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New conditions for independence of events

by

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Abstract: It is well known that two events A and B are independent if and only if $P(AB) = P(A)P(B)$. Here we derive a condition for independence not well known. We then extend the condition for independence of n events.

Keywords: Complement, Generalization, Intersection

1 Introduction

Suppose A and B are two events. Standard undergraduate textbooks (for example, Bertsekas and Tsitsiklis (2002)) give the following conditions for independence A and B : $P(AB) = P(A)P(B)$, where AB denotes the intersection of A and B ; $P(A | B) = P(A)$, where $P(A | B)$ denotes the probability of A conditioned on B ; $P(B | A) = P(B)$, where $P(B | A)$ denotes the probability of B conditioned on A .

A condition for independence not well known is the following. We denote the complement of A by $\overline{A}$.

Proposition 1 A and B are independent if and only if

$$P(\overline{AB}) P(\overline{BA}) = P(AB) \{1 - P(B) - P(\overline{AB})\} \quad (1)$$

or equivalently

$$P(\overline{AB}) P(\overline{BA}) = P(AB)P(\overline{AB}). \quad (2)$$

Proof: Note that

$$\begin{aligned} & P(A)P(B) = P(AB) \\ \iff & [P(AB) + P(A) - P(AB)] [P(AB) + P(B) - P(AB)] = P(AB) \\ \iff & P(AB)P(B) + [P(A) - P(AB)] [P(AB) + P(B) - P(AB)] = P(AB) \\ \iff & P(AB)P(B) + P(AB) [P(A) - P(AB)] + [P(A) - P(AB)] [P(B) - P(AB)] = P(AB) \\ \iff & [P(A) - P(AB)] [P(B) - P(AB)] = P(AB) \{1 - P(B) - [P(A) - P(AB)]\} \\ \iff & P(\overline{AB}) P(\overline{BA}) = P(AB) \{1 - P(B) - P(\overline{AB})\}. \end{aligned}$$

The result follows.

The condition (2) has been noted before in the last four lines in Section 2 of Joarder and Al-Sabah (2002). The condition (1) appears to be new.

The aim of this note is to generalize Proposition 1. Section 2 generalizes it for n events. Section 3 notes some particular cases of the result of Section 2.

2 Main result

Suppose there are n events. For simplicity of notation, let $P(i)$, $i = 1, 2, \dots, n$ denote the probability that the i th event occurs. Let $V = P(123 \dots n)$ denote the probability of the intersection of the n events. Let $\tilde{P}(i) = P(i) - V$ for $i = 1, 2, \dots, n$.

Theorem 1 generalizes Proposition 1 for n events.

Theorem 1 *With the notation as above, the n events are independent if and only if*

$$\begin{aligned} \tilde{P}(1)\tilde{P}(2) \dots \tilde{P}(n) = V \Big\{ & 1 - P(2)P(3) \dots P(n) - \tilde{P}(1)P(3)P(4) \dots P(n) \\ & - \tilde{P}(1)\tilde{P}(2)P(4)P(5) \dots P(n) - \dots - \tilde{P}(1)\tilde{P}(2) \dots \tilde{P}(n-1) \Big\}. \end{aligned}$$

Proof: Note that

$$\begin{aligned} & P(1)P(2)P(3) \dots P(n) = V \\ \iff & [V + P(1) - V][V + P(2) - V] \dots [V + P(n) - V] = V \\ \iff & VP(2)P(3) \dots P(n) + [P(1) - V][V + P(2) - V]P(3)P(4) \dots P(n) = V \\ \iff & VP(2)P(3) \dots P(n) + V[P(1) - V]P(3)P(4) \dots P(n) \\ & + [P(1) - V][P(2) - V][V + P(3) - V]P(4)P(5) \dots P(n) = V \\ & \vdots \\ & \vdots \\ \iff & VP(2)P(3) \dots P(n) + V[P(1) - V]P(3)P(4) \dots P(n) \\ & + V[P(1) - V][P(2) - V]P(4)P(5) \dots P(n) + \dots \\ & + V[P(1) - V][P(2) - V] \dots [P(n-2) - V]P(n) \\ & + [P(1) - V][P(2) - V] \dots [V + P(n) - V] = V \\ \iff & VP(2)P(3) \dots P(n) + V[P(1) - V]P(3)P(4) \dots P(n) \\ & + V[P(1) - V][P(2) - V]P(4)P(5) \dots P(n) + \dots \\ & + V[P(1) - V][P(2) - V] \dots [P(n-2) - V]P(n) \\ & + V[P(1) - V][P(2) - V] \dots [P(n-1) - V] \\ & + [P(1) - V][P(2) - V] \dots [P(n) - V] = V \\ \iff & [P(1) - V][P(2) - V] \dots [P(n) - V] = V \Big\{ 1 - P(2)P(3) \dots P(n) \\ & - [P(1) - V]P(3)P(4) \dots P(n) - [P(1) - V][P(2) - V]P(4)P(5) \dots P(n) - \dots \\ & - [P(1) - V][P(2) - V] \dots [P(n-1) - V] \Big\}. \end{aligned}$$

Hence the result.

3 Particular cases

Here, we state particular cases of Theorem 1 for $n = 2, 3, 4$. When $n = 2$, Theorem 1 reduces to

$$\tilde{P}(1)\tilde{P}(2) = P(12) \left\{ 1 - P(2) - \tilde{P}(1) \right\},$$

which is equivalent to Proposition 1. When $n = 3$, Theorem 1 reduces to

$$\tilde{P}(1)\tilde{P}(2)\tilde{P}(3) = P(123) \left\{ 1 - P(2)P(3) - \tilde{P}(1)P(3) - \tilde{P}(1)\tilde{P}(2) \right\}.$$

Finally, when $n = 4$, Theorem 1 reduces to

$$\tilde{P}(1)\tilde{P}(2)\tilde{P}(3)\tilde{P}(4) = P(1234) \left\{ 1 - P(2)P(3)P(4) - \tilde{P}(1)P(3)P(4) - \tilde{P}(1)\tilde{P}(2)P(4) - \tilde{P}(1)\tilde{P}(2)\tilde{P}(3) \right\}.$$

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