VYGOTSKY’S THEORY OF SCIENTIFIC CONCEPTS
AND CONNECTIONIST TEACHING IN
MATHEMATICS

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DAVID SWANSON
SCHOOL OF ENVIRONMENT, EDUCATION AND DEVELOPMENT
Manchester Institute of Education
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ABSTRACT

David Swanson
The University of Manchester
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Vygotsky’s theory of scientific concepts and connectionist teaching in mathematics

This thesis can be described in various terms. It is a translation of Vygotsky's theory of scientific concepts, in reality a theory of development, into a theory of mathematics teaching and learning. It is a theorisation, and development, of connectionist pedagogy in mathematics (a relatively underdeveloped, yet exemplary, amalgam of various reform/progressive /meaningful approaches to teaching). And, it is an investigation of the elements and processes involved in mathematical concept development, and the mediating role which classroom tasks can play. Alongside this, these understandings are embedded within a wider understanding of society, schooling, mathematics and mathematics teaching which help explain the current dominant practice in the classroom, and in doing so add to the understandings already described. In sum, the thesis therefore represents the beginnings of a systematic Marxist perspective of mathematics education which can cohere analysis at the multiple levels of society, schooling, classroom teaching and learning, and individual concept development. As such it is also, as should always be the case with Marxist perspectives, a guide to action for critical mathematics educators.

The thesis begins with context, motivation and strategy, an overview of relevant literature, and an explanation of the methodology and methods used within. The relationship between Vygotsky's theory of concept development and connectionist teaching is then outlined and developed. The wider societal perspective follows, with an emphasis on generalised commodity production as the key shaper of schools and classrooms. Both of these themes are then developed in relation to the example of vocational mathematics, both providing evidence of the existence and nature of scientific activity and concepts, and connecting their absence to the obstacles related in the previous section. The thesis continues by exploring a pedagogical development based on Vygotsky's theory, looking at the explicit problematising of generalisation, and analysing classroom dialogue in relation to this. In the other direction, a theoretical development is then made, following an illustration of the pedagogical and theoretical framework through the development of a particular concept. Finally, conclusions are drawn and future work outlined.
DECLARATION

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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ACKNOWLEDGEMENTS

A fundamental aspect of the approach and content of this thesis is that the social is analytically prior to the individual. That has certainly been the case with the production of this work, and if joint authorship were to be taken seriously from this perspective the title page would be longer than the thesis.

Of those to thank for their contribution, the most important is Julian Williams. I will be eternally grateful to Julian for providing me with refuge from the storms of Further Education, and being such a model supervisor (in inverse proportion to my modelling of studentship). Particularly important have been the thousands of conversations we have had about my work, his work, and everything else over the last few years, and his forensic and challenging reviews of the content of this thesis.

Many others have shaped the ideas herein, particularly those taking part in numerous reading groups and seminars, the participants on the social theories of learning course at the University of Manchester, and all those who have worked in B4.10 of the Ellen Wilkinson Building, especially Sophina, Maria, Darinka and Steph. My thanks also extend to the many political activists and friends who have shaped my view of the world over the years, and all the teachers and others I have worked alongside, who have helped form my thoughts on education and mathematics.

Final thanks must go to Shirin who has read more of this, and more times, than anyone ever really should, and whose consistent positivity has made the process both possible and bearable.
THE AUTHOR

David Swanson taught mathematics for many years in Further Education. His recent research experience includes roles in the ESRC funded Teleprism project, the Royal Society’s Vision for Science and Mathematics education project, the NCETM’s multiplicative reasoning project, and he currently works on an ESRC funded impact project connecting Q-Step university social science statistics with post-16 Core Mathematics provision in schools and colleges. He also currently lectures on the PGCE Secondary mathematics course at The University of Manchester, and works with the local NW1 Maths Hub on teacher professional development, mainly focussed around lesson study activity.

Recent Publications

Swanson, D. (under review) “I don't mind doing brackets”: Commodities, alienation and education.


Selected Conferences


1 Introduction

The purpose of this thesis is to begin to develop a coherent theoretical framework for mathematics education from a Marxist perspective. Such a framework should encompass every essential aspect of the object of enquiry including a philosophical understanding of the nature of mathematics; the role of mathematics, schooling and mathematics education within society, and the history of that role; what it means to learn mathematics; how mathematical understanding develops; and the practical issues of pedagogy\(^1\) in relation to these issues. To complete all this within a thesis is of course impossible. However, it is arguably possible to begin, or to develop the outlines of, such a framework, and that is the intention of what follows. Common sense would argue that there is an inverse proportional relationship between breadth and depth in any study, so the approach taken here requires some justification. Some of that justification appears in this introductory section, but much of it is postponed until later chapters (particularly that on methodology).

This introduction consists of four elements. First, the context and motivation of the research is outlined, initially from the personal perspective of the author, and then in more general terms. Second, an overview is given of the strategy of investigation and the rationale for that strategy. Third, a justification for submitting the thesis by alternative format is presented along with an account of the construction of the thesis. The introduction then ends with an outline of the chapters which follow.

1.1 Context and motivation

This inquiry is motivated by two intersecting concerns of the author. The first arising from some of the frustration experienced while teaching mathematics over a period of fifteen years. The other arising from the frustrations of a much longer

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\(^1\) The term pedagogy is used here and throughout the thesis to refer to both teaching and learning (and their unity), due to the lack of such a word in English (there is such a word in Russian for example- ‘obuchenie’). It is used also despite awareness of the term androgogy (the teaching and learning of adults), which can be used in contrast to pedagogy (of children). Here, pedagogy is used in its more general sense to cover both.
period living in a world dominated by poverty, war, inequality and oppression. The latter may seem incongruous in a thesis on the learning of mathematics; however, explanations for its relevance will follow.

The former, which is, on the surface, more central to the concerns of this study, is perhaps best summed up by the experience of teaching a particular course in further education, a GCSE mathematics equivalent course under the auspices of the Open College Network. This course was taught to adults on Access courses, courses which enable progression to university via equivalent level qualifications rather than the standard GCSE and A-Level route. Access courses are in general very inspiring to teach on, involving the development in confidence and consciousness of adults, who for various reasons had failed at school or who found themselves in uninspiring jobs and wanting something better from life.

The GCSE equivalent mathematics course was frustratingly designed however. It was evidence based and broken up into six credits with no final exam (although assessment of each credit did include a classroom based test). In order to claim equivalence with GCSE mathematics the rules insisted that students show evidence of capability of every aspect of the content. The six credits were therefore broken up into around forty-five smaller skills to be so evidenced. In order to provide sufficient opportunity to achieve each criterion, within a system that would be clear for students to follow (and not overcomplicated to mark), assessments were designed based on the separation of those criteria. The time taken by frequent assessment, combined with the atomised nature of the elements assessed, left little space for anything but teaching those elements as atomised skills.

As a teacher, I tried to make this process more meaningful, but overall something felt wrong. The skills I was teaching were in general separated from the real world, and even more so, separated from each other. At annual moderation meetings, where those teaching the course across local colleges would come together, I would raise this as a problem of the course: that the structure was driving out any meaning from the mathematics we were teaching, and if mathematics is its connections to the real world and to other mathematics, then we were really driving mathematics out of the course. After many years of raising these objections, the only significant change to occur was in the opposite direction, with the raising of the number of criteria to be assessed on the course to over eighty, exacerbating the problem.
This experience, observation and analysis of mathematics teaching is not unique, either in time (e.g. see the 1858 commission report quoted in Howson & Rogers, 2014, p.259) or space (e.g. see Schoenfeld, 1988). In fact such teaching dominates in the UK (Pampaka et al., 2012a; Pampaka, Wo, Kalambouka, Qasim & Swanson, 2012b) and perhaps globally. For example, in Singapore, a country with a reputation for recent initiatives in mathematics education which emphasise creativity and innovation (e.g. see Lesh & Zawojewski, 2007, p.764), and where problem solving is at the heart of the official curriculum (Leong et al., 2011), the reality in the classroom is somewhat different:

Observation revealed that a typical mathematics lesson started with teacher’s lecture-style talking followed by students individually working on exercises assigned by the teacher and later the teacher would provide answer checking and/or feedback. All these classroom activities simply involved transmitting knowledge and practising procedural routines... The classroom teacher was the major source of knowledge and played an authoritative role concerning knowledge... It was seldom observed that teachers encouraged students to question or offer their own thinking or opinions... Most... oral responses only consisted of one single syllable, “yes” or “no”, or simply a numerical answer without giving any substantial explanations. As in oral responses, there was also little emphasis on engaging students in sustained writing, such as giving justifications. Instead, students often worked on pre-designed worksheets which contained a large number of repeated exercises which were meant for drill and rote practice (Fan & Zhu, 2007, p.498).

This direct experience, and general situation, raises the key questions which frame the study: Why is mathematics teaching the way it is? And how can it be made more meaningful for teachers and students? These questions are not new and answers exist, particularly to the second. There is a long history of pedagogical attempts, certainly since Pestalozzi in the early 1800s (see Bjarnadóttir, 2014, p.443) and perhaps further, to develop methods which move beyond rote learning toward an active role for students and deeper understanding. Much of the positive history of such developments is encapsulated within what is termed connectionist teaching in mathematics (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997, p.31; Swan 2006, p.66). Although no doubt rooted in pedagogical experience, mathematics educational research (MER), theories of pedagogy and wider theories, connectionism as a practice remains under-theorised. The thesis therefore takes connectionist teaching as a starting point, and translates the second framing question above to
another – How do we deepen and make more coherent our understanding of how and why connectionism might work?

The second motivation of the study, touched on in the first paragraph of this section, contributes further questions, and also key aspects of the strategy of investigation. The historically developed systematic response to the inequality, exploitation, and oppression which are among the negative features of the modern form of societal organisation called capitalism, is, of course, Marxism. Classical Marxism, as an intellectual tradition, is grounded in activism rather than the academy. It is a philosophy which, in Sartre's (1963) words is,

simultaneously a totalization of knowledge, a method, a regulative idea, an offensive weapon, and a community of language... [a] "vision of the world"... [and] an instrument which ferments rotten societies (p.6).

A central feature of this offensive weapon and instrument of fermentation is an obsession with consciousness. In broad terms, how people come to understand the world better in order to change it. For Marxism (it should be stressed here, that this term is used in the thesis as a convenient shorthand for the revolutionary Marxist tradition, without, as seeming to here, denying the existence of disagreements within it, or even other Marxisms, in academia or elsewhere), this concerns the development of a conscious, coherent, systematic and scientific understanding of the world in interrelation with human activity in concrete circumstances, both collectively and for particular individuals. Framed in this way, the overlap with the study of the learning of mathematics is noticeable.

Mathematics too is a coherent, systematic and scientific body of knowledge which emerges and develops in relation to human activity. To see that the particularities and generalities of the development of mathematical consciousness and class consciousness can contribute to an understanding of each other is an example of the 'totalization of knowledge' that Sartre speaks of.

This means, again, that the thesis does not start from scratch in creating a systematic approach to mathematics education. It takes as a basis the general understandings of consciousness in the works of Marx, Lukacs, Gramsci, Lenin and others. Beyond this, it also takes as a basis attempts to develop and extend this general approach to wider questions of development and education. The work of Lev Vygotsky, writing in post-revolutionary Russia, represents, in the author’s view, the high-water mark of such attempts, and as such, necessarily becomes the starting point of further exploration. This work brought scientific
concept (or word-meaning) development to the centre of the thesis. Much of the study therefore involves the working out of the particular mediations involved in the development of mathematical concepts through the bringing together of Vygotsky's theory, wider understandings of mathematics pedagogy, and concrete activities within mathematics classrooms.

Marxism also brings a critique of society which helps explain the wider structures that mathematics education is embedded within, assisting in the understanding of why mathematics teaching is the way it is, and what the obstacles may be to implementing more meaningful approaches in the classroom. Some emerging critiques within the field of mathematics educational research have questioned even attempts to make mathematics teaching and learning more meaningful (see, for example, Lundin 2012; Pais & Valero, 2012). Seeing in those attempts an effective echoing and reproduction of the dominant ideology due to their depoliticisation and decontextualisation from the realities of the capitalist context. Contrary to this, it will be argued here that by integrating a political critique of society with an understanding of concept development, a basis is laid for a genuinely critical mathematics pedagogy. Given that notions of problem-solving and modelling or 'reform' pedagogy in mathematics have come to increasingly appear (at least superficially) within official policy, this existential and reflexive question for critical pedagogy and research is an important part of any attempt to map out a guide to action in the educational field. But it will take the content of the entire thesis to seriously address this question, and the issue will be returned to in the conclusion.

The final key contribution of Marxism to the thesis is in taking what can be called a totality approach to the question of mathematics education. The methodological origins and implications of this position are outlined in the next chapter. Here, the relevance to the nature of this thesis is briefly outlined. In broad terms, it is taken here that reality is hierarchically structured, that systems of interrelations exist at each level and that each level has no meaning outside of the general relationships within the more general hierarchical structure, but also that each level is semi-autonomous, with its own relationships and cannot be simply reduced to the levels 'above' or 'below'. Understanding too must reflect this hierarchical, systemic structure and its mediating interrelations. The classic example of this within science is the relationship between physics, chemistry and biology (although it can then extend to psychology and sociology). Despite reductionist attempts, understandings of chemistry cannot be reduced to those of
physics (see, for example, the necessary role of the environment in the formation of pyramidal structure in some molecules, Chibbaro, Rondini & Vulpian, 2014, p.134) but neither does chemistry operate independently of the processes at the quantum mechanical level. This relationship between levels, between parts and wholes, applies at any level of investigation. As biologist Richard Levins (2007) puts it, 'problems are larger than we have imagined and we should extend the boundaries of a question beyond its original limits' (p.108). For example, in traditional, limited understandings of diabetes,

analysis of the regulation of blood sugar may include the interactions among sugar itself, insulin, adrenaline, cortisol, and other molecules but is unlikely to include anxiety, or the conditions that produce the anxiety such as the intensity of labor and the rate of using up of sugar reserves, whether or not the job allows a tired worker to rest or take a snack.

A more fully developed understanding (and one that would no doubt aid actual treatment of diabetes) must include all these interrelations. Richard Lewontin (1991), Levins' close collaborator, argues in more general terms,

It is not that the whole is more than the sum of its parts. It is that the properties of the parts cannot be understood except in their context in the whole. Parts do not have individual properties in some isolated sense, but only in the context in which they are found (p.81).

This outlook has two relevant implications. The first is that it is important to understand systems of interrelations at various levels (here, it should be noted, systems, as will be seen later with particular concepts, are not intended to be seen as either closed or static formations). The focus in the thesis is on such a system, or rather on two interrelated systems – the systemic relations of the elements within Vygotsky's theory of scientific concepts, and the systemic relations between elements of connectionist teaching practice. Together these are woven into a system of theorised pedagogy (or practically-implementable theory). The second implication is that in order to understand such a system attention must be paid both to the wider systems of which it is a part, the narrower systems which it includes, and the ways that these various levels mediate each other. For a theory of mathematical pedagogy, the wider 'whole' of which it is a part is of course schooling, and beyond that, the form of societal organisation which schooling is shaped by. But alongside that is also the wider system of mathematics and its usage outside of school. These wider systems are discussed in chapters 5 and 6.
The most useful levels ‘below’ the pedagogical system, as it were, are those of the development of particular concepts, and the concrete level itself – the empirical practice of teaching and learning in a real classroom. Chapters 7 and 8 have these levels, and the interrelation with the elements of Vygotsky’s theory and connectionist teaching as their focus.

On the surface, taken together, this appears to contravene the usual recommendation for a sufficiently (though not overly) narrow focus within a thesis (e.g. see Dunleavy, 2003, p.18). Often a PhD may, for example, aim to answer a particular hypothesis, within a particular context, aiming for a richness and depth on the particular topic. Here, some particular aspects are treated similarly. But overall, as the object of enquiry is one of systemic relationships, depth can only be achieved through breadth. Attaining a fuller understanding of the interrelation of theory and pedagogy relies on this totality approach. The central two questions of the thesis (which should really be thought of as a unified single question) however remain,

1. How can Vygotsky’s theory of scientific concepts be used to theorise connectionist pedagogy in mathematics?; and

2. How can connectionist practice in mathematics be used to concretise Vygotsky’s theory of scientific concepts?

1.2 Rationale for alternative format, thesis structure, and account of thesis construction

The reasons for submitting the thesis by alternative format are more general than specific. Given the increasing importance of publication for academic communication, university ranking and individual employability, it is increasingly useful to publish PhD results in academic journals as individual papers. This is particularly so given that there is less provision than in the past for post-doctoral funding which is primarily aimed at converting PhD’s into publishable material.

However, in particular, the work in this PhD is of a form that is structurally suitable for publishing as individual papers, having been constructed on this basis. So far, one article (ch.6) has been published in Educational Studies in Mathematics, a second (ch.4) has been through a review process and resubmission is imminent, and a third (ch.5) is the basis of two publications, one
(essentially the chapter version) currently under review for the International Journal of Educational Research, and the other (with expanded empirical data, and in relation to the methodology of Walter Benjamin) to be published as a chapter in the book 'The disorder of mathematics education', a collection of articles from a critical mathematics education perspective. The final two chapters (7 & 8) are currently being prepared for submission. Chapter 8 is in reality much longer than is suitable for a journal article. This is partly due to one role of the work, in illustrating multiple aspects of a pedagogical system and concept development over time, and partly due to taking advantage of the thesis format to present a satisfying amount of empirical data. In reality it will provide the basis for two papers.

Although each part of chapters 4-8 stands alone as a distinct publication (or potential publication), together they form a unity. At the heart of the thesis is chapter 4 on the relationship between Vygotsky’s theory of scientific concepts and connectionist pedagogy in mathematics. Chapter 5, on the influence of the commodity form on education embeds the work on concept development and pedagogy within a wider, societal, understanding of schooling under capitalism. The necessity of doing this, of taking a totality perspective, is discussed more fully in the chapter on methodology (ch.3). In simpler terms here, chapter 5 outlines the societally induced obstacles to pedagogy and concept development while, at the same time, illustrating the connections between psychology and sociology. Chapter 6, on workplace mathematics and vocational schooling plays two roles. Firstly, it provides a basis for the thesis in presenting evidence that scientific concepts and activity do appear both within and outside of schooling. And second, it further develops the connections between understandings of how scientific concepts can arise and the societal obstacles to this, in effect reinforcing the links between chapters 4 and 5. The final chapters, (ch.7 & ch.8), then develop the analysis of the central chapter (ch.4), essentially in two directions. Chapter 7, on the role of generalisation, uses the developed theory to suggest practical implications for pedagogy. Chapter 8, on the other hand, analyses concept development within classroom dialogue to suggest theoretical developments to Vygotsky’s theory.

Although the periods of development of the five articles (ch.4 to ch.8) in the thesis overlapped in complex ways, in essence this can be simplified to two periods. The central article, chapter 4, was developed over several years, from development of the initial ideas, through periods of engagement with Vygotsky’s
writings, wider theory and literature on mathematics pedagogy, the design and
teaching of the short course ‘Teaching and Learning Mathematics’ (TLM), and
early analysis of dialogue within that class. Enclosed within this period, pre-
analysis of the TLM course, chapter 4 and chapter 5 were developed, primarily in
relation to other data. This included a reworking of data from an earlier MSc on
the mathematics of darts players, pilot data from a connectionist classroom, and
data from the Teleprism project on secondary mathematics schooling. The
second phase of the research develops the earlier stage through analysis of
dialogue from the TLM course (ch.7 & ch.8).

Ch. 6, ‘Making mathematics concrete in and out of school’ was co-authored with
Prof. Julian Williams (JW). Joint authorship is a complex process. To untangle
the contributions of the individuals and describe the process of development fully
would represent a thesis in itself, but an outline of the process is necessary here.
JW was invited to contribute at speed to a paper to a special issue on vocational
mathematics, and suggested to myself (DS) that the thesis pilot work within a
connectionist Mathematics Enhancement Course may fit. After ongoing dialogue,
the plan of the article evolved to include a previous case study (JW) and a
relevant study of the mathematics of darts players (DS). The key contributions of
DS included the initial draft; the bulk of the analysis of the two larger case studies
(MEC and darts); the review of previous literature; and, following feedback from
the SI editor; a redrafting of the introductory and concluding sections (which also
more closely aligned the article with this thesis). DS also contributed the revisions
required by the review process. JW contributed an outline based on an initial
period of dialogue, of which the first draft was an expansion; the second case
study analysis; contributions to the first and third case study analyses and initial
drafts of the conclusion. JW also contributed major experience and
understanding in the overall construction of a paper which guided the process. All
aspects of the final paper were the subject of frequent joint review over the period
of its preparation, including multiple redrafts of sections by both JW and DS.

A final point on style: Some alterations to the formatting of the papers which
make up the results chapters (4 to 8) have been made to assist the reader of this
unified thesis. These chapters still refer to themselves as papers or articles to
emphasise their origins but cite other results chapters as chapter or section
references; chapter, section and page numbering has been altered to reflect a
single document; and fonts and font sizes have been changed for consistency.
References for each of the chapters 4 to 8 are self-contained, while those for the surrounding chapters appear at the end of the thesis.

1.3 Outline of chapters

The following two chapters, as is convention, outline first a review of relevant literature (ch.2), and second an overview of the methodology and methods involved in the construction of the thesis (ch.3). Both of these chapters have unconventional aspects to them. Given that the thesis discusses the entirety of mathematics education (not literally, of course, but rather orientated towards such a totality, within practical limits), from the history of schooling to the generalisation of particular mathematical concepts, a literature review is necessarily only going to be partial in its coverage. The focus in this chapter is therefore on the key literature utilised, and engaged with, in the thesis.

Chapter 3 on methodology discusses the relationship between theory and practice, questions on a totality approach, and the particular relationship between form and content within the thesis. Vygotsky’s theory of scientific concepts is a general one relating to the development of scientific understanding and as such can, with modification, be applied to educational research as much as the learning of mathematics and many of the features of the theory appear therefore as elements in the overall methodological approach. Following this discussion the chapter goes on to relate the methods used, as well as to discuss issues such as validity and ethics.

The results chapters begin with chapter 4 which shows the relationship between Vygotsky’s theory and connectionist pedagogy. Acting as a cornerstone to a totality-orientated understanding of mathematics education, this chapter both theorises connectionist pedagogy, and concretises Vygotsky’s theory in relation to the particular mediations of mathematics pedagogy. This allows further development in both directions. First, in chapter 7, the pedagogical framework is extended based on analysis of Vygotsky’s theory and its emphasis on generalisation and mathematical connections. A classroom task involving the explicit problematising of generalisation is explored in some detail to illustrate and deepen understanding of the place of the general approach within the wider framework. Then, in chapter 8, the entire pedagogical and theoretical framework
is illustrated and further explored through analysis of a particular concept within a
series of classroom tasks. This leads to a suggested addition to the Vygotskian
framework in its translation to mathematics teaching and learning.

Between these chapters, chapter 5 embeds the framework within a wider
understanding of society, schooling, mathematics, and mathematics teaching and
learning, based upon Marx's theory of alienation and the negative impact of the
dominance of the commodity form of production. This negative impact is seen to
underlie the dominance of the opposites of the elements of the pedagogical
framework based on Vygotsky's theory. In chapter 6 the different themes of
chapters 4 and 5 are then further developed in relation to the example of
vocational mathematics, both providing evidence of the existence and nature of
scientific activity and concepts, and connecting their frequent absence to the
obstacles related in the chapter on alienation.

Following the results chapters, chapter 9 revisits the results and shows how they
integrate into the totality of the thesis. Perspectives are also given on directions
of future study, and a general conclusion in relation to the framework, and its
relation to critical mathematics education, is outlined.
2 Review of literature

As the thesis takes a totality approach (with the caveats stated in the previous chapter), it covers a range of ground, at various levels, related to mathematics education. This includes the societal factors influencing mathematics, schooling, and the teaching and learning of mathematics; the relationship between Vygotsky's theoretical understanding of concept development and pedagogical approaches in mathematics; and analysis of the development of particular concepts within real classroom dialogue in relation to the developed framework of theoretically informed pedagogy. In such circumstances, any literature review will necessarily be partial in comparison to what could potentially be included. For practical reasons therefore, this review is primarily restricted to the literature which plays a role in the papers contained within the thesis.

The key focus of the thesis is on the relationship between Vygotsky's theory of scientific concepts and connectionist pedagogy in mathematics. The review therefore begins with the first element: Vygotsky and his theory. This is then extended to look at some of the uses to which Vygotsky's work has been put within mathematics educational research, thus beginning to explore the second key element: mathematics pedagogy. The particular approach to pedagogy addressed in the thesis, connectionism, is then discussed. Two other influential theories within MER are then related. Realistic Mathematics Education, and situated cognition (along with other aspects of outside school mathematics). There are two main reasons for their inclusion (along with constructivism and social constructivism, discussed in sec.2.2). They represent the main signifiers around which ideas about mathematics education have collected in recent years. As such these approaches have contributed greatly to the general pool of ideas within MER from which connectionism has emerged. There are therefore significant overlaps with connectionism, and influences on it, from these theories, and so to discuss connectionism is to discuss them too. As general theories (of varying levels of coherence and reach) though, they are also those which an emerging theoretical perspective on mathematics pedagogy must be tested against, and their weaknesses can provide an impetus for theoretical and practical developments.

The interrelation of Vygotsky’s theory with connectionist pedagogy will be continued and developed within the thesis through investigating actual concept
development in the classroom. To allow sufficient depth of analysis, only a few concepts, and their development, will be focussed upon. These are: heuristics, the particular heuristic of being systematic, and the concept of proportion. The thesis does not aim to significantly contribute to knowledge of these concepts from an MER perspective (other than through situating them within its proposed framework), but it is important to outline the key features of the existing literature related to them which are relevant to the ideas contained in the later chapters.

The wider societal understanding, in which the thesis embeds its Vygotskian approach to connectionism, is then addressed. This is done through a discussion of Vygotsky's relationship to Marxism, and an exploration of the relevant literature on Marx's theory of alienation and the commodity form.

Due to the emphasis on literature relevant to the papers contained within the thesis, and as is to be expected with an alternative format thesis, there will be some reiteration of elements of this review in later chapters (In particular, the following sections can essentially be mapped to one another, although often with minor alterations: the final paragraph of 2.2 and the first paragraph of 4.1; the final two paragraphs of 2.4.1 and the last part of 7.3.1; 2.4.2 and the first part of 6.2; 2.5.1 and the first part of 8.3.1; 2.5.2 and 7.4.2; and, the latter parts of 2.6 and 5.3.1).

2.1 Vygotsky and scientific concepts

There are two key works by Vygotsky available in English on the subject of concepts. The first is the chapter on 'The development of thinking and formation of concepts in the adolescent' from 'Pedology of the adolescent' (1998), written around 1930-31 (p.319). A section of this appears as chapter 5 of 'Thinking and Speech' (1987). The second is chapter 6 of Thinking and Speech (1987), written around 1933-34 (p.376) on 'The development of scientific concepts in children'. This second work extends the arguments of the earlier writing and introduces the distinction between everyday and scientific concepts.

Although not explicitly focused on the psychology of concept development, there are other works by Vygotsky which are useful in developing a more complete and historical view of the theory. For example, an earlier work, 'The historical meaning of the crisis in psychology' (1997a), written around 1926-27 (p.14),
raises some themes in relation to the development of a general science of psychology which re-emerge in the discussion of scientific concepts. Also, in ‘The history of the development of higher mental functions’ (1997b), written approximately in 1931 (p.279), the wider psychological system of which thinking in concepts is part is discussed more fully. In this, these higher aspects, such as voluntary attention, logical memory and volition, are seen to join with the development of concepts as part of a complex interrelating whole, the development of which is stimulated by the social and cultural with a key role played in the process by signification and, in particular, words.

Significant discussion of the content of Vygotsky's work on concepts is postponed until chapter 4. However, a central claim of the thesis is that these works contain a richer and broader potential relationship to understandings of mathematics pedagogy than have generally been taken to be the case. The next section relates some of these earlier attempts to connect Vygotsky to mathematics education.

2.2 Uses of Vygotsky

Some early attempts to relate Vygotsky’s work to MER included some of his key general ideas: the relationship of inner speech and verbal speech within mathematical problem situations (Brown, 1971); the role of play as an intermediary toward abstract thinking detached from object dependence (Tamburrini, 1974); the central role of language in concept development (Austin & Howson, 1979); cultural internalisation, the difference and relation between spontaneous and scientific concepts, and the relationship between learning and development through the zone of proximal development (ZPD) (Fuson, 1980).

The notion of ZPD is perhaps Vygotsky’s most influential idea within education; a particular form of his general understanding that the social leads development. This concept emerged in relation to a critique of IQ testing (see van der Veer, 1990) which Vygotsky (1998) argued only attended to already matured development:

What the child can do today in cooperation and with guidance, tomorrow he will be able to do independently. This means that by ascertaining the child’s potentials when he works in cooperation, we ascertain in this way the area of maturing intellectual functions that in
the near stage of development must bear fruit and, consequently, be transferred to the level of actual mental development of the child. Thus, in studying what the child is capable of doing independently, we study yesterday's development. Studying what the child is capable of doing cooperatively, we ascertain tomorrow's development. The area of immature, but maturing processes makes up the child's zone of proximal development (p.202).

A key underpinning element of ZPD is what Chaiklin (2003) refers to as the assistance assumption (p.43) – simply, that individuals can achieve more in conjunction with more capable others (teacher or peers), or, more importantly, an understanding of how this process works and leads to development. At heart this rests on Vygotsky’s (1997b) emphasis that,

[E]very function in the cultural development of the child appears on the stage twice, in two planes, first, the social, then the psychological, first between people as an intermental category, then within the child as an intramental category (p.106).

When viewed in this broader way, the social origin of mental processes does not necessarily require a more competent other. Society may simply pose new problems, and collective responses, which the individual is part of, can outstrip those of the individual on their own. However, in general, and particularly within education, more developed, cultural forms of concepts pre-exist the individual and their introduction is mediated by the concepts of the teacher or of peers. As Vygotsky (1994a) puts it in general terms;

[I]n child development, that which it is possible to achieve at the end and as the result of the developmental process, is already available in the environment from the very beginning. And it is not simply present in the environment from the very start, but it exerts an influence on the very first steps in the child's development (p.347).

Within the classroom, this becomes a very conscious process. Problems are deliberately put before the child which contain, draw out and develop particular concepts, for example. Alongside this conscious element, say, in explicitly outlining a concept, is another sometimes less conscious process – the developmental impetus which comes from joint use of words at different levels of understanding, say, between an adult and a child. Akin to the processes involved in the uneven and combined development of nation states, advanced elements are incorporated into less developed systems where, 'the very process of assimilation acquires a self-contradictory character' (Trotsky, 1977, p.27). These contradictions can then become one element in encouraging further development. In chapter 5 of Thinking and Speech (1987, p.121), Vygotsky outlines a
(theoretical) anatomy of forms of generalisation through childhood as this process unfolds. This places dialogue in a central role in relation to the process of social factors leading development, whether in general terms or for particular concepts.

Elements of this aspect of Vygotsky's work were an important influence on the theoretical trend within mathematics educational research of social constructivism (Nesher, 2015). Social constructivism (see e.g. Lerman, 1996; Jones, Jones & Vermette, 2010) builds on constructivism, which was heavily influenced by Piaget's work (see von Glaserfield, 1984), and retains its emphasis on an active role for the individual in constructing their knowledge. Constructivism's limitations lie in its other emphasis, the claim that learners are not discovering knowledge of a pre-existing world (which may or may not exist) independent of the subject, but that knowledge, instead, can only be the individual's organisation of their own experiential world (Kilpatrick, 1987). This means that the theory begins and ends with the individual. The rising interest in Vygotsky's work brought an alternative perspective – that the social is the starting point for understanding the development of the individual. This connected with work from a constructivist perspective which emphasised the role intersubjectivity and communication play in mathematics learning (see, e.g. Cobb, Wood & Yackel, 2002, p.162). Social constructivism, underpinned by Vygotsky's position, then argued against an 'underlying model for the socially isolated individual mind. Instead, the underlying metaphor is dialogical or persons-in-conversation, comprising socially embedded persons in meaningful linguistic and extra-linguistic interaction and dialogue' within joint activity (Ernest, 2010, p.43). In many ways this position still remained trapped in the individualist paradigm of constructivism, it was the interaction of individuals which was seen to play a part in development, rather than something more fundamentally social i.e. where the social is taken as analytically prior to individuals (see Lerman, 1996, for a similar argument).

The relating of Vygotsky to mathematics education through social constructivism was also, perhaps, limited by access to texts. Much of the work relied on earlier, reduced, and often fragmented, versions of Vygotsky's texts. For example, Ernest (1998, p.301) relies on an earlier version of Thinking and Speech rather than the fuller version in the Collected Works which was available in 1987 (and the volume with additional extensions of chapter 5 which was published in 1997). This perhaps meant that Vygotsky's fuller, detailed account of concept
development was not explored sufficiently or incorporated into the work of social constructivists.

Yet, social constructivism contributed greatly to the sharing of some of Vygotsky's general ideas within MER – particularly in relation to the developmental role of dialogue.

Various aspects of Vygotsky's theory of scientific concepts have continued to prove fruitful in mathematics education research since social constructivism, particularly in regard to the interrelation of scientific and everyday concepts (e.g. see Yoshida, 2004), and the importance of social dialogue (e.g. Bussi, Boni, Ferri & Garuti, 1999; Renshaw & Brown, 2007). However, in general, and despite some very useful empirical work, most attempts to connect Vygotsky's ideas to mathematics education remain at too high a level of generality in terms of the theory. There is generally insufficient mediation or detail of Vygotsky's theories to enable a developmental interrelation of the theory with practice.

2.3 Connectionist pedagogy

The dominant form of mathematics teaching (Pampaka et al., 2012a; Pampaka et al., 2012b), 'transmissionism', involves an emphasis on the ability to memorise and reproduce a disconnected collection of set routines and procedures, transmitted clearly by the teacher, with little attention to reasoning (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997 p.33). Such methods have a long history, and have a key early theorisation in the behaviourism and associationism of Thorndike (e.g. 1931), an approach critiqued by Vygotsky (1987, p.133) as inadequate when extended to the complexities of concept formation.

Opposition to transmissionism in teaching mathematics also has a long history (see, for example, Howson & Rodgers, 2014). In recent years, a key focal point for, and influence on, the movement for more meaningful forms of pedagogy in mathematics has been the National Council for Teachers of Mathematics (NCTM) in the U.S. and their teaching 'Standards' (CCSSI, 2010; NCTM, 1989, 1991, 1995, 2000). Although the standards emphasise a balance of factual and

2 Outside of MER there are, of course, examples of serious engagement with the details of Vygotsky’s theory on concepts. See, in particular, Blunden, 2012, for a perspective with much in common to that taken here.
procedural knowledge with conceptual understanding, it is their ‘process’
standards which most clearly emphasise positive pedagogical elements. For
example, the standards for mathematical practice (CCSSI, 2010, p.6) include

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning

These standards imply forms of pedagogy which encourage problem solving and
modelling, dialogue, making connections in mathematics and metacognition, for
example, even if the standards insist on being non-prescriptive in terms of
teaching methods (p.5). The NCTM standards emerged under the heavy
influence of constructivism (see Bossé, 1995) in its varying formulations. In
practice, constructivists drew on a range of compatible pedagogical approaches
with ‘a level of specificity to guide instructional practice’ (Confrey & Kazak, 2006,
p.306) that the more general philosophical approach itself lacked. ‘Reform’
mathematics, the term regularly applied to the approach of the NCTM standards,
includes these, and further developments in MER to the present day, in its
designation.

Reform mathematics is arguably the common sense of many researchers and
some practitioners within mathematics education. Yet due to these layers of
influence it arguably remains relatively incoherent in terms of its theorisation.
Although some would argue, from a pragmatist perspective, that this is a strength
(see e.g. Lesh & Sriraman, 2010, p.143), here an opposite position is taken.
Following Gramsci (1971), it is argued that one must strive to make common
sense more coherent and more systematic (p.327). The thesis therefore
approaches the elements of reform mathematics in this way, seeing in them the
basis for a critical pedagogy and beginning a totalising theorisation of some of its
key features.

Rather than use reform mathematics as a terminological and practical starting
point in the thesis, one of its forms to surface recently in the UK, connectionism,
is preferred. This is partly for negative reasons: One of the key historical uses of
the word ‘reform’ refers to attempts to improve the situation of ordinary people
within the constraints of capitalist society (see, e.g., Luxemburg, 1966); however, with the rise of neo-liberalism a new almost Orwellian twist has been added to the word. This is seen, for example, in phrases such as 'labour-market reform', which refer to processes entailing greater insecurity and lower wages etcetera for workers, precisely the opposite of the word’s old meaning. Reform mathematics, as a term, similarly contains these contradictory elements (see, for example Apple, 1992), encompassing both a desire to increase particular national economic competitiveness via improved educational means but also perspectives which are more socially critical.

There are more positive reasons for adopting connectionism as a starting point. Although the term shares a lack of clear definition and under-theorisation with reform mathematics, its first significant usage (Askew et al., 1997) included its differentiation not just from transmissionism but also from discovery learning (p.35), showing a superior effectiveness to both (p.28; see also Swan, 2006; Pampaka et al., 2011). This usefully distances connectionism from forms of pure constructivism, emphasising the positive role of teachers in guiding and shaping learning (see also Kirschner, Sweller & Clark, 2006, for further evidence of the limitations of pure or minimally guided discovery). The second positive benefit of using connectionism as a term is its obvious stress on connections. As will be seen in chapter 4, it is useful to view a concept as the ensemble of its connections, to the real world, to other concepts within the system of mathematics, and to students’ own current conceptions. The term therefore has practical benefits in almost sloganising the key emphases required within the classroom.

Connectionism, like reform, does have a range of historical connotations, not all of which are progressive, or helpful. Thorndike (1931) also described his behaviourist pedagogy as connectionism (p.122), based on the repeated reinforcing of associative connections (see also Fehr, 1953, p.13). As will be seen in chapter 4, such a simplistic view of connections is not implied here. Instead what is intended is a more dialectical, mutual mediation within relationships which are embedded in meaningful activity. The term connection remains a useful first approximation for such relationships however. Connectionism has also been used more recently in relation to modelling learning and cognition as emergent properties of neural networks. Seen as a critical alternative to cognitivism (Bauersfeld, 1992, p.19), this form of connectionism is relatively undeveloped in relation to education applications (Shultz, 2007, p.1497),
and often reinforces aspects of the old behaviourist argument (p.1501), even according to its advocates.

In Askew et al.'s (1997) report connectionism is loosely defined, firstly as contrasting with transmissionism and discovery approaches, and secondly in terms of some common pedagogical characteristics of connectionist teachers. The key themes which emerge from analysis of these common characteristics include an emphasis on:

- Solving realistic problems (p.32). Concepts and their application are learned together (p.36) through challenges which require reasoning (p.36);
- Connecting mathematics to the real world through applying to new situations (p.32);
- Connecting to students' own understandings. Sharing, comparing and building on students' own methods and conceptions (p.32), making conscious, and challenging, misconceptions (p.32);
- Connecting to other mathematics. Not just learning set procedures but a meta approach including openness to different methods which can be compared for their effectiveness and appropriateness in context (p.31). Connecting between different parts of the curriculum (p.31). Reasoning and justification (p.32);
- Genuine dialogue between teacher and students. Purposeful interpersonal activity (p.35) jointly exploring understandings (p.36). 'A high degree of focused discussion between teacher and whole class, teacher and groups of pupils, teacher and individual pupils and between pupils themselves' (p.46).

These key themes – genuine problems; connecting mathematics to the real world, students’ existing conceptions and other mathematics; dialogue; and reflexivity – are taken as the central features of meaningful mathematics teaching and learning to be explored further and theorised within the thesis. They are not unique to connectionist teaching. Some or many of these features exist to some extent within some of the philosophical and pedagogical approaches mentioned
so far, such as social constructivism and constructivism, and within looser assemblages of reform/progressive/meaningful mathematics pedagogy\textsuperscript{3}.

### 2.4 Other key aspects of mathematics educational research

Other theories of mathematics learning, pedagogical and otherwise, play a role within the loose assemblages described above, and within the thesis. Some of these are now discussed.

#### 2.4.1 Realistic Mathematics Education (RME)

Realistic Mathematics Education originated with, and was dominated by, the towering intellect and personality of Hans Freudenthal, a working mathematician (Est, 1993) with a long-standing interest in the teaching and learning of mathematics (Goffree, 1993). RME has an emphasis on contextual or meaningful problems and the progressive mathematisation (Treffers & Vonk, 1987) of children’s own mathematical productions through guided reinvention, often via carefully chosen problems and models. A key feature of the approach is didactical phenomenology (Freudenthal, 1983), the dissection of a mathematical topic, ‘describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind...as far as this description is concerned with the learning process of the young generation’ (p.ix). That is, it addresses mathematical topics and concepts not just in terms of their formal logical structure, but also from the perspective of the human activities from which these generalisations arise, alongside the particular issues of their development within teaching and learning.

RME also emphasises the importance of particular models. Streefland has shown how some, such as the bar model in relation to proportion problems, can be effective in encouraging generalisation from below (see Van Den Heuvel-

\textsuperscript{3}It is assumed here that many of these pedagogical approaches, and ideas within MER over the historical period before 1997, were influential on the authors who originated the term ‘connectionist’, being, as they were, key actors in the debates around mathematics education at the time. Similarly, the influences of Vygotsky’s work, both directly and indirectly, on the authors is likely not to have been negligible.
Panhuizen, 2003). The structure of such models lie close to the structure of initial problems and contain key aspects of the structure of more formal mathematical approaches, supporting the classic transition from being a ‘model of’ to a ‘model for’ as it is used in an increasing variety of contexts. More generally within RME, a key part in the increasing formalisation of vertical mathematisation is played by reflective abstraction – student reflection on their own (and other students’) concrete mathematical productions (e.g. Streefland, 1991, p.31), particularly through acts of comparison (e.g. p.319).

These elements of RME resonate well with connectionist practice. However, one drawback of RME is its uni-directional view of mathematical development. This is seen within views of progressive vertical mathematisation (Treffers & Vonk, 1987, p.62) or the influential Van Hiele (2004) level model (the Van Hieles were PhD students of Freudenthal). This assumes that lower levels of mathematics must be mastered before more complex forms are attempted (see e.g. Freudenthal, 2002, p.97). The reflective reproblematisation (e.g. through a comparison of models), which can make practice more conscious through requiring generalisation, is often in danger of being postponed too long, perhaps indefinitely within a unidirectional vision of ‘gradual progression’ (e.g. Streefland, 1991, p.19).

2.4.2 Situated cognition and outside school mathematics

Current understandings of outside school mathematics are rooted in the challenge by Lave (1988) and others to the dominant cognitivist separation of cognition from activity: “what you learn is bound up with what you have to do” (Scribner, 1985, p.203). Theories of situated cognition argue that the form that mathematical activity takes is highly situation dependent and distributed across mind, body, activity, other people, artefacts, setting and so on. The richer meaning and complexity of activity outside of school therefore meant there was a need to reverse the relative marginalisation, or outright dismissal, of the mathematical activity of everyday life. The arithmetic that, for example supermarket shoppers (Lave, Murtaugh, & de la Rocha, 1984), or street-sellers (Carraher, Carraher, & Schliemann, 1985), engaged in was not only qualitatively different but was also found to be more accurate and fool-proof than when the same subjects engaged in apparently isomorphic school mathematics. In the outside world, people were more “in control of their activities, interacting with the
setting, generating problems in relation with the setting and controlling problem solving processes” (Lave, 1988, p.70), using other resources more, and arithmetic less, but in a more integrated and meaningful way.

Studies of mathematics in the workplace were integral to this wider category of everyday or street mathematics and were the source of many findings. For example, Nunes, Schliemann, and Carraher (1993, p.126) found “both flexibility and transfer were more clearly demonstrated for everyday practices than for the school-taught proportions algorithm,” when investigating proportional knowledge in the workplace. Such practices could utilise and preserve meaning due to their derivation from activity which has a purpose, allowing social and empirical rules to be utilised alongside logical relationships, thus increasing the complexity that could be dealt with and decreasing the errors. Similar findings have been noted in a variety of vocations, for example, within nursing (Noss, Pozzi, & Hoyles, 1999), where a practical meaning of the notion of an average is seen to be more efficient and effective than the school mathematics versions due to it being ‘webbed' together with practical and professional expertise.

In general, in these types of mathematical thinking which are tied to perception and action, conscious awareness and control of the mathematics tends to be very limited. Abstraction or elements of abstract systems are seen, however, to aid limited forms of generalisation to contexts with similar elements, as in Nunes et al. (1993), or in the situated abstractions of Noss, Hoyles, and Pozzi (2002). The question that arises from this is whether more conscious mathematical activity can arise, either outside school or within schooling itself.

In contrast to everyday and workplace mathematical activity, the situated cognition literature above has variously characterised school mathematics as inauthentic, procedural, calculation driven, detached from meaning, passive, formal, formulaic, algorithmic, learned by rote, and lacking specific purpose. In reality, these accurate descriptions of transmissionist pedagogy point also to similarities between school and situated mathematics (see Greiffenhagen & Sharrock, 2008). Repetitive practice of short meaningless procedures encourages situated associative learning rather than conscious and critical thought.

The positive impact of the criticisms of schooling from a situated cognition perspective has been the encouragement of efforts to bring the concrete reality of the workplace into schools as curriculum tools so as to encourage more
meaningful and purposeful activity there (Wake & Williams, 2000; Williams & Wake, 2007a), although with the understanding that transition between contexts is problematic (Nicol, 2002; Straesser, 2000). Such approaches are then also seen as better preparation for the reality of the workplace (e.g., Bakker, Kent, Derry, Noss, & Hoyles, 2008). In addition, vocational education can design approaches and tools which more efficiently develop situated knowledge within the workplace (e.g., Bakker, Groenveld, Wijers, Akkerman, & Gravemeijer, 2014).

However, alongside this it has been suggested that changes in the demands of the modern workplace require a rethinking of the relationship with school, at least for a minority. The black-boxing of mathematics in artefacts (Latour, 1987) and in activity systems, (see, e.g., Williams & Wake, 2007a) is seen to be problematic as the nature of demands on employees changes. “Making the invisible visible,” by opening up these black boxes, could then help improve efficiency, production and profitability (Bakker, Hoyles, Kent, & Noss, 2006). It is argued, for example, that a key skill-deficit among mid-level employees is not so much in performing calculations, but in understanding systems, particularly when development or communication with others is required: There is at times a need to “understand, at some level, the model behind a given symbolic artifact” (Hoyles, Noss, Kent, & Bakker, 2010, p.173). Although this view of the modern workplace can be questioned, it does again open opportunities for encouraging more conscious mathematical activity inside and outside school.

2.5 Particular concepts

There are two concepts explored in greater depth within the thesis, through analysis of their development within classroom dialogue, in relation to the elements of Vygotsky's theory of scientific concepts and connectionist pedagogy. These are the heuristic concept of being systematic, and the general concept of proportion.

2.5.1 Being systematic

Polya’s 'How to solve it' (2004), originally published in 1945, is usually taken as the starting point of a conscious attention to heuristics within mathematical problem solving (although he himself traces some of their earlier intellectual roots,
Described as ‘mental operations typically useful in solving problems’ (p.130), and ‘provisional and plausible reasoning’ (p.113), heuristics are contrasted with the certainty of complete solutions and deductive proofs but are seen as providing the scaffolding for such proofs (p.113). ‘Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science’ (p.117). This places heuristics not just at the heart of mathematical activity but also at the heart of the formal mathematical system itself. This is even more so given Lakatos’ (1976) arguments that the deductivist style of presentation of proofs provides only an illusion of certainty, and hides their heuristic development (and potential future development) through, for example, counterexamples, refutations and criticism (p.142).

Since Polya, Schoenfeld’s work on explicit instruction in heuristics has perhaps been the most influential in the field of mathematics education (see Schoenfeld, 1992, for an overview within the wider problem solving context). This has shown that Polya’s heuristics are more complex than they initially appear and so are often too general to be implemented without reduction in their complexity (see, e.g., Schoenfeld, 1992, p.353), but also that explicit training in the use of heuristics does have the potential to improve problem solving performance (see, e.g., Schoenfeld, 1979). Schoenfeld (2013) continues to defend the role of explicit heuristic instruction as part of a more developed approach to problem solving; however, doubts have been raised about its effectiveness (English, Lesh, & Fennewald, 2008). It is suggested that there is a problem of concentrating on rule governed processes in problem solving, that the relationship between concept development and heuristics, beliefs, dispositions or processes is underdeveloped, and that there is an assumed ordering where concepts are learned first, then heuristics and processes, and only then can these be put together (and therefore that in practice this point is rarely reached). Thus English et al. argue for the criticality of research into the nature of the relationship between concept development and the development of problem solving competencies. Problem solving is viewed as ‘integral to the development of an understanding of any given mathematical concept or process’ (p.7) and a modelling perspective (which includes as part of it ‘identifying, integrating, modifying, or refining sets of mathematical concepts drawn from various sources’, p.8) is viewed as central to any problem solving activity. All these points can be agreed with. One path toward this, suggested here, is a simple one: to view
heuristics themselves as concepts in the Vygotskian sense. They are
generalisations of problem solving activity which includes within them their
relationships to both mathematical systems and practical (and mathematical)
experience. As will be seen (ch.4), in Vygotsky’s terms they are scientific
concepts as their relationship to the object (here, mathematical and practical
activity) is mediated by other concepts within a definite system.

The particular heuristic concept of ‘being systematic’ appears within Polya and
Schoenfeld’s work. Polya points out that analogical conclusions drawn from
parallel cases are stronger if those cases are systematically arranged rather than
random collections. Varying the problem in a systematic way (illustrated by Polya
with an example where n is taken as 1, 2, 3 etc.) is shown to be a key form of
induction, which can lead to (the similarly named but different process of)
mathematical proof by induction (Polya, 2004, p.43). Schoenfeld echoes Polya
within his own problem solving strategies (1979, p.79), with encouragement to
attempt this approach in the presence of an integer parameter (this requirement
is necessary for the basic form of proof by induction but not for the wider process
of induction e.g. integer values can be drawn from measurements in order to aid
pattern spotting). A second use of ‘being systematic’ in this context is as a means
of ensuring completeness, and avoiding repetition, when attempting to exhaust
all possible cases (see Schoenfeld, 2010, p.99). This usually involves using
structure and order to break down the cases into more manageable groups.

2.5.2 Proportion

There has been much research on the conceptual intricacies of, and pedagogical
approaches to, multiplicative and proportional reasoning. Two of the most useful
long-term projects are that of the Rational Number Project in the U.S (see, for
example, Behr, Harel, Post & Lesh 1992) and the work of the Concepts in
Secondary Mathematics and Science Project in the U.K., and its follow-up
projects (see Hart, 1981 & 1984; Hart, Johnson, Brown, Dickson & Clarkson,
1989). A third major influence on thinking in this area is the didactical
phenomenology of Realistic Mathematics Education (RME) (Freudenthal, 1983;
Streefland, 1991), and it is from this work that features relevant to the classroom
are drawn.
Of the many insights into fraction, ratio and proportion derived from this approach, the following are particularly relevant to the analysis of classroom dialogue which follows in chapter 7:

i. There is a common, early, practical-based sense of proportion and comparing ratios, (see e.g. Brink & Streefland, 1979);

ii. It is possible to equate two ratios without contemplation of reducing those ratios to a number or magnitude (Freudenthal, 1983, p.180);

iii. The processes involved in i) and ii) are easier in the visual field than with magnitudes (p.189);

iv. There is a strong conceptual interrelation of fractions and ratio (p.134). Indeed, the interrelation of a wide range of concepts in this field (including part-whole, decimal, ratio, proportion and linearity etc.) is hugely important (see Kieren, 1976; Vergnaud, 1988);

v. Ratio represents a higher level concept than fraction because comparison is an essential part of the concept (Freudenthal, 1983, p.180);

vi. Fractions are the ‘phenomenological source’ of rational numbers (p.134) and often dominate, and reduce the complexity (and understanding) of, classroom work on ratio and proportion;

vii. Conceptualisation of fractions is in turn dominated in practice (p.145) and in pedagogy (p.147) by the limited part-whole relationship;

viii. There is an important distinction, and shift in difficulty, between ratio comparisons which remain within one system of measurement (internal) and those which also involve a transformation between measurement systems (e.g. from time to distance – external);

ix. The early sense insight of ratio and proportion can become blocked by the learning of algorithms which are prone to automisation (p.209).

These insights are only a small part of the vast literature on what is a central topic in school mathematics. They appear here simply because they have some direct relation to the discussions of chapter 7. That chapter is not particularly intended to be a contribution to the knowledge of this field, but rather aims to use
and integrate existing knowledge within a more systematic approach to pedagogy (and similar points could be made in relation to heuristics and chapter 8). The quality and detail of the insights therefore also serve to emphasise the importance of previous MER within a general strategy of systematisation.

The first three points in the list indicate that there are early forms of practical understanding of ratio, yet ratio is seen in the fifth point to be a higher level concept. This hints that such higher level concepts may have a development which overlaps temporally with the development of lower level concepts thus opening up the potential to a bi-directional relationship between them, and the possibilities within pedagogy of utilising this relationship to assist concept development.

### 2.6 Marxism, alienation and the commodity form

To arrive at a full understanding of concept development or mathematics pedagogy, attention to cognitive processes or processes of learning and teaching must be embedded within the wider contexts of schooling and capitalist society. The perspective taken here on these questions, Marxism, provides a natural complement to the included psychological theories. Vygotsky, and his writings, must be seen as emerging from, and remaining within the wider Marxist tradition (see for example, Wertsch, 1985, p.10; Blanck, 1990, p.40). He was explicit in his support for the concerns of the then recent workers’ revolution in Russia, both in private (Van der Veer & Zavershneva, 2011, p.466), and in his writing (see e.g. Vygotsky, 1994b), and his collected works are littered with quotations from, and engagement with, the ideas of Marx, Engels, Lenin, Plekhanov and others (see e.g. Au, 2007, on the relationship between Vygotsky’s and Lenin’s writings on consciousness; or to Marxism more generally, Fu, 2012). It is therefore not difficult to embed Vygotsky's efforts in psychology within the wider understandings of society provided by Marxism – he himself took such understandings for granted.

In the 1844 manuscripts Marx (1992) famously discusses some of the implications of satisfying human needs through a system of generalised commodity production. Instead of a process of engaging in direct, conscious, collective activity to satisfy needs, as in, say, humanity's pre-historic hunter-gatherer past, the current form of social organisation of production mediates the
link between activity and needs. Workers produce, but the product they produce is not theirs to use. The majority sell their ability to work to the minority who have the means to organise the production of commodities, in essence turning themselves, or their ability to work, into a commodity. In exchange workers get money, with which they can go and buy other commodities. Marx argues that this alienation from the product of labour also means alienation from the process of labour (through its disconnection from immediate needs, and the lack of control of the process), from what it is to be human (i.e. conscious social producers), and therefore from other humans (whether within production or between either side of the production consumption divide). Many of these aspects of alienation also apply in earlier forms of class society where the product of labour is taken from the producer, for example, when a peasant works part of the year on the estate of their feudal lord. As Marx argues in Capital (1982), what differentiates capitalist society is that the social relations involved in production become hidden:

The mysterious character of the commodity-form consists therefore simply in the fact that the commodity reflects the social characteristics of men's own labour as objective characteristics of the products of labour themselves (p.164) ... It is nothing but the definite social relation between men themselves which assumes here, for them, the fantastic form of a relation between things (p.165) ... they do not appear as direct social relations between persons in their work, but rather as material relations between persons and social relations between things (p.166).

This *commodity fetishism*, in a society dominated by commodity production, and where human beings are themselves objectified in the commodity labour power, has a profound effect on all aspects of human culture, including education. '[T]he problem of commodities must not be considered in isolation or even regarded as the central problem in economics, but as the central, structural problem of capitalist society in all its aspects' (Lukacs, 1971:83). A final factor relevant to education, that itself arises from the alienation of genuine social relations, and the competitive nature of capital, is the state, or what Marx calls the ‘illusory community’ (Marx & Engels, 1974, p.83). The state gives the impression of standing over society and being neutral, of representing the common good, but in reality it represents the interests of capital against other classes, as well as mediating conflicts between capitals when necessary. It also performs, or attempts to perform tasks that are in the general interests of capital, the things that companies cannot, or do not want to do individually, such as ensuring the
general availability of the commodity labour power at sufficient levels and in sufficient quantities.

Ideas within society are fundamentally shaped by practice (Marx & Engels, 1974, p.47). The generalisations made from experience are intertwined with the dominant ideology (the generalisations from practice made from the perspective of those who dominate society, p.65) to form what Gramsci (1971) calls 'common sense' (p.419). Although the process is both complex and non-mechanical, it is far-reaching. The dominance of the commodity form of production and the resultant objectification, alienation and atomisation underpin modern conceptions of the individual and theories which take the individual as their starting point (Meszaros, 1970, p.254). They also have a reciprocal relationship with the rationalism and reductionism of the scientific revolution which accompanies the rise of capitalism (Lukacs, 1971, p.230). And finally, they help to reduce our understanding of knowledge, an active relationship with the world, to that of an object, something that can be taken out of one person's head and slotted into another's.

Some of the implications of this for education and mathematics education in particular have been well explored by Lave and McDermott (2002), Williams (2011) and Jones (2011). However, it is proposed here that there is a need to restructure such understandings more systematically around the unifying factor of commodity production, in order to allow a more systematic integration with a developed Vygotskian understanding of mathematics pedagogy.

2.7 Summary

The main conclusions to be drawn from this review are that:

1. The relationship between Vygotsky’s theory of scientific concepts and mathematics education remains relatively underdeveloped despite the significant influence of Vygotsky’s writings on MER.

2. Connectionist pedagogy remains relatively under theorised and unsystematic, in part reflecting the relatively unsystematised and less coherent state of MER taken as a whole.

3. Given the situated cognition literature, the question of the existence of scientific concepts, outside or inside of school, requires some attention,
including explanations for their common absence.

4. Understandings of alienation within education may be restructured around the unifying factor of commodity production, and this may provide a basis for more systematic integration of these ideas with the theoretical understandings of pedagogy, and scientific concept development within the thesis. This may be particularly so given their common Marxist theoretical origins.

5. Much work exists within MER on particular concepts, and more generally, which can inform and be integrated into a systematic approach to concept development and pedagogy. At the same time this work can raise contradictions, or exhibit weaknesses, which can provide an impetus for theoretical development.

6. Higher level concepts have a history within individual development which may overlap with the development of lower level concepts raising the possibility of utilising the potential bi-directional relationship between them within pedagogy.

In what follows, chapter 4 addresses points 1 and 2 through the integration of Vygotsky’s theory of scientific concepts with understandings of connectionist pedagogy. Chapter 5 begins to address point 4 by outlining a view of alienation structured around the unifying factor of commodity production, and showing the relationship between this view and educational research. Chapter 6 continues this through further discussion of the obstacles to concept development, while also incorporating the understandings of chapter 4 to discuss the existence of scientific concepts.

Chapter 7 builds on the work of chapter 4 to explore the potential within pedagogy of consciously developing the bi-directional possibilities mentioned in point 6, and analysing the empirical data arising from just such an approach. Chapter 8 then builds on chapters 4 and 7 to illustrate and further develop the theoretical and pedagogical approach, through the tracing of the development of a particular concept over a series of lessons with one class.
3 Methodology and methods: Theory and practice, form and content

3.1 Introduction

In this chapter the methodological approach of the thesis is described. First in more traditional terms, and then, more deeply, in relation to Vygotsky’s theory of scientific concepts and the wider Marxist approach in which it is embedded. This deeper perspective situates the study in relation to questions of the relationship of theory and practice, and to totality approaches. It also suggests a particular relationship between form and content herein, where both educational research and mathematics education are seen as examples of the general case of development of scientific concepts. Thus the content and the method of this work overlap to a great degree. If the focus in the thesis is on concretising Vygotsky’s theory in relation to the learning of mathematics, then the methodological position must include establishing the mediations of the general theory which are particular to research activity.

3.2 Methodology, in standard educational research terms

In general terms, this study utilises a qualitative multiple-case study approach (Yin, 2003) due to the analytic benefits of the method and its potential for aiding theoretical generalisation. Various case studies contribute to the thesis: the outside-school mathematical activities of darts players, the views of mathematics and mathematics teaching of secondary school students, the classroom activities of potential mathematics teachers on a pre-teacher training mathematics enhancement course, and, central to the thesis, an undergraduate course on the teaching and learning of mathematics. This latter course was both effectively a pre-pre-teacher training course for potential primary school teachers and a vocational course for potential social workers whose future jobs may intersect in some way with the world of education, through support roles inside and outside formal schooling.
The first two of these cases involved an ethnography-light approach (i.e. in adopting ethnographic practices but without full immersion or long term engagement). Such an approach, when used by some theoretical anthropologists, has been termed ‘shabby fieldwork’ (Greenwood, 2000, p.178). To help overcome dangers of superficiality, the method is combined here with a suitable intensity of empirical data collection and analysis. With this proviso, this method opens up research to many of the benefits of ethnography, in being empirically grounded and open to discovery in the field (Baszanger & Dodier 1997). Within mathematics educational research it allows the productive interrelation of socio-cultural theory and attention to cognitive processes, and therefore has the potential to connect such methodology to future practical interventions (Eisenhart 1988). A more direct relationship to practical intervention is seen in the final listed case study. This case, the central case of the thesis, used an adapted form of instructional design methodology⁴ (see Cobb, Confrey, Lehrer & Schauble, 2003; Van den Akker, Gravemeijer, McKenney, & Nieveen, 2006).

The remaining case, which focuses on secondary school students, relied on a simpler qualitative interview approach. This data was only one element of the original mixed-method project from which it emerged. The methodological uses it is put to here are distinctive, and are outlined and discussed in chapter 5.

### 3.3 Theory and practice and totality in Marxism and educational research

Questions of methodology are at heart questions of the relationship of theory to practice. Marxism has a distinctive position in regard to this question. The starting point for this can be taken from the famous theses on Feuerbach, firstly (thesis ii),

> The question whether objective truth can be attributed to human thinking is not a question of theory but is a practical question. Man must prove the truth — i.e. the reality and power, the this-sidedness of his thinking in practice. The dispute over the reality or non-reality of thinking that is isolated from practice is a purely scholastic question (Marx, 2005, p.422).

⁴ The adaption lay primarily in avoiding some assumptions which may come with the terminology i.e. in not viewing instructional design as directly comparable to a science or engineering experiment (see, for example, the discussion on critical realism, sec.3.7) and without a pragmatist orientation toward the relationship between theory and practice (see, sec.3.3).
And then (thesis xi),

The philosophers have only interpreted the world, in various ways; the point is to change it (p.423).

This second is less a rejection of philosophy as such, but a reinforcing of the second thesis’s emphasis on a dialectical unity of theory and practice (Gramsci, 1995, p.384). For the natural world, Engels (1941) highlights experiment and industry, as two key means by which humanity can come to understand something through trying to change it:

If we are able to prove the correctness of our conception of a natural process by making it ourselves, bringing it into being out of its conditions and making it serve our own purposes into the bargain, then there is an end to the Kantian ungraspable “thing-in-itself” (Engels, 1941 p.22).

The “thing-in-itself” became a thing for us (p.23).

Within educational research this perspective is on the surface similar to the approach of pragmatism. But this approach can either reject theory in favour of ‘what works’, or can extend ‘what works’ to theory itself, and refuse to systematise or make coherent, theoretical perspectives. For example, within the ‘turn to phronesis’ (Kessels & Korthagen, 2001), there is a rejection of formal abstract theory or rules and an emphasis instead on practical wisdom or perceptual knowledge. This is echoed within some instructional design methodology literature, which is often explicit in its pragmatist origins (e.g. Cobb, Confrey, Lehrer & Schauble, 2003), and its eschewing of grand theories in favour of practical-based knowledge. Lesh and Sriraman (2010), for example, contest that, ‘No single “Grand Theory” is likely to provide realistic solutions to realistically complex problems’ (p.126). Cobb (2007) is most explicit about this perspective as being the pragmatism of Dewey. He takes as examples experimental psychology, cognitive psychology, sociocultural theory, and distributed cognition and argues they are contradictory but all useful in their own way depending on what is being looked at, and encourages ‘bricolage’ – that is, taking what you need when you need it.

Dewey himself was contradictory in relation to the role of theory. See, for example, his (1904) views on teacher education where he argues for an initial avoidance of practical experience lest the wrong practical wisdom be learned. A Marxist approach, although sharing an emphasis on practice with pragmatism, is clear on not rejecting theory in favour of generalisations from immediate practice. ‘All science would be superfluous if the form of appearance of things directly
coincided with their essence (Marx, 1981, p.956). Science must go beyond the 'visible and merely apparent' to trace in a detailed way the inner connections between things (p.428).

The accumulated knowledge which is the result of scientific activity must then be taken as the basis of any future investigations, but with two provisos. First, with an awareness that even the most formal of scientific results, in mathematics or logic still ultimately derive from practice,

[M]an’s practice, repeating itself a thousand million times, becomes consolidated in man’s consciousness by figures of logic. Precisely (and only) on account of this thousand-million-fold repetition, these figures have the stability of a prejudice, an axiomatic character (Lenin, 1976, p.216).

And second, that such findings must then be relatable back to practice:

[E]very truth, even if it is universal, and even if it can be expressed by an abstract formula of a mathematical kind (for the sake of the theoreticians) owes its effectiveness to its being expressed in the language appropriate to specific concrete situations. If it cannot be expressed in such specific terms, it is a byzantine and scholastic abstraction, good only for phrasemongers to toy with (Gramsci, 1971, p.201).

This perspective also does not reject the idea of one grand theory, but instead strives to totalise knowledge, on the grounds that reality itself is an interrelating whole:

The totality of all sides of the phenomenon, of reality and their (reciprocal) relations—that is what truth is composed of. The relations (= transitions = contradictions) of notions = the main content of logic, by which these concepts (and their relations, transitions, contradictions) are shown as reflections of the objective world. The dialectics of things produces the dialectics of ideas, and not vice versa (Lenin, 1976, p.196).

Any particular investigation should as far as possible 'examine all... facets,... connections and mediacies...[and] a full “definition” of an object must include the whole of human experience, both as a criterion of truth and a practical indicator of its connection with human wants (Lenin, 1965, p.94). Ultimately, the attempt to move beyond contradictory theories and incorporate them in a unified perspective is also a struggle to go beyond contradictions in reality (Gramsci, 1971, p.445). For Marxism, the primary concern is overcoming the internal contradictions of capitalist society, and a truly total perspective can only be
attained by the majority as a collective force in the process of transforming that society (see Lukacs, 1971).

This appears some distance from the immediate concerns of the thesis. However, many echoes of this understanding of theory, practice and totality will reappear reformulated in Vygotsky's own work on scientific concepts (for his relationship with Marxism see sec.2.6). Also, this understanding is reflected within the methodology of the study. The thesis avoids a purely contemplative approach, and interrelates theory and practice, in three key ways. First, through the main case study involving the design and teaching of a course based on theoretical understanding. Second, in the further development of the theoretical framework through the experience of that design and teaching, and through analysis of the empirical data produced. And third, through embedding the project within the wider Marxist framework of societal transformation and taking a critical mathematics perspective. In general terms this third aspect acted as an overarching factor mediating all empirical and theoretical work, and, in more particular terms shaped the emphasis in the thesis on Vygotsky's theory and its integration with theories of alienation and the commodity form. The development of this work is implicitly subordinated at all points to the wider question of what those who are critical of society are to do within the field of mathematics education, and this issue is addressed more explicitly within the conclusion.

3.4 Form = content, Vygotsky's theory of scientific concepts

This thesis explores and theorises the teaching and learning of mathematics. Although embedded in wider understandings of schooling and society it argues for a view of learning mathematics as primarily a form of the development of scientific concepts. As such it places Vygotsky's theory of scientific concepts at its heart and develops it into a theorisation of meaningful pedagogy. Vygotsky's theory is not specifically a theory of pedagogy, but of psychological and cognitive development (from a social and cultural perspective). Education does play a key role in the theory, and discussions of schooling and classroom learning are threaded throughout his works on concepts, but the theory requires some translation, and the reshaping through different mediations, to turn it into a theoretical underpinning of pedagogy. This work is done throughout the thesis, but centrally, in chapter 4. In shorthand, as will be seen there, the development of
scientific concepts can be viewed as the dialectical mutual mediation of systems of abstract thought with the concrete, that is, the real world of activity, objects, perception and such.

Educational research, equally, can be viewed as the development of scientific concepts. If learning is viewed as not just the memorisation of the formal results of the thinking of others, but as a more active, investigative process, then the development of new understandings in research can be seen as a similar process to learning but with greater cultural novelty. Again, it involves the dialectical mutual mediation of systems of abstract thought with the concrete (a view which incorporates elements of both the interrelation of theory and practice, and, through emphasis on the systemic nature of thought, totality). Vygotsky’s general theory is therefore useful here also, with suitable adaptation to the particular mediations of research. This leads to a synergy in the thesis between its form, or methodology and its content, leading to a richer coherence.

To explore the implications of Vygotsky’s theory in relation to research requires pre-empting some of the later theoretical results of the thesis (particularly those of ch.4). This breaks somewhat with the normal linear, serial presentation of a thesis, but such linear presentations are artificial. In practice, questions and answers evolve and develop together, as do readings of literature, developments of methodology and findings. Fuller explanations of the pre-emptions follow in later chapters.

The essence of Vygotsky’s theory, in relation to the development of scientific concepts, is made up of the following elements:

1) That concepts form and develop within genuine problems and real social activity.

2) That concepts represent a merging of abstract and concrete.

3) The importance of the fact of a scientific concept’s place within a definite system.

4) The relationship between everyday and scientific concepts.

5) The key role of social dialogue

6) The relationship of conscious awareness to generalisation within systems.

For the first element, in the most general terms the thesis addresses the genuine problem of making sense of mathematics education, and slightly less generally, of both theorising connectionist pedagogy, and through mathematics pedagogy,
developing existing theory. Already, this brings in the third element: Vygotsky’s work and wider Marxist theory exemplify a pre-existing definite system to which developments are being made. Similarly, the intended outcome is a systematisation of concepts of meaningful pedagogy. Genuine problems and real activity also bring in the second element of the relationship of abstract and concrete. The concrete here, again in general terms, is schooling and teaching, particularly the author’s own experience. This pattern exists too in more particular terms, in the more specific problems through which these more general questions are approached. For example, the specific problem of evidencing the existence of something beyond situated thinking within outside school (and school) activity, or of probing the elements of concept development within classroom dialogue. Each genuine problem involves a fusion of the developing abstract system with rich concrete detail.

At each stage of examining or engaging in practice and the concrete one is met with generalisations made from that practice. From the sense that is made by darts players or school children of their own activity, through to the common sense of progressive teachers and educationalists, and beyond to other theorisations of pedagogy, these mediate the relationship between the concrete world of activity and the more coherent and systematic generalisations of Vygotskian theory. This is element four within the outlined system. How does this mediation occur? Only through element five, dialogue. To which can be added the dialogue specific to the academic world, the presentation and discussion of papers, the reading groups, the attempts to formulate in writing, all of which help fuse the abstract and concrete, and systematise it. The result of all of these elements is element 6, consciousness and understanding.

This analysis of research, and in particular the research conducted here, is of course only an overview. The thesis is concerned with the theorising of teaching and learning of mathematics not the theorising of educational research, and there is not the space to do otherwise. However, the aim of this overview is to stress the coherence of the approach taken here. The argument that the form and content of the research are in many ways the same is itself an example of the generalisation within systems and the embedding of the particular within a totality. And as such it aims to add weight to the arguments which follow within the study.

This question of the multiple levels at which Vygotsky’s theory works within the thesis is returned to when the main case study of the research is described in
more detail below. The methods of data collection and analysis used there, and within the other case studies, is now related.

3.5 Methods

There are four case studies which contribute to the thesis. The first, a study of the mathematics of darts players, was conducted as a contribution to a Master’s degree in Educational Research, but was simultaneously a part of the wider project of which this thesis plays a central role. The second study, an investigation of a connectionist classroom of adult learners on a pre-initial teacher training mathematics enhancement course, was designed as a pilot for the collection and analysis of classroom dialogue and other student output. The third case arose from a separately funded project which investigated links between pedagogy and student dispositions within secondary school mathematics classrooms, and which therefore mainly encountered and discussed transmissionist practices. The final and main case study involved the design and teaching of a short course for undergraduates on the teaching and learning of mathematics. This course featured engagement with connectionist pedagogical activities and reflections upon that activity. The methods of data collection and analysis for each of these cases are outlined in the following sections.

3.5.1 Ethnography of darts players

This study adopted an ethnography-light approach (i.e. in adopting ethnographic practices but without full immersion or long term engagement). Darts players were observed during practice and tournaments, and interviewed both in relation to incidents which occurred during observation and regarding their wider practice. Autobiographies of professional darts players, online discussion forums, cultural artefacts and televised high-level darts matches also provided useful material for analysis, context and triangulation of findings.

Brief individual and group individual interviews were conducted during observations of tournaments, club matches and practice sessions of darts players at varying levels of ability. Five, longer, semi-structured interviews were conducted with participants, two of which were followed up with a second
interviews. A focus on interviews followed Scribner’s (1985) approach to ethnography of workplace mathematics, and incorporated elements of Witzel's (2000) problem-centred approach, in form, and, in its aim of theory development (see, Flick, 1998). The interviews often centered on breakdown moments seen in observation or recalled from the participant’s history, to make the reasoning involved more explicit (as in other workplace studies including Pozzi, Noss & Hoyles, 1998; Williams, Wake & Boreham, 2001; Williams and Wake 2007a, b). Data analysis borrowed from grounded theory the pattern of coding the categories and their interconnections, establishing the key categories and generating theory (Robson 2002: 493). However, the typically strict rules for coding in this methodology were rejected as too mechanical. Also, theory played a far more conscious role from the beginning of the study, both in terms of the developed understandings of situated cognition, and the understanding of the potential of cognition which lay beyond the situated based on Vygotsky’s notion of scientific concepts. For example this shaped the original decision to search in a vocational area where the mathematics seemed potentially complex in comparison to previous studies in situated cognition.

3.5.2 A connectionist classroom

This case study focused on adult mathematics on a Mathematics Enhancement Course (MEC). MECs are aimed at adults who wish to progress onto a PGCE Mathematics course (or other secondary mathematics ITT pathway) but who (generally) have a degree in a subject other than mathematics. The students typically spend five days a week, for 26 – 36 weeks (although much shorter courses do exist), primarily learning mathematics, but with some classes on other educational aspects and forms of teaching practice activities.

The data comes from an MEC group undergoing a module in investigative, problem-solving mathematics, chosen for study because of the connectionist pedagogical approach of the tutor. With a total of 23 students sat around four or five tables, tape recorders were placed on all group tables to capture both whole class and group discussions. Each class was also observed with notes taken of participant position, board activities, the nature of the activity, levels of engagement and moments of potential research interest. Audio recording led to large quantities of data for potential analysis. Each class lasted approximately
two and a quarter hours and up to half of each class involved discussion within groups. This process produced around six hours of material for each session, with fourteen sessions in total.

Audio recording does limit the access to students’ non-verbal communication including body and facial gesture and writing. In the first instance, instructional sequences where writing was at a minimum were investigated to simplify the analysis. Beyond this, the choice of which sequences to analyse was primarily informed by theoretical concerns, aided by a detailed breakdown of the form and content of activity in each session.

After initial analysis based on the theoretical framework, brief interviews were held with both the class tutor and the participants. Again these were conducted under the influence of Witzel’s approach to problem-solving interviews (2000). Discussions with the tutor and students allowed for correction of errors in transcription, the ability to challenge any inferences within the analysis, and some collective analytical development. These interviews also provided extra layers of primary data both in seeing the state of students’ conceptualisation in a later and different context, and by bringing in deliberate meta-cognition in the form of reflexivity of the form and content of the discussion under analysis. Alongside this, all other relevant materials, including artifacts, and student output were also collected or recorded where possible.

3.5.3 Transmissionist practice – Teleprism

This third of the minor case studies relies on data from the Teleprism project, a large scale ESRC funded mixed-method project (award ref: RES-061-25-0538 www.teleprism.com) investigating the relationship between pedagogy and the dispositions and attitudes of students in relation to mathematics. The project includes a large scale survey, representing the views of over 13 000 secondary school mathematics students, and their teachers, from over 40 schools. Alongside this, the project includes a range of qualitative work, primarily interviews and classroom observations.

In such a mixed method study there is a danger that the qualitative research, rather than bringing in new dimensions or depths which narrow quantitative instruments cannot provide, is instead overly shaped and restricted in the image of the quantitative instrument. As a counter to this, an initial attempt was made to
identify elements of the interviews which did not relate to, or went beyond, the questions and themes of the quantitative questionnaire. From this process a series of fragments emerged which evolved into the data found in chapter 5. As that chapter engages substantially with methodological questions, including greater detail of the process above, further description of methods is postponed until later.

3.5.4 A course in Teaching and Learning Mathematics

A short course for undergraduates in teaching and learning mathematics provided the main data collection activity for the thesis. Vygotsky’s theory interrelated with the research, design and activities of the course at three key levels, along the lines described in section 3.4. First, in relation to development of the thesis, the overarching aim when designing the course was to concretise the theory as developed to that point, with the assumption that doing so would lead to further development in the theoretical perspective. Designing a course from a theoretical perspective a) was a genuine problem; b) involved the interrelation of the systemic abstract with the concrete (for example, in that the reasons for the course existing within the university structure, and the needs of the students within that, asserted themselves as a priority over the author’s need to produce a PhD); c) involved dialogue – with the students, co-tutors, peers, supervisors, and indeed with existing literature, both about the planning and the outcomes of the course; d) was done (as was the data analysis that followed) in relation to a theoretical system, (which) e) allowed reflection, generalisation and meta-cognition (see, for the results of this process, ch.4, 7 & 8).

The second level at which the theory operated was in relation to exploring the teaching and learning of mathematics. The students on the course faced the real problem of trying to understand the processes involved in mathematical activities (and to plan and write about such activities); this was done in relation to concrete experience – that is, through engaging collectively in those mathematical activities; this primarily involved dialogue as a class, in groups and in pairs; it was done in relation to a system (i.e. the interrelating themes and concepts of the course); and this produced reflection and generalisation in relation to mathematics, their own teaching and learning, and teaching and learning in general (see ch.7).
This pattern was repeated at the third level of the mathematical activities the students engaged with as part of the course. These primarily took the form of problem solving activities, often relating to physical or social experience and involving dialogue and justification, and with an emphasis on mathematical connections. And, again this was designed to produce generalisation and conscious awareness within mathematics (see ch.7 & 8).

3.5.4.1 Course details

The teaching and learning mathematics course was aimed at two particular groups of undergraduates. The first group were students on an English language and literacy course who were aiming to progress onto a PGCE course in primary education following their degree. These students were particularly concerned to develop their thinking in relation to mathematics (often a concern for primary teachers (see, e.g. Uusimaki, & Nason, 2004; Peker, 2009), but were also interested in wider processes of teaching and learning. The second group were from a youth and community work course. In their intended future job roles (and already within the work placements on their course) these students expected to assist those who were struggling with mathematics or schooling in general. This could occur through informal situations in the community, semi-formal support connected with schools or more formal support within classrooms, or separately in conjunction with a trained teacher. All students on the course had achieved at least a grade C in GCSE mathematics, although often this had been some years ago, and many had low confidence in their mathematical abilities. In broad terms, the potential primary teachers were younger and had a higher level of mathematical attainment. The youth and community workers were generally older, were less confident mathematically, but brought a higher awareness of social issues related to education due to the nature of their course and their own personal experience.

The theoretical framework shaped the structure of the course and the nature of the activities undertaken. It was assumed that the types of activities undertaken within the class and the emphasis on dialogue would lead to rich data for the thesis. Therefore, the ultimate motivation of thesis production could, to a large extent be put to one-side during the detailed planning and teaching of the course so as to distort the educational experience as little as possible. The theoretical
framework, however, naturally continued to inform the author during both design and implementation as the understandings and concerns of any teacher would.

The course contained seven weekly two-hour sessions. The first five sessions were designed around the key themes of the theoretical perspective. The final two sessions took place several weeks later and were concerned with consolidation and assistance with the students’ design and analysis of their own mathematical activities linked to the assessment task for the course.

Session 1:

The first session focussed on two key aspects: i) problem solving and modelling, and ii) concepts. The first aspect was addressed through the mathematical activity of making a bungee jump from elastic bands. In groups, students were given weights and a small number of elastic bands to experiment with before making a prediction of the number of bands they would need for a larger drop to be effective. The second key aspect, of concepts, was addressed through groups first attempting to make as many connections as possible to the word ‘circle’, before a loose classification, reflecting some of the themes of the course, was attempted. These mathematical activities were followed (as in all sessions) by discussion and reflection on the mathematical activities, and on issues which arose in relation to teaching and learning in general. At different points in the course such reflection took place in pairs, groups and as a class, as well as individually in reflective diaries.

Session 2:

The overall theme of this session was connecting mathematics to the real world, which involved four main tasks: 1) Students discussed three things they had learned, in different aspects of life, and how they had learned them. 2) A problem from the Die Hard series of films, where exactly four litres of water had to be made using containers which held three and five litres. This problem had multiple extensions, and students were initially provided with model containers and water. 3) A class discussion on how much mathematics we actually use in the real world. Students had been asked in the previous class to bring examples from their own
experience to this session. 4) A class task of making a throwing game fair leading to the emergence of a circle. This was extended through various shape drawing tasks, including the construction of parabolas.

Session 3:

The theme of this session was dialogue, communication and justification. The tasks included: 1) Communicating what they had learned over the last week about parabolas in poster form. 2) Communicating and replicating a pattern of blocks, or drawings with geometric features (of varying difficulty), in pairs, while sat back to back. Levels of freedom to communicate for the person replicating were varied from not being allowed to talk, to being allowed to question, to full dialogue being permitted. 3) A problem of sharing out chocolate bars, through choosing the table that would lead to receiving the most. This involved various problems of communication and justification, but particularly around how to convince someone that three-quarters is larger than two-thirds.

Session 4:

This session was the first of two on the connections within mathematics. Its main focus was on heuristics in problem-solving. The main tasks were 1) Finding the number of squares on a chessboard. 2) Matching a list of heuristics to problems previously encountered on the course. 3) Making bridges from spaghetti. Groups were given some weights and some strands of spaghetti to experiment with, then had to predict how much weight a bridge made of twenty strands of spaghetti would be able to hold. In this session, students also attempted to teach a partner a mathematical concept which they had learned about over the previous week.

Session 5:

The tasks in this second session on the connections and systemic nature of mathematics included: 1) Physically grouping together on the basis of finding connections between concepts, with each student using their concept from the previous session. 2) Problem of $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ in context of exam questions 3)
Generalising proportion problem, based on finding similarities between various proportion related GCSE maths topics 4) Making a concept map or mind map of the course using themes, concepts and problems.

Alongside the problems outlined above, and the reflection and discussions on teaching and learning which punctuated them, each week students were provided with some key readings which related to, illustrated, or extended the themes of the course.

3.5.4.2 Data collection and analysis

The most important data collection in this case involved the audio recording of all dialogue produced during the run of the course. Students were grouped on four or five tables during each class and a recorder (or two) was placed on each table. Other student output was also collected including images of work, a reflective diary produced by each student and student assessments. The final elements of data included the author's production and design notes for the course and reflections on each session.

The recordings collected represented a vast amount of audio from six two hour classes, large portions of which involved group discussions on individual tables. A period of immersion in the data then followed and a categorised database of what was contained within the recordings drawn up including, for example, reference to the mathematical, pedagogical and social nature of activities, the levels and range of engagement, critical moments, and key utterances. This immersion combined with three particular areas of interest arising from ongoing theoretical engagement and development to select the data used in the thesis. These were: 1) a concern to explore concept development over time, 2) the importance of generalisation, and 3) the role and interrelation of three aspects of metacognition – attitudes and dispositions towards mathematics and learning, metacognition of the process of doing mathematics, and mathematical generalisation itself. The first two areas led to the selection of particular concepts as explored in chapters 7 and 8. Due to reasons of space, exploration of the third topic will be explored and developed outside of the thesis.

In analysing the classroom dialogue a decision was taken to not over-categorise the dialogue in the first instance. Some important and useful work on mathematics dialogue has taken detailed categorisation as a basis (see for
example Wells & Arauz, 2006). To utilise a pre-existing categorisation such as
this was viewed as leading to the imposition of too many conscious and
unconscious theoretical assumptions on the analysis, and also to an
unnecessarily closed analysis, limiting the flexibility of a more open approach
open to multiplicity of meaning and interrelations. Any attempt to invent a new
categorisation at a similar level of detail would suffer similarly. The key
categorisation used involves instead the more general elements of Vygotsky's
theory, combined with the usual human sense-making abilities, given sufficient
contextual detail and objective utterances. As a more long term project develops
around the themes of this thesis, further more detailed mediations and
categorisations are expected to continue to arise, as they begin to do in later
chapters.

3.7 Validity

Given the complexity of the issues discussed in the thesis, and the inability to
replicate the investigations, doubt is inevitably raised regarding validity (see
Foster and Hammersley, 1998, p.610). Standard conceptualisations of
generalisability are unattainable using the methods employed here, for example,
in sampling based on a mixture of perceived theoretical need and convenience
(Schofield, 1993, 92), therefore the research remains at an exploratory level and
any external validity relies on its theoretical cogency (Silverman, 2005, p.136). An
alternative, but in practice equivalent perspective, would be to view any
generalisations as analytic rather than statistical (Yin, 2003, p.32) with their
validity reliant on the insight they bring to case studies (p.38).

Of the philosophical perspectives on validity more common academically than the
form of Marxism expressed in sec.3.3 and 3.4, critical realism perhaps has most
in common (see Bhaskar & Callinicos, 2003 for a discussion on their
complementarity). This offers a fuller justification of seeing validity arise through
the explanatory value of the theory produced. Critical Realism emerges within the
philosophy of science with the publication of Bhaskar's (2008) 'A realist theory of
science'. It argues that objective generative mechanisms, or structures, underlie
the events experienced by humans; that in interacting these tendencies generally
do not come to fruition in pure form; but that their existence can be inferred
through deliberate human interaction with the world, generally through
experiment (Callinicos, 2008, p.569). In extending the critical realist approach to social science (Bhaskar, 1998), it is again assumed that objectively real social structures exist antecedent to human actors which both constrain and empower them (Archer, 1995, p.42). However, social science is seen to be more limited in its ability to form closed systems (in which to test generalisations and uncover generative structures) in comparison to physical sciences which can do so through, for example, experimentation (Callinicos, 2008, p.572). Social structures cannot be separated in the same way from the events and activities they govern, nor from the conceptions of the human actors involved. Therefore, although theories within social science can and should be tested empirically, it is argued that an emphasis in judging them must shift from predictive accuracy to explanatory power (Bhaskar, 1998, p.194).

3.8. Ethics

The research herein conformed to the Revised Ethical Guidelines for Educational Research (BERA, 2004). Although the research deals only with adults (except the case study explored in chapter 5, a separate project which underwent its own ethical process and provided this work with secondary data), relates to non-contentious issues, and in its main case involved the author's own professional practice, it is nonetheless vitally important to ensure issues such as voluntary informed consent, right to withdraw, and right to privacy are at all times stressed where appropriate.

The central case study, the design and teaching of an undergraduate course in the teaching and learning of mathematics involved much collective investigation and reflection on pedagogy and mathematics. The idea of researching such activity therefore seemed natural to the students, and this greatly assisted in voluntary participation in the project. A central ethical danger of this case was the author's potential conflict of interest in the dual aims of assisting the students in their learning, and the production of adequate empirical data for the thesis. However, the course was designed as it would have been had the needs of empirical data been absent, and the needs of the thesis ignored during the teaching of the course. Connectionist practice, as the course was both about and using, encourages dialogue and reflection, therefore there seemed little risk to the thesis in pursuing this approach.
All participants were provided with information about the research in clear non-academic language, and informed consent was obtained using standard consent forms as seen in appendix 1. Paper, computer files and audio recordings have been subject to all reasonable security measures to protect confidentiality and privacy, and pseudonyms have been used at all stages of the recording and reporting of the data. Findings of the study will be shared with participants in an appropriate form.
4 Vygotsky's theory of scientific concepts and connectionist teaching in mathematics

David Swanson

Abstract: Various aspects of Vygotsky's theory of scientific concepts have proven fruitful in mathematics education research, particularly in regard to the interrelation of scientific and everyday concepts and the role of dialogue. It is argued here that Vygotsky's theory offers more than this however, in that it provides the outlines of a systematic and coherent theoretical basis for connectionist teaching practices, and a justification for why such a theoretical basis would be useful.

The elements of connectionist mathematics teaching, which are common to many approaches which oppose the dominant transmissionism, include: an emphasis on problem solving; connecting mathematics to physical and social experience; a focus on the connections within mathematics; the use of dialogue and justification; making connections with students' current understandings; and the encouragement of reflexivity. These pedagogical elements are seen to map neatly across to the central aspects of the theory of scientific concepts. This paper outlines and explores those aspects of Vygotsky's theory, relying on his own words where possible, and draws out their connections to mathematics teaching. In addition a key question arising from previous literature linking Vygotsky’s theory to the mathematics classroom is addressed in order to deepen the analysis: the seeming contradiction between cultural internalisation and student agency.

A proposed outline of a systematic theoretical basis for connectionist (and other non-transmissionist) teaching, based on Vygotsky’s theory, emerges through this engagement with his own and other’s writings, and in conclusion the benefits to pedagogy and research of having such a basis are drawn out.

Keywords: Vygotsky, scientific concepts, mathematics pedagogy, connectionist teaching
4.1 Introduction

Various aspects of Vygotsky's theory of scientific concepts have proven fruitful in mathematics education research; particularly in regard to the interrelation of scientific and everyday concepts and the importance of social dialogue (e.g. see Yoshida, 2004; Renshaw & Brown, 2007). It is argued here that Vygotsky's theory offers more than this however, in that it provides the outlines of a systematic and coherent theoretical basis for progressive pedagogical practices, and a justification for why such a theoretical basis would be useful. The much maligned, yet dominant in practice, rote-learning devoid of meaning which has been exacerbated in recent times by an increased emphasis on exam results and atomisation of the curriculum (e.g. see Gainsburg, 2012), was clearly prevalent in Vygotsky’s (1994) time too:

Educational experience, no less than theoretical research, teaches us that, in practice, a straightforward learning of concepts always proves impossible and educationally fruitless. Usually, any teacher setting out on this road achieves nothing except a meaningless acquisition of words, mere verbalization in children, which is nothing more than simulation and imitation of corresponding concepts which, in reality, are concealing a vacuum. In such cases, the child assimilates not concepts but words, and he fills his memory more than his thinking. As a result, he ends up helpless in the face of any sensible attempt to apply any of this acquired knowledge (p.356).

But more than sharing an awareness of the problem, Vygotsky also developed a theory containing key elements which can be seen to map across to the main aspects of pedagogical approaches which oppose the dominant transmissionism and encourage more meaningful activity in the mathematics classroom. There have, of course, been many such approaches, at different times and in different places, in both research and practice. One such approach, connectionist teaching (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997), although relatively undeveloped theoretically, exemplifies many of the elements found in these and provides a terminology which fits aptly with the theory to be discussed below. To illustrate this, the main aspects emphasised by connectionist teaching are listed (on the left) next to the central strands of Vygotsky’s theory of scientific concepts:

- Meaningful problems
- The relationship of concepts to activity and real problems
Connecting mathematics to the real world

The relationship between, and merging of, abstract and concrete in concepts

The connections within mathematics

The importance of the scientific concept’s place within a definite system

Connecting to students’ existing knowledge

The relationship between everyday and scientific concepts

Dialogue, reasoning and justification

The key role in concept formation played by the word and dialogue

Reflexivity

The relationship of conscious awareness and control to generalisation within systems

An initial comparison of these two lists may already provoke an awareness of the similarities between them. The aim of what follows is to draw out and deepen these connections and, in doing so, to show how Vygotsky’s theory is well placed to provide an effective theoretical justification for, and explanation of, the relative success of connectionist, and similar, teaching methods (Askew et al., 1997; Swan, 2006; Pampaka et al., 2012). To assist in that process, substantial use will be made of Vygotsky’s own writings. The use of such extensive quotations perhaps requires some justification. First, it acts as evidence for this central claim of the article. In what follows there is already a translation in focus, from Vygotsky’s central concern of understanding how social and cultural factors lead child development, to the issue of concept development within the classroom. To add an additional layer of reinterpretation of the particular points he makes would create an unnecessary distance from his work. In addition, Vygotsky’s writings are quite accessible, and it is hoped that direct experience of this encourages others to engage more substantially with the original writings.

The analysis is then supplemented with discussion of a question arising from previous attempts to connect Vygotsky’s theory to mathematics teaching: the seeming contradiction between cultural internalisation and student agency. In total, it is argued that Vygotsky’s theory provides a systematic theoretical basis for connectionist teaching, and in the conclusion the benefits to pedagogy, and research, of having such a basis are drawn out.
4.2 Scientific concepts

The term *scientific* can often carry implications of rationalism, empiricism or, perhaps, a narrow restriction to the physical sciences. This was not Vygotsky’s intention, as is partly indicated by his inclusion of the terms ‘exploitation’ and ‘class struggle’ under the designation (Vygotsky, 1987, p.215). What Vygotsky does mean by scientific, however, will take time to unfold through all that follows.

For Vygotsky (1998), thinking in concepts represents ‘a new and higher form of thinking’ which appears in adolescence due to the influence of external culture, and in particular, schooling:

> [W]hen educational material is assimilated that consists for the most part of general positions that express some law or rule, through the influence of speech, attention is diverted more and more in the direction of abstract relations and thus leads to the formation of abstract concepts (p.38).

This new content ‘of necessity requires [the adolescent’s] transition to new forms, and places before her problems that can be resolved only through the formation of concepts’. This embedding of concepts, and the integrated development of thought, attention, memory and will (see Vygotsky, 1997b), in wider social processes and relationships prevents the theory from becoming cognitivist despite its clear focus on cognition.

The importance of the social world posing problems in leading development is an approach shared by connectionist pedagogy, and this is a useful starting point for exploring the links between connectionism and Vygotsky’s theory.

4.2.1 Meaningful problems

In teaching mathematising ‘the real world’ is represented by a meaningful context involving a mathematical problem. ‘Meaningful’ of course means: meaningful to the learners. Mathematics should be taught within contexts, and I would like the most abstract mathematics taught within the most concrete context (Freudenthal 1981, p.144).

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5 Liberties have been taken here to occasionally modify the original translations of general pronouns in Vygotsky’s work, from he/his to she/her, for a more gender-balanced approach.
The importance of using stimulating problems, whether real in the literal sense of real-world or real in the sense of meaningful and genuine is hopefully obvious in terms of helping to engage and motivate learners. For Vygotsky (1987), it is also essential for concept development, in creating a need for the concept:

The concept does not live in isolation... it is not a congealed, static formation but a formation that is always encountered in the vital and complex process of thinking. A concept always fulfils some function in communication, reasoning, understanding, or problem solving (p.123).

This functional role of the concept means that it cannot appear simply through associative connection:

[T]he concept arises and is formed in a complex operation that is directed toward the resolution of some task... In itself, learning words and their connections with objects does not lead to the formation of concepts. The subject must be faced with a task that can only be resolved through the formation of concepts (Vygotsky, 1987, p.124).

It has frequently been illustrated within the situated cognition (Lave, 1993) literature that humans can, and frequently do, find all sorts of associative connections to more simply and effectively achieve their aims (see, for example, Hoyles, Noss & Pozzi, 2001, on how nurses deal with proportion and medicine in practice). To go beyond this, amongst other things, requires a good reason or a failure of those associative methods to work (e.g. see the investigative mathematical practices of some darts players, ch.6). It can be argued that the transmissionist approach in the classroom with its repetitive practice of short, meaningless questions generally encourages a situated, associative approach by students as this is usually sufficient for the task in hand. Greiffenhagen & Sharrock (2008) for example, have shown the similarity of school and situated mathematics (however, without the argument here that there is something that lies beyond both). The use of meaningful problems, on the other hand, is more likely to stimulate the need for, as well as providing the motivation for, concept development.

4.2.2 The abstract-concrete relationship

Attempting to connect formal mathematics to the real world within pedagogy has interrelated aims – to aid motivation by making the mathematics more meaningful, and to increase understanding (see, for example, Gainsburg, 2008). Ignoring
typical word problems, which are often artificial (Verschaffel, Greer, & De Corte, 2000) and where the content tends to be ignored by students (Greer, 1997), uses of the real world can range from Freirean links to social inequality (e.g. Gutstein, 2008), or connections to other subjects of interest to students' everyday lives (e.g. McIntyre, Rosebery & González, 2001). It can also include modelling or grounding mathematics directly in social, physical and visual experience (as in, for example, children experiencing the concept of circle as the loci of fair throwing points in a game, see Doig, Groves & Fujii, 2011).

In Vygotsky's theory, the relationship between the abstract and the concrete is central to understanding the nature of concepts. By concrete, Vygotsky essentially refers to physical, spatio-temporal and visual perception and direct experience of social activity:

[F]or the child in particular, the concept is linked with sensual material, the perception and transformation of which gives rise to the concept itself. This sensual material and the word are both necessary for the concept's development (Vygotsky, 1987, p.111).

The emphasis Vygotsky places on the relationship between abstract and concrete in concept development is pre-figured in his earlier writings on the science of psychology:

[To] every ultimate concept, even to the most abstract, corresponds some aspect of reality which the concept represents in an abstract, isolated form. Even purely fictitious, not natural-scientific but mathematical concepts ultimately contain some echo, some reflection of the real relations between things and real processes, although they did not develop from empirical, actual knowledge, but purely a priori, via the deductive path of speculative logical operations. As Engels demonstrated... even such an obvious fiction as zero, i.e., the idea of the absence of any magnitude, is full of properties that are qualitative, i.e., in the end they correspond in a very remote and dissolved form to real, actual relations. Reality exists even in the imaginary abstractions of mathematics (Vygotsky, 1997a, p.248).

At the same time,

[E]ven the most immediate, empirical, raw, singular natural scientific fact already contains a first abstraction... Physical body, movement, matter – these are all abstractions. The fact itself of naming a fact by a word means to frame this fact in a concept, to single out one of its aspects; it is an act toward understanding this fact by including it into a category of phenomena which have been empirically studied before... Everything described as a fact is already a theory (p.249).
This emphasis on the role of the system in which a concept belongs appears more strongly in the later phase of Vygotsky's work on concepts, where a differentiation occurs between everyday and scientific concepts. In the earlier phase, experimental analysis leads Vygotsky (1987) to an analytical categorisation of pre-conceptual forms in development which are distinguished by the nature of the generalisation involved. This includes syncretic generalisation which is based on subjective factors, various forms of complexes which rely on objective connections in perception and activity, and pseudo-concepts which are externally similar to concepts but still based on concrete connections. ‘True’ concepts are then distinguished by generalisation on the basis of abstraction.

The unique intellectual formations present in the pre-adolescent period are, in fact, functionally equivalent to the true concepts that mature later. They fulfil a function similar to that of concepts and function in the resolution of similar tasks. However, experimental analysis indicates that their psychological nature, their constituents, their structure, and their mode of activity differ significantly from those of the true concept (p.130).

In tracing these earlier forms the merging roles of generalisation and abstraction are uncovered:

The concept ... develop[s] along two different channels. First... the function of combining or connecting a series of separate objects through a common family name is basic to the child's complexive thinking. This constitutes the first of the two channels... [P]otential concepts, concepts which are based on the isolation of several common features, develop in parallel with complexes and constitute the second channel. These two forms constitute the dual roots of concept formation (Vygotsky, 1987, p.165).

In developed conceptual thought these two paths of abstraction and generalisation merge. What is important for Vygotsky about this form of abstraction is that it should not be seen, as in the typical view of abstraction, as simply the shedding of non-essential features:

the claim that the abstract thinking of the adolescent breaks away from the concrete, and the abstract from the visual is incorrect: the movement of thinking during this period is characterized not by the intellect's breaking the connections to the concrete base which it is outgrowing, but by the fact that a completely new form of relation between abstract and concrete factors in thinking arises, a new form of their merging and synthesis (Vygotsky, 1998, p.37).
The concrete still remains part of the abstract but a new relationship based on the inner connections between things is also fused with it. In this understanding, the development of abstract concepts necessarily requires the existence of rich, concrete experience from which to abstract and generalise from. Pedagogies which attempt to transmit an understanding of formal mathematics without this inevitably suffer:

It is completely clear that if the process of generalizing is considered as a direct result of abstraction of traits, then we will inevitably come to the conclusion that thinking in concepts is removed from reality. … Others have said that concepts arise in the process of castrating reality. Concrete, diverse phenomena must lose their traits one after the other in order that a concept might be formed. Actually what arises is a dry and empty abstraction in which the diverse, full-blooded reality is impoverished by logical thought (p.53).

Instead:

A real concept is an image of an objective thing in its full complexity. Only when we recognize the thing in all its connections and relations, only when this diversity is synthesized in a word, in an integral image through a multitude of determinations, do we develop a concept. According to the teaching of dialectical logic, a concept includes not only the general, but also the individual and the particular.

In contrast to contemplation, to direct knowledge of an object, a concept is filled with definitions of the object; it is the result of a rational processing of our experience, and it is a mediated knowledge of the object. To think of some object with the help of a concept means to include the given object in a complex system of mediating connections and relations disclosed in determinations of the concept. Thus the concept does not arise from this as a mechanical result of abstraction – it is the result of a long and deep knowledge of the object (p.53).

This important perspective will be returned to, and reinforced, in a later section on the ontology of concepts. However, counter-arguments exist to enriching mathematical activity with strong concrete context. One is the ability of a minority of students to succeed in its absence, or, the activity of the many professional mathematicians who get by just fine without any thought of the real world (see, e.g. Burton, 1998). This is arguably because devoting large amounts of time to mathematics (see Butterworth, 2006) is enough to make it as richly experiential and concrete as the world outside of mathematics is for the majority. Most though are alienated from mathematics long before this can occur.
A second more complex counter argument is that an over-reliance on the concrete in classroom settings (e.g. through the manipulation of collections of objects, or other physical and visual models) can inhibit the need for developing specifically mathematical or logical forms of thought (see for example, Davydov, 1990, p.59). Again, the situated cognition literature reinforces this point with its many examples of the success of thinking tied to context which can remove the necessity of going further. Careful choice of models which are more likely to be generalised (see Van Den Heuvel-Panhuizen, 2003), or of problems which require more developed concepts to solve, offer two potential paths for moving beyond this. Another, though, lies in encouraging conscious attention to generalisation and the system to which concepts belong.

### 4.2.3 Connections in a system

Ideas required for understanding a particular topic turn out to be basic for understanding many other topics too...Unfortunately the benefits which might come from teaching them are often lost by teaching them as separate topics, rather than as fundamental concepts by which whole areas of mathematics can be interrelated (Skemp, 1976).

In a recent research project involving the author, one secondary school student, when asked which area of mathematics she enjoyed the most, responded, 'I don't mind doing brackets'. Aside from the admirable unwillingness to be too enthusiastic, the notion that 'brackets' are an area of mathematics offers an illustrative reflection of the atomised nature of the curriculum. Meaningful, open-ended problems are one method of overcoming this 'bite-size' approach, for example, in giving the opportunity to compare alternative methods. Simply making the connections between various aspects of the curriculum would in itself be a helpful starting point though. To give one example, by rough calculation around half of the GCSE mathematics curriculum in the UK directly involves proportional thinking. This includes equivalent fractions, pie charts, similar triangles, best buys, trigonometry and other traditional topics. Yet the potential connections between these are missed in a fragmented approach to the curriculum.

Vygotsky's theory points to the importance of these systemic connections in concept development. Scientific concepts are seen not only to merge abstract
and concrete, as described in the previous section, but also to include the relationships between concepts:

The concept does not emerge in the child’s mind like a pea in a sack. Concepts do not lie alongside one another or on top of one another with no connections or relationships…In contrast to what is taught by formal logic, the essence of the concept or generalization lies not in the impoverishment but in the enrichment of the reality that it represents, in the enrichment of what is given in immediate sensual perception and contemplation. However, this enrichment of the immediate perception of reality by generalization can only occur if complex connections, dependencies, and relationships are established between the objects that are represented in concepts and the rest of reality. By its very nature, each concept presupposes the presence of a certain system of concepts. Outside such a system, it cannot exist (Vygotsky, 1987, p.224).

One simple example which illustrates this:

The group of nine on playing cards is richer and more concrete than our concept of ‘9’, but the concept ‘9’ involves a number of judgements which are not in the nine on the playing card; ‘9’ is not divisible by even numbers, is divisible by 3, is $3^2$, and the square root of 81; we connect ‘9’ with the series of whole numbers, etc. Hence it is clear that psychologically speaking the process of concept formation resides in the discovery of the connections of the given object with a number of others, in finding the real whole. That is why a mature concept involves the whole totality of its relations, its place in the world, so to speak….We see that the concept is a system of judgements brought into a certain lawful connection: the whole essence is that when we operate with each separate concept, we are operating with the system as a whole (Vygotsky, 1997a, p.100).

At the heart of systematisation is again the act of generalising. To give the most simple example, if the word ‘dog’ represents the generalisation through abstraction of the things that are dogs, the word ‘animal' in turn generalises this generalisation.

A new stage in the development of generalization is achieved only through the reformation—not the nullification – of the previous stage. The new stage is achieved through the generalization of the system of objects already generalized in the previous stage, not through a new generalization of isolated objects. The transition from preconcepts (e.g., the school child’s arithmetic concept) to true concepts (e.g., the adolescent’s algebraic concept) occurs through the generalization of previously generalized objects (Vygotsky, 1987, p.230).
Building this intricate, interrelated conceptual system is a key task in mathematics education. Systematising generalisations, such as algebra, are seen by Vygotsky as allowing more freedom of movement in thought:

The process involved in the liberation from links with the numerical field occurs differently than the process involved in the liberation from links with the visual field. The growth in freedom that occurs with the emergence of the algebraic generalization is explained by the potential for reverse movement from the higher stage to the lower that is inherent in the higher generalization; the lower operation is already viewed as a special case of the higher. Arithmetic operations are preserved even after algebra is learned...Research indicates that the adolescent views the arithmetic concept as a special case of the more general algebraic concept. Research also indicates that operations with the arithmetic concept become freer. Because of its independence from particular arithmetic expressions, it is applied in accordance with a more general formula (p.230).

In a later section, this positive effect of systematisation is explored further, first, in terms of how the higher generalisations reach down and reshape what is generalised, and, second, in Vygotsky's equating of generalisation with conscious awareness and control. Before moving on however, it is worth reflecting on some points related to the following quote:

In school, the child does not learn the decimal system as such. He learns to write numbers, add, multiply, and solve problems. Nonetheless, some general concept of the decimal system does develop (Vygotsky, 1987, p.207).

This generalisation 'from below' is important in two senses. For Vygotsky, higher level generalisations are introduced to children primarily through schooling, and this forms part of the more general process of social and cultural factors leading development. However, there is also an awareness of the active role that students play which means the theory isn't solely about internalisation of pre-existing cultural forms. An overemphasis on one direction of this process and the downplaying of the other can lead to problems (see sec.3.6 for a fuller discussion). Generalisation from below also has implications for understanding more limited pedagogies. For analytical purposes, this article as a whole paints a picture of two extremes in pedagogy, connectionism and transmissionism. In practice, many of the aspects attributed to connectionism appear in its portrayed opposite in diluted form. So, for example, in terms of systematisation even the worst textbook will structure the material in some kind of logical way, building on
what has come before, and connected topics will be met temporally close to one another even if the links between them are not made explicit. Given the 'real problem' of making sense of the material, some students will be able to generalise to some extent (see, e.g. Davydov, 1990, p.61). So the real argument contained here is not that generalisation, or for that matter understanding mathematics, is impossible with transmissionist practices, but rather that there are more effective means of encouraging generalisation from below and understanding.

4.2.4 Word, Dialogue, Justification

Words (and other signs) play a key role in Vygotsky's theory. 'Psychologically, the development of concepts and the development of word meaning are one and the same process' (Vygotsky, 1987, p.180). Word meaning should be viewed 'not only as a unity of thinking and speech but as a unity of generalization and social interaction, a unity of thinking and communication' (p.49). Words themselves are not concepts, but rather are the means by which concepts are formed (p.126). However, in development, it is the changing role of the word which, for Vygotsky, is the key factor in the emergence of conceptual thought, as 'the means through which the adolescent masters and subordinates his own mental operations and directs their activity in the resolution of the tasks that face him' (p.131). Words do this through acting as a 'means of actively directing attention, partitioning and isolating attributes, abstracting these attributes, and synthesizing them' (p.130). The genesis of this, as for all 'higher psychological processes', is social (Vygotsky, 1997b, p.104). Words act as a means of interaction and mutual understanding between more and less-developed conceptual forms (Vygotsky, 1987, p.144) within the genuine problems of communicating and understanding in social activity (p.123), allowing concepts to emerge.

Understandings of the importance of communication and dialogue within the mathematics classroom are well developed within research literature, both in terms of the interrelation of communication and conceptual thought, and in promoting an active and more equal role for students (see, e.g. Sfard, 2008). Within connectionist teaching there is a particular emphasis on dialogue which involves reasoning and justification (Askew et al., 1997, p.32), often as part of problem solving and modelling (see, e.g. Ryan & Williams, 2007). This aspect of
pedagogy connects to the relationship between dialogue and systemic generalisation in Vygotsky's theory:

To communicate an experience of some other content of consciousness to another person, it must be related to a class or group of phenomena... this requires generalization. Social interaction presupposes generalization and the development of verbal meaning; generalization becomes possible only with the development of social interaction (Vygotsky, 1987, p.48).

Although justification and argument aren't particularly discussed, the distinction Vygotsky does make between oral and written speech (Vygotsky, 1987) has relevance. Writing is seen as requiring a higher level of abstraction due to its lacking of intonation and expression, and the absence of an interlocutor with shared contextual knowledge. Also, the more immediate motivational factors present in speech are seen to be absent:

With every moment, the situation that is inherent in oral speech creates the motivation for each turn of speech; it creates the motivation for each segment of conversation or dialogue. The need for something produces the request. The question creates the answer. The expression brings the retort and the failure to understand the clarification. A multitude of similar relationships between speech and motive are fully determined by the situation inherent in real oral speech. Thus, oral speech is regulated by the dynamics of the situation. It flows entirely from the situation in accordance with this type of situational-motivational and situational-conditioning process. With written speech, on the other hand, we are forced to create the situation or – more accurately – to represent it in thought (p.203).

This distinction seems a useful one, and the developmental role of this generalisation of generalisations would apply to written mathematics also. However, some forms of oral speech lie closer to written forms, and similar processes are at work. In reasoning, justification and argumentation immediate contextual factors are present, but these are also related explicitly to elements of mathematical systems. The connections and generalisations which have to be made in justifications act similarly to writing in encouraging representation in thought at a distance from immediate contextual factors. At the same time they retain, among the other benefits of collective engagement, some of the dynamism of oral speech as described by Vygotsky (see e.g. Alrø & Skovsmose, 2002; Weber, Maher, Powell & Lee, 2008, for some rich examples of the interplay of these aspects).
4.2.5 Conscious awareness and control

Intimately connected to dialogue and reasoning within connectionist teaching is an emphasis on reflexivity. Discussions of the appropriateness of strategies and the act of explaining and comparing methods encourage students' conscious awareness and greater control of their mathematical activity, as well as of their learning in a wider sense. This reflexivity, or metacognition, is in turn intimately connected with generalisation and the systemic nature of concepts.

In Vygotsky's terms, conscious awareness represents the generalisation or abstraction of internal mental forms of activity (Vygotsky, 1987, p.190) and therefore systematisation:

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[I]f a higher concept arises above the given concept, there must be several subordinate concepts that include it. Moreover, the relationships of these other subordinate concepts to the given concept must be defined by the system created by the higher concept. If this were not so, the higher concept would not be higher than the given concept. This higher concept presupposes both a hierarchical system and concepts subordinate and systematically related to the given concept. Thus, the generalization of the concept leads to its localization within a definite system of relationships of generality. These relationships are the foundation and the most natural and important connections among concepts. Thus, at one and the same time, generalization implies the conscious awareness and the systematization of concepts (Vygotsky, 1987, p.191).
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This means that,

\[
[O]nly within a system can the concept acquire conscious awareness and a voluntary nature. Conscious awareness and the presence of a system are synonyms when we are speaking of concepts, just as spontaneity, lack of conscious awareness, and the absence of a system are three different words for designating the nature of the child's concept (p.191).
\]

This encourages the practice of paying some attention to the systemic connections of mathematics within pedagogy. The generalisation, for example, of proportional thinking from the various topics mentioned earlier can lead to greater awareness, freedom of operation and control in thinking about those particular topics, in a similar way to Vygotsky's example cited earlier on the relationship between algebra and arithmetic. Mathematical reasoning, contrasting of methods
and also exploration of misconceptions are also part of this relationship between reflexivity and systematisation. For example:

For contradiction to be sensed, the two contradictory judgments must be viewed as particular cases of a single, more general concept. As we have seen, this type of relationship among concepts is absent where concepts are not included in some system. It is, indeed, impossible (Vygotsky, 1987, p.235).

In many ways it is difficult to separate mathematical metacognition, in the sense of hierarchical systemic concepts, from the metacognition of doing, or learning, mathematics. As Veenman et al. (2006, p.5) point out, 'It is very hard to have adequate metacognitive knowledge of one's competencies in a domain without substantial (cognitive) domain-specific knowledge'. The connections in the different forms of metacognition are underlined by Vygotsky’s argument that awareness and control, in a general sense, are brought in, developmentally, through the introduction of scientific concepts, primarily in educational settings. However, it is also worth considering what the relevant system is in the case of learning mathematics, and how much conscious awareness and control of learning can arise if the system is relatively undeveloped. The logic of Vygotsky’s theory implies that educational theory itself requires greater systematisation if it is to assist more in the practice of teaching. Such systematisation could also play a more direct role in learning for students, where discussions on what mathematics is, on what modelling involves, on how mathematics relates to the real world, on comparing strategies in problem solving and on what counts as a good mathematical justification etcetera can both make school mathematics more meaningful, and feed back into the development of content knowledge (see e.g. Lampert, 1990). Such approaches also encourage student agency and lay the basis for a more inclusive pedagogy (see Solomon, 2008, p.178).

4.2.6 The relationship between everyday and scientific concepts

The final connections to discuss are those between mathematics and students’ own knowledge. This has a strong overlap with the section on connections to the real world and the motivations which arise from relating mathematics to relevant aspects of students' lives. It also encompasses students' previous mathematical
conceptualisations, which can either be built upon or challenged. And, finally, it
includes relationships with the generalisations which students have made in their
everyday lives, either through their own activity or in to wider issues.

A starting point for exploring these connections is Vygotsky’s distinction between
everyday, or spontaneous, concepts. The key difference between the two forms
of concept being the presence or absence of a system.

Outside a system, the only possible connections between concepts are
those that exist between the objects themselves, that is, empirical
connections. This is the source of the dominance of the logic of action
and of syncretic connections of impressions in early childhood. Within a
system, relationships between concepts begin to emerge. These
relationships mediate the concept’s relationship to the object through its
relationship to other concepts. A different relationship between the
concept and the object develops. Supra – empirical connections
between concepts become possible (Vygotsky, 1987, p.234).

Everyday concepts generally arise in the immediate presence of the aspect of
reality they represent, even if, for the child, they are introduced by adults.
Scientific concepts on the other hand are initially introduced, particularly in
schooling, mediated by other concepts and when ‘the child’s thought is presented
with different tasks than when his thought is left to itself’ (p.178). This different
relationship to the object of the two types of concept means that the
developmental paths they follow are different.

When the child learns a scientific concept, he quickly begins to master
the operations that are the fundamental weakness of the everyday
concept. He easily defines the concept, applies it in various logical
operations, and identifies its relationships to other concepts. We find
the weakness of the scientific concept where we find the strength of the
everyday concept, that is, in its spontaneous usage, in its application to
various concrete situations, in the relative richness of its empirical
content, and in its connections with personal experience. Analysis of
the child’s spontaneous concept indicates that he has more conscious
awareness of the object than of the concept itself. Analysis of his
scientific concept indicates that he has more conscious awareness of
the concept than of the object that is represented by it (Vygotsky, 1987,
p.218).

However, in development, both types of concept are part of a unified process,
and are not separated in consciousness. The introduction of scientific concepts is
dependent on the sufficient development of the everyday concepts which
mediate their relationship to the object. Then, in the other direction, the ‘system
that emerges in the sphere of scientific concepts is transferred structurally to the
domain of everyday concepts, restructuring the everyday concept and changing its internal nature from above’ (Vygotsky, 1987, p.192). This two-way process is essential to Vygotsky’s argument that schooling (and therefore the social) leads development: ‘conscious awareness enters through the gate opened up by the scientific concept’ (p.191). But it also stresses that the sometimes initial verbalism of scientific concepts can be made more concrete through their use in activity.

For Vygotsky, by a certain point in schooling the two lines of thought have merged and the distinction between the two types of concept seems redundant. However, he also makes a point that seems to contradict this. That is, that ‘more elementary forms continue to predominate in many domains of experience for a long time’ (Vygotsky, 1987, p.160), and, ‘when applied in the domain of life experience, even the concepts of the adult and adolescent frequently fail to rise higher than the level of the pseudo-concept. This indicates that there is scope for the bi-directional process described in development to be an on-going one in education (see, for some examples of mathematical approaches to this, Renshaw & Brown, 2007; Boero, Pedemonte & Robotti, 1997). It also raises the question though of why such forms of thinking generally dominate. The answer to this is perhaps essentially contained in preceding sections. The situated cognition literature has been noted already and comparisons made with the memory-based empty formalism of the transmissionist teaching of mathematics. Schooling, and often everyday life, put tasks before the individual that are not, in general, of a nature to require much development or use of scientific generalisations and systematisation in Vygotsky’s sense. In much of day-to-day life this is not necessarily a negative thing, but it is in schools, which could and should be spaces for developing rich experience of this type of active and developed thought.

4.3 Internalising culture or active construction?

There is a seeming contradiction between two aspects of Vygotsky’s theory: a) the importance of the social and pre-existing culture in leading development, and b) the active role of individuals where concepts represent ‘a complex and true act of thinking’ in meaningful social activity. These have influenced two distinct paths in pedagogy influenced by Vygotsky’s work, one based on developments by
Davydov (1990), the other involving the addition of a more social emphasis to constructivism (e.g. Lerman, 1996; Jones, Jones & Vermette, 2010).

Davydov's pedagogical stance arose under the influence of the mathematical formalisations of Bourbaki around the same time as the New Math, to which it bears a resemblance (but with algebra replacing set theory as a starting point, see Davydov, 1975a). The approach involves beginning mathematics education with the most generalisable (content-based) abstraction, before 'ascending to the concrete' (Davydov, 1990, p.158). In this way aiming to maximise the benefits of the existing advanced culture akin to the impact of scientific concepts on everyday concepts as described in section 2.6. The method has been claimed to have had great success (see Davydov, 1975b, which also contains a more concrete description of the method). Translations into the English speaking world also appear to have been effective (see Schmittau, 2004, 2011; Kinard & Kozulin, 2008).

However, one advocate of the general approach has contrasted its ability to develop metacognitive skills unfavourably to guided discovery, or constructivist, approaches which involve more open problems (Karpov & Haywood, 1998, although this view is tempered, or reversed in later writing, see Karpov, 2003; 2013). Davydov (1990) himself argued that instructional material should be closer to the exposition of scientists' results than the investigative process which led them there, in order to provide students with the most advanced forms of generalisation available (p.160). At best students should be involved in quasi-investigation where 'in compressed, curtailed form, students reproduce the operations that lead, for instance, to delineating an abstract element in a system that is to be studied' (p.161).

There is sufficient evidence of the limitations of pure or minimally guided discovery (e.g. Askew et al., 1997; Kirschner, Sweller & Clark, 2006). However, to avoid any open discovery misses some important points. First, education should have wider aims than skill acquisition, and damage is caused by restricting student agency (Wrigley, 2006). Davydov's approach may move beyond transmissionism but still evokes a sense of filling pails rather than setting fires (to use the common paraphrase of Plutarch on education, 1992: p.50). But, more than this, as seen in section 2.6, Vygotsky's writings argue for a coincidence of systemic generalisation and conscious awareness and control. The metacognition of mathematical structure is intimately entwined with the metacognition of doing mathematics, and this is a two-way process. Attempting
to solve problems in wider contexts by drawing on conceptual structures can strengthen and extend those conceptual structures in qualitative ways. It isn’t so much that constructivism and sociocultural theory provide two halves of a good story that should be pragmatically pieced together (Cobb, 1994), but that sociocultural theory itself does and should contain both these aspects of the dialectic between social and individual, or structure and agency, as it does in Vygotsky’s theory.

This means that in mathematics pedagogy there should be space for both well-designed, highly structured tasks which encourage particular conceptual developments and for rich problems with more than one right answer, or more than one method of getting to the right answer, or space for the investigation to go off in a direction the teacher doesn’t expect. This is particularly true if more open tasks are supplemented with activities which involve paying conscious attention to systemic mathematical connections, by, for example, comparing and justifying solutions, methods, and models (see Boero et al., 1997, for one approach to this).

Finally, it should again be pointed out that differences in pedagogical approach are exaggerated here for analytical purposes. There is, of course, an ‘active side’ (Marx, 1992, p.442) to Davydov’s method (as there even is potentially, although less so, in transmissionism). Schmittau’s (2004, p.62) description of the approach as a ‘series of very deliberately sequenced problems that require children to go beyond prior methods, or challenge them to look at prior methods in altogether new ways’, sounds remarkably similar to, for example, the didactical phenomenology (Freudenthal, 1983) and the progressive mathematisation (Treffers & Vonk, 1987) of realistic mathematics education. In general, the similarities between non-transmissionist pedagogical approaches are, in practical terms, perhaps more important than their differences.

4.4 Conclusion

The aim of this paper has been to show, first, that the various elements of Vygotsky’s theory form an interrelated whole. His analysis of the origin and development of thinking in concepts leads to a view of scientific concepts as primarily representing generalisation through abstraction, where the abstract is not empty formalism but contains within it both the rich concrete which has been
generalised and systemic relationships with other concepts; that such
generalisations only arise when there is a real need in problem solving, justifying
or communicating in social activity; and that this generalisation represents
conscious awareness and allows conscious control over our thought and activity.

Secondly, this interrelated theory has been shown to map across to the key
elements of connectionist pedagogy, helping to begin to explain why and how
these elements work, or why learning may be less effective in their absence. In
doing so, an interrelated pedagogical system has been presented. This is not
intended to be prescriptive. There are many ways to bring genuine problems,
aspects of the real world, dialogue and attention to generalisation and systemic
connections into teaching. However, this is an argument for ensuring all those
elements are present and to see the ‘systems’, that is, of mathematics, of doing
mathematics and of learning mathematics, as interrelated and mutually beneficial.

The paper has used Vygotsky's writings as a basis, and in minor ways adapted
his analysis, in order to begin to structure a systematised understanding of the
learning of mathematics. In the process much has been ignored. First, in the
presentation of Vygotsky's ideas, the basis in experiments and the theoretical
justifications in his work have, in the main, been sacrificed in order to present the
structural essence of his argument. In discussing pedagogy, some statements
have been made rather too boldly about what is good and what is not. From
pedagogy also, certain key aspects have not been discussed, such as the use of
models (their mediating role between concrete and abstract, and, concept and
system deserve more space than is available here).

The justification for the overall approach has been to sustain an emphasis on
both the connections between the elements of the theory and the connections
between theory and practice. These equate to the two central strands of
Vygotsky's theory, the interrelation of abstract and concrete and the role of
systems, which it is suggested here apply to all forms of scientific activity (for
Vygotsky the interrelation of the systemic abstract with concrete practice, and
through that their mutual development, is scientific activity). In these general
terms the theory provides a basis for understanding not only learning, teaching
and doing mathematics, but also mathematic educational research.
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Addendum to chapter 4

This section was originally part of the preceding paper but was removed for publication purposes due to the length of the paper. It remains an important part of the overall argument of the thesis however, so is reproduced here.

Ontology

This paper has so far assumed that concepts exist, but what is the nature of their existence, and does it matter? One author who has produced useful work on the relationship between Vygotsky’s theory and mathematics pedagogy, Gordon Wells (e.g. 1994), has argued that scientific concepts can only be possessed by the wider culture and not by individuals (Wells, 2008). This argument is utilised for noble intentions, essentially to insist that students should be taught how to think for themselves, rather than be taught the end product of other people’s thinking. If we accept that individuals cannot possess concepts then this helps undercut notions of transmissionism in pedagogy and justifies an emphasis on motivated enquiry and problem solving.

However, there are problems with the starting point of this argument and Vygotsky’s theory itself provides an alternative perspective that leaves the useful conclusion intact. Wells’ discussion rests on accepting Popper’s (1978) division of the world into world 1 (physical objects), world 2 (thought processes) and world 3 (thought content), with the activity of individuals restricted to world 2. But this division simply extends dualism (‘dualism is not enough. We have to recognise world 3’, p.164) and leaves us with a transcendental platonic realm which is philosophically unpalatable.

As an alternative, section 2.2 outlined Vygotsky’s view of the abstract not as a shedding of the concrete but instead the bringing in of its relationships to other concepts. This allows a view of a cultural concept not as separated from the activities it arises in and from but instead as the structured totality of its relationships both to practice and other concepts (Hegelian immanence rather than Kantian transcendence). This is precisely the point Marx (1992) makes in regard to the human essence as being ‘no abstraction inherent in each single individual’. Instead, ‘In its reality it is the ensemble of human relations.’ (p.422). With words and signs playing a unifying role, we are able to reify and generalise
complex processes and relationships, and through this form and use concepts at a cultural level. But we do this also at an individual level. In this sense then it can be said that individuals have concepts. This still means that they are not objects that can be transmitted from one head to the other, but it does mean that their formation is inseparable from activities such as motivated enquiry.
Abstract: This article argues, first, that education is shaped fundamentally, and in a negative way, by the fact that human needs are currently met through a generalised system of commodity production; and second, that this understanding should be taken not as an optional addition, but as integral to educational research. The article begins with a methodological issue related to categorisations in mixed-method projects. Attempts within one study to overcome this unearthed many examples of interview fragments which expressed alienation. The influences of commodity production on mathematics education are described, and a sample of the data discussed in this light. Finally, various methodological conclusions are drawn, and it is argued that improvements in mathematics education require a wider critical perspective on society.

Keywords: Alienation; Commodities; Mathematics Education; Categorisation; Mixed-Method; Totality
5.1 Introduction

[T]he problem of commodities must not be considered in isolation or even regarded as the central problem in economics, but as the central, structural problem of capitalist society in all its aspects (Lukacs, 1971, p.83).

In modern society human needs and wants are met (or, often not met) through a generalised system of commodity production. This article will argue, first, that education is fundamentally shaped, and shaped in a negative way, by this fact; and second, that this understanding should be taken not as an optional addition, but as integral to educational research. This argument is illustrated through a particular research project, one investigating the relationship between the pedagogy and dispositions of U.K. secondary school mathematics students.

The starting point lies in some methodological issues which can arise in large-scale mixed-method studies, of which this project was an example. These problems relate to the narrowness of quantitative instruments and the limitations of qualitative categorisations which arise in relation to them, but these issues are generalised to wider questions of what can be missed when studies are restricted to a particular categorisation or focus. Addressing these methodological problems within this project led to a mass of qualitative data fragments which spoke of alienation and the influence of the commodity form. In what follows, Marx’s theory of alienation, and the role of commodities within it, is outlined, along with the various means by which commodity production influences education. Some representative fragments are then discussed in order to illustrate the project data’s relation to Marx’s theory. Through this exploration it is then argued that the data not only represents aspects of the context which connect to or influence the main concerns of the original study, but that questions of alienation in fact permeate the issues at the heart of the project: mathematics pedagogy, student dispositions, and their relation.

In conclusion, three levels of methodological advice are proposed, in order of difficulty of acceptance: 1) that mixed-method studies should build in attempts to escape the categorisations which arise within them; 2) that all educational research, where possible, should take account of the wider systems to which the particular objects of study belong, since these wider systems mediate (that is, do
not lie outside but are inside) the particular; 3) that society is a structured totality, and the nature of that totality is one shaped by the dominance of the commodity form of production. Accepting 3) has implications for those who want to make mathematics more meaningful in the classroom.

5.2 Methodological origins and issues

The source of the data used here is the Teleprism project, a large scale ESRC funded mixed-method project (award ref: RES-061-25-0538 www.teleprism.com) investigating the relationship between pedagogy and the beliefs and attitudes of students in relation to mathematics. The data arose through an attempt to respond to an arguably mundane methodological issue, typical of mixed method projects. At the heart of the project is a large scale survey, involving over 13,000 secondary school mathematics students, and their teachers, from over 40 schools. The data from this survey was used to measure examples of mathematics teaching on a single scale, from transmissionist (or rote-learning) to connectionist (see Askew, Brown, Rhodes, Wiliam, & Johnson, 1997, p.31, essentially ‘reform’-type teaching), and to measure students’ dispositions towards mathematics (including enjoyment and openness to future study etc.), again on a single scale. Reducing the complexities of classroom practice and human feelings to single measures creates rather blunt instruments, but ones which seem a necessary first step toward establishing the existence of an objective relationship between styles of teaching and student dispositions.

Alongside the survey, the project included a range of qualitative work, primarily interviews and classroom observations. The key motivations for this qualitative work are triangulation and complementarity (see Greene, Caracelli & Graham, 1989). First, it aims to check the validity of the quantitative instrument and its findings. Do people do and think what they claim to do and think within the survey? Secondly, it aims to add layers of depth and meaning to the themes which emerge from the survey. Generally in such projects, it is these (hoped for) rich examples which are also used to illustrate and communicate, in easily understandable form, the findings of the project.

This overall process, although not unusual for such projects, contains certain methodological dangers. If quantitative instruments can be seen to be relatively blunt or narrow, both of the motivations of triangulation and complementarity may
act to transfer this bluntness and narrowness into the qualitative part of the study, thus losing some of the suggested benefits of qualitative work. To achieve validity and deeper exploration or illustration, the design of the qualitative phase has to reflect in some detail what has been asked in the main survey. The survey will shape what questions are asked in interviews, what is looked for in observations, which data is analysed, how the data is analysed and how it is coded. There are approaches to overcoming such problems within mixed method studies, and these were used within the discussed project, for example, increasing the openness of the design by asking more open questions within interviews. However, here, these methods felt insufficient, with the needs of triangulation and complementarity always acting to reassert the categorisations arising from the quantitative instrument.

Before outlining the additional approach taken within the project it is worth stepping back and generalising the methodological issue faced. Here, a particular categorisation imposes itself on qualitative research which may limit what is analysed or restrict the nature of analysis. But, such dangers exist with categorisation and qualitative research at all levels. Any initial categorisation of qualitative data requires an ‘investigator responsiveness’ (which may not be forthcoming), relying on ‘the researcher’s creativity, sensitivity, flexibility and skill’ (Morse, Barrett, Mayan, Olson & Spiers, 2002, p.17) to go beyond it if necessary. Any initial, or developed, categorisation will be shaped by expectations (see Lakatos, 1978, p.15) or, often unconscious, assumptions (see, for example, Vygotsky, 1997, p.249), thus constraining potential understanding. Stepping back further, any chosen area of study itself imposes limits upon what data will be generated, either through survey or interview. For example, in our project on pedagogy and dispositions, the data collected will be restricted, in the main, to pedagogy and dispositions despite the fact that these may be shaped by many other factors.

There are therefore many reasons to attempt to break free of categorisations. In this project, a simple additional approach was utilised. A specific effort was made in the analysis stage to find data elements which lay outside of the categorisations which had arisen, and for answers which seemed, in some way, to be going beyond the questions asked. Although what arose in this process were isolated fragments, many of these fragments seemed to express alienation, disengagement and lack of control. This prompted a revisiting of Marx’s theory of alienation, and its particular emphasis on the role of commodities. It also
prompted a return to the wider data, unearthing more expressions of the mediations of commodity production within data directly related to pedagogy and mathematical dispositions.

In the following sections, the essence of Marx’s theory is outlined, along with the various ways in which commodity production mediates what goes on within schools and mathematics classrooms. Some of the found expressions are then presented and discussed to illustrate their relation to Marx’s theory. Finally, some interrelated conclusions are drawn on the nature of mathematics education, and on methodological issues in studying it.

5.3 Commodities, alienation and education

There has been much valuable writing in recent times on the relationship between alienation and education and mathematics education in particular (see Lave & McDermott, 2002; Williams, 2011; Jones, 2011). What follows restructures more than adds to this. Here, understandings of alienation within education are structured around the unifying factor of generalised commodity production, tracing the varied influences of this on the classroom.

5.3.1 Marx, alienation and the commodity

There is both an everyday and a more precise conception of alienation. The common sense conception is one of disengagement and the feelings associated with disconnection. Within education, this sense of the term can be expressed by behavioural issues or absence (whether physical or just mental), and emotions such as boredom, stress and anger. The philosophical sense of the word connects well to this everyday sense but goes deeper. Originally a legal/economic term meaning to give away or make another’s, and to create distance or be cut off from (usually God), the developed meaning of the term emerges through translations of German philosophy in the 18th and 19th centuries (Williams, 1976, p.33). The sense this provides is of emptying out and externalising (usually) one’s own essence, and, for the result of that process to then have negative effects on, or control over, you. The classic expression of this is in the critique of religion – ‘the being of man is alone the real being of God’
Marx builds on this further but argues that the real roots of societal alienation lie not in religion but in human labour, and particularly, in the production of commodities in capitalist society (see Marx, 1982; 1992). In this still historically recent form of society, the satisfaction of human needs and wants is organised through the competitive production and sale of commodities. Those who produce are separated from the means to produce and, in order to survive (through purchasing commodities), have to sell their ability to work to those who do have those means. In this, ‘the worker sinks to the level of a commodity, and moreover the most wretched commodity of all’ (Marx, 1992, p. 322). Human collective abilities to transform the world, to create, and to reproduce themselves, become subsumed in a process which is not directly aimed at the satisfaction of needs, but the production of profits and the accumulation of capital necessary for competitive advantage. This process therefore reproduces and strengthens the worker's lack of means of production. So, human labour is externalised and objectified in commodities, alienated, and the alienated product then has negative effects, and control over the labourer.

Marx (1992, p.325) argues that this alienation from the product of labour also means alienation from the process of labour (through its disconnection from immediate needs, and the lack of control of the process), from what it is to be human (i.e. conscious social producers, as activity, self, body and nature become just a means to the end of individual survival), and therefore from other humans (whether within production or between either side of the production consumption divide).

Externalisation of labour in objects and the appropriation of those objects occurred in previous forms of class society also (see Sayers, 2011, for a discussion on the relevance of alienation to say, feudal, society). As Marx makes clear in Capital (1982, p.165), the form of alienation in capitalist society also requires the exchange of those ‘objects’ (commodities) in the marketplace, and the equalisation of value (and the equalisation of the multiplicity of forms of labour) which occurs there. Social relations between producers only surface in and through this exchange, and because of this it comes to appear as if it is the commodities themselves which have the social relations, hiding the real social relations between people within them:

The mysterious character of the commodity-form consists therefore simply in the fact that the commodity reflects the social characteristics
of men's own labour as objective characteristics of the products of labour themselves (p.164)... It is nothing but the definite social relation between men themselves which assumes here, for them, the fantastic form of a relation between things (p.165)... they do not appear as direct social relations between persons in their work, but rather as material relations between persons and social relations between things (p.166).

The social dominance of this process of reification, and the masking of real relationships and processes behind objects (and the relationships between objects), has wide reaching implications in society, both direct and indirect, many of which have some relevance when it comes to looking at schooling and mathematics education (see below).

In discussing education, an additional relevant factor, which itself arises from the alienation of genuine social relations, and the competitive nature of capital, is the state, or, what Marx (Marx & Engels, 1974) calls the 'illusory community' (p.83). The state gives the impression of standing over society and being neutral, of representing the common good, but in practice it represents the interests of capital against other classes and mediates conflicts between capitals when necessary. It also performs, or attempts to perform tasks that are in the general interests of capital, the things that companies can't, or don't want to do individually, such as ensuring the general availability of the commodity labour power at sufficient levels.

### 5.3.2 Capitalism and education

Before capitalism, in what was to become the UK, institutionalised education was very much a minority pursuit. In 1072, for example, even the king of England still had to sign documents with a cross (see Poplawski et al., 2008, p.15). The earliest schools were religious vocational schools attached to cathedrals and monasteries, at a time when the church was a central part of the ruling class (Williams, 1961, p.148), but from the 13th century independent schools such as Winchester and Eton were formed, and these became the schools of the rising capitalist class (p.151). The poor went largely uneducated. For example, even with an increase in schooling through the industrial revolution, only around half of children attended school in 1816, and this generally only on one day a week, focussed on moral education and for a brief period (the average duration of
schooling was one year even by 1835, see p.157). There was rising pressure from industry to avoid the burden of training the minimally literate and numerate supply of labour power required to remain competitive, but this met resistance. For example, one failed attempt to increase the spread of schooling in 1807 met this response in the UK parliament:

However specious in theory the project might be of giving education to the labouring classes of the poor, it would, in effect, be found to be prejudicial to their morals and happiness; it would teach them to despise their lot in life, instead of making them good servants in agriculture and other laborious employments to which their rank in society had destined them; instead of teaching them the virtue of subordination, it would render them factious and refractory, as is evident in the manufacturing counties; it would enable them to read seditious pamphlets, vicious books and publications against Christianity; it would render them insolent to their superiors; and, in a few years, the result would be that the legislature would find it necessary to direct the strong arm of power towards them (Giddy, 1807. See also Lincoln, 1859; Graff, 1991, p.22 in relation to parallel arguments in the U.S.).

This was not far from the truth as pressure for education also grew from below, from a radicalised working class developing its own educational practice, particularly following the rise of the Chartist movement. One small illustrative example is the Lord Street Working Men's Reading Room in Carlisle, where,

[F]ifty men, anxious to read about the European revolutions of 1848, clubbed together to buy newspapers. A year later, with 300 members and 500 books, it had far outgrown its premises, a borrowed schoolroom. A new Elizabethan-style building was constructed in 1851, with congratulatory messages from Charles Dickens and Thomas Carlyle. Governed by a committee of workingmen, it charged a subscription of only 1d. a week, and even that was waived for the unemployed (Rose, 2002, p.65).

The needs of industry, combined with pressure from below (see Simon, 1974), led eventually to the development of the mass compulsory education seen today, through various parliamentary acts from the 1870s onward. Schools were shaped by the needs of factory production, and still reflect a similar influence today. The needs of companies and the profitable selling of commodities are still seen by the state as the primary motivation for education, as can generally be seen in speeches on the subject by politicians. To give a flavour, here is recent UK Conservative Prime Minister David Cameron explaining the main reason for attempting to lower truancy levels:
We want to create an education system based on real excellence, with a complete intolerance of failure. Yes, we’re ambitious. But today, we’ve got to be. We’ve got to be ambitious if we want to compete in the world. When China is going through an educational renaissance, when India is churning out science graduates…any complacency now would be fatal for our prosperity (Cameron, 2011).

5.3.3 Schooling, mathematics education and the commodity

For simplicity, the relationship of schooling, and within that mathematics education, to alienation and the commodity can be divided into three (in reality, interrelated categories): 1) The functional role that education plays for the system of commodity production, particularly in ensuring the adequate supply of the commodity labour power, suitably differentiated and with preferred attributes; 2) The replication of the experience of the workplace within schools, a process accelerated as education itself becomes, or more closely resembles, a commodity; and 3) The wider generalisation of the impact of commodity production to ideas, and understandings of thought and learning.

5.3.3.1 The functional roles of schooling

The argument that schools play key functional roles for capital is a topic which has been well explored in radical educational literature (see, for example, Reimer, 1971; Gintis, 1972; Bowles & Gintis, 1976). The first, simple, role schools play is in the custodial care of children that would otherwise have to be cared for by their working parents. It is estimated that if schools in the UK were closed for one week, the economic cost due to absenteeism from work would amount to around £1 billion (Sadique, Adams & Edmunds, 2008). Industrial action by teachers can also be seen to have a more generalised impact on the economy than that in other sectors (e.g. see Mason, 2011).

A second, even more vital role is in the development of the commodity labour power, that is, in the preparation of children to enter the workforce as adults. On one level this means the development of skills, from basic literacy and numeracy up to more technical skills. This role can be exaggerated, since, arguably, 'most workplace learning takes place on the job rather than off the job' (Eraut, 2004,
What matters more is the overall development, and reproduction, of a hierarchically, and otherwise, differentiated workforce to approximately match the societal division of labour. This role is partially fulfilled through differentiation of schools where intake can be shaped by fees, selection and class-related geography, and through the exam system, a fairly effective means of reproducing class inequalities (see, e.g. Sirin, 2005).

The additional benefit of the exam system is that it masks this reproductive role, convincing winners and, more importantly, losers that their relative success or failure rests on individual merit (see, Bourdieu & Passeron, 1990). This introduces an aspect of the ideological role schooling plays for capitalism. Part of developing the commodity labour power lies in developing human beings to see themselves as competitive individuals (and ultimately, as marketable commodities). A diet of testing, measurement, and examination trains children to think of themselves this way, but also to internalise the results as objective characteristics of themselves as individuals.

Another aspect of ideological training students get is that of being a good and submissive worker. Workers are unusual commodities in that what is purchased is the ability to labour rather than a fixed quantity of labour. Because human beings are not passive or fixed objects, both sides can attempt to renegotiate the terms of the purchase on an ongoing basis. Workers can, for example, carve out time for themselves during the working day (see, e.g., Hamper, 1991) or organise collectively to protect themselves. Companies can hire managers to watch over, or add pressure on, workers to maximise their productivity, or devise new ways to increase their workload. The sit still, be quiet and do what you’re told culture within schools assists in ingraining these habits of subordination in the future workforce. Just as the internalisation of competitive individuality fostered by schooling can later act to undermine collective bargaining by workers. Although attempts at more explicit indoctrination, in the sense of consciously convincing individuals of the dominant ideology (Marx & Engels, 1974, p.64), do take place within schools (see e.g. Mansell, 2013), it is this shaping of expectations through lived experience which is perhaps more fundamental.

5.3.3.2 Education as commodity

Recent years have seen a relentless drive towards privatisation across all
education sectors of the UK (for recent developments in Higher Education see McGettigan, 2013, and for the impact on schools, and beyond, see Ball, 2007; 2013). Even where education is not formally a business it is increasingly run as if it is. But in education, what is the product? What is the commodity? As both Williams (2011) and Jones (2011) point out, students themselves are not engaged in productive labour which generates surplus value and the grades and certificates they achieve are not commodities produced by them. However, certificates can be seen instead as the commodities which are being sold to them. (In reality, they are sold the process of education and potential success in an exam, but this is similar to other commodities such as say film, where customers do not give their money back if they fail to pay attention when watching the film).

The fact that ‘certificates belong to the student, not the teacher or school, and are not sold or exchanged’ (Jones, 2011, p.369) also does not exclude them as commodities. Again this could be said about many things that would not be denied the status of commodity, haircuts being one random example. Although, perhaps more than haircuts, certificates do potentially influence the future exchange rate of the commodity labour power for those holding them.

According to Jones (2011, p.369) teachers in the state sector are not productive in the Marxist sense. However, it is certainly arguable that they are indirectly productive in that state education increases the potential productivity of the system as a whole (see Harman, 2009, p.135) – through reducing the socially necessary labour time required for production within the capitalist firms serviced by the particular state. Teachers within private education, on the other hand, are directly involved in the production of commodities (see Rubin, 1973, 265). The impact of the neo-liberal era, with the development of transitional semi-public, semi-private forms in schools, colleges and universities alongside the pseudo-marketisation of publicly funded education, has been to transfer the social relations of commodity production into the state sector. It is in this sense that it is meaningful to talk about the commodification of qualifications (such as in Warmington, 2007). The logic of this process has led to exams increasingly shaping how education is delivered, an increase in managerialism and control in schools, fixed and atomised curricula, set lesson plans for teachers, and set ways to teach disciplined by inspection regimes. In the words of the radical slogan, ‘the school is a factory’, and this is so for the students too who are caught up in the production process. Even if teachers want to produce a different product, for example, well-rounded, critical thinking, self-confident, social human beings
with a depth of understanding of their subject, there are increasing mechanisms to ensure they focus on exam results instead (see Paton, 2013).

5.3.3.3 Knowledge (and mathematics) as object

The final point to make is that this commodification, the reification of a social process of learning into its outcome object, the qualification, is an example of a generalised feature of capitalism affecting wide areas of life (for an example, see Badiou, 2012, on aspects of the commodification of love via dating websites). This generalised process of reification extends even to how we think, and how we think we think. There are two key elements to this. First, 'ideas do not fall from heaven and nothing comes to us in a dream' (Labriola, 1966, p.155), that is, just as in the indoctrination element of schooling, where the understanding and accepting of how the world is rests primarily on the lived experience of schooling, ideas are fundamentally shaped by practice (Marx & Engels, 1974, p. 47). The generalisations which are made from experience become intertwined with the dominant ideology (the generalisations from practice made from the perspective of the dominant class in society, p.65) to form what Gramsci (1971) calls 'common sense' (p.419). Although the process is both complex and non-mechanical, it is far-reaching. The dominance of the commodity form of production and the resultant objectification, alienation and atomisation underpin modern conceptions of the individual and theories which take the individual as their starting point (Meszaros 1970, p.254). They also underpin the rationalism and reductionism of the scientific revolution which accompanies the rise of capitalism (Lukacs 1971, p.230). And finally, they help to reduce our understanding of knowledge, an active relationship with the world, to that of an object, something that can be taken out of one person's head and slotted into another's.

Mathematics has played a central role in many of these developments:

It is anything but a mere chance that at the very beginning of modern philosophy the ideal of knowledge took the form of universal mathematics; it was an attempt to establish a rational system of relations which comprehends the totality of the formal possibilities, proportions and relations of a rationalised existence with the aid of which every phenomenon – independently of its real and material distinctiveness – could be subjected to an exact calculus (Lukacs,
Mathematics education is therefore particularly shaped by this and the accumulated effects of all the processes and features of alienation described above, with many forces acting in a direction to encourage transmissionist pedagogy and limit the potential for more meaningful approaches to teaching. Some examples: 1) The division of labour into mental and manual, and the further separation of knowledge into individual subjects discourage the use of meaningful problems and the relating of mathematics to everyday knowledge or physical and social experience; 2) The needs of capital for a stratified and differentiated labour force, the reification and commodification of knowledge in certificates, and a view of knowledge as object shapes atomised curricula which divide mathematics into narrow process skills, and lose the systemic connections so central to understanding; 3) The individualisation of the commodity labour power and the competitive demands of exams discourage emphasis on the social dialogue which is so fruitful in concept development; And 4) The practicalities of custodial care, the time pressures of the ‘production line’, the training of submission, and the idea of knowledge as transferable object encourage passive drill and memorisation rather than active and reflective thought.

5.4 Expressions of alienation and the dominance of the commodity

The dominance of the commodity form of production has been argued to mediate schooling and mathematics teaching in three key ways. First, through the functional roles schooling plays within capitalism: at its most basic, custodial care but mainly, the development of the individual as a commodity, labour power. This development includes the stratified distribution of that commodity to match the division of labour in society; the convincing of individuals that their place in that stratification is based on merit, and a general training in submission in preparation for the workplace. The second factor is the replication of the relationships of the workplace within classrooms with this having been reinforced by the increasing commodification of education and the spread of neo-liberal managerialism to public services. The final factor is the generalised impact on thought, and views of knowledge and learning, as these have come to express the lived experience of society based on the production of commodities.

It is these mediations of education by commodity production which were found to
be expressed within the interviews with school mathematics students in the Teleprism project. A small selection of these interview fragments are now presented and discussed (see appendix 2 for a wider selection) in order to illustrate the type of connections with the themes of alienation which were found in the data.

Sarah (14) I've really enjoyed school... most people, they probably don't like school, or they dislike it, but I've never hated school because if we don't have it there's really nothing else for us to do.

This first quote on the surface represents a neat (possible) expression of the custodial care function of schooling – ‘there’s really nothing else for us to do’. It may alternatively, or simultaneously, represent a need for the sort of (at least superficially) purposive activity that school represents, or the social relationships which are found and developed there. It is always worth remembering when discussing alienation that when it is said that genuine social relationships and activity become reified and atomised, that does not mean that they do not still exist. Instead, it is that they become distorted, dominated by and hidden behind the resulting objectifications.

This quote also helps to illustrate what could have led to a particular piece of data becoming selected in the original methodological approach described in earlier sections. Here, the very existence of schooling is questioned, at least as a rejected imagined alternative. This goes very far beyond a narrow search for the relationship between pedagogy and dispositions which must assume the existence of schooling. It is a long time since the questioning of schooling as such was a serious (at least minority) position with debates on education (see, for example, Illich, 1971). Such quotes as these therefore open perspectives and in doing so bring wider issues in to the narrower questions being researched.

Interviewer Tell us a little about yourself.

Grace (15) ...I work well in a team.

This second quote captures the process of reification of social relations, the production of the individual, and the transformation of the individual into a commodity. Of course, human beings do generally work well in a team; we are social creatures after all. And this student may simply be expressing the positive experience of being part of social activity. However, in the phrasing this social activity has become an attribute of the individual – ‘I work well in a team’. More
than this, the phrase is one that sells that individual. It is a common utterance, but usually in a restricted domain, that of job applications. It is a promise to be a good worker. Here, (presumably) it is the school which has inculcated the student with the phrase, and with the attendant belief in being a competitive individual which must sell itself. There are many social forces, of which school is only one, which help create an individual as an individual (the interviewer’s question here being one micro example). However, schooling is the prime site of the development of the competitive individual and the stratification which results, as expressed in the following:

Interviewer Why don't you think you're going to get As?

Callum (13) It's just like mission impossible for me.

Interviewer Why?

Callum ‘Cause I'm not bright, I'm not intelligent.

Interviewer Why do you say these things?

Callum ‘Cause I'm not.

This is a representative illustration of how the process of competition for grades translates into students seeing their lack of success as an objective quality of themselves as an individual. First, the processes and relationships of developing an understanding of the world are reduced to, and objectified in, exams and grades, and then students are reduced to, and become, their grades. The results of the production of these intertwined commodities (or quasi-commodities), the individual (labour power) and the qualification, come to dominate the real processes and relationships involved. Pedagogy becomes shaped toward passing exams, and students internalise their failures and engage with the world as those failures.

Interviewer And do you think Mathematics will be useful in your future life?

Patrick (15) Yeah it's big GCSE isn't it.

This quote expresses the commodification of learning itself. The interviewer asks if mathematics will be useful in later life. Perhaps, because the teaching of
mathematics is detached from meaningful connections with real life (reduced to
the memorisation of an empty formalism rather than being the interrelation of
abstract systems with rich concrete experience), the student doesn’t contemplate
the intended meaning of the question. Instead the assumed meaning is the
importance in life of passing the exam. It is this process of commodification which
turns school into a factory. It is not that teachers intend to train students how to
behave appropriately in their future workplace, but that the processes in work and
in school are analogous. The logic of production is the same, and the alienation
from product, process, self and others follows (which is why Lave & McDermott’s,
2002, translation of Marx’s ‘estranged labour’ to ‘estranged learning’ works so
well).

Stuart I’ve been told off a couple of times, but that was
only because I’ve not understood it and I speak to
people if I’ve not understood it, but now I know not
to speak to people and ask the teacher.

Here is an example of the submissiveness of the workplace transferred to the
classroom, and its acceptance as the way things are. Alienation from others
occurs at the expense of learning, as dialogue in social activity is ruled out and
replaced by (a likely) monologic relationship from teacher to student.

Interviewer How did it change with the transition from that
school then to this school, the secondary school?

Shirin (11) It changed because the lessons are shorter, you
have to hurry up to do more work and produce
what you can.

Interviewer Can you give me an example?

Shirin Well in English we used to write the date, the title
and the learning objective in a maximum of 2
minutes, whereas I used to write the date, then
underline it, then write the title, then underline it.
So now I just write it and do it and she moves on
to the next slide, so it is just getting into that way of
what to do.

The theme of ‘school as factory’ is continued in this quote, along with a sense of
a speeding up of the production line between primary and secondary education.
One can imagine, perhaps, the care and attention this student used to take, and
the pride or pleasure she took from her aesthetics and creativity. As she is
increasingly alienated from the product and any say over the process of production, so she is alienated from herself.

Interviewer Are there any topics you like more than others?

Rosa (12) ... I don’t mind doing brackets.

This quote contains within it, perhaps, the whole of mathematics education. Only in the world of school mathematics would brackets (i.e. multiplying out brackets) be considered a topic in itself. Just as the active process of learning is objectified in a certificate or grade, or human relationships are reified in the atomised individual, the dominance of the commodity form also shapes our understanding of knowledge, turning processes and relationships into objects. Within school mathematics this process is reinforced by, among other things, the shaping of curricula to fit the ends of the final product, the exam. What results is a mathematics disconnected from human experience and activity, and disconnected also from other mathematics. The result is atomised processes to be ‘done’, to be memorised without meaning through repetition. This is reinforced by a different student, discussing (coincidentally) the different ways of multiplying out brackets: ‘Yeah there’s like different methods… my teacher now uses eyebrows and smiley face and it’s like it looks like that, but the one in year seven used a crab claw and it looked like a crab claw.’ Comparison of methods is a great way within problem solving to make connections between, and generalise, mathematics in relation to the real world. Here (while acknowledging the best efforts and creativity of the teachers involved), we simply have attempts to create meaning from meaninglessness in the aid of memorisation.

If the original quote says much about mathematics pedagogy, it also says as much about student dispositions. When asked about ‘liking’, the student instead offers a grudging ‘I don’t mind’. This refusal to fully identify with the meaningless of school mathematics speaks of the everyday sense of the word alienation, of disengagement. This disconnection is rooted in, and arises through, the processes described in the more theoretical understanding of alienation, through the various mediations of the dominance of commodity production in schooling. The grudging ‘I don't mind’ also signals hope however. Distance and expressions of distance can also be the first steps towards alternatives. Whether by students, teachers or researchers, it is expressions of shared disaffection that provide the potential basis for joint activity (see, Volosinov, 1976, p. 89) which can begin to challenge the alienation of education.
5.5 Conclusions

There are conclusions which can be drawn from the above in relation to both content and methodology. In fact, as will be shown, these conclusions are inextricably linked, as at a certain point, methodology becomes content and content becomes methodology. In content, the dominance of the commodity form has been argued to be the unifying factor in an understanding of alienation within the classroom. The various ways in which commodity production influences education have been traced: 1) in the functional roles of schooling for capital, including custodial care, the development of the individual as a commodity, labour power, the stratified distribution of that commodity, and the convincing and training of individuals to accept their place in various ways; 2) in the role of increasing commodification and the associated managerialism in sharpening these factors; and 3) in the general objectification and atomisation of knowledge which results as cultural ideas follow lived experience. These influences have then been argued to have been expressed within fragments of interviews with school mathematics students, showing that they are not just external influences on, for example, pedagogy and dispositions, but are instead internal to these.

In terms of methodology, three conclusions are suggested from the experience of the project and the analysis above, in order of difficulty of acceptance. The first is that one way of testing and challenging categorisations of data, (whether through the dominance of quantitative instruments in mixed method studies or otherwise), is to actively look for what lies outside that categorisation. This can mean literally examining the uncategorised, or looking, for example, at interview answers which in some way seem to go beyond the question asked. Here, this led to the opening of horizons which then fed back in to the particular concerns of the study, deepening our understanding of pedagogy, dispositions and their relationship. Such a methodological suggestion is hopefully uncontroversial, and perhaps even trivial.

The second methodological suggestion is a generalisation of the first. That is, however narrow the focus of particular research is, it should always take account of the wider systems in which the focus is embedded in. As Lewontin (1991) argues in relation to biology, ‘the properties of... parts cannot be understood except in their context in the whole. Parts do not have individual properties in
some isolated sense, but only in the context in which they are found' (p. 81). In the example here, mathematics pedagogy and dispositions have been argued to be parts of various wholes (such as the economic organisation of society, schooling, and philosophies of mathematics) which mediate them in ways which are essential to understanding them. This methodological suggestion is perhaps more controversial. Reductionism in science has been very successful, and its influence strong for the reasons outlined in this paper (here, content and methodology start to align). The arguments and evidence outlined above can at best represent one small contribution toward the counter-perspective of seeing wider systems as integral to any research focus.

The final methodological suggestion (where method and content coincide) is a far more controversial argument. It is that human society is a structured totality which is fundamentally shaped by the dominance of commodity production, and that this understanding should be integral to educational research. One objection to this perspective is that it wrongly reduces complex human behaviour to economics. It can only be stated that that is not the intention, that the relation between culture and economics is taken as both dialectical and highly complex. The writer Walter Benjamin aptly compares the influence of the economic base on the superstructure (see Marx, 1992, p. 425) to the expressions in a sleeper's dream content caused by an overfull stomach (Benjamin, 1999, p. 392). This article has attempted to trace the origin of the equivalent of such expressions, and thus is unlikely to be sufficient on its own to justify the methodological perspective. Again, it can only represent one small contribution to the argument for this methodological perspective.

If it is correct, however, it has important implications for those who not only want to understand aspects of education, but to change them. Attempts to improve pedagogy, and to make life in the classroom more meaningful for students, can cut against alienation (at least in it the common sense meaning of the word). This is particularly so if pedagogy involves doing the opposite of what commodification encourages. Teachers can subvert the space they are in by giving more control to students, encouraging social, dialogic and critical problem solving, connecting mathematics to the real world and students' lives, and encouraging the development of connections within mathematics. However, ultimately the forces described in this paper will assert themselves, as curricula must be covered and exams passed for teachers to survive in their roles. In order to sustain meaningful activity in the classroom therefore, some consciousness of the obstacles can be
helpful. Ultimately, ways must be found to connect critical pedagogy to critical perspectives on society, in ideas but also in practice.

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**References**


6 Making abstract mathematics concrete in and out of school

David Swanson & Julian Williams

Abstract We adopt a neo-Vygotskian view that a fully concrete scientific concept can only emerge from engaging in practice with systems of theoretical concepts, such as when mathematics is used to make sense of outside school or vocational practices. From this perspective the literature on mathematics outside school tends to dichotomise in- and out- of school practice, and glamorises the latter as more authentic and situated than academic mathematics. We then examine case study ethnographies of mathematics in which this picture seemed to break down in moments of mathematical problem solving and modelling in practice: (i) when amateur or professional players decided to investigate the mathematics of darts scoring to develop their ‘outing’ strategies, and (ii) when a prevocational mathematics course task challenged would-be mathematics teachers’ concept of fractions. These examples are used to develop the Vygotskian framework in relation to vocational and workplace mathematics. Finally, we propose that a unified view of mathematics, outside and inside school, on the basis of Vygotsky’s approach to everyday and scientific thought, can usefully orientate further research in vocational mathematics education.

Keywords Vygotsky, abstract and concrete mathematics, vocational mathematics, in and out of school
6.1 Introduction

In this paper we approach vocational mathematics education from a neo-Vygotskian perspective, centred on a particular view of what constitutes scientific conceptualisation and activity. The experimental and theoretical work of Vygotsky (1986) and his colleagues led to a distinction between scientific concepts, everyday concepts and various forms of pre-conceptual thought. He stressed, however, that this categorisation was a theoretical one and that, in practice, for adults, these forms can co-exist, that movement frequently occurs from one form to the other, and that no clear dividing line exists between them. What differentiates conceptual thought (whether everyday or scientific) from the prior developmental forms, in this framework, is argued to be a qualitative change in the relationship between abstract and concrete (1998, p.37).

We take the term abstract in this context to refer to the unity of two processes: First, the act of grouping or synthesis, and second, the act of separating, or analysis (the basis on which things are grouped). This second process also, at times, provides a narrower usage for the term abstraction in Vygotsky’s work. Concrete, in its loosest sense, we take as the real world but, more specifically, the “perceptual”, or “practical, action-bound thinking” based on sensory impressions or function respectively (1986, p.138). For Vygotsky, processes of abstraction play a role in development from an early stage, but in true conceptual thought it is necessary to view “the abstracted elements apart from the totality of the concrete experience in which they are embedded” (p.135). “A concept emerges only when the abstracted traits are synthesised anew and the resulting abstract synthesis becomes the main instrument of thought” (p.139).

For Vygotsky, what then signifies scientific thought, in comparison to everyday conceptual thought, is that the concepts are also part of an organised system. The type of system is important here, as everyday concepts can also be part of systems (see Davydov, 1990), for example: brother, sister, mother, et cetera provides an early ‘everyday’ system of relationships. Our view is that, just as in pre-conceptual thought, where meaning remains tied to the perceptual and practical, everyday concepts tend towards systematisation based on relationships in perception and practical activity. The systems of scientific thought, although ultimately rooted in such relationships, are formed instead on the basis of logical and scientific—including mathematical—relationships.
Therefore, everyday concepts, despite in one sense uniting abstract and concrete, are still dominated by the surface relations or connections perceived in everyday practice. Scientific concepts, in contrast, encourage and require “conscious and deliberate control” through being “placed within a system of relations of generality” (Vygotsky, 1986, p.172). Vygotsky’s stress is on the role of schooling in introducing such forms of thought, although he was equally aware of the danger of “empty verbalism” if such systems are learnt (or memorised) without rich development in relation to the concrete (p.150). He paid less attention to the use of scientific concepts outside of school but the implication is that the same bi-directional process between abstract systems and living reality can occur. However, in order for this to happen we would suggest stressing the second half of Marx’s (1992) dictum: “It is not enough for thought to strive for realization, reality must itself strive towards thought” (p.252). That is to say, for a scientific system to become concrete (or vice versa), a need or problem must arise in practice which cannot easily be satisfied or overcome by less conscious, everyday means.

From this perspective, then, we will now consider workplace mathematics (as well as its relation to everyday mathematics, vocational schooling and schooling more generally).

6.2 Mathematics outside school and in the workplace: Selected literature

Our current understandings of workplace mathematics are rooted in the challenge by Lave (1988) and others to the dominant cognitivist separation of cognition from activity: “what you learn is bound up with what you have to do” (Scribner, 1985, p.203). The form that mathematical activity takes was rightly argued to be highly situation dependent and distributed across mind, body, activity, other people, artefacts, setting and so on. The richer meaning and complexity of activity outside of school therefore meant there was a need to reverse the relative marginalisation, or outright dismissal, of the mathematical activity of everyday life. The arithmetic that, for example supermarket shoppers (Lave, Murtaugh, & de la Rocha, 1984), or street-sellers (Carraher, Carraher, & Schliemann, 1985), engaged in was not only qualitatively different but was also found to be more accurate and fool-proof than when the same subjects engaged
in apparently isomorphic school mathematics. In the outside world, people were more "in control of their activities, interacting with the setting, generating problems in relation with the setting and controlling problem solving processes" (Lave, 1988, p.70), using other resources more, and arithmetic less, but in a more integrated and meaningful way.

Studies of mathematics in the workplace were integral to this wider category of everyday or street mathematics and were the source of many findings. For example, Nunes, Schliemann, and Carraher (1993, p.126) found “both flexibility and transfer were more clearly demonstrated for everyday practices than for the school-taught proportions algorithm,” when investigating proportional knowledge in the workplace. Such practices could utilise and preserve meaning due to their derivation from activity which has a purpose, allowing social and empirical rules to be utilised alongside logical relationships, thus increasing the complexity that could be dealt with and decreasing the errors. Similar findings have been noted in a variety of vocations, for example, within nursing (Noss, Pozzi, & Hoyles, 1999), where a practical meaning of the notion of an average is seen to be more efficient and effective than the school mathematics versions due to it being 'webbed' together with practical and professional expertise.

Re-conceptualising all this in Vygotskian terms and within Activity Theory generally (see Blunden, 2010), we suggest that these activities have their motives in production, and mathematics becomes embedded in such activities just to the extent that it is functional to the activity. This fossilisation (Vygotsky, 1997, p.71) of the mathematics — often in physical artefacts, or in procedures, or fused in situated concepts — means that the acting subject is generally barely aware of the mathematics embedded there. It is concrete but not theoretical for them. This does not preclude the existence of abstraction or elements of abstract systems within the process which can aid limited forms of generalisation to contexts with similar elements, as in Nunes et al. (1993), or in the situated abstractions of Noss, Hoyles, and Pozzi (2002). But the dominant system remains the perceptual and action-bound one in which they are embedded, thus limiting conscious awareness and control of the mathematical system per se.

In contrast to everyday and workplace mathematical activity, the situated cognition literature above has variously characterised school mathematics as inauthentic, procedural, calculation driven, detached from meaning, passive, formal, formulaic, algorithmic, learned by rote, and lacking specific purpose. In terms of the framework we have outlined, this could be the “empty verbalism”
described earlier. However, we would argue it illustrates not just the weak
interrelation with the concrete in school mathematics, but also weak attention to
the systemic nature of the abstract formal system too, as school curricula often
atomize topics and limit the systemic connections that can be made in
mathematics (e.g., Gainsburg, 2012).

These criticisms of schooling have encouraged efforts to bring the concrete
reality of the workplace or outside into schools as curriculum tools so as to
encourage more meaningful and purposeful activity there (Wake & Williams,
2000; Williams & Wake, 2007a), although with the understanding that transition
between contexts is problematic (Nicol, 2002; Straesser, 2000). Such
approaches are then also seen as better preparation for the reality of the
workplace (e.g., Bakker, Kent, Derry, Noss, & Hoyles, 2008). In addition,
vocational education can design approaches and tools which more efficiently
develop situated knowledge within the workplace (e.g., Bakker, Groenveld,
Wijers, Akkerman, & Gravemeijer, 2014).

However, alongside this it has been suggested that changes in the demands of
the modern workplace require a rethinking of the relationship with school, at least
for a minority. The black-boxing of mathematics in artefacts (Latour, 1987) and in
activity systems, (see, e.g., Williams & Wake, 2007a) is seen to be problematic
as the nature of demands on employees changes. “Making the invisible visible,”
by opening up these black boxes, could then help improve efficiency, production
and profitability (Bakker, Hoyles, Kent, & Noss, 2006). It is argued, for example,
that a key skill-deficit among mid-level employees is not so much in performing
calculations, but in understanding systems, particularly when development or
communication with others is required: There is at times a need to “understand,
at some level, the model behind a given symbolic artifact” (Hoyles, Noss, Kent,
& Bakker, 2010, p.173). This, for us, seems to provide illustration of the type of
scientific activity described by Vygotsky: Such an approach advocates that
mathematical systems and knowledge needs to be consciously brought to bear
on the job in workplace practice—which then heralds a scientific, theoretical
understanding of workplace practice.

Boundary crossing and third space contexts (e.g., Tuomi-Gröhn & Engestrom,
2003; Williams & Wake, 2007a; Hahn, 2014; Akkerman & Bakker, 2011) can thus
also be used to bring academic/theoretical practices and practical/vocational
work practices together, integrating the two types of knowledge, assisted by the
awareness which can arise due to the contrasting approaches (Bakker & Akkerman, 2014).

The purpose of this paper is to show how our neo-Vygotskian perspective provides a unified understanding of mathematics in work, school, and vocational mathematics education. In relation to the overview of the literature above, this perspective leads us to suggest: (a) that the potential for scientific thought/activity can arise more widely than situated cognition perspectives, or even the recent vocational literature, allow for; (b) that these possibilities may be frustrated in practice by the structures of the workplace (and that these restrictions are similar to the ones encountered in schooling); and (c) that, nevertheless, both school and work are potential sites for scientific thought and practice, particularly, if the similar limiting structures in both are consciously challenged.

We illustrate these points, and the wider unifying perspective that Vygotsky’s theory of scientific concepts provides, by exploring some unusual cases of work/outside-school mathematics and vocational schooling from this perspective.

6.3 Methodology

We choose to look at case studies of mathematics education that may be revealing because they are close to the boundary between school and work, either just outside in leisure/work (darts players) or just inside (schooling of would-be mathematics teachers). These cases are special, in that they provide vantage points from which to question and problematise vocational and school mathematics. We also briefly revisit a third vocational case, as a useful corrective, in order to establish doubt about the benign or authentic, situated nature of workplace activity in a traditional setting with a hierarchical division of labour (further details of this case are to be found in Williams & Wake, 2007a, 2007b).

The motivation for the two main cases was their occurrence far from equilibrium, in sites where interesting things may happen and where contradictions are more likely to be exposed. Darts players provide an interesting case of situated mathematics involving people with generally modest academic attainment, dealing with a complexity of mathematics beyond simple everyday arithmetic, where systemic mathematical thinking may become advantageous. Similarly, a case of a connectionist (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997)
mathematics classroom was investigated, where the aim of learners being to become teachers themselves added an additional interrelating layer to the students’ perceptions and discussions of concrete—but imagined—vocational practices.

Both studies adopted an ethnography-light approach (i.e., in adopting ethnographic practices but without full immersion or long term engagement). Darts players were observed during practice and tournaments, and interviewed both in relation to incidents which occurred during observation and regarding their wider practice. Autobiographies of professional darts players, and online discussion forums for players, also provided useful material and context. Similarly, the adult student, would-be teachers were recorded during their classes, and interviewed along with their tutor, both in relation to observations and otherwise. All other relevant materials, including artefacts, were also collected or recorded where possible.

Our general approach to knowledge is to take a totality perspective (see Lukacs, 1967). We believe that integrating analyses across diverse case studies can, at times, allow us to go beyond the inherent limitations of more partial viewpoints, and can also help subject generalised theory to the appropriate stress of demanding cases.

### 6.4 Calculating at the oche: darts in leisure and work

Darts is a game where players stand at the oche (a line marked 2.369 m from the face of the dartboard, measured horizontally) and throw small sharp sticks called darts at a board which has been carefully segmented into sections worth different points (see Fig.6.1). Historically, this game became popular in working class clubs and pubs, but recently, with world television deals, the game has become seriously competitive and professional.

The different radial sections of the dart board target score one to twenty points, *doubles* and *triples* of those numbers in the outer ring and inner ring respectively, and twenty-five and fifty for the central rings known as the *bull’s eye*. The most common form of the game involves each player starting at a score of 501 and taking it in turns to throw three darts, until one player reduces their score to
exactly zero—but crucially ending on a dart that counts double; that is, *going out* with a double.

![Dartboard layout: The outer ring scores double, inner ring treble, and bulls eye 25 and 50.](image)

The game in fact offers excellent opportunities for practice with numbers in a fun context. However, the most interesting mathematics of the game occurs for players beyond a certain minimal skill level as the end of the game approaches. Here, the aim shifts from simply throwing at the section which gives the highest points—typically, treble twenty—to a more complex strategy which weighs up the most useful section to throw at that makes finishing easiest. For example, when a player gets down to a score of 67, they could go out by scoring treble 19 then double 5, say. But if they start with three darts it would be better to go for treble 17 then double 8, because a narrow miss on treble 17 may score a single 17, and they would still have a double (the bulls eye counts as a double) to aim at. Or if, instead, they get treble 17 but miss the double 8, hitting a single 8, that would still leave double 4 to go out with. On such slim differences a game and a match might be decided. The factors are multiple, but include (a) the probability of hitting the section aimed at, (b) what is likely to be hit if the dart misses (such as a single when aiming for a double or treble), and (c) whether the chosen path leads to a good double to end on — that is, one practised often (commonly, 20, 16, or 8) but, more importantly, a number than can be divided by two repeatedly, since, for example, missing double 16 is likely to score single 16 leaving double 8, and so on.

In our investigations we found the mathematics of this part of the game to be classically situated for players: The development of know-how in any situation
takes place over a long period of time as skill increases, is highly concrete and context dependent, integrated with artefacts, and involves apprenticeship in the society of those with more advanced knowledge. In fact, most darts players reported little or no memory of learning their numerical strategies with "you just pick it up" being a common account. Furthermore, much of this know-how has been crystallized in the outs table that players can download from the internet and carry in their pockets (see Fig.6.2).

![Darts Outs Chart](image)

*Figure 6.2. A typical outs chart showing one way to go out with three darts.*

However, despite the classically situated and embedded nature of the mathematics for most people, there are two key ways in which something beyond this develops. The first comes through engagement and interest in the game, and through that, the mathematical aspects. This could perhaps be termed *intrinsic motivation* although given its interrelation with practice in the real world this term never quite feels adequate. We will rather call it the *scientific motivation*: the interest to get to the bottom of the theory of a situation of interest, which motivates genuinely theoretical, scientific intellectual development of a practical phenomenon. It is also often associated with prior mathematical confidence, as one top professional illustrates in his autobiography:

> At school I was always good at maths; my maths got me into Hackney Downs Grammar School when I took the eleven plus exam.
Darts was an extension of this. I'd spend hours working out different permutations to finish on, all that sort of thing.

They let me leave [school] six months early, well before my sixteenth birthday. There was no point in staying: I wouldn't have passed my exams because I didn't do any work. I was just mad at school, nuts. I told the teachers I didn't need an education because I was going to play darts, so that's why they signed me off for the last six months and told me I could go home. (Bristow, 2010, p.22)

The second, and more likely, path for situated mathematics to develop is in the emergence of a need which cannot be satisfied in the old way. This is often focused around a **breakdown** moment, whereby actions which have become automatic fail in some way and thus become the subject of conscious attention (for a description rooted in Leontiev's work see Williams & Wake, 2007a; and a similar approach in Pozzi, Noss, & Hoyles, 1998). For example, one of the non-professional players we interviewed reported:

It were a bit scary, when I started playing in the leagues. I were only 18, and I realised once, in a match, nobody's allowed to tell you which way you can go. You can only ask what's left. So I remember once hitting a strange treble and I looked and I knew I wanted... I can't remember the number, maybe it was 67 left, and I just didn't know which way to go for that... and everybody is looking at me, and I just didn't know... and I just went, “Wow, I don't know what to do now, I've got two darts left in me hand and I'll have to go for something!”.

I were keen, I just didn't want the embarrassment of not knowing what to throw for, so I taught myself. I went to a lot of bother to find out simplest way to finish under 150. I had to ask which way would you go for that and why, and then eventually I'd work my own ways out. I thought, “No I'm better off doing that, because that leaves me treble 20 and bull or treble 18 and bull”.

Bobby George, a professional player, relates a similar tale in his autobiography:

“There's no way you can lose playing like that”, Roy [his friend and fellow player] said. What he didn't know, of course, was that I still couldn't count to save my life. I was faced with a 90 out-shot to beat Roger and win the title but I didn't have a clue how to go about it. “Treble 18, Bob”, he shouted from the floor. Well, I hit treble 18 but my mind was still a complete blank about what I should do next. Nerves sometimes make moments like that even worse and I just stood at the oche bewildered, looking for help. .. It is no wonder some of the older players despaired of me. I admits it was a bloody ridiculous state of affairs.

Deep down I knew I had to rely on myself to progress. Another Essex player, Glen Lazer, and I worked out each and every possible
permutation... My game improved almost overnight. I saw how trebles and singles that sit next to each other on the board can work in your favour... I was never any good at mathematics at school but I found that darts is more about remembering numbers and combinations. I had to crack this and it took some time... To this day, I don't do any form of arithmetic when I play darts. I just know how all the numbers work...Working out all those combinations gave me confidence. (George, 2007, p.55)

As can be seen in both cases, deliberate attempts to memorise can play a role in such processes, and, from the outside, the end result may be indistinguishable from that of being simply memorized (see Vygotsky, 1978, p.64). But, again in both cases, breakdown moments have led to a move beyond the situated, to an active working out involving systematic mathematical work. In this conscious process, systemic relationships (of the number system, the layout of the board and rules of the game) are united with the individual’s history, concrete experience and everyday conceptualisations. For example, George explains:

I have always liked 121 as an out-shot for the simple reason that there are so many ways to hit it. Just think about it. Numbers 14 and 11 are next to each other so immediately there is one large treble target to hit. After that, there are lots of options—treble 14 leaves you treble 19 and double 11. Treble 11 leaves you treble 20 and double 14. Single 11 leaves you treble 20, bull, and single 14 leaves you treble 19, bull. You can never have too many options in darts (George, 2007, p.57)

Whereas the growth of mathematical expertise of the player in the early/middle stages appears gradual and incidental, picked up as they go, when the player gets seriously competitive (whether as a competitive club player or a professional, it makes little difference), there comes the moment when the strategy needs to be perfected, and the mathematics must be personally and individually tailored and mastered to achieve a high level of competence. Even at this point, a player is likely to go to the default darts outs table for an answer (see Fig.6.2.). But growth of expertise requires that this table be generalised (to include ALL potential combinations) and specialised (to the tastes and skills of the player, e.g., some prefer to aim at the bull rather than certain trebles, etc.). It is at this moment that the mathematical work of the darts player approaches that of an investigation of the type likely to be conducted in scientific mathematics in academe or school: We hypothesise that these moments, while perhaps rare in practice, can arise in almost all leisure and workplace activities. We argue this involves a genuinely scientific conceptualization in practice in
Vygotsky’s sense. Notice, though, that after the mathematical work has been done, the know-how may again become automated in the darts player’s own memory as their personal outs repertoire.

The experience of mathematics used by the darts players here offers a case where leisure meets professional work: The mathematics arose from a perceived need to develop the darts expertise required to perform at the top level, whether as a club player or a professional and champion. The motive for the activity is therefore almost identical in both cases, but actually raises potential differences between mathematics outside school and in work, due to this being such a special occupation. The professional darts player is a rare profession in that it emerges from a leisure activity, and the player goes professional in the game at a high level of special expertise. In other words, they are rather unlike those workers who have to sell their labour in a mass market of employees. We suspect that the majority see their work as a means to an end rather than as a vocation (in the sense of a calling) and are far less likely to be motivated to engage in such processes of working out problems mathematically. And that even when they are so motivated the division of labour is likely to, directly or indirectly, frustrate them (for some evidence of workplace disengagement, see Gallup, 2013). Thus, while we celebrate this case we are aware that all too often, in workers’ experiences, mathematics may rarely present itself as the answer to a question. We therefore now revisit a study of just such a workplace example.

6.5 Mathematics in a hierarchical workplace

One of the main findings of Williams and Wake (2007a) was that workplace systems historically structure and hide the mathematics in two ways: First, as is well known, work processes tend to hide mathematics in artefacts and tools; second, the division of the labour process in the workplace tends to produce knowledge boundaries inside the workplace institution. These two processes work together to reduce the mathematical demands of the labourer in general, and to privilege the activity of some few workers whose status may be related to their position of power/knowledge and also their competence (including mathematical competences).

Williams and Wake (2007a) provided details of one professional engineer in a large power plant, whom we called “Dan,” whose status as the expert in charge
of the spreadsheet formulae was maintained by a division of labour in which those operatives below him were cut off from access to the meaning of these formulae: Their own work was reduced to reading dials and filling in record sheets that supplied Dan with the data needed to predict the plant’s consumption of power. On the other hand, there are also boundaries around management who also are sometimes cut off, or black-boxed, from the knowledge of what is going on lower down in the industrial hierarchy. Such a division of labour appears to work—as long as there are no obvious breakdowns. But when breakdowns occur, suddenly the need to cross boundaries requires the black boxes to be opened and the automated mathematics to be re-vivified, which is quite often non-trivial (Williams & Wake, 2007b; see also Noss, et al., 2002).

However, the power relations and competition within the workplace that allow for a specialist to acquire special authority may not be functional at the broader, system level, as they tend to encourage a lack of transparency and a lack of openness. In general, unequal power relations imply a lack of equal sharing of knowledge in a common enterprise that would normally be expected in a community of practice (Wenger, 1998). Upon examination Dan’s spreadsheets and recording instruments seemed to the outsider-researchers to be unhelpfully opaque—and yet Dan’s special status perhaps depended on these procedures being arcane. In general Williams and Wake (2007a, b) found that tools in the workplace are often idiosyncratically designed, as there seems to be less motivation to produce clarity of expression than one might expect (cf. non-experts reading a computer manual).

Finally, we can reflect that workplaces as not so very different from schools regarding mathematics. Based on cases such as Dan’s, the operator classes are given routine and unchallenging mechanical tasks to do, while only those who are more privileged are likely to be given work that is challenging and highly valued (see Braverman, 1998). Indeed, we can add to this evidence even from ethnographies of industrial scientific practices: Case studies by Latour and many others have shown how power and politics are engaged in the construction of science and scientific knowledge due to competitive divisions of labour (Latour & Woolgar, 1986).

From our reading of the literature we conclude that workplaces are often far from benign as places of authentic learning, and that vocational mathematics is often structured by the power relations associated with a classed division of labour in ways that may be alienating for some workers, or would-be workers. The
exigencies of practice in the workplace can cut workers off from thinking and communicating mathematically, and therefore from developing the sort of scientific conceptualisations that Vygotskian theory describes. However, there is the parallel point that school is also sometimes—perhaps as often—just as alienating for the majority of students, but for opposite reasons: The theoretical concepts of mathematics are, at least potentially, present, but aspiring workers may not become engaged in a productive, motivating practice, other than the passing—or failing to pass—regulatory tests.

We will now consider a case in which some scientific mathematical work arose, apparently authentically, in schooling: Our intention here is to complicate this story of the alienation of workers and students from mathematics.

### 6.6 Would-be teachers troubled by fractions

As discussed earlier, academic and school mathematics are rightly viewed as being formal. However, we would also agree that there is a didactic transposition between mathematics as a professional practice to mathematics for the classroom (Chevallard, 1988). What results is school—and institutionally—situated (Ozmanter & Monaghan, 2008), and, often involves a situated formalism involving transmissionist teaching and learning, with the memorization and repetition of atomized process skills. Such pedagogy is contested by, for example, connectionist approaches (Askew et al., 1997), which stress, among other aspects, the connections within mathematics, connections to the real world (and students’ own knowledge), and the use of dialogue in meaningful problem solving. Here, we explore a case where such an approach seemed to support a move from the situated formalism of most schooling to something more scientific. We do this by looking at a particular case of pre-vocational mathematical schooling provided for potential teachers of mathematics.

That the mathematics required for mathematics teaching is a form of (pre-)vocational mathematics may seem incongruent to some, but the arguments for seeing school mathematics as situated must be assumed to apply just as much to its teaching. This is to some extent recognised within the teacher education literature, where mathematical knowledge for teaching is seen as “a kind of professional knowledge of mathematics different from that demanded by other
mathematically intensive occupations, such as engineering, physics, accounting, or carpentry” (Ball, Hill, & Bass, 2005, p.17). What is different is often the paramount need for clarity of communication, and this requires an understanding of the audience or learner. This leads, for instance, to the need for multiple explanations and the ability to see or anticipate difficulties, misconceptions, and errors. Arguably, these are, or should be, as important to the programmer of a computer operating system, an engineer trying to explain the potential dangers of extremely cold weather to the Challenger’s O-rings (which led to the NASA space shuttle disaster in 1986), or in the case of Dan’s spreadsheet formula. Yet, in industry there is an ever present bottom line of functioning that does not always prioritise transparency, clarity, and even functionality.

6.6.1 Mathematics Enhancement Courses

Mathematics Enhancement Courses (MEC) are aimed at adults in the UK who wish to progress onto a secondary mathematics, pre-service Initial Teacher Training pathway, but who have been judged to have insufficient or insufficiently recent mathematical qualifications. The courses vary in length and pattern of attendance, depending on the previous mathematical experience of the students (in the case below this was 5 days a week for 36 weeks). The students primarily learn mathematics, but the courses can also include some classes on wider aspects of education and forms of teaching practice activities, with some limited contact with schools. Courses are funded by the UK Training and Development Agency for Schools, and thus there are no fees and a relatively reasonable weekly bursary which attracts students from a wide range of previous backgrounds.

The government specification for MEC stresses “connectedness as against fragmentation” and “deep and broad understanding of concepts, as against surface procedural knowledge” (Stevenson, 2008, p.103). Beyond this, there are fewer curriculum and assessment constraints for individual institutions than when they provide more traditional prevocational mathematical qualifications. This may allow a space for teachers to teach in a less regulated way and, through this, to provide alternative instructional practices. It is also relevant that the tutor in this case was an experienced mathematics teacher, educator, and researcher whose
beliefs are consistent with connectionism. We now turn to a study of one instructional sequence in relation to the wider issues of this article.

6.6.2 A case of fractions: troubling the theoretical concept in practice

The students, in this and a previous class, had been exploring the early Egyptian method of dealing with fractions, primarily through investigative and discursive problem solving activity. The issue of addition of fractions arose in this context and, in particular, the fraction addition $1/2 + 1/3 = 5/6$. A student raised a common error, suggesting that $1/2 + 1/3 = 2/5$. This mistake is thought to arise from an overemphasis on procedure at the expense of theoretical understanding, and may be related to treating the form of fraction multiplication as analogous to addition. Numerator is added to numerator, and denominator added to denominator. The MEC tutor saw an opportunity in this discussion, and raised the possibility of a pupil having, instead, based their answer on their experience with test results. If, say, they had been given “one out of two” for the first question in a test, and “one out of three” for a second question, how much would they have scored altogether? Two out of five (or two fifths, or 40%) would be regarded as the correct answer in this context.

This led to some audible confusion among the students as they realised that this was true, yet went against their normal understanding of how fractions work. The tutor then invited them to discuss the problem in their different groups. The setting up of a contradiction within the students’ conceptualisation (or the drawing to attention of a misconception) is a deliberate strategy, commonly used by the tutor as an aid to concept development.

Already in this situation we can see some interesting combinations. We have both school and vocation—the students are learning mathematics, but are doing so in a context of becoming teachers themselves at some point. Often, within the MEC class we studied, this second factor is implicit as the students go about their mathematical activity, but when it surfaces and becomes explicit, as it does in the dialogue that follows, it does so naturally, and there is never a sense that the class is now engaged in a different type of activity.

Our example contains both school mathematics and vocational mathematics. We have the formal rules of addition of fractions, yet the class is also addressing the
way test scores are represented as fractions in class. Furthermore, this example engages with a classic element of mathematics for teaching, the understanding of misconceptions and how teachers might potentially respond. Finally, we also have formal, situated mathematics, alongside, at least potentially, a deeper conceptual understanding of the mathematics involved in test-scoring practices.

Following Roth (2014) we could ask the question: Where are the boundaries between these aspects if they are all present in the classroom, potentially at any point? However, we—perhaps contra-Roth—would resist the idea that the subjective individual is the key unit of analysis for the unifying of the various factors in such cases. Many aspects of the activity bring these elements together, including, here, the instructional example itself. Although it is more obvious in this MEC case, we would argue that this is true, to an extent, in more general school/vocational cases (e.g., through the mathematics contained within classroom tools—see Williams & Goos, 2013).

Returning to the problem before the class, the contradiction—that a half plus a third equals five sixths, yet, one out of two and one out of three, combined in a test, scores two out of five—is recognisably a question of modeling, but also, conceptually, of unitisation, a topic which has been well explored (see, e.g., Lamon, 1996). However, as the concept of unitisation could easily remain at a formal level, our own preferred understanding is one that relates both to the real world and a meta-understanding of what mathematical theory is. One version of this would be as follows: Mathematics is a modeling practice which abstracts from the patterns and regularities of real, practical experience in the world. The fractions (and, more generally, numbers) that we use arose from this process historically, and therefore originally represent fractions and numbers of something (we note here the related and valid points made by Schmittau, 2003, on the relation of fractions to measurement rather than counting). As a culture we have reified this process to produce numbers as objects in themselves, and happily add 2 to 3 to get 5, or teach that 1/2 plus 1/3 equals 5/6 without reference to any mediating context other than the symbolic manipulation itself. What then becomes implicit and hidden is that in most real practices such numbers have to be tied to a unit of something, and, in particular, when adding, the added numbers or values, and the total, need to refer to the same unit element (even in group or ring theory), and so of the same thing. So 1/2 of something added to 1/3 of the same thing equals 5/6 of the same thing (even if that same thing is just the
number ‘one’). In the contested example above, we could write $\frac{1}{2}$ of 2 marks + $\frac{1}{3}$ of 3 marks = $\frac{2}{5}$ of 5 marks which is also, in this context, true.

In the MEC class, we recorded a group of students on one table, where an initial discussion of the problem was followed by the following dialogue:

7 Beth [Attempted explanation:] Because that [one out of two] isn’t half. Because that’s not a real half. It’s only half of your two possible marks, isn’t it?
8 Dora It is a half though
9 Beth ...But it’s not a half of the whole test though, is it.
10 Elise Yeah. The five questions are divided into two questions.
11 Beth There are five questions
12 Dora But this is like one over five isn’t it and this is one over five. [writing $\frac{1}{2} = \frac{1}{5}$ and $\frac{1}{3} = \frac{1}{5}$]
13 Elise Oh that’s just confusing. A half is equals a half. What? A half is equal to one fifth, isn’t a half equal to a half?

We find this dialogue fascinating for many reasons. Beth, at line 7, already seems to have a concrete form of the conceptualisation we suggested above, although it includes a dismissal of “one out of two” as not being a “real” half. This reflects the formalism that students can bring with them when they first enter a path toward teaching. The generalised concept of unitisation is not explicitly expressed here, and Dora can counter the explanation with the obvious, “it is a half though”. Yet Dora too, in line 12, can express a concrete form of the solution. Elise then asks a pertinent question—one that begs to be mathematised with multiple units.

This generalisation doesn’t materialise in dialogue here, though, and the discussion is inconclusive; there is a pause and then a slight reformulation of the initial disagreement occurs:

22 Beth Yeah, it’s, sorry, it is two over five but it’s not half
23 Dora But then they will say, “ok so…”
24 Beth Yeah it is two over five but it’s not half plus a third is it.
25 Dora No... no but...
26 Beth ‘Cause then you start looking at it going...
27 Dora I’m looking at it and thinking yeah...
28 Beth It’s right
29 Dora I agree [laughs] [pause]

What is important for us here in this continuing to-and-fro between the formal and the contradictory concrete example is the line, “But then they will say”. The “they”
here refers to the idealised future students, the ones initially referenced by the
tutor, who here mediate the MEC students’ own expressions of doubt, and of
needing to be convinced of their own argument. Potential student teachers of
mathematics often come with an experience of transmissionist classrooms and a
model of the teacher as expert, one who primarily must explain to others, and
therefore one whose efficacy depends precisely on being convincing, if only to
themselves (and it is perhaps in this sense that teachers will often say “I never
really understood this concept until I had to prepare myself to teach it”). Here, we
suggest, this imagined future demand sustains the dialogue beyond the point
where the participants might otherwise have been motivated. The dialogue
continues with Dora’s reference to the test papers: They will see, in which each
page of the test has a box for the marks scored on that page written as a fraction,
the kids will see these boxes and know that they add the fractions up in just this
way to get their total score:

30 Dora It’s just that if you tell them to add them together. Of that one I
get one and that one ...and you know how those test papers are, do you remember? There’s a box, usually, that does say that. So it’s like if you tell them to look at those as fractions. I never thought of that.

31 Beth Who would look at that?

32 Dora Kids will...

33 Beth Will they? Really?

34 Anna But you do. You’ll get your result as a fraction, your teacher will write it as a fraction.

The conversation continues, retracing the previous argument about the fractional
parts being different units:

41 Anna See what I mean it’s not proportional.

42 Beth [very quickly] Yeah that’s not half the marks for the whole test, that’s half out of those two questions.

43 Dora Yeah, and that would be the difference

44 Beth But it’s actually quite a hard concept to explain why it doesn’t work, but you see easy why it’s two fifths [pause]

The students here seem almost to have produced a satisfactory argument, that
is, in a form that almost convinces them, but there remains an awareness that the
concept is still not in an explicit form that resolves the contradiction adequately.
The idea is still considered hard, and so perhaps too hard for them. There is a
sense here of an ambiguity of meaning in the words they or them: the students
as both learners (finding this argument hard to put together) and as imagined
teachers (whose learners will expect them to be able to explain simply and
clearly). This puts new heightened demands on their own learning, as they are learning, ultimately, to teach, which implies the argument must be better than almost persuasive.

An essentially transmissionist view of teaching, one based on the centrality of explaining, has here mediated students’ awareness of a vocational form of mathematics required for teaching. At the same time, despite this transmissionist outlook, this orientation has affected their own learning, motivating them towards gaining a perhaps deeper understanding of the mathematical concepts involved.

What do we mean here by a deep understanding? We argue that this is precisely the dialectical synthesis of the systemic, theoretical mathematics of fractions with the concrete practice of teaching, or explicit explaining as required for teaching. We argue that the discourse above shows these students as struggling to achieve this. The fact that this synthesis is perhaps not fully or adequately achieved is clear in their finishing in some doubt. The articulation of our own understanding preceding the above analysis was our attempt to articulate the sort of deep understanding that teaching practice might require.

The school-work of these would-be teachers is special for two major, and distinct, reasons: First, it is characterised as schooling as the activity is clearly structured by the object of gaining a qualification from the MEC to enter teacher-training. En route, the students’ main task is to learn/acquire sufficient mathematics knowledge and skills to begin teaching. On entry to a teacher-training course they will still be students, but their activity becomes hybrid, as they will enter classrooms and practice teaching (even while not yet being fully qualified and employed). Thus, these would-be teachers are as close to becoming workers/professionals as one can get while still being firmly students, that is, without actually practising their work/profession. What we find significant about this activity is that the students are already in role as imaginary, would-be teachers—that is, they envision themselves as teachers of the subject even while they are learning it. This role can be sufficiently strong to provide a motive for their learning activity, and it deserves to be called a vocational motive, even though they are not yet practising their vocation. We have seen this in other school contexts, where mathematics became embedded in the vocational work of engineering students (Black, Hernandez-Martinez, Davis, & Wake, 2010). But the act of imagination is equally valid when a leisure, or other, activity is made real in the classroom—such as that of shopping, where the so-called situated intuition of
purchasing and giving change can become useful in learning subtraction (see Williams, Linchevski, & Kutscher, 2008).

6.7 Conclusion

Our aim has been to show how Vygotskian perspectives can help us to see how genuine scientific activity can arise, whether in school, work, or vocational mathematics education. In each of these cases we have also suggested how such activity can be frustrated by institutional structures, and how such structures might alienate learners and workers from scientific activity and thought. Mathematics can be ritualized, and even fossilised in practices in both vocational and academic contexts. It was suggested that this arises from the embedding and automisation of mathematics in artefacts and operational procedures in production (and schooling) systems, and that these evolve historically in systems with divisions of labour that black box mathematics socially as well as materially.

But we have also seen in case studies how this fossilization of mathematics in practice can break down too, and lead to activity of a mathematically scientific nature in Vygotsky’s sense. In our perspective, mathematical authenticity is visible when the abstract “rises to the concrete” (Marx, 1973, p.101). That is, formal theoretical–mathematical concepts are made concrete in practice by learners or workers solving concrete tasks, in meaningful social practices—whether in school or in workplaces.

A caveat: We allow that the activity of the university academic in, for example, ring theory is a practice and that proving new theorems is one socially-meaningful, concrete product of mathematics. In the same way we can have no problem with school children or construction workers being engaged in problem solving of a genuinely mathematical nature, as well as mathematics that enhances their scientific understandings of practices in the rest of life. We emphasise that it is the rules of the institutions (whether work or school) that may prevent workers and students from engaging in such activity. Why so?

In both cases it seems the object of the institutional activity may conflict with genuinely useful and functional scientific learning with mathematics. In workplaces the worker may be told, “It is not your job to think (about the O-rings), we have managers who will decide (whether the space shuttle Challenger flies);”
while, in school, students may be told “Never mind why (minus times minus is plus), this is how to get the answer and pass your exams.”

As researchers, we conveniently chose a workplace case study of the darts player, where the knowledge seemed less alienating as the work was close to fun. But we also conveniently chose an academic environment in which students were professionally motivated by understanding mathematics for teaching. Perhaps these are the exceptions, where, even in school, understanding and having to explain the mathematics becomes a priority, and getting the mathematics straight is crucial to being a top darts player. But we conveniently chose to make the point: That the dialectical opposition and synthesis of the systemic abstract with the concrete is what makes the difference in making mathematics scientific, whether it occurs in school or in work/leisure. In both cases those involved were motivated to do so.

In conclusion, we would suggest that there are valuable forms of thought that, although never unsituated, do go beyond the situated to become scientific, in Vygotsky’s sense of a conscious, mutually mediating systemic synthesis of the abstract and concrete. This is not to reverse the gains of situated cognition and devalue the most-of-the-time everyday practice of “just plain folks” (Lave, 1988), which we still hold as essential sources of scientific inquiry. It does, however, involve critiquing the institutionalised, situated formalism of most mathematics schooling, where test-taking rituals have sometimes substituted for scientific goals. A similar institutional critique can, in many cases, be extended to the workplace. Both schools and workplaces are contested arenas. If, at the moment, schools are far from ideal in developing scientific conceptual learning, workplaces are also often far from being ideal places where it can be expressed, or even developed. We believe it would aid the vocational mathematics literature to relate these two understandings more directly.

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7 Generalising proportion: The potential role of problematising the mathematical system in concept development

David Swanson

Abstract: The importance to learning of making connections between different parts of mathematics is often stressed. Here, a Vygotskian perspective on the development of scientific concepts is taken which sees such connections as involving generalisation within systems, and which stresses the importance of the bidirectional developmental relationship between generalisations and what is generalised. A variety of forms and approaches to generalisation and making connections within mathematics, from practice and research, are surveyed from this perspective.

An additional approach, involving the explicit problematisation of generalisation is then discussed and its application within a classroom context is explored in order to further concretise the Vygotskian perspective. Some conclusions on the featured method and the theoretical approach and suggestions for further study are made.

Keywords: Vygotsky, Mathematical connections, Generalisation, Proportion
7.1 Introduction

The pedagogical importance of making connections between different areas of mathematics is often stressed in policy. For example, the current UK national curriculum states:

Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas. The programmes of study are, by necessity, organised into apparently distinct domains, but pupils should make rich connections across mathematical ideas to develop fluency, mathematical reasoning and competence in solving increasingly sophisticated problems (DfE 2014); (Similarly, see NCTM, 2000, p.15).

Research also shows that effective teachers of mathematics place a high emphasis on these connections (Askew, Brown, Rhodes, Johnson & Wiliam, 1997, p.3). In classroom practice however, Ofsted (2008) report that the making of connections is infrequent (p.6). Similar comments are found in the mathematics education literature, for example, ‘curricula have been chopped into small pieces, which focus on the mastery of algorithmic procedures as isolated skills’ (Schoenfeld, 1988, p.163), and the associated transmissionist approach to teaching is found to dominate (see e.g. Pampaka et al., 2012). The reasons for the perpetuation of this transmissionism are powerful and deep-lying (in high-stakes examinations, in the associated curricula and in the nature of schooling in capitalist society, see ch.6), therefore there is a need for the development of practical approaches to focus pedagogy on systemic connections in the mathematical classroom, alongside the development of theoretical understandings which are grounded in research and practice.

This article presents a case study of a small-scale attempt to implement one such practical pedagogical approach to developing connections – in relation to drawing the links between various topics involving proportional reasoning. First, however, section 7.2 outlines elements of Vygotsky’s (1987) theory of scientific concepts and the relationship of those elements to making mathematical connections through pedagogy. Some understandings of the role that attention to systemic connections can play in concept development flow from this analysis, as do some implications for the form that pedagogical approaches can usefully take. Various methods of making connections between practice and research are then surveyed and examined in relation to this theoretical framework. An additional method of explicitly problematising generalisation is then introduced through case
study data and a classroom dialogue is used to concretise the analysis. Some conclusions on the featured method and the theoretical approach are then drawn and suggestions for further study made.

7.2 Vygotsky and scientific concepts

The theoretical perspective of this article is grounded in Vygotsky’s (1987) writings on concept development. There is a strong connection between the interrelating aspects of Vygotsky’s theory of scientific concepts and various ‘reform’ or progressive pedagogies in mathematics (see ch.4). In relation to making mathematical connections, there are general and particular aspects of the theory which are relevant. Of the general points, the first is that Vygotsky stresses the weakness of purely verbal attempts to transfer knowledge – that concepts only arise (or develop) for an individual as part of attempts to solve genuine problems (Vygotsky, 1987, p.123) – ‘the concept is formed only with the emergence of a need that can be satisfied in the concept, only in the process of some meaningful goal-oriented activity directed on the attainment of a particular goal or the resolution of a definite task’ (p.127). Allied with his understanding of the importance of dialogue in social activity, this connects the theory to the emphasis on problem solving and dialogue which, rightly, lies at the heart of most meaningful approaches to mathematics education. The first conclusion to be drawn in relation to making and generalising connections within mathematics is that these too need to be necessitated by a problem situation, as opposed to merely being pointed out by a teacher. Although students can themselves generalise spontaneously even when connections are simply stated or implicit (in general terms, see Davydov, 1990, p.62), explicitly problematising the connections themselves ought to maximise the developmental potential of the mathematical activity. This point is reinforced by what follows and the analysis of the next section.

The second relevant general point is that concepts (including scientific concepts) develop (Vygotsky, 1987, p.167). If scientific concepts are viewed as the structured totality of their relationships with both other concepts and with physical and social activity, then development here means qualitative changes in this ensemble of relationships. Such development occurs through the introduction of new factors (in concept development, primarily through new social demands.
being placed on the individual, p.132), but also alongside and in interrelation with the dialectical emergence of qualitative change from accumulated quantitative changes (see Vygotsky, 1997, p188), that is, through accumulation, and interrelation, of the concept’s inner connections. The drawn out nature of this process allows space for higher and lower levels\textsuperscript{6} of generalisation, and hierarchical or other connections between concepts, to develop alongside each other. And this complex and overlapping process plays a key role in concept development.

Many, generally positive, approaches to mathematics education include, explicitly or implicitly, a uni-directional view of mathematical development. For example, the Van Hiele (2004) level model, or the progressive vertical mathematisation of realistic mathematics education (RME) (Treffers & Vonk, 1987, p.62), assume that lower levels of mathematics must be mastered before more complex forms are attempted (see e.g. Freudenthal, 2002, p.97). To an extent this is true within Vygotsky’s theory too: The abstract contains and therefore needs rich concrete experience (Vygotsky, 1998, p.37), scientific concepts (those in a definite system) require a certain level of development of everyday concepts (Vygotsky, 1987, p.219), and higher levels of complexity of generalisation rely on lower levels of generalisation (p.230). But for Vygotsky this is not the end of the story. The key to development is the existence of more developed cultural forms of concepts being brought into relation with an individual’s current conceptual forms (primarily through instruction). Structuring progressive mathematisation through carefully chosen problems is certainly one means by which this can occur, but it is not necessary, useful, or even possible, to wait for the lower levels to fully develop before engaging with more complex processes. There is some evidence of this in practice: for example in geometry, the area in which Van Hiele first developed his level model, students often operate at multiple levels, even within one problem (Burger & Shaughnessy, 1986, p.45). The different levels can exist simultaneously, at varying levels of development, for any individual. But also they continue to develop in relation to each other. For Vygotsky, ‘instruction always builds on a foundation that has not yet fully matured’ (Vygotsky, 1987, p.231), and ‘is only useful when it moves ahead of development’. More complex generalisations are seen to place demands on the individual that require and

\textsuperscript{6}Here ‘level’ refers both to levels of complexity/difficulty (akin to Van Hiele, 2004, see following paragraph) and, more narrowly, to hierarchical levels of generalisation.
stimulate development of the lower level generalisations (p.213). At the same
time the process of generalisation creates a structured connection between lower
and higher level concepts which then becomes part of the lower level concept,
qualitatively changing it.

The final point to make regarding scientific concepts, or generalisations within
definite systems, is their relationship to conscious awareness. For Vygotsky, such
generalisations and conscious awareness are synonymous and allow voluntary
control (Vygotsky, 1987, p.191; the fact that one can be consciously aware of
everyday concepts too hints therefore at an element of systematicity within them
and the less than sharp distinction between these forms in practice – see the
discussion in sec.8.4). Conscious awareness is distinct from consciousness, for
example,

I tie a knot. I do it consciously. I cannot, however, say precisely how I
have done it. My action, which is conscious, turns out to be lacking in
conscious awareness because my attention is directed toward the act
of tying, not on how I carry out that act... However, it can become the
object of consciousness when there is conscious awareness.
Conscious awareness is an act of consciousness whose object is the
activity of consciousness itself (p.190).

This metacognition occurs, for Vygotsky, through generalisation, which creates a
new relationship to the process or object, effectively isolating it, and allowing
conscious awareness and control. For scientific concepts, where this ‘relationship
is mediated through other concepts that themselves have an internal hierarchical
system of interrelationships’ (p.191) this ability to isolate is more pronounced,
allowing a metaphorical grip on concepts7. The introduction of such concepts
through schooling effectively ‘opens the gate’ for conscious awareness, and
leads development in childhood. More immediately relevant here though, is the
reinforcing of the earlier point regarding the importance of making mathematical
connections, and their generalisation, the explicit focus of problems. It is possible

7 Vygotsky refers to his isolation metaphor as crude. A slightly more complex version is
to say that concepts are not objects but processes and relationships, moments in activity
as it were. The process of concept formation and development is that of reification, or
objectification, of these processes and relationships, enabled by words and signs.
Concepts are effectively sculpted into ‘objects’, not by the removal of anything but by
the bringing in of relationships (particularly structured relationships such as hierarchical
generalisation) through the addition of different perspectives on them. For more on the
ontology of concepts, see addendum to chapter 4.
to make connections within a mathematical problem without the connections themselves being the focus of attention. Such activity can lead to spontaneous generalisation of those connections, from below as it were, but there is no necessary generalisation involved and the mathematical activity can remain situated and unaware. However, if some form of generalisation of the connections is the explicit problem then this can create a new perspective on the concepts involved, and allow the various mediations of those concepts, such as the context, the problem and previous understanding to interrelate while conscious attention is focused upon them.

Vygotsky himself suggests that one weakness of his theory of scientific concepts is that it is too general (Vygotsky, 1987, p.240). In using primarily social science concepts as a prototype for scientific concepts the particular nature of different types of scientific concepts and the particular structure of relations for each is undeveloped. To reach a more particular understanding of the development of mathematical concepts requires not just attention to the formal logical relations of the mathematical system itself but to all the potential mediating factors – the processes of learning, teaching and doing mathematics and the objects, experiences and activities which are part of particular mathematical abstractions, and which these abstractions are part of. To begin to make this more concrete the next section briefly examines various ways of making mathematical connections in relation to the general themes addressed here of problem solving, generalisation, conscious attention and bi-directional development. The sections which follow then delve deeper into a particular attempt to problematise generalisation in order to see the micro-processes at work within classroom dialogue.

7.3 Pedagogical approaches to generalising connections

This section surveys a range of approaches to making connections in mathematics in relation to the theoretical perspective outlined above. In order to categorise pedagogical approaches to connections it is perhaps tempting to begin with the various types of connections possible within the formal logical structure of mathematics (see for example, the nevertheless useful, Businskas, 2008). However, this would concede too much to the myth of abstract mathematics’ detachment from concrete human social activity, particularly when
addressing the teaching and learning of mathematics from a problem solving perspective. This is not to underestimate the importance of the particularities of those formal mathematical structures and logical relations but rather to see those particularities as secondary to more fundamental processes. Here instead analysis begins with a loose distinction between mathematics (including structure, models, representations, and processes) and problems (including context). This leads to the following connections-related distinctions, each of which is explored in its variety.

### 7.3.1 Same mathematics, different problems

Although the main focus in this article is on the connections between parts of mathematics which are usually taught separately, there are similar processes of generalisation at work when it is primarily the problem or context which changes and the mathematics remains within the same area. Even attempts to replicate a given process on a new example within transmissionist teaching can, with some generosity, be viewed as problem-solving for the student. Questions help structure answers and there is mutual mediation between this structuring under the circumstance of slightly different numbers and context and the attempt to replicate a given mathematical procedure. The ensuing slight variation in the procedure then provides an opportunity for generalisation. Beyond transmissionism, for example, in generalising number patterns into algebraic forms, similar variation in examples, particularly with well-chosen examples (see Zazkis, Liljedahl & Chernoff, 2008) can help stimulate the process of generalisation. However, in such examples explicit generalisation is the focus of the mathematical task.

To make generalisations through the connections made between varied examples without this explicit focus is less likely (see, for example, in a different context, Gentner, Loewenstein & Thompson, 2003). As in Vygotsky’s simple example of knot tying, attention is generally focused on the act of replicating the process rather than the process itself, and therefore there is no necessary generalisation involved (however, see the method of procedural variation, e.g. Lai & Murray, 2012, for perhaps the maximal encouragement of generalisation through variation within transmissionism). In more genuine problem solving, that is, when the process to be followed is not essentially given, this route becomes
closed off, and more conscious, active processes can arise (see for example Hiebert et al., 1996). Importantly, such awareness can then be strengthened through their reproblematisation in the acts of communicating, comparing and justifying solutions (see ch.4). These latter processes position concepts in new mediating problem structures, providing new perspectives on them which expose inner connections and contradictions, and encourage generalisation.

A more fruitful aspect of transmissionist curricula is their cumulative nature. Often students may use some mathematics they have learned in a wider, or more complex, process. For example, having memorised the rules for the arithmetic of negative numbers they may eventually apply this in multiplying out two sets of brackets. It is difficult to recall and use the process for, say, multiplying two negative numbers without having some awareness that you are doing it. This, in a sense, allows viewing such arithmetic from a different perspective and therefore offers increased potential for conscious generalisation. However, the practice of embedding one merely memorised process within another again generally acts to limit the connections made.

Theming by content in transmissionist teaching, for example, by putting all algebraic work in a block, also allows some potential for generalisation from below due to the temporal proximity of the work. Again though, without a need for students to develop connections or generalise to higher level concepts this suffers from similar limitations. A step up from this approach is to instead theme material by concept (see, for example, Bosse, 2003). Generalising concepts across different mathematical domains and making rich connections at least seems more likely in such an approach, but could equally relapse into normal practice if such connections are not explicitly focused on within problems.

In some ways similar to theming by concept, is the use of one model across different real-world or mathematical contexts. Within the RME tradition, Streefland has shown how some models, such as the bar model in relation to proportion problems, can be effective in encouraging generalisation from below (see Van Den Heuvel-Panhuizen, 2003). The structure of such models lie close to the structure of initial problems and contain key aspects of the structure of more formal mathematical approaches, supporting the classic transition from being a ‘model of’ to a ‘model for’ as it is used in an increasing variety of contexts.

More generally within RME, a key part in the increasing formalisation of vertical mathematisation is played by reflective abstraction – student reflection on their
own (and other students’) concrete mathematical productions (e.g. Streefland, 1991, p.31), particularly through acts of comparison (e.g. p.319). These elements of RME resonate well with Vygotskian theory; however, there are differences between the two frameworks. First in that the Vygotskian perspective suggests that there is perhaps a possibility that particular models can easily remain at an unconscious level, and therefore simply be a superior rote-learned process for dealing with particular types of questions. But second, that the reflective reproblematisation (e.g. through a comparison of models), which can make practice more conscious through requiring generalisation, is in danger of being postponed too long, perhaps indefinitely within a unidirectional vision of ‘gradual progression’ (e.g. Streefland, 1991, p.19).

7.3.2 Same problem, different mathematics

Having made the loose distinction between problem/context and mathematics it may be clear from the above that this categorisation is already beginning to dissolve. The problem/context may itself be other mathematics (e.g. as in when theming by concept), using some previous mathematics within a more complex procedure in a new problem mixes bits of mathematics together, and problem activities may be evolved towards multiple mathematics (e.g. in comparing models). However, this section persists with the distinction to look at further situations where multiple forms of mathematics are consciously brought together within a single problem.

Encouraging the use of different methods, representations, models or procedures when tackling a mathematical problem (allowing ‘cognitive autonomy’, see Stefanou, Perencevich, DiCintio & Turner, 2004) can increase engagement and motivate students toward developing deeper understanding. Particularly in more open and social problem solving this also provides rich opportunities for generalisation (of mathematical structure and strategies for solving – see Hiebert et al., 1996) between previous mathematical experience, other students contributions and problem structure. And this is especially so when generalisation is explicitly required through the comparison or justification of methods (e.g. see Rittle-Johnson & Star, 2007). This problematising of the comparison of methods is extended in Boero’s ‘voices and echoes’ game (Boero, Pedemonte & Robotti, 1997; Boero, Pedemonte, Robotti, & Chiappini, 1998), which comes from a
similar Vygotskian perspective, and with similar concerns to this article. In this, students are introduced to the 'voices' of important mathematicians and scientists through texts and given problems which in some way utilise those voices. This leads to acts of generalisation between students' own understandings, the task and the 'voices' bringing contradictions and similarities to the fore and making them more conscious of the structure of their own methods.

Although connections between different areas of mathematics are rarely made within schooling, the one area where connections between different representations are most explicit is in the relationship between algebra and graphs. Again, without explicit attention to generalising these connections they may remain at a procedural level though. One way to go beyond this might be in the use of computing tools which integrate visual and symbolic forms. This allows a freer flow between the representations than when using pencil and paper and seems to increase the possibilities for generalisation of the relationship between them (Healy & Hoyles, 1999). Combining this approach with problems which explicitly discuss the connections between representations raises conscious awareness and control over those connections and improves operational ability within the different representations compared to normal pedagogical methods (Porzio, 1994).

7.3.3 Problematising mathematics

There are commonalities in the higher forms of activity encountered in the previous two sections. In both sections deliberate conscious generalisation of the mathematics appeared, for example, through reproblematising the mathematics used in a problem through the requirement to communicate or justify, or, through comparing or justifying different models or approaches in relation to one problem. Problem solving is at the heart of meaningful mathematical learning and should dominate activity in the classroom, and any attempt to generalise mathematical structure assumes some previous use of the mathematics within a problem or problems. This means explicit generalisation can often be integrated within other problems, but sometimes generalisation can be introduced as a more distinct task. One such method is when students are asked to draw concept, mind or knowledge maps (see Brinkmann, 2003; 2005) of a mathematical area. These are seen to aid students in visualising and structuring their existing knowledge
but the process can also lead to the development of new connections and concepts. One suggested drawback in this approach is the confusion which can arise for students due to the many possible connections which can be drawn.

A similar approach to mind-mapping is taken in the problem given to students in the data analysis below, except here the focus is narrowed to one particular hierarchical relationship rather than on mapping the relationship of many concepts. In this, students are explicitly asked to generalise across various mathematical tasks which involve proportional thinking. The original inspiration for the problem actually lies in Davydov’s (1990) critique of simplistic empiricist views of generalisation – for example the idea that the concept of red arises through spotting ‘redness’ in a variety of objects, isolating this property and generalising to the abstraction red (p.34). One problem with this view is that the initial step appears to require the concept which is to emerge. Nevertheless, any objects or concepts which are generalised are in some sense the same (and, at the same time, in other senses different). Preselecting the ‘objects’ combined with awareness that the generalised concept will likely exist in less developed form already for some students was assumed to provide the potential for isolating the similarities and generalising them within a concept. The students were therefore simply told that the different mathematical topics were somehow connected, or ‘the same’ and asked to figure out why, or in what way.

The task is therefore an explicit problematisation of the generalisation of connections. The dialogue which emerged from the problem is used to illustrate the type of discussions which can arise in such problems, and is used to further concretise the theoretical framework discussed so far. Before exploring the data however, some relevant particularities of the group of students and the topic of proportion are outlined.

7.4. Generalising proportion

7.4.1 Teaching and learning mathematics

The project drawn on here is a short course on teaching and learning mathematics for undergraduates. This course is primarily aimed at two types of students, those who are aiming to enter initial teacher training in primary
education following their degree, and youth and community work students who feel that knowledge of teaching and learning may assist them in their future job roles, either through direct support within educational establishments or in more informal situations. The course is designed to introduce the main elements of progressive forms of teaching which put an emphasis on conceptual understanding (here labelled, connectionist teaching, see Askew, Brown, Rhodes, Wiliam, & Johnson, 1997), and is theoretically informed by Vygotsky’s theory of scientific concepts (see ch.4). These elements include: what mathematics is (or should be), centred on problem solving, modelling and concepts; the various types of connection between mathematics and the real world; the connections within mathematics (both in terms of the mathematical system and the practice of doing mathematics e.g. heuristics); dialogue and justification; and reflexivity and metacognition. The teaching and learning mathematics course is also taught in a connectionist way, with much mathematical problem solving activity, and with dialogue and reflexivity about those activities in relation to this systemic approach to teaching and learning.

The dialogue analysed here is taken from a brief classroom activity as part of the general theme of connections in mathematics. As described in the section above, the task was an attempt to test an alternative approach to problematising the mathematical system through explicitly generalising mathematical connections. The students in each of four groups were presented with a variety of proportion related mathematical topics on separate sheets (including similar triangles, equivalent fractions, pie charts, converting fractions to percentages, sharing in ratios and trigonometry, see appendix 2). Each sheet contained an explanation of how to tackle problems taken, with deliberate irony, from a classic example of the disconnected curriculum, the BBC Bite-size revision website (BBC, 2014). Each sheet also had one or two problems for students to try. The students were asked to work through one topic each, preferably one where they already had some confidence or familiarity. Although all students in the class had mathematical experience up to GCSE level, for many this experience was at least several years in the past and few of the students expressed much confidence in their mathematical abilities. The three tutors in the classroom therefore assisted with the process of re-familiarisation on the various topics with particular attention to those who appeared to be facing the most difficulties. After around ten minutes of this the students were then told to work out within their group what connected the different problems and to find what they had in common.
The first thing to note is that none of the students expressed immediate recognition that the topics were all representations of proportional thinking. On one level this is not astonishing. Transmissionist teaching of atomised curricula is the norm in schools (Pampaka et al., 2012) and therefore it is unlikely that any of the students would previously have been encouraged to see the connections between the materials. However, given the preponderance of proportion in the school mathematics curriculum this fact should be astonishing. For example, an unscientific review by the author of some recent GCSE exam papers revealed that around 90% of questions involved multiplicative reasoning of some form, and approximately 50% related directly to proportional thinking.

7.4.2 Some aspects of proportional thinking

There has been much research on the conceptual intricacies of, and pedagogical approaches to, multiplicative and proportional reasoning. Two of the most useful long-term projects are that of the Rational Number Project in the U.S (see, for example, Behr, Harel, Post & Lesh 1992) and the work of the Concepts in Secondary Mathematics and Science Project in the U.K., and its follow-up projects (see Hart, 1981 & 1984; Hart, Johnson, Brown, Dickson & Clarkson, 1989). A third major influence on thinking in this area is the didactical phenomenology of Realistic Mathematics Education (RME) (Freudenthal, 1983; Streefland, 1991), and it is from this work that features relevant to the classroom dialogue below are drawn.

RME has an emphasis on contextual or meaningful problems and the progressive mathematisation of children’s own mathematical productions through guided reinvention, often via carefully chosen problems and models. Didactical phenomenology of a mathematical topic involves ‘describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind...as far as this description is concerned with the learning process of the young generation’ (Freudenthal, 1983: ix). That is, it addresses mathematical topics and concepts not just in terms of their formal logical structure, but also from the perspective of the human activities from which these generalisations arise, alongside the particular issues of their development within teaching and learning. Of the many insights into fraction, ratio and proportion derived
from this approach, the following are particularly relevant to the analysis of classroom dialogue which follows:

i. There is a common, early, practical-based sense of proportion and comparing ratios, (see e.g. Brink & Streefland, 1979);

ii. It is possible to equate two ratios without contemplation of reducing those ratios to a number or magnitude (Freudenthal, 1983, p.180);

iii. The processes involved in i) and ii) are easier in the visual field than with magnitudes (p.189);

iv. There is a strong conceptual interrelation of fractions and ratio (p.134). Indeed, the interrelation of a wide range of concepts in this field (including part-whole, decimal, ratio, proportion and linearity etc.) is hugely important (see Kieren, 1976; Vergnaud, 1988);

v. Ratio represents a higher level concept than fraction because comparison is an essential part of the concept (Freudenthal, 1983, p.180);

vi. Fractions are the ‘phenomenological source’ of rational numbers (p.134) and often dominate, and reduce the complexity (and understanding) of, classroom work on ratio and proportion;

vii. Conceptualisation of fractions is in turn dominated in practice (p.145) and in pedagogy (p.147) by the limited part-whole relationship;

viii. There is an important distinction, and shift in difficulty, between ratio comparisons which remain within one system of measurement (internal) and those which also involve a transformation between measurement systems (e.g. from time to distance – external);

ix. The early sense of ratio and proportion can become blocked by the learning of algorithms which are prone to automatisation (p.209).

7.4.3 Attempts at generalisation
The discussions of the four groups in the classroom are now surveyed, with particular attention paid to the forms of connections, contradictions and conflicts (i.e. engaged disagreement) which arose within the groups.

7.4.3.1 Group 1

This first group is perhaps least developed in terms of their response. At first they bounce around some of the mathematical features seen in the processes before them – division, ratio, fractions and multiplication – which are all key aspects of the proportion concept, and are concepts which proportion can be a key aspect of. Some of these features are confirmed to be in the different topics, but it is unclear whether any of the students are at least implicitly connecting the different features. They then voice a generalisation at a broader level than intended, and recognition that the generalisation is too broad is seen in their laughter.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Jon</td>
<td>Both of ours have made division... to get the answer.</td>
</tr>
<tr>
<td>Leanne</td>
<td>Oh this has got division [too].</td>
</tr>
<tr>
<td>Sal</td>
<td>This is ratio.</td>
</tr>
<tr>
<td>Kylie</td>
<td>And fractions [This may represent a connection of ratio to fractions for the student but this is not explicit].</td>
</tr>
<tr>
<td>Leanne</td>
<td>And fractions yeah.</td>
</tr>
<tr>
<td>Sal</td>
<td>There’s a triangle, let’s do the triangle.</td>
</tr>
<tr>
<td>Leanne</td>
<td>The percentages have got multiplying and fractions as well. Like you were saying they have. [pause] It’s all maths. [Laughter].</td>
</tr>
<tr>
<td>Jon</td>
<td>Good answer.</td>
</tr>
</tbody>
</table>

Following this, a tutor appears briefly to check their progress, and Leanne suggests the connection is, ‘They all seem to have fractions in them’. This is a step forward which unites all the problems around one feature. The common appearance of fractions relates to the a/b=c/d representation of proportion which is present in many of the examples. The fraction concept can contain, and therefore be, an undeveloped proxy for the concept of proportion, but, arguably,
here it seems stuck at the level of being a recognised common object in the process. The tutor encourages them to develop this by suggesting they look at what is being done with the fractions. Alone again, the group suggest some possibilities: ‘Some form of sum or multiplication or whatever has had to have been done to…’; ‘Like problem solving?’ (a reasonable guess given the nature of much of the mathematical activity on the course); and then,

<table>
<thead>
<tr>
<th>Leanne</th>
<th>Formulas?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jon</td>
<td>Oh yeah.</td>
</tr>
<tr>
<td>Leanne</td>
<td>Like for example for the ratio and the percentage…</td>
</tr>
<tr>
<td>Jon</td>
<td>Good call.</td>
</tr>
<tr>
<td>Leanne</td>
<td>… there was like a pattern, like you had to follow a set formula to get the answer.</td>
</tr>
<tr>
<td>Jon</td>
<td>And this one’s got a formula, going 2,3,4 5, etc.</td>
</tr>
<tr>
<td>Leanne</td>
<td>This one [pause].</td>
</tr>
<tr>
<td>Sal</td>
<td>Yeah, this is a formula as well, to get sine.</td>
</tr>
<tr>
<td>Leanne</td>
<td>There you are.</td>
</tr>
<tr>
<td>Jon</td>
<td>And you say you’re not good at this...</td>
</tr>
</tbody>
</table>

With this the group seems satisfied and they drift off into discussions of their previous mathematical experience. Again, like ‘it's all maths’, the generalisation is too broad, and this discussion barely moves to the conceptual level of the process level which dominates mathematics education. Nevertheless the group have come up with some forms of generalisation – a common element which is intimately connected with ratio and proportion, fractions, and a common activity, following formulas and given processes.

7.4.3.2 Group 2

The second group spend almost as little time on their discussion, again quickly finding generalisations which satisfy them, but this time they are the ones that the tutors hoped for. This group had slightly higher GCSE grades in mathematics in
comparison to the other groups, and had attended school more recently. Here, a tutor is at first continuing to explain the trigonometry example to one student and this merges into the group discussion on connections. Trigonometry is perhaps the least likely of the topics present to have been related to proportion in previous school experience. Why there is a common sine ratio for any particular angle is a question often deemed unnecessary given the mnemonic SOHCAHTOA and the rote learned procedures which can deliver correct answers.

![Trigonometry Diagram](image)

*Figure 7.1. Excerpt from trigonometry worksheet*

<table>
<thead>
<tr>
<th>Tutor A</th>
<th>[To Jan] Although, it [the trigonometry question] actually boils down to a question of ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Tutor A</td>
<td>Because that’s 2 to 1 [see Fig.7.1].</td>
</tr>
<tr>
<td>Sam</td>
<td>[listening, excited] Oh, that’s like that one that’s ratio [sharing in ratio problem], and this [similar triangles] works down to ratio, because it’s like 3 to 4.5 and 4 to 6 [see Fig.7.2], what the difference is.</td>
</tr>
<tr>
<td>Jan</td>
<td>Oh yeah. That is it isn’t it, ratio.</td>
</tr>
<tr>
<td>Sally</td>
<td>Mine’s is like fractions into percentages, so.</td>
</tr>
<tr>
<td>Liz</td>
<td>That’s what this is.</td>
</tr>
<tr>
<td>Sally</td>
<td>So you’re scaling it.</td>
</tr>
<tr>
<td>Sam</td>
<td>Oh right, so it’s all about, like, scaling things.</td>
</tr>
</tbody>
</table>
Although the origins of this collective realisation of ‘scaling’ perhaps lie initially with the tutor, the power of the concept of ratio leads to a rapid generalisation across most of the students and their tasks. The concept of being in proportion (ratio) connects equally rapidly in the transcript to the concept of a process which keeps things in proportion while changing size (scaling). As in group 1 there is a generalisation which seems satisfying to the group and the conversation moves on, again, to previous school experience of mathematics. Later, in a whole class discussion on what the groups have found, Sam and Kath talk to each other to remind themselves about what they had found:

<table>
<thead>
<tr>
<th>Kath</th>
<th>You said a word for it. Like, proportion. Not proportion.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sam</td>
<td>Oh, proportional.</td>
</tr>
<tr>
<td>Kath</td>
<td>Was it?</td>
</tr>
<tr>
<td>Sam</td>
<td>Yeah, proportional.</td>
</tr>
</tbody>
</table>

Figure 7.2. Excerpt from similar triangles worksheet

The concept ‘proportional’ appears as another equivalent to the general conceptual connection found. The group then contributes to the whole class discussion:

<table>
<thead>
<tr>
<th>Sam</th>
<th>They were all, like, about proportion.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutor B</td>
<td>Yeah? Can you explain?</td>
</tr>
</tbody>
</table>
So the interconnected concepts of ratio, proportion and scaling provide a generalisation across the different problems at an appropriate level which can encompass comparison of numerical objects including measurements, comparison of visual objects, and the activity of transforming between these objects. These concepts clearly pre-exist to an extent for the students, and exist integrated with the process of numerical calculation in at least some of the separate domains, but there is still room for development in the act of generalising the concepts across different mathematical topics. Even here though, the various concepts remain, at least explicitly, at a level of a loose conglomeration. Further focussed attention on the relationship between concepts such as ratio and proportion, mediated by the mathematical examples before the students, may have provided a useful next step here, given more time.

7.4.3.3 Group 3

<table>
<thead>
<tr>
<th>Type of vehicle</th>
<th>Number of vehicles</th>
<th>Calculation</th>
<th>Degrees of a circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cars</td>
<td>140</td>
<td>(140/270) x 360</td>
<td>= 187</td>
</tr>
<tr>
<td>Motorbikes</td>
<td>70</td>
<td>(70/270) x 360</td>
<td>= 93</td>
</tr>
<tr>
<td>Vans</td>
<td>55</td>
<td>(55/270) x 360</td>
<td>= 73</td>
</tr>
<tr>
<td>Buses</td>
<td>5</td>
<td>(5/270) x 360</td>
<td>= 7</td>
</tr>
</tbody>
</table>

*Figure 7.3. Excerpt from pie chart worksheet*

In this group and the next more interesting things occur as conscious attention to systemic connections begins to draw out contradictions in the conceptual understanding of the mathematics. Here, the meaningful activity of scaling, or,
transforming while keeping in proportion, which can exist separate from numerical calculation, clashes with the details of the mathematical process.

<table>
<thead>
<tr>
<th>John</th>
<th>So what have we got to do? Look at what?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sonja</td>
<td>What’s similar between them all? What’s yours?</td>
</tr>
<tr>
<td>John</td>
<td>Mine is just pie charts, and the angles that it’ll give you. The degrees innit [isn’t it]. So like here, the example was 270 makes the total, so whatever the amount was you divided it by the whole amount and timesed it by 360 which will give you the degrees.</td>
</tr>
<tr>
<td>Sonja</td>
<td>Right, so you basically have got to take from 360 and… [The use of the word ‘take’ here may motivate John’s continued description of the procedure as it could mean ‘take away’, which is not seen to be part of the process. However, as seen in Sonja’s later contributions she may be using the term in a much looser sense, either literally as ‘take’ a share or part of, or in some other way connected to that process.]</td>
</tr>
<tr>
<td>John</td>
<td>Nah, cause the survey, there was 270 vehicles but 140 were cars so to find out how much that should be on the pie chart…</td>
</tr>
<tr>
<td>Sonja</td>
<td>You have to take it from 360.</td>
</tr>
<tr>
<td>John</td>
<td>You do 140 times, divided by 270 times 360 that’ll give you the answer.</td>
</tr>
<tr>
<td>Sonja</td>
<td>Oh right.</td>
</tr>
<tr>
<td>John</td>
<td>So down here I had to write…</td>
</tr>
<tr>
<td>Sonja</td>
<td>I don’t understand…</td>
</tr>
<tr>
<td>John</td>
<td>You could use a calculator like…</td>
</tr>
<tr>
<td>Sonja</td>
<td>No, what I’m saying is like, when you look at, it’s like percentages, it’s like you’re making, it’s the equivalent, that is just a different way of writing that isn’t it.</td>
</tr>
<tr>
<td>John</td>
<td>Hmm, yeah, this is, yeah.</td>
</tr>
<tr>
<td>Sonja</td>
<td>And that’s what this is. I’m doing, a fraction, three quarters is the same as nine twelfths.</td>
</tr>
<tr>
<td>John</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Sonja</td>
<td>So, that’s sort of similar, and,… I don’t know what she’s [Chloe] doing there…</td>
</tr>
</tbody>
</table>
A more generalised, deeper understanding of proportion would incorporate both the aspects on display in the conversation above. Sonja seems to see the meaningful overview of what is going on in each example, that under transformation something is essentially staying the same. This illustrates Freudenthal's point about the ability to equate two ratios without calculation. The pie chart proportions are 'equivalent' which in turn is 'like percentages' and 'three quarters is the same as nine twelfths'. What is the same is not made explicit and the word proportion is not used, but there is a clear sense of something being the same. This aspect of generalisation, that two things are the same despite being different in other respects is part of all the separate proportional problems. This overview level seems useful in being able to generalise the generalisations in a single concept of proportion. Despite its lack of explicitness Sonja's focus on this level has already allowed her to articulate a connection between the different problems. John, on the other hand, is focused on the process of how you transform a share of a total to an angle in a pie chart. This reflects the dominant view of understanding in mathematics education that of knowing the process that leads to the answer. If all the examples were explored at this level then commonalities may emerge, and this too may become a route to generalisation, for example, through the form $a/b=c/d$. Without mediating concepts, such as scaling, however, this path seems likely to become distracted into the small differences in the procedures of each example. The discussion between John and Sonja continues, for example:

<table>
<thead>
<tr>
<th>John</th>
<th>So if there was 50 people, well it would have been 50 divided by 60, do you know what I mean. So whatever the amount is you divide it by the people who are involved and then times it by 360.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sonja</td>
<td>See, I don't understand all that bit. I just know that you've got to get a pie chart and that fractions out of 360 innit. So, yeah I understand but, I can't be arsed with it, it just does my head in.</td>
</tr>
</tbody>
</table>

Sonja says she is a dyslexic student, and that she struggles with long chains of process in a noisy classroom environment, and wanting to avoid the details is a useful motivation within mathematics for generalisation. However, the integration of the procedure and the overview (e.g. the ability to explain each through the other) cannot be completed by ignoring the details of the procedure. Sonja's
suggested answer, when a Tutor appears, seems pregnant with the proportion concept, and does contain elements of the numerical procedure:

| Sonja | I think it’s just different ways of writing the same…[pause] they’re fractions and these are fractions and it’s just different ways; it’s equivalent fractions, it’s the same. See that’s what I’m saying, I just simplify it, it’s just that you’re changing that into a fraction out of 360. All the bits that you’ve done in the middle does my head in but, you’re just changing the denominator that’s all. |

![Figure 7.4. Excerpt from equivalent fractions worksheet](image)

In a larger group, with a wider range of problems, the words proportion or scaling may have emerged from one of the problems, as in the previous group, to provide a handle for this emerging concept. The third person in this group who has been quietly working away on her own is brought into the discussion by the tutor:

| Tutor C | What’s that got to do with similar shapes? [pause][to Chloe:] Can you tell us about the similar shapes then? What you’ve figured out so far. |
| Chloe | I don’t think I can, em…[Pause]…so, they’re the same, the same shape but with different… |
| Sonja | Well that’s similar to, where’s mine, where did I put mine, [moving things around]. Oh yeah, sorry. That is similar to that… to me. [Pause 20 secs as people look]. I don’t know but it just kinda seems similar, because the three… [Pause] that is bigger isn’t it though, isn’t it, that is bigger…bigger shape but, the same thing. |

Again, Chloe has a sense of ‘the same but different’ without being able to make explicit what is the same. Sonja recognises the connections between the problems and can extend the description of similar triangles to ‘bigger’ ‘but the
same’. And, interestingly, the word 'similar' seems to transfer from the particular example of similar triangles to Sonja's description of the general connection (yet this appears to be without conscious awareness). However, the conversation then moves on to an explanation of how to work out the lengths in similar triangles and doesn't return to the question of generalising in the short time remaining, so the emerging concept does not explicitly emerge or develop further.

7.4.3.4 Group 4

In this group a dispute about what constitutes a fraction intersects with the attempt to find a connection between the problems. Initially the group splits into two smaller groups to compare their examples. In the first group, Huan, who is the only student in the class with achievement at A-level, attempts to explain the trigonometry example to Janet. After a while Janet says:

<table>
<thead>
<tr>
<th>Janet</th>
<th>I don’t know how you relate to me because I’m doing converting into percentages.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Huan</td>
<td>I think we need some, something in the middle, like the concepts.</td>
</tr>
<tr>
<td>Janet</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Huan</td>
<td>Fractions?</td>
</tr>
<tr>
<td>Janet</td>
<td>Triangles?</td>
</tr>
<tr>
<td>Huan</td>
<td>Yeah.</td>
</tr>
</tbody>
</table>

Consciousness of the need for mediating concepts is a notable surprise in the dialogue. Meanwhile Mary, Mike and Diane discuss their examples. Mike explains that his involves simplifying fractions, and Mary connects this to the fractions in her pie chart example:

<table>
<thead>
<tr>
<th>Mary</th>
<th>I think, mine is pie charts so it has to be divided up into fractions for pi, it could be related</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mike</td>
<td>Cause, you’re dividing it aren’t you, you’re dividing up the area</td>
</tr>
</tbody>
</table>
Mary: [To Diane] What’s yours? Similar triangles... I don’t know how that relates to these two [laughs].

Mike: Do you have to do something else?

Diane: Eh, [laughing] I can’t, what that side is.

Mary: The length of the side?

Diane: Yeah.

Mary: Can fractions be used to help that kind of thing?

Mike: X equals y and all that?

Diane: Em.

Mike: [Pointing at a calculation on the similar triangle sheet] Well, it is, we’ve got fractions there.

Diane: Well the answers there but I don’t understand how. I get, right, up there it’s like three over four, and then that’s four point five over the x, which is a six.

Mike: Three over four equals four point five...

Diane: Cause that, do you know what I mean it’s just the same side.

Mike: Oh you use both of them.

Diane: So that’s that one, and that’s...

Mary: What’s that one and that one got to do with that one? [Pause].

Mike: Three over four, so that’s that times that, yeah.

Mary: Well there’s fractions and your fractions so there’s something to do with...

Mike: Is that divided by...?

Diane: Are these fractions though? [Pointing at 3 over 4 and 4.5 over 6].

Mike: Yeah.

Their first attempt is to extend the fraction link to the question of similar triangles, 'Can fractions be used to help that kind of thing?' And Mike spots what looks like fractions on the sheet, 'We've got fractions here' (\( \frac{3}{4} = \frac{4.5}{x} \), see Appendix 2). This is essentially the a/b=c/d form of proportion applied to pairs of sides of similar triangles, but the calculations are confusing Diane. As in the discussions of group 3, Diane has a sense of what is going on in her example when she says, 'it's just
the same side', but this expression of being in the same proportion doesn’t appear to resonate sufficiently. Mary then reasserts the fraction connection between them all, but there is an obstacle to this emerging candidate for what connects the problem: ‘Are these fractions though?’ A tutor sits down briefly as this question is asked:

<table>
<thead>
<tr>
<th>Tutor C</th>
<th>They look like fractions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diane</td>
<td>But they’re not they, they’re just written differently.</td>
</tr>
<tr>
<td>Tutor C</td>
<td>Yeah, well [tone implying this is a difficult question] they’re not fractions.</td>
</tr>
<tr>
<td>Diane</td>
<td>I wouldn’t say that was a fraction, but I don’t know.</td>
</tr>
</tbody>
</table>

Then after the three students briefly check the length of the missing side with the tutor, and explain how they got that answer, they return to the question of fractions:

<table>
<thead>
<tr>
<th>Mary</th>
<th>They’ve used this. They’ve used fractions to break down this.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diane</td>
<td>But they’re not fractions really? It’s just the way it’s written out. See what I mean, cause how’s that a fraction?</td>
</tr>
<tr>
<td>Mary</td>
<td>What are they meant to be then? Three of four is three fourths.</td>
</tr>
<tr>
<td>Diane</td>
<td>Yeah, but how’s that got anything to do with three quarters, it’s just like... Do you see what I mean?</td>
</tr>
</tbody>
</table>

From this last exchange, it seems Diane may have a restricted view of fractions as the sharing of a unit. ‘The fraction as part of something is of such a convincing and fascinating concreteness that one is easily satisfied with this one phenomenological approach and forgets about all others. In all examples, whether visualised or not, one restricts oneself to fracturing’ (Freudenthal, 1983, p.147). It could also be that Diane has a sense that something has been lost in reducing the ratios of sides in triangles to fractions [here ¾ and 4.5/6]. The others may be more comfortable extending the use of fractions in practice but are not sufficiently conscious in their usage to be able to offer an explanation. There are three important generalisations that may be involved here. The first is perhaps
one of the most important types of generalisation in mathematics, to see a process and the object resulting from the reification of that process as two aspects of a unity (see Sfard, 1991). With fractions this generally means being able to move between the process of dividing a unit into parts and taking so many of them, and the fraction as mathematical object. The second generalisation is seeing the equivalence between that process with regard to a unit and the process of dividing multiple units into a number of parts (e.g. the equivalence between finding $\frac{3}{4}$ of a unit and dividing 3 units into 4 equal parts). This second generalisation can aid a movement between division in any context and fraction activity. The third generalisation, which can link with the others, is that between ratio and fraction. Tutor C, above, is of course technically correct to say that, for example, $4.5/6$ is not a fraction, but is clearly hedging the answer with the tone of his response. To deal with the question may have been seen to distract from the central problem the students are addressing. With the tutor now gone, the two groups compare what they have found, but quickly return to this question.

<table>
<thead>
<tr>
<th>Janet</th>
<th>[To Mike, Mar and Diane] What are yous doing?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mike</td>
<td>We’re trying to work out this [similar triangles].</td>
</tr>
<tr>
<td>Diane</td>
<td>Are these fractions?</td>
</tr>
<tr>
<td>Mike</td>
<td>Yep.</td>
</tr>
<tr>
<td>Huan</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Mike</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Diane</td>
<td>They’re not. How’s it a fraction?</td>
</tr>
<tr>
<td>Huan</td>
<td>Is it about similar…?</td>
</tr>
<tr>
<td>Diane</td>
<td>It’s about, like, this… [Shows sheet].</td>
</tr>
<tr>
<td>Huan</td>
<td>Ok…yeah. [pause] And what? What do they want you to do?</td>
</tr>
<tr>
<td>Diane</td>
<td>I don’t really understand [laughs].</td>
</tr>
<tr>
<td>Janet</td>
<td>Let’s have a look.</td>
</tr>
</tbody>
</table>

Janet: [To Mike, Mar and Diane] What are yous doing?

Mike: We’re trying to work out this [similar triangles].

Diane: Are these fractions?

Mike: Yep.

Huan: Yeah.

Mike: Yeah.

Diane: They’re not. How’s it a fraction?

Huan: Is it about similar…?

Diane: It’s about, like, this… [Shows sheet].

Huan: Ok…yeah. [pause] And what? What do they want you to do?

Diane: I don’t really understand [laughs].

Janet: Let’s have a look.
The two students Mike and Huan are quite happy to label the numbers on the sheet fractions, but no justification is offered. To speculate, this may be because the generalisation exists in practice rather than in conscious understanding. With more time, and greater awareness of the debate on behalf of the tutors this question could have become an opportunity for wider classroom discussion, and through that potentially greater conscious awareness of the generalisations involved.

The students turn again in the wider group to the similar triangles sheet. Janet reads out some of the explanation on the sheet, which discusses how to work out missing lengths using algebra.

<table>
<thead>
<tr>
<th>Janet</th>
<th>Oh! [To Huan] This is similar to yours.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Huan</td>
<td>Yes…yes.</td>
</tr>
<tr>
<td>Mike</td>
<td>Then it’s similar to mine cause it’s fractions.</td>
</tr>
<tr>
<td>Huan</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Janet</td>
<td>And mine’s fractions. Mine’s converting into percentages using fractions.</td>
</tr>
<tr>
<td>Mike</td>
<td>So that is, [pointing at pie chart sheet] that is percentages as well.</td>
</tr>
<tr>
<td>Huan</td>
<td>Ok.</td>
</tr>
<tr>
<td>Janet</td>
<td>Right.</td>
</tr>
<tr>
<td>Mary</td>
<td>Yeah, this is degrees of a circle, which works out fractions.</td>
</tr>
<tr>
<td>Janet</td>
<td>This is something to do; [To Mike] was this what you were doing where length is in proportion to this length.</td>
</tr>
<tr>
<td>Mike</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Huan</td>
<td>Yeah it’s a proportion, that word? Yeah. So got it?</td>
</tr>
</tbody>
</table>

Again it is fractions that are taken to be the connecting link initially. Mike’s reference to pie charts as involving percentages, and his grouping of these 'percentages' together with fractions seems interesting, and may indicate a merging of various representations of the sharing of a total in an implicit generalisation. Janet is the first to use the word proportion however. This appears to come from the sheet on similar triangles. It's important to realise here that the students have not been asked to create the concept of proportion from
scratch. All of them are assumed to have some sort of concept of proportion and scaling already, although this will be limited to particular representations, and some of the materials before them mention these words. Janet may be beginning to connect up all the problems with the concept of proportion here, but Huan certainly seems to see it as the answer they are looking for. Unfortunately the others don’t explicitly respond to this idea and Mike re-asks the initial question:

<table>
<thead>
<tr>
<th>Name</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mike</td>
<td>So, what’s the connection?</td>
</tr>
<tr>
<td>Mary</td>
<td>Fractions.</td>
</tr>
<tr>
<td>Mike</td>
<td>Fractions and percentages.</td>
</tr>
<tr>
<td>TutorB</td>
<td>[appears and sits down] Did you come up with an answer?</td>
</tr>
<tr>
<td>Mike</td>
<td>Em, we said fractions and percentages.</td>
</tr>
<tr>
<td>TutorB</td>
<td>Hm?</td>
</tr>
<tr>
<td>Huan</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Mike</td>
<td>Is that right?</td>
</tr>
</tbody>
</table>

Huan’s ‘Yeah’ here despite having seen proportion as the answer already is not surprising as fractions and percentages are concrete instantiations of the proportion concept. In fact, fraction is arguably ‘the word by which the rational [as in ratio-nal] number enters’ (Freudenthal, 1983, p.134). And the students and groups who spotted the common fractional form were therefore on a potential path to the generalisation required. Although the ‘overview’ of seeing that scaling occurs as in group 3 seems an easier path to the generalisation than detailed comparison of the process and its elements, either could play a part in the process, and both are necessary for a fully developed understanding. Again with further time for this task, it could have been possible to return from the generalisations of proportion and scaling to the level of the various similar processes involved in the problems to develop a more integrated understanding.

The tutor is not satisfied with fractions and percentages as an answer and asks for more, prompting Huan to talk of proportion. However, the running debate about what counts as a fraction reasserts itself briefly.
<table>
<thead>
<tr>
<th>Tutor B</th>
<th>Something even deeper than that.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Huan</td>
<td>Proportion.</td>
</tr>
<tr>
<td>Tutor B</td>
<td>Yeah, what do you mean by proportion?</td>
</tr>
<tr>
<td>Mary</td>
<td>It’s not the connection, but are they fractions?</td>
</tr>
<tr>
<td>Tutor B</td>
<td>Which?</td>
</tr>
<tr>
<td>Mary</td>
<td>How it’s laid out, that’s laid out like a fraction is it?</td>
</tr>
<tr>
<td>Tutor B</td>
<td>Yes and no.</td>
</tr>
<tr>
<td>Diane</td>
<td>But it’s not a fraction is it. Cause how’s that a…</td>
</tr>
<tr>
<td>Tutor B</td>
<td>Yeah, you’re told you can’t put a half in a fraction. But, a fraction is just a division, and a division is just a fraction, and…</td>
</tr>
<tr>
<td>Mary</td>
<td>So dividing is the answer.</td>
</tr>
<tr>
<td>Mike</td>
<td>Say, dividing yeah.</td>
</tr>
<tr>
<td>Tutor B</td>
<td>Kind of. [To Huan] What were you saying? What word did you use?</td>
</tr>
<tr>
<td>Janet</td>
<td>Proportion.</td>
</tr>
<tr>
<td>Huan</td>
<td>I think it’s the word proportion.</td>
</tr>
</tbody>
</table>

The tutor's response of 'Yes and no' bears the typical hallmark of generalisation, when things are both the same and different at the same time. There is a grudging acceptance of the technical rules before making a casual but explicit generalisation between division and fractions, but this just prompts those fixated on the process to switch answers to division. The tutor tries to avoid this and re-prompts Huan, and both Janet and Huan express the looked for concept. A discussion then follows where the tutor tries to draw out the scaling that is occurring in each of the problems. By this stage recognition quickly follows, for example Mike says, 'Oh yeah. So in mine’s it’s scaled it up'. An emphasis in problem solving pedagogy on an active role for students does not of course lead to all students making a particular generalisation at the same time. There will always be unevenness, even where the generalisation appears collaboratively (e.g. as in group 2). Therefore an individual student or even a tutor can sometimes successfully 'transmit' a generalisation to others. However, this process should not be cut short and substituted with pure transmissionist teaching in general. The engagement with a genuine problem, immersion in the
task and the searching for an answer all help provide material that can be crystallised by such interventions within collaborative problem solving.

7.4.4 Summary

Problematising the mathematical system, by attempting to force a generalisation of the concept of proportion and connections between normally separated mathematical topics can be seen to have led to a range of potential and genuine mathematical developments. In group 2 it led to a conscious generalisation of ratio, proportion and scaling across a range of previously distinct mathematical topics, at least to some extent and for some students. In group 3 it raised a fertile contradiction between an emphasis on meaning and process which with further development could have linked proportional awareness with the formal \( \frac{a}{b} = \frac{c}{d} \). In group 4 the problem both echoed some of the developments in the other groups and provoked questions on the nature of fractions and their relationship to ratios, which again with more time and space to develop could have led to further development of inner connections between the students’ mathematical concepts of fractions, ratio and proportion. Even Group 1’s narrow focus on mathematical element and process, although limited, could have played a part in the wider debates in the class.

7.5 Conclusion

Analysis of Vygotsky’s theory of scientific concepts suggested the following aspects of a pedagogical approach to making connections within mathematics: i) that a genuine need for the connection must arise within a real problem; ii) that attempts to generalise connections are most effective in developing conscious awareness and control; and iii) that generalisation of connections need not wait for the full development of the concepts and connections which are generalised but in fact aid that development. A brief review of the literature on various types of pedagogical task dealing with connections indicated some success for similar methods as part of a more general problem solving approach in the classroom. The classroom dialogue analysed in the final sections illustrated in greater detail some of the processes at work when this type of mathematical task is attempted.
Although the usual health warnings should apply when such a comparatively brief snapshot of classroom data is presented, there are some claims that can justifiably be made. First, importantly, it is possible to have classroom discussions such as these (i.e. of quite an abstract nature). Although these are adult students and their mathematical activities are mediated by also learning about teaching and learning, the students are not particularly confident in mathematics and so such activities should be generalisable beyond this cohort. Second, aspects of the concept to be generalised do clearly exist already to some extent for at least some of the students (variation in this seems to make the task easier for some than others). And different aspects may make generalisation easier (e.g. the non-calculation based sense of scaling versus the numerical procedures), or in fact may be generalised earlier (allowing different levels to develop at the same time and in relation to each other). Third, the activity does focus conscious attention on concepts and connections, subjecting them to real scrutiny, with contradictions and connections arising from this. Fourth, there is also some evidence of the beginnings of the higher level generalisation developing the lower (e.g. in group 2’s recognising of various proportion related concepts at work in their remaining topics). However, any longer term conceptual development, where the hierarchical connection becomes a useful part of the various concepts, at whatever level is not something that this classroom dialogue can provide any evidence for.

As suggested in the previous section there are various ways in which this task could have been extended with more classroom time devoted to it. There are also many other areas of mathematics where similar generalisation of connections is possible (even within particular proportional topics, questions of ‘what is the same’ and ‘what is different’ may provide a useful activity). The method’s use as an occasional pedagogical approach therefore seems worthy of further investigation and development. Theoretically also there is much that needs to be developed. For example, how the various mediations at work in such activity (the problem itself, the systemic mathematical structure, existing student knowledge, the contributions of others, the type of generalisation to be made etc.) interrelate requires further investigation. And, in particular, more evidence is required of the longer term development of mathematical concepts, seen through a variety of tasks, in order to justify the claim that higher levels (of complexity, of generalisation) lead the development of lower levels.
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8 Being systematic: The development of a heuristic concept in a connectionist mathematics classroom

David Swanson

Abstract: This paper explores mathematical concept development from a Vygotskian perspective through tracing the development of a particular concept, that of the heuristic—being systematic—among a class of students on an undergraduate course on teaching and learning mathematics. First, Vygotsky's theory of the development of scientific concepts, and its relationship to connectionist teaching in mathematics, are outlined. The particular concept explored here, and the nature of the context, a course itself shaped by Vygotsky's theoretical understanding and connectionist pedagogical practice, are then described. The story of the emergence and development of the heuristic concept within the class is then related in some detail in order to illustrate the interrelation of the theoretical elements and their practical application.

In conclusion, some further questions arising from the data in relation to developments of the theoretical and pedagogical system are discussed. The issues include the influences on, and dynamics of, individual utterances within dialogue; the pedagogical balance between cultural transmission and individual agency; and, the relationship between everyday and scientific concepts. For this latter question, some potential problems in Vygotsky's presentation of the relationship between everyday and scientific concepts are outlined, and some formulations of the Italian Marxist Antonio Gramsci are introduced as a useful supplement to Vygotsky's approach.

Keywords: Concept development, heuristics, Vygotsky, Gramsci, Mathematics education


8.1 Introduction

This paper explores mathematical concept development from a Vygotskian perspective through tracing the development of a particular concept, that of the heuristic *being systematic*, among a class of students on an undergraduate course on teaching and learning mathematics. Vygotsky’s theory of the development of scientific concepts and its relationship to connectionist teaching (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997) in mathematics are first outlined (see ch.4 for a fuller account). Then, the particular concept investigated here, and the nature of the context, a course itself shaped by Vygotsky’s theoretical understanding and connectionist pedagogical practice (see sec.3.5.4 for more details), are described. The story of the heuristic concept within the class is then related in some detail in order to illustrate the interrelation of the theoretical elements and their practical application.

In addition, the analysis of classroom data leads to further discussion in the conclusion on the use of Vygotsky’s theory to inform and underpin a systematic framework for mathematics pedagogy. Some questions arising from the data, suggesting the need for further developments in Vygotsky’s theory when applied to classroom practice, are discussed. The issues include an understanding of the influences on the formation of particular utterances within mathematical dialogue; the relative weight given to, and interrelation of, cultural transmission and student agency within pedagogy; and the relationship between everyday and scientific concepts. For this latter question, some potential problems in Vygotsky’s presentation of the relationship between everyday and scientific concepts are outlined, and some formulations of the Italian Marxist Antonio Gramsci are introduced as a useful supplement to Vygotsky’s approach.

8.2 Vygotsky, scientific concepts and connectionist teaching in mathematics

The theoretical framework used here is that of Vygotsky’s (1987) theory of scientific concepts, which, in general, concerns the development of the relationship between systems of abstraction and the concrete in social activity (here, the concrete is taken loosely as the real world and experience of it, but more specifically as “perceptual”, or “practical, action-bound thinking” based on
sensory impressions or function respectively, see Vygotsky, 1986, p.138). The theory describes key elements in the development of scientific concepts which can be mapped across to important practices in the mathematics classroom (see ch.4 for more detail).

Vygotsky (1987) stresses a) that concepts only arise and develop within attempts to solve genuine problems (including problems of communicating or reasoning) where there is a need for the concept (p.123); b) that scientific concepts are generalisations which arise through a bi-directional process of abstraction, where abstraction is seen not as the shedding of the rich concrete, or the lower level abstractions, which are generalised, but instead as the bringing in of other connections, that is, the systemic relationships to other concepts (p.224); c) that there is a bi-directional process of development between lower and higher levels of generalisation (or, between the everyday generalisations people make within activity and scientific generalisations) through this process; and d) that this process depends on words, and other signs, (necessary to synthesise the multiplicity of connections which a concept represents), and social dialogue, which brings together varied and different levels of understanding, thus allowing potential development.

These stresses can be related to teaching in some clear ways. For example, point a) fits very well with the problem solving and modelling approach at the heart of most progressive or reform pedagogies. Point b) speaks against traditional approaches to teaching which assume that the abstract can be transmitted in some way detached from the rich concrete which is an essential part of it. It also therefore provides a justification for methods which relate mathematics to visual, physical and social experience. Alongside these connections to the real world, it also suggests the importance of emphasising the connections within mathematics, particularly through acts of generalisation. Point c) reinforces these aspects of point b) but draws attention toward the role of students’ own understandings, either that generated through visual, physical or social experience, or through previous mathematical experience. It is these that provide a basis for, and are themselves reshaped by, developing generalisations.

More developed cultural forms of concepts are also part of this bi-directional process, and must to an extent come from outside the individual. Given that transmissionism is discouraged, this still leaves a key role for the teacher in shaping the environment through which concepts can be developed, primarily through the selection of problems, and the steering of developments in solving
them. Finally, point d) relates to the social dialogue which must be encouraged within the classroom to assist in these processes, particularly through reasoning and justification which connect the concrete or lower level generalisations to abstract systems.

The need for genuine problems, relating mathematics to the real world, to student understandings and to other mathematics, and the role of dialogue, justification and reflection therefore all have a theoretical underpinning in Vygotsky's theory. These pedagogical practices are those seen as central to connectionist teaching (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997), as shown in chapter 4. Vygotsky’s theory therefore provides a theoretical underpinning to this underdeveloped pedagogical approach (essentially until now a somewhat disconnected list of progressive practices drawn from a variety of pedagogical and theoretical perspectives) which can enable its more systematic application. It also brings with it implications for necessary developments of the connectionist approach, primarily in an increased emphasis on explicit generalisation and the connections of the mathematics system.

This system of theoretical and pedagogical understanding informed the design of mathematical activities (as well as those activities related to understanding teaching and learning) in the class described below, in which the particular concept analysed develops. There are classroom tasks which connect the mathematics to the real world, and to student understandings, tasks which develop connections in the system of mathematics, and tasks which explicitly problematise generalisation itself in order to increase conscious awareness and control of the concepts involved. The development of a particular concept within the classroom is traced through these activities, in order to illustrate and deepen understanding of the outlined perspective, and, to challenge and develop that perspective further.

8.3 Concept and context

The theoretical perspective and its relation to aspects of pedagogy outlined above is now illustrated, and explored further, through its relation to examples of actual classroom dialogue and practice. The particular concept looked at here, the mathematical heuristic of being systematic, is traced in its development within an undergraduate classroom. To set some context before getting to the data,
there follows a brief discussion on the nature of heuristics within mathematics and mathematics education, and some further detail on the educational setting, and the students involved.

8.3.1 Problem solving and heuristics

Polya (2004) originally published in 1945, is usually taken as the starting point of a conscious attention to heuristics within mathematical problem solving (although he himself traces some of their earlier intellectual roots, p.112). Described as ‘mental operations typically useful in solving problems’ (p.130), and ‘provisional and plausible reasoning’ (p.113), heuristics are contrasted with the certainty of complete solutions and deductive proofs but are seen as providing the scaffolding for such proofs (p.113). ‘Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science’ (p.117). This places heuristics not just at the heart of mathematical activity but also at the heart of the formal mathematical system itself. This is even more so given Lakatos’ (1976) arguments that the deductivist style of presentation of proofs provides only an illusion of certainty, and hides their heuristic development (and potential future development) through, for example, counterexamples, refutations and criticism (p.142).

Since Polya, Schoenfeld’s work on explicit instruction in heuristics has perhaps been the most influential in the field of mathematics education (see Schoenfeld, 1992, for an overview within the wider problem solving context). This has shown that Polya’s heuristics are more complex than they initially appear and so are often too general to be implemented without reduction in their complexity (see, e.g., Schoenfeld, 1992, p.353), but also that explicit training in the use of heuristics does have the potential to improve problem solving performance see e.g. Schoenfeld, 1979). Schoenfeld (2013) continues to defend the role of explicit heuristic instruction as part of a more developed approach to problem solving; however, doubts have been raised about its effectiveness (English, Lesh, & Fennewald, 2008). It is suggested that there is a problem of concentrating on rule governed processes in problem solving, that the relationship between concept development and heuristics, beliefs, dispositions or processes is underdeveloped, and that there is an assumed ordering where concepts are learned first, then heuristics and processes, and only then can these be put together (and therefore
that in practice this point is rarely reached). Thus English et al. argue for the criticality of research into the nature of the relationship between concept development and the development of problem solving competencies. Problem solving is viewed as ‘integral to the development of an understanding of any given mathematical concept or process’ (p.7) and a modelling perspective (which includes as part of it ‘identifying, integrating, modifying, or refining sets of mathematical concepts drawn from various sources’, p.8) is viewed as central to any problem solving activity. All these points can be agreed with. One path toward this, suggested here, is a simple one: to view heuristics themselves as concepts. They are generalisations of problem solving activity which includes within them their relationships to both mathematical systems and practical (and mathematical) experience. In Vygotsky’s terms they are, in general, scientific concepts as their relationship to the object (here, mathematical and practical activity) is mediated by other concepts within a definite system. This means that the methods for developing mathematical concepts within the classroom can be applied to heuristics too. That is, there should be a need for the concept to arise or develop within a genuine problem, connections must be made with both the real world and other parts of mathematics, there should be an emphasis on dialogue, justification and reflection to assist these processes, and there should, at times, be specific attention paid to the concepts themselves, such as through problematising their generalisation (see ch.7).

The particular heuristic concept examined here, that of ‘being systematic’ appears within Polya and Schoenfeld’s work. Polya points out that analogical conclusions drawn from parallel cases are stronger if those cases are systematically arranged rather than random collections. Varying the problem in a systematic way (illustrated by Polya with an example where n is taken as 1, 2, 3 etc.) is shown to be a key form of induction, which can lead to (the similarly named but different process of) mathematical proof by induction (Polya, 2004, p.43). Schoenfeld (1979) echoes Polya within his own problem solving strategies (p.79), with encouragement to attempt this approach in the presence of an integer parameter (this requirement is necessary for the basic form of proof by induction but not for the wider process of induction e.g. integer values can be drawn from measurements in order to aid pattern spotting). A second use of ‘being systematic’ in this context is as a means of ensuring completeness, and avoiding repetition, when attempting to exhaust all possible cases (see
Schoenfeld, 2010, p.99). This usually involves using structure and order to break down the cases into more manageable groups.

As a concept, therefore, being systematic connects to other heuristics, and to the mathematical system, in theory and in practice, in definite (and, perhaps, in yet to be defined) ways. At the same time it connects to practice through more everyday conceptions. Some of the aspects of the mathematical concept are also seen within everyday usage of the term systematic. For example, dictionary definitions (OED, 2014) include ‘arranged or conducted according to a system, plan, or organised method’, ‘regular’, ‘methodical’, and synonyms of systematic (Chambers, 2014) include ‘logical’ and ‘ordered’. It is to be expected therefore that the students featured in the classroom discussions below may have at least some aspects of this everyday usage of the concept, and perhaps even of its more mathematical usage.

8.3.2 Teaching and learning mathematics

The data which follows is taken from a short undergraduate course on the teaching and learning of mathematics designed on the basis of Vygotsky’s theory and its relation to connectionist teaching practice. The course was aimed at two particular groups of students. The first included students who were aiming to progress from their current degree to a PGCE (primary) teacher training course. For these students it represented an introduction to thinking about teaching and learning (and mathematics) undistracted by other issues they may face on a future PGCE, such as classroom management. The second group involved youth and community work students. These students also potentially faced dealing with mathematics education in their future careers, either through direct support roles within schools, or more informally when supporting individuals outside of school. It was thought useful to them to understand some of the difficulties students may face in traditional classrooms, and how they could perhaps assist in moving beyond these.

The students came to the course, therefore, with differing, if interrelated, needs. They also came with a wide variety of mathematical abilities (with many unconfident in mathematics despite their GCSE qualifications). The central aim of the course was therefore not to develop particular skills in either classroom teaching or mathematics, but instead to encourage students to develop some
understanding of what learning involves, and to develop the ability to critically reflect on teaching, learning and their own activity i.e. to provide the framework and tools to assist them in consciously developing the particular skills required in their own future contexts.

In order to do this, the course taught the elements of connectionist teaching and learning using connectionist methods. The themes of the sessions were framed using these elements: What mathematics is (including an emphasis on problem solving, modelling and concepts), the various ways in which mathematics connects to the real world, how mathematics connects to other mathematics (including both the systemic nature of mathematics itself and other connections in doing mathematics, such as heuristics), the role of dialogue and justification, and reflexivity. The sessions primarily involved engaging in mathematical problem solving activities and then stepping back to discuss and reflect on those activities in terms of the themes of the session. In this way the learning about what learning is followed a similar pattern to the presented aspects of the learning of mathematics, in that key concepts of learning were related to direct experience and to each other within a wider system, through reflection, dialogue and justification.

The concept of being systematic arose explicitly within four mathematical activities over several sessions of the class. These tasks and the dialogue arising from them are now explored in some detail. In what follows, each section of dialogue is related along with some initial analysis before it is then more directly related to Vygotsky's theory and the interrelated pedagogical system.

8.3.3 Task 1: Elastic band bungee jump

The first classroom task where the concept of 'being systematic' appeared was in a problem chosen as an introduction to problem solving and modelling. The task was to work out the number of elastic bands required to make an effective bungee jump. The students were given a 'person' (a weight), a small number of elastic bands and a metre ruler and they were asked to use the materials to make a prediction of the number of elastic bands they would need for a bigger drop in a stairwell outside the classroom. The task was not chosen with the aim of developing the particular concept, but rather because it simply represented a
genuine, and enjoyable, problem which does not require complex mathematics to solve.

There are several reasons why being systematic (i.e., here, starting by testing one elastic band, then two, etc.) is a useful approach in the context of this problem. First, it helps in finding the type of relationship between the number of elastic bands and the distance dropped. Previous practical experience may hint at a linear relationship, as does the pedagogical situation (asking these students to find a more complex relationship is unlikely), but whether testing that hypothesis or simply searching for any pattern, having as much data as possible, in a consistent form will aid that process. At the same time it assists in finding the actual numerical values of the linear relationship based on the step increase, it allows testable predictions, and, with multiple samples, it increases accuracy through allowing the averaging out of random fluctuations. Arguably, another more everyday sense of being systematic in this task could be seen in measuring the drop for each number of elastic bands several times and averaging out the results (overcoming the haphazard or random through using a system); And also, perhaps more trivially, making sure that the weight is dropped in a similar way each time.

The students in the class are grouped on four tables. Selections from three of the conversations as they worked on the problem are now discussed (a fourth is omitted due to its overwhelming similarity with the others).

8.3.3.1 Group 1

Group 1’s initial approach was to try dropping the weight with all seven elastic bands they had initially been given tied to it. At first they measure a drop of 60cm (twice) then 75cm (twice) for the case of seven elastic bands, but rather than trying to average these out in some way, the latest figure they find is assumed to be the correct one (although they do repeat again to confirm).

They implicitly assume a proportional relationship exists between the number of elastic bands used and the height dropped by the weight (this is mediated through the particular act of doubling). In fact when eventually told that the desired drop is 333cm, they multiply their answer by 4 (through repeated addition) to reach 300cm (and this gives them 4 times 7 = 28 elastic bands) seemingly
without the awareness of the more complex calculations that could encompass the additional 33cm drop.

Early on in their discussion (and before they knew the length of the required drop), one student raises the possibility of variation in the behaviour of individual elastic bands.

<table>
<thead>
<tr>
<th>SN</th>
<th>We have to… I think we have to try different, like, because we have to know the elastic consistency.</th>
</tr>
</thead>
</table>

And then a minute later in the discussion, this point seems to be repeated,

<table>
<thead>
<tr>
<th>SN</th>
<th>So we have to know the height.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JA</td>
<td>So if the height was 120cm we would need 14 elastic bands? Is that right yeah? [This stems from their initial measurement of 60cm for 7 elastic bands].</td>
</tr>
<tr>
<td>SN</td>
<td>Basically, because each of them is not the same, the elastic.</td>
</tr>
<tr>
<td>JA</td>
<td>Yeah, that’s what I mean, yeah.</td>
</tr>
</tbody>
</table>

Another minute later, JA is now expressing the same point,

<table>
<thead>
<tr>
<th>JA</th>
<th>About 750 [mm]. So that’s about 75cm. So if the height was double that we might need…</th>
</tr>
</thead>
<tbody>
<tr>
<td>LH</td>
<td>You’d need to double the elastic bands.</td>
</tr>
<tr>
<td>JA</td>
<td>..we might need, like, fourteen elastic bands, roughly. But it’s a bit hard because each elastic band will have a different elastic, because they’re not all the same.</td>
</tr>
</tbody>
</table>

This seems to challenge the assumed linear relationship, or at least the ability to make reasonably accurate predictions, depending on the variation in elasticity (and the required length of drop). Collecting more data at this point could establish how significant the variation is, but instead the conversation continues, as talk of random variation seems to spark talk of a different type of variation. The possibility is raised that acceleration may affect the behaviour. Longer chains of elastic bands may stretch disproportionately due to the increased speed the
weight will be travelling at. This is first rejected with confidence by one student on the grounds that the weight and elastic bands remain the same.

| JA     | The greater the drop the more speed it’s going to build up innit. |
| SN     | The higher? |
| JA     | Yeah the higher the drop. |
| SN     | And if the weight is thicker, it would be quicker. |
| LH     | Is that going to make a difference to our… |
| SN     | No because this is the same and this is the same [Pointing at weight then elastic bands]. |
| JA     | Ah right so it doesn’t matter. |
| SN     | The weight of the thing is the same, the elastic limit of the…the elastic bands are the same, so it doesn’t matter what is the height. |

A little later, the tutor intervenes with the whole class to create doubts about the initial strategy of most groups: ‘But how do we know that the same thing will happen every time you have an extra elastic band? How do we know the proportion stays the same?’ and suggesting that using all the elastic bands initially may not be the best strategy. One group in the class discuss how they started with 8 and then tried 4 to check precisely this, and that it was approximately proportional. Within group 1 this wider class conversation then sparks suggestions for the need for a more systematic approach.

| JA     | 28 elastic bands, what do you say? |
| MM     | Say, just try it with one, maybe two. |
| JA     | That’s long that. |
| SN     | But if we get longer, the stretch of the elastic will get different. |
| LH     | That’s what I was trying to ask before but I couldn’t phrase it. |
| SN     | Yeah, it’ll be different because… |
| LH     | So how do you work that out then? |
| SN     | Shall we try…? I’m thinking one and two and three before we calculate |
Student MM (who had entered late and missed most of the group’s earlier work) suggests, ‘just try it with one, maybe two’. Student JA balks at the effort that might involve, ‘that’s long that’. Student SN, who had initially raised the issue of inconsistency in the bands, but who had also been the one in the group to reassure the others that increasing speed wouldn't be a problem now says, ‘But if we get longer, the stretch of the elastic will get different’. And, ‘Shall we try. I’m thinking one and two and three before we calculate the difference but… [Laughs] sounds silly’. Therefore their own doubts about the proportional relationship, now reinforced by the brief class discussion have encouraged an explicitly more systematic approach from two of the students, but this is clashing with a general sense of time running out for the group. As a compromise they try 4 elastic bands (without separating them from the other elastic bands) and get 420mm, then try with one band several times (perhaps the trickiest to measure accurately) and get 110mm. There is no explicit attempt to fit this data to a linear pattern or to include accounting for their measurement to the nearest 10cm each time (e.g. an average drop of 106mm for each elastic band would explain each of the figures they have found so far). But there is a sense that it isn’t different enough to alter their prediction. So in the final stages of their discussion they decide to ‘gamble’ on 28 elastic bands because 300cm is sufficiently under the required 333cm, this number of bands is then decreased slightly due to nagging worries about the potential extra stretching due to the speed of a greater drop and this reduces their estimate to 25. Finally, some quick testing from different heights, but still within the classroom (technically cheating), as they add extra elastic bands encourages them to go back to 28 (their theoretical model) and even possibly add one or two more. This seems a reasonable approach given their time constraints.

Analysis:

In these selections of dialogue there is much to discuss in relation to Vygotsky's theory. For reasons of space and clarity however, here, and in the discussion of later sections, analysis is (and will be) restricted to some key points.

A key starting point for the theory is the need for genuine problems: that concepts only develop where there is a need for them to do so. This task is a genuine
problem (within the social context of the students' level of engagement), in that, the answer is not immediately obvious and requires some effort to solve. There is certainly some impetus for using the concept of proportionality within it (the group has this concept in concrete form – more on this below). There is also some impetus for development of this concept through addressing the problem. At a minimum level, even unconscious transfer of the concept to a new practical situation could be considered a qualitative change and thus potentially a development, however, a more substantial development, a more conscious use of the concept, is also motivated by the needs of the problem. The competitive element with other groups and the need for the bungee jump to get as near to the ground without touching it as possible, both encourage attention which may elicit a testing, or questioning as happens here, of proportionality.

As seen in an earlier section (8.3.1), connections with the concept of linear proportionality are central to this introduction to the heuristic concept of being systematic (which is probably the most common relationship deductively established through the process). In the above discussion, the implicit questioning of linearity by the group cries out for its testing in practice via a systematic approach. Again therefore there is a need for the concept within the problem, at least at a concrete level (i.e. the practice of testing one, two, three elastic bands rather than conscious awareness that this is a heuristic, 'being systematic'). In the end, for the students in this group, it takes a nudge from outside, an external questioning of only testing their full amount of bands, for this concrete concept to emerge ('I'm thinking one and two and three before we calculate the difference…'), but it does emerge, even if they don't act on it due to time constraints. In Vygotskian terms, this implicit sense of linear proportion, this act of testing one, then two etc., and the practical relationships between these, are precisely 'the concrete' which the scientific concept is built from as development continues.

In the above we see the importance of the heuristic concept being within the problem, of it being needed in its solution. We also see the importance of the physical situation itself – for example, it is the actuality of these elastic bands which provoke uncertainty as to their consistency. However, both of these aspects only have meaning in relation to the thinking of the students themselves. Their conceptualisations, in however concrete a form they occur, must be rooted in previous experiences whether physical, practical, discursive or mathematical even if these roots cannot be traced. In a sense the key elements of the concept
already exist for these students and are brought out by engagement with the problem and the activity. This is seen in three key places – first in the assumption of linearity (at least as a model), then in the questioning of linearity, and then in the leap to being systematic. In terms of Vygotsky’s theory these could perhaps be seen as the students’ everyday concepts which can then be interrelated with scientific concepts in schooling, but here they feel more than this, some explicitly containing aspects of mathematics and science (with reference to ‘elastic consistency’, and relationships such as larger drop ⇒ more speed, more weight ⇒ more speed, even if the science is sometimes incorrect). This raises questions which will be returned to later in the article.

8.3.3.2 Group 2

This group starts by seeing how many elastic bands are required to drop exactly one metre. Again, quite early on doubt is cast on there being a linear relationship.

<table>
<thead>
<tr>
<th>BC</th>
<th>Shall we try it with one less? Because if you drop it, the more elastic bands it actually has, the further it gets pulled down. So I don’t know, just try it, and then see.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JN</td>
<td>Yeah, yeah, so that would, the force of that would be pulling it more.</td>
</tr>
<tr>
<td>SO</td>
<td>No, no it definitely won’t. I’ll just hold it.</td>
</tr>
<tr>
<td>BC</td>
<td>We’ll just try it and see [pause].</td>
</tr>
</tbody>
</table>

And,

| JN       | If you drop that, the more gravitational pull, what it’s going to stretch down?                                                                                                                        |
| SO       | I think the longer the distance… [Pause] it’s got gravity to take into account.                                                                                                                        |
| JN       | Yeah exactly.                                                                                                                                                                                          |
| SO       | The gravitational pull is bigger the further it drops isn’t it?                                                                                                                                           |
| JN       | Yeah.                                                                                                                                                                                                  |

Although, as with the last group, their physics concepts stray from the dominant model (the force of gravity remains the same, as does the acceleration due to
gravity – it is the speed that is increasing) this provides a reason for checking the relationship.

**JN**  
I’m just thinking right..., if..., so, saying about gravity, the further it goes the more it pulls, so we couldn’t really judge it with just one metre. So if we’re going to have to take that into account then… Maybe each metre or whatever that we measure…

Here the concept of being systematic begins to appear, not through increasing the number of elastic bands one at a time, but in increasing the distance dropped instead. Given the number of elastic bands they have though, it isn't possible for them to check two metres, but

**JN**  
I’m going to try [something else i.e. throwing it upwards]. Did you see that? That almost went to the bottom.

**SO**  
Yeah, but you can’t, you’re chucking it up first.

**JN**  
True, but if you drop it, you know what I mean, you’re just…

**SO**  
But we’ve got to drop it from the thing though haven’t we. We’ve got to drop it from there.

**JN**  
But when I do that.

**SO**  
That’s making it more than a metre.

**JN**  
True, but that’s showing that from a higher distance.

**SO**  
Oh yeah. I know what you mean yeah. Do that again.

**JN**  
Alright.

**SO**  
Yeah, so we’re right. We were right, weren’t we?

Throwing the weight upwards (equivalent to having a faster initial downward velocity) seems to make the weight stretch further, and this convinces them that the greater speed of longer chains will lead to a relative increase in the stretch for those bands. Their initial approach is just to estimate the effect this will have when predicting for a greater drop.

**JN**  
See, I think that by each metre we do we’re going to have to take an elastic band off, so say we did 6 for the first metre, the second metre
After the brief class discussion, where the idea is raised of trying a different number of elastic bands to check proportionality, they also shift to being more systematic. First they try 3 elastic bands but then:

<table>
<thead>
<tr>
<th>JN</th>
<th>Yeah, I want to see how many cm one is doing, and then we can see, say, if like 2 elastic bands goes like 12cm, so we can see that’s going 10cm from the top, 20cm and so on.</th>
</tr>
</thead>
<tbody>
<tr>
<td>SO</td>
<td>Oh yeah.</td>
</tr>
<tr>
<td>JN</td>
<td>It's just easier to plus 3.</td>
</tr>
<tr>
<td>SO</td>
<td>So how many elastic bands is that, just 1?</td>
</tr>
<tr>
<td>BC</td>
<td>Yeah start with one.</td>
</tr>
<tr>
<td>SO</td>
<td>What's on it?</td>
</tr>
<tr>
<td>JN</td>
<td>Just do two. Start from two.</td>
</tr>
<tr>
<td>BC</td>
<td>Ok.</td>
</tr>
</tbody>
</table>

They proceed to check from two to seven elastic bands, and go back to recheck one anomalous result.

| JN          | Could you do 5 again?                                                                                                                                                                                |

And then,

<table>
<thead>
<tr>
<th>SO</th>
<th>The thing is if you divide it by the elastic each time, you'll find how much it'll go up with each elastic band.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutor</td>
<td>[who is nearby, listening] How much does it go up by each time?</td>
</tr>
<tr>
<td>JN</td>
<td>So it went up by 110 [mm], roughly every time.</td>
</tr>
<tr>
<td>Tutor</td>
<td>It seems a bit different there though.</td>
</tr>
<tr>
<td>JN</td>
<td>Yeah, that's the acceleration. I don't know.</td>
</tr>
<tr>
<td>BC</td>
<td>We might need to check the answer.</td>
</tr>
<tr>
<td>Tutor</td>
<td>Because that one looks the dodgiest?</td>
</tr>
</tbody>
</table>
They then go on to retest all their results and their prediction is the most accurate of the different groups when it comes to trying the final large drop of 330cm outside the classroom.

Analysis:

In this group, the same elements are seen as in the first group. The problem and physical situation interrelate with students’ previous understandings to draw out concrete versions of the concept. Again scientific terminology and relationships are brought in (force, gravitational pull, larger drop ⇒ more force), and in direct relationship to the relationship (between elastic bands and drop) they are trying to find. And again, therefore, the conceptions the students bring feel like something more than everyday concepts. The students then test the relationship, first through trying drops from different heights (a form of being systematic), and then, because this is impractical, throwing the object upwards, a method which is unlikely to offer accurate predictions. The initial decision to test the number of elastic bands per unit dropped, rather than the other way round, perhaps hampers their progress. This approach is quickly revised when they pick up what the rest of the class is doing.

This illustrates one key aspect of the importance of dialogue (another central element of Vygotsky’s theory and any pedagogical system based on it) – the existence of multiple voices, and therefore multiple opportunities for someone to come up with a leap or breakthrough which can either be adopted wholesale or incorporated in a more complex way into the thinking of others. Although here this takes place in a class wide basis, and is reinforced by the tutor as the assumed approach, this takes place also on a group level here and in all the other dialogues. The multiplicity of dialogue can also lead to disagreements. For example, here there is a small example when one student starts throwing the object upwards. Such disagreements, as in this case, can lead to greater consciousness, for both sides because they often require justification and reasoning. Although in this example explanation by one student was sufficient, at other times such reasoning can continue in argumentation, testing and developing concepts further in relation to the problem situation.
Again, in this group, a concrete version of the concept of being systematic emerges, and the approach is accepted as obvious once it is introduced. Its emergence and acceptance hint at some form of related past experience, or at least that individual student understandings can merge with elements of other key factors (the problem, the physical reality, the thinking of others) and crystallise in a concrete understanding of the approach. In dialogue, of course, the processes and connections underlying such leaps are often invisible. The objective elements in particular cases can be identified to an extent (as is done here), but much further study beyond particular cases is required to deepen understanding of this essential aspect of concept development.

Alongside a clearer verbalisation connecting the strategy of being systematic to the practical task (see student JN, ‘I want to see how many cm one is doing…’), there are further mathematical connections explicitly drawn here (see student SO, ‘The thing is if you divide it by the elastic each time, you’ll find how much it’ll go up with each elastic band). Of the different groups this one contains the students who have most recently completed school, and who on average have higher GCSE grades. This perhaps provides more resources to draw on, and therefore leads to slightly more conscious mathematical connections being made within verbalised reasoning. It should be remembered that most in this class have not been in school for several years, and are in any case new to a problem solving approach to mathematics. The need for dialogue, reasoning and justification is itself a heuristic of social problem solving which requires development over time.

8.3.3.3 Group 3

The third group begins by testing to see how far they can stretch individual elastic bands (i.e. using a different force than gravity) but don't know how to proceed from there. They see others testing out by dropping weights and try that approach instead, but similar to group 2, they test how many elastic bands are required to fall one metre. They begin by trying all eight elastic bands, and then quickly one student suggests:

<table>
<thead>
<tr>
<th>JK</th>
<th>Half that, so do only 4.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ST</td>
<td>Hmm?</td>
</tr>
</tbody>
</table>
JK | So do it with 4 elastic bands. No, I mean take them off so there’s only 4.
---|---
ST | Why?
JK | Ok, hold this.
ST | Why?
JK | Trust me.
ST | No, but why?

This seems to represent an aspect of the concept of being systematic emerging, the testing of different values, but also with the choice of 4 an element of systematicity (4, 8, 12 etc. is equivalent to 1, 2, 3…) in the form of a half/double proportionality. The meaning behind the student JK’s reluctance to explain why has multiple possibilities, but could represent an inability to easily put into words the reasoning behind an approach which can be described and practically carried out. The group proceed without acting on the suggestion though. They continue, implicitly assuming a linear relationship, but a question is raised about scaling up using this method.

ST | So 8 is right. 8 is a metre without him smashing his face on the floor.
---|---
JK | But then if you put 16 on, plus another one, ’cause don’t forget that [the gap between how far it falls with 8 elastic bands and a metre]’s still going about…
ST | Yeah, when we go measure outside how thingy it is, we know for each metre that we measure we need eight elastic bands.
JK | But there’s still that much gap so if you put another metre on that it’s going to be…
ST | But don’t worry because we don’t want to smash his face on the floor.
JK | No, what I’m saying is, if it goes higher and higher there’s that gap that you still need to fill.

Again a linear relationship is assumed, but given they have tested the number of bands per drop they are concerned about the scaling up of the gap (eight elastic bands didn’t quite reach one metre). The group explains their strategy to the tutor, who then explicitly asks about whether the relationship is proportional.
<table>
<thead>
<tr>
<th>ST</th>
<th>[Tutor appears] We were going to do it, we’ve just measured with the 8 elastic bands, it goes just short of the metre, so we were going to multiply the 8 by the metres.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutor</td>
<td>Ok, but do we know that the same thing will happen if we multiply?</td>
</tr>
<tr>
<td>JK</td>
<td>No.</td>
</tr>
<tr>
<td>ST</td>
<td>No, till we do it, till we practice.</td>
</tr>
<tr>
<td>Tutor</td>
<td>Is there something else we could do?</td>
</tr>
<tr>
<td>ST</td>
<td>Add another 8 on.</td>
</tr>
<tr>
<td>Tutor</td>
<td>Or?</td>
</tr>
<tr>
<td>ST</td>
<td>Subtract.</td>
</tr>
<tr>
<td>Tutor</td>
<td>Because we only have 8, so yeah. Maybe.</td>
</tr>
<tr>
<td>ST</td>
<td>See what the four does.</td>
</tr>
<tr>
<td>JK</td>
<td>We’ve done that.</td>
</tr>
<tr>
<td>ST</td>
<td>No, we haven’t actually done it.</td>
</tr>
</tbody>
</table>

The tutor is therefore just confirming an idea which was already present in the group. Student JK’s comment that ‘We’ve done that’ is not literally true but is in a sense psychologically true, at least for this student. It is a confirmation that this was his previous intention. This brief exchange will then ripple through the rest of the classroom when they go on to test 4 bands and then explain what they’ve done in the class discussion. As seen above, the other groups pick up on trying a smaller number of elastic bands and this then develops into a systematic approach. The actual contribution to the class discussion was,

<table>
<thead>
<tr>
<th>ST</th>
<th>[from group 3] We just took half of ours off to see if it was the same. We started with eight and took four off to see if it was half a metre because with eight it went a full metre.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JA</td>
<td>[from group 1] Did it go half?</td>
</tr>
<tr>
<td>ST</td>
<td>Yeah.</td>
</tr>
<tr>
<td>JA</td>
<td>Just short, ah sweet.</td>
</tr>
</tbody>
</table>

Because this group are convinced that the behaviour is proportional and that they can scale up, they don't try any further examples. They scale up more accurately
than group 1 did but they still have their early problem with the gap but they simply estimate the extra they might need.

Analysis:
Again in this group we see various elements of Vygotsky’s theory at play. The interrelation of the problem situation, direct experience with the real world objects, student understanding and dialogue. Again, a concrete form of the concept of being systematic, or at least aspects of it, emerges from the students, even though it is not originally acted upon (perhaps here, if the student who considered this option had justified it to the others it may have been taken up).

The role of the tutor is more prominent here, in directing attention, and nudging the students toward the need to test linear proportionality. Within Vygotsky’s theory there is a seeming contradiction (see sec.4.4) between an emphasis on the social leading development through higher forms of thought (e.g. the full cultural version of scientific concepts) interacting with lower forms (e.g. students’ everyday concepts) and an equally essential emphasis on an active role for students. Attention only to the first aspect can encourage viewing Vygotsky’s theory as more suited to transmissionist teaching in mathematics (as, for example, one reviewer of an article by the author has claimed), where teachers merely show how to follow processes and students then copy. The opposite side of this dichotomy would be pure discovery approaches or an extreme constructivism. However, both elements are necessary, despite their seeming contradiction.

It has already been shown that here to a large extent the problem posed already contains the solution, and the concept, in its structure (although not in isolation from the conceptualisations the students bring to it) – primarily in the need to test and find the linear relationship in a reliable way. This is arguably the primary way that higher levels of culture are introduced and mediated through teaching. Teacher questioning plays a similar role. Yet there is a wide range of possibilities within both problems and questioning in terms of how much active effort is required by students (at one extreme a question can become a statement). It has also been seen here that dialogue inevitably includes individuals making leaps which can crystallise understanding for other students while actively engaged in solving a problem. Teachers can at times play a similar role even if they are trying to avoid falling into the trap of transmissionism. The balance required, and
the form of interrelation required between, cultural ‘transmission’ and an active role for students is not something which can be fully answered here however. This is another key area for future investigation requiring much testing and empirical data beyond the classroom dialogue presented here.

This first classroom task involved a problem which requires the concept of being systematic (either consciously or concretely) to solve. Each of the groups begins to form and use this concept at various levels of explicitness. Their bringing in of scientific terminology and understanding when faced with finding a relationship (particularly the testing of linearity) expresses some of the key aspects of the concept of being systematic, and in this they seem clearly to be going beyond the everyday version of this (and other) concept(s). The dialogue above also raises questions regarding the intricacies of the formation of particular utterances, and the balance in pedagogy between cultural transmission and individual student agency. All of these questions require further attention, however, only the first, on the relationship between everyday and scientific concepts, will be addressed within this paper, when some thoughts on the issue are discussed in the conclusion.

The next task builds on the development of the concept of being systematic which has occurred so far, through introducing heuristics and ‘being systematic’ more explicitly.

8.3.4 Task 2 – introducing the heuristic of being systematic explicitly through a particular problem

This second classroom task took place in the fourth session of six (task 1 took place in the first session). This is one of two sessions which focussed on the connections within mathematics, and mainly introduced problem solving heuristics – i.e. approaches to try when solving problems which have general applicability across various contexts. Heuristics were explicitly introduced not by definition but in relation to a particular problem – finding how many squares there are on a chess board (this task was originally taken from Mason, Burton & Stacey, 1982, but with an awareness of its potential use in introducing heuristics via Toh, 2007). A series of typical heuristics were introduced by the tutor. Each was projected on the whiteboard as the class worked through the problem, and the
Figure 8.1. Squares on a chessboard. The problem as posed to the class.

tutor would immediately translate the heuristic (or ask the students to translate the heuristic) into how it could be immediately applied within the problem. These heuristics were 1) Think of a related problem; 2) Draw a picture; 3) Try a simpler problem; 4) Be systematic 5) Make a list or table 6) Look for patterns; 7) Guess and check; and 8) Extend or generalise.

Having covered heuristics 1 to 3, the students have individually worked out simpler problems – the cases of a 1 by 1 square and a 2 by 2 square. The tutor then makes two explicit references to the heuristic of ‘being systematic’ while the phrase 'be systematic' is projected on the board. The first in asking 'so we want to be systematic, what size square should we try next?' and confirming that a 3x3 square (followed by a 4x4 square) would be next. The second, while 'be systematic' was still on the board, followed the students working on the 3x3 problem and one student having shared their solution with the class. This helped restructure the solution, by breaking down and ordering the different sized squares to be found within it.

| Tutor | … let’s maybe start by working out how many 1 by 1 squares there are, then how many 2 by 2s there are, and then how many 3 by 3s there are. So we can break it down like that and be nice and systematic about it. |
Analysis:

It should be noted that there is no student dialogue to discuss for this task which directly related to the concept under investigation. Students mainly worked individually, and inter-student and whole class discussion focussed on other matters. This, the form of introducing heuristics, and the above quote introducing a method all therefore seem redolent of a transmissionist approach which was usually avoided on the course. There were of course reasons for this in the time constraints of the course and the particular session. With more time it would have been useful, for example, for the various heuristics to emerge within the class through a meta-analysis of their own approaches in various problems. This (in practice not overly concerning) drawback will be seen to be compensated for in the following task. A saving grace of the approach here is that the heuristics are not introduced via dry definition but mediated by a particular problem, and that there is a light (almost subliminal) touch to their introduction.

The key element of Vygotsky’s theory introduced here is that of generalisation and the importance of a scientific concept's place within a system of concepts. This appears in three ways. First, in the simple hierarchical generalisation of 'heuristics', of which being systematic is an example. This brings the connected meta-ideas of problem solving and approaches to problem solving into the concept. The more intricate complexities of how being systematic connects to the other heuristics within a process of problem solving (for example in finding patterns in processes with often linear relationships, often with integer increases in the independent variable etc.) are introduced, not explicitly, but mediated by the concrete example of the particular problem. Finally, being systematic is itself presented as a generalisation (of two different processes, with different purposes, i.e. step increases in the size of the chessboard with the aim of finding a pattern, and increasing the found inner squares by one each time to ensure completeness).

So, being systematic is introduced as part of a framework of heuristics, but as a concept it is only briefly mentioned. It is the practice of being systematic which dominates as students are expected to continue increasing the size of the square, and this practice is later connected to the practices of producing an ordered list and spotting a pattern. How much impact this has at the time on the students is impossible to tell. The next classroom task perhaps offers more insight into this, as students are given the problem of explicitly generalising the concept to previous classroom activities they have engaged in.
8.3.5 Task 3: Explicitly problematising generalisation

Task 3 immediately followed task 2 in the fourth session of the class. In groups, students were asked to match a given list of heuristics to various mathematical problems they had encountered over the previous three sessions. (In those sessions and here, groups vary from the groups formed for task 1, and not all students attended all sessions). The list of heuristics, slightly different from the list in task 2, included: Use models; Draw a picture; Try a simpler problem; Act it out; Use a suitable notation; Make a systematic list; Look for patterns; Guess-and check; Use algebra; Extend or generalise. Here the concepts from the previous list of ‘being systematic’ and ‘make a list’ are merged – as they often are in practice (results need to be written down, and this assists in spotting patterns or in obtaining completeness).

Following the first two tasks, the students will have some concept of ‘being systematic’. In the first task it was used concretely within the problem and aspects of it reached explicitness (such as the need to test one, then two, etc), without the concept itself ever being expressed explicitly. In the second they have acted out being systematic within a problem, and in concrete relation to the acting out of other heuristics, while the concept itself was simultaneously named and, at least, verbally embedded within the generalisation of heuristics. Here, the concept the students have will be drawn out and perhaps developed as it is tested against other problem solving experiences they have encountered (through making a mathematical problem out of explicit generalisation of the concept).

Each group of students selected three problems from those they had engaged with over the sessions. For two problems two groups said yes they had used a systematic list, but this seemed obvious to them and there was no additional dialogue on the question. Similarly, for another problem one group said they hadn’t used the heuristic, but again with no discussion. Although such generalisations of the concept may be assumed to play a role in the concept’s development, without further data which shows the students’ reasoning there is little that can be said about these, other than that there are contexts where the concepts use is already obvious (or obvious in its absence). More interesting to analyse are the examples where some additional justification occurred as the
concept was explained, or stretched, in the act of generalisation. This occurred in relation to three tasks.

1) Bungee jump

This was task 1, which has previously been discussed. Both groups who discuss this task make explicit the approach of trying one, then two, etcetera. Although for the first group it is enough to verbalise the initial one, with the progressive examples implicit, yet understood:

<table>
<thead>
<tr>
<th>BC</th>
<th>We made a list.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JA</td>
<td>Hmm.</td>
</tr>
<tr>
<td>BC</td>
<td>We did like one is so many metres.</td>
</tr>
<tr>
<td>JA</td>
<td>Yeah. So we looked for patterns.</td>
</tr>
<tr>
<td>BC</td>
<td>And the list.</td>
</tr>
<tr>
<td>JA</td>
<td>And makes a systematic list, yeah.</td>
</tr>
</tbody>
</table>

In the above dialogue BC was in a group, 2, which did use a systematic approach in task 1, whereas JA’s group, 1, discussed doing so but did not in practice. The following group contain three students from the earlier group 3 (who did test 8 and 4 elastic bands but did not extend this further) and two from group 1 (who discussed but did not implement). But:

<table>
<thead>
<tr>
<th>ST</th>
<th>Then we did the table.</th>
</tr>
</thead>
<tbody>
<tr>
<td>LH</td>
<td>Yeah that one.</td>
</tr>
<tr>
<td>DC</td>
<td>Yeah I didn’t write it down or…</td>
</tr>
<tr>
<td>LH</td>
<td>No.</td>
</tr>
<tr>
<td>ST</td>
<td>But we did make a…</td>
</tr>
<tr>
<td>DC</td>
<td>Did everybody though?</td>
</tr>
<tr>
<td>Tutor</td>
<td>This was like, somebody had one elastic bands, two elastic bands,</td>
</tr>
<tr>
<td>J</td>
<td></td>
</tr>
<tr>
<td>ZL</td>
<td>Yeah, yeah.</td>
</tr>
</tbody>
</table>
They are using table as a synonym of systematic list here. A tutor explicitly checks that they are referring to the pattern of 1, 2, etcetera, which they confirm, despite not having structured it this way themselves. The group 3 students (ST, DC and ZL) may of course have drawn a table with the two results they did have (DC remembers not doing that), but this somehow seems to merge with a memory of the collective class activity where some groups did implement this systematic approach.

2) Describe and copy a drawing

This was a task from session 3 of the course, which explored dialogue, reasoning and justification in mathematics. The problem was based on a popular task used within many educational settings. This involved students in pairs sitting back to back, with one describing images (here mostly made up of geometrical shapes), and the other attempting to recreate those images, where levels of communication allowed for the drawer varied from listening only, to yes/no questioning, to full dialogue. Two groups who attempted to match heuristics to this problem (unexpectedly) generalised the concept of being systematic (or here ‘make a systematic list’) in similar ways.
we’ll be like ‘don’t know’ [laughter].

JG  No, I’ve got it, I’ve got a reason.

KH  You’ll be like ‘it wasn’t a list and I didn’t do it but it’s still there’.

JG  It was a mental list. It doesn’t say it has to be written.

And;

VG  Make a systematic list.

SM  Kind of, ‘cause we like, when you start you start at a point and then go from there don’t you. So yeah.

The sense that they are generalising is of breaking the task down, doing one small step at a time, and not missing anything out. These are aspects of the concept, closely tied to the everyday sense of systematic. Usually, in applying this heuristic, there is a purpose of making it easier to spot a pattern in data, so the form of a list or table will usually appear in close connection. This causes the confusion in the first extract of dialogue, given this absence. The attempt to generalise the list structure alongside systematicity in the form of a mental list (presumably in having an order in which the individual parts will be described) is particularly interesting. This also causes the phrase ‘kind of’ in the second extract, which shows an awareness that the form is not identical, but that key aspects of the content can be carried across into this context.

3) Sharing chocolate bars

Again this task took place in the third session of the course and related to reasoning and justifying. And again, it used a common problem in mathematics, where different tables have different numbers of chocolate bars on them and students have to work out going to which table will give them the biggest share of chocolate. The group from the last piece of dialogue find a similar way to extend and generalise the concept of being systematic:
SM | Make a systematic list. Yeah, we did because we went one at a time up to the front didn’t we.
VG | Yeah.
AK | Yep, yep.

Here, there is a parallel with the sense of increasing by one and getting a different result each time. Within the chocolate bar task there was no attempt to go further and spot a pattern in these results, or use the systematic increase in any way but it is still interesting to see a connection like this being made, generalising an aspect of the form.

Analysis:
The aim of a task like this is to focus conscious attention on the concept itself. In terms of Vygotsky’s theory, it aims for generalisation and systemic connections to be made (thus allowing greater conscious awareness and control of the concept) while at the same time staying true to the need for genuine problems. The act of generalisation is itself the genuine problem here (see ch.7 for a wider perspective on such tasks). Here, although within a brief activity, it does seem to test and explore the concept in a conscious way, drawing certain aspects to the surface in relation to memories of recent mathematical activity. Also, the students easily relate the concept to the activity of task 1, a more typical example of the concept's usage, and thus seem to have more consciousness awareness of that activity, than most perhaps had during the task itself.

With more time this task could have been extended to look at the connections between the heuristics used in each task, and why each could be used within it, to deepen understanding of the concept and its relations to practical situations and other concepts. However, even if in truncated form, this task completes a cycle of activities which touch on the key aspects of Vygotsky's theory and the associated elements of pedagogy based upon it. The concept has been involved in a range of genuine problems; it has been connected to the real world, to students' understandings, and to other mathematics within a system, with all of this taking place through dialogue in social activity. The final task returns to the beginning, to see if anything has changed in the approach or understanding of the students following this process.
8.3.6 Task 4: Spaghetti bridges

So far the concept of being systematic has appeared in various activities: 1) a situation where a need for the concept arises and starts to take form, 2) Where it is explicitly named and illustrated and applied through an example, and 3) Where it is consciously generalised to other contexts. In the final task described here the students faced a similar problem to that of task 1. Students were given some sticks of spaghetti and some weights to experiment with. By making a bridge between two tables using the spaghetti they were to predict how much weight a bridge made of twenty pieces of spaghetti (in parallel) would hold. The weights they were given were not sufficient to break such a bridge. It was hoped that the students would hypothesise and explore a linear relationship between the number of strands of spaghetti and the weight they could hold, through applying some of the heuristics they had learned (including that of being systematic). The tutor initially showed them that one strand of spaghetti could hold a surprising amount of weight before breaking. Then they were given the problem of making a prediction and encouraged, in general terms, to use some of the heuristics they had learned. Unlike in section 8.3.3, here, further analysis is postponed until after a description of the attempts of all four groups.

8.3.6.1 Group A

Group A contains three students who were absent from the first session and so did not encounter the similar task 1, and one student SM who was in group 3 of the earlier analysis (the group who tested 8 and 4 elastic bands in that example). Initially SM questions whether it will be a linear relationship due to a quite reasonable proposition:

| SM | But it doesn’t work like that because it’ll accumulate… So if we’ve got one strand of spaghetti, that’ll hardly hold anything will it. No, because it doesn’t work that way because when you put them together as a whole they will hold more weight than as an |
individual. So, say you’ve got one strand of spaghetti, that’ll hold 30g. 3 strands of spaghetti will have more strength together than they would as individuals. It’s like the sum of the whole is greater than the sum of the parts, basically.

| AK   | Right so this is 20g. |
| SM   | Right, is this table wobbling? Do you want to move yours over a little bit? Thanks. Right so... one strand of spaghetti. So it obviously holds that. |
| VG   | This is tense. |
| AK   | Hmm. |
| SM   | Right so we’ve got 50. |
| AK   | Did you mean...? |
| SM   | Oh, no 40. It held 40. |
| VG   | I thought they held 30. |
| SM   | No that’s 20, ...30, 40. |
| VG   | Right. |
| SM   | Right, so if we try two strands. Right, that will easily hold 40,... [Trying different weights] 50’s fine, ...60’s fine, ...70’s fine, ...80’s fine,... 90’s fine. No, 90 snapped. |
| AK   | So 80. |
| SM   | It was fine at first, and we’ve moved the table a little bit then. |
| AK   | How much are we putting 80 or 90? |
| SM   | Two strands is... |
| SM   | Two strands is 80... So 3 strands... so that was 80. |
| AK   | It was 90 wasn’t it? |
| SM   | No it was 80... [Trying 3 strands] 20, 30, 40, 50, 60, 70, 80, 90, 100,...110,... 120,... 130... |
| AK   | That’s going to go up to 160. |
| SM   | 140, no it’s not going to go up to 160 I don’t think...140... 150? [Breaks] 140... |
That was 140 weren’t it?

AK  So it wasn’t 40g.

VG  He [the tutor]’s going to do a lot of hoovering [the spaghetti makes a mess as it breaks].

SM  One, two, three, four strands now.

So one student, SM, raises a possibility that the relationship is not directly proportional, and leads the way in practice in applying a systematic approach. This approach is not questioned by the others and (perhaps for that reason) is not made explicit. A little later, after testing 5 though:

SM  Right so I think we can do this two ways, we can either do it, so we can do 5 now, and then we can do another 5, and then we’ve done 20, or do we do it as like, the thing, and after we’ve done like say 6 or 7 then we do the sums to keep multiplying it up. Which way do we want to do it?

They decide to test another 5 strands, deferring initially to SM, but then

LK  Why have you done 5 and not gone to 6?

AK  Why didn’t you just do 6?

SM  [Counting up].

LK  Why have you done 5 twice?

SM  Give me a minute! Right, so what we’ve got there is we’ve got 20 strands all together that have been tested…

LK  Oh right

SM  …in a smaller group. Or what we can do is, we can go up to 5 and then try and work out a pattern from that to get to 20, but what we got then is like a number that sort of gives us an idea of what it could be. So I’ve got roughly 1820g and 20 strings of spaghetti…but then we can do the algebra to double check it…
Do it the other way if you want... but we’re not going to have enough weights to go up to 20 strings, because we’ve only got that many left now, after we’ve done 5... We’ve just got to do the maths now.

AK
Right so that’s... looking at figures, so there doesn’t seem to be any pattern really of how much these are... Unless we do the experiment again. From the beginning. And check our results.

So, because their first five results aren’t sufficient to convince them of a pattern, they first retrace their steps, and they also go on to extend the number of strands of spaghetti tested (they go up to seven), and they manage to make a more accurate prediction.

8.3.6.2 Group B

This group have three students from the earlier group 3 (the group who tried 4 and 8 elastic bands), two from group 1 (who discussed but didn’t implement a systematic approach), and one student who missed the first session. Again one of the students immediately launches into a systematic approach:

Do you want to start at 2? We know one’s 30; we need to see what 2 is.

I think the smaller the gap, the more weight it’ll hold.
Yeah.

This insight by MC should make them restart with one strand of spaghetti to test for the specific gap between their tables, but they never make that leap. ST notices one of the groups beginning with twenty sticks of spaghetti and adding weights. It is at this point that the tutor points out to the class that the weights they have will not be enough to break twenty strands. In response to this ST makes explicit to the group the strategy of ‘being systematic’:
<table>
<thead>
<tr>
<th>ST</th>
<th>[To group] Right we’ll have to start like I said at 2. Let’s start at 2 and work our way up. Let’s be systematic like he said.</th>
</tr>
</thead>
<tbody>
<tr>
<td>DC</td>
<td>Right so we’re going to start at 2</td>
</tr>
<tr>
<td>ST</td>
<td>Yeah ’cause we know one’s 30.</td>
</tr>
<tr>
<td>DC</td>
<td>So say if that’s 60.</td>
</tr>
<tr>
<td>ST</td>
<td>We know it holds 40 so try 50.</td>
</tr>
<tr>
<td>TJ</td>
<td>Do you think it’s always the same though?</td>
</tr>
<tr>
<td>ZL</td>
<td>Too many variables.</td>
</tr>
</tbody>
</table>

Again, the question of whether it is ‘always the same (i.e. goes up by the same amount each time) arises in relation to the complexities of the situation which may confound that. And again they test this by being systematic and increasing the number of strands by 1 each time. The pattern of increases this gives them is 40, 30, 40, (this data is consistent with an amount of weight per stick somewhere around 35-37g).

| ZL    | I’m worried just adding one by one won’t do.                                                                                                                                              |
| DC    | I think they’re trying to see if there’s a pattern.                                                                                                                                     |
| ZL    | I mean you’ve got so many variables like little mistakes will…; you’ll need more spaghettis to reduce mistakes.                                                                   |
| SN    | You mean you need to repeat your experiment.                                                                                                                                           |
| ZL    | No, I mean,… Sorry I can’t explain it                                                                                                                                                     |
| MC    | We’ll do just one more this way and then we’ll double it up. So we’ll do up to 5 this way and then we’ll go up to 10, or the way you were suggesting.                                           |
| MC    | Could be 40, 30, 40, 30.                                                                                                                                                                 |
| ST    | That’s what we’ve got written here.                                                                                                                                                       |
| MC    | Is that what yous have got? If that doesn’t work then we’ll jump it to 10.                                                                                                                |
If there is a linear relationship then MC is right, increasing the number of strands of spaghetti will narrow down the range of values of weight per strand (assuming they are reasonably consistent). But first they need to be convinced that there is a relationship. When they test five strands the pattern they have found continues.

<table>
<thead>
<tr>
<th>ST</th>
<th>Should we double up should we try 10 now? Is there enough weight?</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>If we were going by this here pattern so if it were 10 it should be [calculates by counting up by 40s and 30s]… 350.</td>
</tr>
</tbody>
</table>

So being systematic has led them to spot a pattern, which they then progress toward making a prediction that they can test (although the pattern remains at an oscillating increase rather than an actual value per unit increase). In practice they don’t have enough weight to test ten strands so do for nine, based on the pattern they have found and they make a reasonably accurate prediction for twenty strands (by continuing the oscillating increase).

8.3.6.3 Group C

This group contains 2 students from the previous group 1, and one each from groups 3 and 4. The majority of them have therefore been through the experience of applying a systematic approach to a similar problem. Again, one student begins by suggesting a systematic approach:

<table>
<thead>
<tr>
<th>BC</th>
<th>So would you start at 1 and then work your way up and make a list.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JA</td>
<td>I think, have we got 20 pieces of string, 20 spaghetti?</td>
</tr>
</tbody>
</table>

The tutor at this point says to the class that they don’t have enough weight to break twenty strands so:
You know what you need to do, you need to know what breaks one.

Yeah.

Yeah.

And then…

And then what breaks 2 and,

We already worked that out though didn’t we.

Yeah, but it’ll be different because of the width of the table.

So, if we get one…

[They start testing].

Or should we jump the gun and should we do 5, what breaks 5?

No I don’t reckon, go with 1.

1, work our way up.

You can see I was a naughty kid can’t you [laughter].

Shall we start with 1?

Yeah.

It appears that the systematicity of going up by one each time is so established as to be a rule so that it becomes ‘naughty’ for JA to suggest breaking it. After testing one and two strands they begin making predictions to test:

I’m reckoning 3 will get you 140. I reckon you can increase it by 50 each time.

They essentially proceed this way improving their prediction each time. Some anomalous results make them retest and be more precise about repeating how they test each time and they manage to make a reasonable prediction.
8.3.6.4 Group D

Group D has four students from the previous group 4 in the bungee jump task, who did apply a systematic approach in practice, plus two students who missed the first session. Again, one student immediately proposes a systematic approach.

<table>
<thead>
<tr>
<th>LE</th>
<th>Ok, so, one.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JH</td>
<td>Hold on a minute, we just starting with twenty? Start with twenty and just try to put the weights on.</td>
</tr>
<tr>
<td>LE</td>
<td>Oh right ok.</td>
</tr>
</tbody>
</table>

It is reasonable to test twenty strands just in case, but as they are doing so the tutor points out they won’t be able to do this. So again a systematic approach is explicitly adopted, but with some debate about the details:

<table>
<thead>
<tr>
<th>JH</th>
<th>So yeah we need to do what you said then.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JG</td>
<td>See how much it breaks 10 on or…</td>
</tr>
<tr>
<td>JD</td>
<td>So we’re going to have to go up by 1 then create a table or a graph.</td>
</tr>
<tr>
<td>JH</td>
<td>Er, you don’t have to go up by one you could go up by like two. Hold on do you mean go up to 20 by that? So start with 10 pieces?</td>
</tr>
<tr>
<td>LE</td>
<td>You need to do one, then two then three.</td>
</tr>
<tr>
<td>KH</td>
<td>Yeah one two three four.</td>
</tr>
<tr>
<td>JH</td>
<td>So start with 10 pieces?</td>
</tr>
<tr>
<td>LE</td>
<td>No start with one.</td>
</tr>
<tr>
<td>KH</td>
<td>One.</td>
</tr>
<tr>
<td>JH</td>
<td>Oh one piece?</td>
</tr>
</tbody>
</table>

And then this group proceeds much as the others did.
8.3.6.5 Analysis

There is little need for analysis in this section. Recounting all four groups here perhaps labours the point, but it does so in order to provide indisputable evidence of the development which has taken place. Faced with a new problem with structural similarities to a task they had faced in a previous session, all groups quite consciously, and with varying levels of explicitness apply the concept of being systematic. In a brief class discussion following the task the tutor asks how doing the task had differed from the bungee jump task they'd tried in the first session. One student mentions heuristics,

| ST  | I think today’s was easier knowing what these were. Perhaps if we had done this before the bungee jump, because I was really, really, I kept using the word systematic, look for patterns wherever I could but with that one we just did it and see what happened. |

In some ways, this is a little too neat. It is worth stressing that the concept is not fully developed for the students (if it ever can be for anyone), and they will vary in their understanding of it. However, for many it will have some rich connections with their experience of the real world problems, with the previous understandings that they brought to those problems, and will be part of a network of related concepts.

The tasks and classroom practice which have led to this can be described as connectionist, but a particular form of connectionism, one theoretically underpinned by Vygotsky’s theory of scientific concepts. In practice this has meant an increased attention to concepts in its design, and a particular view of concepts, as the structured totality of their connections to the real world and to other concepts. This latter understanding encourages the use of genuine problems which contain the concept; relating the concept directly, or indirectly, to real world experience (from physical to mathematical); relating the concept to student conceptions, through building on and developing them (and, as has been seen, these can often contain key aspects of the scientific concept already); and attention to generalisation of the concept (here, through trying to find it within
other, previous problem solving activities, and its embedding within a wider system of heuristics). The result is a dialectical process of concept development which, at least here, and for some students, has led to the emergence and conscious awareness of the concept, and an ability to apply it consciously within other problem solving situations.

8.4 Conclusions and starting points

The starting point of this article was that Vygotsky’s theory of scientific concepts provides a theoretical basis for connectionist mathematics pedagogy. The key pedagogical features justified by the theory include the necessity of genuine problems for concept development, an emphasis on connections within those problems (to real world activity, to student understandings and to other mathematics), and the important role of dialogue, reasoning and justification. The classroom dialogue analysed above illustrates both the workings of the pedagogical elements drawn from Vygotsky’s theory and aspects of the theory itself. The results of this short sequence of classroom activities give a taste of the nature of scientific concepts in Vygotsky's terms: a merging of the formal abstract with rich concrete experience; conscious awareness and control via the concept's place within a system; arrived at through collective attempts at solving genuine problems which involve much dialogue and reasoning.

The second aim of this article was to use this act of illustration of Vygotsky’s theory to raise further questions, and to develop the theoretical and pedagogical framework through engagement with empirical data. Three main issues have arisen in this respect through the analysis presented above. The first is a lack of understanding of the process of formation of actual utterances by students, particularly the leaps and crystallisations which represent qualitative shifts on the path to solving problems. A rough picture can be painted of the mutually mediating elements which play a part: the problem and context, the social (including the words of others), and the prior understandings that individuals bring. A richer understanding of these processes would be beneficial, however given the complexities of this question it is one that must be postponed for further study.

Related to this, a second issue lies in the relative weight and interrelation of cultural transmission (the bringing in of higher level cultural concepts through
problems, questioning and direct transmission, by teacher or peers) and the active role of individual students. Discussion of these first two issues requires further study and is, for now, postponed.

The third issue is one raised by the nature of the students' attempts seen in the first task. These include assumptions of linearity, questioning of linearity (often bringing in scientific arguments and terms, such as elastic limit, gravity, acceleration and force), applications of the concept of being systematic in relation to testing linearity (even if to varying degrees of concreteness and consciousness). These bring in aspects of scientific concepts either directly or in their systemic connectedness to other concepts in practice. This hints that a dichotomy between everyday concepts (those representing generalisations from below, from practice, which are not systematised) and scientific concepts (those which exist in the culture, are usually brought to students through schooling and are in systems) which is present in Vygotsky’s work is too simplistic. Although this did not inhibit a theorisation of connectionism and the development of practice based on that theorisation, there is a potential danger in the perspective. It could encourage transmissionist practices, as the emphasis could shift to the need to directly give students the scientific concepts which they lack. However, in the case study above, and I would argue more generally in practice, the dichotomy between everyday and scientific concepts is exaggerated. Students often come with aspects of the developed concept including some of its systemic connections, even if these are in more practical or concrete form. This extended conclusion therefore ends by exploring the relationship between everyday and scientific concepts in Vygotsky a little further, and introduces an alternative formulation within the work of Gramsci, that may orientate classroom practice more effectively.

8.4.1 Vygotsky and scientific and everyday concepts

Although Vygotsky touches on teaching in his development of the theory, his central focus is rather on the role such concepts play in wider processes of psychological development. This focus necessitates a clear logical distinction between scientific and everyday concepts which helps clarify the process of development but which, in effect, exaggerates their practical difference, particularly given their interrelation following the introduction of scientific concepts (see Vygotsky himself, 1987, p.220). This leads to some contradictions with the
statements Vygotsky does make about teaching, and with the coherent pedagogical approach which can be derived from the wider theory.

The key distinctions between the two forms of concepts made by Vygotsky are that scientific concepts are part of a definite system and that they are used consciously, whereas everyday concepts lie outside a system and are spontaneous (Vygotsky, 1987, p.220). In addition he argues that scientific concepts are introduced in schools by definition and then concretised (moving in the opposite direction to everyday concepts) (p.168), and that their introduction must be mediated by other concepts (and their systemic relations), as they have no direct relationship to the object (p.223).

There are some problems with each part of this, which mean that the distinction between the two forms of concept is less clear in practice. First, as Davydov (1990) points out, everyday concepts appear in systems too (p.188). Take the everyday concept of brother, an example used by Vygotsky (1987, p.218). There are obvious systemic relations with other concepts such as sister, mother, and family etcetera for this concept, which may be less formal and precise than say relations between mathematical concepts but the difference is quantitative rather than qualitative, and those relationships still mediate the relationship between concept and object. This is particularly so once scientific concepts have been introduced for any individual and begin to influence the everyday, developing their potential for definition and conscious use. Before this, it may be the practical relationships (e.g. the actual relationship between a mother and daughter) which instead mediate, but even here there are close relationships between the practical and conceptual forms. Some of the participants in the practical relationship (e.g. the mother) will be acting under the influence of the cultural conceptual form and therefore the actual relationship will itself be mediated by this. In the other direction, the conceptual form represents a generalisation of a myriad of such practical relationships (see e.g. Vygotsky, 1997, p.248) and is immanent to the ensemble of such relations. Vygotsky himself recognises this a) in his understanding of the relationship between abstract and concrete (where the abstract does not represent the shedding of the concrete but instead the bringing in of new relationships see, Vygotsky, 1998, p.37), and, b) in his tracing of developmentally earlier, pre-conceptual forms of generalisation based in practical activity (see e.g. Vygotsky, 1987, p.139) and his awareness that higher forms generalise and build on those earlier forms (p.229).
The second problem in Vygotsky's position is with the idea that scientific concepts must first be introduced by definition (and then made more concrete). Again Davydov (1990, p.88) points out that everyday concepts may also be introduced this way (Vygotsky though also noted the role of instruction in concept development pre- and outside-school, see 1987, p.238). More importantly though, this appears to espouse or accept a transmissionist approach to teaching. And in this it contradicts Vygotsky himself elsewhere, for example, where he points to the fruitlessness of purely verbal attempts to transfer knowledge (1994a, p.356), or where he stresses that concepts can only be formed or develop within problems and tasks where there is a genuine need for them (1987, p.124).

The third problem, which connects intimately with the second, is the idea that scientific concepts can only have an indirect relationship to their object. The mediation of this relationship by other concepts and their systemic connections is of course fundamental to the nature of such concepts, but there is a sense in which more direct relationships can exist alongside (and part of) the mediated relationship. There is evidence of a direct relation for particular complex concepts, such as proportion for example, where children have awareness, particularly in the visual field, of preservation or non-preservation of proportion before any more scientific version of the concept is developed (see, e.g. Brink & Streefland, 1979). This may be viewed as the everyday form of the concept which then becomes integrated into the more developed scientific version, or it may be viewed already as an amalgam. Either way, such direct connections can play a key role in the generalisation of mathematical relations (see ch.4), and again, the relationship between everyday and scientific concepts is seen as a more complex one.

These seeming contradictions in Vygotsky’s work, when viewed from an educational perspective of developing actual concepts in the classroom, necessitate a slight reframing of the relationship between everyday and scientific concepts which puts greater emphasis on their interconnectedness and that prevents any slide towards transmissionist teaching. To assist this, it is helpful to supplement Vygotsky’s analysis with some formulations in the work of the Italian Marxist, Antonio Gramsci.
8.4.2 Gramsci, contradictory consciousness and common sense

On the surface it may appear as if the concerns of Gramsci and Vygotsky are far apart. Vygotsky is dealing with psychology and development, while Gramsci is concerned with class consciousness, political organisation and revolution. However, both are writing about consciousness from within the same Marxist intellectual tradition. On one hand, Vygotsky is explicit about his support for the concerns of the recent revolution in Russia, both in private (Van der Veer & Zareshnega, 2011, p.466) and in his writing (see e.g. Vygotsky, 1994b), and his collected works are littered with quotations from, and engagement with, the ideas of Marx, Engels, Lenin, Plekhanov and others (see e.g. Au, 2007, on the relationship between Vygotsky’s and Lenin’s writings on consciousness). On the other hand, Gramsci views Marxism as a scientific understanding of the world, and as a coherent system (see e.g. Gramsci, 1971, p.419) and is concerned with how such a view is developed. The complication within Gramsci’s work is that it has to take account of two competing systems of ideas, the dominant ideology developed from the class position of those who control society, and the alternative ideological system from the perspective of the oppressed and exploited. Nevertheless, there are insights within his views on the relationship of both to everyday conceptions which have more general applicability.

The starting point for Gramsci is contradictory consciousness:

The active man-in-the-mass has a practical activity, but has no clear theoretical consciousness of his practical activity, which nonetheless involves understanding the world in so far as it transforms it. His theoretical consciousness can indeed be historically in opposition to his activity. One might almost say that he has two theoretical consciousnesses (or one contradictory consciousness): one which is implicit in his activity and which in reality unites him with all his fellow-workers in the practical transformation of the real world; and one, superficially explicit or verbal, which he has inherited from the past and uncritically absorbed (p.333).

This contradiction between thought and action is seen not as self-deception (p.326) as it is not the adoption of the fully worked out versions of ruling ideologies but the incorporation of explanations which relate most closely to concrete activity (p.420). Gramsci describes the result of this as common sense, a conception of the world that combines the contradictions in consciousness, which are 'fragmentary, incoherent and inconsequential' and 'half-way between folklore properly speaking and the philosophy, science, and economics of the
specialists (p.326). This means, though, that here is a 'healthy nucleus that exists in common sense, the part of it which can be called good sense' (p.328). Elements of a more scientific philosophy are seen as part of this fragmentary collection of ideas, and therefore 'science' and 'common sense' are inseparable (p.328). Gramsci goes on to discuss how to develop a scientific understanding of the world more fully:

First of all … it must be a criticism of "common sense", basing itself initially, however, on common sense in order to demonstrate that "everyone" is a philosopher and that it is not a question of introducing from scratch a scientific form of thought into everyone's individual life, but of renovating and making "critical" an already existing activity (p.330).

The argument for taking common sense as a starting point and through criticism making this more ordered, systematic and coherent (p.327) reflects Gramsci's emphasis (shared with Vygotsky) on the importance of the interrelation of theory and practice. For example,

The popular element "feels" but does not always know or understand; the intellectual element "knows" but does not always understand and in particular does not always feel...The intellectual's error consists in believing that one can know without understanding and even more without feeling and being impassioned (not only for knowledge in itself but also for the object of knowledge) (p.418).

8.4.3 Vygotsky, Gramsci and being systematic

In relation to concept development in mathematics education, Gramsci's views may offer a more adequate description of what exists in the classroom. It is perhaps rarely the case that particular concepts of interest must be 'introduced from scratch'. They will likely exist to some extent within the class already, even if in fragmentary and incoherent form, both due to the previous mathematical experience of students, and through their sharing of the everyday cultural conceptions that relate to, and are part of, those concepts. The key then is to begin with this relatively incoherent and unsystematic amalgam of the everyday and scientific and through criticism make it more coherent and systematic. Both of these aspects require genuine problem solving activity.

The classroom dialogue analysed in this article illustrate this perspective. The first task involved a genuine problem with a need for the concept of being
systematic. The students’ common sense understandings are evident in their approach to the task, with practical and everyday understandings mingling with elements of scientific and mathematical understandings. The needs of the task combined with some of the students’ perspectives and the nudging of the teacher encourage a sense of the concept to emerge. In task 2, through some heavily scaffolded problem solving activity, the tutor explicitly but loosely attaches names to some problem solving activities, including that of being systematic. This gives a name to a concept which has already been present within the class through earlier problem solving activities including task 1. Gramscian criticism of the students’ common sense conceptions of the concept could then be one way of viewing the explicit focus of task 3, through the problematising of the generalisation of the concept to previous problem solving activity. Finally, within task 4, a very similar task to task 1, there appears a more conscious and coherent application of the concept.

Gramsci’s perspective therefore provides a useful supplement to Vygotsky’s theory. But it is a supplement and not a replacement. Much of what Gramsci argues aligns closely with Vygotsky’s theory, and even with many of his statements about the interrelation of everyday and scientific concepts. Also Vygotsky’s theory is much richer in its possible relation to pedagogical practices due to its more detailed understanding of generalisation and conscious awareness, the relation of abstract and concrete, and the role of language. The main thrust of this article has been to explore how those elements of Vygotsky’s theory work in practice when translated into pedagogical practice and real concept development within a classroom. Gramsci (1971), however, adds a more realistic framing to the process of concept development which more usefully orientates mathematics teaching (further inoculating it from transmissionism), and which fits more naturally with connectionist practice and its attention to developing students’ own productions.

The personality is strangely composite: it contains Stone Age elements and principles of a more advanced science, prejudices from all past phases of history at the local level and intuitions of a future philosophy which will be that of a human race united the world over. To criticise one’s own conception of the world means therefore to make it a coherent unity and to raise it to the level reached by the most advanced thought in the world (p.324).
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References


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9 Conclusion

This thesis began with two key elements, Vygotsky’s theory of scientific concepts, and the practice of connectionist pedagogy in mathematics, and asked how the first could be used to theorise the second, and simultaneously how the second could concretise the first in relation to mathematics education. It was also seen from the literature that connectionist pedagogy was under-theorised in its various expressions, essentially representing a list of positive aspects of meaningful teaching, garnered from experience and perhaps many pedagogical traditions; and, that the details of Vygotsky’s theory remain underdeveloped in relation to mathematics education, despite the obvious influence of Vygotsky’s work on mathematics educational research.

The key contribution of knowledge of the thesis lies then in:

i) Making Vygotsky’s theory of scientific concepts concrete in mathematics education in new ways, for example, in a) showing how the teaching of a problem solving heuristic concept can develop students’ own conceptions toward a more systematic and conscious concept, and b) showing how the explicit problematisation of a generalisation, such as proportion, is both possible and laden with possibilities for concept development.

ii) Providing a theoretical framework for connectionism, showing how its various elements fit together in Vygotsky’s theory of concepts in ways that inform pedagogy, for example, in encouraging a focus on concepts and seeing concepts as their connections (and thus emphasising the embedding of concepts in genuine problems, real world activity and the mathematical system).

In the following sections, these key contributions will be detailed and qualified. In particular this will include drawing out some limitations, and potential extensions of both mathematics pedagogy and Vygotskian theory. In what follows, a summary and synthesis is given of the findings and the contributions to knowledge of the individual sections. As each section is summarised and discussed, its role within the wider thesis is related, and limitations, omissions and directions for future work are also discussed. This is followed by an overview of how those individual sections fit together within a framework for understanding and developing mathematics pedagogy, a synthesis which represents the more general contribution to knowledge of the thesis. Directions for future work are then outlined in relation to development of the framework as a whole. And, finally,
implications are drawn regarding strategies for critical mathematics educators in current circumstances.

9.1 Summary and contribution to knowledge (individual chapters)

9.1.1 Vygotsky's theory of scientific concepts and connectionist pedagogy

Chapter 4 is the cornerstone of the thesis. It shows how the main aspects of Vygotsky's theory of scientific concepts can map onto the main pedagogical elements of connectionist teaching.

The thesis began with two sets of assumptions: First that various elements of reform/non-transmissionist/problem-solving and modelling pedagogy are essential for developing successful, meaningful and engaging mathematical activity and learning in the classroom, and that connectionist pedagogies of the sort variously described in the literature are exemplary in their inclusion of these elements. Secondly, that Marxist understandings of consciousness, and its relation to activity, represents the most advanced approach to such questions. Given that concept development is at the heart of learning mathematics, Vygotsky's work, representing the most developed Marxist approach to concept development to date was taken as a key starting point.

Both these assumptions contain their own difficulties. 1) Connectionist pedagogy remains relatively underdeveloped as an approach within research, being in essence a collection of activities, emphases and beliefs (see Askew et al., p.35). These elements of connectionism require coherence and development, and as an essential part of that process, a theoretical underpinning. 2) Vygotsky's theory of concept development is a general one, and although at times he touches on mathematics, the bulk of the work and examples he outlines are in relation to other areas. Also, although schooling and pedagogy play a key role in the theory, in essence it is not a theory of pedagogy but of cultural-psychological development. There is therefore a need to both translate and particularise the general theory to make it of use within the field of mathematics education. Elements of Vygotsky's work have, to some extent, been related to mathematics education by others, but primarily at the level of partial (if important) insights, rather than in more systematic ways (see sec.2.2).
The contribution of knowledge of this chapter is therefore in beginning to resolve these two issues. In showing the intimate connections between the key elements of Vygotsky's theory and the central aspects of connectionist pedagogy, it both offers an outline of the necessary theorisation of connectionist practice, and the concretisation of the theory in the activity of mathematics. In doing so it begins to develop both Vygotsky's theory and the pedagogical approach. It also lays the basis for the further developments, in both these directions, seen in later chapters.

The interrelation of Vygotsky’s theory and connectionist pedagogy can be summed up as follows:

1) In both, there is a key emphasis on concept development only occurring as part of the solution to meaningful problems;

2) The connections between mathematics and the world of experience and activity relate to Vygotsky's understanding of the relationship between abstract and concrete;

3) The connections between mathematics and student's existing understandings relate to Vygotsky's relationship between everyday and scientific concepts;

4) Word and dialogue play an essential role in both;

5) Connections within a system are the key distinguishing feature of scientific concepts, where the relationship between concepts mediates the relationship between concept and activity leading to greater conscious awareness and voluntary control of the concepts in use. This relates to making connections within mathematics, particularly through generalisation, and the emphasis on allowing space for reflexivity within the classroom.

A connectionist approach to pedagogy, with a Vygotskian theorisation, would therefore view mathematics teaching and learning as involving the development of mathematical concepts (and systems of mathematical concepts). An essential aspect of this is a view of concepts as being the structured ensemble of their connections both to the real world and to other concepts. It may seem recursive, and therefore unintuitive, to define some 'thing' or object as its connections. It is the existence of an objective element, that is, the word or sign, which allows this
though, somewhat like a stick in the making of candyfloss, or a speck of dirt in
the production of a pearl. For example, in simplistic terms, the concept ‘circle’ is
formed through the connections between the word ‘circle’ and situations involving
some aspect of circularity, with their mutual existence and relationship, their
mutual mediation, within a wider situation representing the ‘connection’ between
geometry and practical activity.

Concepts should also be seen as generalisations, or, in particular, as the result of
generalisation (grouping), on the basis of abstraction (the isolation of common
and distinctive features). Remaining with a simplistic example still, the concept
‘circle’ therefore represents a generalisation of circular objects, or rather of
physical, spatio-temporal, or visual experience of such objects. In more complex
cases (and even for circle) it may not even be objects (or experience of them)
which are being generalised. In many scientific concepts this can mean it is often
other generalisations, or relationships between generalisations, which are
generalised.

For mathematics education, this view of concepts means that they cannot be
formed in isolation from their connections with rich concrete experience. Even in
the case of generalisations of generalisations (or at even higher levels of
abstraction), reality and experience of it are an essential part of concepts. This
cuts against a more common view of abstractions which sees them as a
shedding of the concrete (some of the causes of this way of thinking are seen in
ch.5). This extracts everything from mathematics but a thin formal shell leaving
little room for anything more than transmission, and memorisation through
repetition, in teaching and learning. A Vygotskian perspective on concepts
therefore helps explain aspects of the need for genuine problem solving within
teaching, and the success of methods which emphasise this and connecting
mathematics to physical or social experience, either directly or via students’ prior
experiences.

The approach also reinforces that there must be a need for the concept within the
problem for it to form or develop. The lesson of the situated cognition literature is
that if a problem can be solved in a simpler way (i.e. without concept
development) it usually will be. It is also possible that the concept can be used in
solving a problem without consciousness of it. For example, the cultural concept
may be so embedded in objects or in social practices and therefore so much of
the thought process distributed in the environment that consciousness of it is
unnecessary. Therefore, in addition there often must be other processes, and
other problems, to draw the concept into conscious use. Much of this can occur in social dialogue within problem solving. The problem of communicating, reasoning, justifying, challenging or contradicting embed the concept in a different kind of problem, one which require generalisation of the concept and connection to the wider mathematical system. In simple terms it is these higher level generalisations of the concept, in combination with the concept’s own generalisation (of objects, processes, relationships or concepts) which in effect carve out the concept, giving it an object like quality, and with that the potential for conscious awareness and control.

The Vygotskian framework therefore adds to those pedagogies based on a problem solving approach an additional emphasis on explicitly problematising generalisation and connections, and motivation to develop further practical methods of doing this within the classroom (this attention to connections is already often present within discussion of progressive pedagogies, but for reasons explored in ch.7, often fails to materialise in practice). As these higher level generalisations develop they then mediate and reshape the lower levels. To continue the simplistic example above in order to illustrate: The generalisation ‘circle’ allows consciousness of circular (or approximately circular) objects (and processes and much else), but the generalisation shape (only one possible generalisation among many wider systems that the concept circle can be placed within) allows consciousness of the concept ‘circle’ and therefore allows more conscious use of the term. Then the systemic relationships within the distinction of different shapes (e.g. straight versus non-straight sides, circle as the limit of regular x-sided polygons as x tends to infinity etc.), or the properties common to all shapes, mediate the relationship between the concept circle and circular objects, reshaping consciousness of those objects.

Additional, more minor contributions of this chapter include using the overall framework to overcome the persistent dichotomy in pedagogy and literature between the importance of student agency and the necessity of internalising pre-existing advanced culture, to show that both aspects are essential and interrelated. Also, an ontological viewpoint of concepts is presented which is consistent with Vygotsky’s general theory, allowing a focus on the development of the concepts of individuals while at the same time maintaining their social and relational nature.
A restriction of focus to the essential elements of pedagogy, and of Vygotsky’s theory, aids clarity of the connections between the two and, at the same time, provides a simple cognitive model which can mediate theoretical understandings of pedagogy and assist its operationalisation within teaching. However, this restriction of emphasis is also the main weakness of the chapter. Many details of both Vygotsky's theory and meaningful mathematics pedagogy are of course omitted. To take just one key example, the role and understanding of models is well-developed within research and practice. There are many points of contact between Vygotsky's theory and aspects of models. Models can range from physical objects, to activities, to concepts, systems of concepts, formal mathematical theories and more. They can provide a connection between similar or different levels of reality and/or cognition via generalisation of essential structure and therefore can be a pedagogical bridge between activity, everyday understandings, and scientific understandings of activity. None of these, or other, aspects of models are discussed in detail within the chapter, and no serious, developed theoretical understanding of pedagogy can exist without this. And similar arguments could be made about other aspects of pedagogy or theory. Although there are additional developments contained within later chapters, it is important to stress therefore that this chapter only represents the beginnings of a systematic theorisation of pedagogy (and one which takes concepts as a starting point, where alternative starting points, such as 'models' are possible). However, it arguably, provides a coherent basis for further developments.

9.1.2 Expressions of the commodity form

Chapter 5 embeds the Vygotskian understanding of the development of scientific concepts within a wider societal perspective. The chapter both outlines the manifold mediations between commodity production and the mathematics classroom, and provides evidence for these mediations as they are expressed in the words of secondary school mathematics students.

Recently, on the XMCA (2014) discussion list there was a debate regarding the most useful 'unit of analysis' for mathematics education. Vygotsky himself was explicit about the usefulness of having a unit of analysis (see e.g. Vygotsky, 1987, p.46), essentially the smallest element of analysis possible which contains within it all essential aspects of the situation under analysis and their relationships. In
thinking and speech that unit is taken as word meaning, or, concept, which is seen as equivalent (p.140). As mathematics education is about conceptual development it seems unnecessary therefore to invent a new unit – what is instead required is to find the particular mediations of this more general case.

This is something Vygotsky is himself explicit about in his own criticisms of his results (p.239). The idea for a unit of analysis is taken by Vygotsky from Marx's capital (Marx, 1982, p.90), a work he wanted to emulate within the field of psychology (Vygotsky, 1997a, p.330). It is possible to argue that Vygotsky focussed too narrowly on psychological factors, often omitting the particular form of societal organisation from his analysis (see, e.g. Williams, 2012, p.58; Roth & Lee, 2007, p.189, for forms of this view). Does this mean the unit of analysis chosen by Vygotsky is also too narrow to grasp all the factors at work within mathematics education? This depends on the focus of the analysis. It is sometimes useful to abstract from particular social forms to understand the more general processes involved, but those social forces can't be ignored completely. However, there is already another useful 'unit of analysis' which helps understand these wider processes. That is Marx's own 'unit of analysis', the commodity. From the commodity Marx unfolds both the workings of the economy and touches on their wider social and psychological impact via shaping the form of alienation under capitalism. This understanding has been developed further by others such as Lukacs (1971) and Meszaros (1970). Therefore, again, when investigating the social aspects of education there is little need to invent a new unit of analysis. It is far more useful to mediate the general understanding of the impact of the commodity form in relation to the particular field of education. Beyond this, there is also a need to show where and how the two systems of understanding, based on where and how the two units of analysis meet.

Although written as distinct paper, and thus having its own developmental logic which strays to some extent from the central needs of the thesis, this chapter’s role is to show where the analysis based on these two units of analysis (the commodity and the concept) meet. By outlining the key influences of commodity production on mathematics, and showing the expression of these influences in fragments of interviews with school students, it offers an explanation for the transmissionist practice whose elements directly contradict those suggested as central to connectionist practice. The impact of commodity production, mediated by the functional roles of schooling for capitalism, the exam system, and understandings of knowledge etcetera lead in the classroom to memorisation and
copying of processes rather than genuine problems; the detachment of
mathematics from the real world and the systemic connections of mathematics; a
focus on individual rather than social, dialogic activity; and disregard for the
student’s own conceptions and productions. Or, in short, to activity which hinders
concept development.

The main contributions to knowledge of this chapter are therefore a) to refocus
understandings of alienation within educational research on the role of the
commodity form, and in doing so, b) to embed understandings of mathematics
education, pedagogy and concept development, and particularly the Vygotskian-
informed connectionist pedagogy described herein, within wider theories of
society. In adding to the analysis of scientific concepts by outlining the societal
factors which interfere and block their development, it reinforces the importance
of the essential aspects of pedagogy outlined in chapter 4 for meaningful activity
within the classroom, and, therefore, it positions the pedagogical/theoretical
approach in the thesis as critical mathematics education.

Other contributions of this chapter include:

- Suggesting a critique of the methodology involved in many mixed-method
  research projects, that is, to point out the dangers that relatively blunt
  quantitative instruments can transfer their bluntness and narrowness to
  the qualitative methods which are supposed to add sharpness and
  richness.

- To raise a methodological solution to such problems, that is, in general
  terms, to pay attention to data which breaks free from any developed and
  potentially imposed categorisation. And in narrower terms, to propose
  always integrating attention to the wider systems to which a research
  focus belongs into educational research. And, in yet narrower terms, to
  propose integrating the particular understandings of Marx’s theory of
  alienation into educational research.

The limitations of this chapter lie within the restricted nature of its evidence base.
Although the various mediations between commodity production and the
mathematics classroom are outlined in a coherent way and empirical evidence
for their expression presented, this can only be a small contribution to
understanding of alienation in relation to schooling and mathematics. Further
work is required to trace these mediations in greater detail. One useful path for
such investigation would include mapping the mathematics education field in
terms of organisations and individuals, the freedoms and constraints they face, along with their understandings of the role of education and mathematics within society, and their views on pedagogy.

9.1.3 Scientific concepts in and out of school

Chapter 6 shows that there is a type of cognition, both in and out of school, which in some sense lies beyond situated cognition (i.e. cognition which is distributed across the individual, the place, the activity, objects, others etc.). Some advanced aspects of the mathematics of some darts players reflect a form of cognition which takes place outside of the immediate activity being considered, which merges more abstract structures with aspects of concrete reality, and which is particularly conscious. This activity fits well with Vygotsky’s view of scientific concepts as the conscious interrelation of abstract systems with the concrete within social activity and in relation to a genuine problem.

Similar activities are also seen within an example of vocational schooling where aspects of students’ formal conceptualisations of fractions seem in contradiction with an imagined real world situation. In this situation the problem of overcoming the contradiction is sustained and given extra impetus by the imagined needs of their future vocational role. This example also shows the key potential role of social dialogue for concept development as part of the processes described by Vygotsky’s theory.

Although cognition can never be unsituated, these examples represent cognitive activity qualitatively different from either side of the oft-perceived dichotomy between distributed cognition and empty formalism. The two merge together within scientific concepts. This form of cognition is often unnecessary in daily life, but its rarity in situations where it may be helpful must be explained. This is done here by integrating the understandings of chapter 5. Parallels are drawn between how these processes work themselves out and act as obstacles to meaningful or scientific cognitive activity in workplaces and in schooling.

This contributes a more integrated approach to vocational and school mathematics learning which can benefit both. However, the main contribution of knowledge of this chapter is to argue for and provide evidence of the existence of scientific conceptual activity. As such it is of fundamental value to the thesis.
9.1.4 Generalising proportion

Chapter 7 takes Vygotsky’s theory of scientific concepts and its interrelation with elements of progressive pedagogical systems as a starting point. Vygotsky’s emphasis on concepts being generalisations, and the place of scientific concepts within a system, are argued to have the least developed equivalents upon translation to reform approaches to pedagogy. It is also argued that part of the reason for this may lie in the assumption of a unidirectional nature to the development of mathematical knowledge, where lower levels of understanding are required to be fully formed before generalisation to a higher level of complexity becomes possible. This can then mean that in practice the connections are never consciously made. For example a bar model (a pictorial approach utilising rectangles which can be divided up, compared etc. See, e.g., Van Den Heuvel-Panhuizen, 2003) may be used across a variety of contexts without students ever getting to the stage of conscious discussion of what it is about those contexts which allow the bar model to be used in all of them. The bar model could then remain at a situated, process level even with its potential for spontaneous generalisation.

Vygotsky’s theory although retaining some necessity of development of lower levels before higher stages of complexity are reached, also assumes a) development over time of all generalisations, and potential for temporal overlap in development of lower and higher levels of complexity; b) that a developed form of any generalisation includes its relation within a system, and thus the lower level requires aspects of the higher level in its fuller development; and c) that development of the higher level ‘reaches down’ and further develops the lower level.

The chapter looks at various forms of approach to generalisation within mathematics pedagogy in relation to the outlined Vygotskian framework and suggests an additional simple method. This method, a form of problematising generalisation itself involves comparison of two or more methods, models, calculations or contexts to find what is the ‘same (but different)’ between them. It then looks at an example of the method in relation to proportion, and explores the sort of dialogue which can arise among students when given the task of generalising. The classroom discussion indicates that such an approach is
feasible. Given the class's general lack of confidence, recent lack of experience, and general history of transmissionist practice they successfully sustain a relatively abstract dialogue which raises many potential aspects for potential further discussion. The dialogue also indicates that higher level generalisations, such as a concept of proportion which can unite a range of different applications, can exist for students in both an everyday (or mathematically less developed) sense (e.g. in a sense of scaling while remaining somehow the same), and within some of the particular applications, allowing potential for its further generalisation and mathematical enrichment.

This chapter therefore extends the work of the thesis in further developing and testing a pedagogical approach based on the theoretical work of chapter 4. Its primary contribution to knowledge lies in this, the theoretical perspective which underlies it, and the additional categorisation and incorporation of other pedagogical approaches to generalisation into the framework. Problem solving and modelling rightly lie at the heart of meaningful approaches to teaching mathematics, and although there are some opportunities for conscious generalisation within such activities, an additional emphasis on this, and additional ways of doing so are both possible and necessary.

The limitations of this chapter lie in the limitations of the example, a brief classroom task which was relatively undeveloped from the perspective of the outlined approach. Many more such examples are necessary in order to deepen understanding, examples which cover a wider range of mathematical generalisations (between different variables, methods, models, areas of mathematics etc.), examples which build on and develop the student insights which arise, and examples which trace the longer term impact and the future process of interrelation of higher order generalisations with particular instantiations. This also therefore outlines the direction of future work arising from the chapter.

9.1.5 The development of a heuristic concept

Chapter 8 explores the development of a particular concept, that of the problem solving heuristic 'being systematic', tracing its evolution through multiple tasks over several sessions of an undergraduate class. In doing so it both illustrates and tests the developed theoretical framework, and the influence of the
interrelated system of pedagogical elements outlined in chapter 4 and extended in chapter 7.

The key features seen within the dialogue include:

1) The importance of genuine problems first in engaging and motivating students, but primarily in providing a need for concepts. This is seen in both the problems as such and the attendant problems of communicating, justifying and reasoning within them. The task which required explicit generalisation of the concept reinforces the lesson of chapter 7 that meaningful problems can successfully include those which focus on explicit generalisation.

2) The importance of connecting mathematics to the real world, or, to the lower level mathematics or concepts which are generalised, including students’ prior conceptions and those which emerge in relation to the problem situation. In Vygotsky's terms these are the rich connections with the concrete which make up concepts. In the dialogue, concrete forms of the concept are seen for example within the problem situation, the physical objects and the concrete forms of solution offered by the students.

3) The importance of systematic connections in mathematics, seen here in the tasks which for example connect the concept within a system of heuristics, or generalise the concept to other mathematical situations and problems. These systemic connections enter and mediate the relationship between the concept and the concrete, and in doing so help form the scientific concept, allowing it to be used more consciously.

In illustrating, and further exploring the theoretical and pedagogical frameworks in relation to Vygotsky's theory and the connected pedagogical framework, the chapter also extends the work of the thesis in relation to the question of everyday and scientific concepts. Given the nature of the student contributions, it seems too simplistic to say that no elements of systematicity exist already for them, and that they bring only everyday concepts to the situation, even if the scientific concepts they bring are underdeveloped, or if they only have elements of the scientific concept (e.g. a tendency to certain connections rather than others, or concrete versions of more abstract connections). This means that in practice the distinction made between everyday and scientific concepts, which is nonetheless useful, and particularly so in Vygotsky’s tracing of development over childhood, is perhaps overstated and less useful when it comes to educational practice. More useful in providing a psychological orientation for teachers, which is also less
open to transmissionism than a view that scientific concepts must always be introduced and defined from outside (and hence by the teacher), is Gramsci's notions of common sense and good sense. That is, it is more useful to assume that students bring a more complex amalgam of everyday and scientific concepts, and to see the task of teachers as facilitating the increasing systematicity and coherence (as well as generalisation to wider areas of concrete reality) of thought through the selection (and guidance) of appropriate problems.

The chapter's main contribution to knowledge is in its illustration, testing, and exploration of the thesis's central claims, but its secondary contribution is in proposing this useful orientation.

The limitations of the chapter, as usual, lie mainly in the fact that it makes generalisations from a limited selection of empirical data. Such limitations can only be overcome through an extended program of research. Within the chapter there are some suggestions for where that future research may focus initially. These include, first, a closer analysis of the particular utterances within dialogue, their relation to each other and to the other aspects of the situation, and, second, investigation into the relative form and weight of cultural transmission and student initiative within problems. In general such work was avoided here due to the already extended length of the chapter.

9.2 The thesis as a whole

In summary then, the thesis presents the beginnings of a systematic, Vygotskian theoretical approach to connectionist teaching of mathematics. In doing so it coheres, develops and theorises connectionist pedagogy, and at the same time, translates Vygotsky’s developmental theory into a theory of teaching and learning, and of mathematics teaching and learning in particular (thus extending the existing Vygotskian influence on mathematics education to a more systematic perspective). The key elements of this pedagogical framework include the need to connect mathematics to real world experience (i.e. beyond the real world of sitting in a classroom and only practising mathematical procedures); to students’ prior understandings; and, to other mathematics. The framework also emphasises that these connections must be made within the attempt to solve genuine problems, whether genuine real world or mathematical, or problems of communication, reasoning and justification in relation to these (and thus with an
emphasis on social dialogue and reflexivity to facilitate this). This approach is underpinned theoretically by two key theoretical aspects in particular. One is a notion of the abstract as not separate from the concrete, and thus not an empty formalism which can be transmitted. The second is the emphasis on the systematic connections within mathematics, a view of concepts as generalisations, and an understanding of the bi-directional processes of development that can occur in the relation between higher and lower (or more complex and less complex) forms of conceptions.

The thesis develops connectionist practice in its particular emphasis on the role of problematising generalisation as an essential element of conceptual development, (which can and should take place earlier than is commonly practised). It also further develops Vygotskian theory in relation to the developed pedagogic framework by introducing the Gramscian perspective on common sense and good sense. This providing a more useful orientation to teaching than the everyday/scientific concept distinction which can potentially encourage dismissal of students’ already developed thought and the straight transmission of culturally advanced concepts.

The thesis embeds this pedagogic understanding in a wider societal understanding of schooling, mathematics and mathematics education. First in showing that scientific activity and concepts can and do arise both inside and outside of schooling, but also in showing the obstacles to this occurring. The analysis which says that society is a totality which is fundamentally and negatively shaped by the existence of a generalised system of commodity production meets and merges with the Vygotskian pedagogic framework directly by explaining why the opposites of the elements of the framework dominate in classroom practice. The functional role schooling plays for capitalism in developing the commodity labour power (i.e. in developing human beings into self-selling, competitive commodities, distributed to fit the division of labour in society, accepting of their place within that and trained in submission), the production of the pseudo-commodity (and more and more real commodity) exam certificates, and the general influence of living within a system of commodity production on ideas of knowledge, mathematics and learning all combine to detach mathematics from the real world, atomise different aspects of mathematics, encourage individual rather than social approaches, monologism over dialogism, and limit students’ active role in the learning process.
For the thesis as a whole, and the total theoretical and pedagogical framework, four tests can be suggested. These relate to issues of coherence, development, reach and functionality.

1) Internal coherence. Does the thesis fit together as a coherent whole?

Although written as separate articles, the five results chapters of the thesis form a coherent, systematic (if not fully developed) perspective on mathematics education. This perspective integrates a Marxist perspective on society and schooling, Vygotsky’s theory of scientific concepts, others’ theories and research on mathematics education, accumulated wisdom of pedagogic practice, and micro-analysis of concept development within classrooms. Much of this conclusion so far has been dedicated to showing this integration, so that work won’t be repeated here. Throughout it has been argued that a totality perspective and an integration of the various levels at work in mathematics education is essential for a fuller understanding of any individual part of it. It can only be added that Vygotsky’s theory of scientific concepts itself emerges from and adds further weight to this perspective. Theory and research too are the development of systems of scientific concepts in relation to rich concrete activity, and with suitable alterations for the particularities of research, Vygotsky’s theory can also be applied. The systemic connections and mutual mediation of the various levels related to conceptual understanding of concept development (from classroom conversation through to the workings of the economy), and the close interrelation of theory and practice within the thesis provide an unusual coherence to the thesis. Here form (or method) and content are one.

2) Does the thesis’ framework provide a sufficient basis for further development?

The evidence that it does exists particularly in the developments of chapters 7 and 8, where the practically informed interrelation of Vygotsky’s theory and systems of pedagogy is extended both theoretically and pedagogically. Evidence also exists in the numerous suggestions for future work within this chapter which have arisen in the course of the frameworks internal development. Some
additional extensions to the framework as a whole, and for future work in the medium term, not noted so far, include:

- The connections and potential of development within transmissionism;
- Pedagogic concepts and a theory of teacher development;
- The role of organisation in mediating conceptual development;
- How concept development mediates dispositions toward mathematics;
- The relation of mathematical and other metacognition; and
- Identity as a concept of self.

3) Can the framework incorporate insights from, or critique, other literature and findings in MER?

Subjectively (for the author at least) it can. Objectively, some evidence exists for this in the breadth of the thesis and the integration of a wide range of perspectives within it. Further evidence could be provided through an analysis of a range of recent articles from a range of leading mathematics education journals, showing how findings from these can be integrated, or understandings can be challenged through the framework. Reasons of time and space prevent such work within this thesis however.

4) Does it provide a guide to action for those involved in mathematics education and particularly critical mathematics educators?

For teachers, the framework provides some simple guidelines for pedagogy, as the theory can be operationalised around the key important elements. Connectionism already contains within its name an orientating perspective on emphasising connections with the real world, student understandings and other mathematics. The framework as a whole is also one that can be shared (via connectionist methods) in a not overly theoretical form. The particular undergraduate course in teaching and learning mathematics, which provided most of the classroom data for the thesis, represents an early attempt at doing this, and therefore provides some evidence toward this claim.
In addition, subjectively, the framework very adequately informs the author’s own teaching within mathematics, initial teacher training and teacher professional development. In simple terms, whether the content of the instruction is logarithms, encouraging dialogue in the classroom, or how to conduct a lesson study analysis session, the method orientates first on thinking through what the key concepts are, and then on finding real problems within which those concepts are helpful (only rarely does one have the luxury of approaching these in the other direction – beginning with an interesting problem and thinking through the concepts within it). The next stage is to, where possible, create experience of the problem situation (connecting to the real world), for example, by enacting a mini-lesson when carrying out lesson study analysis. Then there is a focus on dialogue in order to draw out students own conceptions. This also begins the process of ‘criticism’ of those conceptions, and the beginnings of generalisations within systems, as responses build on or contradict each other in relation to each other. Finally (although really this process is iterative), there is attention to the question of how this can be generalised further, through, for example, connecting to something more distant (mathematically or otherwise), or embedding in a wider system (e.g. how does experience of school observations interfere with conducting lesson study analysis?), with the understanding that further attempts at generalisation must themselves involve genuine problems.

For researchers the thesis provides an example of applying a totality perspective, and therefore of the benefits in this approach. It also provides a rich potential future research agenda, at least for the author, but also perhaps for others who would like a more systematic approach to research.

To discuss the potential of the thesis/framework as a guide to action for critical mathematics educators, whether teachers or researchers, is to return to the questions raised in the introduction to the thesis. This therefore deserves its own section, one which concludes the thesis as a whole.

9.3 Critical mathematics education

There were two intersecting motivations for developing the thesis which were raised in the introduction. The first, based on years of teaching experience, reflected a desire to understand mathematics teaching and learning, and particularly, to assist in developing and spreading more meaningful activity in the
classroom. The second, based on years of life experience, was to understand mathematics education from a perspective critical of the way society is organised. It was also raised in the introduction that some have argued that these two motivations could be contradictory. That improving mathematics education could simply be reinforcing and reproducing the system which negatively dominates our lives.

However, the results of the thesis contain within it a different perspective. First, chapter 5 shows how capitalist interests are served by a form of organisation of schooling and a type of teaching which directly contradict and limit the options for the more meaningful forms of teaching described in other chapters. Even if some elements of the dominant system would like to see deeper understanding of mathematics, among at least a minority of students, to feed into the development of products and markets for capital based in the nation state (and some do), the other roles schooling plays for capitalism are too important for this to happen. After all, meaningful mathematics education, or the development of scientific concepts is really a form of critical thought and activity, which works against all the aspects of domination described in chapter 5. Schooling which involved social, collective problem solving, and students confident to think for themselves and to challenge the ideas of others would have problematic consequences when those students later entered the workforce. Although it overestimates the role of education to think this in itself would change the world (it would require much wider social movements), it would of course be helpful to those who wish to challenge the way society is organised if school was like this. This is why anyone coming from a critical perspective looks to subvert whatever space they are in, whether classroom, research or elsewhere to encourage the development of critical thought and activity.

On the other hand the real limitations which schooling imposes on work of this nature means that it is difficult to sustain critical educational activity if it is solely limited to the individual classroom (or university corridor). It requires organised networks, and, in tandem, a collective development of ideas from within those networks, to both spread and sustain such activity. Taken together these points mean both that it is in the interests of critical mathematics educators to encourage meaningful activity in the classroom and to work alongside others who wish to do this, and, in the interests of those who want more meaningful activity in the classroom to work with those who have a critical perspective on society (because they bring an understanding of the obstacles, and, usually, experience
in organising networks). The main task therefore for those who are critical of society and who work within mathematics education, is to help create, develop and shape organisational forms which encompass both aspects. That means, in the UK for example, connecting up members of the National Union of Teachers with members of the Association of Teachers of Mathematics, and those who fight against privatisation of universities with those researchers who have critical perspectives on learning mathematics. Ultimately, this is the key future work which emerges from this thesis.
References (for chapters 1, 2, 3 & 9 and addendum to chapter 4)


APPENDIX 1: Participant information sheet and consent form

The role of situated and meta-cognition in the development of mathematical concepts.

Participant Information Sheet

You are being invited to take part in a PhD study based at the University of Manchester, and funded by the Economic and Social Research Council (ESRC). The study involves observations of mathematics lessons, recording of course output and interviews. Before you decide it is important for you to understand why the research is being done and what it will involve. Please take time to read the following information carefully and discuss it with others if you wish. Please ask if there is anything that is not clear or if you would like more information. Take time to decide whether or not you wish to take part. Thank you for reading this.

Who will conduct the research?

David Swanson

Room B4.10, Ellen Wilkinson Building
School of Education
University of Manchester
Oxford Road
Manchester, M13 9PL
Email: david.swanson@manchester.ac.uk
Tel: 07931640392

Title of the Research

The role of situated and meta-cognition in the development of mathematical concepts.

What is the aim of the research?

The study aims to develop understanding of how connections to real world experience and connections within mathematics interrelate in the development of mathematical thinking. It also aims to understand how this process relates to students’ motivation and attitude toward mathematics. New understandings of how mathematics pedagogy can be more meaningful, less alienating and yet deliver on outcomes will be important to mathematics education research, policy and practice.
Why have I been chosen?

You have been chosen to take part because your short course at the University of Manchester offers rich examples of the particular aspects of teaching and learning of interest to the study. You should note however that your participation is entirely voluntary.

What would I be asked to do if I took part?

If you agree to take part in the case study, the following activities are likely to take place:

1. We will observe your mathematics classes. These lessons will be audio or video recorded for analysis purposes.
2. We may take photographic recordings of your written output in your mathematics class.
3. We may also conduct an interview with you about your story as a mathematics learner and your perceptions of mathematics in general.

If at any time you feel uncomfortable about the recording, or our presence, in the lesson, we will directly stop any of our activities.

What happens to the data collected?

Once collected, data will be kept in secure databases. The video/audio recordings of the class observations and the audio recordings of the interviews will be transcribed and anonymised.

How is confidentiality maintained?

The interviews and observations will be transcribed and anonymised and any identifiers of the person will be removed. After the completion of the project all audio and video files will be deleted.

What happens if I do not want to take part or if I change my mind?

It is up to you to decide whether or not to take part. If you do decide to take part you will be given this information sheet to keep and be asked to sign a consent form. If you decide to take part you are still free to withdraw at any time without giving a reason and without detriment to yourself.

Will I be paid for participating in the research?

Unfortunately, there are no resources available to provide compensation for taking part in this study. You will however be receiving free mathematics tuition as a result of the study.

What is the duration of the research?

Your involvement in this part of the study will entail

- Observations of your classes during the course.
- Potentially, an interview during the course (maximum thirty minutes) and/or a brief interview (maximum twenty minutes) after the end of the course.

Where will the research be conducted?

The research will take place at the University of Manchester.

Will the outcomes of the research be published?

The details of the study will contribute to a PhD thesis, and it is expected that the outcomes of the research will also be published in peer reviewed scientific journals, conference proceedings and presentations.

Written feedback can also be provided to you upon request.

Contact for further information

For further information, please contact David Swanson:

Room B4.10, Ellen Wilkinson Building
School of Education
University of Manchester
Oxford Road
Manchester, M13 9PL

David Swanson
Email: david.swanson@postgrad.manchester.ac.uk
Tel: 07931640392

What if something goes wrong?

If there are any issues regarding this research that you would prefer not to discuss with members of the research team, please contact the Research Practice and Governance Co-ordinator by either writing to ‘The Research Practice and Governance Co-ordinator, Research Office, Christie Building, The University of Manchester, Oxford Road, Manchester M13 9PL’, by emailing: Research-Governance@manchester.ac.uk, or by telephoning 0161 275 7583 or 275 8093
The role of situated and meta-cognition in the development of mathematical concepts

PARTICIPANT CONSENT FORM

If you are happy to participate please complete and sign the consent form below

1. I confirm that I have read the attached information sheet on the above study and have had the opportunity to consider the information and ask questions and had these answered satisfactorily.

2. I understand that my participation in the study is voluntary and that I am free to withdraw at any time without giving a reason.

3. I understand that classes will be video or audio-recorded.

4. I understand that interviews will be audio-recorded.

5. I agree to the use of anonymous quotes

6. I agree that any data collected may be passed to other researchers

7. I agree that any data collected may be published in anonymous form in academic books or journals.

I agree to take part in the above project

Name of participant:  [ ] Date:  [ ] Signature:

Name of person taking consent:  [ ] Date:  [ ] Signature:
**APPENDIX 2: Teleprism interview fragments**

R = a researcher, S = a student (with age and gender in brackets).

<table>
<thead>
<tr>
<th>No.</th>
<th>Role</th>
<th>Transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>S (13F)</td>
<td>I've really enjoyed school like most people, like, they probably don't like school or they dislike it but I've never hated school because if we don't have it there's really nothing else for us to do.</td>
</tr>
<tr>
<td>2)</td>
<td>R</td>
<td>Can you tell me a bit about yourself...</td>
</tr>
<tr>
<td></td>
<td>S (13m)</td>
<td>... I've had long hair all my life.</td>
</tr>
<tr>
<td>3)</td>
<td>R</td>
<td>Tell us a little about yourself.</td>
</tr>
<tr>
<td></td>
<td>S (15f)</td>
<td>... I work well in a team.</td>
</tr>
<tr>
<td>4)</td>
<td>R</td>
<td>Ok. So...do you think in general you’re a good student?</td>
</tr>
<tr>
<td></td>
<td>S (12f)</td>
<td>Yeah, generally I always hand my homework in on time, present myself smartly and always try my hardest with my work.</td>
</tr>
<tr>
<td>5)</td>
<td>R</td>
<td>So what do you want to do with your life after school, have you thought about that?</td>
</tr>
<tr>
<td></td>
<td>S (11m)</td>
<td>Well I have always wanted to be a pioneer, but there is nowhere to pioneer any more is there? ... I don't know what else I can do apart from that, because I don't want to be here for the rest of my life sat a desk, just sorting out files like asking someone to fax this to the next country; I don't want to do that... That is not my type of thing – sitting at a desk with coffee doing that all day and then coming home... Yes, because if you want to be a pioneer ... you set your mind to it and OK I have set my mind to it, I want to be a pioneer, you do your exams and then it is can I be a pioneer? 'No'. How do you be a pioneer, do you have to do this sort of exam, then you have to do that and that, and it is just like, I am only going to countries, just looking around.</td>
</tr>
<tr>
<td>6)</td>
<td>S (14f)</td>
<td>I have been set on being a lawyer for about three quarters of a year now.</td>
</tr>
<tr>
<td>7)</td>
<td>R</td>
<td>Does his (the student’s father) work involve Maths?</td>
</tr>
<tr>
<td></td>
<td>S (13m)</td>
<td>No I don't think so.</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>What does he do?</td>
</tr>
<tr>
<td></td>
<td>S</td>
<td>Suitcases.</td>
</tr>
<tr>
<td>8)</td>
<td>R</td>
<td>Do you want to go to University?</td>
</tr>
</tbody>
</table>
S (12m) Yeah
R And what would you do there?
S I don’t even know what you do at university.

9) S (13f) I don’t think I’m going to get a job to be honest.
S (13m) Why?
S (f) Because I’m getting low grades in everything.
S (m) You don’t think you’re going to get a job? You’re still going to get a job just work in McDonalds.

10) R Why don’t you think you’re going to get As?
S (13m) It’s just like mission impossible for me.
R Why?
S ’Cause I’m not bright, I’m not intelligent.
R Why do you say these things?
S ’Cause I’m not.
R When did you start thinking of yourself like that?
S Like if you get something wrong you think I knew that but I weren’t thinking I weren’t concentrating.

11) R Hmm how do you feel about GCSEs now?
S (13f) I’m scared I’m so scared.
R Yes but you are quite good really, you are progressing very well aren’t you?
S In that way I don’t believe in myself, GCSEs, I am a lot more confident in maths but GCSEs I just don’t believe in myself.
R You have another two years you know before you, you know.
S I know that’s the scary thing.

12) R What’s your level then in Maths?
S (12m) 6E.
R 6E ... that’s pretty okay. What’s your target level?
S 6A.
R 6A, yeah so you’re below your target level?
S I’ve not moved. I’ve not moved up a level or a sub level since
year seven. The end of year seven.

R The end of year seven so you're still on 6E? And how you feel about that?

S Not proud of it.

R Yeah.

S It’s alright.

13) R And do you think Mathematics will be useful in your future lives?

S (15m) Yeah, it’s big, GCSE, isn't it.

14) S (11f) I’m definitely going to college but I don’t know about University because it’s really expensive.

15) S (13f) Some teachers the way they like speak to kids the way they approach them and the way they speak and you know shout the words I've never believed in any of that like you shout at me I’ll shout back you, speak to me calmly I’ll give you a calm response, do you know what I mean?

16) S (11f) I’ve been told off a couple of times, but that was only because I’ve not understood it and I speak to people if I’ve not understood it, but now I know not to speak to people and ask the teacher.

17) S (11f) I think I have had my ideal lesson, (it wasn't intentional) it was when our teacher, who has really bad asthma, so one day she came in and said she wanted everyone to be quiet. She explained it, because it was something she had done from the lesson and she gave us sheets on what to do; and everyone just quietly finished their work; it doesn't have to be anything fancy I can work with noise, but everyone was just quiet, so that was nice.

18) R ...primary school, did you have any good memories or bad memories?

S (15f) Yeah, I think I have good memories, but I can’t remember.

19) R And how did you find primary school?

S (11f) It was alright. It’s a lot different to high school but...

R In what ways is it different?

S Like people here called the teachers Miss and Sir but in primary school you’ve got to call them by their name, like Miss Clarke or something, so that’s changed a lot.
20) R | So what about school, what can you remember from primary school, can you remember much?  
| S (11m) | When we were in Year 4 from Reception you would get 4 breaks, but when you go into juniors – above Year 4 you get 3 breaks, morning, dinner and end.  

21) R | How did it change with the transition from that school then to this school, the secondary school?  
| S (11f) | It changed because the lessons are shorter, you have to hurry up to do more work and produce what you can.  
| R | Can you give me an example?  
| S | Well in English we used to write the date, the title and the learning objective in a maximum of 2 minutes, whereas I used to write the date, then underline it, then write the title, then underline it. So now I just write it and do it, and she moves on to the next slide, so it is just getting into that way of what to do.  

22) S (11f) | My worst subject...I like art but I don’t like the lesson. I mean I like doing my own art, I don’t like people telling me what to do in art.  

23) S (13m) | I decided to pick drama, science and French...I took drama for a lesson for me to get you know, you know a free lesson, ‘cause I like drama...  
| R | So it’s easy for you?  
| S | Yeah, ‘cause with too much on...’cause I thought a lot of work to do on paper, I needed a break so...  

24) S (13f) | In Citizenship they were saying, erm, some things you want to change and I said homework and I got the highest score out of the whole class saying you should ban homework and I did really good reasons and I got a 6A.  

25) R | Okay can you describe to me the lesson a little bit?  
| S (13m) | She puts like an objective on the board and then like she reads it out and then she goes through examples what we’ve got to do and then she explains what we’ve got to do and then she asks us to try and work out the answer and at the end she like shouts one of us up and then we've got to write the answer up on the board and then if we get it wrong she like explains how we got it wrong and stuff.  

26) S (14f) | My old teacher used to explain why something works, but my new teacher doesn't do that, I quite like to see how things work. Because now when we are told something we just have to accept that it works.
**27)** R | Anything that's been particularly good about Maths this year?
---|---
S (13f) | Hmm don't know. I don't, I’d say everything’s been good in Maths, like I'm not, we went outside a couple weeks ago we had to do angles and we had to like measure from the bottom of a certain, like we went over to the side and we had to measure from the floor and say what the angle was from the floor to the top of that.
R | I see, was that in Maths?
S | Yeah we all did it in groups together.
R | Okay so how did you find that?
S | That was mental.

**28)** S (11f) | ...on ‘Fun Friday’s’ which is today, erm, Miss gets us like sheets where we’ve got to work out what we’ve been doing in class like practice on times tables or something else.

---

**29)** R | So can you think of examples where you use this algebra in real life?
---|---
S (12m) | ...like...I don’t know actually, I think like...you can do it for sharing but I don’t know how, if you add three pizzas...no actually I can’t think of anything.

---

**30)** R | In an ideal world as we say, how would you like the lesson to be?
---|---
S (15m) | A lot more interactive.
R | Can you tell me an example?
S | Like...in the classrooms they’ve got that board, like...that...I don’t even know what it’s called, you know what I mean?
R | The interactive whiteboard, yes.
S | Teachers just sit there and they write on that and then you’ve got to copy, but it’d be better if like other people could come up and attempt stuff on it if you know what I mean.

---

**31)** S (11m) | The other thing I find hard is when you have to get like say minus seven add minus four I don’t get it it’s like you have to swap them around and like add it up and take it away and it’s just a bit... 
R | So do you know why you’re doing swapping them around and adding them up and taking them away?
S | I kind of know it’s like the method but it’s just...
R | So you don’t mean all the...
S | No.
<p>| | |</p>
<table>
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<tbody>
<tr>
<td>R</td>
<td>Like the real?</td>
</tr>
<tr>
<td>S</td>
<td>Yeah</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>32)</th>
<th>S (14f)</th>
<th>Yeah there's like different methods and things so that could be quite difficult to get used to.</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>So what is the difference between this teacher and before?</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>Um, well for factorising, use that as an example, I know the teacher I had in year seven used a different method than the one I use now.</td>
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<tr>
<td>R</td>
<td>Can you tell me what exactly was different?</td>
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</tr>
<tr>
<td>S</td>
<td>Well when you have like the two brackets next to each other.</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>Yeah.</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>You like supposed to factorise it out and my teacher now uses eyebrows and smiley face and it's like it looks like that but the one in year seven used a crab claw and it looked like a crab claw.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>33)</th>
<th>R</th>
<th>Which is your favourite or best topic in mathematics?</th>
</tr>
</thead>
<tbody>
<tr>
<td>S (12m)</td>
<td>Probably times.</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>And the most difficult one, the least?</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>Divide.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>34)</th>
<th>R</th>
<th>Are there any topics you like more than others?</th>
</tr>
</thead>
<tbody>
<tr>
<td>S (12f)</td>
<td>... I don’t mind doing brackets.</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX 3: Proportion worksheets

Similar Triangles

If two triangles are ‘similar’ then the three sides of each triangle are in proportion:

So if we take two of the sides in A and the equivalent two sides in B, then \( \frac{3}{4} = \frac{4.5}{6} \)

Also, if we take an equivalent side in A and B, and another equivalent side in A and B, then \( \frac{3}{4.5} = \frac{4}{6} \)

This means we can work out any missing lengths. Say the 6cm length was missing from B. We could have worked out what it should be by using algebra:

\[
\frac{3}{4} = \frac{4.5}{x} \quad \text{so} \quad \frac{4}{3} = \frac{x}{4.5} \quad \text{so} \quad 4.5 \times \frac{4}{3} = x \quad \text{so} \quad x = \frac{18}{3} = 6
\]

Or \( \frac{3}{4.5} = \frac{4}{x} \) so \( \frac{4.5}{3} = \frac{x}{4} \) so \( 4 \times \frac{4.5}{3} = x \) so \( x = \frac{18}{3} = 6 \)

Or we could see that \( 1.5 \times 3 = 4.5 \) so \( 1.5 \times 4 = 6 \)

Or we could see that \( 3 \div 3 = 1 \) and \( 1 \times 4.5 = 4.5 \) so \( 4 \div 3 = \frac{4}{3} \) and \( \frac{4}{3} \times 4.5 = 6 \)

Try this?

Triangles PQR and XYZ are similar.
What is the length of XZ?
Equivalent Fractions

A fraction can be written in different ways and still mean the same thing. These are called equivalent fractions. Look at the shaded areas in this rectangle.

\[
\frac{3}{5} \quad \text{and} \quad \frac{6}{10}
\]

So \( \frac{3}{5} = \frac{6}{10} \)

You can produce lots of equivalent fractions by multiplying or dividing the top and bottom by the same number.

\[
\frac{3}{4} \quad \xrightarrow{\times 2} \quad \frac{6}{8} \quad \xrightarrow{\times 3} \quad \frac{9}{12} \quad \xrightarrow{\times 4} \quad \frac{12}{16}
\]

If you are asked to fill in a missing number, remember that the top and bottom must be multiplied or divided by the same number.

Try these?

1. \( \frac{3}{4} = \frac{9}{?} \)
2. \( \frac{2}{3} = \frac{?}{15} \)
3. \( \frac{24}{36} = \frac{?}{3} \)
Pie Charts

Look at this record of traffic travelling down a particular road.

To draw a pie chart, we need to represent each part of the data as a proportion of 360, because there are 360 degrees in a circle. For example, if 55 out of 270 vehicles are vans, we will represent this on the circle as a segment with an angle of:

\[(55/270) \times 360 = 73\text{ degrees}.\]

This data is represented on the pie chart below.

<table>
<thead>
<tr>
<th>Traffic Survey 31 January 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of vehicle</td>
</tr>
<tr>
<td>Cars</td>
</tr>
<tr>
<td>Motorbikes</td>
</tr>
<tr>
<td>Vans</td>
</tr>
<tr>
<td>Buses</td>
</tr>
<tr>
<td>Total Vehicles</td>
</tr>
</tbody>
</table>

This will give the following results:

<table>
<thead>
<tr>
<th>Traffic Survey 31 January 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of vehicle</td>
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<tr>
<td>Motorbikes</td>
</tr>
<tr>
<td>Vans</td>
</tr>
<tr>
<td>Buses</td>
</tr>
</tbody>
</table>

Try these?

Sixty people were asked to name their favourite season. The results are given below:

<table>
<thead>
<tr>
<th>Season</th>
<th>Number of people</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring</td>
<td>15</td>
</tr>
<tr>
<td>Summer</td>
<td>25</td>
</tr>
<tr>
<td>Autumn</td>
<td>16</td>
</tr>
<tr>
<td>Winter</td>
<td>4</td>
</tr>
</tbody>
</table>

If a pie chart is to be drawn, calculate the angle required for one person.

If a pie chart is to be drawn, calculate the angle required for the summer.
Converting to a percentage

A percentage is a fraction of 100.
30% (30 in each 100) as a fraction is $\frac{30}{100}$

To converting a fraction into a percentage, we want to convert the fraction into something out of 100.
So to change $\frac{3}{4}$ into a percentage, we need to find $x$ in $\frac{3}{4} = \frac{x}{100}$
If we multiply both sides by 100 we get (on the right hand side) $x$, and on the left hand side $\frac{3}{4} \times 100 = 75$
So $\frac{3}{4} = 75\%$

The simple way to change a fraction into a percentage is therefore to just multiply by 100.
7/10 is equivalent to $\frac{7}{10} \times 100 = 70\%$

Try these?

If 8 out of 10 cats prefer something, what is that as a percentage?
What is $\frac{5}{16}$ as a percentage?

\[ \text{Answer: } 31.25\% \]
Ratios

A ratio is a way to compare amounts of something. Recipes, for example, are sometimes given as ratios. To make pastry you may need to mix 2 parts flour to 1 part fat. This means the ratio of flour to fat is 2:1.

Ratios can be used to solve many different problems - for example, with recipes, scale drawings or map work.

Example 1:
Sam does a scale drawing of his kitchen. He uses a scale of 1:100. He measures the length of the kitchen as 5.9m. How long is the kitchen on the scale drawing? Give your answer in mm.

The answer is 59mm. You need to convert 5.9m to mm, then divided by 100 to give the answer.
If you did not get the correct answer, remember that the scale of 1:100 means that the real kitchen is 100 times bigger than the scale drawing.

5.9m = 590cm (multiplied by 100) = 5900mm (multiplied by 10)
So the scale drawing would be 5900 ÷ 100 = 59mm.

Example 2:
A recipe to make lasagne for 6 people uses 300 grams of minced beef. What is the ratio of people to minced beef? How much minced beef would be needed to serve 8 people?

The ratio is 6: 300 = 1: 50 (300÷6). So 8 people would need 8 x 50 = 400g of minced beef (or horse).

Try these?

A map scale is 1:20000. A distance on the map is measured to be 5.6cm. What's the actual distance in real life? Give your answer in metres.

If three apples cost 45p, how much would five apples cost?
Sine of 30°

The sides of the right-angled triangles are given special names - the hypotenuse, the opposite and the adjacent.

The hypotenuse is the longest side and is always opposite the right angle. The opposite and adjacent sides relate to the angle under consideration.

\[ \sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{O}{H} \]

The classic 30° triangle has a hypotenuse of length 2, an opposite side of length 1 and an adjacent side of \( \sqrt{3} \)

So \( \sin(30°) = \frac{1}{2} = 0.5 \)

Try these?

A)

Find the length of side BC.

B) If a road has a grade of 30°, this means that its angle of elevation is 30°. If you travel 1.5 miles on this road, how much elevation have you gained

Answer: 25.2m