-1}, A_{t_{i-1}}, Y_s) - G^{n,m,\varepsilon}(t_{i-1}, A_{t_{i-1}}, Y_{s-}) - G^{n,m,\varepsilon}_x(t_{i-1}, A_{t_{i-1}}, Y_{s-}) \Delta Y_s \\
&+ \sum_{t_{i-1} < s \le t_i} G^{n,m,\varepsilon}(s, A_s, Y_{t_i}) - G^{n,m,\varepsilon}(s-, A_{s-}, Y_{t_i}).
\end{aligned}
$$

By summing over the partition, we obtain $G^{n,m,\varepsilon}(t, A_t, Y_t) - G^{n,m,\varepsilon}(0, A_0, Y_0)$. Then re-changing the order of integration, we may use continuity of the convolution approximation to allow the mesh size of the partition to tend to zero. The full expression is

$$
\begin{aligned}
&G^{n,m,\varepsilon}(t, A_t, Y_t) - G^{n,m,\varepsilon}(0, A_0, Y_0) = \\
&\quad \int_0^t n\, F(s-, A_{s-}, 2\varepsilon + \tilde{b}^m_{s-})\, \rho(n(Y_{s-} - 2\varepsilon))\, \mathrm{d}Y_s \\
&\quad + \frac{1}{2} \int_0^t n^2\, F(s-, A_{s-}, 2\varepsilon + \tilde{b}^m_{s-})\, \rho'(n(Y_{s-} - 2\varepsilon)) \\
&\qquad\qquad + n\, F_x(s-, A_{s-}, 2\varepsilon + \tilde{b}^m_{s-})\, \rho(n(Y_{s-} - 2\varepsilon))\, \mathrm{d}[Y,Y]^c_s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-\infty}^{\infty}\Bigg\{\int_0^t F_s(s-,A_{s-},Y_s-z/n+\tilde b^m_{s-})\,\mathbb{1}_{\{Y_s-\frac{z}{n}>2\varepsilon\}}\,\mathrm{d}s\\
&\qquad+\int_0^t F_a(s-,A_{s-},Y_s-z/n+\tilde b^m_{s-})\,\mathbb{1}_{\{Y_s-\frac{z}{n}>2\varepsilon\}}\,\mathrm{d}A^c_s\\
&\qquad+\int_0^t F_x(s-,A_{s-},Y_s-z/n+\tilde b^m_{s-})\,\mathbb{1}_{\{Y_s-\frac{z}{n}>2\varepsilon\}}\,\mathrm{d}\tilde b^{m,c}_s\\
&\qquad+\frac12\int_0^t F_{xx}(s-,A_{s-},Y_{s-}-z/n+\tilde b^m_{s-}))\,\mathbb{1}_{\{Y_{s-}-\frac{z}{n}>2\varepsilon\}}\,\mathrm{d}[Y,Y]^c_s\\
&\qquad+\int_0^t F_x(s-,A_{s-},Y_{s-}-z/n+\tilde b^m_{s-})\,\mathbb{1}_{\{Y_{s-}-\frac{z}{n}>2\varepsilon\}}\,\mathrm{d}Y_s\Bigg\}\;\rho(z)\,\mathrm{d}z\\
&+\sum_{0<s\le t}G^{n,m,\varepsilon}(s-,A_{s-},Y_s)-G^{n,m,\varepsilon}(s-,A_{s-},Y_{s-})-G^{n,m,\varepsilon}_x(s-,A_{s-},Y_{s-})\Delta Y_s\\
&+\sum_{0<s\le t}G^{n,m,\varepsilon}(s,A_s,Y_s)-G^{n,m,\varepsilon}(s-,A_{s-},Y_s).
\end{aligned}
\tag{2.68}
$$

We now substitute $Y=X-\tilde b^m$. Then, we may replace terms X_s by their respective left-limits X_{s-}, as the jump set of X is at most countable, and the continuous measures such as dA^c assign zero measure to countable sets. We also change the order of integration freely. Combining the above with the H approximations, we obtain

$$
\begin{aligned}
&(G^{n,m,\varepsilon}+H^{n,m,\varepsilon})(t,A_t,X_t-\tilde b^m_t)-(G^{n,m,\varepsilon}+H^{n,m,\varepsilon})(0,A_0,X_0-\tilde b^m_0)=\\
&\int_0^t n\Big[F(s-,A_{s-},\zeta+\tilde b^m_{s-})\,\rho(n(X_{s-}-\tilde b^m_{s-}-\zeta))\Big]^{2\varepsilon}_{\zeta=-\varepsilon}\,\mathrm{d}X_s\\
&+\int_0^t n\Big[F(s-,A_{s-},\zeta+\tilde b^m_{s-})\,\rho(n(X_{s-}-\tilde b^m_{s-}-\zeta))\Big]^{2\varepsilon}_{\zeta=-\varepsilon}\,\mathrm{d}\tilde b^m_s\\
&+\frac12\int_0^t\Big[n^2F(s-,A_{s-},\zeta+\tilde b^m_{s-})\,\rho'(n(X_{s-}-\tilde b^m_{s-}-\zeta))\Big]^{2\varepsilon}_{\zeta=-\varepsilon}\\
&\qquad\qquad+\Big[n\,F_x(s-,A_{s-},\zeta+\tilde b^m_{s-})\,\rho(n(X_{s-}-\tilde b^m_{s-}-\zeta))\Big]^{2\varepsilon}_{\zeta=-\varepsilon}\,\mathrm{d}[X,X]^c_s\\
&+\int_{-\infty}^{\infty}\Bigg\{\int_0^t F_s(s-,A_{s-},X_{s-}-z/n)\,\mathbb{1}_{\{X_{s-}-\tilde b^m_{s-}-\frac{z}{n}\notin(-\varepsilon,2\varepsilon]\}}\,\mathrm{d}s\\
&\qquad+\int_0^t F_a(s-,A_{s-},X_{s-}-z/n)\,\mathbb{1}_{\{X_{s-}-\tilde b^m_{s-}-\frac{z}{n}\notin(-\varepsilon,2\varepsilon]\}}\,\mathrm{d}A^c_s\\
&\qquad+\frac12\int_0^t F_{xx}(s-,A_{s-},X_{s-}-z/n)\,\mathbb{1}_{\{X_{s-}-\tilde b^m_{s-}-\frac{z}{n}\notin(-\varepsilon,2\varepsilon]\}}\,\mathrm{d}[X,X]^c_s\\
&\qquad+\int_0^t F_x(s-,A_{s-},X_{s-}-z/n)\,\mathbb{1}_{\{X_{s-}-\tilde b^m_{s-}-\frac{z}{n}\notin(-\varepsilon,2\varepsilon]\}}\,\mathrm{d}X_s\Bigg\}\;\rho(z)\,\mathrm{d}z
\end{aligned}
\tag{2.69}
$$

$$+ \sum_{0<s\le t} \int_{-\infty}^{\infty} \Big\{ F(s, A_s, X_s - z/n) - F(s-, A_{s-}, X_{s-} - z/n)$$
$$- F_x(s-, A_{s-}, X_{s-} - z/n)\, \Delta X_s \Big\}\, \mathbb{1}_{\{X_{s-} - \tilde{b}^m_{s-} - \frac{z}{n} \notin (-\varepsilon, 2\varepsilon]\}}\, \rho(z)\, \mathrm{d}z$$
$$+ n \Big[F(s-, A_{s-}, \zeta + \tilde{b}^m_{s-})\, \rho(n(X_{s-} - \tilde{b}^m_{s-} - \zeta)) \Big]_{\zeta=-\varepsilon}^{2\varepsilon} \Delta X_s .$$

The square parentheses above represent the difference of the expression they contain, evaluated at the subscript and superscript limits.

Now we may pass to the limit as $\varepsilon \to 0$. Decomposing $X = K + M + X_0$ as in (2.36), we employ the assumption (2.52). Letting $\varepsilon \to 0$, we may then use (2.51) and continuity of F. We are left with

$$(G^{n,m} + H^{n,m})(t, A_t, X_t - \tilde{b}^m_t) - (G^{n,m} + H^{n,m})(0, A_0, X_0 - \tilde{b}^m_0) =$$
$$\int_0^t \Big\{ \int_{-\infty}^{\infty} H(s-, A_{s-}, X_{s-} - z/n)\, \mathbb{1}_{\{X_{s-} - \frac{z}{n} \neq \tilde{b}^m_{s-}\}}\, \rho(z)\, \mathrm{d}z \Big\}\, \mathrm{d}\lambda(s)$$
$$+ \int_0^t \Big\{ \int_{-\infty}^{\infty} F_x(s-, A_{s-}, X_{s-} - z/n)\, \mathbb{1}_{\{X_{s-} - \frac{z}{n} \neq \tilde{b}^m_{s-}\}}\, \rho(z)\, \mathrm{d}z \Big\}\, \mathrm{d}M_s \tag{2.70}$$
$$+ \sum_{0< s \le t} \int_{-\infty}^{\infty} \Big\{ F(s, A_s, X_s - z/n) - F(s-, A_{s-}, X_{s-} - z/n)$$
$$- F_x(s-, A_{s-}, X_{s-} - z/n)\, \Delta M_s \Big\}\, \mathbb{1}_{\{X_{s-} - \frac{z}{n} \neq \tilde{b}^m_{s-}\}}\, \rho(z)\, \mathrm{d}z.$$

The indicator function may be removed as $\{n(X_{s-} - b^m_{s-})\}$ has zero Lebesgue measure in the z variable. Using (2.60) and the dominated convergence theorem, we may now take the limit as $m \to \infty$, then let $n \to \infty$, to give the result. □

Proof of Theorem 2.4.1. The proof of the local time on curves formula for jump processes, Theorem 2.4.1, follows in the same way as the proof of the extended Itô formula, Theorem 2.5.1, with the following considerations. Recall that the approximation $\tilde{b}^m$ is simply b in this setting.

Note that the absence of condition (2.51) means that the first-derivative boundary term associated to $\mathrm{d}[X,X]^c_s$ in (2.69),

$$\frac{1}{2} \int_0^t \Big[n\, F_x(s-, A_{s-}, \zeta + \tilde{b}^m_{s-})\, \rho(n(X_{s-} - \tilde{b}^m_{s-} - \zeta)) \Big]_{\zeta=-\varepsilon}^{2\varepsilon} \mathrm{d}[X,X]^c_s , \tag{2.71}$$

does not vanish as $\varepsilon \to 0$. To introduce the local time, we prove weak convergence of the measures $d_t J_t^n$ given below to the measure $d_t \ell_t^0(X-b)$. Define J^n by

$$J_t^n = \int_0^t n\,\rho(n(X_{s-} - b_{s-}))\,\mathrm{d}[X-b, X-b]_s^c. \tag{2.72}$$

Note that the measure $\mathrm{d}[X-b, X-b]_s^c$ assigns zero measure to countable sets, and as X and b are cádlág processes, they have at most countably many jumps. Thus we may replace $X_{s-} - b_{s-}$ by $X_s - b_s$ above.

For fixed $\omega \in \Omega$, define the function

$$s \mapsto g(s) = F_x(s, A_s, b_s+) - F_x(s, A_s, b_s-). \tag{2.73}$$

We know that g is right-continuous with left limits (for almost-all ω) by (2.40). By the standard theory of regulated functions, g admits a uniform approximation by right-continuous step functions. That is, for fixed $\varepsilon > 0$ there exists h such that $\|h - g\|_\infty < \varepsilon$ on $[0, t]$, with

$$h(s) = \sum_{i=0}^{N} h_i\,\mathbb{1}_{[a_i, a_{i+1})}(s) + h_{N+1}\mathbb{1}_{a_N}(s), \tag{2.74}$$

where $N \in \mathbb{N}$, the h_i are real-valued constants, and $0 = a_1 < a_2 < \cdots < a_N = t$. We employ again the time-dependent occupation time formula, yielding

$$\begin{aligned}&\int_0^t g(s)\,n\,\rho(n(X_s - b_s))\,\mathrm{d}[X-b, X-b]_s^c = \\ &\int_{-\infty}^{\infty}\left(\int_0^t g(s)\,\mathrm{d}_s\ell_s^a\right) n\,\rho(na)\,\mathrm{d}a \;= \int_{-\infty}^{\infty}\left(\int_0^t g(s)\,\mathrm{d}_s\ell_s^{c/n}\right)\rho(c)\,\mathrm{d}c.\end{aligned} \tag{2.75}$$

This gives us the estimate

$$\begin{aligned}&\left|\int_0^t g(s)\,\mathrm{d}_s\ell_s^{c/n} - \int_0^t g(s)\,\mathrm{d}_s\ell_s^0\right| \\ &\qquad \le \left|\int_0^t (g(s) - h(s))\,\mathrm{d}_s\ell_s^{c/n} - \int_0^t (g(s) - h(s))\,\mathrm{d}_s\ell_s^0\right| \\ &\qquad\quad + \left|\int_0^t h(s)\,\mathrm{d}_s\ell_s^{c/n} - \int_0^t h(s)\,\mathrm{d}_s\ell_s^0\right|.\end{aligned} \tag{2.76}$$

For any $\delta > 0$, we can take M large enough such that the final term is bounded above by δ, for all $0 < c/n < 1/M$. This follows from the convergence of $d_s\ell_s^{c/n}$ to $d_s\ell_s^0$ for each indicator function in h. The first term is bounded by

$$2\varepsilon \sup_{0<c<1/M} \ell_t^c. \tag{2.77}$$

Each of these terms can be made arbitrarily small. So we have shown

$$\lim_{n\to\infty} \frac{1}{2}\int_0^t (F_x(s-, A_{s-}, b_{s-}+) - F_x(s-, A_{s-}, b_{s-}-)) \\ n\,\rho(n(X_{s-}- b_{s-}))\,\mathrm{d}[X-b, X-b]^c_s \qquad (2.78)$$
$$= \frac{1}{2}\int_0^t (F_x(s-, A_{s-}, b_{s-}+) - F_x(s-, A_{s-}, b_{s-}-))\ \mathrm{d}_s\ell^0_s(X-b),$$

where convergence holds almost surely. Finally, when letting $n\to\infty$, we must employ (2.42), or the corresponding condition of Remark 2.4.7, to ensure that the limiting jump-terms form an absolutely convergent sum. This completes the proof of Theorem 2.4.1. □

The following remark establishes the 'local time on surfaces' formula under strong smoothness conditions when the process obeys (2.42).

Remark 2.5.2. Observe that the conditions on the function F in the 'local time on surfaces' formula under strong smoothness conditions, equation (2.20), immediately give us that the function $(t,a)\mapsto F_x(t,a,b(t,a)+) - F_x(t,a,b(t,a)-)$ is jointly continuous, and $(b_t)_{t\geq 0}$ is a semimartingale by assumption.

Note that strong smoothness immediately gives local boundedness of F_t, F_a and F_{xx}, and ensures that their left limits exist everywhere. Combining Remarks 2.4.2 and 2.4.6, we may immediately see that the left-hand side of (2.43) is already of the required form. We assume (2.42) holds, giving the result.

We can also establish that the first formula under weaker conditions, (2.12), follows as a special case of the previous Theorems.

Remark 2.5.3. Consider equation (2.12). In this case, there is no bounded variation process A. Note that $t\mapsto F_x(t,b(t)\pm\varepsilon)$, are continuous functions in t for each fixed $\varepsilon>0$, and converge uniformly. This ensures that the map $t\mapsto F_x(t,b(t)+) - F_x(t,b(t)-)$ is continuous.

Take λ to be the Lebesgue measure. Note that by the occupation times formula the set $\{s\in[0,t]\,|\,X_s = b(s)\}$ is λ-null, almost surely. Outside this set, the infinitesimal generator appearing in (2.13) is continuous, and so has left limits. Employing Remark 2.4.3, and taking H to be the expression in (2.13), the theorem follows.

We note next an important consideration regarding limits in the integral terms, which is presented in [46, p. 15 Rmk. 2.4].

Remark 2.5.4. If

$$\mathbb{P}(X_{s-} = b_{s-}) = 0, \tag{2.79}$$

for all $0 < s \leq t$, then we find that

$$\int_0^t \mathbb{1}_{\{X_{s-}=b_{s-}\}} \, \mathrm{d}s = 0 \tag{2.80}$$

almost surely, and so we may introduce this indicator function into the time integral, eliminating the limits in the space variable. In fact, if X solves an SDE such as (2.11) then (2.79) is satisfied, and furthermore

$$\int_0^t \mathbb{1}_{\{X_{s-}=b_{s-}\}} \, \mathrm{d}X_s = 0. \tag{2.81}$$

This may be shown using the extended occupation times formula and Fubini theorem. Hence we may eliminate limits in the stochastic integral in the same way. Similar considerations apply to other integral terms.

Chapter 3

Local Time-Space Integration

3.1 Introduction

Recall the Itô-Tanaka formula

$$F(X_t) = F(X_0) + \int_0^t F'_-(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} \ell_t^a\,\mathrm{d}F'_-(a), \tag{3.1}$$

where $\mathrm{d}F'_-$ is the Lebesgue-Stieltjes measure associated to the left derivative of F. In addition to the local time on curves formula of Chapter 2, many authors have continued the work of developing local time correction terms by adding a time dependence to the function F. Eisenbaum [18] and Ghomrasni and Peskir [36] derived a change of variables formula of the form

$$F(t,X_t) - F(0,X_0) = \int_0^t F_t(s,X_s)\,\mathrm{d}s + \int_0^t F_x(s,X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t\int_{\mathbb{R}} F_x(s,a)\,\mathrm{d}\ell_s^a, \tag{3.2}$$

where the final term is an expression which aims to unify correction terms depending on the local time.

It was pointed out by Ghomrasni and Peskir [36] that a formal integration-by-parts of this final term, the so-called 'local time-space integral', yield many of the other Itô formulae involving local time discovered by other researchers, including the local time on curves formula. In particular, by a formal integration-by-parts in a we see that

$$\int_0^t\int_{\mathbb{R}} F_x(s,a)\,\mathrm{d}\ell_s^a = -\int_0^t\int_{\mathbb{R}} \ell_s^a\,\mathrm{d}_a F_x(s,a), \tag{3.3}$$

which immediately transforms (3.2) into (3.1) in the time-independent case. In the time-dependent case it remains to establish the sense of the right hand side of this equation, but we can go beyond established results as will be seen in Definition 3.3.3 and onwards. The corresponding integration by parts in s and two-dimensional analogue in (s, a) are formulas already established in special cases in the literature, which we will examine in Section 3.2, but again we can improve on these results.

As a final remark, Ghomrasni and Peskir state that 'it is an interesting problem to establish these formulas rigorously under natural conditions'. In this chapter we will take up this challenge, and show how we can prove further results in the same way. Aside from unifying previous Itô formulae involving local time, the local time-space integral can also extend the results of Le Gall [42] on stochastic differential equations involving local time (SDELT). The main sticking point in generalising to the time-dependent case was the lack of a time-dependent Itô-Tanaka formula, a crucial component of Le Gall's method. The local time-space integral overcomes this limitation. Chapter 4 forms a full account of these equations and the author's extensions, but we must first perform a rigorous construction of the integral and examination of its properties which is the subject of this chapter. Note that the local time is generally of unbounded variation in the a variable, so the Lebesgue-Stieltjes construction cannot be applied, and other methods are required.

First we review various results of other authors on these correction terms in Section 3.2. Then, beginning from simple functions and employing a discrete integration by parts, in Section 3.3 we demonstrate three representations for the local time space integral. Representation (3.13) has appeared before in the special case when the measure μ is Lebesgue measure, but the generality here is new. Representation (3.15) is well known in the case of Lebesgue measure. Though it has been established by Eisenbaum [18] in the case of Brownian motion, and by Peskir [46] for the case of a Dirac measure, we treat the significantly more general setting of semimartingales and general Radon measures.

Of course it is necessary to develop a change-of-variables formula, Theorem 3.4.1, in Section 3.4. In Section 3.5 we will consider the local time-space integral as an operator and look at properties of the map $(H, X) \mapsto \int_{\mathbb{R}} \int_0^T H(s, a)\, \mathrm{d}\ell_s^a(X)$. Importantly, the final term in (3.2) is of bounded variation in many cases, meaning that the change of variables formula provides the semimartingale decomposition of the resulting process. Theorem 3.5.2, giving sufficient conditions for a type of dominated convergence theorem, is new but specific to our approach. Theorem 3.5.6 is new to the author's knowledge, proving continuity as a function of the semimartingale. To provide a sufficient condition for this, we also derive a condition for convergence in probability of the local time of a sequence of semimartingales, Lemma 3.5.7. Again, this is new to the author's knowledge. An important condition on the sequence of semimartingale is uniformity in $\mathcal{H}^1$, which we introduce in Definition 3.5.3.

In Section 3.6, we demonstrate several results which show how the local time-space calculus can be applied. Theorem 3.6.1 is the local time on curves formula of Chapter 2 in its most simple form. Here we derive it quickly using local time-space calculus. Theorem 3.6.2 was established by Ghomrasni [35], along with other results, and we repeat it in our setting with the same method of proof. Finally, Theorem 3.6.3 was previously established directly, but we may derive the result using local time-space calculus and an application of Theorem 3.5.6, showing the power of this approach.

3.2 Previous works

We establish here the results of previous authors on correction terms involving local time. We will see later (Theorem 3.4.1) that these formulae all appear as special cases of the local time-space integral.

3.2.1 Bouleau-Yor

The Bouleau-Yor formula provides a prototype for the time-dependent version that we will introduce. This was first established in [10], but a transparent proof and further comments can be found in [49]. Given a function $F : \mathbb{R} \to \mathbb{R}$, which is absolutely continuous with locally-bounded derivative f, we have that

$$F(X_t) - F(X_0) = \int_0^t f(X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_{\mathbb{R}} f(a)\,\mathrm{d}_a\ell_t^a. \tag{3.4}$$

The final integral is defined as a limit in probability of approximating sums, which we will introduce in an analogous manner to construct a time-dependent version of this formula.

3.2.2 Al-Hussaini and Elliot

Using the Bouleau-Yor formula (3.4) above, Al-Hussaini and Elliot [1] proved that for a suitably regular function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned} F(t, X_t) - F(0, X_0) = &\int_0^t F_t(s, X_s)\,\mathrm{d}s + \int_0^t F_x(s, X_s)\,\mathrm{d}X_s \\ &- \frac{1}{2}\int_{\mathbb{R}} F_x(t, a)\,\mathrm{d}_a\ell_t^a + \frac{1}{2}\int_0^t \int_{\mathbb{R}} F_{xt}(s, a)\,\mathrm{d}_a\ell_s^a\,\mathrm{d}s, \end{aligned} \tag{3.5}$$

where the $\mathrm{d}_a\ell_s^a$ integral is the Bouleau-Yor integral. The final expression is the one defined in (3.17) below, in the special case of Lebesgue measure. Notably the work [1] also contains some approximation results for the local time.

3.2.3 Eisenbaum

The formula established by Eisenbaum in [18] is of the same form as (3.7) below. The construction of the local time-space integral for Brownian motion follows from the representation

$$\int_0^t \int_{\mathbb{R}} f(s, a)\,\mathrm{d}\ell_s^a = \int_0^t f(s, B_s)\,\mathrm{d}B_s - \int_0^t f(s, B_s)\,\mathrm{d}^* B_s, \tag{3.6}$$

where the final integral is a backwards stochastic integral. A further interesting result is the representation of the local time-space integral for Brownian motion as an iterated integral with respect to a Radon measure in the same form as (3.15), which stems from the work of Azema, Jeulin, Knight and Yor [3]. The idea of approximating by Riemann-type sums, in the same spirit as our construction, is also explored.

In [19] and [20], the above representation is extended to Lévy processes and reversible semimartingales respectively, with bounded variation of jumps. This is extended to general one-dimensional Lévy processes by Eisenbaum and Kyprianou in [21], where it is used to characterise the domain of the extended generator of such processes. The construction of the local time-space integral makes use of the Föllmer-Protter-Shiryayev formula given in [32], which depends on time-reversal of the process and hence cannot easily be extended. The class of admissible functions f is however particularly broad, and the corresponding change of variables formula requires only the existence of first-order Radon-Nikodym derivatives in each argument.

3.2.4 Ghomrasni-Peskir

The local time-space formula of Ghomrasni and Peskir [36] is fundamentally the same as the one we obtain. If $F\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is C^1 then

$$F(t,X_t) - F(0,X_0) = \int_0^t F_t(s,X_s)\,\mathrm{d}s + \int_0^t F_x(s,X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t \int_{\mathbb{R}} F_x(s,a)\,\mathrm{d}\ell_s^a. \quad (3.7)$$

The construction of the final integral makes use of the local time on curves formula, which instead appears in our approach as a consequence of the local time-space calculus, Theorem 3.6.1. The latter part of [36] is devoted to demonstrating, mainly through non-rigorous manipulations, many of the other formulae that appear in this work.

3.2.5 Elworthy-Truman-Feng-Zhao

In [22], Elworthy, Truman and Zhao construct an integral of the form (3.17) below, and prove a corresponding change of variables formula. Feng and Zhao generalise this formula to higher dimensions in [29], using single parameter integrals which contain the components of the process along with their one-dimensional local times. Further, via a generalisation of the multi-parameter Young integral, Feng and Zhao directly construct the integral

$$\int\int_{[0,T]\times\mathbb{R}} H(s,a)\,\mathrm{d}_{(s,a)}\ell_s^a. \quad (3.8)$$

We refer the reader to [28] for this approach, which is not taken up in this thesis. In [30] and [31], the local-time integral in (3.4) is expressed as a rough-path integral, for continuous semimartingales and a certain class of Lévy processes respectively.

3.3 Constructing the local time-space integral

Below we will define the local time-space integral. This is motivated by the work of Ghomrasni and Peskir [36], where it was noticed that an integration-by-parts procedure applied to the formal expression

$$\int_0^T \int_{\mathbb{R}} F_x(s,x)\,\mathrm{d}\ell_s^x \quad (3.9)$$

yields many of the previous expressions given in Section 3.2. We take this as inspiration for the construction, starting from the definition for simple sums and then employing discrete integration by parts and approximation by Riemann-type sums. It is remarkable that the naive equation (3.11), introduced by Eisenbaum [18], turns out to be the correct definition to unify the other Itô formulae with local time correction terms.

We continue in the setting of the previous section. First let us note that the Bouleau-Yor integral is well defined by means of (3.4).

Definition 3.3.1. Given a locally-bounded function $f : \mathbb{R} \to \mathbb{R}$, we define the Bouleau-Yor integral of f by

$$\int_{\mathbb{R}} f(a)\,\mathrm{d}_a\ell_t^a = 2\left[F(X_t) - F(X_0) - \int_0^t f(X_s)\,\mathrm{d}X_s\right], \tag{3.10}$$

where F is any antiderivative of f.

Note that versions of Fubini's theorem and the dominated convergence theorem for the $d_a\ell_s^a$ integral follow from the stochastic and deterministic versions.

The fundamental definition of the local time-space integral of a product of indicator functions, as introduced by Eisenbaum [18], is

$$\Lambda(\mathbb{1}_{(s,t]}\,\mathbb{1}_{(x,y]}) = \int_0^T \int_{\mathbb{R}} \mathbb{1}_{(s,t]}\,\mathbb{1}_{(x,y]}\,\mathrm{d}\ell_s^x = \ell_t^y - \ell_s^y - \ell_t^x + \ell_s^x. \tag{3.11}$$

This is extended to simple functions by linearity. We will use either of these two expressions to denote the local time-space integral.

By considering the right-hand side of (3.11) above as a Lebesgue-Stieltjes integral, or Bouleau-Yor integral given by (3.10), we are led to the following three definitions. If a function H satisfies the conditions of any of these definitions, we say that H is *local time-space integrable* (with respect to the semimartingale X).

Definition 3.3.2. Assume that $H\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is left continuous in each argument when the other is fixed. Further assume that $t \to H(t,a)$ admits a density h with respect to some Radon reference measure μ. That is

$$H(t,a) - H(s,a) = \int_s^{t-} h(u,a)\,\mathrm{d}\mu(u), \tag{3.12}$$

for each $s,t \in [0,T]$ and $a \in \mathbb{R}$. Let h be also left continuous in a for each fixed u. Then we define the local time space integral $\int_0^T \int_{\mathbb{R}} H(s,x)\,\mathrm{d}\ell_s^x$ of H to be

$$\int_{\mathbb{R}} H(T,a)\,\mathrm{d}_a\ell_T^a - \int_0^{T-} \left(\int_{\mathbb{R}} h(u,a)\,\mathrm{d}_a\ell_u^a\right)\mathrm{d}\mu(u), \tag{3.13}$$

where the inner integral is the Bouleau-Yor local time integral (3.10).

Definition 3.3.3. Let $H\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be left continuous in each argument when the other is fixed. Further, assume that H admits a density g in its latter argument, with respect to some Radon reference measure ν. That is

$$H(t,y)-H(t,x)=\int_x^{y-} g(t,a)\,\mathrm{d}\nu(a), \tag{3.14}$$

for each $t\in[0,T]$ and $x,y\in\mathbb{R}$. Also let g be left continuous with right limits in t for each fixed a. Then the local time space integral $\int_0^T\int_{\mathbb{R}} H(s,x)\,\mathrm{d}\ell_s^x$ of H is defined by

$$-\int_{\mathbb{R}}\int_0^T g(u,a)\,\mathrm{d}_u\ell_u^a\,\mathrm{d}\nu(a). \tag{3.15}$$

Definition 3.3.4. Let $H:[0,T]\times\mathbb{R}\to\mathbb{R}$ be left continuous in each variable when the other is fixed. Assume that H is of locally-bounded Vitali variation, meaning for each $L>0$

$$\sup_{\pi}\left[\sum_{i,j=0}^{n-1}\Big|H(t_{i+1},x_{j+1})-H(t_i,x_{j+1})-H(t_{i+1},x_j)+H(t_i,x_j)\Big|\right]<\infty, \tag{3.16}$$

where the supremum is taken over all finite disjoint collections of rectangles π with vertices $(t_i\,,x_j)$ which cover $[0,T]\times[-L,L]$. Further, assume that the map $a\mapsto H(0,a)$ is of locally bounded one-dimensional variation. We define the local time space integral $\int_0^T\int_{\mathbb{R}} H(s,x)\,\mathrm{d}\ell_s^x$ of H as

$$-\int_{\mathbb{R}}\ell_T^a\,\mathrm{d}_a H(T,a)-\iint_{[0,T]\times\mathbb{R}}\ell_u^a\,d_{(u,a)}H(u,a), \tag{3.17}$$

where the final integral is a two-parameter Lebesgue-Stieltjes integral.

It can easily be seen that the stipulation that $a\mapsto H(t,a)$ be of bounded variation for a single fixed t, along with bounded two-dimensional variation, ensures that $a\mapsto H(t,a)$ is of bounded variation for all fixed $t\in[0,T]$.

The following lemma establishes the equivalence of the different representations for the local time space integral, starting from (3.11). It is also useful to establish convergence of the local time-space integral of smoothed functions in order to develop a change-of-variables formula, Theorem 3.4.1.

Lemma 3.3.5. *Definitions 3.3.2, 3.3.3 and 3.3.4 agree with each other on their common domains of definition.*

Proof. The proof follows by initially considering Riemann-type sums of smoothed functions, extending (3.11). As all the expressions agree by definition for simple functions, we take limits (in probability) to obtain the result. Uniqueness of limits in probability confirms that the definitions agree when mutually defined.

By localisation of the underlying process, we may assume that the field of local times $(\ell_t^a)_{a\in\mathbb{R},0\leq t\leq T}$ is compactly supported. Let $[-K,K]\subset\mathbb{R}$ be a closed interval, such that $[0,T]\times[-K,K]$ contains the support of ℓ. Limits of integration are taken to be consistent with left continuity of H if not explicitly described. We define $H(s,a)$ for negative s by $H(0,a)$, to preserve left continuity and simplify the proof.

First, mollify H by taking a function $\rho\in C^\infty(\mathbb{R})$, supported on $[0,1]$, such that $\int_{\mathbb{R}}\rho(x)\,\mathrm{d}x=1$. Define

$$\begin{aligned} H^{m,n}(t,a) &= \int_{\mathbb{R}}\int_{\mathbb{R}} H(t-y/m,a-z/n)\,\rho(y)\,\rho(z)\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} mnH(r,q)\,\rho(m(s-r))\,\rho(n(a-q))\,\mathrm{d}r\,\mathrm{d}q, \end{aligned} \tag{3.18}$$

for each $m,n\in\mathbb{N}$. Then $H^{m,n}$ is smooth in both variables, and it is straightforward to show that $H^{m,n}$ converges to H pointwise as $m\to\infty$ then $n\to\infty$ due to the left continuity of H.

We fix an arbitrary partition of $[0,T]\times[-K,K]$ into rectangles, and label the vertices (t_i,x_j) for $i=0,\dots,N$ and $j=0,\dots,M$. We ensure $x_0=-K$, $x_M=K$, $t_0=0$ and $t_N=T$ by adding points if necessary.

Take $\tilde{H}^{m,n}\colon[0,T]\times\mathbb{R}\to\mathbb{R}$ to be the left-endpoint approximation of H, namely

$$\tilde{H}^{m,n}(t,x)=\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} H_{i,j}^{m,n}\mathbb{1}_{(t_i,t_{i+1}]}(t)\mathbb{1}_{(x_j,x_{j+1}]}(x), \tag{3.19}$$

where $H_{i,j}^{m,n}=H^{m,n}(t_i,x_j)$. By definition, the local time-space integral of $\tilde{H}^{m,n}$ is

$$\Lambda(\tilde{H}^{m,n})=\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} H_{i,j}^{m,n}\left[\ell_{t_{i+1}}^{x_{j+1}}-\ell_{t_i}^{x_{j+1}}-\ell_{t_{i+1}}^{x_j}+\ell_{t_i}^{x_j}\right]. \tag{3.20}$$

Note that this expression agrees with (3.11). By rearranging this sum in the style of a discrete integration by parts, we may express (3.20) in any of the following three ways:

$$\begin{aligned} \Lambda(\tilde{H}^{m,n}) &= \sum_{i=0}^{N-1} H_{i,M-1}^{m,n}\left[\ell_{t_{i+1}}^{x_M}-\ell_{t_i}^{x_M}\right]-\sum_{i=0}^{N-1} H_{i,0}^{m,n}\left[\ell_{t_{i+1}}^{x_0}-\ell_{t_i}^{x_0}\right] \\ &\quad -\sum_{i=0}^{N-1}\sum_{j=1}^{M-1}\left[H_{i,j}^{m,n}-H_{i,j-1}^{m,n}\right]\left[\ell_{t_{i+1}}^{x_j}-\ell_{t_i}^{x_j}\right]; \end{aligned} \tag{3.21}$$

$$\begin{aligned}\Lambda(\tilde{H}^{m,n}) = &\sum_{j=0}^{M-1} H^{m,n}_{N-1,j}\left[\ell^{x_{j+1}}_{t_N} - \ell^{x_j}_{t_N}\right] - \sum_{j=0}^{M-1} H^{m,n}_{0,j}\left[\ell^{x_{j+1}}_{t_0} - \ell^{x_j}_{t_0}\right] \\ &- \sum_{i=1}^{N-1}\sum_{j=0}^{M-1}\left[H^{m,n}_{i,j} - H^{m,n}_{i-1,j}\right]\left[\ell^{x_{j+1}}_{t_i} - \ell^{x_j}_{t_i}\right];\end{aligned} \tag{3.22}$$

$$\begin{aligned}\Lambda(\tilde{H}^{m,n}) = &H^{m,n}_{N-1,M-1}\ell^{x_M}_{t_N} - H^{m,n}_{0,M-1}\ell^{x_M}_{t_0} - H^{m,n}_{N-1,0}\ell^{x_0}_{t_N} + H^{m,n}_{0,0}\ell^{x_0}_{t_0} \\ &- \sum_{i=1}^{N-1}\ell^{x_M}_{t_i}\left[H^{m,n}_{i,M-1} - H^{m,n}_{i-1,M-1}\right] + \sum_{i=1}^{N-1}\ell^{x_0}_{t_i}\left[H^{m,n}_{i,0} - H^{m,n}_{i-1,0}\right] \\ &- \sum_{j=1}^{M-1}\ell^{x_j}_{t_N}\left[H^{m,n}_{N-1,j} - H^{m,n}_{N-1,j-1}\right] + \sum_{j=1}^{M-1}\ell^{x_j}_{t_0}\left[H^{m,n}_{0,j} - H^{m,n}_{0,j-1}\right] \\ &+ \sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\ell^{x_j}_{t_i}\left[H^{m,n}_{i,j} - H^{m,n}_{i-1,j} - H^{m,n}_{i,j-1} + H^{m,n}_{i-1,j-1}\right].\end{aligned} \tag{3.23}$$

Note that local times at x_M and x_0 vanish by compactness of support.

We deal first with Definitions 3.3.2 and 3.3.3. Assume, for now, that H satisfies the conditions of Definition 3.3.2. By standard manipulations,

$$\begin{aligned}H^{m,n}(t,a) - H^{m,n}(s,a) = &\\ &\int_{\mathbb{R}}\int_{\mathbb{R}} mn\, H(r,q)\Big[\rho(m(t-r)) \\ &\qquad - \rho(m(s-r))\Big]\rho(n(a-q))\,\mathrm{d}r\,\mathrm{d}q \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} mn\, H(r,q)\int_s^t m\rho'(m(u-r))\,\mathrm{d}u\,\rho(n(a-q))\,\mathrm{d}r\,\mathrm{d}q \\ &= \int_s^t\int_{\mathbb{R}}\int_{\mathbb{R}} m^2 n\, H(r,q)\,\rho'(m(u-r))\,\rho(n(a-q))\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}u.\end{aligned} \tag{3.24}$$

Now applying properties of Lebesgue-Stieltjes integrals,

$$\begin{aligned}H^{m,n}(t,a) - H^{m,n}(s,a) &\\ &= \int_s^t\int_{\mathbb{R}}\int_{\mathbb{R}} -mn\, H(r,q)\,\rho(n(a-q))\,\mathrm{d}_r\rho(m(u-r))\,\mathrm{d}q\,\mathrm{d}u \\ &= \int_s^t\int_{\mathbb{R}}\int_{\mathbb{R}} mn\,\rho(m(u-r))\,\rho(n(a-q))\,\mathrm{d}_r H(r,q)\,\mathrm{d}q\,\mathrm{d}u \\ &= \int_s^t\int_{\mathbb{R}}\int_{\mathbb{R}} mn\, h(r,q)\,\rho(m(u-r))\,\rho(n(a-q))\,\mathrm{d}\mu(r)\,\mathrm{d}q\,\mathrm{d}u,\end{aligned} \tag{3.25}$$

where for consistency we have chosen $h(r,q) = h(0,q)$ for negative r, and $\mu((-\infty,0))$ equal to zero.

We now follow the same steps when H instead obeys Definition 3.3.3, which yields

$$\begin{aligned} H^{m,n}(t,y)-&H^{m,n}(t,x) \\ &= \int_x^y \int_{\mathbb{R}}\int_{\mathbb{R}} mn\, g(r,q)\, \rho(m(t-r))\, \rho(n(p-q))\, \mathrm{d}\nu(q)\, \mathrm{d}r\, \mathrm{d}p. \end{aligned} \tag{3.26}$$

Again, we proceed in the situation of Definition 3.3.2. Following from (3.22),

$$\begin{aligned} \Lambda(\tilde{H}^{m,n}) &= \sum_{j=0}^{M-1} H^{m,n}_{N-1,j} \left[\ell^{x_{j+1}}_{t_N} - \ell^{x_j}_{t_N}\right] - \sum_{j=0}^{M-1} H^{m,n}_{0,j} \left[\ell^{x_{j+1}}_{t_0} - \ell^{x_j}_{t_0}\right] \\ &- \sum_{i=1}^{N-1}\sum_{j=0}^{M-1} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}}\int_{\mathbb{R}} mn\, h(r,q)\, \rho(m(u-r)) \\ &\qquad\qquad \rho(n(x_j-q))\, \mathrm{d}\mu(r)\, \mathrm{d}q\, \mathrm{d}u \left[\ell^{x_{j+1}}_{t_i} - \ell^{x_j}_{t_i}\right]. \end{aligned} \tag{3.27}$$

Incorporating indicator functions, we see

$$\begin{aligned} \Lambda(\tilde{H}^{m,n}) &= \sum_{j=0}^{M-1} \int_{\mathbb{R}} H^{m,n}_{N-1,j}\, \mathbb{1}_{\{x_j < a \le x_{j+1}\}}\, \mathrm{d}_a \ell^a_{t_N} \\ &- \sum_{i=1}^{N-1}\sum_{j=0}^{M-1} \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} mn\, h(r,q)\, \rho(m(u-r)) \\ &\qquad \rho(n(x_j-q))\, \mathbb{1}_{\{t_{i-1}\le u < t_i\}}\, \mathbb{1}_{\{x_j < a \le x_{j+1}\}}\, \mathrm{d}\mu(r)\, \mathrm{d}q\, \mathrm{d}_a \ell^a_{t_i}\, \mathrm{d}u. \end{aligned} \tag{3.28}$$

Letting the mesh of the time partition tend to zero, using continuity of local time in the time variable, we obtain

$$\begin{aligned} &\sum_{j=0}^{M-1} \int_{\mathbb{R}} H^{m,n}(T,x_j)\, \mathbb{1}_{\{x_j < a \le x_{j+1}\}}\, \mathrm{d}_a \ell^a_T \\ &- \sum_{j=0}^{M-1} \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} mn\, h(r,q)\, \rho(m(u-r))\, \rho(n(x_j-q)) \\ &\qquad \mathbb{1}_{\{x_j < a \le x_{j+1}\}}\, \mathrm{d}\mu(r)\, \mathrm{d}q\, \mathrm{d}_a \ell^a_u\, \mathrm{d}u. \end{aligned} \tag{3.29}$$

Now let the mesh of the space partition tend to zero. By smoothness of ρ, and left continuity of H in space,

$$\begin{aligned} \Lambda(H^{m,n}) &= \int_{\mathbb{R}} H^{m,n}(T,a)\, \mathrm{d}_a \ell^a_T \\ &- \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} mn\, h(r,q)\, \rho(m(u-r))\, \rho(n(a-q))\, \mathrm{d}_a \ell^a_u\, \mathrm{d}\mu(r)\, \mathrm{d}q\, \mathrm{d}u. \end{aligned} \tag{3.30}$$

Let us state the corresponding result for H satisfying Definition 3.3.3. Recall that the local time is right continuous, and pointwise convergence of $\ell^{x_j}_s$ to ℓ^a_s implies weak convergence of the measures $\mathrm{d}\ell_s^{x_j}$ to $\mathrm{d}\ell_s^a$ for each fixed a. Thus, after following the same procedure in that case, with obvious modifications, we obtain

$$\Lambda(H^{m,n}) = -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^T mn\, g(r,q)\,\rho(m(s-r))\,\rho(n(p-q))\,\mathrm{d}_s\ell_s^a\,\mathrm{d}\nu(q)\,\mathrm{d}r\,\mathrm{d}p. \tag{3.31}$$

Now we take limits as $m \to \infty$, then $n \to \infty$. Making the change of variables $z = m(u-r)$ in (3.30), we see

$$\begin{aligned}\Lambda(H^{m,n}) &= \int_{\mathbb{R}} H^{m,n}(T,a)\,\mathrm{d}_a\ell_T^a \\ &\quad - \int_{-mr}^{m(T-r)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} n\,h(r,q)\,\rho(n(a-q))\,\mathrm{d}_a\ell_{r+z/m}^a\,\rho(z)\,\mathrm{d}\mu(r)\,\mathrm{d}q\,\mathrm{d}z.\end{aligned} \tag{3.32}$$

Taking a limit as $m \to \infty$, we note that continuity of local time in the time variable ensures the Bouleau-Yor integrals converge. Indeed, this follows from the continuity of the underlying process X and continuity of the stochastic integral. Further, unless $0 \le r < T$ the limits of integration both diverge in the same direction, yielding zero by compactness of the support of ρ. So we have

$$\Lambda(H^n) = \int_{\mathbb{R}} H^n(T,a)\,\mathrm{d}_a\ell_T^a - \int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} n\,h(r,q)\,\rho(n(a-q))\,\mathrm{d}_a\ell_r^a\,\mathrm{d}\mu(r)\,\mathrm{d}q. \tag{3.33}$$

Finally, changing variables as before and taking a limit as $n \to \infty$, we obtain

$$\Lambda(H) = \int_{\mathbb{R}} H(T,a)\,\mathrm{d}_a\ell_T^a - \int_0^T\int_{\mathbb{R}} h(r,a)\,\mathrm{d}_a\ell_r^a\,\mathrm{d}\mu(r). \tag{3.34}$$

Again performing the same operations with obvious modifications, we see that (3.31) becomes

$$\Lambda(H) = -\int_{\mathbb{R}}\int_0^T g(s,q)\,\mathrm{d}_s\ell_s^a\,\mathrm{d}\nu(q). \tag{3.35}$$

Now let us deal with Definition 3.3.4. We will require Lemma 3.3.6, which is proved separately. Beginning from (3.23), we see

$$\begin{aligned}\Lambda(\tilde{H}^{m,n}) &= -\sum_{j=1}^{M-1}\int_{\mathbb{R}} \ell_T^{x_j}\,\mathbb{1}_{\{x_{j-1}\le a<x_j\}}\,\mathrm{d}_a H^{m,n}(t_{N-1},a) \\ &\quad - \sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\iint_{[0,T]\times\mathbb{R}} \ell_{t_i}^{x_j}\,\mathbb{1}_{\{t_{i-1}\le u\le t_i\}}\,\mathbb{1}_{\{x_{j-1}\le a<x_j\}}\mathrm{d}_{(u,a)}H^{m,n}(u,a).\end{aligned} \tag{3.36}$$

Exchange the order of summation and integration, then take limits as the mesh of the time and space partitions converge to zero. Using the joint right continuity of local time and the dominated convergence theorem, we see

$$\Lambda(H^{m,n}) = \int_{\mathbb{R}} \ell_T^a\,\mathrm{d}_a H^{m,n}(T,a) - \iint_{[0,T]\times\mathbb{R}} \ell_u^a\,\mathrm{d}_{(u,a)}H^{m,n}(u,a). \tag{3.37}$$

Using Lemma 3.3.6, fix $\varepsilon > 0$ and choose a right-continuous step function ψ such that $\|\ell - \psi\|_\infty < \varepsilon$. Then

$$\begin{aligned}
&\left| \iint_{[0,T]\times\mathbb{R}} \ell_u^a \,\mathrm{d}_{(u,a)} H^{m,n}(u,a) - \iint_{[0,T]\times\mathbb{R}} \ell_u^a \,\mathrm{d}_{(u,a)} H(u,a) \right| \\
&\le \left| \iint_{[0,T]\times\mathbb{R}} \ell_u^a - \psi(a,u) \,\mathrm{d}_{(u,a)} H^{m,n}(u,a) \right| \\
&\quad + \left| \int_{[0,T]\times\mathbb{R}} \psi(a,u) \,\mathrm{d}_{(u,a)} (H^{m,n} - H)(u,a) \right| \\
&\quad + \left| \int_{[0,T]\times\mathbb{R}} \psi(a,u) - \ell_u^a \,\mathrm{d}_{(u,a)} H(u,a) \right| \\
&\le \varepsilon \Big[\mathrm{TV}(H^{m,n}) + \mathrm{TV}(H) \Big] + \Bigg| \sum_{i=1}^m \sum_{j=1}^n a_{i,j} \Big[H^{m,n}(t_i, x_j) - H(t_i, x_j) \\
&\qquad - H^{m,n}(t_{i-1}, x_j) + H(t_{i-1}, x_j) - H^{m,n}(t_i, x_{j-1}) + H(t_i, x_{j-1}) \\
&\qquad + H^{m,n}(t_{i-1}, x_{j-1}) - H(t_{i-1}, x_{j-1}) \Big] \Bigg|.
\end{aligned} \tag{3.38}$$

By pointwise convergence of $H^{m,n}$ to H, each term in this final sum can be made arbitrarily small. A simpler version of the same procedure also shows that the former integral in (3.37) converges.

Finally, let us note that the procedure we followed yields the same result independently of the choice of sequence of partitions. □

The following is a technical result which was needed in the previous lemma, but will also be useful later.

Lemma 3.3.6. *The field of local times admits a uniform approximation by jointly right continuous step functions. That is, for all $K > 0$ and $\varepsilon > 0$ there is a function ψ of the form*

$$\psi(s,a) = \sum_{i=1}^m \sum_{j=1}^n a_{i,j} \, \mathbb{1}_{[t_{i-1}, t_i)}(s) \, \mathbb{1}_{[x_{j-1}, x_j)}(a), \tag{3.39}$$

such that $\|\ell - \psi\|_\infty < \varepsilon$ on the domain $[0,T] \times [-K,K]$.

Proof. Fix $\varepsilon > 0$. Let $A_\varepsilon = \{(t,a) \in [-K,K] \times [0,T] \,:\, \text{there exists } \psi \text{ right continuous such that } \|\ell - \psi\|_\infty < \varepsilon \text{ on } [0,t] \times [-K,a]\}$.

As $a \mapsto \ell_0^a$ is right continuous with left limits (in fact the zero function), it admits a uniform approximation by right continuous step functions (see Lemma A.2.9). Thus $[-L,L] \times \{0\} \subset A$.

Choose such a uniform approximation with error at most $\frac{\varepsilon}{2}$. Precisely, take $\phi : [-L, L] \to \mathbb{R}$ such that

$$\phi(a) = \sum_{i=1}^{n} a_i \, \mathbb{1}_{\{x_{i-1} \le a < x_i\}}, \tag{3.40}$$

and $\|\phi - \ell\| < \frac{\varepsilon}{2}$. Now we may choose $\delta > 0$ such that for all $a \in [-L, L]$ and $s \in [0, \delta)$, we have $|\ell_s^a - \ell_0^a| < \frac{\varepsilon}{2}$. If this were not the case, there would be some sequence (a_k, s_k) such that $0 < s_k < 1/k$ and $|\ell_{s_k}^{a_k} - \ell_0^{a_k}| > \frac{\varepsilon}{2}$. The set $\{a_k\}$ must have an accumulation point, and so by passing to a subsequence, we may assume $a_k \to a$ monotonically for some point a, meaning $(a_k, s_k) \to (a, 0)$ monotonically in a. This contradicts the joint right continuity with left limits in space, and continuity in time, of the local time.

We may now extend ϕ by defining

$$\theta(a, s) = \sum_{i=1}^{n} a_i \, \mathbb{1}_{\{x_{i-1} \le a < x_i\}} \, \mathbb{1}_{\{0 \le s < \delta\}}, \tag{3.41}$$

and noting that $\|\theta - \ell\|_\infty \le \|\phi - \ell_0\|_\infty + \|\ell - \ell_0\|_\infty < \varepsilon$. Thus we know $[-L, L] \times [0, \delta) \subseteq A$.

Define $\mathcal{T} = \sup\{\, t \mid [-L, L] \times [0, t) \subseteq A\}$. Arguing by contradiction, assume $\mathcal{T} < T$. By the same procedure as before, using joint regularity of the local time, we may choose $\delta > 0$ such that $\|\ell_{\mathcal{T}}^a - \ell_s^a\| < \frac{\varepsilon}{3}$, uniformly in a, for all $s \in (\mathcal{T} - \delta, \mathcal{T} + \delta)$. By definition of $\mathcal{T}$, we may choose some jointly right continuous step function θ such that $\|\theta - \ell\| < \frac{\varepsilon}{3}$ on $[-L, L] \times [0, \mathcal{T} - \frac{\delta}{2})$. Finally, as $a \mapsto \ell_{\mathcal{T}}^a$ is regulated, we choose some step function $\phi : [-L, L] \to \mathbb{R}$ such that $\|\phi - \ell_{\mathcal{T}}\| < \frac{\varepsilon}{3}$. We extend the definition of θ by defining

$$\tilde{\theta}(a, s) = \begin{cases} \theta(a, s) & 0 \le s < \mathcal{T} - \frac{\delta}{2} \\ \phi(a) & \mathcal{T} - \frac{\delta}{2} \le s < \mathcal{T} + \frac{\delta}{2} \end{cases} \tag{3.42}$$

Again by the triangle inequality, we may show that $\|\tilde{\theta} - \ell\| < \varepsilon$. Thus we have $\mathcal{T} = T$.

Finally we may show that $[-K, K] \times [0, T] = A$ by the same procedure, as $a \mapsto \ell_T^a$ is regulated, and replacing $(\mathcal{T} - \delta, \mathcal{T} + \delta)$ by $(T - \delta, T]$ in the argument just given.

□

Remark 3.3.7. The previous definitions and lemmas of this section hold also for random functions $H : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ which obey the conditions pathwise almost surely. For the Bouleau-Yor integral to be well defined in Definition 3.3.1, it is sufficient that $(\omega, s) \mapsto H(s, X_s, \omega)$ be predictably measurable.

Remark 3.3.8. We may generalise Definitions 3.3.2 and 3.3.3 to the case when H can be described by the sum of a finite number of measures and densities. That is,

$$H(t,a) - H(s,a) = \sum_{i=1}^{n} \int_{s}^{t-} h^i(u,a)\,\mathrm{d}\mu^i(u), \tag{3.43}$$

where each h^i and μ^i obey the required conditions, and the analogous extension holds for Definition 3.3.3. The expression for the local time-space integral of such a function changes in the obvious way.

Remark 3.3.9. We chose to use left limits and the right local time, which is right continuous in space. These may be replaced by right or symmetric limits and the left or symmetric local time, provided the corresponding conditions (mostly right continuity) are satisfied. Note that the left local time has a modification which is left continuous in space. Also note that if one uses right limits with left local time, then (3.13) should be an integral with respect to ℓ_u^{a-} instead.

3.4 A local time-space change of variables formula

Our aim in constructing the local time-space integral was to unify various representations of the correction term in generalisations of Itô's formula. The following change of variables formula provides this connection.

Theorem 3.4.1. *Let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, given by $(t,x) \mapsto F(t,x)$ be left continuous in t and continuous in x when the other argument is fixed. Also assume that F admits the following:*

1. *A density in the time variable t with respect to some Radon reference measure μ, denoted F_t, which is left continuous in the space variable x. That is,*

$$F(t,x) - F(s,x) = \int_{s}^{t-} F_t(u,x)\,\mathrm{d}\mu(u), \tag{3.44}$$

   for each $s,t \in [0,T]$ and $x \in \mathbb{R}$.

2. *A partial derivative in the space variable x, denoted F_x, which is local time-space integrable in any of the given forms.*

Then we have

$$\begin{aligned} F(T, X_T) - F(0, X_0) = &\int_0^{T-} F_t(s, X_s)\, \mathrm{d}\mu(s) + \int_0^T F_x(s, X_s)\, \mathrm{d}X_s \\ &- \frac{1}{2} \int_0^T \int_{\mathbb{R}} F_x(s, a)\, \mathrm{d}\ell_s^a, \end{aligned} \tag{3.45}$$

for all $T \geq 0$.

Proof. We first prove the result for functions of class C^2. We have that

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}} F_x(s, a)\, \mathrm{d}\ell_s^a = \frac{1}{2} \left(\int_{\mathbb{R}} F_x(T, a)\, \mathrm{d}_a \ell_T^a - \int_0^T \left(\int_{\mathbb{R}} F_{xt}(s, a)\, \mathrm{d}_a \ell_s^a \right) \mathrm{d}s \right), \tag{3.46}$$

which follows from Definition 3.3.2. We expand the right-hand side using (3.10), giving

$$\begin{aligned} &F(T, X_T) - F(T, X_0) - \int_0^T F_x(T, X_u)\, \mathrm{d}X_u \\ &- \int_0^T \left(F_t(s, X_s) - F_t(s, X_0) - \int_0^s F_{xt}(s, X_u)\, \mathrm{d}X_u \right) \mathrm{d}s. \end{aligned} \tag{3.47}$$

Now apply the stochastic Fubini theorem and fundamental theorem of calculus, to give

$$\begin{aligned} &F(T, X_t) - F(0, X_0) - \int_0^T F_t(s, X_s)\, \mathrm{d}s \\ &- \int_0^T F_x(T, X_u)\, \mathrm{d}X_u + \int_0^T \int_u^T F_{xt}(s, X_u)\, \mathrm{d}s\, \mathrm{d}X_u. \end{aligned} \tag{3.48}$$

Another application of the fundamental theorem of calculus yields the result.

We then establish (3.45) by approximating functions in the domain of definition of each representation by functions in C^2. Specifically, mollify F in the same form as (3.18). Then we have

$$\begin{aligned} F^{m,n}(t, a) &= \int_{\mathbb{R}} \int_{\mathbb{R}} F(t - y/m, a - z/n)\, \rho(y)\, \rho(z)\, \mathrm{d}y\, \mathrm{d}z \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} mnF(r, q)\, \rho(m(t - r))\, \rho(n(a - q))\, \mathrm{d}r\, \mathrm{d}q. \end{aligned} \tag{3.49}$$

The stochastic and classical dominated convergence theorems and the proof of Lemma 3.3.5 give convergence of the stochastic and local time-space integrals. Only the time integral remains. We can see that

$$\begin{aligned}
&\int_0^T F_t^{m,n}(s, X_s)\,\mathrm{d}s \\
&\quad= \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} mnF(r,q)\,\rho(m(s-r))\,\rho(n(a-q))\,\mathrm{d}r\,\mathrm{d}q\right)\bigg|_{a=X_s}\mathrm{d}s \\
&\quad= \int_0^T \left(\int_{\mathbb{R}}\int_{\mathbb{R}} m^2nF(r,q)\,\rho'(m(s-r))\,\rho(n(a-q))\,\mathrm{d}r\,\mathrm{d}q\right)\bigg|_{a=X_s}\mathrm{d}s \\
&\quad= \int_0^T \left(\int_{\mathbb{R}}\int_{\mathbb{R}} -mnF(r,q)\,\rho(n(a-q))\,\mathrm{d}_r\rho(m(s-r))\,\mathrm{d}q\right)\bigg|_{a=X_s}\mathrm{d}s \\
&\quad= \int_0^T \left(\int_{\mathbb{R}}\int_{\mathbb{R}} mnF_t(r,q)\,\rho(m(s-r))\,\rho(n(a-q))\,\mathrm{d}\mu(r)\,\mathrm{d}q\right)\bigg|_{a=X_s}\mathrm{d}s \\
&\quad= \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} mnF_t(r,q)\,\rho(m(s-r))\,\rho(n(X_s-q))\,\mathrm{d}\mu(r)\,\mathrm{d}q\,\mathrm{d}s.
\end{aligned} \tag{3.50}$$

Now make the substitution $u = m(s-r)$. We get

$$\begin{aligned}
&\int_0^T F_t^{m,n}(s, X_s)\,\mathrm{d}s \\
&\quad= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-mr}^{m(T-r)} nF_t(r,q)\,\rho(u)\,\rho(n(X_{r+u/m}-q))\,\mathrm{d}u\,\mathrm{d}\mu(r)\,\mathrm{d}q.
\end{aligned} \tag{3.51}$$

Now take a limit as $m \to \infty$. Recalling that X is continuous, we also note that the limits of the u integral diverge to the same limit unless $0 \le r < T$. Thus we have

$$\int_0^T F_t^n(s, X_s)\,\mathrm{d}s = \int_{\mathbb{R}}\int_0^T nF_t(r,q)\,\rho(n(X_r-q))\,\mathrm{d}\mu(r)\,\mathrm{d}q. \tag{3.52}$$

Exchanging the order of integration, changing variables, then taking the limit as $n \to \infty$ and using the left continuity of F_t in space gives

$$\int_{\mathbb{R}}\int_0^T nF_t(r,q)\,\rho(n(X_r-q))\,\mathrm{d}\mu(r)\,\mathrm{d}q \to \int_0^T F_t(r, X_r)\,\mathrm{d}\mu(r). \tag{3.53}$$

This completes the proof. □

Remark 3.4.2. We may generalise (3.44) by replacing μ and F_t by a finite sum of arbitrary Radon measures and densities, which are left continuous in space, and obey the obvious analogue of (3.44).

3.5 Properties of local time-space integration

A key consequence of the classical Itô formula (1.6) is that the semimartingale decomposition of $F(B_t)$ is given explicitly. Specifically, the quadratic variation integral is of bounded variation. The same holds for the local time-space integral, except for the integral with respect to the space variable of local time, which cannot be of bounded variation in time in general.

Theorem 3.5.1. *If H is local time-space integrable, then expression (3.15), both integrals in (3.17) and the two-variable integral in (3.13) are of bounded variation.*

Proof. Take a partition π of $[0,T]$, such that $t_0 = 0$ and $t_N = T$. Dealing with (3.15),

$$\begin{aligned}\sum_{t_i\in\pi}\left|\int_{\mathbb{R}}\int_{t_i}^{t_{i+1}} g(u,a)\,\mathrm{d}_u\ell_u^a\,\mathrm{d}\nu(a)\right| &\le \int_{\mathbb{R}}\sum_{t_i\in\pi}(\ell_{t_{i+1}}^a-\ell_{t_i}^a)\sup_{t_i\le u\le t_{i+1}}|g(u,a)|\,\mathrm{d}|\nu|(a)\\ &\le \int_{\mathbb{R}}\ell_T^a\sup_{0\le u\le T}|g(u,a)|\,\mathrm{d}|\nu|(a).\end{aligned} \tag{3.54}$$

By continuity of the underlying semimartingale, the local time is almost surely compactly supported. Thus the right-hand integral may be taken over a compact set, and so the right hand side is finite and independent of the partition.

The variation of (3.17) over any partition is bounded by the supremum of the local time, which is almost-surely bounded pathwise, and the Vitali variation of H.

Finally, as h is bounded in (3.13), and μ is of finite total variation on compacts, the two-variable integral is of bounded variation. □

It is natural to be concerned with properties of the map $H \mapsto \Lambda(H)$, meaning continuity or a type of dominated convergence theorem.

Theorem 3.5.2. *Let H^n and H be a sequence of local time-space integrable functions satisfying any one of the following conditions:*

1. *H^n and H satisfy Definition 3.3.2, and there exists a measure γ such that $\nu^n \ll \gamma$ and $\nu \ll \gamma$, where the densities $g^n\frac{\mathrm{d}\nu^n}{\mathrm{d}\gamma}$ are uniformly locally bounded and converge pointwise to $g\frac{\mathrm{d}\nu}{\mathrm{d}\gamma}$;*

2. *H^n and H satisfy Definition 3.3.3, and there exists a measure γ such that $\mu^n \ll \gamma$ and $\mu \ll \gamma$, where the densities $h^n\frac{\mathrm{d}\mu^n}{\mathrm{d}\gamma}$ are uniformly locally bounded and converge pointwise to $h\frac{\mathrm{d}\mu}{\mathrm{d}\gamma}$;*

3. *H^n and H satisfy Definition 3.3.4, with uniformly locally-bounded total variation, and $H^n \to H$ pointwise.*

Then we have

$$\int_{\mathbb{R}} \int_0^T H^n(s,a)\,\mathrm{d}\ell_s^a(X) \to \int_{\mathbb{R}} \int_0^T H(s,a)\,\mathrm{d}\ell_s^a(X), \tag{3.55}$$

where convergence holds uniformly on compacts in probability.

Proof. The final case follows in the same way as the final part of the proof of Lemma 3.3.5, namely (3.38) with $H^{m,n}$ there replaced by H^n. The other two cases are straightforward applications of the deterministic and stochastic dominated convergence theorems. □

We are motivated by [50] to also consider some form of continuity depending on the underlying semimartingale. Let us first introduce a type of localisation. Given a sequence of semimartingales $(X^n)_{n\geq 1}$, we will refer to the bounded variation and local martingale components of X^n as A^n and M^n respectively (which vanish at time zero almost surely). We define $\mathcal{H}^1$ to be the set of continuous semimartingales X, with decomposition $X_0 + M + A$, such that $\|X\|_{\mathcal{H}^1} = \mathbb{E}\left[\langle M, M\rangle_\infty^{1/2} + \int_0^\infty |\mathrm{d}A_s|\right]$ is finite.

Definition 3.5.3. We say that a sequence of semimartingales $(X^n)_{n\geq 1}$ are uniformly locally in $\mathcal{H}^1$ if there exists a sequence of stopping times T_m, increasing to infinity almost surely, such that for all $n \in \mathbb{N}$ we have $\|X^n_{\cdot\wedge T_m}\|_{\mathcal{H}^1} \leq K(m)$ for some constant $K(m)$ independent of n.

Note that all continuous semimartingales are locally in $\mathcal{H}^1$. Asking that a sequence be uniformly locally in $\mathcal{H}^1$ prevents the following situation. Take X^n to be a Brownian motion starting at zero, reflected between barriers at 0 and $1/n$. The quadratic variation of each X^n is independent of n, and as $n \to \infty$ the X^n converge uniformly on compacts in probability to the zero process. One may see that $\int_0^t |\mathrm{d}A_s^n|$ is unbounded in n for any fixed t, so the sequence does not obey Definition 3.5.3. In general, however, we have the following result, due to Barlow and Protter [4].

Lemma 3.5.4 ([4] Thm 1, Corr 2)**.** *Let X^n be a sequence of semimartingales in $\mathcal{H}^1$, such that $\int_0^\infty |\mathrm{d}A_s^n| \leq K$ for some non-random constant K. Assume that there is some process X such that $\mathbb{E}\left[(X^n - X)^*\right] \to 0$. Then X is a semimartingale, and both $\lim_{n\to\infty} \|(M^n - M)^*\|_{\mathcal{H}^1} = 0$ and $\mathbb{E}\left[(A^n - A)^*\right] \to 0$, with $\int_0^\infty |\mathrm{d}A_s| \leq K$.*

We now impose stronger conditions on a sequence of semimartingales.

Definition 3.5.5. A sequence of semimartingales X^n will be called admissible if;

1. For each $n \in \mathbb{N}$, $X_0^n = 0$ almost surely.

2. There exists a sequence of stopping times T_m such that $T_m \uparrow \infty$ almost surely, and
$$\sup_{s \in \mathbb{R}_+} \left| X^n_{T_m \wedge s} \right| < K(m) \tag{3.56}$$
almost surely, for some constant $K(m)$ independent of n.

3. The sequence X^n is locally uniformly in $\mathcal{H}^1$.

We now determine when convergence of an admissible sequence implies convergence of the associated local time-space integrals.

Theorem 3.5.6. *Let X^n be a sequence of admissible semimartingales converging to a semimartingale X uniformly on compacts in probability. Let H be local time-space integrable in the sense of Definition 3.3.3 or Definition 3.3.4. Assume that $|\mathrm{d}A^n|$ converges weakly in probability to some (random) measure λ, meaning*
$$\int_0^t f(s) |\mathrm{d}A^n_s| \to \int_0^t f(s)\, \mathrm{d}\lambda(s) \tag{3.57}$$
in probability, for each continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ and $t \in \mathbb{R}_+$. If for all $a \in \mathbb{R}$
$$\mathbb{E}\left[\int_0^t \mathbb{1}_{\{X_s = a\}}\, \mathrm{d}\lambda(s) \right] = 0 \tag{3.58}$$
for all $t \in \mathbb{R}_+$, then we have
$$\int_{\mathbb{R}} \int_0^T H(s,a)\, \mathrm{d}\ell^a_s(X^n) \to \int_{\mathbb{R}} \int_0^T H(s,a)\, \mathrm{d}\ell^a_s(X), \tag{3.59}$$
where convergence holds uniformly on compacts in probability.

Proof. Take a sequence of stopping times T_m with $T_m \uparrow \infty$ almost surely, such that
$$\sup_{s \in \mathbb{R}+} \left| X^n_{s \wedge T_m} \right| < K(m) \qquad \text{and} \qquad \| X^n_{\cdot \wedge T_m} \|_{\mathcal{H}^1} < K(m) \tag{3.60}$$
almost surely for some $K(m)$ independent of n. Fix $T \in \mathbb{R}_+$ and take $M \in \mathbb{N}$ such that $\mathbb{P}(T_m < T) < \varepsilon$ for all $m \geq M$. On the complement of this set, that is on $\{T_m \geq T\}$, $\ell^a_t(X^n)$ is compactly supported in $[0,T] \times [-K(M), K(M)]$. Again on $\{T_m \geq T\}$, using the Tanaka formula and triangle inequality, we can show
$$C = \sup_n \left(\sup_{[0,T] \times [-K,K]} \mathbb{E}\left[\ell^a_t(X^n) \right] \right) < \infty. \tag{3.61}$$

It follows that the same bound holds for X in place of X^n on the set $\{T_m \geq T\}$ with the same constant C by Lemma 3.5.4.

Beginning with Definition 3.3.3, by elementary bounds we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_0^T g(s,a)\,\mathrm{d}_s\ell_s^a(X^n)\,\mathrm{d}\nu(a) - \int_{\mathbb{R}}\int_0^T g(s,a)\,\mathrm{d}_s\ell_s^a(X)\,\mathrm{d}\nu(a)\right| \\ &\leq \int_{\mathbb{R}}\left|\int_0^T g(s,a)\,\mathrm{d}_s(\ell_s^a(X^n) - \ell_s^a(X))\right|\,\mathrm{d}|\nu|(a).\end{aligned} \tag{3.62}$$

Restricting to the set $\{T_m \geq T\}$, take an expectation and apply Fubini's theorem. We then determine the limit as $n \to \infty$ of

$$\mathbb{E}\left[\left|\int_0^T g(s,a)\,\mathrm{d}_s(\ell_s^a(X^n) - \ell_s^a(X))\right| \mathbb{1}_{\{T_m \geq T\}}\right], \tag{3.63}$$

for each fixed a. Fix $\delta > 0$. As g is regulated in s, we may choose some left continuous step function ψ such that $\|g(\cdot,a) - \psi\| < \delta/2C$. Using the triangle inequality, we may bound (3.63) by

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^T \big(g(s,a) - \psi(s)\big)\,\mathrm{d}_s(\ell_s^a(X^n) - \ell_s^a(X))\right| \mathbb{1}_{\{T_m \geq T\}}\right] \\ &+ \mathbb{E}\left[\left|\int_0^T \psi(s)\,\mathrm{d}_s(\ell_s^a(X^n) - \ell_s^a(X))\right| \mathbb{1}_{\{T_m \geq T\}}\right].\end{aligned} \tag{3.64}$$

The former term is bounded by 2δ. It is straightforward to see that the latter term is bounded and converges to zero in probability as $n \to \infty$, giving convergence in expectation. Now we may apply the dominated convergence theorem to the expression

$$\int_{\mathbb{R}} \mathbb{E}\left[\left|\int_0^T g(s,a)\,\mathrm{d}_s(\ell_s^a(X^n) - \ell_s^a(X))\right| \mathbb{1}_{\{T_m \geq T\}}\right] \mathrm{d}|\nu|(a), \tag{3.65}$$

to obtain convergence to zero. This gives convergence in probability of (3.62).

A more straightforward version of the same method gives the result for Definition 3.3.4. □

Now we establish conditions for convergence in probability of the local times. Afterwards, we shall consider ways to verify the assumptions of the following lemma which are useful for us, but by no means exhaustive.

Lemma 3.5.7. *In the setting of Theorem 3.5.6, we have* $\ell_s^a(X^n) \to \ell_s^a(X)$ *uniformly on compacts in probability in* s*, for each fixed* $a \in \mathbb{R}$*.*

Proof. First let us assume that X^n and X are uniformly bounded in $\mathcal{H}^1$, and that $\mathbb{E}\left[(X^n - X)^*\right] \to 0$. This gives that $\mathbb{E}\left[\langle M^n - M, M^n - M\rangle_\infty^{\frac{1}{2}}\right] \to 0$ and $\mathrm{d}A^n \to \mathrm{d}A$ weakly in expectation. Furthermore, $\mathrm{d}A^n$ and $\mathrm{d}A$ have total variation uniformly bounded by some deterministic constant K.

Expanding via the Tanaka formula, we see

$$\begin{aligned}
&|\ell_t^a(X^n) - \ell_t^a(X)| \\
&= \Big| \Big(|X_t^n - a| - |X_0^n - a| - \int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}X_s^n\Big) \\
&\quad - \Big(|X_t - a| - |X_0 - a| - \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s\Big)\Big| \\
&\leq \Big||X_t^n - a| - |X_t - a|\Big| + \Big||X_0^n - a| - |X_0 - a|\Big| \\
&\quad + \Big|\int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}X_s^n - \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s\Big|
\end{aligned} \tag{3.66}$$

The first two terms converge to zero uniformly on compacts in in probability by uniform convergence in expectation of X^n to X, and the continuous mapping theorem. Using the decompositions of X^n and X, the difference of the stochastic integrals with respect to the local martingale parts is bounded by

$$\begin{aligned}
&|\int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}M_s^n - \int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}M_s| \\
&+ |\int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}M_s - \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}M_s|.
\end{aligned} \tag{3.67}$$

Dealing with each part separately, we have

$$\begin{aligned}
&\mathbb{E}\left[|\int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}M_s^n - \int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}M_s|\right] \\
&= \mathbb{E}\left[\left(\int_0^t \operatorname{sgn}(X_s^n - a)^2\,\mathrm{d}\langle M^n - M, M^n - M\rangle_s\right)^{\frac{1}{2}}\right] \\
&\leq \mathbb{E}\left[\left(\int_0^t \mathrm{d}\langle M^n - M, M^n - M\rangle_s\right)^{\frac{1}{2}}\right] \\
&= \mathbb{E}[\langle M^n - M, M^n - M\rangle_t^{\frac{1}{2}}] \leq \mathbb{E}[\langle M^n - M, M^n - M\rangle_\infty^{\frac{1}{2}}].
\end{aligned} \tag{3.68}$$

This converges to zero by assumption, and is independent of t.

Expanding the other part, fixing some $\varepsilon > 0$, we see

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}M_s - \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}M_s\right|^2\right] \\
&= \mathbb{E}\left[\int_0^t (\operatorname{sgn}(X_s^n - a) - \operatorname{sgn}(X_s - a))^2 \,\mathrm{d}\langle M, M\rangle_s\right] \\
&\leq \mathbb{E}\bigg[\int_0^t (\operatorname{sgn}(X_s^n - a) - \operatorname{sgn}(X_s - a))^2 \left(\mathbb{1}_{\{X_s \notin (a-\varepsilon, a+\varepsilon)\}}\right. \\
&\qquad\qquad\qquad\qquad \left. + \ \mathbb{1}_{\{X_s \in (a-\varepsilon, a+\varepsilon)\}}\right) \mathrm{d}\langle M, M\rangle_s\bigg] \\
&\leq \mathbb{E}\left[\int_0^t (\operatorname{sgn}(X_s^n - a) - \operatorname{sgn}(X_s - a))^2 \ \mathbb{1}_{\{X_s \notin (a-\varepsilon, a+\varepsilon)\}} \,\mathrm{d}\langle M, M\rangle_s\right] \\
&\quad + \mathbb{E}\left[\int_0^t \mathbb{1}_{\{X_s \in (a-\varepsilon, a+\varepsilon)\}} \,\mathrm{d}\langle M, M\rangle_s\right].
\end{aligned} \tag{3.69}$$

For each fixed $\varepsilon > 0$, as $n \to \infty$ the former term converges to zero by uniform convergence of X^n to X in expectation. The latter term may be re-expressed using the occupation time formula (A.10) as

$$\mathbb{E}\left[\int_{\mathbb{R}} \mathbb{1}_{\{x \in (a-\varepsilon, a+\varepsilon)\}} \ell_t^x \,\mathrm{d}x\right]. \tag{3.70}$$

Letting now $\varepsilon \to 0$, this converges to zero. Uniformity in t follows after noting that the integrands are positive, and the measures $\mathrm{d}\langle M, M\rangle$ are non-negative.

We follow the same procedure for the Lebesgue-Stieltjes integrals. Let $U_0 = \{(s,\omega) \in \mathbb{R}_+ \times \Omega : |X_s - a| \geq 1\}$ and $U_i = \{(s,\omega) \in \mathbb{R}_+ \times \Omega : 2^{-(i+1)} \leq |X_s - a| < 2^{-i}\}$ for each $i \in \mathbb{N}$. Employing Tonelli's theorem,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \operatorname{sgn}(X_s^n - a)\,\mathrm{d}A_s^n - \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}A_s^n\right|\right] \\
&= \mathbb{E}\left[\left|\int_0^t \sum_{i=0}^{\infty} \Big(\operatorname{sgn}(X_s^n - a) - \operatorname{sgn}(X_s - a)\Big) \mathbb{1}_{U_i} \,\mathrm{d}A_s^n\right|\right] \\
&\quad + \mathbb{E}\left[\left|\int_0^t \Big(\operatorname{sgn}(X_s^n - a) - \operatorname{sgn}(X_s - a)\Big) \ \mathbb{1}_{\{X_s = a\}} \,\mathrm{d}A_s^n\right|\right] \\
&\leq \sum_{i=0}^{\infty} \mathbb{E}\left[\sup_{0 \leq s \leq t} \Big|\operatorname{sgn}(X_s^n - a) - \operatorname{sgn}(X_s - a)\Big| \mathbb{1}_{U_i} \operatorname{TV}\left(\mathrm{d}A_s^n\,|_{U_i}\right)\right] \\
&\quad + \mathbb{E}\left[\left|\int_0^t \Big(\operatorname{sgn}(X_s^n - a) - \operatorname{sgn}(X_s - a)\Big) \ \mathbb{1}_{\{X_s = a\}} \,\mathrm{d}A_s^n\right|\right]
\end{aligned} \tag{3.71}$$

Uniform convergence of X^n to X in expectation implies that the sum converges to zero as $n \to \infty$, after employing the dominated convergence theorem. Again this is uniform in t. The final term is zero by assumption (3.58) for each t. Finally,

$$\mathbb{E}\left[\left|\int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}A_s^n - \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}A_s\right|\right] \to 0 \tag{3.72}$$

as the set of possible discontinuity points $\{X_s = a\}$ is λ-null, again by (3.58). One can also observe this by writing the signum function as a difference of two indicator functions, and considering measures of the sets $\{X_s \geq a\}$ and $\{X_s < a\}$. This is independent of t by assumption.

Finally, let X^n and X be as in the statement of the theorem. Then there exists a sequence of stopping times T_m which increase to infinity almost surely, such that $\|X^n_{\cdot\wedge T_m}\|_{\mathcal{H}^1} < K(m)$, and $\|X_{\cdot\wedge T_m}\|_{\mathcal{H}^1} < K(m)$, with each of the stopped processes bounded. Fix $t_0 > 0$ and positive constants ε and δ. Take M such that $\mathbb{P}(T_M \leq t_0) < \varepsilon$. Then it follows

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\leq t\leq t_0} |\ell^a_t(X^n) - \ell^a_t(X)| > \delta\right) \\ &\leq \mathbb{P}\left(\sup_{0\leq t\leq t_0} |\ell^a_t(X^n_{\cdot\wedge T_m}) - \ell^a_t(X_{\cdot\wedge T_m})| > \delta\right) + \mathbb{P}(T_M \leq t_0) \end{aligned} \tag{3.73}$$

Applying the former part of this proof to the stopped semimartingales $X^n_{\cdot\wedge T_m}$ and $X_{\cdot\wedge T_m}$, noting that convergence in probability and uniform boundedness implies convergence in expectation, we obtain the result. □

The next result is a probabilistic version of a general result in the theory of functions of bounded variation, namely that convergence in L^1 and convergence of total variations in $\mathbb{R}$ implies weak convergence (see Lemma A.3.7). This helps to verify (3.58) when $\lambda = |\mathrm{d}A_s|$.

Lemma 3.5.8. *Let A^n be a sequence of processes of locally uniformly bounded variation, meaning that there exists a sequence of stopping times T^m increasing to ∞ almost surely, such that $\int_0^\infty |\mathrm{d}A^n_{s\wedge T_m}| < K(m)$ for constants $K(m)$. Further assume there exists a process of locally bounded variation A such that $\int_0^t |(A^n_s - A_s)|\,\mathrm{d}s \to 0$ in probability for each $t \in \mathbb{R}_+$. If we have $\int_0^t |\mathrm{d}A^n_s| \to \int_0^t |\mathrm{d}A_s|$ in probability for each $t \in \mathbb{R}_+$, then also $|\mathrm{d}A^n| \to |\mathrm{d}A|$ weakly in probability.*

Proof. By taking a minimum, we assume that T^m also localises $\int_0^\infty |\mathrm{d}A_s|$ with the same constant $K(m)$. Fix $\varepsilon > 0$ and $t_0 > 0$ and take M such that $\mathbb{P}(T_M \leq t_0) < \varepsilon$. Fixing a continuous f, by straightforward bounds we find

$$\mathbb{E}\left[\left|\int_0^{t_0} f(s)|\mathrm{d}A^n_{s\wedge T_M}| - \int_0^{t_0} f(s)|\mathrm{d}A_{s\wedge T_M}|\right|^2\right] \leq 4\,K(M) \sup_{0\leq s\leq t_0} |f(s)|^2\,, \tag{3.74}$$

meaning that this sequence of random variables is uniformly integrable and lies in $L^1(\mathbb{P})$. Thus it is relatively sequentially compact in the weak topology by the Dunford-Pettis theorem. Given any convergent subsequence, by passing to a further subsequence, we obtain a subsequence converging almost surely. The deterministic Theorem [2, Prop 3.15, p. 126] applies to this subsequence, which therefore converges to zero in expectation. As any subsequence has a convergent subsequence with the same limit, we obtain that the original sequence converges to zero in the weak topology. By using the bounded test function $\mathbb{1}_{\{\Omega\}}(\omega)$ we obtain the result. □

Remark 3.5.9. Note that condition (3.58) is satisfied when λ admits a density with respect to Lebesgue measure almost surely, and $\mathbb{P}(X_s = a) = 0$ for each $0 < s \leq t$, by applying Fubini's theorem. In particular, this holds in the important special case when X is a one-dimensional Itô process with strictly positive diffusion coefficient.

3.6 Applications

We derive the local time on curves formula under strong smoothness conditions, in its original form (2.10) using the right local time.

Theorem 3.6.1. *Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale, with corresponding field of local times ℓ_s^a, and let $b : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function of bounded variation. Define*

$$C = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x < b(t)\},$$
$$D = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > b(t)\}.$$

Suppose we are given a continuous function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that

$$F \text{ is } C^{1,2} \text{ on } \bar{C},$$
$$F \text{ is } C^{1,2} \text{ on } \bar{D}.$$

The precise meaning of this condition is that the restriction of F to C can be extended to a $C^{1,2}$ function on the whole of $\mathbb{R}_+ \times \mathbb{R}$, and likewise for D. Then the following *change of variable formula holds*

$$\begin{aligned}F(t,X_t) = F(0,X_0) &+ \int_0^t F_t(s,X_s-)\,\mathrm{d}s + \int_0^t F_x(s,X_s-)\,\mathrm{d}X_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\ \mathbb{1}_{\{X_s\neq b(s)\}}\,\mathrm{d}\langle X,X\rangle_s \\ &+ \int_0^t \frac{1}{2}\Big(F_x(s,b(s)+) - F_x(s,b(s)-)\Big)\ \mathbb{1}_{\{X_s=b(s)\}}\,\mathrm{d}_s\ell^b_s(X).\end{aligned} \tag{3.75}$$

Equivalently, we may express this as

$$\begin{aligned}F(t,X_t) = F(0,X_0) &+ \int_0^t F_t(s,X_s-)\,\mathrm{d}s + \int_0^t F_x(s,X_s-)\,\mathrm{d}X_s \\ &+ \frac{1}{2}\int_{\mathbb{R}}\int_0^t F_{xx}(s,a+b(s))\ \mathbb{1}_{\{a\neq b(s)\}}\,\mathrm{d}_s\ell^{b+a}_s(X)\,\mathrm{d}a \\ &+ \int_0^t \frac{1}{2}\Big(F_x(s,b(s)+) - F_x(s,b(s)-)\Big)\ \mathbb{1}_{\{X_s=b(s)\}}\,\mathrm{d}_s\ell^b_s(X),\end{aligned} \tag{3.76}$$

where ℓ^{b+a}_s is the local time of $X-b$ at a.

Proof. The equivalence of the representations follows immediately from the time-dependent occupation time formula Theorem A.1.4. Write $G(s,x) = F(s,x+b(s))$, and note that $Y = X - b$ is a semimartingale as b is of bounded variation. Then $F(s,X_s) = G(s,Y_s)$ by definition.

We can see that G admits a second space derivative as a density with respect to the measure ν, given by

$$\nu(A) = \delta_0(A) + \mathrm{Leb}(A), \tag{3.77}$$

where δ_0 is the Dirac mass at 0, and

$$G_{xx}(s,a) = \begin{cases} F_x(s,b(s)+) - F_x(s,b(s)-) & a = 0, \\ F_{xx}(s,a+b(s)) & a \neq 0. \end{cases} \tag{3.78}$$

This means G_x is local time-space integrable in the form of Definition 3.3.3. As b is of bounded variation, it generates a Lebesgue-Stieltjes measure $\mathrm{d}b$. Denoting by $\mathrm{d}b^c$ and $\mathrm{d}b^\perp$ the Lebesgue-continuous and Lebesgue-singular part of $\mathrm{d}b$ respectively, we have by the Lebesgue-Stieltjes chain rule

$$\begin{aligned}G(t,x) = G(s,x) &+ \int_s^t F_t(u,x+b(u)-) + F_x(u,x+b(u)-)\,\frac{\mathrm{d}b^c}{\mathrm{d}u}\,\mathrm{d}u \\ &+ \int_s^t F_x(u,x+b(u)-)\,\mathrm{d}b^\perp(\mathrm{d}u).\end{aligned} \tag{3.79}$$

So we have something as described in Remark 3.4.2, with left-continuous integrands in space.

Now apply Theorem 3.4.1 to G and Y. Substituting in the expressions of G and its derivatives in terms of F, we obtain

$$\begin{aligned} F(T, X_T) - F(0, X_0) = & \int_0^T F_t(s, X_s-)\,\mathrm{d}s + \int_0^T F_x(s, X_s-)\,\mathrm{d}b(s) \\ & + \int_0^T F_x(s, X_s-)\,\mathrm{d}(X-b)_s \\ & - \frac{1}{2}\int_{\mathbb{R}}\int_0^T F_{xx}(s, a+b(s))\,\mathbb{1}_{\{a\neq 0\}}\,\mathrm{d}_s\ell_s^a(Y)\,\mathrm{d}a \\ & + \int_0^T (F_x(s, b(s)+) - F_x(s, b(s)-))\,\mathrm{d}_s\ell_s^0(Y). \end{aligned} \tag{3.80}$$

The $\mathrm{d}b(s)$ integrals cancel with each other. Finally, note that the measure $\mathrm{d}_s\ell_s^a(Y)$ gives full measure to the set $\{s \,|\, Y_s = a\}$, which is the same as $\{s \,|\, X_s = a + b(s)\}$. This gives (3.76) and completes the proof. □

In Ghomrasni and Peskir's work [36], it is observed that the a similar formula can be derived by formal manipulations in the case when F is instead non-smooth over a curve $c : \mathbb{R} \to \mathbb{R}_+$, taking values in the time parameter. Indeed, they also mention that a 'similar candidate formula' can be formally derived for a general curve $\gamma : [0,1] \to [0,T] \times \mathbb{R}$ over which F is non-smooth. It may be possible to derive a partial result in this direction from Definition 3.3.4. However, taking $\gamma(t) = (t,t)$, it is easily seen that the function

$$G(s,x) = \begin{cases} 1 & \text{if } x > s \\ 0 & \text{otherwise} \end{cases} \tag{3.81}$$

is not of bounded variation in the sense we described in Definition 3.3.4, so the present method does not apply. The author hopes to establish a more general time-space local time on curves formula in future work, but also see Chapter 5 for more ideas in this direction.

The following theorem is a result from Ghomrasni [34, 35], which we present here using the same idea of proof, but with an alternative construction of the local time-space integral. Other results in Ghomrasni [35] follow directly using the same methods.

Theorem 3.6.2. *Let $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ be local time-space integrable. Then*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T H(s, X_s) - H(s, X_s - \varepsilon)\,\mathrm{d}\langle X, X\rangle_s = \int_{\mathbb{R}}\int_0^T H(s,a)\,\mathrm{d}\ell_s^a, \tag{3.82}$$

where the limit is taken in probability.

Proof. Define $H_\varepsilon(s,a) = \frac{1}{\varepsilon}\int_{a-e}^{a} H(s,x)\,\mathrm{d}x$. Note that $H_\varepsilon \to H$ pointwise as $\varepsilon \to 0$. Note also that $\frac{\partial}{\partial a}H_\varepsilon(s,a) = \frac{1}{\varepsilon}\left(H(s,a) - H(s,a-\varepsilon)\right)$. Thus by the occupation time formula,

$$\begin{aligned}\Lambda(H_\varepsilon) &= -\int_{\mathbb{R}}\int_0^T \frac{H(s,a)-H(s,a-\varepsilon)}{\varepsilon}\,\mathrm{d}_s\ell_s^a\,\mathrm{d}a \\ &= \frac{1}{\varepsilon}\int_0^T H(s,X_s) - H(s,X_s-\varepsilon)\,\mathrm{d}\left\langle X,X\right\rangle_s.\end{aligned} \tag{3.83}$$

Assume that H satisfies the conditions of Definition 3.3.3. Then we may write

$$\int_{\mathbb{R}}\int_0^T \frac{H(s,a)-H(s,a-\varepsilon)}{\varepsilon}\,\mathrm{d}_s\ell_s^a\,\mathrm{d}a = \int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \frac{g(s,x)}{\varepsilon}\,\mathbb{1}_{\{a-\varepsilon\le x<a\}}\,\mathrm{d}\nu(x)\,\mathrm{d}_s\ell_s^a\,\mathrm{d}a. \tag{3.84}$$

Rewriting the indicator function and exchanging the order of integration, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^T \frac{g(s,x)}{\varepsilon}\,\mathbb{1}_{\{x<a\le x+\varepsilon\}}\,\mathrm{d}_s\ell_s^a\,\mathrm{d}a\,\mathrm{d}\nu(x) = \int_{\mathbb{R}}\frac{1}{\varepsilon}\int_x^{x+\varepsilon}\int_0^T g(s,x)\,\mathrm{d}_s\ell_s^a\,\mathrm{d}a\,\mathrm{d}\nu(x). \tag{3.85}$$

As g is a regulated function in the time variable, right-continuity of local time in the space variable allows us to determine that the function $\phi(a) = \int_0^T h(s,x)\,\mathrm{d}_s\ell_s^a$ is right continuous at x. Taking a limit as $\varepsilon \to 0$, we obtain the result.

The corresponding result when H satisfies instead Definitions 3.3.2 or 3.3.4 follow by integration by parts in s then a similar method. □

We now recast a result of Protter and San Martin [50] in terms of local time-space integration. We require the local time for a general semimartingale, and recall the details of Section A.1.3.

Theorem 3.6.3. *Fix a semimartingale X. Let H be a continuous process, and θ a continuous process of bounded variation. Assume that*

$$\int_0^t \mathbb{1}_{\{X_s=\theta_s\}}\,\mathrm{d}A_s = \int_0^t \mathbb{1}_{\{X_s=\theta_s\}}\,\mathrm{d}\theta_s = 0 \tag{3.86}$$

for each $t \in \mathbb{R}_+$ almost surely. Then for a refining sequence of partitions $(\pi_n)_{n\ge1}$ of $[0,t]$, whose mesh converges to zero, we have

$$\lim_{n\to\infty}\sum_{t_i\in\pi_n} H_{t_i}\left(\ell_{t_{i+1}}^{\theta_{t_i}}(X) - \ell_{t_i}^{\theta_{t_i}}(X)\right) = \int_0^t H_s\,\mathrm{d}_s\ell_s^\theta(X), \tag{3.87}$$

where convergence is uniformly on compacts in probability.

Proof. For each n, define the processes $H^n(s,a) = \sum_{t_i \in \pi_n} H_{t_i} \mathbb{1}_{\{t_i < s \leq t_{i+1}\}} \mathbb{1}_{\{a>0\}}$ pathwise and $X^n_s = X_s - \sum_{t_i \in \pi_n} \theta(t_i) \mathbb{1}_{\{t_i \leq s < t_{i+1}\}}$. It is easily confirmed that X^n are locally uniformly in $\mathcal{H}^1$. Then we may express the sum on the left hand side as

$$\int_{\mathbb{R}} \int_0^t H^n(s,a) \, \mathrm{d}_s \ell^a_s(X^n) \, \mathrm{d}\delta_0(a). \tag{3.88}$$

By the triangle inequality,

$$\begin{aligned} &\left| \int_0^t H^n(s,a) \, \mathrm{d}_s \ell^a_s(X^n) - \int_0^t H_s \, \mathrm{d}_s \ell^\theta_s(X-\theta) \right| \\ &\leq \left| \int_0^t \big(H^n(s,a) - H(s,a)\big) \, \mathrm{d}_s \ell^a_s(X^n) \right| + \left| \int_0^t H(s,a) \, \mathrm{d}_s \big(\ell^a_s(X^n) - \ell^a_s(X-\theta)\big) \right|. \end{aligned} \tag{3.89}$$

The former integral is bounded by

$$\sup_{0 \leq s \leq t} |H^n(s,a) - H(s,a)| \, \mathrm{TV}\left(\mathrm{d}_s \ell^a_s(X^n)\right) \leq \sup_{0 \leq s \leq t} |H^n(s,a) - H(s,a)| \, \ell^a_t(X^n). \tag{3.90}$$

Taking an expectation, this converges to zero by uniform convergence of H^n to H, and boundedness in expectation of the sequence of local times.

We now verify convergence in probability of local times to apply Theorem 3.5.6. Note that the bounded variation component of X^n is $A - \sum_{t_i \in \pi_n} \theta(t_i) \mathbb{1}_{\{t_i \leq s < t_{i+1}\}}$, and the bounded variation component of $X - \theta$ is $A - \theta$. Using Lemma 3.5.8, we see that

$$\left| \mathrm{d}_s \left(A_s - \sum_{t_i \in \pi_n} \theta(t_i) \mathbb{1}_{\{t_i \leq s < t_{i+1}\}} \right) \right| \to |\mathrm{d}_s (A_s - \theta_s)| \tag{3.91}$$

weakly in probability. Now we must show (3.58). Given a set $V \in \mathcal{B}(\mathbb{R}_+)$ and a Borel measure μ on $\mathcal{B}(\mathbb{R}_+)$, the collection $\mathcal{N}_V = \{U \in \mathcal{B}(\mathbb{R}) \,|\, \mu(U \cap V) = 0\}$ is a monotone class. We now note that (3.86) implies $(u,v) \in \mathcal{N}_{\{s \,:\, X_s = A_s\}}$ for all $u < v$, where $\mu = \mathrm{d}A_s$ or $\mathrm{d}\theta_s$. By the monotone class theorem, $\mathcal{N}_V = \mathcal{B}(\mathbb{R})$.

Finally as $X^n \to X$ uniformly on compacts in probability, we obtain the result. □

Chapter 4

Stochastic Differential Equations Involving the Local Time

4.1 Introduction

The most fundamental stochastic differential equation involving local time (SDELT) is given simply by

$$X_t = X_0 + B_t + \beta \ell_t^a(X). \tag{4.1}$$

The solution to this equation is the well-known skew Brownian motion with parameter β (denoted SkBM), first described in this form by Harrison and Shepp [37]. The SkBM models a particle which travels like a Brownian motion, except at the origin where it hits a permeable barrier with unequal probability of transmission or reflection. If the probability of transmission or reflection is time-dependent, then a natural extension of the SkBM would be a process solving an equation of the form

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} \int_0^t h(s, a)\,\mathrm{d}_s\ell_s^a(X)\,\mathrm{d}\nu(a). \tag{4.2}$$

In this Chapter, we prove results on existence and pathwise uniqueness of equations of this form. In Section 4.2 we provide some motivation and recap work of other authors on equations of this form. By extending the method of Le Gall, given in the time-independent case, we establish a bijective correspondence between equations of the form (4.2) and standard Itô equations of the form (4.4) in Lemma 4.3.1. This requires an extended definition of the local time-space integral, Lemma 4.3.2, and a change of local time result, Lemma 4.3.3. Finally we obtain an existence and uniqueness result, Theorem 4.3.5, in the case when $b = 0$. We then extend this to include a drift term under relatively strong conditions in Theorem 4.3.7. Finally we discuss the close connections with SDELTs with local time on curves, of the form (4.6) below.

4.2 Previous work

The study of such equations was initiated by Stroock and Yor [55] in the time homogeneous case, where we have

$$X_t = X_0 + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} \ell_t^a(X)\,\mathrm{d}\nu(a). \tag{4.3}$$

Their aim was to study the 'purity' of certain martingales. Le Gall provided a general treatment of the time homogeneous case in the paper [42], relying upon what has been called the 'method of local times' for stochastic differential equations which he introduced in [41]. It should be noted that choosing $b = 0$, $\sigma = 1$ and ν a point mass at zero, we obtain (4.1) as a special case. See [43] for a thorough survey of constructions of SkBM.

The results of Le Gall are based upon a bijective transformation which removes the local time component by means of the Itô-Tanaka formula. Using similar machinery, Rutowski [54] and Bass and Chen [6] weakened the conditions on the coefficients of (4.3). We can establish existence and uniqueness results for the transformed equation using the method of local times. The bijection transforms these solutions back to the original equation and so preserves existence and uniqueness.

The general conditions for existence and pathwise uniqueness of the solutions to stochastic differential equations of the form

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \tag{4.4}$$

are the Lipschitz and linear growth conditions on both the drift and diffusion coefficients. However in dimension one, in the time-homogeneous case, there are two well-known generalisations attributed to Yamada-Watanabe and Nakao. The Yamada-Watanabe conditions are effectively Hölder conditions, and are known to be sharp. The Nakao condition is more suited to our requirements, as it allows a discontinuous diffusion coefficient, and is effectively a bound on its quadratic variation. We will use a modification of Le Gall's [41] proof of this result.

The first development in the time inhomogeneous case is due to Weinryb [56], generalised by more recent work by Ouknine and Bouhadou [9] and Étoré and Martinez [26], who examined the equation

$$X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t \beta(s)\,\mathrm{d}_s\ell_s^0(X). \tag{4.5}$$

More recently, Étoré and Martinez [27] have extended this to the case when the final term is the local time on a curve, namely by the equation

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s + \int_0^t \beta(s)\,\mathrm{d}_s\ell_s^r(X), \tag{4.6}$$

for some curve r of class C^1. This uses a slight generalisation of the 'local time on curves' formula of Theorem 3.6.1, originally due to Peskir [46], instead of the Itô-Tanaka formula. We will discuss this in Section 4.4.

The treatment which we present here uses the change of variables formula Theorem 3.4.1 in place of the Itô-Tanaka formula, which is crucial in the time homogeneous case. We establish weak existence and pathwise uniqueness results for equations of the form (4.2), which provides existence of a unique strong solution.

4.3 Existence and uniqueness results

Throughout, we let $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be bounded and measurable. Further, we assume that $\sigma \geq \varepsilon > 0$ for some $\varepsilon > 0$. This is sufficient [52, Co. IX 1.14] to ensure existence and uniqueness in law for the equation (4.4).

Allowing σ to approach zero is outside the scope of our method, but it should be noted that if one also appropriately constrains the 'soujourn time' at points where σ vanishes, then existence, uniqueness and the Markov property can be obtained for SDEs of the form (4.4), see [25].

We further assume that ν is a finite a regular finite Borel measure, $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is bounded and measurable, and that $|h(s, a)\nu(\{a\})| < 1/2$ for all $(s, a) \in \mathbb{R}_+ \times \mathbb{R}$. Essentially this condition cannot be relaxed, as if $|h(s, a)\nu(\{a\})| > 1/2$ then even a weak solution cannot exist in general, and if $|h(s, a)\nu(\{a\})| = 1/2$ then the equation may admit a weak solution but no strong solution. This was first discussed by Harrison and Shepp [37], and the paper of Engelbert and Blei [24] provides a thorough account of these cases in the time homogeneous setting.

We assume that for some strictly increasing function ρ, the diffusion coefficient σ obeys

$$(\sigma(t,y) - \sigma(t,x))^2 \leq |\rho(y) - \rho(x)|\,, \tag{4.7}$$

for all $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2$.

Given h and ν obeying the previous assumptions, we now define the function F which will transform away the local time term of (4.2). This is a natural modification of the method of Le Gall [42] to the time dependent case, using exactly the same construction with time-dependent coefficients. His proof uses the Itô-Tanaka formula, whereas ours uses the corresponding Theorem 3.4.1. Let

$$F(t,x) = \int_0^x \exp\left(\zeta(t,a)\right)\psi(t,a)\,\mathrm{d}a, \tag{4.8}$$

where the functions ζ and ψ are defined by

$$\zeta(t,x) = \int_{-\infty}^x h(t,z)\,\mathrm{d}\nu^c(z), \tag{4.9}$$

where ν^c is the continuous part of ν, and

$$\psi(t,x) = \prod_{z<x}\Big(1 - h(t,z)\nu(\{z\})\Big), \tag{4.10}$$

where an empty product is taken to be 1. We let F_x denote the left derivative of F, and define the function G by

$$G(t,y) = \left[F(t,\cdot)\right]^{-1}(y). \tag{4.11}$$

To compress notation, we define $\tilde{G}(t,y) = (t, G(t,y))$ and $\tilde{F}(t,x) = (t, F(t,x))$. Then $F \circ \tilde{G}$ and $G \circ \tilde{F}$ are identity maps. We stipulate now that h is differentiable in its former argument when the latter is fixed, with h_t being continuous in time and admitting left limits in space. This ensures that F_t exists and is continuous in time with left limits in space. We can also establish that G can be represented as

$$G(t,y) = \int_0^y \frac{1}{\exp\Big(\zeta \circ \tilde{G}(t,a)\Big)\,\psi \circ \tilde{G}(t,a)}\,\mathrm{d}a. \tag{4.12}$$

Lemma 4.3.1. *The process X solves the stochastic differential equation with local time*

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} \int_0^t h(s,a)\,\mathrm{d}_s\ell_s^a(X)\,\mathrm{d}\nu(a), \tag{4.13}$$

if and only if the process $Y_t = F(t, X_t)$ solves the stochastic differential equation

$$Y_t = Y_0 + \int_0^t F_{t-} \circ \tilde{G}(s, Y_s)\,\mathrm{d}s + \int_0^t (F_x\,\sigma) \circ \tilde{G}(s, Y_s)\,\mathrm{d}B_s, \tag{4.14}$$

where F_{t-} represents the left limit of F_t in the space variable.

Proof. Proving the forward direction consists of applying Theorem 3.4.1, using representation (3.15). The local time component of (4.2) cancels with the local time-space integral of F by construction.

We show the reverse direction. Note that $X_t = G(t, Y_t)$ by definition. Using the Lebesgue-Stieltjes chain rule then pushing forward,

$$\begin{aligned} G_y(t,y) = G_y(t,x) &- \int_x^{y-} \frac{1}{(F_x \circ \tilde{G}(t,a))^2}\,\mathrm{d}_a(F_x \circ \tilde{G}(t,\cdot)) \\ &+ \sum_{x<a<y} \frac{1}{(F_{x+} \circ \tilde{G}(t,a))} - \frac{1}{(F_x \circ \tilde{G}(t,a))} \\ &\qquad - \frac{1}{(F_x \circ \tilde{G}(t,a))^2}\Big(F_{x+} \circ \tilde{G}(t,a) - F_x \circ \tilde{G}(t,a)\Big) \\ &= \int_{G(t,x)}^{G(t,y)-} \frac{h(t,a)}{F_{x+}(t,a)}\,\mathrm{d}\nu(a), \end{aligned} \tag{4.15}$$

where F_{x+} refers to the right-hand space derivative of F. Applying Theorem 3.4.1 to the function G and semimartingale Y, using the representation (4.21) below, we see

$$\begin{aligned} G(t, Y_t) = G(0, Y_0) &+ \int_0^t G_{t-} + G_y \cdot \big(F_{t-} \circ \tilde{G}\big)(s, Y_s)\,\mathrm{d}s \\ &+ \int_0^t G_y \cdot (F_x\,\sigma \circ \tilde{G})(s, Y_s)\,\mathrm{d}B_s + \int_{\mathbb{R}}\int_0^t \frac{h(s,a)}{F_{x+}(s,a)}\,\mathrm{d}_s\ell_s^{F(s,a)}(Y)\,\mathrm{d}\nu(a). \end{aligned} \tag{4.16}$$

It then remains to show that

$$G_{t-} + G_y \cdot \big(F_{t-} \circ \tilde{G}\big) = 0, \tag{4.17}$$

$$G_y \cdot (F_x\,\sigma \circ \tilde{G}) = \sigma \circ \tilde{G}, \tag{4.18}$$

$$\int_{\mathbb{R}}\int_0^t \frac{h(s,a)}{F_{x+}(s,a)}\,\mathrm{d}_s\ell_s^{F(s,a)}(Y)\,\mathrm{d}\nu(a) = \int_{\mathbb{R}}\int_0^t h(s,a)\,\mathrm{d}_s\ell_s^a(X)\,\mathrm{d}\nu(a). \tag{4.19}$$

The first two equations follow by passing derivatives under the integral in (4.12), and the one-sided chain rule. To prove the final equation, we use Lemma 4.3.3. □

It is apparent from the previous proof that we require an extension to the notion of local time-space integral to allow limits which may vary in time. The following extension suffices for our purposes, and demonstrates how further specific cases could be obtained.

Lemma 4.3.2. *Let $H\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be left continuous in each argument when the other is fixed. Further, assume that H can be written as*

$$H(t,y)-H(t,x)=\int_{\Phi(t,x)}^{\Phi(t,y)-} g(t,a)\,\mathrm{d}\nu(a), \tag{4.20}$$

for each $t\in[0,T]$ and $x,y\in\mathbb{R}$. Also let g be left continuous in t for each fixed a. Assume that Φ is invertible as a function of its latter argument for each fixed t, and write $\Psi(t,z)=\left[\Phi(t,\cdot)\right]^{-1}(z)$. Further assume that Ψ is a continuous function of bounded variation in t for each fixed z. Then the local time space integral $\int_0^T\int_{\mathbb{R}}H(s,x)\,\mathrm{d}\ell_s^x$ of H is defined by

$$-\int_{\mathbb{R}}\int_0^T g(u,a)\,\mathrm{d}_u\ell_u^{\Psi(\cdot,a)}\,\mathrm{d}\nu(a). \tag{4.21}$$

This expression extends the previous representations and the corresponding version of Theorem 3.4.1 holds.

Proof. We follow the proof of Lemma 3.3.5. By pushing forward, we may write (4.20) as

$$H(t,y)-H(t,x)=\int_x^{y-} g(t,\Phi(t,a))\,\mathrm{d}(\Psi(t,\cdot)_{\#}\nu)(a). \tag{4.22}$$

Now we may replace the right hand side of equation (3.26) by

$$\int_x^y\int_{\mathbb{R}}\int_{\mathbb{R}} mn\,g(r,\Phi(t,q))\,\rho(m(t-r))\,\rho(n(p-q))\,\mathrm{d}(\Psi(t,\cdot)_{\#}\nu)(q)\,\mathrm{d}r\,\mathrm{d}p. \tag{4.23}$$

By pushing forward by Φ, we see this becomes

$$\int_x^y\int_{\mathbb{R}}\int_{\mathbb{R}} mn\,g(r,q)\,\rho(m(t-r))\,\rho(n(p-\Psi(t,q)))\,\mathrm{d}\nu(q)\,\mathrm{d}r\,\mathrm{d}p. \tag{4.24}$$

The proof continues until we take the limit as $n\to\infty$, where we have

$$\Lambda(H^n)=-\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^T n\,g(s,q)\,\rho(n(p-\Psi(s,q)))\,\mathrm{d}_s\ell_s^p(X)\,\mathrm{d}\nu(q)\,\mathrm{d}p. \tag{4.25}$$

Now it remains to show that this is equal to

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^T n\,g(s,q)\,\rho(np)\,\mathrm{d}_s\ell_s^p(X-\Psi(s,q))\,\mathrm{d}\nu(q)\,\mathrm{d}p. \tag{4.26}$$

By expanding the local time using (A.3), we see that (4.25) is equal to

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^T n\, g(s,q)\, \rho(n(p-\Psi(s,q)))\, \mathbb{1}_{\{p\le X_u\le p+\varepsilon\}}\, \mathrm{d}\langle X,X\rangle_s\, \mathrm{d}\nu(q)\, \mathrm{d}p. \tag{4.27}$$

Now exchanging the order of integration and translating, then taking the limit back inside, after noting $\langle X,X\rangle_s = \langle X-\Psi(\cdot,q), X-\Psi(\cdot,q)\rangle_s$ for each fixed q, we obtain the right hand side. Finally we allow $n\to\infty$ which finishes the proof. □

The above representation is also useful in establishing the following 'change of local time' result. This is a natural extension of the time homogeneous case (see [52, Ex. VI 1.23])

Lemma 4.3.3. *Let $G:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ be continuous, and strictly increasing in its latter argument when the former is fixed. Define*

$$F(t,x) = [G(t,\cdot)]^{-1}(x). \tag{4.28}$$

Assume G obeys the conditions of Theorem 3.4.1, with the partial derivative G_y satisfying Definition 3.3.3. Assume that $t\mapsto F(t,a)$ is of bounded variation for each fixed a. For a semimartingale Y, define $X_t = G(t,Y_t)$. Then we have

$$\int_0^t G_y(s,F(s,a)+)\, \mathrm{d}_s\ell_s^{F(\cdot,a)}(Y) = \ell_s^a(X), \tag{4.29}$$

for each $a\in\mathbb{R}$.

Proof. The proof follows by considering, for each fixed a, the maps

$$x\mapsto |x-a|, \tag{4.30}$$

$$H(t,z) = |G(t,z+F(t,a))-a| = [G(t,z+F(t,a))-a]\,\mathrm{sgn}(z), \tag{4.31}$$

applied to the semimartingales X and $(t, Y-F(\cdot,a))$ respectively. We may then expand using the Tanaka formula and Theorem 3.4.1, where the local time-space integral of G takes the form of Definition 3.3.3.

We expand the function $\bar{G}(t,z) = G(t,z+F(t,a))$. By mollifying, taking small increments, then using the Lebesgue-Stieltjes chain rule and taking limits, one can establish that

$$\bar{G}(t,z)-\bar{G}(s,z) = \int_s^t G_t(u,z+F(u,a))\,\mathrm{d}u + \int_s^t G_y(u,z+F(u,a))\,\mathrm{d}_uF(u,a). \tag{4.32}$$

We can establish likewise, with some technical manipulations, that

$$\begin{aligned} H(t,z) - H(s,z) = &\int_s^t \operatorname{sgn}(z) G_t(u, z + F(u,a))\,\mathrm{d}u \\ &+ \int_s^t \operatorname{sgn}(z) G_y(u, z+F(u,a))\,\mathrm{d}_u F(u,a). \end{aligned} \tag{4.33}$$

Further, note that

$$\begin{aligned} H_z(t,y) - H_z(t,x) &= \int_x^y \operatorname{sgn}(z) G_{yy}(t, z + F(t,a))\, \mathbb{1}_{\{z\neq 0\}}\,\mathrm{d}\nu(z) \\ &\quad + [G_y(t, F(t,a)+) + G_y(t, F(t,a)-)]\,\delta_0(z) \\ &= \int_x^y \operatorname{sgn}(z) G_{yy}(t, z+F(t,a))\,\mathrm{d}\nu(z) \\ &\quad + 2\,[G_y(t, F(t,a)+)]\,\delta_0(z). \end{aligned} \tag{4.34}$$

After using the Tanaka formula and expansion of $X_t = G(t, Y_t)$, the result follows. □

Remark 4.3.4. The above Lemma 4.3.3 holds also when G_y satisfies (4.20), with obvious modifications to the proof.

Now that we have established the bijective correspondence of Lemma 4.3.1, we can prove the main existence and uniqueness result.

Theorem 4.3.5. *Given the previous assumptions on σ, h and ν, we have the existence of a unique strong solution for the equation (4.2).*

Proof. By the one-to-one correspondence established in Lemma 4.3.1, this reduces to considering existence and uniqueness for the equation (4.14). We prove the existence of a weak solution and then pathwise uniqueness of solutions. Using the argument of Yamada and Watanabe, these two things together imply the existence of a unique strong solution.

As there is some $\varepsilon > 0$ such that $\sigma \geq \varepsilon$, and F_x is bounded below by a positive constant due to finiteness of ν, we see that $F_x\sigma > \delta$ for some $\delta > 0$. Thus weak existence and uniqueness holds for (4.2).

By (4.7) and direct examination of the expression for F_x, we see there is some $\tilde{\rho} : \mathbb{R} \to \mathbb{R}$ such that

$$|(F_x\sigma)(t,y) - (F_x\sigma)(t,x)|^2 \leq |\tilde{\rho}(y) - \tilde{\rho}(x)| \tag{4.35}$$

for all $(x, y) \in \mathbb{R}^2$. Then it follows that

$$\left|(F_x\sigma) \circ \tilde{G}(t, y) - (F_x\sigma) \circ \tilde{G}(t, x)\right|^2 \leq \left|\tilde{\rho} \circ G(t, y) - \tilde{\rho} \circ G(t, x)\right|, \tag{4.36}$$

for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2$. By the theory of [41], it suffices to show that

$$\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z_s>0\}} Z_s^{-1}\, \mathrm{d}\langle Z, Z\rangle_s\right] < \infty, \tag{4.37}$$

where $Z = \bar{Y} - Y$ for any two solutions $\bar{Y}, Y$ of (4.14). We see

$$\begin{aligned}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z_s>\delta\}} Z_s^{-1}\, \mathrm{d}\langle Z, Z\rangle_s\right] &\leq \\ &\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z_s>\delta\}} Z_s^{-1} \left(\tilde{\rho} \circ G(s, \bar{Y}_s) - \tilde{\rho} \circ G(s, Y_s)\right) \mathrm{d}s\right].\end{aligned} \tag{4.38}$$

Define processes $\bar{X}$ and X by $\bar{X}_s = G(s, \bar{Y}_s)$ and $X_s = G(s, Y_s)$. Take $Z^u = X + u(\bar{X} - X)$ and note that Z^u is a semimartingale. Take a bounded sequence of increasing C^∞ functions f_n which converge pointwise to f except possibly on the discontinuity set of F. Then note that

$$\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z_s>\delta\}} Z_s^{-1} \left(\tilde{\rho}(\bar{X}_s) - \tilde{\rho}(X_s)\right) \mathrm{d}s\right] \leq K\mathbb{E}\left[\int_0^t \int_0^1 \frac{\partial f_n}{\partial x}(Z_s^u)\, \mathrm{d}u\, \mathrm{d}s\right], \tag{4.39}$$

where K is the Lipschitz constant of G in the space variable. We employ the occupation time formula with respect to Z^u to see

$$\mathbb{E}\left[\int_0^1 \int_0^t \frac{\partial f_n}{\partial x}(Z_s^u)\, \mathrm{d}s\, \mathrm{d}u\right] \leq \frac{1}{\varepsilon^2} \int_0^1 \int_{\mathbb{R}} \frac{\partial f_n}{\partial x}(a) \mathbb{E}\left[\ell_t^a(Z^u)\right] \mathrm{d}a\, \mathrm{d}u. \tag{4.40}$$

We note that, using Lemma 4.3.3, $\mathbb{E}\left[\ell_t^a(Z^u)\right]$ is uniformly bounded in (a, u) by some constant $C_t < \infty$, and thus the expression (4.39) is bounded by

$$\frac{2KC}{\varepsilon^2} \max_a |f(a)|. \tag{4.41}$$

The conclusion follows after letting $n \to \infty$ and $\delta \to 0$, noting that the discontinuity set of f is at most countable. □

The analogue of (4.2) obtained by replacing the left local time with symmetric local time appears in the literature, and may also produce a more realistic model. We employ the following trick, originally demonstrated in [6], to transfer our results to the symmetric case without repeating the original analysis.

Lemma 4.3.6. *Assume that X solves the SDE (4.2). Then the symmetric local time $\tilde{\ell}$ is related to the right local time by the relation*

$$\tilde{\ell}_t^a = \int_0^t \Big[1 - h(s,a)\nu(\{a\})\Big] \, \mathrm{d}_s \ell_s^a(X). \tag{4.42}$$

Proof. By definition, the symmetric local time is

$$\tilde{\ell}_t^a = \frac{1}{2}\left(\ell_t^a + \ell_t^{a-}\right). \tag{4.43}$$

For any semimartingale Z, with canonical decomposition $V + M$ into an adapted process of bounded variation and a local martingale respectively, we see from [52, Thm. VI 1.7] that

$$\ell_t^a - \ell_t^{a-} = 2\int_0^t \mathbb{1}_{\{Z_s = a\}} \, \mathrm{d}V_s. \tag{4.44}$$

Using the SDE (4.2), we see this is

$$\begin{aligned} \ell_t^a - \ell_t^{a-} &= 2\int_{\mathbb{R}}\int_0^t h(s,z)\, \mathbb{1}_{\{X_s = a\}} \, \mathrm{d}_s \ell_s^z(X) \, \mathrm{d}\nu(z) \\ &= 2\int_{\{a\}}\int_0^t h(s,z) \, \mathrm{d}_s \ell_s^z(X) \, \mathrm{d}\nu(z) = 2\int_0^t h(s,a)\nu(a) \, \mathrm{d}_s \ell_s^a(X). \end{aligned} \tag{4.45}$$

The conclusion then follows by substituting this expression into (4.43). □

We conclude by including a classical drift term. Assume now that σ, ν and h obey all the previous assumptions. We deal with the SDE (4.2). Assume that b and σ satisfy the following:

1. $\frac{b}{\sigma^2}(t, \cdot) \in L^1$ for each t;
2. $t \mapsto b(t,a)$ is C^1 for each fixed a, with $b_t(t,a)$ left continuous and admitting left limits in space;
3. $t \mapsto \sigma(t,a)$ is C^1 for each fixed a, with $\sigma_t(t,a)$ left continuous and admitting left limits in space;

Theorem 4.3.7. *Given the previous assumptions on σ, b, h and ν, we have existence and uniqueness for the equation (4.2).*

Proof. As $\sigma \geq \varepsilon$, we note that

$$\int_0^t b(s, X_s) \, \mathrm{d}s = \int_0^t \frac{b}{\sigma^2}(s, X_s) \cdot \sigma^2(s, X_s) \, \mathrm{d}s = \int_{\mathbb{R}}\int_0^t \frac{b}{\sigma^2}(s,a) \, \mathrm{d}_s \ell_s^a \, \mathrm{d}a, \tag{4.46}$$

by the occupation time formula. We replace h by $h + b/\sigma^2$, except possibly on some set of Lebesgue measure zero in the a-variable, and then apply Theorem 4.3.5. □

4.4 SDEs with local time on curves

As mentioned in the introduction, the author's treatment is partially inspired by the work of Étoré and Martinez [27] on stochastic differential equations with local time on curves, which demonstrates a contemporary interest in such equations. The authors used the SDE results therein to prove results in PDEs by applying a kind of generalised Feynman-Kac formula. Here we examine their SDE results and how they connect with the results in Section 4.3.

Equation (4.2) admits a special case of the form

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s + \int_0^t h(s)\,\mathrm{d}_s\ell_s^a. \tag{4.47}$$

This is a diffusion which is skewed by an interface or barrier between two media whose permittivity changes in time, in the spirit of Weinryb [56]. Étoré and Martinez have extended this to the case when the curve may also be time dependent, and to the case of multiple non-intersecting curves. The proof uses an extension of the local time on curves formula, which follows straightforwardly from the work of Du Toit [16], and hence from Theorem 2.4.1. One of their main contributions is to prove existence and uniqueness for SDEs with a discontinuous diffusion coefficient which is not covered by the traditional Nakao condition. That is, we assume that X solves the SDE

$$\mathrm{d}X_t = b(s, X_s)\mathrm{d}s + \sigma(s, X_s)\mathrm{d}B_t, \tag{4.48}$$

where b is bounded and measurable, and σ is measurable and bounded below and above by positive constants. Let $y : [0, T] \to \mathbb{R}$ be of bounded variation and denote its image $y([0, T])$ by Δ. Assume that:

1. $\sigma \in C^{0,1}(\mathbb{R} \setminus \Delta)$;

2. For some constant $C > 0$, we have

$$\sum_{x \le z \le y} \left|\sigma^2(t, z+) - \sigma^2(t, z-)\right| \le C \int_0^T \sum_{x \le z \le y} \left|\sigma^2(s, z+) - \sigma^2(s, z-)\right| \,\mathrm{d}s < \infty \tag{4.49}$$

   for all $x, y \in \mathbb{R}$ and $t \in [0, T]$.

These hypotheses are denoted $\mathbf{H}^{(y)}$ and $\mathbf{AJ}^{(y)}$ respectively. Under these conditions, we have a unique strong solution for (4.48).

Let us present the following example. Define the curve $b : [0,1] \to \mathbb{R}$ by $b(t) = t$. Take the function $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ to be

$$\sigma(t,z) = \begin{cases} 1 & \text{if } z \le t, \\ 2 & \text{if } z > t. \end{cases} \tag{4.50}$$

We can see directly that it is impossible to find a bounded increasing function f such that

$$|\sigma(t,y) - \sigma(t,x)|^2 \le |f(y) - f(x)|, \tag{4.51}$$

for all $(t,x,y) \in [0,1] \times \mathbb{R}^2$ where σ is as given in (4.50), as in this case f should have a discontinuity of size 1 at every point in $[0,1]$. Fixing $\varepsilon > 0$ and letting $y = \frac{1}{2} + \varepsilon$, $x = \frac{1}{2} - \varepsilon$, we see

$$\sum_{x \le z \le y} \left| \sigma^2(1/2, z+) - \sigma^2(1/2, z-) \right| = 3. \tag{4.52}$$

Then

$$\int_0^1 \sum_{x \le z \le y} \left| \sigma^2(s, z+) - \sigma^2(s, z-) \right| \mathrm{d}s = 6\varepsilon. \tag{4.53}$$

By letting $\varepsilon \to 0$ it is apparent that we cannot bound (4.52) by some constant multiple of (4.53). This demonstrates that the $\mathbf{AJ}^{(y)}$ hypothesis, which must hold for all points $x < y$, is highly restrictive, and excludes all but constant level curves. However, finitely many constant level curves are already covered by the traditional Nakao condition (4.51).

By applying a space transformation of the same form as (4.8) with only the 'discrete' component ψ, in combination with a generalised local time on curves formula, a variation of the proof of Theorem 4.3.5 gives a unique strong solution for equations of the form (4.6) under the additional assumptions that $\beta \in C^1$ takes values in $(-1,1)$ and the curve b is of class C^1. Further, σ should obey appropriate conditions for existence and uniqueness after transformation.

It is straightforward (except for some technical details) to allow multiple curves under sensibly adapted conditions, provided they do not intersect or asymptotically approach each other. This is essentially the problem with allowing a kind of time variation of the parameter a in (4.2) - if ν is Lebesgue-singular (see Theorem A.2.4), then the support of ν may contain a Cantor component with many (even dense) accumulation points. Allowing these points to vary independently with time creates a very complex picture which the transformation (4.8) and machinery of the local time-space integral is not equipped to deal with. On the contrary, the support of such ν is generally an uncountable set, and has a complex geometry which does not arise as some kind of limiting version of SDEs of the form (4.6). Equations (4.6) and (4.2) seem to represent fundamentally different phenomena, which overlap in the case of a constant-in-time barrier between two fundamentally different phases. If the barriers vary together in time, by replacing X by $X - b$ for some curve b, we can reduce the problem back to the constant-in-time case.

Chapter 5

Higher Dimensions

5.1 Introduction

In Chapter 3 we stayed strictly in the time-space case, a two-dimensional setting where only one of the components is a full semimartingale. In Chapter 2, we worked with n-dimensional semimartingales whose first $n-1$ components were of bounded variation. A natural next step is to try and extend our results to include multiple full semimartingales.

The most obvious stumbling block in this endeavour is the lack of existence of a higher dimensional local time. Given a two-dimensional Brownian motion $(X_t, Y_t)_{t \geq 0}$, we may follow (A.1) and define its occupation measure

$$\nu_t(A) = \int_0^t \mathbb{1}_{\{(X_s, Y_s) \in A\}} \, \mathrm{d}s. \tag{5.1}$$

This (pathwise defined) measure is concentrated on the paths of the process, which have Lebesgue measure zero almost surely. Any hope that this measure might have a density with respect to two dimensional Lebesgue measure, and thus give a kind of higher dimensional occupation time formula, is immediately lost.

Being indefatigable, we may realise that one way to avoid this issue is to work locally - the definition via a limiting occupation measure is local in space, and additivity in time means we only need to work on small time intervals. Problems which can be reduced locally to a one-dimensional analogue are still fair game. Localising in space automatically gives a kind of localisation in time, albeit over random time intervals. This is the approach taken in Section 5.2, yielding Theorem 5.2.2, a kind of two-dimensional local time on curves formula.

Another way to approach the problem is to abandon the idea of having a global occupation density, but to separate the n-dimensional state space into nice $(n-1)$-dimensional subsets, then consider the occupation time to each of these subsets. The space variable of such an 'occupation density' would be an index of the $n-1$ dimensional subspaces. Such decomopositions are known as foliations in the smooth case. The irregularity of semimartingale paths requires some inspiration from geometric measure theory. This is the approach followed by [8], which we will describe in Section 5.3.

It was shown by Meyer [45] that convex functions preserve semimartingales in $\mathbb{R}^n$. To find their semimartingale decomposition, and hopefully obtain a local time or some other object of interest along the way, one might try to follow the same procedure in higher dimensions as Tanaka did in dimension one. In one dimension, this relies on the representation of convex functions as a linear term plus an integral with respect to their distributional second derivative. In higher dimensions a similar analogue is available, and so we collect together the results of other researchers that might form a reasonable approach, and which the author hopes will form an interesting research direction in the future.

5.2 Two dimensional local time on curves

In order to work locally, we define which curves can be acceptably localised.

Definition 5.2.1. Let $b : [0,1] \to \mathbb{R}^2$ be a continuous closed curve. We say that b is LDCF (locally the difference of convex functions) if we can express it locally as the graph of a difference of two single-variable functions.

Precisely, fix $t \in [0,1]$. Then b is LDCF at t if there is some $U_{b(t)}$, a neighbourhood of $b(t) \in \mathbb{R}^2$, and a function $f_t : \mathbb{R} \to \mathbb{R}$, such that $f_t = F - G$ for convex functions F and G, and such that $\mathrm{graph}(f_t) \cap U_{b(t)} = \mathrm{im}(b) \cap U_{b(t)}$ where $\mathrm{graph}(f_t)$ is the image of $(x, f_t(x))$ or $(f_t(y), y)$.

We say b is LDCF if it is LDCF at each $t \in [0,1]$.

If b is LDCF, then it cannot self-intersect, as then b would have no local expression as a graph at the intersection point. Thus any such b is a simple closed curve, hence a

Jordan curve. We apply the Jordan curve theorem to establish that b divides $\mathbb{R}^2$ into two disjoint, open regions I and E, which are respectively bounded and unbounded, such that $I \cup E = \mathbb{R}^2 \setminus \mathrm{im}(b)$. We can imagine I and E as the interior and exterior respectively.

Theorem 5.2.2. *Let $b : [0,1] \to \mathbb{R}^2$ be a closed curve which is LDCF. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function which is C^2 on $\mathbb{R}^2 \setminus im(b)$. Then for any continuous two-dimensional semimartingale (X, Y), we have*

$$\begin{aligned} F(X_t, Y_t) =& F(X_0, Y_0) + \int_0^t F_x(X_s, Y_s)\,\mathrm{d}X_s + \int_0^t F_y(X_s, Y_s)\,\mathrm{d}Y_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(X_s, Y_s)\,\mathrm{d}\langle X, X\rangle_s + \int_0^t F_{xy}(X_s, Y_s)\,\mathrm{d}\langle X, Y\rangle_s \\ &+ \frac{1}{2}\int_0^t F_{yy}(X_s, Y_s)\,\mathrm{d}\langle Y, Y\rangle_s + A_t, \end{aligned} \tag{5.2}$$

for an increasing process A_t. which admits the pathwise representation on the set $U_{b(t)}$

$$A_t = \frac{1}{2}\int_0^t \Big[F_y(X_s, Y_s+) - F_y(X_s, Y_s-)\Big]\,\mathrm{d}_s \ell_s^b(Y), \tag{5.3}$$

or alternatively

$$A_t = \frac{1}{2}\int_0^t \Big[F_x(X_s+, Y_s) - F_x(X_s-, Y_s)\Big]\,\mathrm{d}_s \ell_s^{b^{-1}}(X). \tag{5.4}$$

The following result establishes the necessary localisation (in time) which allows us to apply the Local Time on Curves formula in a locally (in space), yielding the representations (5.3) and (5.4).

Lemma 5.2.3. *Let X be a two-dimensional semimartingale, and $\{U_n \,|\, n \in \mathbb{N}\}$ be a countable open cover of $\mathbb{R}^2$. Then there exist stopping times $T_1, T_2, \ldots,$ such that*

1. *$T_i \uparrow \infty$ almost-surely.*
2. *On the stochastic interval $[T_i, T_{i+1}] \cap \{T_{i+1} > T_i\}$, we have that $X_t \in U_n$ for some $n \in \mathbb{N}$.*

Proof. The existence of such a sequence of stopping times is proved in [23]. □

We are now ready to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. For each $t \in [0,1]$, choose some open neighbourhood of $b(t)$, denoted $U_{b(t)}$, such that b is a difference of convex functions on $U_{b(t)}$. Note that $\{U_b(t) \,|\, t \in [0,1]\}$ is an open cover of the range of b. By assumption, the range of b is compact, and so there is a finite sub-collection $\{U_1, \ldots, U_N\}$ which covers $\mathrm{im}(b)$. Adjoin the open set $\mathbb{R}^2 \setminus \mathrm{im}(b)$ to this collection. Then by Lemma 5.2.3, we obtain a sequence of stopping times $(T_i)_{i=1}^{\infty}$.

Note that every finite open cover of $\mathbb{R}^2$ admits a partition of unity which is subordinate to that cover. Denote the (smooth) functions in this partition by ϕ^n, for $n = 1, \ldots N$, where ϕ^N is subordinate to $\mathbb{R}^2 \setminus \mathrm{im}(b)$. Then we define $F^n(x,y) = F(x,y)\,\phi^n(x,y)$ for each $n = 1, \ldots, N$.

We adopt the notation X^{T_i} to represent the semimartingale X stopped at T_i. Now we consider stochastic increments $F^n(X_t^{T_{i+1}}) - F^n(X_t^{T_i})$. This is zero unless both $t \in [T_i, T_{i+1}]$ and $X_t \in U_n$ on this same interval.

We now apply the Local Time on Curves formula, then sum over n and i, which gives the result. □

5.3 Local time on a foliation

This section is based on the work of Bevilacqua and Flandoli [8]. By disintegrating the occupation measure, it is possible to show in certain cases the existence of a 'geometric local time' over a foliation by $n-1$ dimensional subsets of a domain in $\mathbb{R}^n$. Regularity of this local time can be obtained by finding a representation as the local time of a one-dimensional process.

The following is entirely derived from [8]. Let X be an R^N-valued continuous semimartingale, and define $g_t^{jk} = \mathrm{d}\left\langle X^j, X^k \right\rangle_t / \mathrm{d}t$ when this exists, noting that g defines a matrix-valued process. We write $A \leq B$ for two matrices A and B if $B - A$ is non-negative definite. Let $\phi \in C^2(\mathbb{R})$ satisfy $\inf_{x \in A} |\nabla \phi(x)| > 0$ on some open set $A \subseteq \mathbb{R}^N$.

Theorem 5.3.1 ([8] Thm. 1)**.** *With X, g and ϕ as above, if $cI_n \leq g_t \leq CI_n$ for constants $C > c > 0$ and all $0 \leq t \leq T$, then there exists a random, bounded, compactly supported non-negative function $\mathcal{L}^a_{i,A,\phi}$ and random probability measures $\mathcal{Q}^i_{A,\phi}(a, \mathrm{d}x)$*

concentrated on $\Gamma_a = \{x \in A : \phi(x) = a\}$ *for a.e.* $a \in \mathbb{R}$ *such that*

$$\int_0^T \mathbb{1}_{\{X_t \in A\}} f(X_t)\,\mathrm{d}\langle X^i, X^i\rangle_t = \int_{\mathbb{R}} \left(\int_{\Gamma_a} f(x)\, \mathcal{Q}^i_{A,\phi}(a, \mathrm{d}x) \right) \mathcal{L}^a_{i,A,\phi}\,\mathrm{d}a. \tag{5.5}$$

The Theorem asserts the existence of a disintegration of the occupation measure in a jointly measurable way, over the leaves of the foliation (smoothly varying $N-1$ dimensional subsets) given by $\Gamma_a = \phi^{-1}(a)$. Most importantly, the measure controlling the disintegration has a density with respect to Lebesgue measure, namely $\mathcal{L}$. In general, the measures $\mathcal{Q}$ are irregular.

An important property of the classical local time is right continuity in the space variable. Bevliacqua and Flandoli have shown this in a limited form ([8, Thm. 3]), but establishing regularity in the general case is open.

Working in the time-space case, we may fix a continuous curve of bounded variation $b : [0, T] \to \mathbb{R}$ and semimartingale (t, X_t), then compare with the local time defined simply by $\ell^b_t(X) = \ell^0_t(X - b)$ as we have before. Unfortunately there is an incompatibility between the two definitions.

Example 5.3.2 (Restatement of [8, Ex. 43])**.** Given a unit vector $v \in \mathbb{R}^N$ let b be the surface orthogonal to v passing through the origin. Then

$$\ell^b_t = \sqrt{1 + |a|^2}\, \mathcal{L}^0_{A,\phi}, \tag{5.6}$$

where $\phi(x) = \langle x, v\rangle$ and a is any vector such that b is defined by $x_n = \sum_{i=1}^{n-1} a_i x_i$.

If we perform a linear dilation or contraction of $\mathbb{R}^n$ which leaves v fixed, then the local time ℓ^b_t changes but $\mathcal{L}^0_{A,\phi}$ remains fixed. This can be inferred from the direct expression for $\mathcal{L}$ which is obtained in [8].

5.4 Towards a multidimensional Tanaka formula

The lack of existence of a higher dimensional local time naturally leads us to ask how the classical construction fails. Though the occupation time formula is fundamental to our intuition of local time, it is not the defining characteristic, and arises instead by comparison of Itô's formula and the Itô-Tanaka formula. By imitating the steps of the Tanaka construction in higher dimensions, we can hope to reveal an alternative approach which yields an interesting theory. The results collected here take a step in this direction.

We recall Theorem A.2.7 which states that any convex function $f : I \to \mathbb{R}$ on some interval $I \subseteq \mathbb{R}$ has the representation

$$f(x) = \frac{1}{2} \int_I |x - a| \, \mathrm{d}\mu(a) + \alpha_I x + \beta_I, \tag{5.7}$$

for constants α_I, β_I, where $\mu = f''$ in the distributional sense is a positive Radon measure. Further, for any positive Radon measure μ, there is a convex function f such that $\mu = f''$ and the above formula holds. It then follows that

$$f'_-(x) = \frac{1}{2} \int_I \operatorname{sgn}(x - a) \, \mathrm{d}\mu(a) + \alpha_I. \tag{5.8}$$

This is the key to proving the Itô-Tanaka formula from the Tanaka formula - we substitute the stochastic expansion of $|X_t - a|$ and apply the above.

Recall that a function $\phi \in C^2(\mathbb{R}^n; \mathbb{R})$ is harmonic if and only if $\Delta\phi = 0$. Thus in dimension one, harmonic functions and linear functions coincide. Further, a function $\phi \in C^2(\mathbb{R}^n; \mathbb{R})$ is subharmonic if and only if $\Delta\phi > 0$, and so in dimension one, C^2 subharmonic functions and convex functions coincide. We now define subharmonic functions more generally, and observe now that this will imply that they coincide with convex functions in dimension one.

Definition 5.4.1. A subharmonic function ϕ on a domain $D \subseteq \mathbb{R}^n$ is an upper semi-continuous function taking values in $\mathbb{R}$ such that, for any relatively compact $A \subset D$ and any harmonic function $\psi : \bar{A} \to \mathbb{R}$ such that $\phi \leq \psi$ on ∂A, we have $\phi \leq \psi$ on A.

In higher dimensions, convex functions are a subset of subharmonic functions. We recall the following representation theorem for subharmonic functions which parallels Theorem A.2.7.

Theorem 5.4.2. *Let ϕ be a subhamonic function on a domain $D \subseteq \mathbb{R}^n$. Then there is a unique positive Borel measure μ on D such that for any $K \subset D$ relatively compact, there is a harmonic function v on K such that*

$$u(x) = v(x) + \int_K |x - y|^{2-n} \, \mathrm{d}\mu(y) \tag{5.9}$$

if $n = 2$, and

$$u(x) = v(x) + \int_K \log|x - y| \, \mathrm{d}\mu(y) \tag{5.10}$$

if $n \geq 3$, for any $x \in K$.

The key difficulty in extending the one-dimensional method is that the potential functions, the integrands above, are generally unbounded and so cannot be approximated uniformly by C^∞ functions. However, an approximation by C^∞ functions does preserve the semimartingale decompositions, as was shown by Carlen and Protter [13]. Their work shows that for a semimartingale X in $\mathcal{H}^1$ (see the paragraph above Definition 3.5.3), and an approximating sequence F^n converging uniformly on compacts to a convex function $F : \mathbb{R}^n \to \mathbb{R}$, the semimartingale decomposition $F^n(X)$ converges to that of $F(X)$ in the sense described in Lemma 3.5.4. Localisation gives the result for general semimartingales.

Finally let us consider some results from the excellent and concise paper of Dudley [17]. Given a convex function $F : \mathbb{R}^n \to \mathbb{R}$, its second derivatives exist in the distributional sense as Radon measures. The $n \times n$ matrix of these measures, denoted A, is symmetric and nonnegative definite on each set U with compact closure. Further, each element of A is absolutely continuous with respect to $n-1$ dimensional Hausdorff measure. The trace of this matrix μ is the distributional Laplacian of F, effectively the measure μ in Theorem 5.4.2. It follows that each of these measures is absolutely continuous with respect to the trace, meaning there is a decomposition $A = M\mu$ for some nonnegative, symmetric matrix-valued function M with entries bounded by 1.

The author hopes that these results will be useful in developing a higher dimensional theory of local time, continuing the work of the many great contributors to the classical one-dimensional case.

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Appendix A

A Review of Local Time and Integration

A.1 Fundamentals of local time

A.1.1 Intuitive discussion

A common representation of the local time is as an 'occupation time density' as follows. Given a continuous function $F : \mathbb{R} \to \mathbb{R}$, we may define the occupation measure ν of any Borel set A of $\mathbb{R}$ by

$$\nu_t(A) = \int_0^t \mathbb{1}_{\{F(s)\in A\}}\, \mathrm{d}s$$

Intuitively, $\nu_t(A)$ is a measure of the time spent by a particle which travels according to F in the set A up to time t. Consider the following special cases.

1. F is a constant function, taking value $c \in \mathbb{R}$. Then its occupation measure is a Dirac measure at c. It follows that the occupation measure of any function which is constant on some interval of time contains a discrete part (see A.2.4).

2. F is C^1 and strictly increasing. Then by a change of variables we find that the generated measure has a density with respect to Lebesgue measure.

We may generalize this idea to a continuous semimartingale X path-by-path, defining its occupation measure by

$$\nu_t(A) = \int_0^t \mathbb{1}_{\{X_s\in A\}}\, \mathrm{d}\langle X, X\rangle_s\,. \tag{A.1}$$

Recall that X is constant on some interval if and only if $\langle X, X\rangle$ is too. Thus even if X remains constant on some interval there is no singularity in the occupation measure. The question remains whether ν_t has a density with respect to Lebesgue measure; that is, whether there exists a random function ℓ_t^a such that

$$\nu_t(A) = \int_A \ell_t^a \,\mathrm{d}a = \int_{\mathbb{R}} \mathbb{1}_{\{a\in A\}}\, \ell_t^a \,\mathrm{d}a.$$

If so, it then follows (by the monotone class theorem) that for all bounded Borel functions $f : \mathbb{R} \to \mathbb{R}$,

$$\int_0^t f(X_s)\,\mathrm{d}\langle X, X\rangle_s = \int_{\mathbb{R}} f(a)\,\ell_t^a \,\mathrm{d}a. \tag{A.2}$$

We may then use Lebesgue's differentiation theorem to see that for almost all $a \in \mathbb{R}$,

$$\ell_t^a = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\mathbb{R}} \mathbb{1}_{\{a<x<a+\varepsilon\}}\,\ell_t^a\,\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{\{a<X_s<a+\varepsilon\}}\,\mathrm{d}\langle X, X\rangle_s\,. \tag{A.3}$$

The result (A.2) is known as the occupation time formula. This encodes a crucial intuitive property of the local time; it allows us to transform functions of the semimartingale in time (or rather with respect to the rescaled time $\langle X, X\rangle$) to functions of space, or vice-versa. Result (A.3) gives justification for use of the term 'local time'.

Using (A.2), we may write formally using the Dirac delta δ_a

$$\ell_t^a = \int_{\mathbb{R}} \delta_a(x)\,\ell_t^a\,\mathrm{d}x = \int_0^t \delta_a(X_s)\,\mathrm{d}\langle X, X\rangle_s\,.$$

Distributionally, the derivative of the absolute value function is the sign function, and the derivative of the sign function is twice the Dirac delta at zero. So, using Itô's formula without justification, we arrive at the equation

$$|X_t| = |X_0| + \int_0^t \mathrm{sgn}(X_s)\,\mathrm{d}X_s + \ell_t^a. \tag{A.4}$$

This formula (A.4) is known as Tanaka's formula, and is in fact a more suitable candidate for the rigorous definition which we may now introduce.

A.1.2 Local time for continuous semimartingales

The following results are due to [52]. Throughout this section we assume X is a continuous semimartingale. We first provide the basis for extending Itô's formula.

Theorem A.1.1. *Let f be a convex function and X a continuous semimartingale. Then $f(X)$ is a semimartingale, and there exists an adapted, continuous, increasing process A^f such that*

$$f(X_t) - f(X_0) = \int_0^t f'_-(X_s)\,\mathrm{d}X_s + A_t^f,$$

where f'_- is the left derivative of f.

Noting the important asymmetry, define

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{A.5}$$

This is the left derivative of the function $f(x) = |x|$.

Definition A.1.2. We define the (right) local time ℓ_t^a to be the increasing process A^f associated to the shifted absolute value function $f_a(x) = |x - a|$ in Theorem A.1.1, namely

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s + \ell_t^a. \tag{A.6}$$

A natural extension of this formula is the Itô-Tanaka formula, which can be obtained once one knows that difference of convex functions are essentially superpositions of the absolute value function (see A.2.7).

Theorem A.1.3. *If f is the difference of two convex functions, then*

$$f(X_t) - f(X_0) = \int_0^t f'_-(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} \ell_t^a\,\mathrm{d}f'_-(a). \tag{A.7}$$

Comparing this with Itô's formula gives us the following identity for $f \in C^2$,

$$\int_0^t f''(X_s)\,\mathrm{d}\langle X, X\rangle_s = \int_{\mathbb{R}} f''(a)\,\ell_t^a\,\mathrm{d}a. \tag{A.8}$$

Using density of C^2 functions and the monotone class theorem, we can then rigorously obtain the occupation time formula.

Theorem A.1.4. *For all bounded (or positive) Borel-measurable $\Phi : \mathbb{R} \to \mathbb{R}$ and $t \ge 0$,*

$$\int_0^t \Phi(X_s)\,\mathrm{d}\langle X, X\rangle_s = \int_{\mathbb{R}} \Phi(a)\,\ell_t^a\,\mathrm{d}a \tag{A.9}$$

almost-surely. For all bounded (or positive) Borel-measurable $\Phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$\int_0^t \Phi(s, X_s)\,\mathrm{d}\langle X, X\rangle_s = \int_{\mathbb{R}}\int_0^t \Phi(s, a)\,\mathrm{d}_s\ell_s^a\,\mathrm{d}a \tag{A.10}$$

almost-surely.

We have proceeded rigorously by defining the local time pathwise in t for fixed a. It is natural to ask corresponding questions about a and consider the joint the regularity properties of ℓ_t^a.

Theorem A.1.5. *There exists a modification of $\{\ell_t^a : t \in \mathbb{R}_+, a \in \mathbb{R}\}$ such that the map $(a,t) \mapsto \ell_t^a$ is almost-surely jointly continuous in t and càdlàg in a.*

Moreover, if $X = X_0 + M + V$ is the decomposition of X into a random variable X_0, a continuous local martingale M with $M_0 = 0$ almost-surely, and a continuous bounded variation process V with $V_0 = 0$ almost-surely, then

$$\ell_t^a - \ell_t^{a-} = 2\int_0^t \mathbb{1}_{\{X_s = a\}}\, \mathrm{d}V_s = 2\int_0^t \mathbb{1}_{\{X_s = a\}}\, \mathrm{d}X_s. \tag{A.11}$$

In particular, if X is a local martingale then there is a bi-continuous modification of the family ℓ_t^a of local times.

This regular modification allows us to determine that

$$\ell_t^a = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \mathbb{1}_{\{a<x<a+\varepsilon\}}\, \ell_t^x\, \mathrm{d}x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{a<X_s<a+\varepsilon\}}\, \mathrm{d}\langle X, X\rangle_s\,. \tag{A.12}$$

We also make mention of the left and symmetric local times, which are defined by ℓ_t^{a-} and $\left(\ell_t^a + \ell_t^{a-}\right)/2$ respectively. We denote the symmetric local time by L_t^a. The above Theorem A.1.3 extends using the right or symmetric derivative of f respectively, and we have a modification which is jointly continuous in t and left or symmetrically continuous in a. The analogue of expression (A.12) also holds.

A.1.3 Local time for discontinuous semimartingales

We follow [49]. The case of a discontinuous semimartingale X can be divided into two sub-cases. The first is that X has bounded variation of jumps almost surely, that is

$$\sum_{0<s\le t} |\Delta X_s| < \infty \tag{A.13}$$

almost-surely. Then we retain the results above with minor modification. In the second case, when X does not necessarily have jumps of bounded variation, we lose the regularity results. The existence of local time for a general semimartingale follows a similar procedure to the continuous case.

Theorem A.1.6. *Let f be a convex function and X a semimartingale. Then $f(X)$ is a semimartingale, and there exists an adapted, right-continuous, increasing process A^f such that*

$$f(X_t) - f(X_0) = \int_{0+}^{t} f'_-(X_s)\,\mathrm{d}X_s + A_t^f, \tag{A.14}$$

where f'_- is the left derivative of f. Moreover $\Delta A_t = f(X_t) - f(X_{t-}) - f'_-(X_{t-})\Delta X_t$.

Again using Tanaka's formula, we make the following definition;

Definition A.1.7. Let $f_a(x) = |x-a|$. Define the local time ℓ_t^a by

$$\ell_t^a = A_t^{f_a} - \sum_{0<s\le t} \left\{ f_a(X_s) - f_a(X_{s-}) - \operatorname{sgn}(X_{s-} - a)\Delta X_s \right\}, \tag{A.15}$$

where $A_t^{f_a}$ is the increasing process associated to f_a from Theorem A.1.6.

The occupation time formula carries over from the continuous time case. Noting $[X,X]^c = [X^c, X^c]$, the following is immediate.

Theorem A.1.8. *For all bounded (or positive) Borel-measurable $\Phi : \mathbb{R} \to \mathbb{R}$ and $t \ge 0$,*

$$\int_0^t \Phi(X_s)\,\mathrm{d}[X,X]_s^c = \int_{\mathbb{R}} \Phi(a)\,\ell_t^a\,\mathrm{d}a \tag{A.16}$$

almost-surely. Moreover, for all bounded (or positive) Borel-measurable $\Phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\int_0^t \Phi(s, X_s)\,\mathrm{d}[X,X]_s^c = \int_{\mathbb{R}} \int_0^t \Phi(s,a)\,\mathrm{d}_s\ell_s^a\,\mathrm{d}a \tag{A.17}$$

almost-surely.

In the case that X obeys (A.13), then analogous versions of Theorem A.1.5 and equation (A.12) hold.

There is an explicit counterexample for regularity of the local time in the discontinuous case. Azema's martingale [49, Sec. IV 8] has unbounded variation of jumps, and a local time which is zero for all $a \in \mathbb{R}$ *except* at $a = 0$, where its distribution is the same as the local time at 0 of standard Brownian motion.

A.2 Integration, measure and convexity on $\mathbb{R}$

The following definitions are adapted from Bass [7] unless otherwise stated. We define a Radon measure for our purposes.

Definition A.2.1. A Radon measure on a topological space is a regular, signed measure on the Borel sigma algebra generated by the open sets of that space.

Now we establish a definition of functions of bounded variation on $\mathbb{R}$. Note that there are other common equivalent definitions.

Definition A.2.2. We say a function $f : [a, b] \to \mathbb{R}$ is of bounded variation if it can be expressed as the difference of two increasing functions.

There is a straightforward connect between measures and functions of bounded variation. This is equivalent to the idea of a distribution function in probability theory.

Definition A.2.3. Let f be a right-continuous function of bounded variation. Then the pre-measure $\mathrm{d}f$ defined by

$$\mathrm{d}f\left((x, y]\right) = f(y) - f(x) \tag{A.18}$$

extends to a bona-fide measure $\mathrm{d}f$ on the Borel sets of $[a, b]$.

Further, any finite Radon measure μ on the Borel sigma algebra on $[a, b]$ defines a càdlàg function of bounded variation by

$$f(x) = \mu\left((a, x]\right), \tag{A.19}$$

with $f(a) = 0$.

We refer to the measure associated to such f as its Lebesgue-Stieltjes measure. For Radon measures on $\mathbb{R}$, we have an important special decomposition.

Theorem A.2.4. *Given a finite Radon measure μ on the Borel sets of $\mathbb{R}$, we may write $\mu = \mu^c + \mu^d + \mu^{ca}$ where*

1. *μ^c is absolutely continuous with respect to Lebesgue measure;*

2. *μ^d is discrete, meaning assigns full measure to a countable set;*

3. *μ^{ca} is singular with respect to Lebesgue measure and also assigns zero measure to countable sets.*

We first note that convex functions on a connected domain in $\mathbb{R}$ are necessarily continuous. There is a profitable connection between convexity, bounded variation and finite Radon measures on $\mathbb{R}$ which we now describe.

Theorem A.2.5. *A function $f : [a, b] \to \mathbb{R}$ is convex if and only if it admits a left and right derivative at each point, denoted f'_- and f'_+ respectively, which are both nondecreasing functions.*

The following theorem now follows by combining the previous results.

Theorem A.2.6. *A function $f : [a, b] \to \mathbb{R}$ is the difference of two convex functions if and only if it admits a left derivative on $(a, b]$ of bounded variation, if and only if its second distributional derivative exists as a finite signed measure.*

The following Theorem from [52] is also the key to the Itô-Tanaka formula.

Theorem A.2.7 ([52] Appendix Prop. 3.2)**.** *Let $f : I \to \mathbb{R}$ be a convex function on some interval I. Then the second derivative of f in the distributional sense f'' is a positive Radon measure.*

Conversely, for any Radon measure μ on $\mathbb{R}$, there is a convex function f such that $f'' = \mu$ and for any interval I and $x \in int(I)$ we have

$$f(x) = \frac{1}{2}\int_I |x - a| \, \mathrm{d}\mu(a) + \alpha_I x + \beta_I, \tag{A.20}$$

$$f'_-(x) = \frac{1}{2}\int_I sgn(x - a) \, \mathrm{d}\mu(a) + \alpha_I, \tag{A.21}$$

where α_I and β_I are constants.

Another serendipitous equivalence will be useful to us. First we define regulated functions.

Definition A.2.8. A function $f : [a, b] \to \mathbb{R}$ is called regulated if it admits left and right limits at each point in the open interval (a, b), a left limit at b, and a right limit at a.

It turns out that this is equivalent to the existence of a uniformly approximating sequence of step functions.

Lemma A.2.9. *A function $f : [a,b] \to \mathbb{R}$ is regulated if and only if for all $\varepsilon > 0$ there is a function ψ_n such that $\|f - \psi_n\|_\infty < \varepsilon$ and*

$$\psi_n = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{A_i}, \tag{A.22}$$

where $m \in \mathbb{N}$, the α_i are constants and the sets A_i are open, closed or half-open intervals, or singletons.

Further, if f is right continuous, the sets A_i may be taken to be intervals of the form $[x, y)$. If f is left continuous they can be taken to be intervals of the form $(x, y]$.

Note that regulated functions can only have countably many points of discontinuity.

A.3 Weak convergence of functions and measures

The following definitions and theorems are adapted from [2] unless otherwise indicated.

Given an interval $[a, b]$, we know from Section A.2 that càdlàg functions of bounded variation and finite signed measures are in correspondence. Further, the decomposition of measures on $\mathbb{R}$ allows us to make use of special cases which are not treated by the general theory of weak convergence. First we cover the standard results.

For a Radon measure μ on a topological space $\mathcal{M}$, we define $|\mu|$ to be its total variation measure, and $\mathrm{TV}(\mu) = |\mu|(\mathcal{M})$. Despite the fact that our measures are often signed measures rather than probability measures, we retain the probabilistic terminology and say that a sequence of Radon measures μ^n converges to μ *weakly* if $\mu^n(g) \to \mu(g)$ for all continuous functions f of compact support. In functional analytic terms, this is local weak-* convergence.

The portmanteau theorem is a well-known result for probability measures which asserts an equivalence between weak convergence and other more easily verifiable properties. The notion of continuity requires a topology, which distinguishes weak convergence from stronger notions such as convergence in total variation. Essentially, the mass of a weakly converging sequence of measures can escape from an open set to its (topological) boundary, or enter a closed set from its complement at the boundary. In the case of signed measures, the positive and negative masses of the sequence can in fact cancel in the limit, which is easily seen by a sequence such as

$$\mu_n = \delta_{1/n} - \delta_{-1/n}. \tag{A.23}$$

In this case, the positive mass approaching the origin from above nullifies the negative mass approaching from below. Thankfully, only such destruction of mass is allowed, and no spontaneous creation of mass is allowed.

Theorem A.3.1. *The map $\mu \mapsto TV(\mu)$ is lower semicontinuous with respect to the topology of weak convergence.*

We will often require the following result, which ensures that if no mass is destroyed in the limit at the boundary of a set then its measure converges, and its functional equivalent.

Theorem A.3.2. *Let μ_n be a sequence of measures converging weakly to μ. If $|\mu_n| \to \lambda$ weakly, then $TV(\lambda) \geq TV(\mu)$, and if E is relatively compact such that $\lambda(\partial E) = 0$ then $\mu_n(E) \to \mu(E)$.*

Further if g is bounded and compactly supported and its discontinuity set is λ-null then $\mu_n(g) \to \mu(g)$.

The one-to-one correspondence of functions of bounded variation and measure on $\mathbb{R}$ becomes its definition in higher dimensions.

Definition A.3.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is of bounded variation if its distributional derivative is an R^n-valued Radon measure. That is,

$$\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} \mathrm{d}x = \int_{\mathbb{R}^n} \phi(x) \mu_i(\mathrm{d}x) \tag{A.24}$$

for Radon measures μ_i and all $\phi \in C^\infty(\mathbb{R}^n; \mathbb{R})$.

Convergence of functions f_n is often easy to check and provides a candidate limit for compactness results on the measures μ_i.

Definition A.3.4. We say a sequence of functions of bounded variation f_n converge weakly to f if $f_n \to f$ in L^1 and the distributional derivatives of f_n converge weakly to that of f.

Given a sequence of measures which arise as distributional derivatives of a sequence of functions of bounded variation, it is often much easier to check that the corresponding functions converge in L^1 than to directly check weak convergence. The following Lemma makes the remaining task much easier.

Lemma A.3.5. *If $f_n \to f$ in L^1 and the distributional derivatives of f_n are uniformly bounded in total variation, then $f_n \to f$ weakly.*

In making use of Theorem A.3.2, it can be difficult to establish the limiting measure λ of the sequence of total variation measures. First we introduce another mode of convergence.

Definition A.3.6. We say f_n converges strictly to f if $f_n \to f$ in L^1 and the total variations of the f_n (as numbers) converge to that of f.

Strict convergence implies weak convergence as the convergence of total variations implies their boundedness. The arguments above should convince the reader of the truth of the following Lemma.

Lemma A.3.7. *If $f_n \to f$ strictly, then the total variation measures of the distributional derivatives of f_n converge weakly to that of f.*

We present the following example which will be highly useful to us.

Example A.3.8. Assume f_n are continuous functions of bounded variation on $[a, b]$, converging strictly to some continuous function of bounded variation f. Let μ_n and μ be the associated Lebesgue-Stieltjes measures. Then $\mu_n(g) \to \mu(g)$ for all regulated functions g, as such functions can only have at most countably many discontinuities, and μ assigns zero measure to countable sets.

If the f_n and f are increasing and continuous with $f_n \to f$ pointwise, then pointwise convergence implies strict convergence, and so again $\mu_n(g) \to \mu(g)$ for all regulated functions g.