Amplitude Bifurcation in the Peeling Relaxation ELM Model

by

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Abstract

School of Physics and Astronomy

Doctor of Philosophy

by Peter F. Devoy

In this report we build on existing theories of Edge Localised Modes (ELMs) that incorporate the theory of Magnetic Relaxation, firstly by introducing the effects of the bootstrap current and secondly by applying the relaxation model to the field of ELM mitigation. The bootstrap current is a toroidal plasma current that arises spontaneously as a result of the confinement of the plasma. It is shown that the effect of introducing this current is to split the ELM widths associated with this model into two classes; one group with large ELM widths, the other with small ELM widths. An analytical study is then carried out which provides a theoretical basis for this bifurcation of results. Finally the idea of ELM mitigation using Resonant Magnetic Perturbations will be introduced before the Relaxation Theory of ELMs is used to give new insight into ELM mitigation results . . .
Declaration of Authorship

I, Peter Devoy, declare that this thesis titled, ‘Amplitude Bifurcation in the Peeling Relaxation ELM Model’ and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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Many thanks to everyone who has made this thesis possible, with special thanks to my Dad, my brother John, and my supervisor Philippa...
Chapter 1

Introduction

1.1 Nuclear Fusion

Nuclear fusion is the process by which two light nuclei fuse together to produce a heavier nucleus. The difference in mass between the initial nuclei and the end product leads to either the release or absorption of energy, depending on the position of the reactants on the binding energy curve. It is apparent that if the reactants can be chosen so that energy is released during the process then it would be a potential method of power generation. While there are several combinations of nuclei that will fuse together to release energy the most likely fuels for such a reaction are the isotopes of hydrogen, Deuterium and Tritium. By fusing one of each of these nuclei to make a Helium nucleus and a neutron (the reaction shown in 1.1.1), around 17 Mev of energy are released per reaction:

\[
{^2}_1D + {^3}_1T \rightarrow {^4}_2He + {^1}_0n. \tag{1.1.1}
\]

Deuterium is naturally abundant, it can be extracted from sea water, and Tritium can be produced by bombarding Lithium (one of the most common metals on Earth) with neutrons. Therefore nuclear fusion offers a practically limitless source of energy with none of the long term radioactive by-products associated with fission power generation.

One of the major stumbling blocks on the path to controlled nuclear fusion is that, in order for the nuclei to fuse, they must pass close enough to each other that the attractive, but short range, Strong Nuclear force overcomes the electrostatic repulsion between the two positively charged nuclei. In order for the nuclei to overcome this electrostatic repulsion, they must have a great deal of kinetic energy and this in turn requires very high temperatures. It can be shown that in order for the nuclei to fuse a temperature of around 100 Kev is required. This in turn creates
the problem of ensuring that enough reactions take place to give a net energy gain. The ideal solution would be that sufficient energy is produced by the fusion reactions that they become self sustaining, a situation known as ignition.

The achievement of ignition depends mainly on the density of the fuel, the temperature at which the reactions take place and length of time over which the reactions can take place. In the early days of fusion research these factors were combined to form the Lawson criterion [38] shown below, which states the conditions required for ignition to occur:

$$nt \geq 1.5 \times 10^{20},$$

where $n$ is the density of reactants and $t$ is the energy confinement time. Although the exact numbers in the Lawson criterion have been revised over the years the general principle remains true. Unfortunately producing conditions where the Lawson criterion is satisfied is extremely challenging as the extremely hot fuel is not easily contained. Any contact with the walls of a conventional container would both damage the container and reduce the temperature of the fuel to a point where fusion reactions would be impossible. Therefore some other way of containing the fuel must be found. Attempts at fusion are defined by the method of confinement of the reactants, of which there are two broad approaches. The first method attempts to compress the fuel to a very high degree while simultaneously heating it to the required temperature and thus using the inertia of the fuel itself to contain the reactions. The second method uses magnetic fields to ‘bottle’ the hot fuel, making use of the quasi-neutral properties of a gas at high temperatures (a state of matter known as plasma). The two approaches are known respectively as inertial confinement and magnetic confinement.

1.1.1 Inertial confinement

In Inertial confinement experiments small amounts of fuel are simultaneously greatly compressed and heated to an extremely high temperature [3]. This heating and compression can be generated by high powered lasers or an intense x-ray pulse. At the high temperatures and pressures found in the target fuel pellet, conditions could potentially be suitable for fusion to occur, although on a very short timescale. There has been much recent progress in the field of inertial confinement. However, a more in depth review of this topic is beyond the scope of this report which will concentrate on magnetic confinement techniques.
1.1.2 Magnetic Confinement and Plasma Physics

When heated to a high enough temperature the electrons in a gas become detached from their parent nuclei, forming what is known as a plasma. Plasma is a quasi-neutral fluid made up of electrons and ions, meaning that while its constituent parts are charged, on large enough length scales the plasma appears neutral as the positive and negative charges cancel each other out. Any separation of charge generates an electric field which acts to oppose and eliminate the charge separation. Due to the extremely high temperatures required for fusion reactions to take place, the nuclear fuel is hot enough to be in the form of a plasma. It was noted early on in fusion research that as plasma is made of these charged components it would be possible to manipulate this plasma using magnetic fields, and if this manipulation were sophisticated enough it might allow the plasma to be contained entirely within a ‘magnetic bottle’. This containment is not, however, straightforward. The state of the plasma is constantly changing and factors as diverse as density fluctuations, plasma rotation caused by the injection of fuel, and the presence of impurities can all lead to radical alterations in plasma behavior.

Referring back to the Lawson Criterion we can derive a general measure of how well the plasma is contained by looking at both the duration of plasma containment and the density achieved at a temperature high enough for reactions to take place. Achieving high $n$ and $t$ in 1.1.1 has two distinct components. The first is that the number of particles lost after undergoing collisions and scattering out of the plasma is as low as possible. The second is that the plasma must not undergo any large scale bulk movements out of the magnetic bottle. Theoretically modeling how these two aims are achieved is in general covered by two branches of plasma physics, Kinetic Theory (overview given in [49]) and Magneto Hydro Dynamics (MHD) (originally discovered by [2], see [28] for a comprehensive overview) respectively.

Kinetic theory describes the plasma on very small length scales by examining individual particle dynamics and particle distributions. Kinetic theory is thus mostly outside the scope of this thesis, which will be concerned chiefly with looking at the plasma on larger scales where the movement of individual particles merge into fluid behavior. However some important results of kinetic theory such as the bootstrap current are central to the theme of this report.

1.2 General Plasma Confinement Considerations

A magnetic confinement configuration is only useful for fusion if it fulfills certain criteria, the first of which is that the plasma should be confined so that the number of particles escaping the magnetic jar is low enough to allow sufficiently high densities and temperatures for fusion to be maintained for a suitable time period. Though particle losses are determined by kinetic theory and thus largely outside the scope of this thesis, it can be shown that the particle losses
along magnetic field lines are far greater than losses perpendicular to the field lines. Therefore open ended systems where particles are free to flow out of the plasma along the field lines are unsuitable for fusion experiments. Most experiments are configured so that the magnetic field lines loop back upon themselves. This forms a toroidal system where the only losses are perpendicular to the field lines.

The second consideration is that the system should have a static equilibrium state such that the plasma is in force balance and not in motion. The state of this equilibrium is described by the time-independent MHD equations [28][32], as given in the next section.

The third criterion is that the equilibrium state of the plasma is stable. A plasma is stable when it returns to the equilibrium state after a perturbation, and unstable if any displacement from equilibrium grows and is reinforced. Plasma stability is one of the most important areas of current investigation, with fusion plasmas being subject to a wide range of instabilities, many of which can cause severe disruptions to plasma containment. The central focus of this thesis will be on instabilities that occur in the edge region of Tokamak plasmas.

1.3 Tokamaks

Most current fusion experiments use a magnetic containment configuration known as a Tokamak [51] (an example of which is shown in Figure 1.1 below), where the plasma is contained by both external magnetic fields and one that is generated by running a large current through the plasma itself. Tokamak design has evolved considerably over time, with a general progression towards larger machines. Most current Tokamaks have an average minor radius (the radius of the poloidal cross-section of the torus) of $\sim 0.5\text{m}$ and a toroidal current of $\sim 2 - 3\text{MA}$, with larger Tokamaks (such as JET[50]) having a minor radius of $\sim 1\text{m}$ and toroidal current of $\sim 7\text{MA}$.

The next generation of Tokamak design is led by ITER [19, 17], currently under construction in Cadarache, France. ITER is substantially larger than current Tokamaks with a minor radius of $\sim 2\text{m}$ and a plasma current of $15\text{MA}$. This increase in size, along with advances in technology, will allow modes of operation in ITER much closer to that of a working fusion reactor.

To reach temperatures suitable for fusion reactions to take place, several different methods are used to heat the plasma. The primary source of heating in a Tokamak is the ohmic heating generated by the toroidal current, which is induced in the plasma by a large transformer inside the center of the the torus. Unfortunately the resistivity $\eta$ of a plasma decreases with temperature according to $\eta \propto T^{-\frac{3}{2}}$ [27]. This means that at temperatures above $\approx 3\text{ Kev}$ ohmic heating becomes ineffective and secondary means of heating the plasma must be employed. The two current forms of secondary heating are neutral beam injection (NBI) [45], and Radio Frequency (RF) [35] heating. In NBI heating neutral atoms of fuel (eg. deuterium), which pass through the
Tokamak’s magnetic fields, are 'fired' into the plasma at high velocity and transfer their energy to the plasma through collisions. In RF heating, radio waves that match one of the resonant frequencies of the plasma are beamed into the plasma, generating heat in a similar fashion to a microwave oven. How much of each form of heating is used, and in the case of RF heating which one of the plasma’s resonant frequencies is targeted, depends on the temperature of the plasma. A more detailed examination of plasma heating is beyond the scope of this thesis, however it remains an area of current research [18].

In a Tokamak the plasma is distributed in a toroidal configuration, the exact nature of which can vary depending on the individual experiment, with layouts ranging from a ringed cylinder to something closer to a sphere (the spherical Tokamak). The plasma itself is separated from the containing vessel of the Tokamak by a vacuum region. The containing vessel must be rigorously designed to resist the large amount of energy that can be deposited on its wall by particles escaping from the plasma. This is especially true in the region of the Tokamak called the divertor, located beneath the main body of the reactor. The divertor is the target of all the plasma which escapes containment from the magnetic bottle. The boundary of this magnetic bottle is called the separatrix, which divides the closed magnetic field lines of the 'bottle' with the open field lines that exist outside of it. Any plasma that moves outside the separatrix is deflected to the divertor, which is thus exposed to a great deal of heat flux. The problems of heat flux have meant that the walls of the containment vessel in most modern experiments are made from carbon based materials which have suitable thermal properties. Unfortunately as carbon is chemically reactive it can bind with the radioactive Tritium fuel, meaning that over long periods of time layers of
radioactive carbon soot can build up on the walls of the vessel. In response to this problem future experiments such as ITER will have Beryllium walls and divertors comprised of Tungsten (as part of a major upgrade, JET has been refitted with ITER-like walls and divertor). This choice of materials means that limits for heat flux in ITER are very strict [12].

The need for the vessel walls, and the divertor in particular, to be able to withstand large energy depositions is especially true when relating to the phenomena of Edge Localised Modes (ELMs). Elms are instabilities which occur at the edge of the plasma and can deposit sizable amounts of energy on the vessel walls. As fusion experiments become larger and operate at higher temperatures (increasing the energy available for ELMs) this energy deposition could become potentially detrimental to the integrity of the vessel walls. An understanding of ELMs is therefore of great importance to continued fusion research. In order to explain the origins of ELMs it is first necessary to understand the various regimes in which Tokamaks operate. These are defined primarily by the temperature and pressure of the plasma in the reactor. The different properties of these regimes can explain some of the features of ELMs and also why they occur.

1.3.1 L-Mode

The first regime of Tokamak operation is the L-mode or low mode of operation. This was the first regime to be explored as it is observed at relatively low heating powers and temperatures. When in low mode the plasma has some instabilities but does not suffer from ELMs and is thus of limited interest to this report. During the transition from L-mode to H-mode (see below) some ELM like events do occur, called dithering cycles, but as these do not occur at the temperatures required for sustained fusion reactions they are again not discussed in this report.

1.3.2 H-Mode

Although the most readily reached regime, L-mode is not suitable for fusion experiments close to break even levels as the temperatures achieved and the confinement times are both too low. Instead the H-mode of operation is now the favoured mode of operation for the more advanced test reactors. This mode occurs when the heating power put into the plasma reaches a critical point and the plasma moves out of L-mode [8]. H-mode was first discovered on the ASDEX Tokamak in 1982 [13] and subsequently achieved on all large Tokamaks. The use of secondary heating sources, particularly NBI heating, was key to accessing H-mode.

When H-mode is reached a ‘transport barrier’ [14] is formed at the edge of the plasma which stops particles escaping to the vacuum that surrounds the plasma. Figure 1.2 shows the difference in pressure between H-mode and L-mode as well as the position of the edge transport barrier. The improvement in confinement time after the transition is approximately twofold.
This in turn leads to the build up of large pressure gradients at the edge of the plasma known as the H-mode pedestal which is instrumental in the formation of ELMs. These pressure gradients also create proportional edge current gradients. Both the current and pressure gradients can lead to instabilities. Large pressure gradients are associated with ballooning mode instabilities whereas edge current gradients are linked to peeling mode instabilities. Both modes will be discussed in more detail later in this paper.

The H-mode is unstable to different types of ELMs depending on the temperatures running in the reactor at that point. For example at high temperatures Type I ELMs (see below) are the only type of ELM present, whereas at low temperatures and pressures Type-III ELMs are prevalent. Commonly the plasma will be unstable to a number of different instabilities as the heating power put into the plasma increases during the reactor shot. This can be seen in 1.3 [9], which shows the progression from small Type-III ELMs to large Type I ELMs as the NBI power increases.

1.4 Edge Localized Modes

Edge Localized Modes are instabilities which occur in the edge of Tokamak plasmas. Review articles containing details of established ELM theory can be found in [53], [7]. They are characterized by periodic discharges of both particles and energy from the edge of the plasma to
the surrounding vacuum and reactor walls. Each burst occurs on a timescale that is short compared to the time between bursts so that they appear as distinct events, with the time between events remaining approximately constant. This is because ELMs are thought to be caused by the gradual build up of gradients (pressure or edge current) within the edge region. These gradients are removed with each ELM event but they then build up again until they are sufficient to trigger another event. Elms are therefore a product of the transition to H-mode and the formation of the H-mode pedestal at the edge of the plasma. This unfortunately means that most of the plasma configurations of interest are susceptible to ELMs. Elms are categorised into many distinct types which occur under different conditions in the plasma, however for the purposes of this thesis only the most common two categories of ELM will be discussed; Type I Elms (sometimes called giant ELMs) and Type-III ELMs.

**Figure 1.3:** Graph taken from [9] showing the transition from small Type-III ELMs to Large Type I ELMs as heating power is increased on the MAST spherical Tokamak

### 1.4.1 Type I ELMs

Type-I ELMs were the first type to be observed in a reactor. They consist of large emissions of energy and particles over a very short time scale (200\(\mu\)s) at widely spaced intervals. Each event can emit as much as 20% of the energy stored in the pedestal [11] (it is thought that the large
proportion of energy released is because the edge plasma becomes briefly linked to the core region, allowing for a much higher energy flux out of the plasma, whereas in other ELMs types the flux comes only from the edge region). This means that as Tokamaks have grown larger, and the amount of energy stored in the pedestal has grown accordingly, the amount of energy released in Type I ELMs has increased to a point where serious damage can be caused to plasma facing components by sustained periods of ELM activity. With ITER now being constructed, there is an urgent need to understand how Type I ELMs can be eliminated or reduced in magnitude. This area of study is called ELM mitigation and will be covered more thoroughly in Chapter 5.

Type I ELMs are produced by a combination of two types of magnetohydrodynamic instability [15]; the ballooning mode and the peeling mode. While these instabilities will be covered in more detail in Chapter 2, it is sufficient here to state that ballooning modes are caused by edge pressure gradients and peeling modes are driven by edge current densities. The pedestal of an H-mode plasma has both high edge pressure gradients, and also high edge current densities (generated by the Bootstrap Current, which will be introduced in the next chapter). When a plasma is at the limit of both peeling and ballooning stability, the two modes couple to form the peeling-ballooning stability boundary. Type I ELMs occur when the peeling-ballooning boundary is crossed. The large ejection of matter and energy by the ELM lowers both the edge pressure and the edge current (the ELM ‘crash’), stabilising the plasma to the trigger instabilities. The edge pressure then begins to rise, in turn generating increasing edge current densities via the bootstrap mechanism, until the stability limit is again reached, triggering another ELM. The cycle of a Type-I ELM is shown in figure 1.4.

1.4.2 Type-III ELMs

Type-III ELMs are much smaller in terms of energy released (approximately 1-5% of the total plasma energy is released) than type I ELMs. However they are much more frequent and for this reason are sometimes called grassy ELMs. They occur just after the transition from L-mode to H-mode so the edge temperature and pressure associated with them is lower than for Type-I ELMs. This means that the pressure gradients are too low for ideal ballooning instabilities to be the cause of the increased transport to the surrounding vacuum. However it is thought that resistive ballooning modes, which can occur at lower pressures, do have some connection with Type-III events. Evidence for this comes from the high level of magnetic activity which occurs just before each burst. These resistive ballooning modes are though to operate in conjunction with peeling mode instabilities. Peeling modes are the result of the destabilising edge current present in the H-mode pedestal.

Type-III ELMs are heavily affected by heating power, with their frequency decreasing as heating power increases. At a sufficiently high heating power Type-III events are suppressed altogether.
Figure 1.4: Diagram showing the progression of an ELM cycle. The plasma reaches the peeling-ballooning stability limit, causing an ELM crash. This reduces both the edge pressure gradient and the edge current density. The plasma then enters the recovery phase, where edge gradients again build up, until the next ELM is triggered.

It is thought that this relates to a critical edge temperature. However, the suppression of Type-III ELMs usually leads to the evolution of Type-I ELMs.

1.5 Summary and Thesis Outline

In this report we will introduce a theory of ELMs which builds on the Taylor relaxation approach proposed by Gimblett et al [31], by incorporating the bootstrap current to the initial state. The
first aim is to find the effect a localized current density peak, generated by the bootstrap mechanism, has on the size of the area disrupted by ELMs. Computer modeling will be used to relate ELM width to the size and shape of the bootstrap. Furthermore small ELM width approximations will be used to establish a mathematical basis for the ELM behavior found in the computer models. The second aim is to use this model to explain aspects of Resonant Magnetic Perturbation (RMP) field based ELM mitigation results, and potentially predict future results.

This thesis will thus be broken down into the following sections; Chapter 2 will cover the background theory concerning plasma equilibrium, plasma stability, and introduce Taylor Relaxation. Chapter 3 will present the Taylor relaxation model of ELMs and cover some of the results of the model. Chapter 4 will examine how the inclusion of the bootstrap current alters this model of ELMs.

Chapter 5 then shifts to the topic of ELM mitigation and examines how the Taylor relaxation model of ELMs offers possible insight into experimental ELM mitigation results. Finally a summary of the thesis is given in Chapter 6, along with possible future work.
Chapter 2

Background Theory

In this chapter an outline of the background physics required for this thesis will be given. First a brief overview of transport theory will be given with a view to introducing the bootstrap current, which will play an important role in the E.L.M. theory described in this thesis. Then ideal MHD will be introduced and the ideal MHD equations stated, though a full derivation will not be given. Once this basis has been established the problem of finding the equilibrium state of a magnetically defined plasma will be introduced and the stability of such an equilibrium examined using stability theory. The principles of stability theory will then be applied to ELMs to describe the most common underlying instabilities associated with ELM disruptions. Finally the process of Taylor relaxation will be introduced.

2.1 Tokamak Geometry

The Tokamak is a toroidal containment system which is axi-symmetric in the toroidal direction (the long way round the torus) [51]. For convenience the Tokamak is usually described using the cylindrical coordinate system shown in Figures 2.1 and 2.2. The radial distance from the center of the middle of the torus is called the major radius and is labeled $R$. The angular distance around the center point is called the toroidal angle and labeled $\phi$. The vertical distance from the mid-plane of the torus is labeled $Z$. It can also be useful to split coordinates into toroidal $\phi$ and poloidal $\theta$ directions In this case $z$ and $R$ are translated to $\theta$ and $r$. The distance from the centre of the torus to the mid point of the plasma is labeled $R_0$, and the distance from the mid point of the plasma to its edge is labeled $a$.

In addition it is possible to simplify the physics of a Tokamak by using the large-aspect-ratio approximation. This is applicable when the ratio $\varepsilon$ of the minor radius $r$ and the major radius $R_0$ (the inverse of the aspect ratio) is small so that
Figure 2.1: Diagram showing a poloidal cross section of basic Tokamak geometry

Figure 2.2: Diagram showing basic Tokamak geometry from above
\[ \varepsilon = \frac{r}{R_0} \ll 1, \]  

(2.1.1)

In this ordering we can define several other useful quantities which describe aspects of Tokamak physics. The first of these quantities is the safety factor \( q(r) \). The safety factor is important when assessing the stability of a toroidal plasma as it is a measure of the plasma’s susceptibility to current-driven instabilities. These will be introduced in Section 2.5. In a toroidal system with a magnetic field that has both \( B_\theta \) and \( B_\phi \) components the resulting magnetic field lines will be helical, curving round around the flux surfaces. The field lines return to their starting poloidal positions after a certain change in toroidal angle \( \phi \) (\( \Delta \phi \)). The safety factor is defined so that

\[ q(r) = \frac{\Delta \phi}{2\pi}, \]  

(2.1.2)

For example a field line that returns to its initial position after 4 circuits of the torus (\( \Delta \phi = 8\pi \)) would have a \( q \) value of 4. It can be shown that the \( q \) value is distinct for each magnetic surface.

In the large-aspect-ratio ordering the safety factor can be approximated to

\[ q(r) \approx \frac{rB_\phi}{R_0B_\theta} \approx \varepsilon \frac{B_\phi}{B_\theta}, \]  

(2.1.3)

The safety factor is usually \( \approx 1 \) in most Tokamak experiments, so it can be inferred from Equation 2.1.3 that \( \frac{B_\phi}{B_\theta} \approx \varepsilon \), i.e. the poloidal magnetic field \( B_\theta \) is small in comparison with the toroidal field \( B_\phi \).

### 2.2 Particle Dynamics and Transport in a Plasma

Transport theory is the study of the movement of particles and energy within a plasma, which necessitates examining the plasma at a very small scale [27] [51]. The aim of this study is to determine how particles, and the energy associated with them, can be contained by the magnetic fields found in a fusion experiment such as a Tokamak. Failure to achieve a sufficient level of containment leads to energy losses that would render a fusion device uneconomical or unable to reach sufficiently high temperatures for sustained fusion reactions to take place. Losses can occur as a result of particles being scattered out of the plasma by coulomb collision, or through micro instabilities.

There are three regimes associated with transport: classical transport, neoclassical transport, and anomalous transport. Each of these regimes will be discussed, with a focus on neoclassical
transport which gives rise to the bootstrap current. Classical and neoclassical transport are determined by coulomb collisions, while anomalous transport is dependent on micro-instabilities.

However, before discussing particle collisions, it is helpful to introduce two aspects of particle motion in a magnetic field that will be of use later on in this section; magnetic moment and mirror trapping. Figure 2.3 shows how a particle moves in a magnetic field; simultaneously moving forward whilst orbiting the magnetic field line with orbital radius \( r_L \) (the Larmor radius), resulting in helical motion around the field line. It can be shown that in a slowly varying (adiabatic) magnetic field the magnetic moment of a particle, \( \mu \) is constant and given by

\[
\mu = \frac{mv_{\perp}^2}{2B},
\]

(2.2.1)

where \( m \) is the particle mass, \( v_{\perp} \) is the component of velocity perpendicular to the magnetic field and \( B \) is the magnetic field strength. Taking the square of the particle’s velocity results in

\[
v^2 = v_{\parallel}^2 + v_{\perp}^2,
\]

(2.2.2)

where \( v_{\parallel} \) is the velocity parallel to the magnetic field. When 2.2.2 is combined with 2.2.1 the following is obtained

\[
v^2 = v_{\parallel}^2 + \frac{2\mu B}{m},
\]

(2.2.3)
This implies that in the case when the magnetic field becomes large, so that \( \frac{2\mu B}{m} > v^2 \), the particle reverses direction parallel to the magnetic field so that \( v^2_\parallel \) does not become negative. Therefore, at the point where \( v^2_\parallel = 0 \) the particle is reflected. If a particle is between two strong magnetic fields it is possible for the particle to be continually reflected by these fields in a process called mirror trapping. The condition for trapping is obtained by examining the magnetic moment of a particle at the point of lowest field strength \( B_{\text{min}} \) so that

\[
\mu = \frac{mv^2_{\perp \text{min}}}{2B_{\text{min}}},
\]

(2.2.5)

where \( v^2_{\perp \text{min}} \) is the perpendicular velocity at the magnetic field minimum. As \( \mu \) is invariant Equation 2.2.4 can be evaluated at the point of reflection, \( v^2_\parallel = 0 \), so that

\[
v^2 - \frac{v^2_{\perp \text{min}}B_{\max}}{B_{\text{min}}} = 0,
\]

(2.2.6)

\[
\Rightarrow
\]

\[
\frac{v^2}{v^2_{\perp \text{min}}} = \frac{B_{\max}}{B_{\text{min}}},
\]

(2.2.7)

where \( B_{\max} \) is the largest magnetic field strength that the particle will experience. This is the condition for particle trapping in a varying magnetic field.

### 2.2.1 Classical Transport

The most basic form of transport is that which arises as a result of coulomb collisions between particles, resulting in particles being knocked out of magnetic confinement. The calculation of transport due to collisions in a linear system is called classical transport. As a plasma consists of many charged particles all interacting, modeling even classical transport is a non-trivial process, and is largely beyond the scope of this thesis. However, it will be useful to list a few of the general results of classical transport theory that will be referred to during this thesis.

- Transport of particles via collisions only occurs when unlike particles collide (for example electron-ion collisions). Conservation of momentum results in no change in position of
particles during like particle interactions (for example electron-electron collisions). However, collisions between like particles can still transfer energy meaning they play a key role in the transfer of energy between particles (thermal diffusion).

- Interactions between charged particles are only considered significant if the velocity of the incident particle is changed by more than 90°. Collisions between unlike particles that meet this criterion result in changes of particle position of order $r_L$.

- The rate of particle diffusion for electrons, $D_e$, is the same as that for ions $D_i$. However, the rate of thermal transport, $K_{e,i}$ for electrons and ions respectively, is not equal and is usually dominated by ions.

- Unfortunately transport calculations show that mirror trapping alone is not viable as a confinement method; collisions quickly scatter trapped particles into freely moving particles, which then escape confinement.

### 2.2.2 Neoclassical Transport

While a simple linear device would be attractive from the point of view of equilibrium and stability calculations, such a device would however be open-ended, allowing the magnetic field lines to exit the ends of the device. It can be shown from transport calculations that losses along magnetic field lines are several orders of magnitude greater than losses perpendicular to the magnetic field lines. Due to the unsustainable reduction in pressure and energy that these end losses would entail open ended linear systems are not feasible for a realistic fusion device. Almost all approaches to fusion favour a closed, toroidal system eliminating end losses. Neoclassical transport is a correction to classical transport theory that arises from the toroidal nature of a device. These effects are far more significant than would be expected from a simple correction to a cylindrical system. In a Tokamak neoclassical effects dwarf losses from classical transport, with particle transport being $\approx 30$ times greater in the neoclassical case, even for passing particles. The reason for this dramatic increase in transport is due to a combination of particle drifts and a small fraction of trapped particles, both of which are a product of the magnetic field distribution in toroidal systems. In a Tokamak the externally applied toroidal field is generated by a series of electro-magnets which encircle the plasma containment vessel. The current flowing in these coils generates a toroidal magnetic field given by

$$B_\phi = B_0 \frac{R_0}{R},$$  \hspace{1cm} (2.2.8)

where $B_\phi$ is the toroidal component of the magnetic field, and $B_0$ is the magnetic field strength at the center of the plasma. It has already been shown that in a varying magnetic field particles with
sufficiently low velocity parallel to the magnetic field \((v_\parallel)\) become trapped, being repeatedly reflected between regions of strong magnetic field strength. As the magnetic field in a torus varies according to \(1/R\), a particle that starts at the outer edge of a toroidal device (large \(R\) and small magnetic field) and follows the helical magnetic field lines, this particle will eventually move to a region of higher magnetic field strength in the inner region of the torus (low \(R\)). Particles of a sufficiently low value of \(v_\parallel\) will become mirror trapped on the outboard side of the torus. These trapped particles no longer complete full orbits but are instead reflected back and forth in what are called ‘bounce’ orbits. The size of these orbits is governed by the \(v_\parallel\) speed of the particles; those with low \(v_\parallel\) are confined to small areas on the outside of the torus and are called deeply trapped, whereas those with \(v_\parallel\) near the trapping limit come close to completing full orbits, and are referred to as weakly trapped. Figure 2.4 shows the bounce orbits of two trapped particles in a toroidal device (one weakly trapped and the other deeply trapped), along with a graph of toroidal magnetic field strength \(B_\phi\) against major radius \(R\).

![Diagram showing the bounce orbits of two trapped particles projected onto the poloidal plane. The red path represents the orbit of a weakly trapped particle and the blue path represents the orbit of a strongly trapped particle. A graph showing the dependance of \(B_\phi\) on radial distance \(R\) is shown on the same X-axis.](image)

The criterion for mirror trapping in a Tokamak can be derived as followed. Using the large aspect ratio approximation \(1/\varepsilon\) introduced before and referring to Figures 2.1, 2.2, and 2.4, the
magnetic field can be written

\[ B = \frac{B_0}{1 + \left( \frac{r}{R_0} \cos \theta \right)} \]  

(2.2.9)

Therefore,

\[ B \simeq B_0(1 - \varepsilon \cos \theta), \]  

(2.2.10)

making use of small value approximations. Using conservation of \( \mu \) and kinetic energy, and using the subscript \( \text{min} \) to denote values at the outboard side (\( \theta = 0 \)), we can write

\[ v^2 = v^2_\parallel + v^2_\perp = v^2_{\parallel\text{min}} + v^2_{\perp\text{min}}, \]  

(2.2.11)

and

\[ \mu = \frac{m v^2_\perp}{2B_0(1 - \varepsilon \cos \theta)} = \frac{m v^2_{\perp\text{min}}}{2B_0(1 - \varepsilon)}. \]  

(2.2.12)

Combining Equations 2.2.11 and 2.2.12 gives

\[ v^2_\parallel = v^2 \left( 1 - \frac{v^2_{\perp\text{min}}}{v^2} \left[ 1 + 2\varepsilon \sin^2 \left( \frac{\theta}{2} \right) \right] \right), \]  

(2.2.13)

\[ \implies \]  

\[ \frac{v^2_\parallel}{v^2_{\perp\text{min}}} \leq \sqrt{2\varepsilon}, \]  

(2.2.14)

is the condition for trapping in a toroidal system. These trapped particles perform 'banana' orbits (see Figure 2.5), bouncing between the inboard and outboard side of the plasma. The poloidal projection of this movement resembles a banana shape, the width of which is labeled \( \delta_{rb(e,i)} \).

It can be shown that this fraction of trapped particles in a Tokamak enhance transport significantly, even compared to drift-affected passing (untrapped) particles. Part of this increase in transport is due to the ease with which trapped particles are scattered into becoming passing particles. This increase in transport is due to the radial dependency of \( B_\phi \), which generates a vertical particle drift, with electrons and ions drifting in opposite directions. This drift (combined with a similar drift due to curvature) means that when untrapped particles collide with
one another, the resulting particle displacement (for shifts in velocity greater than 90°) is of the same order as this drift distance. This displacement is much larger than $r_L$, leading to increases of of $\approx 30$ and $\approx 10$ compared to classical transport for $D_{e,i}$ and $K_{e,i}$ respectively.

A measure of this is the collisionality of a plasma, given by

$$\nu^* = \frac{t_b}{\tau},$$

(2.2.15)

where $t_b$ is the time taken to complete one banana orbit and $\tau$ is the average time taken for a trapped particle to scatter and become a passing particle.

In addition to the increase in transport neoclassical effects also generate several currents in a Tokamak, currents which arise from the interaction between trapped and freely moving particles. The largest of these currents is called the bootstrap current and can form a significant percentage of the main toroidal current. The bootstrap current is important both in that a large bootstrap current will make a fusion reactor more economically viable and also that such a large contribution to the toroidal current will significantly affect the stability of the plasma, as will be examined later in this thesis. There now follows a brief overview of how the bootstrap current is generated and a general outline of its main properties.

### 2.2.3 The Bootstrap Current

The bootstrap current is a current that arises spontaneously from the variations in density and magnetic field strength found in a Tokamak [41]. The current flows parallel to the induced plasma current inside the torus, therefore contributing to the total toroidal plasma current. In a Tokamak the plasma is confined in part by poloidal magnetic fields generated by an induced toroidal current. Therefore as the bootstrap adds to this induced current it helps generate the poloidal field. As the bootstrap current is a natural consequence of the confinement of the plasma a further increase in confinement is seemingly gained for nothing, hence the term bootstrap. The bootstrap current is a purely neoclassical phenomenon as it is generated by the interaction between trapped and passing particles.

It has been noted previously that there is radially decreasing magnetic field in a Tokamak (as show in figure 2.4) There is also a corresponding density gradient, the plasma density decreasing with $R$ on the outboard side of the Tokamak. It can be shown that in the presence of this density gradient both trapped and freely moving particles generate magnetization currents. These currents can be visualised by considering a flux surface located on the density gradient. This flux surface is tangential to two particle orbits, one with $v_\parallel < 0$ at the flux surface and one with
\( v_{\parallel} > 0 \), as shown in Figure 2.5 for two trapped particles. These particles therefore orbit in opposite directions. Due to the density gradient, the \( v_{\parallel} < 0 \) orbit will be more densely populated, leading to a net flow of electrons in one direction at the flux surface, thus generating a current, called a magnetisation current (as the trapped particles’ bounce orbits are banana shaped, the trapped particle magnetisation current is often referred to as the banana current). It can be shown [27], [51] that such currents are generated for both trapped and passing particles with magnitudes given by

\[
J_{\text{Trapped}} \approx -q \left( \frac{r}{R_0} \right)^{1/2} \frac{T}{B_0} \frac{\partial n}{\partial r},
\]

and

\[
J_{\text{Passing}} \approx -q \frac{T}{B_0} \frac{\partial n}{\partial r},
\]

These currents are both in the same direction as the main toroidal current in a Tokamak, and form part of the bootstrap current. However, the greater part of the bootstrap current is generated as a result of collisions between trapped and passing particles. As momentum is conserved in collisions between electrons, any net momentum change arising from collisions for either the trapped or passing particles must be balanced by an opposite change in momentum for the other type of particle.

It can be shown that the change in parallel momentum due to collisions for the passing particles is

\[
\Delta p_{\parallel(\text{passing})} \approx qT \frac{\partial n}{\partial r} \frac{\nu_{ee}}{\omega_{ce}},
\]

where \( \nu_{ee} \) is the electron-electron collision frequency and \( \omega_{ce} \) is the electron cyclotron frequency.

While there are fewer trapped particles \( (n_{\text{trapped}} \approx \varepsilon \quad n_{\text{passing}}) \), they occupy a small region of velocity space, and so it is easy for collisions to scatter them from this velocity space and convert them to passing particles. Thus the collision frequency for trapped particles is higher for trapped particles \( (\nu_{\text{trapped}} \approx (1/\varepsilon) \nu_{ee}) \) and the total change in momentum for trapped particles due to collisions is

\[
\Delta p_{\parallel(\text{trapped})} \approx qT \left( \frac{1}{\varepsilon} \right)^{1/2} \frac{\partial n}{\partial r} \frac{\nu_{ee}}{\omega_{ce}},
\]
By comparing the change in momentum for both the trapped and passing particles it can easily be seen that

\[ \Delta p_{\parallel(\text{passing})} \approx (\varepsilon)^{\frac{1}{2}} \Delta p_{\parallel(\text{trapped})}. \]  

(2.2.20)

This implies that there is a net transfer of momentum from the trapped particles to the passing particles. The result of this transfer of momentum is that the passing particles have an additional velocity parallel to the magnetic field \( u_b \). By modeling the passing particles as now having a Maxwellian distribution shifted by \( u_b \) it is possible to reformulate the parallel momentum change for passing particles as

\[ \Delta p_{\parallel(\text{passing})} \approx m_e \left( -\frac{J_{\text{Passing}}}{e} + n_{\text{passing}} u_b \right) v_{ee} \]  

(2.2.21)

As the momentum change for passing particles is now equal to that of trapped particles, Equations 2.2.19 and 2.2.18 can now be equated to give an expression for the bootstrap current \( J_B \) (where \( J_B = -en_{\text{passing}}u_{\text{bootstrap}} \)), so that

\[ J_B \approx q \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2}} \frac{T}{B_0} \frac{\partial n}{\partial r} \]  

(2.2.22)

By comparing 2.2.22 with \( J_{\text{passing}} \) and \( J_{\text{trapped}} \), it is possible to see that the bootstrap current is substantially larger than both of these currents (by factors of \( \frac{1}{\varepsilon} \) and \( (\frac{1}{\varepsilon})^{\frac{1}{2}} \) respectively).

### 2.2.4 Anomalous Transport and H-Mode

In addition to normal transport and neoclassical effects there is another mode of transport that accounts for the majority of losses in a Tokamak. As this form of transport cannot be explained by classical coulomb collision models it is referred to as anomalous transport. Anomalous transport is not fully understood but it is thought that micro-instabilities (instabilities that occur on the length scale of a Debye length) create turbulence that boosts transport. While an in-depth study of anomalous transport is outside of the scope of this thesis it is thought that suppression of this mode of transport might be linked to the transition to H-mode of operation in a Tokamak. The transition to H-mode occurs when the plasma reaches a critical temperature, at which point a dramatic drop in transport from the edge region develops leading to a roughly 2-fold improvement in confinement. It is thought that an increase in shear flow velocity caused by the higher temperatures stabilises the micro instabilities that drive anomalous transport. The drop in transport and thus improvement in confinement at the edge is referred to as a transport barrier,
which leads to a build up of edge pressure, and due to the large pressure discontinuity between the edge region and the surrounding vacuum, large pressure gradients. As has been seen above large pressure gradients generate a correspondingly large bootstrap current, and this is true of the edge region in H-mode. Thus in the edge region there are large toroidal currents and large pressure gradients both of which are of vital importance when considering the stability of the edge region and consequently when considering ELMs.

### 2.3 Brief Introduction to Ideal MHD

The field of Magnetohydrodynamics (MHD) [29] [32] is the theoretical framework that predicts the bulk properties of a plasma on the length scales associated with a nuclear fusion experiment (for example a Tokamak). The centerpiece of MHD is the set of MHD equations. These equations are the basis for establishing the equilibrium state of a magnetically contained plasma and also for determining the stability of a given equilibrium. While a complete derivation of the MHD equations is not relevant here (detailed derivations of the MHD equations can be found in [29]), there follows a general outline of the process which concludes with the complete set of ideal MHD equations. Ideal MHD is based on the assumption that the plasma is perfectly...
conducting; the inclusion of resistivity in the plasma requires resistive MHD. For this thesis the plasma will be considered to be a perfect conductor and thus described by the ideal MHD equations. While an experimental plasma will of course not be an ideal conductor, the use of the ideal MHD equations is still acceptable as any plasma equilibrium that is unstable to ideal MHD instabilities will also be resistively unstable. Also ideal MHD instabilities are usually much more disruptive to the plasma than resistive instabilities. For these reasons it is almost always more important to ensure ideal stability than resistive stability.

It is possible to derive the MHD equations from the full set of Boltzmann kinetic distribution functions for both the electrons and ions contained in the plasma. These equations combined with Maxwell’s equations completely describe the behavior of the plasma but due to their complexity cannot be easily solved, if at all, and contain too much information to be useful. A good deal of simplification is required to move from these distributions to the MHD equations. A simpler and more intuitive method of deriving the MHD equations is to consider equations describing conservation of particle number, momentum, and energy for a fluid made up of electrons and ions, then combining these relations with Maxwell’s equations. The resulting set of equations is called the two-fluid model and can be extrapolated into the MHD equations by defining the length scales that are to be examined and by making some approximations.

2.3.1 The Two-Fluid Model

2.3.1.1 Particle/Mass Conservation

The first step is to look at mass conservation for a fluid of electrons and ions. As the overall number of particles, \( N \), in a closed system must remain constant then

\[
\frac{dN}{dt} = \frac{dN_i}{dt} = \frac{dN_e}{dt} = 0, \tag{2.3.1}
\]

where the subscripts \( i \) and \( e \) denote ions and electrons respectively (and will do so for all terms in this derivation). By considering the rate of change of particles of a volume element \( \Delta V \) moving at velocity \( \mathbf{u} \) it can be shown that

\[
\frac{dN}{dt} = \Delta V \left( \frac{dn}{dt} + n \nabla \cdot \mathbf{u} \right) = 0, \tag{2.3.2}
\]

where \( n \) is the number density of the plasma. Converting the time derivative to the Eulerian form so that

\[
\frac{dn}{dt} = \left( \frac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n \right), \tag{2.3.3}
\]
and using the vector relation

\[ \nabla \cdot (n\mathbf{u}) = n\mathbf{\nabla} \cdot \mathbf{u} + \mathbf{u} \cdot \nabla n, \quad (2.3.4) \]

Equation 2.3.2 can now be written in the following form for electrons and ions respectively

\[ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (2.3.5) \]

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0. \quad (2.3.6) \]

This is the first set of two fluid equations, describing the conservation of both electrons and ions.

### 2.3.1.2 Momentum Conservation

Using a similar method to that used for particle conservation, it is possible to write the conservation of momentum for a volume element \( \Delta V \) as

\[ \frac{d}{dt}(\text{Momentum}) = \Delta V m_e n_e \left( \frac{d}{dt} + \mathbf{u}_e \cdot \nabla \right) \mathbf{u}_e = F, \quad (2.3.7) \]

where \( F \) is the force acting on the fluid element. By considering the different forces which act on the plasma (the force resulting from the electric field \( \mathbf{E} \), the force resulting from the magnetic field \( \mathbf{B} \), the force due to pressure gradients (\( \nabla p \)), and the drag forces caused by particle collisions) it is possible to show that

\[ m_e n_e \left( \frac{d}{dt} + \mathbf{u}_e \cdot \nabla \right) \mathbf{u}_e = -e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla p_e - m_e n_e \nu_{ei}(\mathbf{u}_e - \mathbf{u}_i), \quad (2.3.8) \]

\[ m_i n_i \left( \frac{d}{dt} + \mathbf{u}_i \cdot \nabla \right) \mathbf{u}_i = -e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i - m_i n_i \nu_{ei}(\mathbf{u}_i - \mathbf{u}_e), \quad (2.3.9) \]

where \( \nu_{ei} \) is the electron-ion collision frequency. These are the conservation of momentum equations for electrons and ions.
2.3.1.3 Energy Conservation

In a similar manner to the previous sections the conservation of energy for a fluid element will now be examined. However due to the many different sources of energy gain and energy loss for a plasma fluid element, an abbreviated form of the energy conservation equation will be given, with the explanation that many of the terms involved will be shown to be negligible on the timescales examined by MHD. Therefore it can be shown that the energy conservation equations are given by

\[
\frac{3}{2} n_e \left( \frac{\partial}{\partial t} + u_e \cdot \nabla \right) T_e + p_e \nabla \cdot u_e + \nabla \cdot q_e = S_e, \tag{2.3.10}
\]

\[
\frac{3}{2} n_i \left( \frac{\partial}{\partial t} + u_i \cdot \nabla \right) T_i + p_i \nabla \cdot u_i + \nabla \cdot q_i = S_i, \tag{2.3.11}
\]

where \( q \) is the heat flux vector, \( S_e \) is a composite term that includes energy loss/gain terms for the electrons (and will disappear in the MHD regime), and \( S_i \) is the composite energy loss/gain term for the ions, and which will similarly become negligible in the MHD regime.

2.3.1.4 Maxwell’s Equation

The two fluid model is completed by the addition of Maxwell’s equations, given by

\[
\nabla \times E = - \frac{\partial B}{\partial t}, \tag{2.3.12}
\]

\[
\nabla \times B = \mu_0 e (n_i u_i - n_e u_e) + \frac{1}{c^2} \frac{\partial E}{\partial t}, \tag{2.3.13}
\]

\[
\nabla \cdot E = \frac{e}{\varepsilon_0} (n_i - n_e), \tag{2.3.14}
\]

\[
\nabla \cdot B = 0, \tag{2.3.15}
\]

2.3.2 Simplifications to the Two-Fluid Model

There are several approximations that greatly simplify the two-fluid model. These simplifications can be justified by examining the length, time, and velocity scales of interest when seeking
to describe plasma behaviour in a fusion device. For example if an approximation is valid when
the length scales being described are greater than the electron Larmor radius, then that approx-
imation is easily met when looking at a Tokamak, whose length scales are on order $a$. By doing
this for the two fluid equations it can be shown that the following approximations are valid for
fusion plasma’s of interest.

- The momentum of the electrons in the fluid is approximated to 0 and the electron mass is
  considered very much smaller than the mass of the ions ($m_i \gg m_e$). The plasma density $\rho$
is then determined solely by the ions so that

$$\rho = m_i n_i. \quad (2.3.16)$$

From Equation 2.3.16 the momentum of a plasma element can now be written as

$$\Delta V \rho \mathbf{v} = \Delta V (m_i n_i \mathbf{u}_i). \quad (2.3.17)$$

where $\mathbf{v}$ is the plasma velocity, implying that

$$\mathbf{v} = \mathbf{u}_i. \quad (2.3.18)$$

- The velocities of interest in an MHD plasma are very much smaller than the speed of light.
  This in turn implies that the displacement current term in Equation 2.3.13 ($\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$) can be
  ignored.

- The previous simplification can be used to show that the plasma is quasi-neutral. This is
done by taking the divergence of the new form of Equation 2.3.13 to give

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mu_0 e (n_i \mathbf{u}_i - n_e \mathbf{u}_e)) = \mu_0 \nabla \cdot \mathbf{J} = 0, \quad (2.3.19)$$

where $\mathbf{J}$ is the current density. Then, by multiplying the two particle number conservation
equations 2.3.8 and 2.3.9 by their appropriate electric charges and adding them together,
the following expression for charge conservation is obtained

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (e (n_i \mathbf{u}_i - n_e \mathbf{u}_e)) = \frac{\partial \sigma}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (2.3.20)$$

where $\sigma$ is the net charge of the plasma. Equations 2.3.19 and 2.3.20 together imply that
the rate of change of net charge $\frac{\partial \sigma}{\partial t}$ is zero, therefore the plasma is quasi-neutral, and
$n_e = n_i = n$. 


• The Plasma gains and loses heat on time scales that are very long compared to those in MHD. It can be shown that under these conditions the $S_e$ and $S_i$ terms in Equations 2.3.10 and 2.3.11 can be ignored, along with the $\nabla \cdot \mathbf{q}$ terms in both equations. It can be shown that with these simplifications Equations 2.3.10 and 2.3.11 can be rewritten as

$$\frac{d}{dt} \left( \frac{p_e}{\rho^\gamma} \right) = 0, \quad (2.3.21)$$

and

$$\frac{d}{dt} \left( \frac{p_i}{\rho^\gamma} \right) = 0, \quad (2.3.22)$$

where $\gamma$ is the adiabatic index.

Once these approximations have been integrated into the two-fluid equations, the equations for electrons and ions can then be combined to give the single-fluid equations. The single-fluid equations together with the low frequency forms of Maxwell’s equations, are the basis of the MHD equations.

2.3.3 The MHD equations

2.3.3.1 The Mass Equation

This equation is obtained by multiplying Equation 2.3.6 by $m_i$, and substituting in Equations 2.3.16 and 2.3.18 to give the MHD mass equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0. \quad (2.3.23)$$

2.3.3.2 The Momentum (Force) Equation

Setting the electron momentum to 0 in Equation 2.3.8, then substituting Equations 2.3.16 and 2.3.18 into the two-fluid momentum equations, gives

$$0 = -e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla p_e - m_e n_e v_{ei} (\mathbf{u}_e - \mathbf{u}_i), \quad (2.3.24)$$

for electrons and

$$\rho \frac{d\mathbf{v}}{dt} = e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i - m_i n_e v_{ei} (\mathbf{u}_i - \mathbf{u}_e), \quad (2.3.25)$$
for ions. Adding Equations 2.3.24 and 2.3.25 together, using charge neutrality \( n_e = n_i \), and the fact that the electron and ion pressures simply add to form the total pressure \( p \), gives the MHD momentum Equation

\[
\rho \frac{dv}{dt} = J \times B - \nabla p. \tag{2.3.26}
\]

### 2.3.3.3 The MHD Energy Equation

The MHD energy equation is obtained by adding Equations 2.3.21 and 2.3.22 together to give

\[
\frac{d}{dt} \left( \frac{p \rho}{\gamma} \right) = 0, \tag{2.3.27}
\]

### 2.3.3.4 Ohm’s Law

It can be shown that the two-fluid electron momentum Equation 2.3.24 can be rewritten in the form

\[
E + v \times B = \frac{1}{en} (J \times B - \nabla p_e) + \eta J, \tag{2.3.28}
\]

where \( \eta \) is the plasma’s resistivity. For an ideal plasma \( \eta = 0 \). It can also be shown that the term \( \frac{1}{en} (J \times B - \nabla p_e) \) is small in the MHD regime, thus 2.3.28 becomes the ideal Ohm’s law for MHD

\[
E + v \times B = 0. \tag{2.3.29}
\]

### 2.3.3.5 Ampere’s Law

As described above, MHD use the low frequency version of Ampere’s Law with no displacement current

\[
\nabla \times B = \mu_0 J. \tag{2.3.30}
\]

### 2.3.3.6 Faraday’s Law

Faraday’s Law is unchanged so that
\[ \nabla \times E = -\frac{\partial B}{\partial t}. \] (2.3.31)

### 2.3.3.7 No Magnetic Monopoles

\[ \nabla \cdot B = 0. \] (2.3.32)

This completes the full set of MHD equations. However it should be remembered that the MHD equations are only valid as long as the approximations made in Section 2.3.2 are valid, so that in a plasma where the length scales are much greater or much shorter than the minor radius $a$, or when studying a plasma on very short or long time scales, the MHD equations are not appropriate and must be modified.

### 2.4 Equilibrium

Due to the very high temperature of the plasma in a fusion experiment keeping the plasma away from the containing structure of the experiment is of great importance. Therefore maintaining the plasma in a static equilibrium position away from the surrounding wall is one of the first steps in a fusion device. Determining the equilibrium state of a plasma is equivalent to calculating the conditions required for the plasma to be in force balance.

The first step in calculating the equilibrium state is to use the time-independent MHD equations, the MHD equations when the plasma is at rest and in force balance. These equations are given below.

\[ \nabla \cdot B = 0, \] (2.4.1)

\[ J \times B = \nabla p, \] (2.4.2)

\[ \nabla \times B = \mu_0 J, \] (2.4.3)

An immediate result can be derived from equation 2.4.2 by taking the scalar product of 2.4.2 with either $B$ or $J$, as shown below

\[ B \cdot \nabla p = 0, \] (2.4.4)
\[ \mathbf{J} \cdot \nabla p = 0, \quad (2.4.5) \]

This shows that the magnetic field lines and the plasma current lie on surfaces of constant pressure, in other words there is no change in pressure in the direction of either the magnetic field or the plasma current. In cylindrical geometry, for example a long cylinder or a 'straightened' torus, these surfaces of constant pressure are an infinite set of inwardly concentric cylinders, often referred to as 'nested' surfaces. Figure 2.6 shows an example of this below.

![Diagram showing nested flux surfaces. Also seen are example magnetic field (B) and current lines (J)](image)

As the magnetic flux is defined by the magnetic field contained within any surface it is common to label these surfaces of constant pressure flux surfaces. Magnetic flux will be discussed more when plasma geometries of two dimensions are introduced later.

### 2.4.1 Equilibrium in 2-D systems and The Grad Shafranov Equation

In 1-dimensional cylindrical systems the only force that needs to be balanced to maintain equilibrium is the radial outward plasma pressure. However, as it has been shown that 1-D systems are unfeasible due to end losses, it is required that the long straight cylinder be bent round into a toroidal shape. There are unfortunately several new forces which arise due to the transition to a 2-D system and which must be balanced in order to maintain equilibrium. Figures 2.7 to 2.8 below show the 2 main forces that occur in 2-D geometry; the tyre-tube force and the hoop force (based on diagrams from [27] and [43], respectively).

Together these produce an outwards (i.e. away from the centre of the torus in the direction of the major radius R) net force on the plasma, which left unchecked will push the plasma
Figure 2.7: Diagram showing the tyre tube force. The outer plasma surface at $R = R_2$ (Surface $S_2$, marked by green dashes) is larger than that of the inboard side at $R = R_1$ (Surface $S_1$, marked by red dashes). As both surfaces are on a surface of constant pressure ($p$), there is a net outwards force. Based on diagram from [27]

into the containing wall. This force must be balanced either by surrounding the plasma with a highly conductive wall or by applying an external vertical magnetic field. The result of either of these options is to reinforce the magnetic field on the outboard side which produces a restoring, inwards force.

The conditions for equilibrium in 2-D can be written in terms of the poloidal flux function $\psi$. This expression is called the Grad-Shafranov equation [33][44] and contains two other surface functions, the pressure $p$ and the total poloidal current $f$. It can be shown that both of these terms are functions of $\psi$ so that $p = p(\psi)$ and $f = f(\psi)$. However the forms of both $p(\psi)$ and $f(\psi)$
Figure 2.8: Diagram showing the 'hoop' force. Two plasma elements, with currents $J_1$ and $J_2$ respectively, generate magnetic fields ($B_1$ and $B_2$ respectively) which act to repel each other when at an angle of $\pi$. This produces an outwards radial force analogous to a current carrying circle of wire. Based on diagram from [43]

are arbitrary and suitable expressions for them are usually inferred from transport calculations or experimental results.

The initial step of deriving the Grad Shafranov equation is to split equations 2.4.1 to 2.4.3 into poloidal components (labeled with the subscript $P$) and toroidal components (subscript $\phi$) [43]. The poloidal component is comprised of the combined $Z$ and $R$ components in Tokamak geometry introduced earlier (see Figure 2.1). Equation 2.4.2 thus becomes

$$JJJ_p \times i_\phi B_\phi + J_\phi i_\phi \times B_P = \nabla_P(\psi),$$  \tag{2.4.6}$$

It can be shown that the poloidal field and current can be written in terms of $\psi$ so that

$$B_p = \frac{1}{R}(\nabla \psi \times i_\phi),$$  \tag{2.4.7}$$

and

$$J_p = \frac{1}{R}(\nabla f(\psi) \times i_\phi).$$  \tag{2.4.8}$$
Where $i_\phi$ is the unit vector in the toroidal direction. Substituting 2.4.7 and 2.4.8 into 2.4.6 leads to

\[
\frac{B_\phi}{R} \nabla f + \frac{i_\phi}{R} \nabla \psi = \nabla p, \quad (2.4.9)
\]

After some algebraic work 2.4.9 can be simplified to

\[
j_\phi = R \frac{dp}{d\psi} + B_\phi \frac{df}{d\psi}, \quad (2.4.10)
\]

It can also be shown that the current function $f$ can be written

\[
f = \frac{RB_\phi}{\mu_0}, \quad (2.4.11)
\]

which when substituted into 2.4.10 gives

\[
j_\phi = R \frac{dp}{d\psi} + \frac{\mu_0}{R} f \frac{df}{d\psi}, \quad (2.4.12)
\]

which together with Ampere’s law (equation 2.4.3) gives

\[
R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 R^2 p'(\psi) - \mu_0^2 f(\psi)f'(\psi), \quad (2.4.13)
\]

This is the Grad-Shafranov Equation.

### 2.5 Stability Theory

Once an equilibrium state that effectively separates the plasma from the Tokamak containment vessel has been calculated, the stability of that equilibrium must be found. The stability of a system is determined by how it reacts to any deviation from its equilibrium position. A system that returns to its equilibrium state after a perturbation is defined as stable, whereas a system in which the perturbation is reinforced and grows is unstable. There are many different types of instability; which, if any, of these instabilities affect a plasma depends on the plasma configuration and the regime in which it operates. There are two major underlying causes for plasma instability; currents parallel to the magnetic field, and pressure gradients in areas of unfavourable magnetic curvature. Once the process of determining stability has been outlined below these root causes of instability will be explored further.
The first step in determining stability mathematically is to rewrite the MHD equations in terms of an equilibrium component and a small perturbed quantity. For example the magnetic field in the presence of a small perturbation is given by [27]

\[ B = B_0 + B_1, \]  

(2.5.1)

where \( B_0 \) is the equilibrium magnetic field and \( B_1 \) is the perturbation field. This perturbation quantity is taken to be small compared with the equilibrium value. This means that any term in the MHD equations which is nonlinear in any combination of perturbed quantities is considered negligible. This process of removing nonlinear terms is called linearisation. In addition any terms in the MHD equations that are composed of purely equilibrium quantities are removed, as they are already exact solutions of the MHD equations. An example of both of these processes is shown below where

\[ B \cdot J = (B_0 + B_1) \cdot (J_0 + J_1) = B_0 \cdot J_0 + B_1 \cdot J_1 + B_0 \cdot J_1 + B_1 \cdot J_0, \]

(2.5.2)

becomes

\[ B_0 \cdot J_1 + B_1 \cdot J_0, \]

(2.5.3)

after the removal of non linear perturbed terms and purely equilibrium quantities.

The next step is to introduce the perturbed plasma displacement \( \xi \), defined so that the perturbed velocity of the plasma \( v_1 \) is given by

\[ v_1 = \frac{d\xi}{dt}, \]

(2.5.4)

The plasma equilibrium is considered static so that \( v_0 = 0 \).

As the behavior of the plasma will change with time once it is disturbed from equilibrium, all perturbed quantities are functions of time. As all perturbed terms are linear, this time dependance can be given by the normal mode expansion

\[ Q_1(r,t) = Q_1(r)e^{-i\omega t}, \]

(2.5.5)

where \( Q_1 \) is any perturbed quantity (e.g. \( B_1 \)), and the angular frequency \( \omega \) is introduced.
All the MHD equations except the momentum equation can now be rewritten in terms of the perturbed quantities $Q_1(\text{r})e^{-i\omega t}$ and $\xi e^{-i\omega t}$. After some simplification the following expressions are obtained

\begin{align*}
\rho_1 &= -\nabla \cdot (\rho \xi), \\
P_1 &= -\xi \cdot \nabla P - \gamma P \nabla \cdot \xi, \\
B_1 &= \nabla \times (\xi \times B), \\
J_1 &= \frac{1}{\mu_0} \nabla \times (\nabla \times (\xi \times B)),
\end{align*} \tag{2.5.6-2.5.9}

where the zero subscript on equilibrium terms has been dropped for convenience, and the time component $e^{-i\omega t}$ cancels out in all expressions. Equations 2.5.6-2.5.9 are now substituted into the perturbed version of the momentum equation (Equation 2.3.26) to produce an expression for force balance in the perturbed plasma

\[ \rho \frac{\partial \xi}{\partial t^2} = -\omega^2 \rho \xi = F[\xi], \tag{2.5.10} \]

where $F$ is the force operator given by

\[ F[\xi] = \nabla (\xi \cdot \nabla P + \gamma P \nabla \cdot \xi) + \frac{1}{\mu_0} (\nabla \times B_1) \times B + \frac{1}{\mu_0} (\nabla \times B) \times B_1. \tag{2.5.11} \]

This is an eigenfunction problem with eigenvalues of $\omega^2$. It can be shown that the force operator $F$ has the property of being self-adjoint, which in turn can be used to show that the eigenvalues of $\omega^2$ are purely real. This implies that if $\omega^2 < 0$ then the time progression component $e^{-i\omega t}$ will increase exponentially. Thus any perturbation of the plasma will continue to grow, rendering the system unstable. However if $\omega^2 > 0$ then the time component represents an oscillation about the equilibrium and therefore the system is stable. The point of marginal stability can now be found by calculating $\omega^2 = 0$. 
2.5.1 The Ideal MHD Energy Principle

It is also possible to state the conditions for stability in terms of the energy of the perturbed plasma. If a plasma moves a distance of $\xi$ away from the equilibrium position and in doing so changes its energy by $\delta W$, then it can be shown that $\delta W = 0$ is equivalent to having eigenvalues of $\omega^2 = 0$ in Equation 2.5.10. A plasma is at marginal stability when $\delta W = 0$, with $\delta W < 0$ indicating instability and $\delta W > 0$ indicating stability. This implies that any perturbation of the plasma which leads to a lower energy level than that of the equilibrium, results in the plasma equilibrium being unstable [27][51].

To derive this condition for stability in terms of energy, the change in energy of the system after a displacement is calculated, so that

$$\delta W = -\frac{1}{2} \int F[\xi] \cdot \xi dV. \quad (2.5.12)$$

In situations when the plasma is separated from the containment vessel wall by a vacuum it is possible to calculate the energy contributions of the plasma ($P$), surface ($S$), and vacuum ($V$) sections to $\delta W$, so that

$$\delta W = \delta W_P + \delta W_S + \delta W_V, \quad (2.5.13)$$

Using Equation 2.5.12 to calculate $\delta w$ for each term in Equation 2.5.13 results in

$$\delta W_P = \frac{1}{2} \int_P \left[ \frac{|B_1|^2}{\mu_0} \right] + \gamma p |\nabla \cdot \xi|^2 - \xi \cdot (J \times B_1) + (\xi \perp \cdot \nabla p)(\nabla \cdot \xi \perp) dV, \quad (2.5.14)$$

$$\delta W_S = -\frac{1}{2} \int_S \left( \gamma p \nabla \cdot \xi - \frac{B \cdot B_1}{\mu_0} \right) \xi \perp \cdot dS, \quad (2.5.15)$$

$$\delta W_V = \frac{1}{2} \int_V \frac{|B_1|^2}{\mu_0} dV, \quad (2.5.16)$$
The change in energy of the plasma, Equation 2.5.14, can be rewritten as

\[
\delta W_p = \frac{1}{2} \int_P dV \left[ \frac{|B_{1\perp}|^2}{\mu_0} + \frac{B^2}{\mu_0} \nabla \cdot \xi + 2 \xi_{\perp} \cdot k \right]^2 + \gamma P |\nabla \cdot \xi|^2 - 2 (\xi_{\perp} \cdot \nabla p ) (k \cdot \xi_{\perp}^\bot) - B_1 \cdot (\xi_{\perp} \times b) J_{||},
\]

(2.5.17)

Each line of the integral on the RHS of Equation 2.5.17 represents a distinct contribution to the stability of the plasma. Looking at these contributions and calculating whether the term is stabilising or destabilising gives physical insight into the major causes of instabilities in a plasma.

The first term of the integral in 2.5.17,

\[
\frac{|B_{1\perp}|^2}{\mu_0},
\]

(2.5.18)

represents the energy required to bend magnetic field lines. Any displacement of the plasma that bends magnetic field lines requires energy to do so, therefore this term is always positive and is thus a stabilising effect on the plasma.

The next two terms in 2.5.17 give the energy required to compress magnetic field lines and the plasma itself, respectively. These terms are also positive and therefore help stabilize the plasma.

The next two terms can be either positive or negative and represent the source of the most important instabilities in a plasma. \(-2 (\xi_{\perp} \cdot \nabla p ) (k \cdot \xi_{\perp}^\bot)\) represents the combination of pressure gradients (\(\nabla p\)) in the plasma with the curvature vector \(k\). The plasma can have either good curvature or bad curvature, depending on the direction of \(\nabla p\) in relation to \(k\). When the pressure gradient is in the same direction as the curvature vector \((k \cdot \nabla p > 0)\) there is bad curvature and pressure gradients reinforce any displacement from the equilibrium and are destabilising, whilst when \((k \cdot \nabla p < 0)\) then pressure gradients work to stabilise the plasma. In uniform magnetic fields modes driven by this combination of bad curvature and pressure gradients are called interchange modes and are usually easy to avoid in practice. However a non-uniform magnetic field (such as that found in a Tokamak) can produce a type of pressure driven mode called a ballooning mode. These are one of the major forms of instability in the Tokamak edge region and are a key component of ELM generation.
The final term $-\mathbf{B}_1 \cdot (\mathbf{\xi}_\perp \times \mathbf{b}) J_\parallel$ shows the influence on plasma stability of currents parallel to the magnetic field. Current driven instabilities are the second major form of instability in a plasma. They are usually split into external and internal modes, depending on whether the plasma surface itself is moved (external) or not (internal). An ideal current driven instability is called a peeling mode and will be the focus of this thesis. In Chapter 3 the stability of a Taylor relaxed plasma edge region to peeling modes will be examined in detail.

### 2.6 Taylor Relaxation

The theory of plasma relaxation developed by Taylor [47][48], building on earlier work by Woltjer [52], describes how a turbulent plasma can transition into a particular low energy state, called a relaxed state. This relaxed state is largely independent of the initial conditions of the plasma.

It is observed in many physical systems that, under appropriate conditions, the system returns to a preferred final configuration. This process is called self-organisation. The theory of Taylor relaxation was developed to explain self-organisational behaviour exhibited by the Reverse Pinch Field (RFP) machine. The RFP is a toroidal confinement system where the plasma is contained by a small eternally generated toroidal field and a poloidal field generated by a large induced current. This poloidal field compresses (pinches) the plasma and the initial toroidal field lines. Due to this field line compression, the toroidal magnetic field is weak near to the walls of the containment vessel, and can, under some circumstances, reverse direction. When this field reversal occurs it is observed that the plasma first undergoes a brief period of turbulence, with energy and particles lost to the container walls, before transitioning to a phase with little to no turbulence. This is often referred to as a quiescent phase. It was also observed that the same final quiescent state was reached with differing initial conditions.

It was suggested that the quiescent periods were a consequence of the plasma relaxing to a particular minimum energy state. The plasma has energy $W$ given by the sum of the energy associated with the plasma pressure, and that associated with the magnetic pressure, so that

$$W = \int_\Omega \left( \frac{B^2}{2\mu_0} + \frac{p}{\Gamma - 1} \right) dV,$$  \hspace{1cm} (2.6.1)

where $B$ is the magnitude of the magnetic field, $p$ is the pressure, and $\Gamma$ is the adiabatic invariant. If the relaxed state represents some minimum energy configuration, then minimising Equation 2.6.1 should give a description of that state. In the first step of this process, the plasma pressure was assumed to be small compared to the magnetic energy (i.e. a very low $\beta$ plasma), thus reducing the total energy of Equation 2.6.1 to
\[ W = \int_{V_0} \frac{B^2}{2\mu_0} dV, \quad (2.6.2) \]

If 2.6.2 is minimized without constraint then the result is obviously the trivial state where \( B = 0 \). Therefore some constraints must be placed on the plasma during the minimization process so that the relaxed energy state produced matches the observed plasma behavior. It is known from variational calculus that minimising the integral of an arbitrary function \( F (\int Fdx) \), under the constraint that the integral of another function \( G \) is a constant (\( \int Gdx = C \)), is the same as minimizing the integral

\[ \int F + \lambda G \, dx, \quad (2.6.3) \]

with no constraint, where \( \lambda \) is a Lagrange multiplier. Therefore the key to finding a theoretical model of the relaxed states is to find a suitable invariant (ie. \( \int Gdx = C \)) with which to constrain the energy minimisation. It had been shown previously by Woltjer that for an ideal (i.e. perfectly conducting) plasma with \( l \) number of flux tubes there exist a number of invariants \( K \) given by

\[ K_l = \int_{V_l} A \cdot B \, dv \quad for \quad l = 1, 2, \ldots, \quad (2.6.4) \]

where \( A \) is the vector potential and \( K_l \) (of each \( l \)th flux tube) is the magnetic helicity. These values of the helicity \( K_l \) are known as the Woltjer invariants, and are a measure of the linkage between the \( l \) flux tubes. However minimizing Equation 2.6.2 with respect to the full set of Woltjer invariants would lead to an infinite set of constraints on the relaxed states. This is not what is seen in experiment, where the relaxed states all share similar configurations, independent of the starting conditions of the plasma.

It was suggested by Taylor that instead of the plasma being modeled as perfectly conducting (and thus subject to the full set of Woltjer invariants), if the plasma was modeled as having a very small resistivity, then the magnetic field lines would be free to break and reconnect. While the individual flux tubes would no longer be constrained by Equation 2.6.4, Taylor proposed that the total magnetic helicity \( K_0 \), calculated over the entire volume of the plasma \( (V_0) \) might be conserved, so that

\[ K_0 = \int_{V_0} A \cdot B \, dv, \quad (2.6.5) \]

is unchanged after relaxation. This assumption was based on the idea that the energy of the plasma would decay much more quickly in the presence of a small resistivity than the helicity, so that \( K_0 \) remains effectively constant while \( W \) decays to the minimum level. This type of
selective decay is common in self-organising systems. The unequal rate of decay is due to the spatial scales at which the two quantities are conserved during turbulence [40]. For example if one quantity is accumulated at long wavelengths during turbulence while another quantity is accumulated over short wavelengths, then the introduction of some sort of dissipation (such as resistivity) that occurs at short wavelengths would degrade the latter quantity while leaving the first quantity unchanged. It has been shown [1] that during turbulence magnetic energy accumulates at short scales while magnetic helicity accumulates on long scales. Thus the introduction of a small resistivity would cause energy to decay but allow helicity to remain invariant.

If Equation 2.6.2 is now minimised with the constraint of Equation 2.6.5, the constrained minimum energy $\delta I$ is given by

$$\delta I = \delta W - \lambda \delta K_0 = 0,$$  \hspace{1cm} (2.6.6)

Expanding $\delta W$ and $\delta K_0$ gives

$$\delta W = \frac{1}{\mu_0} \int_{V_0} \delta \mathbf{B} \cdot \mathbf{B} dV = \frac{1}{\mu_0} \int_{V_0} \delta \mathbf{A} \cdot \nabla \times \mathbf{B} dV + \int_{S_0} (\delta \mathbf{A} \times \mathbf{B}) \cdot \hat{n} dS,$$  \hspace{1cm} (2.6.7)

and

$$\delta K_0 = \int_{V_0} (\delta \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \delta \mathbf{B}) dV = 2 \int_{V_0} \delta \mathbf{A} \cdot \mathbf{B} dV + \int_{S_0} (\mathbf{A} \times \delta \mathbf{A}) \cdot \hat{n} dS,$$  \hspace{1cm} (2.6.8)

where $S_0$ is the surface enclosing the plasma volume and $\hat{n}$ is the unit vector perpendicular to the surface. Substituting Equations 2.6.7 and 2.6.8 into Equation 2.6.6 gives

$$\delta I = \int_{V_0} \delta \mathbf{A} \cdot \left( \frac{1}{\mu_0} \nabla \times \mathbf{B} - 2\lambda \mathbf{B} \right) dV + \int_{S_0} (\hat{n} \times \delta \mathbf{A}) \cdot (\mathbf{B} + \lambda \mathbf{A}) dS.$$  \hspace{1cm} (2.6.9)

If the plasma is surrounded by a perfectly conducting surface $\hat{n} \times \delta \mathbf{A} = 0$ and the surface integral in 2.6.9 disappears leading to

$$\delta I = \int_{V_0} \delta \mathbf{A} \cdot \left( \frac{1}{\mu_0} \nabla \times \mathbf{B} - 2\lambda \mathbf{B} \right) dV = 0,$$  \hspace{1cm} (2.6.10)

Equation 2.6.10 implies that, for a non zero potential $\delta \mathbf{A}$, the constrained energy is minimized when

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} - 2\lambda \mathbf{B} = 0,$$  \hspace{1cm} (2.6.11)
As $\lambda$ is a constant by definition, the relaxed state has a form of

$$\nabla \times \mathbf{B} = \mu \mathbf{B},$$  \hspace{1cm} (2.6.12)

defined by only single term $\mu$, where $\mu = 2\mu_0\lambda$, and is independent of the initial conditions of the plasma.

Solutions to this equation can be found and compared to experimental results to see how well relaxation theory matches observed plasma behaviour. If the toroidal system is approximated to a cylinder then solutions to Equation 2.6.12 are found to be of the form

$$B_r = 0,$$  \hspace{1cm} (2.6.13)

$$B_\theta = B_0 J_1(\mu r),$$  \hspace{1cm} (2.6.14)

$$B_z = B_0 J_0(\mu r),$$  \hspace{1cm} (2.6.15)

where $J_1$ and $J_0$ are Bessel functions and $B_0$ is a constant (the magnetic field strength at the center of the RFP). One important features of the RFP is that when a value known as the pinch parameter, $\Theta$, exceeds a critical value the magnetic field at the containing wall ($r = a$) reverses direction. $\Theta$ is a measure of the ratio of the total toroidal current, $I$, to the total toroidal flux, $\Phi_{\text{tor}}$, and is given by

$$\Theta = \frac{B_\theta(a)}{\langle B_z \rangle} = \frac{\pi a I}{\mu_0 \Phi_{\text{tor}}},$$  \hspace{1cm} (2.6.16)

where $\langle B_z \rangle$ is the average of $B_z$ over the plasma volume. It can be shown that in the cylindrical approximation Taylor relaxation predicts $\Theta$ to be

$$\Theta = \frac{\mu a}{2},$$  \hspace{1cm} (2.6.17)

and also predicts that when $\Theta > 1.2$ the magnetic field of the will reverse direction at the wall surrounding the plasma. While the value of $\Theta$ at which the field reverses is slightly higher in practice, the general agreement between theory and experiment is good for the RFP.
Chapter 3

The Taylor relaxation model of ELMs

3.1 Calculating the Peeling Stability of a Cylindrical Plasma

Now that the basic theories of ELMs and Taylor relaxation have been covered we can introduce the mathematical basis for our model. We will consider the plasma to be in cylindrical configuration. This correlates well with the toroidal configuration in a Tokomak with a large aspect ratio such as JET. This cylindrical nature simplifies the mathematics considerably and allows analytic study of certain quantities which would require the use of complex numerical methods in toroidal geometry. This section will cover the derivation of the model published in [31] but otherwise does not contain original work.

3.1.1 General Outline of Stability in Edge Plasmas

For a toroidal plasma there are two main forces operating. There is a stabilising force due to favourable magnetic curvature operating on the pressure gradients found in the edge and a destabilising force caused by the edge current flowing in the H-mode pedestal. For the plasma to be stable the contribution from the pressure related terms must be greater than that from the destabilising edge current terms [36]. We will now look at this force balance after it undergoes a ELM and the plasma in part of the edge undergoes Taylor relaxation.

When the ELM event occurs it reduces the pressure in the edge effectively to zero, thus removing the stabilising pressure gradients. As the pressure gradients have been removed, curvature is no longer relevant and the plasma can be modeled as a cylinder. The relaxation process caused by the ELM would also raise the average edge current density by flattening out the current profile. The effect of these processes would be to leave a relaxed state with no stabilising forces present and so the peeling mode would continue unabated. This is not the case, however. It is found that the process of Taylor relaxation in a plasma next to a vacuum leads to the formation of a
thin skin or sheet current at the boundary. This process is not restricted to cylindrical plasmas, but is a general property of relaxed plasmas. This was discovered by using variational methods to study relaxed plasmas where it was found that the disappearance of the first variation of the minimisation of the energy and the nature of the second variation implied that the formation of a skin current is energetically a necessity (for a more in depth discussion of this topic see [4, 39, 30, 37]).

This skin current is always stabilising and therefore it provides a counteracting force to that of the edge current. In fact the skin current’s stabilising effect increases with the square of the width of the relaxed region, while the destabilising current density increases linearly with region width. Therefore it is possible for a relaxed region of finite extent to be stable to peeling modes.

### 3.1.2 Stability in a Relaxed Cylindrical Plasma

The plasma is considered as being in some initial state with a safety factor profile \( q_i \). This profile is unstable to peeling modes and an ELM event is triggered. This causes the plasma to undergo Taylor relaxation [47], which starts at the edge of the plasma \((r = a)\) and works its way inwards until a point where the plasma becomes stable to all peeling modes. This inner radius will be labeled \( r = r_e \). Therefore the cross section of the plasma consists of an annular relaxed region between \( r = a \) and \( r = r_e \) with a final safety factor profile \( q_f \). This distribution is shown in Figure 3.1 below.

Now the shape of the plasma region is known we can formulate boundary conditions and force balance equations for the plasma at its boundaries. This will therefore determine the stability of the relaxed state to peeling mode instabilities for our geometry. Note that in order to relate the cylindrical geometry being used to a Tokamak, we imagine a periodic cylinder of length \( L = 2\pi R_0 \) with \( B_z = B_\phi \), and the toroidal mode number \( n \) related to the wave number in the \( z \) direction \((k)\) by \( n = R_0 K \).

Firstly, the boundary condition for the plasma at \( r = a \) will be found in terms of the final and initial safety factor profiles \( q_f \) and \( q_i \). Starting with the equilibrium Equation 2.4.2 we can write

\[
\nabla \times (J \times B) = 0,
\]

(3.1.1)

In line with standard stability theory this equation is now split into equilibrium and perturbation quantities, denoted with subscripts 0 and 1 respectively. Terms consisting of only equilibrium quantities cancel, while non-linear perturbation quantities are considered small and ignored. After this process 3.1.1 becomes...
Figure 3.1: The relaxed region (blue) of a cylindrical plasma configuration in the post-ELM state

\[ \nabla \times [J_1 \times B_0 + J_0 \times B_1] = 0, \quad (3.1.2) \]

Expanding 3.1.2 using vector identities, and making use of \( \nabla \cdot B = 0 \) and \( \nabla \cdot J = 0 \) results in

\[ (B_0 \cdot \nabla) J_1 + (B_1 \cdot \nabla) J_0 = 0, \quad (3.1.3) \]

It can be shown that the last two terms of 3.1.3 consists of only higher order terms in \( \varepsilon \) (the inverse aspect ratio) when expanded out using the large aspect-ratio approximation (valid here as a cylindrical-Tokamak approximation is being used), and are thus ignored. Physically this means that both \( B_0 \) and \( B_1 \) are slowly varying in the direction of \( B \) in this geometry. This leaves

\[ (B_0 \cdot \nabla) J_1 = 0, \quad (3.1.4) \]

Taking the toroidal (\( \phi \)) component of 3.1.4

\[ (B_0 \cdot \nabla) J_{1\phi} + (B_1 \cdot \nabla) J_{0\phi} = 0, \quad (3.1.5) \]
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It is now convenient to introduce the perturbed poloidal flux $\psi_1$, defined so that

$$B_1 = \phi \times \nabla \psi_1, \quad (3.1.6)$$

and

$$\mu_0 J_1 \phi = \nabla^2 \psi_1, \quad (3.1.7)$$

where $\phi$ is the unit vector in the poloidal direction. From 3.1.6 and 3.1.7 the following relationships can be derived

$$B_{1r} = -\frac{1}{r} \frac{\partial \psi_1}{\partial \theta}, \quad (3.1.8)$$

$$B_{1\theta} = \frac{\partial \psi_1}{\partial r}, \quad (3.1.9)$$

Before substituting Equations 3.1.6 to 3.1.9 into 3.1.5 the symmetry of the cylindrical Tokamak can be used to Fourier decompose the perturbed poloidal flux into the following form

$$\psi_1 = \hat{\psi}_1 e^{im\theta - in\phi}, \quad (3.1.10)$$

where $m$ is the poloidal mode number and $n$ is the toroidal mode number. This shows that the perturbed flux can be written in terms of the radial component of the perturbed magnetic field, so that

$$\psi_1 = \frac{irB_{1r}}{m}. \quad (3.1.11)$$

Using Equations 3.1.6 to 3.1.10, 3.1.5 now becomes

$$i \left[ \frac{mB_{\theta}}{r} - \frac{nB_{\phi}}{R} \right] \frac{1}{\mu_0} \nabla^2 \psi - i \frac{m}{r} \psi \frac{dJ_\phi}{dr} = 0, \quad (3.1.12)$$

where the zero subscripts on equilibrium terms have been dropped, as has the 1 subscript on the perturbed flux. Simplifying using the safety factor $q = \frac{rB_{\phi}}{mB_{\theta}}$ and introducing $F$, where $F = \frac{B_{\phi}}{r}(m - nq)$, results in
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3.1.3
\[
\frac{d}{dr} \left( r \frac{d\psi}{dr} \right) - \frac{m^2 \psi}{r} = \frac{m}{F} \mu_0 \frac{dJ}{dr} \psi,
\]
(3.1.13)

This is the governing equation for the plasma in terms of $F$ and the flux. However as was stated earlier we wish to find the stability equation and boundary conditions in terms of the safety factor $q$ so this equation must be rearranged. To do this the boundary condition for $\psi$ must be found at the plasma vacuum interface.

3.2 Modelling ELMs in a Relaxed Cylindrical Plasma

3.2.1 Finding the Stability Equation

As was seen previously, the relaxation process creates a skin current at the plasma/vacuum interface. This means there will be a jump in the magnetic field so that $\psi$ will not be continuous across the join. It will be necessary therefore to calculate boundary conditions for $\psi$ that take account of this jump. To do this first we imagine that the skin current is a uniform step current of small width $\delta$ and height $J_{\text{skin}}$, as shown in 3.2 below.

![Figure 3.2: Skin current at the plasma vacuum interface](image)

By using standard calculus techniques the change in the flux between the end of the plasma at point $P$ and the start of the vacuum at point $V$ and vice versa can be written as follows.
\[ \psi_v = \psi_p + \delta \left( \frac{d\psi}{dr} \right)_{p+}, \]  
(3.2.1)

\[ \psi_p = \psi_v - \delta \left( \frac{d\psi}{dr} \right)_{v-}, \]  
(3.2.2)

where the positive subscript denotes that the differential is taken to the right of the jump point (as seen on the diagram) and to the left for the negative subscript. Then taking jump integrals of Equation 3.1.13 at \( P \) and \( V \) the following equations are obtained

\[ \left[ r \frac{d\psi}{dr} \right]^{p+} - \left[ r \frac{d\psi}{dr} \right]^{v-} = m \mu_0 J_{skin} \frac{\psi_p}{F_p}, \]  
(3.2.3)

and

\[ \left[ r \frac{d\psi}{dr} \right]^{v+} - \left[ r \frac{d\psi}{dr} \right]^{p-} = -m \mu_0 J_{skin} \frac{\psi_v}{F_v}, \]  
(3.2.4)

Solving for equations 3.2.3 and 3.2.4 produces the boundary condition

\[ \psi_v F_p = \psi_p F_v, \]  
(3.2.5)

for \( \psi \) at the plasma/vacuum join. Note that the perturbed flux function \( \psi_1 \) as defined in Equation 3.1.10 has a helical mode dependence, and for this reason is not directly continuous across the interface, but rather the related quantity \( \frac{\psi}{F} \), which follows the helical field lines, is continuous. As \( \frac{\psi}{F} \) is continuous across the skin current we can integrate Equation 3.1.13 between \( P \) and \( V \) as long as \( \frac{\psi}{F} \) is the variable of integration. After some rearrangement this gives

\[ \left[ F^2 \frac{d}{dr} \left( \frac{\psi}{rF} \right) \right]_p = 0, \]  
(3.2.6)

Carrying out the differentiation and expanding the brackets

\[ \left[ F \psi' - \frac{F}{a} \psi - \psi F' \right]_p = 0, \]  
(3.2.7)

where \( a \) is the radial position of the plasma vacuum interface (from here on the subscript \( a \) will be used to denote that the variable or calculation is being determined at the \( P/V \) boundary). Expanding Equation 3.2.7 results in
\[ \frac{\psi_p F_p \psi_v}{\psi_v} - \frac{\psi_p F_v \psi_v'}{\psi_v} - \frac{F_v}{a} \frac{F_p}{F_v} \psi_p + \frac{F_p}{a} \frac{F_p}{F_v} \psi_v - \frac{F_v}{F_p} \psi_p' F_p' + \frac{F_p}{F_v} \psi_v' F_v' = 0, \]  
\[ (3.2.8) \]

where we have again used the boundary condition in Equation 3.2.5. Dividing by \( \psi_v F_p \) we obtain

\[ \frac{F_v}{F_p} \frac{\psi_v'}{\psi_v} - \frac{F_p}{F_v} \frac{\psi_p'}{\psi_p} - \frac{F_v}{F_p a} + \frac{F_p}{F_v a} - \frac{F_v'}{F_p} + \frac{F_p'}{F_v} = 0, \]
\[ (3.2.9) \]

having used the boundary condition again and carried out some simplification. We now wish to express this in terms of the plasma safety factor (this is the variable which we will utilise in later parts of this thesis). To this end we take the derivative of the force in both the plasma and the vacuum, and simplify to obtain (see Appendix A for derivations of these results)

\[ F_v = \frac{m B_0}{R_0} \Delta_a, \]
\[ (3.2.10) \]

\[ F_v' = -\frac{2m B_0}{R_0 a} \frac{1}{q_v}, \]
\[ (3.2.11) \]

\[ F_p = \frac{m B_0}{R_0 a} (\Delta_a - \chi_a), \]
\[ (3.2.12) \]

\[ F_p' = -\frac{m B_0}{R_0 a} \left( \frac{2}{q_p} - I_a \right), \]
\[ (3.2.13) \]

where we introduce the following terms:

1. \( \Delta_a \) defined as

\[ \Delta_a = \left( \frac{1}{q_v} - \frac{n}{m} \right), \]
\[ (3.2.14) \]

which quantifies how the peeling instability behaves at a resonance outside the edge of the plasma.

2. The surface skin current \( \chi_i \) defined as

\[ \chi_i = \left( \frac{1}{q_{i+}} - \frac{1}{q_{i-}} \right), \]
\[ (3.2.15) \]

in terms of the \( q \) profiles at a point \( i \) (where \( i \) is either \( r = a \) or \( r = re \)).
3. The toroidal current density

\[ I = \frac{R_0 \mu_0 J}{B_0} = \frac{1}{r} \frac{d}{dr} \left( \frac{r^2}{q} \right), \tag{3.2.16} \]

The flux and its derivative are now needed for both the plasma and vacuum in order to complete Equation 3.2.9. These are found to be

\[ \frac{\psi'_v}{\psi_v} = -\frac{m}{a}, \tag{3.2.17} \]

\[ \frac{\psi'_p}{\psi_p} = -\frac{m}{a} - \frac{\Delta'_a}{a}, \tag{3.2.18} \]

at the plasma vacuum boundary. We have used the jump condition for the poloidal flux given by

\[ \Delta'_a = \left[ \frac{r}{\psi} \frac{d\psi}{dr} \right]^+ - \left[ \frac{r}{\psi} \frac{d\psi}{dr} \right]^- \tag{3.2.19} \]

evaluated at the plasma/vacuum interface. Note that \( \Delta'_a \) is not the differential of \( \Delta_a \) (they are essentially unrelated despite looking similar in standard notation). Substituting these expressions into 3.2.9 and multiplying by \( a \) gives

\[ -\Delta_a (m + 1) + \frac{\Delta_a - \chi_a}{\Delta_a} (m + \Delta'_a + 1) + 2 \left( \frac{1}{q_v (\Delta_a - \chi_a)} - \frac{1}{q_p \Delta_a} \right) + \frac{I}{\Delta_a} = 0, \tag{3.2.20} \]

where \( \chi_a \) is the skin current formed after the relaxation.

Substituting for \( q_v \) and \( q_p \), and multiplying by \( \Delta_a (\Delta_a - \chi_a) \) we arrive after some simplification at

\[ \chi_a \left[ (m - 1 + \Delta'_a)(\chi_a - 2\Delta_a) + \frac{2n}{m} - I \right] + \Delta_a \left[ \Delta_a \Delta'_a + I \right] = 0, \tag{3.2.21} \]

which is the stability equation in terms of the safety factor \( q \) at the plasma/vacuum boundary \( (r = a) \) in its final form. It can also be shown (see appendix A in [31]) that the left hand side of Equation (3.2.21) is equal to \(-\delta W\), the plasma’s potential energy (see Section 2.5.1 for discussion of plasma potential energy and stability). Therefore if we can write equation (3.2.21) in terms of known quantities we can determine the stability of the plasma. However, to do this we need to see how the formation of the skin current affects the plasma at the boundary between the relaxed region and the non-relaxed region. Therefore repeating the process that led to equation (3.2.21) at the \( r = re \) boundary we obtain
\[ \Delta_{E-} = \left[ \Delta_{E-} \Delta'_{E-} + I_{E-} - I_{E+} \right] + \chi_E \left[ (\chi_E + 2\Delta_{E-}) (\Delta'_{E-} + m + 1) + 2 \frac{n}{m} - I_{E+} \right] = 0, \quad (3.2.22) \]

where the terms with \( E+ \) in the subscript denotes that the term is evaluated on the relaxed side of the boundary and \( E- \) in the subscript denotes that the term is evaluated on the non relaxed side of the boundary (see Appendix B for a more complete derivation of this equation).

The stability conditions have now been calculated at both boundaries of the relaxed region of the plasma edge. However to find the size of the relaxed region the two stability conditions need to be linked. Therefore we need another set of equations. We will take the flux on either side of the relaxed region to be that of a vacuum. This is obviously true for the vacuum region but it is less clear that this is the case below the relaxation boundary. We can make this assumption because most of the peeling modes we will be investigating are of high order in \( m \). Therefore in Equation 3.2.9 the second term on the L.H.S will be much larger than the R.H.S. thus approximating to the vacuum case where \( J=0 \). Therefore the flux will be \( \psi = r^m \) in all areas outside the relaxed region \((-m \text{ for } r > a \text{ and } +m \text{ for } r < r_e)\). In the relaxed region the process of Taylor relaxation leaves a flat current profile \( J \) so the \( \frac{dJ}{dr} \) term in Equation 3.2.9 will be zero. As there are two boundaries to this region the flux is then of the form \( \psi = Ar^m + Br^{-m} \) where \( A \) and \( B \) are constants to be found. Using this we can now calculate \( \Delta'_{E-} \) and \( \Delta'_{a} \) to obtain

\[ \Delta'_{a} = -mAa^m - Ba^{-m} - m, \quad (3.2.23) \]
\[ \Delta'_{E} = mre^m - Xre^{-m} - m, \quad (3.2.24) \]

As these terms depend only on the ratio of \( B/A \) we can substitute \( B/A \) for another constant \( X \). Therefore we rewrite the above expressions as follows

\[ \Delta'_{a} = -mAa^m - Ba^{-m} - m \]
\[ \Delta'_{E} = mre^m - Xre^{-m} - m \]

Manipulating the expression for \( \Delta'_{E} \) to find the constant \( X \) we obtain

\[ X = -re^{2m} \left( \frac{\Delta'_{E} + m}{m} - 1 \right) \left( \frac{\Delta'_{E} + m}{m} + 1 \right), \quad (3.2.25) \]
Therefore we now substitute the constant $X$ into the equation for $\Delta_a'$ producing the following relationship linking $\Delta_E'$ and $\Delta_a'$ is obtained

$$\Delta_a' = -2m \frac{(\Delta_E' + 2m)}{(g \Delta_E' + 2m)},$$

(3.2.26)

where $g$ is a damping term given by $g = 1 - \left(\frac{n}{a}ight)^{2m}$. By using equation (3.2.26) we can now eliminate $\Delta_E'$ from equation (3.2.25) and solve for $\Delta_a'$. This gives $\Delta_a'$ in terms of quantities that can be calculated from the safety profile. Therefore by substituting this into Equation (3.2.21) we can calculate the stability of the relaxed state to peeling modes using only known quantities and therefore see how the stability of the plasma is affected by variations in the initial safety profile $q_i$.

### 3.3 Results of the Model

This model predicted ELM widths for a simple quadratic q-profile. The final ELM widths showed strong dependence on $q_a$ as seen in Figure 3.3

![Figure 3.3: Plot Taken from [31] of ELM width, $d_E$, against $q_a$, showing the resonant nature of $d_E$](image)

The pattern of peaks and troughs visible in Figure 3.3 is generated by the $\Delta_a$ term in 3.2.21, where

$$\Delta_a = \left(\frac{1}{q_a} - \frac{n}{m}\right).$$

(3.3.1)
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As $\Delta_a$ describes the location of a resonance outside the plasma it must be positive. There is also the requirement for the plasma to have been initially unstable to peeling modes (see Chapter 4 for more discussion on this topic), so that

$$I_a - 2m\Delta_a > 0.$$  \hspace{1cm} (3.3.2)

Equation 3.3.2 implies that the plasma is most unstable when $m$ is low, while Equation 3.3.1 implies that if $m$ is low, then $n$ must also be low, so that $\Delta_a$ remains positive. Together these two equations suggest that the plasma is most unstable for low $n$, $m$ pairs (eg, $n = 1$, $m = 4$). However, it can also be seen from Equation 3.3.1 that as $q_a$ increases the minimum value of $m$ for any given $n$ also increases. This suggests that the plasma becomes less unstable as $q_a$ increases, reflected in the decreasing values of $d_e$ with $q_a$ shown in Figure 3.3. This is particularly noticeable in the dips in $d_e$ that occur after each integer value of $q_a$.

The dips shown in Figure 3.3 can be explained by noting when $q_a$ is close to either a whole number or a ratio of low whole numbers (for example near to $q_a = 4$ or $q_a = \frac{9}{2}$) and therefore close to the ratio of the low order $n$ and $m$ numbers that produce the most instability (for example $n = 1$, $m = 4$), then $\Delta_a$ is small and the plasma is less unstable, reducing $d_e$. In other words, after an integer or low-order rational there is a deficiency of mode numbers that fulfill the triple requirements for generating large ELMs, namely:

- The plasma is peeling unstable.
- $\Delta_a$ is positive.
- $\Delta_a$ and $m$ are small.
Chapter 4

Results: Examining Length Scales in ELM widths and the Effects of the bootstrap current

4.1 Examining Length Scales in ELM widths

It is was shown in Figure 3.3 that in a plot of $d_E$ against $q_a$ there are clear signs of resonance structure. Here we will show the dependence of the size of these structure levels on the edge current density $I_a$. Starting with the stability equation for the initial plasma state, where the plasma is initially unstable if $I_a - 2m\Delta_a > 0$ is satisfied (this can be derived from 3.2.21 by setting the skin current $\chi_a$ equal to zero to represent the edge stability before relaxation has taken place). Substituting in the the definition of $\Delta_a$ we have

$$I_a - 2m\left(\frac{1}{q_a} - \frac{n}{m}\right) > 0,$$  \hspace{1cm} (4.1.1)

Rearranging this we get

$$q_a > \frac{m}{\frac{I_a}{2} + 2},$$  \hspace{1cm} (4.1.2)

We also have the requirement, from the definition of an externally resonant peeling mode, that $\Delta_a$ be positive. Therefore we can say

$$\frac{1}{q_a} - \frac{n}{m} > 0 \Rightarrow q_a < \frac{m}{n},$$  \hspace{1cm} (4.1.3)
Taking these results together show that for a given n and m pair the range of $q_a$ values that satisfy all requirements is given by

$$\frac{m}{n} > q_a > \frac{m}{\frac{T_a}{2} + 2}.$$  \hspace{1cm} (4.1.4)

Thus each level of structure in the $q_a$ versus $d_E$ plot corresponds to a single mode (with a specific set of n and m numbers) which produces the largest ELM widths in that range of $q_a$. For example the large peak in $d_E$ before $q_a = 4$ corresponds to the $n = 1, m = 4$ mode.

### 4.1.1 Varying Edge Current

It can also be seen from (4.1.4) that the structure range is also dependent on $I_a$. As it is almost certain that any experimental data would have a different edge current, and therefore length scale, it is desirable to be able to modify $I_a$ without changing any of the parameters of the initial q profile ($q_a$ and $q_0$). We do this by modifying the initial current profile in such away as to insure $q \rightarrow q_a$ as $r \rightarrow 1$ and $q \rightarrow q_0$ as $r \rightarrow 0$. This is done simply by changing the exponent of $r$ in the initial current profile from 2 to a variable $\nu$. Therefore the new current profile is

$$q = q_0 + (q_a - q_0) r^\nu.$$  \hspace{1cm} (4.1.5)

The edge current density is given by Equation 3.2.16 evaluated at $r = a$. By substituting (4.1.5) into this expression gives the dependence of $I_a$ on $\nu$.

$$I_a = \frac{2}{q_a} \cdot \frac{\nu (-q_0 + q_a)}{qa^2}.$$  \hspace{1cm} (4.1.6)

Thus it can be seen that there is a linear relationship between $\nu$ and $I_a$. This relationship is plotted below for fixed values of $q_0$ and $q_a$.

In order to see how varying $I_a$ changes the structure length scales the $d_E/q_a$ plot is now redone for two different values of $\nu$ (1.25 and 1.5).

It can be seen that decreasing $\nu$ increases $I_a$ which in turn increases the length scales on the $d_E/q_a$ plot. This is as expected from (1) as increasing $I_a$ will increase the range of unstable $q_a$ for a given set of n and m.
Chapter 4. Results

Figure 4.1: Dependence of $I_a$ on $\nu$ evaluated at $q_a = 4$ and $q_0 = 1$

Figure 4.2: $d_E$ vs $q_a$ for $\nu = 1.25$
In developing our numerical model we follow the same basic principles as used in the previous Taylor relaxation study of ELMs [31], as outlined in Chapter 3. The major difference, however, is that our initial state will now include a bootstrap current term centered at a distance \( r_b \) from the center of the plasma. Due to the nature of the bootstrap current \( r_b \) will be located close to the edge of the plasma.

It was decided that a \( \tanh \)-like function added to the original \( q \) profile would represent the effect of adding the bootstrap current. This is because the steep change now created in the \( q \) profile corresponds to a peak in the current density profile, defined earlier in Equation 3.2.16 as

\[
I = \frac{R_0 \mu_0 J}{B_0} = \frac{1}{r} \frac{d}{dr} \left( \frac{r^2}{q} \right),
\]

(4.2.1)

where \( I \) is the non-dimensional current density, \( q \) is the safety profile and \( r \) is the radial distance. As the actual bootstrap current formulation requires both toroidal geometry and knowledge of the plasma temperature, density, and so on, the \( \tanh \) function was chosen for mathematical simplicity. However, by varying the terms of the function, the shape of a wide array of bootstrap configurations could be replicated. As a precaution the bootstrap distribution from one configuration of the JET tokamak was compared with the \( \tanh \) function bootstrap, with good qualitative agreement. Therefore the safety profile

\[
q = q_0 + (q_a - q_0) r^2 - d_{ia} r^2 (1 + \tanh \frac{r - r_b}{d_{ia}}),
\]

(4.2.2)
was chosen, where $d_{tq}$ is a measure of the amplitude of the bootstrap term, and $d_{ta}$ is measure of both the height and width of the bootstrap term. This profile allows the results in Section 3.2 to be replicated by setting $d_{tq}$ to zero, thus removing the bootstrap term. A comparison of this new profile with the old profile is shown in Figure 4.4. In this graph $q_0$ and $q_a$ were kept constant for both profiles.

As well as comparing the safety profiles it is also beneficial to compare the corresponding current density profiles as this shows more clearly the effect of the bootstrap term. Figure 4.5 shows the comparison of the respective current density profiles.

It is worth noting the dependence of the size of the bootstrap term on the variable $d_{ta}$. It is clear by inspecting 4.2.2 that $d_{ta}$ will determine the width of the bootstrap distribution. It is not so clear however the dramatic effect that $d_{ta}$ has on the height of the distribution. This is seen by isolating the bootstrap current density from the rest of the current density in equation 4.2.2 and plotting the resulting function against $d_{ta}$. The result of this process is seen in Figure 4.6.

Thus it can be seen that a small variation in $d_{ta}$ will lead to a large change in the magnitude of the bootstrap term. Later in the paper it will be shown that dramatic differences in the behaviour of the model are achieved with only small variations in $d_{ta}$.
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Figure 4.5: Comparison of $I$ profile with bootstrap term and $I$ profile with no bootstrap term for $q_0 = 1$, $q_a = 4.5$, $d_{tq} = 0.4$, $d_{ta} = 0.05$, and $r_b = 0.9$.

Figure 4.6: Maximum amplitude of the bootstrap current plotted against $d_{ta}$, with $dtq$ and $r_b$ fixed at 0.4 and 0.9 respectively.


4.3 Calculating ELM Width

Once an adequate current profile has been chosen it is possible to proceed with calculating ELM widths. This process is broken down into steps described below.

First, fixed values are assigned to $d_{iq}$, $d_{ia}$, and $r_B$ so that the form of the bootstrap term is unchanged throughout the experiment. As edge effects are being studied, an initial value of $r_b = 0.9$ in normalised units was used. A good representation for the bootstrap term is obtained when using $d_{iq} \approx 0.4$ and $d_{ia} \approx 0.05$. The edge $q$ value $q_a$ is the main controlling variable, so will take values between 3 and 7, the same range studied using the non-bootstrap profile. In order for the relaxation process to take effect the initial state must be unstable. Therefore we need to include only those initial states which are initially peeling mode unstable. The instability condition is still given by Equation (3.2.25), which gives a set of unstable $m$ and $n$ modes for every value of $q_a$. Once all the initially unstable (pre-ELM) states have been found, it is necessary to find the corresponding final (relaxed) states in terms of the relaxed radius $r_e$.

To do this, the same energy minimisation calculation as described in Section 2.6 must be carried out for the annular plasma region. In this case, as the plasma has two boundaries, two invariants are required to define the final relaxed state. One of these will be the helicity, as before, but another is now required. It has been shown previously that the magnetic flux for an individual flux tube is invariant in ideal MHD. However, if a very small but finite amount of resistivity is introduced flux tubes are free to break and reconnect with each other and the individual flux tubes are no longer distinguishable, in a similar manner to the loss of individual helicity for flux tubes in the original Taylor theory. However, it will now be assumed that the total poloidal flux for the annular plasma region will remain invariant during relaxation, again in a similar manner to the total helicity. Therefore the two conserved quantities in the relaxation process will be the total helicity $K$ and the total poloidal flux $\Psi_\theta$. The total helicity of this annular relaxed region can be derived from the general total helicity of a relaxed state, which was shown in Chapter 2 to be

$$ K = \int_{V_0} A \cdot B dV. \quad (4.3.1) $$

Splitting Equation 4.3.1 into toroidal ($\phi$) and poloidal ($\theta$) components obtains

$$ K = 4\pi^2 R_0 \int_{r_e}^{a} (A_\theta B_\theta + A_\phi B_\phi) r dr. \quad (4.3.2) $$

Integrating the poloidal component of Equation 4.3.2 by parts and using $B = \nabla \times A$ gives
\[
K_{\text{Poloidal}} = 4\pi^2 R_0 \left[ -A_\phi A_\theta r \right]_r^a + 4\pi^2 R_0 \int_{r_e}^a A_\phi B_\phi r \, dr
\]
\[
= 4\pi^2 R_0 \left[ -A_\phi A_\theta r \right]_r^a + K_{\text{Toroidal}}. \tag{4.3.3}
\]

Then, selecting a gauge for \( A \) so that \( A_\phi = 0 \) at \( r = a \) and \( A_\theta = 0 \) at \( r = r_e \), the poloidal component of helicity becomes equal to the toroidal helicity. Therefore the total helicity is merely twice the toroidal helicity. Therefore

\[
K = 2K_{\text{Toroidal}} = 8\pi^2 R_0 \int_{r_e}^a A_\phi B_\phi r \, dr. \tag{4.3.4}
\]

After integrating by parts and making use of \( B_\phi \approx \text{constant} \) in the cylindrical Tokamak approximation, Equation 4.3.4 becomes

\[
K = 4\pi^2 R_0 \left[ A_\phi B_\phi r^2 \right]_r^a + 4\pi^2 R_0 \int_{r_e}^a B_\phi B_\theta r^2 \, dr. \tag{4.3.5}
\]

Setting \( A_\phi = 0 \) at \( r = a \), and normalizing \( B_\phi \), the final form of \( K \) is given by

\[
K = \int_{r_e}^a \frac{r}{q} (r^2 - r_e^2) \, dr, \tag{4.3.6}
\]

where \( q = \frac{nB_\phi}{R_0 B_\theta} \). By using a similar method, the total poloidal flux \( \Psi_\theta \) is found to be

\[
\Psi_\theta = \int V_0 B_\theta dV = \int_{r_e}^a \frac{r}{q} dr. \tag{4.3.7}
\]

Using the variational process for finding the constrained minimum of a function, described in Section 2.6, the poloidal magnetic energy \( W_\theta \) is minimised with the constraint of invariant helicity and invariant poloidal flux if

\[
W_\theta - \lambda_1 K - \lambda_2 \Psi_\theta = 0. \tag{4.3.8}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are Lagrange multipliers. Substituting in expressions for \( W_\theta, K, \) and \( \Psi_\theta \) into Equation 4.3.8 gives

\[
\int_{r_e}^a \left[ \frac{r^3}{q^2} - \lambda_1 \frac{r}{q} (r^2 - r_e^2) - \lambda_2 \frac{r}{q} \right] \, dr = 0. \tag{4.3.9}
\]
The solution to Equation 4.3.9 can be shown to be of the form

\[ q_f = \frac{r}{Cr + \frac{1}{r}}, \]  

(4.3.10)

where \( q_f \) is the q-profile of the final relaxed state, and \( C \) and \( D \) are constants that must be found. This is achieved by letting \( K_{\text{initial}} = K_{\text{final}} \) and \( \Psi_{\theta-\text{initial}} = \Psi_{\theta-\text{final}} \) and eliminating the unknowns. There is now an added complication however, as the new q profile cannot be integrated analytically. Therefore in order to find the constants \( C \) and \( D \) the entire plasma is broken into discrete units of length 0.001 in normalised units, representing possible values of \( r_e \). Therefore by equating the helicity and flux at every possible value of \( r_e \) we obtain a set of simultaneous equations. As all terms are now no longer dependent on \( r_e \) it is possible to use numerical integration. In this way \( C \) and \( D \) are no longer mathematical expressions but matrices of the numerical solutions to the simultaneous equations.

Now that both \( q_i \) and \( q_f \) are known for every possible value of \( r_e \) it is possible to calculate every term in Equation (3.2.25). Thus it is now possible to plot \( -\delta_w \) against \( d_e \) for every initially unstable \( n \) and \( m \) combination. An example of such a plot is shown in Figure 4.7 below.

The plasma becomes unstable when \( -\delta_w \) becomes negative, and the plot crosses the x axis. To obtain an expression for marginal stability the value of \( d_e \) where \( -\delta_w = 0 \) was found for each one of the \( n, m \) pairs. As each point is just a number, with no mathematical relation to any other point, it is necessary to use linear interpolation between the two points on either side of the x
axis to find the point of marginal stability. In the cases where there were multiple instances of $-\delta_w = 0$ for one m, n pair the smallest value of $d_e$ was chosen. Testing this method against the inbuilt Mathematica route-finder (using an initial state that could be integrated analytically) revealed comparable accuracy to 5 decimal places. We now have marginal $d_e$ results for every n and m pair associated with a particular $q_a$. It is assumed that the largest marginal $d_e$ will correspond to the largest possible ELM width. Therefore for every $q_a$ the m and n pair that produce the largest $d_e$ (now called $d_e(\text{max})$) is chosen.

### 4.4 Results of Numerical Calculations

It is now possible to plot $q_a$ against $d_e(\text{max})$ to reproduce Figure 3.3 for a current profile that includes the bootstrap term. This is shown in Figure 4.8 below.

It can be seen that this graph produces what appears to be two bands, with $d_e(\text{max})$ either being large (between 0.6 and 1) or small (between 0 and 0.2) with no results lying between the two bands. This discontinuity is not present in results from [31] where a non bootstrap safety factor profile was used, leading to the assumption that the added bootstrap term is causing the effect. Therefore we will concentrate on how the bootstrap term might affect the equilibrium of the relaxed state, numerically and graphically in this section, and then analytically in a later section.

We start by looking back at the steps that led to this graph and see if an explanation for this banding of results can be found. We will examine points on the graph that are close in terms of $q_a$ but have very different $d_e(\text{max})$.

The next step back from Figure 4.8 is picking the m, n pair for each $q_a$ that produces the $d_e(\text{max})$. Therefore it is helpful to plot a graph of poloidal mode number m against $d_e$ for two points on either side of the bifurcation; $q_a = 3.96$ which gives a very large ELM width and $q_a = 3.97$ which gives a small ELM width.

There are several important features to be seen in Figure 4.9. The first is that the results are grouped together in curved bands, with an offset between the bands resulting from the difference in $q_a$. This banding stems from the $\Delta_x = (\frac{1}{q_x} - \frac{n}{m})$ in Equation (3.2.25) where x is the radial position of the function. When $\frac{1}{q_x} \approx \frac{n}{m}$ then $\Delta_x$ is small and so the marginal $d_e$ is also small.

Secondly it can be seen that the two $q_a$ produce broadly similar marginal $d_e$ for the majority of the range of m modes. However the lowest m mode for $q_a = 3.96$ produces an extremely large marginal $d_e$, which is an order of magnitude greater than most of the other results. It is also significantly larger than that obtained from the lowest m mode for $q_a = 3.97$.

In order to explain this result it is necessary to go back a step further in the calculations and examine the $d_e$ against $-\delta_w$ plots for the two lowest mode numbers (as shown in Figure 4.10).
Figure 4.8: Plot $q_a$ against $d_e(\text{max})$, with constant $d_{ta} = 0.05$, $q_0 = 1$, and $r_b = 0.9$, but varied $dtq$ with $dtq = 0$ (Blue points), $dtq = 0.1$ (Green points) and $dtq = 0.4$ (Red points).
Figure 4.9: Plot of poloidal mode number $m$ against $d_e$ for $q_a = 3.96$ (red) and $q_a = 3.97$ (blue), with $q_0 = 1$, $dtq = 0.4$, $d_{ta} = 0.05$, and $r_b = 0.9$

Figure 4.10: Plot of $-\delta v$ against $d_e$ for two values of $q_a$: $q_a = 3.96$ (red) and $q_a = 3.97$ (blue) with $q_0 = 1$, $dtq = 0.4$, $d_{ta} = 0.05$, and $r_b = 0.9$
It is now possible to see from these graphs how the huge differences in $d_{e(\text{max})}$ found in Figure 4.8 occur. The shape of the blue plot in Figure 4.10 is such that the $x$ axis is crossed quite quickly with the first dip in the plot. Although the red plot in Figure 4.10 has an almost identical shape to the blue plot, the first dip is too shallow to cross the $x$ axis. Therefore $-\delta_w$ remains positive until the second more pronounced dip takes the plot below the $x$ axis, which only happens at much larger $d_e$. Thus a small change in $q_a$ can lead to a large change in $d_{e(\text{max})}$. In physical terms this means that a plasma with $q_a = 3.96$ will remain unstable to peeling modes until the relaxation process has covered over 60% of the plasma radius. This corresponds to a very large ELM width.

4.5 Analysis of Results Leading to Examination of Special Bootstrap Configurations

In the previous section it was seen that a small change in initial conditions could lead to a large variation in the final ELM width. It is now desirable to ascertain the reasons behind this dependence on initial conditions. Because the banding seen in Figure 4.8 does not occur in the non-bootstrap case the most logical assumption is that there are two distinct cases in the bootstrap model. These are:

1. The plasma becomes stable to peeling modes before the relaxation process reaches the position of the bootstrap. Therefore the stable relaxed radius $r_e$ is greater than $r_b$ (as the relaxation process works from the edge of the plasma inwards). In this case the stability considerations will be near identical to those of the non-bootstrap model, leading to similar ELM widths.

2. The relaxation process reaches the position of the bootstrap before the plasma becomes stable. Therefore the stable relaxed radius $r_e$ is less than $r_b$. In this case the stability considerations are now very different from before, leading to different ELM widths.

However the bootstrap current is not confined to a single point but is spread over a section of the plasma whose size is dependant on the width parameter $d_{ta}$. This leads to the question, at what point does the bifurcation of results seen in Figure 4.8 occur? It can be seen that the gap between the bands of results begins at approximately $d_e = 0.2$ in normalised units. This corresponds to the inboard edge of the bootstrap current as (see red plot in Figure 4.5). This leads to two possibilities; either the inside edge of the bootstrap is the trigger for the jump to large ELM widths, or the trigger is linked to another factor such as total current encountered in the relaxation process and the location of the trigger point is merely coincidental. To proceed further we will examine two special cases, one where the bootstrap current is very large and narrow, an approximation to a delta function, and the other case where the bootstrap current is
small and wide. Plots of $q$ and current profiles for both these cases are included in Figures 4.11 and 4.12.

It can be seen that the inboard edge of the delta function approximation is located at $r \approx 0.88$ whereas the shallower profile’s inside edge is located at $r \approx 0.8$. Therefore the width at which the banding begins for each distribution will highlight whether the inside edge of the bootstrap is indeed the trigger. A plot of both distributions is included below.
Figure 4.13 shows that the banding effect does occur for the "bump" model and that it begins at approximately $r = 0.8$, which matches with the inside edge of the "bump". However no banding occurs for the delta function case and all widths are limited to less than $d e < 0.1$. The cause of this can be seen by examining the $-\delta w$ against $d e$ plot for this case shown in 4.14.

Figure 4.14: Plot $\delta w$ against $d e$ for 'delta' bootstrap configuration with $q_a = 4.46$, $dtq = 0.4$, $d t a = 0.005$, and $r_b = 0.9$.

This plot shows that there are large discontinuities at points in the $-dw$ plot which correspond to the outside and inside edges of the bootstrap distribution. This corresponds to the plasma becoming rapidly stable thus explaining the small $d e_{(max)}$ values in Figure 4.13. The presence of
these discontinuities is concerning as it implies a rapid change in stability for a small change in $d_e$. Some possible explanations for this are:

- In the extreme case of a very rapidly changing current profile the calculation intervals were two small to adequately map plasma behaviour.
- The tanh like function used to represent the bootstrap current was an unsuitable choice to model very steep current profiles
- The results are accurate and that the plasma is rapidly stabilised in regions where the current density changes rapidly.

However it should be noted that the current profile used in this case is extremely unusual and does not represent the current profile in an actual fusion machine. None of the other current profile’s investigated during the course of this research showed similar behaviour, all of which were more representative of the profiles found in experiments. Further study on this sort of unusual current distribution would be advised in any further work.

### 4.6 Analytical Study using Small ELM width expansions

In this section small value expansions will be used to find an analytical expression for ELM width in an analogous manner to that used in the non-bootstrap model [31]. We will imagine that our initial current profile containing the bootstrap is in an unstable state. For the purposes of simplifying the mathematics we will assume that the bootstrap current is distributed as a delta function at some point inside the plasma. There are thus two distinct cases to be investigated. For $r_e > r_b$ the bootstrap current does not affect the calculation at all. This means that the expression for ELM width will be the same as in the non-bootstrap case discussed in [31]. For $r_e < r_b$ the relaxed region encompasses the bootstrap current, and so any expansion of the current must include the delta function term. We will therefore begin with this situation. Firstly the initial toroidal current density $I_0$ is Taylor expanded inwards from $r = a$ to give

$$I_Z = I_a - x(aI_a') + \frac{x^2(a^2I_a'')}{2} + I_B\delta(r - r_b), \quad (4.6.1)$$

where $x$ is the normalised distance from the edge of the plasma given by $r = a(1 - x)$, $I_a$ is the current at $r = a$, $I_B$ is the magnitude of the bootstrap current, and $r_b$ is the position of the delta function. It can be shown that the poloidal magnetic field is related to this current by the expression
\[ rB_\theta = -a^2 \mu_0 \int (1-x) I z dx, \quad (4.6.2) \]

Therefore substituting in Equation 4.6.1 and using small value approximations for \( x \) we obtain

\[ B_\theta = -a \mu_0 \left( I_a \left( x + \frac{x^2}{2} - \frac{a x^2}{2} \right) \right) + |B_{\theta a} - a \mu_0 I_B (1 - x_B)| (1 + x + x^2), \quad (4.6.3) \]

where \( x_B \) is the distance of the bootstrap from the edge and \( B_{\theta a} \) is the poloidal magnetic field at \( r = a \). Using this equation it is possible to derive expressions for the helicity and total poloidal flux of the initial state

\[ \Psi_\theta = a [B_{\theta a} - a \mu_0 I_B (1 - x_B)] (d_e + \frac{d_e^2}{2} - a^2 \mu_0 I_a (\frac{d_e^2}{2} + \ldots) + \mu_0 (a^3 l_a') (\frac{d_e^3}{6} + \ldots), \quad (4.6.4) \]

\[ \frac{K}{a^4 B_0} = \left[ B_{\theta a} - \mu_0 I_B (1 - x_B) \right] (d_e^2 - \frac{d_e^3}{3} - \mu_0 I_a (\frac{d_e^3}{3} - \ldots) + \mu_0 (a l_a') (\frac{d_e^4}{12} + \ldots), \quad (4.6.5) \]

where \( d_e \) is the small normalised ELM width.

The relaxed \( q \) profile of the plasma is still given by \( q_f = \frac{r}{Cr + \frac{B}{\mu_0}} \) and therefore it can be shown that the relaxed helicity and poloidal flux are derived from this expression to be

\[ \Psi_\theta = a^2 \left[ C (d_e - \frac{d_e^2}{2} - \ldots) + D (d_e + \frac{d_e^2}{2} - \ldots) \right], \quad (4.6.6) \]

\[ \frac{K}{a^4 B_0} = \left[ C (d_e^2 - \frac{d_e^3}{3} - \ldots) + D (d_e - \frac{d_e^3}{3} + \ldots) \right], \quad (4.6.7) \]

We now define a new constant \( C_0 \) so that \( C_0 = C + D - \frac{B_{\theta a}}{a} - \mu_0 I_B (1 - x_B) \). Therefore having substituted in this expression, we now invoke the invariant nature of the helicity and poloidal flux and equate 4.6.4 to 4.6.6 and 4.6.5 to 4.6.7 to give the simultaneous equations

\[ (1 + \frac{d_e}{2} + \frac{d_e^2}{4}) C_0 - (d_e + \frac{d_e^2}{3}) C = -\mu_0 I_a \frac{d_e}{2} (1 + \frac{d_e}{3}) + \mu_0 (a l_a') (\frac{d_e^2}{6}), \quad (4.6.8) \]

\[ (1 - \frac{d_e}{3} + \frac{d_e^2}{12}) C_0 - (\frac{2d_e}{3} - \frac{d_e^2}{3}) C = -\mu_0 I_a \frac{d_e}{3} (1 - \frac{d_e}{2}) + \mu_0 (a l_a') (\frac{d_e^2}{12}), \quad (4.6.9) \]

Solving for \( C_0 \) we obtain
\[ C_0 = \frac{-\mu_0 (aI_a')}{12} d_e^2, \quad (4.6.10) \]

It should be noted that while this is the same expression obtained for \( C_0 \) in the non-bootstrap case, the definition of \( C_0 \) itself has changed with the inclusion of the delta function in the expanded current.

We now introduce the surface skin current \( \chi_a \) formed at the plasma edge during the relaxation process. \( \chi_a \) is defined by the edge magnetic fields on either side of the skin current such that

\[ \chi_a = \frac{R_0}{ab_0} (B_{\theta a+} - B_{\theta a-}), \quad (4.6.11) \]

where \( B_{\theta a+} = a(C + D) + a\mu_0 I_B(1 - x_B) - aC_0 \) and \( B_{\theta a-} = a(C + D) \). Therefore substituting 4.6.10 into 4.6.11 we obtain

\[ \chi_a = \frac{R_0}{ab_0} \left( a\mu_0 I_B(1 - x_B) + \frac{\mu_0 a^2 I_a' d_e^2}{12} \right), \quad (4.6.12) \]

This implies that \( I_B \) must be of order \( d_e^2 \) in order for this expansion to be valid. It can be shown (see [31], Appendix C2) that by looking at the ordering of the stability balance of the plasma, the largest ELM width will occur when

\[ \chi_a(\text{max}) = -\frac{l_a^2}{16 n}, \quad (4.6.13) \]

where \( l_a \) is the toroidal current density at \( r_a \) and \( n \) is the toroidal mode number of the peeling instability. Finally, we can equate 4.6.12 with 4.6.13 which leads to an expression for the maximum ELM width in relation to the toroidal current

\[ d_{e(\text{max})}^2 = \frac{-l_a^2}{16 n \left( I_B(1 - x_B) + \frac{aI_a'}{12} \right)}, \quad (4.6.14) \]

Therefore a general expression for any \( r_e \) in the presence of a delta function bootstrap is given by
This result shows that there will be a split in ELM widths depending on whether the delta function is reached by the relaxation process. This is very similar to the banding of results seen in Figure 4.8.

It is possible to see this splitting of ELM width graphically by using the numerical calculations from Section 4.3 to calculate the unknown terms in equation 4.6.15, and then plotting the resulting expressions for \( d_{\text{e}(\text{max})} \). For the bootstrap current density \( I_B \) this first requires that the bootstrap component of the total current density be isolated. This is done by subtracting the current density of the non-bootstrap case from the total current density derived from Equation 3.2.16. The resulting expression gives the magnitude of \( I_B \) as a function of \( x \). Therefore by setting \( x = x_B \) we have an approximation for the \( I_B \) in equation 4.6.15. Table 4.1 shows values for all terms in equation 4.6.15 for a given set of initial conditions.

\[
d_{\text{e}(\text{max})} = \begin{cases} 
\frac{-3I_B^2}{4n(aI_a')} & r_e > r_B \\
\frac{-I_a^2}{16n(I_B(1-x_B) + \frac{aI_a'}{12})} & r_e < r_B 
\end{cases}
\] (4.6.15)

The bootstrap current associated with these parameters is necessarily small as otherwise 4.6.14 will not be valid. Substituting from Table 4.1 into 4.6.15 and plotting gives two graphs; one where \( r_e > r_B \) and one where \( r_e < r_B \). These are plotted on the same axis below in Figure 4.15.
4.7 Comparison of Analytical ELM width Approximation with Numerical Results

In this section the results obtained in the previous section will be compared with the numerical results derived earlier in the report. This is done by substituting in the numbers from Table 4.1 into the numerical Mathematica program and producing a graph similar to Figure 4.8. The analytical plots in Figure 4.15 can then be superimposed onto the numerical results. This is shown below in Figure 4.16.

It can be seen that with a smaller bootstrap term the ELM widths are less than before and are thus more realistic in size, with the largest ELM covering approximately 40% of the plasma. It can also be seen that the analytical plots follow the banding of the results quite accurately for \( q_a < 5 \) but become less accurate at high \( q_a \). Due to the restriction in the analytical approximation to a single value of \( n \), the resonant-like behavior seen in the full numerical calculation is no longer present (The resonant behaviour is a product of the relation between \( \frac{1}{q_a} \) and \( \frac{n}{m} \), so for a single value of \( n \), and with no dependance on \( m \), the resonant behaviour disappears). It should also be remembered that the bootstrap current in the analytical approximation is by necessity of a different form to that in the full numerical calculation. Thus an exact match between the two
models is not expected, but the analytical plot does highlight qualitatively how the bifurcation of ELM widths occurs.

4.8 Implications of ELM width Bifurcation for Fusion Plasmas

It has been shown in this Chapter that the presence of a bootstrap current function in the edge region of the plasma has a pronounced influence on the ELM widths predicted by the relaxation model. Figure 4.8 shows that a large bootstrap current can generate ELMs of enormous size, while even a small bootstrap current can instigate the bifurcation process and produce large ELMs. These results have important implications for fusion plasmas as unmitigated ELMs are predicted to be potentially damaging on the future ITER tokamak (see Chapter 5 for more on this topic), and ITER is expected to have a large bootstrap fraction in certain modes of operation.

The model also predicts that if, as the relaxation process moves radially inwards, enough of the bootstrap current is encountered there is a sudden increase in ELM width. This is shown in the extreme case by the analytical approximation put forward in the previous sections, where the delta-function bootstrap distribution produces a discrete bifurcation of ELM widths. This would
suggest that in fusion plasmas it would be best to locate the peak of the bootstrap current as far from the plasma edge as possible to reduce ELM size. It is also seen from Figure 4.13 that a wide bootstrap distribution is more readily encountered by the relaxation process, thus generating larger ELM widths than would be expected for the total current carried in the bootstrap. Thus the predictions made by the relaxation model suggest that wider bootstrap distributions should be avoided in fusion plasmas.

However it must be stressed that the plasma model in this thesis is greatly simplified compared to one found in a tokamak, lacking toroidal geometry, impurities, x-points etc. so precise predictions are not expected.
Chapter 5

ELM Mitigation

In this chapter the Taylor relaxation theory of ELMs will be used to examine the field of ELM mitigation. The work contained in this chapter has in part been presented in [24] and [26], these papers having been co-authored by the author of this thesis.

In section 5.1 of this chapter the need for ELM mitigation will be addressed, along with the potential methods of mitigation currently being studied, while Section 5.2 will look in more detail at the Resonant Magnetic Perturbation (RMP) field method of ELM control. Section 5.3 will then present ELM mitigation results from the JET tokamak which are not compatible with standard mitigation theories. In Section 5.4 the relaxation model of ELMs is used to analyse these unexpected findings. The role of the author of this thesis was to extend the relaxation model to provide two different methods of explaining the mitigation results from JET. In the first method, ELMs with mode numbers matching those of the RMP fields are removed, and in the second method the effects of varying the target plasma’s edge current were examined. While both of these methods provide possible explanations for the experimental mitigation results, the case is made in this thesis that the second method is more plausible.

5.1 ITER and the need for ELM mitigation

ITER is a substantially larger machine than previous Tokamaks, with a minor radius twice that of the next largest experiment, JET. It is hoped that ITER will run in a low collisionality H mode regime, one that necessarily has a large edge pedestal. This means that the amount of energy stored in the pedestal will be large and thus the amount of energy available for ELMs is also large. In ITER if a Type I ELM released 10-20% of the pedestal energy (a typical value for a Type I ELM), then around 20MJ of energy would be ejected from the edge of the plasma [11], with most of it arriving at the divertor, over a time period of less than a millisecond. This is
obviously much higher than the 0.66MJ per ELM limit that the divertor has been rated to if the divertor’s operational life is not to be compromised [12].

Ensuring uncontrolled Type I ELMs do not occur in ITER is thus of great importance. To this end three broad paths for ELM control are open to researchers[16].

**ELM Free Modes of Operation**

Whilst this is conceptually the easiest method of controlling ELMs in practice all of the viable, known ELM-free H modes are unsuitable for use on ITER due to poor performance characteristics. Research is still ongoing into this possibility but, currently, an ’ELMy’ H-mode is still the preferred mode of operation for ITER.

**ELM Pacing**

In ELM pacing ELMs are triggered prematurely by diverse methods, meaning that the ELMs produced are smaller and more frequent than normal. Methods used to trigger the ELMs include pellet injection and displacing the plasma upwards to give ’vertical kicks’. Some progress is being made on this front and ITER will be built with the capability to use pellets as a trigger mechanism, however, a full description of ELM pacing is beyond the scope of this report.

**ELM suppression or Mitigation using Resonant Magnetic Perturbations (RMPs)**

This method of ELM control uses externally generated magnetic fields to alter the properties of the edge region in such a way as to either completely remove ELMs (suppression) or to reduce their amplitudes to manageable levels (mitigation). The theoretical understanding of how this happens is still developing, and will be the focus of this section.

**5.2 DIII-D and Edge Ergodic Regions**

The initial theory of ELM suppression was based around generating an ergodic (stochastic) region at the edge of the plasma [34]. It was hoped that this ergodic region would increase the electron thermal transport in the H Mode pedestal (thus reducing the temperature profile of the pedestal), leading in turn to a decrease in the edge pressure. This pressure reduction would reduce the pressure gradients in the pedestal to such an extent that the ballooning stability limit would not be reached. In the peeling-ballooning model of ELMs, ensuring that the ballooning stability limit was not reached would therefore stop ELMs from occurring entirely [42].

To generate this stochastic layer, an external magnetic field would be used to to perturb the plasma in such a way that magnetic islands would form in the edge region of the target plasma. If these islands overlapped to a great enough extent then the edge region could be considered
ergodic. A measure of the degree of island overlap is given by $\sigma$, the Chirikov parameter. The Chirikov parameter describes the level of interaction between magnetic islands that form in the plasma when the resistivity $\eta \neq 0$. When these islands interact and overlap with each other to a sufficient extent the magnetic field becomes ergodic. Generally a value of $\sigma$ above 1 is required for a region to be considered ergodic [6] and, if this is the case, there will therefore be a high degree of magnetic island overlap.

The DIII-D Tokamak was one of the experiments used to test the effectiveness of an ergodic edge in suppressing ELMs. DIII-D has an internal set of electromagnetic coils which were tuned to produce a $n=3$ resonant magnetic field. The Tokamak was then run in several operating regimes with the coils active.

The initial set of experiments used a plasma with a high collisionality, $\nu^*$[20]. Partial edge ergodisation was achieved and Type I ELMs were largely suppressed while the coils were active. However a few Type I ELMs still occurred, although at widely space intervals, and periodic edge disturbances appeared in place of the ELMs. These disturbances consisted of a cluster of small ELM like events at periodic intervals. The amount of energy deposition due to these events was much lower than had been the case with the ELMs they replaced. The most surprising result was that the confinement characteristics of the pedestal region were largely unchanged by the activation of the coils. The pressure, temperature, and density profiles remained near normal during periods of ELM suppression, contrary to what was expected. The lack of any significant increase in thermal transport was at odds with the theory behind ELM suppression.

The next series of experiments were undertaken with a low $\nu^*$ plasma [21], much closer to the expected mode of operation in ITER. A different perturbation field was generated in the internal coils leading to a greater degree of ergodisation in the pedestal. Operating the plasma with this coil configuration active resulted in a complete suppression of ELMs. In contrast to the high $\nu^*$ case the ELMs were not replaced with any other disturbances, leading to a quiescent pedestal. Also in contrast to the high $\nu^*$ case, the density profile of the pedestal was substantially lowered by the activation of the coils, leading to a reduction of around 40% in the edge pressure. This increase in particle transport and ensuing drop in pressure has been labeled the ’pump out’ effect. However, and again contrary to theory, the thermal transport of the edge remained almost unchanged, showing no increase in thermal transport.

One of the most interesting discoveries in relation to this thesis was the strong dependency of the ELM suppression effect on the edge $q$ value (due to uncertainties in measuring the exact $q$ values at the edge of a plasma with a separatrix, the $q$ value at a point that encloses 95% of the plasma flux surfaces, $q_{95}$, is used instead). There was strong resonant behaviour so that ELMs were only suppressed when $q_{95}$ passed a fixed ratio of $m/n$ of 11/3. This is the point at which the RMP field becomes resonant with the magnetic field at the edge of the plasma.
These results from DIII-D showed a promising path forwards when considering the suppression of ELMs in ITER. However, there are some complications to overcome. Firstly the 'pump out' effect reduces the plasma density considerably, and while methods of counteracting this effect such as fuel pellet injection [22] and gas puffing [23] are being explored, they bring with them their own set of difficulties. Secondly there has been a failure to completely suppress ELMs in the relevant ITER regime, low $\nu^*$, on other machines (Asdex Upgrade has achieved complete ELM removal but in high $\nu^*$ plasmas [46]). For other machines, such as JET [25] and MAST [10], the ELMs have been mitigated rather than suppressed, meaning ELM amplitudes have been reduced but with corresponding increases in ELM frequencies. We will now look in greater detail at results from JET and explain the possible insight the Taylor relaxation model of ELMs gives on those results.

5.3 ELM mitigation experiments on JET

The JET fusion device is the largest current Tokamak and as such is of great value when examining the predicted behaviour of ELMs in ITER. JET is fitted with sets of external coils designed to correct the error field instability. These Error Field Correction Coils (EFCCs) can be calibrated to produce $n = 1$ or $n = 2$ harmonic perturbation fields. After the success of suppressing ELMs with RMPs in DIII-D, the $n=1,2$ coils were used to investigate the effect of RMP fields on the JET plasma. The details of these experiments are detailed in [25][24] [26] but a brief summary follows.

5.3.1 ELM Mitigation Results from JET

1. The level of ergodisation did not affect mitigation, as ELM mitigation was seen with relatively weak edge ergodisation ($\sigma < 1$), as well as for strong ergodisation ($\sigma > 1$)

2. Neither $n = 1$ or $n = 2$ fields were successful in completely suppressing ELMs. ELMs were instead reduced from $\sim 7\%$ of the total stored energy to less than $2\%$ (the resolution limit of the relevant sensors). Elm frequencies increased from $\sim 30\text{hz}$ to $\sim 120\text{hz}$. However, there was a strong dependance on $q_{95}$ with resonant areas of the $q_{95}$ spectrum showing much higher mitigation effects than non-resonant areas, whose ELM frequency ($f_{elm}$) increased by only a factor of 2. This behaviour has been labeled the 'multi-resonant effect' and suggests two distinct forms of mitigation; one weak global effect and one much stronger, local effect at some values of $q_{95}$.

3. The density pump out effect was again observed, occurring with a more noticeable effect in resonant areas of the $q_{95}$ range. There was again no change in the thermal profiles of the mitigated plasmas.
5.4 Analysing JET results with Relaxation Theory of ELMs

The results from JET do not correlate well with the theory of edge ergodisation. Here we will extend the edge relaxation model of ELMs to possibly better account for the results seen in JET. However, before examining the resonant effects of the RMP fields, the non resonant effect present across all \( q_{95} \) values must be discussed. The pressure and density loss from this effect is significant. It is possible that the partial ergodisation of the edge is responsible for this reduced pump out effect, in a manner similar to that seen on DIII-D. This effect will not be examined further here, as the lack of resonant behaviour suggests it is unrelated to the work described here.

Looking at the resonant effects seen in JET, Figure 5.1 (taken from [26]) shows the spikes in ELM frequency as \( q_{95} \) is varied. The multi-resonant effect, the pattern of repeated peaks in ELM frequency that occurs with the RMP fields active, is clearly visible in these figures. As noted in Section 5.3.1, this multi-resonant effect does not have any relationship with the Chirikov parameter, and therefore is not compatible with edge ergodisation model of mitigation. However, the resonant effects seen do bear a strong resemblance to the predictions of the relaxation model of ELMs (for example see Figure 4.3), if the assumption that the ELM width \( (d_e) \) has an inverse relationship with the ELM frequency \( (f_{ELM}) \) is made. This is a reasonable assumption, as once a relaxed post-ELM state is created the edge plasma then returns to its pre-ELM state via diffusion. In this case the time taken for the plasma to reset would scale with the square of the length scale of the relaxed region (i.e. \( d_e^2 \)) and thus the ELM frequency scales as \( \frac{1}{d_e^2} \). Therefore the experimental results appear to be in general agreement with the model.

The application of the RMP fields does not appear to remove the ELMs, instead it increases their frequency through the reduction in ELM width. However, the ELM frequency does not show multi-resonant behaviour when the RMP fields are not active (as seen by the black crossed data points in Figures 5.1 and 5.2). It will be shown that the relaxation model of ELMs can offer explanations for the decreases in ELM width that occur when the RMP coils are active, as well as explaining the simultaneous enhancement of the multi-resonant effect.

In fact, the model provided two alternative interpretations of the experimental data. The first explanation considered was that of edge mode saturation, described in section 5.4.1, after which a more robust scenario was advanced, where the ELM behaviour was linked to a reduction in edge current. This second interpretation is described in Section 5.4.2, and is the explanation presented in [24] and [26].
Figure 5.1: Figure (taken from [26] showing a composite plot of ELM frequency, $f_{ELM}$, against $q_{95}$ for a series of H-mode shots on the JET Tokamak with an $n = 1$ field (dots) and without the field (crosses).

Figure 5.2: Figure (taken from [26] showing a composite plot of ELM frequency, $f_{ELM}$, against $q_{95}$ for a series of H-mode shots on the JET Tokamak with an $n = 2$ field (dots) and without the field (crosses).
5.4.1 Edge mode saturation

Here we examine the idea that an RMP field with a given toroidal mode number \( n \) saturates instabilities with that mode number. This is done by repeating the analysis set out in Chapters 3 and 4 but with the removal of the relevant toroidal modes from the calculation of ELM width \( (d_e) \). It is now argued that if the field is already perturbed by the RMP, this mode is effectively saturated in the relaxed equilibrium state. There is now no further free energy available for perturbations with this mode number, therefore instabilities with that mode number are removed from the calculation. The \( q \) profile used for these calculations did not incorporate the bootstrap current but instead used the same \( q \) profile as in [31]: \( q = q_0 + (q_a - q_0) r^2 \).

It has been shown previously that the size of the ELM predicted by the relaxation model corresponds to the combination of \( n \) and \( m \) that produces the largest value of \( d_e \), i.e. the pair of \( (n, m) \) where the plasma relaxes inwards the most before becoming stable again. We will label this pair the ‘Primary Pair’. Scanning across a range of \( q \) values, the majority of ELMs are caused by lower \( n \) numbers, with \( n = 1 \rightarrow 3 \) being the cause for most of the largest values of \( d_e \). Figure 4.9 is a good illustration of how ELM size diminishes rapidly beyond the initial unstable \( (n, m) \) pair. It is now suggested that if an external RMP field is introduced it might saturate the corresponding \( n \) mode in the plasma and render it incapable of generating ELMs. This would then leave the next most unstable pair to initiate the ELM, which as seen in Figure 4.9, is substantially lower in magnitude than the primary pair. Also with the removal of the primary pair, ELM size becomes more variable as the higher values of \( (n, m) \) found in the secondary pair dominate smaller sections of the \( q_a \) spectrum than the primary pairs. This can be seen in Figure 5.3 where the ELM widths are shown for a range of \( q_a \) in which several values of \( n \) have been saturated.

As mentioned above, ELM frequency may be used as a proxy for the inverse of the ELM width \( (f_{ELM} \text{ is proportional to } \frac{1}{d_e^2}) \). We may then plot \( f_{ELM} \) against \( q_a \) with several mode numbers saturated. The sharp dips in Figure 5.3 are then translated to peaks in 5.4, with qualitative similarity to Figures 5.1 and 5.2.

This process of mode ‘removal’ does give the increase in ELM-size and \( f_{ELM} \) complexity across the \( q_a \) range seen in results from JET in Figures 5.1 and 5.2. However there are several problems with this model. Firstly the complete ELM suppression seen on DIII-D can not be fully explained by mode removal. Even with low \( n \) mode removal at no point in figure 5.3 are ELMs completely removed. Secondly, in relation to the JET data, results show that ELM mitigation occurs across the entire \( q_{95} \) spectrum, for both \( n = 1 \) and \( n = 2 \) [26]. If mode saturation is the cause of this effect then it would be expected that parts of the \( q_{95} \) spectrum would show no mitigation and that these areas would change depending on the mode number suppressed. It would also predict that \( n = 1 \) RMP fields would be much more effective over a larger part of the \( q_{95} \) spectrum than an \( n = 2 \) field; again this is not seen in the data from JET. It is possible that these
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Figure 5.3: Graph of ELM width ($d_{e\text{ (max)}}$) against $q_a$ with no saturation (black points), n=1 saturation (red points), n=2 saturation (blue points), and n=3 saturation (green points).

Figure 5.4: Graph of ELM frequency ($f_{\text{ELM}}$ in arbitrary units) against $q_a$ with no saturation (black points), n=1 saturation (red points), n=2 saturation (blue points), and n=3 saturation (green points).
discrepancies are due to the variation between true $q_a$ in the theory and $q_{95}$ in the experimental results. However a more promising explanation is found in the effect RMP fields might have on the edge current density of the plasma.

### 5.4.2 Edge Current Density Reduction

It was shown in Section 4.1.1 that the range of $q_a$ where any individual mode combination (an $n/m$ pair) is unstable increases as the edge current density $I_a$ increases, according to Equation 4.1.4. As the resonances seen in the JET ELM data were also dependent on $q_{95}$, we will now consider whether the dependence on $I_a$ seen mathematically in 4.1.4 and visually in Figures 4.2 and 4.3 is linked to the RMP ELM mitigation effect. A physical justification of this postulation is that RMP mitigation decreases the edge density gradients via the ‘pump out’ effect. It was shown that in Chapter 2 that several of the currents in the edge region have a dependence on density gradients, therefore it is suggested that this reduction in edge density gradients would lower the edge current, and thus reduce ELM size.

This is done using the standard non-bootstrap form of the safety profile ($q = q_0 + (q_a - q_0)r^2$), but varying $q_0$ in line with $q_a$ so that the edge current density $I_a$ is fixed in relation to the current density at $r = 0$. Plotting $f_{ELM}$ against $q_a$ for these fixed ratios of $I_a/I_{r=0}$ shows that a higher normalised edge current reduces the number of frequency peaks in the $q_a$ range. This plot can be seen in Figure 5.5.

![Figure 5.5: Plot (taken from [26]) of ELM frequency ($f_{ELM}$ in arbitrary units) against $q_a$ for two different values of normalized edge current densities $I_a/I_{r=0}$; $I_a/I_{r=0} = 0.4$ in Green and $I_a/I_{r=0} = 0.075$ in Blue](Image)

This behaviour is explained by Figure 5.6 where it can be seen that the primary toroidal number $n$ (the mode number that generates the largest ELM) has only a few values for most of the $q_a$ spectrum when $I_a/I_{r=0}$ is large ($n = 1$ and $n = 2$ are the primary mode numbers for 80% of the region $q_a = 4$ to $q_a = 5$). For lower $I_a/I_{r=0}$ there are more varied primary mode numbers, with higher mode numbers correlating to higher frequencies (and thus lower ELM widths).
Chapter 5. ELM Mitigation

Figure 5.6: Plot (taken from [26]) of the primary toroidal mode number $n$ against $q_a$ for two different values of normalized edge current densities $I_a/I_r=0$; $I_a/I_r=0.4$ in Green and $I_a/I_r=0$ in Blue.

The plots of $q_{95}$ vs $f_{ELM}$ in Figures 5.1 and 5.2 show favourable agreement with Figure 5.5, with the high $I_a/I_r=0$ result showing similar behaviour to JET shots where there were no RMP fields and the low $I_a/I_r=0$ case showing similar behaviour to the mitigated RMP shots. The hypothesis derived from these results is that RMP fields lower the edge current density in the target plasma, causing variation in the primary mode number $n$ and thus increasing $f_{ELM}$. Compared to the mode saturation hypothesis this explanation is attractive because it does not depend on any specific $n$ value for the RMP coils, and explains ELM mitigation occurring across the range of $q_{95}$ rather than occurring exclusively in those parts of the $q_a$ spectrum whose primary mode number matches the harmonic of the RMP coils.
6.1 Conclusions

In this Thesis the Taylor Relaxation Theory of ELMs developed in [31] has been extended to cover a plasma where the bootstrap current [41] is present in the plasma edge. The presence of this current has a profound impact on the behavior of ELMs in this model. If the plasma relaxes far enough inwards to encounter a sizable portion of the bootstrap current, large increases in ELM width can occur, the magnitude of this increase depending heavily on the proportions of the bootstrap distribution. If the bootstrap current is large and close to the edge of the plasma, the relaxation process is not halted until a significant proportion of the plasma has been affected. A bifurcation of ELM widths is thus observed; small ELM widths occur when either the relaxation process doesn’t proceed far enough inwards to reach the bootstrap current or if the bootstrap current is small, and large ELM widths occur when a large amount of the bootstrap current is encountered during the relaxation process. This bifurcation was further examined with an analytical model that used a $\delta$-function to represent the bootstrap current. This model showed that a bifurcation of elms widths would occur, with the size of the elms dependent on whether the $\delta$ function was reached by the relaxation process. The results from this analytical model correlated well with numerical calculation using a more realistic bootstrap distribution, as long as the bootstrap current was small.

It is tempting to draw a correlation between this bifurcation of ELM widths and the differences between Type-I and Type-III ELMs, especially as Type-I ELMs are encountered during regimes with large H-mode pedestals and thus large bootstrap currents. However without extending the model to include the effects of ballooning modes, this link would be highly speculative.

The next extension of the relaxation model of elms was to look at elm mitigation using RMP fields. This model provided good insight into how RMP mitigation might work, with two plausible explanations for decreases in elm width in the presence of a perturbing magnetic field;
mode suppression and edge current density reduction. Of these edge current density reduction provided qualitative agreement with results from JET.

### 6.2 Further Work

There are several avenues for further work that arise from this thesis. Firstly the relaxation model could be extended to include ballooning modes as well as peeling modes. This would involve incorporating pressure gradients. However, it has been assumed in this thesis, and in the preceding work presented in [31] and [47], that the initiation of Taylor relaxation would remove any edge pressure gradients. Therefore two avenues for incorporating pressure gradients present themselves.

- **Having the initial trigger for Taylor relaxation be either a peeling mode (as in this paper) or a ballooning mode.** The instability would start the relaxation process, removing the pressure gradients and in doing so removing the need to account for plasma curvature (and thus the need for toroidal geometry). This would mean that while the final limit of the relaxation process would still be determined by the balance of destabilising edge current and stabilising current sheets, the initial plasma configuration would not necessarily have to be peeling unstable. The advantage of this option would be that the existing stability calculation process would still be valid, allowing results to be obtained comparatively easily.

- **Having none-zero pressure gradients in the relaxed region.** This would mean that the stability calculations would have to incorporate pressure gradients and be undertaken in toroidal geometry, as well as altering the process of calculating the Taylor relaxed state to include plasma pressure. This would be an extremely difficult (if not impossible) task, and would required a complete reformulation of the methodology described in this thesis.

Secondly full MHD analyses of the relaxation process could be attempted, using one of the codes developed for stability analysis such as Elite. Thirdly the effect of the bootstrap current on ELM mitigation would be important to examine as the presence of a potentially large current in the edge region would have important repercussions for ELM mitigation.
Appendix A

Derivation of Equations 3.2.10 to 3.2.13

Starting with

\[ F_v = \frac{mB_0}{R_0} \Delta \alpha, \quad (A.0.1) \]

and rewriting in terms of \( q_v \), the safety factor in the vacuum, we have

\[ F_v = \frac{mB_0}{R_0} \left( \frac{1}{q_v} - \frac{n}{m} \right), \quad (A.0.2) \]

From the definition of the \( q \) factor we can write

\[ q_v = \frac{rB_\phi}{R_0B_\theta}, \quad (A.0.3) \]

However, as we are evaluating \( q \) in a vacuum we can write

\[ B_\theta = \frac{C}{r}, \quad (A.0.4) \]

where \( C \) is a constant. Therefore \( A.0.3 \) becomes

\[ q_v = \frac{r^2B_\phi}{R_0C}, \quad (A.0.5) \]

\[ \Rightarrow \]

\[ \frac{dq_v}{dr} = \frac{2rB_\phi}{R_0C} = \frac{2q_v}{r}. \quad (A.0.6) \]
As \( q_v \) is evaluated at the plasma/vacuum boundary, \( r = a \) so A.0.5 and A.0.6 become

\[
q_v = \frac{a^2 B_\phi}{R_0 C},
\]

(A.0.7)

\[
\frac{dq_v}{dr} = \frac{2q_v}{a}.
\]

(A.0.8)

We can now differentiate \( F_V \) with respect to \( r \) using the chain rule to obtain \( F'_v \) in terms of \( q_v \)

\[
F'_v = -\frac{mB_0}{R_0} \frac{1}{q_v^2} \frac{2q_v}{r} = -\frac{2mB_0}{R_0 a} \frac{1}{q_v},
\]

(A.0.9)

Next we evaluate \( F \) on the inside of the plasma/vacuum boundary. We start with

\[
F_P = \frac{mB_0}{R_0} (\Delta_{\alpha} - \chi_a),
\]

(A.0.10)

Rewriting in terms of \( q_p \) results in

\[
F_P = \frac{mB_0}{R_0} \left( \frac{1}{q_p} - \frac{n}{m} \right),
\]

(A.0.11)

Therefore differentiating with respect to \( r \)

\[
F'_P = -\frac{mB_0}{R_0} \frac{1}{q_p^2} \frac{dq_p}{dr},
\]

(A.0.12)

It can be shown that at \( r = a \)

\[
\frac{dq_p}{dr} = \frac{2q_p}{a} - \frac{\mu_0 R_0}{a B_0} I_d q_p^2,
\]

(A.0.13)

Therefore the differential of \( F_P \) becomes

\[
F'_P = -\frac{mB_0}{R_0 a} \left( \frac{2}{q_p} - \frac{\mu_0 R_0 I_d}{B_0} \right),
\]

(A.0.14)
Appendix B

Derivation of Relaxed Region Boundary Condition

Taking the unrelaxed plasma to have the subscript \( v \) and the relaxed plasma \( p \), we start with

\[
\left[F_{\psi'} - \frac{F}{r_e} - \psi F'\right]_v^p = 0 \quad (B.0.1)
\]

Expanding B.0.1

\[
F_p \psi'_p - F_v \psi'_v - \frac{F_p \psi_p}{r_e} + \frac{F_v \psi_v}{r_e} - \psi_p F'_p + \psi_v F'_v = 0 \quad (B.0.2)
\]

\[
\frac{1}{\psi'_v} - \frac{1}{\psi'_p} - \frac{F_p}{F_p r_e} + \frac{F_v}{F_v r_e} - \frac{F_p F'_v}{F_p} + \frac{F_v F'_p}{F_v} = 0 \quad (B.0.3)
\]

Using \( \psi_v F_p = \psi_p F_v \) we obtain

\[
\frac{F_p}{F_v} - \frac{F_p}{F_p} - \frac{F_p}{F_p r_e} + \frac{F_v}{F_v r_e} - \frac{F_p}{F_p} + \frac{F_v}{F_v} = 0 \quad (B.0.4)
\]

We now need to collect together expressions for all terms in equation B.0.4. We start with the definitions stated in Chapter 3

\[
F_v = m \frac{B_0}{R_0} \Delta E_-
\]
Appendix B. Derivation of Relaxed Region Boundary Condition

\[ K_E = \left[ \left[ \frac{1}{q} \right] \right]^p_v = \frac{1}{q_p} - \frac{1}{q_v} \]  
\hspace{2cm} (B.0.6)

\[ F_p = m \frac{B_0}{R_0} \left( K_E + \frac{1}{q_v} - \frac{n}{m} \right) \]  
\hspace{2cm} (B.0.7)

\[ I = \frac{1}{r} \frac{d}{dr} \left( \frac{r^2}{q} \right) = \frac{2}{q} - \frac{r}{q^2} q' \]  
\hspace{2cm} (B.0.8)

From B.0.8 we can write

\[ \frac{d}{dr} q = \frac{1}{r} \left( 2q - Iq^2 \right) \]  
\hspace{2cm} (B.0.9)

We also have the derivative of subscripts at some point \( i \)

\[ \frac{dF_i}{dr} = -B_0 \frac{m}{R_0 q_i^2} \frac{1}{r} \left( 2q_i^2 - Iq_i^2 \right) \]  
\hspace{2cm} (B.0.10)

Substituting B.0.9 into B.0.10 obtains

\[ \frac{dF_i}{dr} = -m \frac{B_0 m}{R_0 q_i^2} \frac{1}{r} \left( 2q_i^2 - Iq_i^2 \right) \]  
\hspace{2cm} (B.0.11)

So for \( F_v \),

\[ F'_v = -m \frac{B_0}{R_0 q_v^2} \frac{1}{r} \left( 2q_v - Iq_v^2 \right) = -\frac{B_0 m}{R_0 q^2} \left( \frac{2}{q_v} - I_{E_v} \right) \]  
\hspace{2cm} (B.0.12)

and similarly for \( F_p \),

\[ F'_p = -\frac{B_0 m}{R_0 q^2} \left( \frac{2}{q_p} - I_{E_p} \right) \]  
\hspace{2cm} (B.0.13)

We next define the term \( \Delta'_{E_v} \) so that

\[ \Delta'_{E_v} = \left[ \left[ \frac{\psi'}{r} \frac{\psi}{\psi} \right] \right]^p_v = re \left( \frac{\psi'_p}{\psi_p} - \frac{\psi'_v}{\psi_v} \right) \]  
\hspace{2cm} (B.0.14)

As \( \psi_v = r^{-m} \) we write
\[
\frac{\psi_v'}{\psi_v} = \frac{m}{re} \quad (B.0.15)
\]

and

\[
\frac{\psi_p'}{\psi_p} = \frac{\Delta_E'}{re} + \frac{m}{re} \quad (B.0.16)
\]

We now substitute expressions for \(F_v (B.0.5)\), \(F_p (B.0.7)\), \(F_v' (B.0.12)\), \(F_p' (B.0.13)\), \(\frac{\psi_v'}{\psi_v} (B.0.15)\) and \(\frac{\psi_p'}{\psi_p} (B.0.16)\) into B.0.4. Performing these substitutions and multiplying through by \(re\) gives

\[
\left( \frac{K_E + \frac{1}{q_v} - \frac{n}{m}}{\Delta_E} \right) (\Delta_E' + m - 1) + \left( \frac{\Delta_E^-}{K_E + \frac{1}{q_v} - \frac{n}{m}} \right) (1 - m)
\]

\[
+ \left( \frac{2}{q_p} - I_{E+} \right) (K_E + \Delta_E^-) - \left( \frac{2}{q_v} - I_{E-} \right) \Delta_E = 0 
\]

(B.0.17)

Multiplying by \(\Delta_E^-(K_E + \Delta_E^-)\) and simplifying

\[
(K_E^2 + 2K_E\Delta_E^- + \Delta_E^-)^2 (\Delta_E' + m - 1) + \Delta_E^- (1 - m)
\]

\[
+ \left( \frac{2}{q_p} - I_{E+} \right) (K_E + \Delta_E^-) - \left( \frac{2}{q_v} - I_{E-} \right) \Delta_E = 0 
\]

(B.0.18)

Inputting the following expressions

\[
\frac{1}{q_v} = \Delta_E^- + \frac{n}{m} \quad (B.0.19)
\]

\[
\frac{1}{q_p} = K_E + \Delta_E^- + \frac{n}{m} \quad (B.0.20)
\]

into B.0.18 results in

\[
(K_E^2 + 2K_E\Delta_E^- + \Delta_E^-)(\Delta_E' + m - 1) + \Delta_E^- (1 - m)
\]

\[
+ 2(K_E + \Delta_E^- + \frac{n}{m} - I_{E+})(K_E + \Delta_E^-) - (2\Delta_E^- + \frac{n}{m} - I_{E-})\Delta_E^- = 0 
\]

(B.0.21)

Expanding B.0.21 gives


$$K_E^2(\Delta'_E + m - 1) + 2K_E\Delta_{E-}(\Delta'_E + m - 1) + \Delta_{E-}^2\Delta'_E + 2K_E^2$$

\[ + 4K_E\Delta_{E-} + \left(\frac{2n}{m} - I_{E+}\right)(K_E + \Delta_{E-}) - \frac{2n}{m}\Delta_{E-} + I_{E-}\Delta_{E-} = 0 \quad (B.0.22) \]

Cancelling and simplifying gives

\[ (K_E^2 + 2K_E\Delta_{E-})(\Delta'_E + m + 1) + \Delta_{E-}^2\Delta'_E + \frac{2nK_E}{m} - I_{E+}(K_E + \Delta_{E-} + I_{E-}\Delta_{E-}) = 0 \quad (B.0.23) \]

After a final round of simplification we obtain the boundary condition at the inner relaxed radius

\[ K_E(\Delta_{E+} + m + 1) + \frac{2n}{m} - I_{E+}\) + \Delta_{E-} \left(\Delta_{E-}\Delta'_E - I_{E+} + I_{E-}\right) = 0 \quad (B.0.24) \]
Bibliography


