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Discrete-time counterparts of the RL and RC multipliers

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ABSTRACT

The use of multipliers is an important technique for absolute stability analysis. In the continuous-time domain, the RL and RC multipliers preserve the positivity of memoryless and monotone nonlinearities. We classify their discrete-time counterparts and analyse their phase properties. The classification of the discrete-time counterparts of the multipliers is richer than that of their continuous-time counterparts. Some classes of multipliers that preserve the positivity of memoryless and monotone nonlinearities are parametrised in term of their zero-pole description. Similarly we discuss classes that preserve the positivity of memoryless, monotone and symmetric nonlinearities. Multipliers that lie outside such classes are illustrated with numerical counterexamples. We give and prove the phase properties of the discrete-time counterparts of the multipliers. The phase properties are less flexible compared to those of their continuous-time counterparts.

KEYWORDS

Stability of nonlinear systems; multipliers; constrained control;

1. Introduction

A multiplier is a transfer function that preserves the positivity of nonlinearities within a specific class. The classes of RL and RC multipliers play an important role in absolute stability theory (Brockett & Willems, 1965; Falb & Zames, 1968; O'Shea, 1967; Zames, 1966). For example they are used in the construction of the off-axis circle criterion (Cho & Narendra, 1968). A multiplier of the form

$$M(s) = \frac{(s + a_1)(s + a_2) \cdots (s + a_n)}{(s + b_1)(s + b_2) \cdots (s + b_n)} \quad (1)$$

is said to belong to the class of RL multipliers if

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n. \quad (2)$$

and it is said to belong to the class of RC multipliers if

$$0 < b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n. \quad (3)$$

Their name derives from the fact that they can be realised with a network of resistors and inductors or of resistors and capacitors respectively. An RL multiplier has positive phase bounded above by the phase of $(s + a_1)/(s + b_n)$ while an RC multiplier has negative phase bounded below by the phase of $(s + a_n)/(s + b_1)$ (Fig 1). The multipliers have the important property that they preserve the passivity of memoryless monotone nonlinearities.

In fact the class of RL multipliers is a subclass of the Zames-Falb multipliers while for every RC multiplier of the form (1) there is an anticausal Zames-Falb multiplier

$$\tilde{M}(s) = \frac{(s-b_1)(s-b_2)\cdots(s-b_n)}{(s-a_1)(s-a_2)\cdots(s-a_n)}, \quad (4)$$

with the same phase (Falb & Zames, 1968). We say the class of RC multipliers is phase-contained within the class of Zames-Falb multipliers. The class of multipliers consists of products of an RL multiplier and an RC multiplier is also phase-contained within the class of Zames-Falb multipliers.

Figure 1. In continuous time an RL multiplier has positive phase bounded above by the phase of $(s+a_1)/(s+b_n)$; an RC multiplier has negative phase bounded below by the phase of $(s+a_n)/(s+b_1)$.

An important application of the continuous-time RL and RC multipliers is the construction of the off-axis circle criterion (Cho & Narendra, 1968). It is also useful to understand their phase properties as it is often the case that hand-tuning multipliers produces superior results to automatic searches (an example is discussed at length in (Carrasco, Turner, & Heath, 2016)).

Another application is to develop Lyapunov methods to introduce design. In (Haddad & Bernstein, 1991a,b, 1994), quadratic Lyapunov functions are used in robust stability design. One possible application is to use a subclass of RL and RC multipliers in the design. This subclass may be out of the class of multipliers obtained by Lyapunov functions and may bring a stronger result. It is recognised that the geometric interpretation of multipliers is important when they are to be used in hand-tuning (Altshuller, 2013).

In this paper we study the phase properties of the discrete-time counterparts of the RL and RC multipliers. Our motivation stems from the non-trivial differences between discrete-time and continuous-time Lur'e systems (also written as Lurie or Lurye systems). In particular there are more restrictive phase limitations for discrete-time Zames-Falb multipliers than for continuous-time Zames-Falb multipliers (Wang, Carrasco, & Heath, 2018) and in discrete-time there are second-order counterexamples to the Kalman conjecture (Carrasco, Heath, & de la Sen, 2015; Heath, Carrasco, & de la Sen, 2015b). First-order discrete-time multipliers and their sums have been discussed in the literature (Haddad & Kapila, 1995; Narendra & Cho, 1968); an example where the understanding their phase properties is crucial for hand-tuning can be found in (Heath & Carrasco, 2015). But more generally Moylan (2014) notes that “the whole issue of multipliers for discrete-time systems remains very much an

open question.” In particular the discrete-time counterparts of the RL and RC multipliers do not appear to have been previously considered. Although it is possible to define classes of discrete-time multipliers similar to the continuous-time classes, there is no longer a corresponding direct physical interpretation. Furthermore, while limited versions of the off-axis circle criterion can be derived (Narendra & Cho, 1968), there is no direct counterpart to the continuous time version (Wang et al., 2018). In this paper we shed further light on the phase properties of these important classes of simple discrete-time multipliers.

Specifically we consider discrete-time transfer functions of the form

$$M(z) = \frac{(z+a_1)(z+a_2)\cdots(z+a_n)}{(z+b_1)(z+b_2)\cdots(z+b_n)} \quad (5)$$

with either

$$-1 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < 1, \quad (6)$$

or

$$-1 < b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n < 1. \quad (7)$$

We find that the classification of discrete-time multipliers that have this structure is richer than that of their continuous-time counterparts.

Some preliminary results of this paper were presented in (Wang, Heath, & Carrasco, 2017).

2. Notation and preliminary results

2.1. Spaces (c.f. Willems, 1971)

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integer numbers and positive integer numbers including 0, respectively. Let $\ell(\mathbb{Z}^+)$ be the space of all real-valued sequences, $h : \mathbb{Z}^+ \rightarrow \mathbb{R}$. Let $\ell_1(\mathbb{Z}^+)$ be the space of all absolute summable real sequences $h : \mathbb{Z}^+ \rightarrow \mathbb{R}$. Let $\ell_2(\mathbb{Z}^+)$ denote the Hilbert space of all square-summable real sequences $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ ($\ell(\mathbb{Z}^+)$ is the extended space of $\ell_2(\mathbb{Z}^+)$). Similarly, we can define $\ell(\mathbb{Z})$ as the space of all real-valued sequences, $h : \mathbb{Z} \rightarrow \mathbb{R}$. Let $\ell_1(\mathbb{Z})$ be the space of all absolute summable real sequences $h : \mathbb{Z} \rightarrow \mathbb{R}$. Let $\ell_2(\mathbb{Z})$ denote the Hilbert space of all square-summable real sequences $f : \mathbb{Z} \rightarrow \mathbb{R}$ ($\ell(\mathbb{Z})$ is the extended space of $\ell_2(\mathbb{Z})$).

In this paper, we use ℓ short for $\ell(\mathbb{Z}^+)$, ℓ_1 short for $\ell_1(\mathbb{Z})$ and ℓ_2 short for $\ell_2(\mathbb{Z}^+)$. Then the inner product on ℓ_2 is

$$\langle g, h \rangle = \sum_{k=0}^{\infty} g_k h_k. \quad (8)$$

Given a sequence $h : \mathbb{Z} \rightarrow \mathbb{R}$ such that $h \in \ell_1$, its ℓ_1 -norm is given by

$$\|h\|_1 = \sum_{k=-\infty}^{\infty} |h_k|. \quad (9)$$

2.2. Memoryless nonlinearities (c.f. Willems, 1971)

Definition 2.1 (Memoryless and monotone nonlinearity). We say $\phi : \ell \rightarrow \ell$ is memoryless and monotone if there is a function $N : \mathbb{R} \rightarrow \mathbb{R}$ such that $(\phi u)(k) = N(u(k))$, $N(0) = 0$, and

$$0 \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2}, \quad \forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2. \quad (10)$$

Definition 2.2 (Odd nonlinearity). A memoryless and monotone nonlinearity is said to be odd if the function N in Definition 2.1 is odd, i.e. $N(x) = -N(-x)$, for all $x \in \mathbb{R}$.

Definition 2.3 (Slope-restricted nonlinearity). A memoryless and monotone nonlinearity is said to be slope-restricted in $S[0, k]$ if the function N in Definition 2.1 satisfies

$$\frac{N(x_1) - N(x_2)}{x_1 - x_2} \leq k, \quad \forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2. \quad (11)$$

2.3. Transfer function classes

Definition 2.4. Let $M(z)$ have the form (5); we say $M(z)$ belongs to the class $\mathcal{M}_+(\eta, \xi)$ if

$$\eta < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \xi; \quad (12)$$

we say $M(z)$ belongs to the class $\mathcal{M}_-(\eta, \xi)$ if

$$\eta < b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n < \xi. \quad (13)$$

In addition we say $M(z)$ belongs to the class $\mathcal{M}_{+-}(\eta, \xi)$ if it can be written $M = M_+ M_-$ with $M_+ \in \mathcal{M}_+(\eta, \xi)$ and $M_- \in \mathcal{M}_-(\eta, \xi)$.

A transfer function is causal if its region of convergence includes the unit circle and its exterior. Briefly we say a transfer function is causal if its poles are all in the unit disc. It is anticausal if its poles are all outside the unit disc. It is noncausal if it is neither causal nor anticausal. If $-1 \leq \eta$ and $\xi \leq 1$ the transfer functions in $\mathcal{M}_+(\eta, \xi)$, $\mathcal{M}_-(\eta, \xi)$ and $\mathcal{M}_{+-}(\eta, \xi)$ are all causal. We will find it useful to define the following two classes of transfer functions which are anticausal when $-1 \leq \eta$ and $\xi \leq 1$.

Definition 2.5. Let $\tilde{M}(z)$ have the form

$$\tilde{M}(z) = \frac{(z^{-1} + b_1)(z^{-1} + b_2) \cdots (z^{-1} + b_n)}{(z^{-1} + a_1)(z^{-1} + a_2) \cdots (z^{-1} + a_n)}, \quad (14)$$

we say $\tilde{M}(z)$ belongs to the class $\tilde{\mathcal{M}}_+(\eta, \xi)$ if (12) is satisfied; we say $\tilde{M}(z)$ belongs to the class $\tilde{\mathcal{M}}_-(\eta, \xi)$ if (13) is satisfied.

Remark 2.6. In this paper we adopt the notational convention that causal multipliers are parameterised by a_i 's in the numerator and b_i 's in the denominator, but vice versa for anticausal multipliers.

A multiplier is a transfer function that preserves the positivity of nonlinearities within a specific class (see below).

2.4. Zames-Falb multipliers

The Zames-Falb multipliers play an important role in the classical analysis of both continuous-time and discrete-time Lur'e systems (Desoer & Vidyasagar, 1975; Willems, 1971). They were first proposed by O'Shea (O'Shea, 1967; O'Shea & Younis, 1967) but their properties were clarified and formalised by Zames and Falb for continuous-time systems (Zames & Falb, 1968) and by Willems and Brockett for discrete-time systems (Willems & Brockett, 1968). There has recently been considerable revived interest, not least because they can naturally be expressed within an IQC framework (Megretski & Rantzer, 1997) and the availability of convex searches (Carrasco, Heath, Li, & Lanzon, 2012; Carrasco, Maya-Gonzalez, Lanzon, & Heath, 2014b; Chen & Wen, 1996, 1995; Turner, Kerr, & Postlethwaite, 2009; Wang, Heath, & Carrasco, 2014). For recent overviews see, for example, (Altshuller, 2013; Carrasco et al., 2016; Fetzner & Scherer, 2017; Veenman, Scherer, & Kroglu, 2016).

Definition 2.7 (Zames-Falb multipliers). *The class $\mathcal{M}$ of discrete-time SISO LTI Zames-Falb multipliers is the class of transfer functions that can be written*

$$M(z) = \sum_{k=-\infty}^{\infty} m_k z^{-k}, \quad (15)$$

where $m_k \leq 0$ for all $k \neq 0$ and $\|m\|_1 \leq 2m_0$. Without loss of generality we may set $m_0 = 1$.

The Zames-Falb multipliers have the property that they preserve the positivity of memoryless monotone nonlinearities. That is to say if ϕ is memoryless and monotone and if $M \in \mathcal{M}$ then

$$\langle x, M\phi x \rangle \geq 0 \text{ for all } x \in \ell_2. \quad (16)$$

If ϕ is odd then a wider class of multiplier is available.

Definition 2.8 (Zames-Falb multipliers for odd ϕ). *The class $\mathcal{M}_{\text{odd}}$ is the class of transfer functions $M(z)$ that can be written (15) where $\|m\|_1 \leq 2m_0$. Without loss of generality we may set $m_0 = 1$.*

If ϕ is memoryless, monotone and odd and if $M \in \mathcal{M}_{\text{odd}}$ then

$$\langle x, M\phi x \rangle \geq 0 \text{ for all } x \in \ell_2. \quad (17)$$

Remark 2.9. $\mathcal{M} \subset \mathcal{M}_{\text{odd}}$.

It is straightforward to show that a causal Zames-Falb multiplier in $\mathcal{M}$ (or $\mathcal{M}_{\text{odd}}$) has all its zeros in the unit disc and an anticausal multiplier has all its zeros outside the unit disc. It can be shown that a noncausal Zames-Falb multiplier in $\mathcal{M}$ (or $\mathcal{M}_{\text{odd}}$) can be factorized into the sum of a causal Zames-Falb multiplier in $\mathcal{M}$ (or $\mathcal{M}_{\text{odd}}$) and an anticausal Zames-Falb multiplier in $\mathcal{M}$ (or $\mathcal{M}_{\text{odd}}$); the sum of two Zames-Falb multipliers in $\mathcal{M}$ (or $\mathcal{M}_{\text{odd}}$) is itself a Zames-Falb multiplier in $\mathcal{M}$ (or $\mathcal{M}_{\text{odd}}$).

2.5. Phase substitution and phase containment

The definitions of phase substitution and phase containment are given in (Carrasco, Heath, & Lanzon, 2013) and (Carrasco, Heath, & Lanzon, 2014a) for continuous-time systems. Corre-

sponding definitions can be given for discrete-time systems.

Let $\mathbf{RH}_\infty$ be the class of bounded causal discrete time transfer functions.

Definition 2.10. The subset $\mathcal{SR} \subset \mathbf{RH}_\infty$ is defined as:

$$\mathcal{SR} = \{G \in \mathbf{RH}_\infty : G^{-1} \in \mathbf{RH}_\infty \text{ and } G(\pi) > 0\}. \quad (18)$$

Definition 2.11 (Phase substitution). Let M_a and M_b be two multipliers and $G \in \mathcal{SR}$. The multiplier M_b is a phase substitute of the multiplier M_a when

$$\operatorname{Re} \{M_a(z)G(z)\} \geq 0, \forall |z| = 1, \quad (19)$$

implies

$$\operatorname{Re} \{M_b(z)G(z)\} \geq 0, \forall |z| = 1. \quad (20)$$

Definition 2.12 (Phase containment). Let $\mathcal{M}_A$ and $\mathcal{M}_B$ be two classes of multipliers. The class $\mathcal{M}_A$ is phase-contained within the class $\mathcal{M}_B$ (we write $\mathcal{M}_A \stackrel{P}{\subset} \mathcal{M}_B$), if given a multiplier $M_a \in \mathcal{M}_A$, there exists $M_b \in \mathcal{M}_B$ that is a phase substitute of M_a .

Definition 2.13 (Phase equivalence). Let $\mathcal{M}_A$ and $\mathcal{M}_B$ be two classes of multipliers. The classes $\mathcal{M}_A$ and $\mathcal{M}_B$ are phase equivalent if $\mathcal{M}_A \stackrel{P}{\subset} \mathcal{M}_B$ and $\mathcal{M}_B \stackrel{P}{\subset} \mathcal{M}_A$.

Given a transfer function M , let $\operatorname{Re}\{M\}$ and $\operatorname{Im}\{M\}$ be the real part and imaginary part of M . The phase of a multiplier at each frequency can be expressed as

$$\angle M(e^{j\omega}) = \tan^{-1} \frac{\operatorname{Im} \{M(e^{j\omega})\}}{\operatorname{Re} \{M(e^{j\omega})\}} \text{ with } \omega \in [0, 2\pi]. \quad (21)$$

In this paper we express phase in degrees.

Lemma 2.14. Let M_a and M_b be two multipliers satisfying

$$\operatorname{Re}\{M_a(z)\} > 0 \text{ and } \operatorname{Re}\{M_b(z)\} > 0 \text{ for all } |z| = 1. \quad (22)$$

The multiplier M_b is a phase substitute for the multiplier M_a , if $\angle M_a(z) = \angle M_b(z)$ for all $|z| = 1$.

Proof. See Appendix. □

3. Properties of the continuous-time RL and RC multipliers

The continuous-time RL multipliers defined by (1) and (2) and the RC multipliers defined by (1) and (3) are used by Cho and Narendra (Cho & Narendra, 1968) when deriving the off-axis circle criterion. Here we derive various properties of the continuous-time RL and RC multipliers with respect to the class of continuous Zames-Falb multipliers. Properties of the causal multipliers shown in Lemma 3.3 appear without proof in (Falb & Zames, 1968).

Let $\mathcal{L}_1$ be the space of absolutely integrable functions on $(-\infty, \infty)$ and let $\mathcal{L}_2$ be the space of square integrable functions on $(-\infty, \infty)$.

Definition 3.1 (Continuous-time Zames-Falb multipliers (Zames & Falb, 1968)). Let $\mathcal{M}_{ZF}$ be the class of convolution operators $M : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ whose impulse response is of the form

$$m(t) = \delta(t) - \sum_{i=0}^{\infty} z_i \delta(t - t_i) - z_a(t) \quad (23)$$

where δ is the Dirac delta function, $t_i \in \mathbb{R}$ for all i , $z_i \geq 0$ for all i , $z_a \in \mathcal{L}_1$, $z_a(t) \geq 0$ for all t and

$$\int_{-\infty}^{\infty} z_a(t) + \sum_{i=0}^{\infty} z_i < 1. \quad (24)$$

Phase substitution is defined by Carrasco et al. (2013, 2014a) in the spirit of Falb & Zames (1968). A sufficient condition for a convolution operator M_B to be a phase substitute of convolution operator M_A is that their phase be equal at all frequencies, i.e.

$$\angle M_B(j\omega) = \angle M_A(j\omega) \text{ for all } \omega \in (-\infty, \infty). \quad (25)$$

Definition 3.2 (Phase containment). A class of operators $\mathcal{M}_A$ is phase-contained within $\mathcal{M}_B$, written

$$\mathcal{M}_A \overset{P}{\subset} \mathcal{M}_B, \quad (26)$$

if for every $M_A \in \mathcal{M}_A$ there is a $M_B \in \mathcal{M}_B$ such that M_B is a phase substitute for M_A .

3.1. Causal multipliers

Let $\mathcal{M}_{RL}$ be a set of transfer functions of the form (1) satisfying (2). Let $\mathcal{M}_{RC}$ be a set of transfer functions of the form (1) satisfying (3). The set of transfer functions $\mathcal{M}_{RL,RC}$ consists of transfer functions in the form of $M_1(s)M_2(s)$ with $M_1(s) \in \mathcal{M}_{RL}$ and $M_2(s) \in \mathcal{M}_{RC}$.

Lemma 3.3. Transfer functions in $\mathcal{M}_{RL}$, $\mathcal{M}_{RC}$ and $\mathcal{M}_{RL,RC}$ preserve the positivity of memoryless and monotone nonlinearities. Specifically

- (a) $\mathcal{M}_{RL} \subset \mathcal{M}_{ZF}$;
- (b) $\mathcal{M}_{RC} \overset{P}{\subset} \mathcal{M}_{ZF}$;
- (c) $\mathcal{M}_{RL,RC} \overset{P}{\subset} \mathcal{M}_{ZF}$.

Proof. See Appendix. □

3.2. Anti-causal multipliers

Every causal first order multiplier given by (1) has an anticausal phase substitute given by (4). Thus we can construct classes of multipliers $\tilde{\mathcal{M}}_{RL}$, $\tilde{\mathcal{M}}_{RC}$ and $\tilde{\mathcal{M}}_{RL,RC}$ that are respectively phase equivalent to $\mathcal{M}_{RL}$, $\mathcal{M}_{RC}$ and $\mathcal{M}_{RL,RC}$. The multipliers have the form (4) with parameters satisfying the same conditions as their phase-equivalent causal counterparts.

Theorem 3.4. Transfer functions in $\tilde{\mathcal{M}}_{RL}$, $\tilde{\mathcal{M}}_{RC}$ and $\tilde{\mathcal{M}}_{RL,RC}$ preserve the positivity of memoryless and monotone nonlinearities. Specifically

- (a) $\tilde{\mathcal{M}}_{RL}^P \subset \mathcal{M}_{ZF}$;
- (b) $\tilde{\mathcal{M}}_{RC} \subset \mathcal{M}_{ZF}$;
- (c) $\tilde{\mathcal{M}}_{RL-RC}^P \subset \mathcal{M}_{ZF}$.

Proof. See Appendix. □

Although Lemma 3.3 and Theorem 3.4 are valid for higher order transfer functions, in Fig 2, we perform the regions of (a, b) corresponding to the class of first order transfer functions in the form of

$$M(s) = \frac{s+a}{s+b}. \quad (27)$$

Both causal and anticausal first order multipliers are included in the same figure.

Figure 2. Classification of continuous-time first order multipliers. Region + corresponds to the class $\mathcal{M}_{RL}$, Region - corresponds to the class $\mathcal{M}_{RC}$ and similarly for Regions $\tilde{+}$ and Region $\tilde{-}$.

4. First order multipliers

We begin by considering first order multipliers of the form

$$M(z) = \frac{z+a}{z+b}. \quad (28)$$

4.1. Causal multipliers

Here we restrict our attention to causal multipliers (i.e. with $|b| < 1$). Let $\mathcal{M}_{1+}$, $\mathcal{M}_{1-}$, $\mathcal{M}_{2+}$, $\mathcal{M}_{2-}$, $\mathcal{M}_{3+}$ and $\mathcal{M}_{3-}$ be sets of transfer functions of the form (28) satisfying respectively:

- $\mathcal{M}_{1+}$: $-1 < a < b < 0$;
- $\mathcal{M}_{1-}$: $-1 < b < a < 0$;
- $\mathcal{M}_{2+}$: $0 < a < b < 1$;
- $\mathcal{M}_{2-}$: $0 < b < a < 1$;

- $\mathcal{M}_{3+}$: $-1 < a < 0 < b < 1$,
and either $2b < a + 1$ or $b < 2a + 1$;
 $\mathcal{M}_{3-}$: $-1 < b < 0 < a < 1$,
and either $2b > a - 1$ or $b > 2a - 1$.

Regions corresponding to these sets are shown in Fig 3.

Remark 4.1. In terms of the classes defined in Definition 2.4 we find $\mathcal{M}_{1+} \subset \mathcal{M}_+(-1, 0)$, $\mathcal{M}_{1-} \subset \mathcal{M}_-(-1, 0)$, $\mathcal{M}_{2+} \subset \mathcal{M}_+(0, 1)$, $\mathcal{M}_{2-} \subset \mathcal{M}_-(0, 1)$, $\mathcal{M}_{3+} \subset \mathcal{M}_+(-1, 1)$ and $\mathcal{M}_{3-} \subset \mathcal{M}_-(-1, 1)$.

Theorem 4.2.

- (a) Transfer functions in $\mathcal{M}_{1+}$ and $\mathcal{M}_{1-}$ preserve the positivity of memoryless and monotone nonlinearities. Specifically $\mathcal{M}_{1+} \subset \mathcal{M}$ and $\mathcal{M}_{1-} \overset{P}{\subset} \mathcal{M}$.
(b) Transfer functions in $\mathcal{M}_{2+}$, $\mathcal{M}_{2-}$, $\mathcal{M}_{3+}$ and $\mathcal{M}_{3-}$ preserve the positivity of memoryless monotone and odd nonlinearities. Specifically $\mathcal{M}_{2+} \subset \mathcal{M}_{\text{odd}}$, $\mathcal{M}_{2-} \overset{P}{\subset} \mathcal{M}_{\text{odd}}$ and $(\mathcal{M}_{3+} \cup \mathcal{M}_{3-}) \overset{P}{\subset} \mathcal{M}_{\text{odd}}$.

Proof. See Appendix. □

Region 3+ and Region 3- are bounded by:

$$l_1 : b = \frac{a+1}{2} \tag{29}$$

$$l_2 : b = 2a + 1 \tag{30}$$

$$l_3 : b = \frac{a-1}{2} \tag{31}$$

$$l_4 : b = 2a - 1. \tag{32}$$

Within Region 3+ and Regions 2-, the sub-regions on the right of l_1 and on the left of l_3 correspond to the class $\mathcal{M}_{\text{odd}}$. The sub-regions on the right of l_2 and on the left of l_4 correspond to the class that is phase-equivalent to $\mathcal{M}_{\text{odd}}$.

It is easy to check that $\mathcal{M}_{2+} \not\subset \mathcal{M}$ and $\mathcal{M}_{3+} \not\subset \mathcal{M}$. Counterexamples include $M(z) = \frac{z+0.5}{z+0.3}$ for $\mathcal{M}_{2+}$ and $M(z) = \frac{z-0.3}{z+0.1}$ for $\mathcal{M}_{3+}$. For both examples, they are Zames-Falb multipliers because $\|m\|_1 < 1$, but $0 > b > a$ is not satisfied and hence not all terms of the impulse responses are non-negative.

Similarly it is easy to check that there are first order transfer functions whose pole and zero are both inside the unit circle but have no phase-equivalent Zames-Falb multiplier. An example is $M(z) = \frac{z-1+\varepsilon}{z+1-\varepsilon}$ with $\varepsilon > 0$. When $\varepsilon \rightarrow 0$, there exists no phase-equivalent Zames-Falb multiplier for $M(z)$ following the phase limitation test given in (Wang, Carrasco, & Heath, 2015). Hence there exist first-order discrete-time transfer functions which do not preserve the positivity of slope-restricted nonlinearities.

4.2. Anticausal multipliers

Every causal first order multiplier given by (28) with $|b| < 1$ has an anticausal phase substitute

$$\tilde{M}(z) = \frac{z^{-1} + b}{z^{-1} + a}. \tag{33}$$

Figure 3. Classification of first order causal multipliers. Region 1+ corresponds to the class $\mathcal{M}_{1+}$ and similarly for Regions 1-, 2+, 2-, 3+ and 3-. The classification is richer than that for continuous-time multipliers where all first order RL and RC multipliers preserve the positivity of all slope-restricted nonlinearities, irrespective of symmetry considerations.

Thus we can construct classes of multipliers $\tilde{\mathcal{M}}_{1+}$ through to $\tilde{\mathcal{M}}_{3-}$ that are respectively phase equivalent to $\mathcal{M}_{1+}$ through to $\mathcal{M}_{3-}$. The multipliers have the form (33) with a and b satisfying the same conditions as their phase-equivalent causal counterparts.

Remark 4.3. In terms of the classes defined in Definition 2.5 we find $\tilde{\mathcal{M}}_{1+} \subset \tilde{\mathcal{M}}_+(-1, 0)$, $\tilde{\mathcal{M}}_{1-} \subset \tilde{\mathcal{M}}_-(-1, 0)$, $\tilde{\mathcal{M}}_{2+} \subset \tilde{\mathcal{M}}_+(0, 1)$, $\tilde{\mathcal{M}}_{2-} \subset \tilde{\mathcal{M}}_-(0, 1)$, $\tilde{\mathcal{M}}_{3+} \subset \tilde{\mathcal{M}}_+(-1, 1)$, and $\tilde{\mathcal{M}}_{3-} \subset \tilde{\mathcal{M}}_-(-1, 1)$.

Regions corresponding to these sets are shown in Fig 4. The boundary in the second quadrant is

$$b = \begin{cases} \frac{2a}{a+1}, & -3 < a < -1 \\ \frac{a}{a+2}, & a \leq -3 \end{cases} \quad (34)$$

and the boundary in the second quadrant is

$$b = \begin{cases} \frac{-2a}{a-1}, & 1 < a < 3 \\ \frac{a}{2-a}, & a \geq 3 \end{cases}. \quad (35)$$

Theorem 4.4.

- (a) Transfer functions in $\tilde{\mathcal{M}}_{1+}$ and $\tilde{\mathcal{M}}_{1-}$ preserve the positivity of memoryless and monotone nonlinearities. Specifically $\tilde{\mathcal{M}}_{1+} \stackrel{P}{\subset} \mathcal{M}$ and $\tilde{\mathcal{M}}_{1-} \subset \mathcal{M}$.
- (b) Transfer functions in $\tilde{\mathcal{M}}_{2+}$, $\tilde{\mathcal{M}}_{2-}$, $\tilde{\mathcal{M}}_{3+}$ and $\tilde{\mathcal{M}}_{3-}$ preserve the positivity of memoryless monotone and odd nonlinearities. Specifically $\tilde{\mathcal{M}}_{2+} \stackrel{P}{\subset} \mathcal{M}_{\text{odd}}$, $\tilde{\mathcal{M}}_{2-} \subset \mathcal{M}_{\text{odd}}$ and $(\tilde{\mathcal{M}}_{3+} \cup \tilde{\mathcal{M}}_{3-}) \stackrel{P}{\subset} \mathcal{M}_{\text{odd}}$.

Proof. See Appendix. □

Figure 4. Classification of first order anticausal multipliers. Region $\bar{1}+$ corresponds to the class $\mathcal{M}_{1+}$ and similarly for Regions $\bar{1}-$, $\bar{2}+$, $\bar{2}-$, $\bar{3}+$ and $\bar{3}-$.

4.3. Numerical examples

We start from causal examples to shown a transfer function in $\mathcal{M}_{1+}$ is in $\mathcal{M}$.

Example 1

Given $M_1(z) \in \mathcal{M}_{1+}$:

$$M_1(z) = \frac{z - 0.5}{z - 0.1}, \quad (36)$$

we can write $M_1(z) = 1 - H_1(z)$ with

$$H_1(z) = \frac{0.4}{z - 0.1}. \quad (37)$$

The impulse response of $H_1(z)$ is

$$h_k = 0.4(0.1)^{k-1}, \forall k > 0. \quad (38)$$

So we have

$$\|H_1\|_1 = \sum_{k=1}^{\infty} 0.4(0.1)^{k-1} < 1 \quad (39)$$

Hence $M_1(z) \in \mathcal{M}$.

Whether an anticausal transfer function is a Zames-Falb multiplier can also be known by calculating the impulse responses and the 1-norm of the responses. For a transfer function in $\mathcal{M}_{1-}$, there exists a phase-equivalent transfer function that is in $\mathcal{M}$.

Example 2

Given $M_2(z) \in \mathcal{M}_{1-}$:

$$M_2(z) = \frac{z - 0.1}{z - 0.5}, \quad (40)$$

there exists a phase-equivalent anticausal transfer function

$$\tilde{M}_2(z) = \frac{z^{-1} - 0.5}{z^{-1} - 0.1} \quad (41)$$

such that we can write $\tilde{M}_2(z) = 1 - \tilde{H}_2(z)$ with

$$\tilde{H}_2(z) = \frac{0.4}{z^{-1} - 0.1}. \quad (42)$$

The impulse response of $\tilde{H}_2(z)$ is

$$h_{-k} = 0.4(0.1)^{-k-1}, \forall k < 0. \quad (43)$$

So we have

$$\|H_1\|_1 = \sum_{k=-\infty}^{-1} 0.4(0.1)^{-k-1} < 1 \quad (44)$$

Hence the phase-equivalent anticausal transfer function $\tilde{M}_2(z) \in \mathcal{M}$.

Differences between the class of transfer functions in $\mathcal{M}$ and $\mathcal{M}_{odd}$ can be illustrated in the following examples.

Example 3

Given a feedback interconnection between

$$G_3(z) = \left(\frac{z - 0.1}{z - 0.9} \right)^2 \quad (45)$$

and a non-odd nonlinearity, the positivity of the nonlinearity can be preserved by

$$M_3(z) = \frac{z - 0.9}{z - 0.1} \quad (46)$$

because $M_3(z) \in \mathcal{M}_{1+} \subset \mathcal{M}$ by Theorem 4.2(a).

Example 4

Given a feedback interconnection between

$$G_4(z) = \left(\frac{z + 0.1}{z + 0.9} \right)^2 \quad (47)$$

and an odd nonlinearity, the positivity of the nonlinearity can be preserved by

$$M_4(z) = \frac{z + 0.9}{z + 0.1} \quad (48)$$

because $M_4(z) \in \mathcal{M}_{2-} \stackrel{P}{\subset} \mathcal{M}_{odd}$ by Theorem 4.2(b). However, $M_4(z)$ does not preserve the positivity of a non-odd nonlinearity. This coincides with the result by phase limitation (Wang et al., 2018). It can be shown that, for $M_4(z)$, there exists a phase limitation of non-odd Zames-Falb multipliers hence the positivity of a non-odd nonlinearity cannot be guaranteed.

5. Higher order multipliers

We are now in a position to consider higher order multipliers. First we consider causal multipliers with positive poles and zeros. We show that, similar to their continuous-time counterparts, they are phase-contained within the class of Zames-Falb multipliers. However there is less flexibility in their phase properties; this is consistent with the absence of a direct discrete-time counterpart of the off-axis circle criterion. The results complement those of (Narendra & Cho, 1968) where the phase properties of sums of first order discrete-time multipliers are considered.

Theorem 5.1.

- (a) $\mathcal{M}_+(-1, 0) \subset \mathcal{M}$;
- (b) $\mathcal{M}_-(-1, 0) \overset{P}{\subset} \mathcal{M}$;
- (c) $\mathcal{M}_{+-}(-1, 0) \overset{P}{\subset} \mathcal{M}$.

Proof. See Appendix. □

Theorem 5.2.

- (a) If $M \in \mathcal{M}_+(-1, 0)$ then

$$0 < \angle M(e^{j\omega}) < (1 - \frac{\omega}{\pi})90^\circ \text{ on } \omega \in [0, \pi]. \quad (49)$$

- (b) If $M \in \mathcal{M}_-(-1, 0)$ then

$$-(1 - \frac{\omega}{\pi})90^\circ < \angle M(e^{j\omega}) < 0 \text{ on } \omega \in [0, \pi]. \quad (50)$$

- (c) If $M \in \mathcal{M}_{+-}(-1, 0)$ then

$$-(1 - \frac{\omega}{\pi})90^\circ < \angle M(e^{j\omega}) < (1 - \frac{\omega}{\pi})90^\circ \text{ on } \omega \in [0, \pi]. \quad (51)$$

Proof. See Appendix. □

The bounds of Theorem 5.2 are illustrated in Fig 5.

Similar results follows for multipliers whose poles and zeros have negative real parts.

Theorem 5.3.

- (a) $\mathcal{M}_+(0, 1) \subset \mathcal{M}_{odd}$;
- (b) $\mathcal{M}_-(0, 1) \overset{P}{\subset} \mathcal{M}_{odd}$;
- (c) $\mathcal{M}_{+-}(0, 1) \overset{P}{\subset} \mathcal{M}_{odd}$.

Proof. See Appendix. □

Theorem 5.4.

- (a) If $M \in \mathcal{M}_+(0, 1)$ then

$$0 < \angle M(e^{j\omega}) < \frac{\omega}{\pi}90^\circ \text{ on } \omega \in [0, \pi]. \quad (52)$$

Figure 5. Phase bounds for $\mathcal{M}_+(-1, 0)$, $\mathcal{M}_-(-1, 0)$ and $\mathcal{M}_{+-}(-1, 0)$ given in Theorem 5.2.

Figure 6. Phase bounds for $\mathcal{M}_+(0, 1)$, $\mathcal{M}_-(0, 1)$ and $\mathcal{M}_{+-}(0, 1)$ given in Theorem 5.4.

(b) If $M \in \mathcal{M}_-(0, 1)$ then

$$-\frac{\omega}{\pi}90^\circ < \angle M(e^{j\omega}) < 0 \text{ on } \omega \in [0, \pi]. \quad (53)$$

(c) If $M \in \mathcal{M}_{+-}(0, 1)$ then

$$-\frac{\omega}{\pi}90^\circ < \angle M(e^{j\omega}) < \frac{\omega}{\pi}90^\circ \text{ on } \omega \in [0, \pi]. \quad (54)$$

Proof. See Appendix. □

The bounds of Theorem 5.4 are illustrated in Fig 6.

We have already seen by first order counterexample that $\mathcal{M}_+(-1, 1) \not\subset \mathcal{M}$ (and indeed $\mathcal{M}_+(-1, 1) \not\subset \mathcal{M}_{\text{odd}}$). We cannot necessarily build up higher order multipliers with a and b

in Region 3+ or Region 3- alone. A counterexample is $M = M_{31}M_{32}$ with $M_{31}(z) = \frac{z-1+3\varepsilon}{z+\varepsilon}$, $M_{32}(z) = \frac{z-\varepsilon}{z+1-3\varepsilon}$ and a positive $\varepsilon \rightarrow 0$. Similarly, We cannot necessarily build up higher-order multipliers with a combination of Region 1, 2 and 3. A counterexample of a combination of Region 1 and 2 is $M = M_1M_2$ with $M_1(z) = \frac{z-1+\varepsilon}{z-\varepsilon}$, $M_2(z) = \frac{z+\varepsilon}{z+1-\varepsilon}$ and a positive $\varepsilon \rightarrow 0$. A counterexample of a combination of Region 1 and 3 is $M = M_1M_3$ with $M_1(z) = \frac{z-1+\varepsilon}{z-2\varepsilon}$, $M_3(z) = \frac{z-\varepsilon}{z+1-3\varepsilon}$ and a positive $\varepsilon \rightarrow 0$.

Nevertheless there are first-order multipliers in $\mathcal{M}_{3+}$ and $\mathcal{M}_{3-}$ whose phase exceeds the bounds given in Theorems 5.2 and 5.4. As an example, Fig 7 shows the phase of the transfer function $M_{3+} = \frac{z-1/3+\varepsilon}{z+1/3-\varepsilon}$ with $\varepsilon = 1/100$, which is a multiplier in $\mathcal{M}_{3+}$.

6. Discussion

While the focus of this paper is the analytical properties of the discrete-time RL and RC multipliers, the motivation is for practical control system design. In continuous-time the applications are two-fold: firstly they are used in the construction of the off-axis circle criterion (Cho & Narendra, 1968) which in turn has found many applications; secondly they are phase-contained within the Zames-Falb multipliers (Falb & Zames, 1968) so understanding their properties can aid understanding of the larger multiplier class.

Consider teleoperation as an example, where it is standard to use passivity in the design (Hokayem & Spong, 2006). Malysz & Sirouspour (2009) use the off-axis circle criterion as part of a design with haptic feedback. More recently Tugal, Carrasco, Falcon, & Barreiro (2017) demonstrate that excellent results can be obtained using the continuous time Zames-Falb multipliers. But Jazayeri & Tavakoli (2013) point out that discretization may jeopardize even simple assumptions of passivity in teleoperations systems.

More generally the off-axis circle criterion continues to be used for continuous-time control design. One recent example is for PID tuning for time delay plants with actuator saturation (Zhang, Mercader, & Carrasco, 2018). Meanwhile Plummer & Ling (2000) propose using the off-axis circle criterion for discrete-time control design with actuator saturation. Yet only a very limited form of discrete-time off-axis circle criterion is known (Narendra & Cho, 1968) and a direct equivalent cannot exist (Wang et al., 2018). The analysis of this paper may lead to a more useful discrete-time version.

There are many potential applications for the Zames-Falb multipliers in discrete time. As an example, the multipliers can be used for input-constrained model predictive control (Heath & G., 2007) which is inherently discrete. Recently several convex searches have been proposed both for the continuous time Zames-Falb multipliers (Carrasco et al., 2012, 2014b; Chen & Wen, 1996, 1995; Turner et al., 2009) and for the discrete time Zames-Falb multipliers (Wang et al., 2014). Nevertheless, it is often the case that choosing multipliers by hand can lead to better results; Carrasco et al. (2016) discuss a continuous-time example at length and a discrete-time example is exploited by Heath & Carrasco (2015). An interesting observation of Wang et al. (2014) is that the parameterization of the optimal multiplier is often sparse; this is not yet properly understood. Hence understanding the phase properties of simple classes of multipliers remains useful in control system design and analysis.

In continuous time the RL and RC preserve the positivity of memoryless monotone nonlinearities irrespective of symmetry properties. In discrete time we see a richer behaviour. A recent study (Heath, Carrasco, & Altshuller, 2015a) has shown that understanding symmetry properties of multipliers is important with respect to exogenous signals. This further motivates further work to explore the practical implications of this paper's results.

7. Conclusion

We have defined classes of discrete-time multipliers in an analogous manner to the RL and RC multipliers in continuous-time. The classes $\mathcal{M}_+(-1, 0)$ and $\mathcal{M}_-(-1, 0)$ are phase-contained within $\mathcal{M}$ and may be thought of as the most natural counterparts of the RL and RC multipliers for continuous-time. However their phase properties are considerably less flexible. The classes $\mathcal{M}_+(0, 1)$ and $\mathcal{M}_-(0, 1)$ are phase-contained within $\mathcal{M}_{odd}$. We have also discussed transfer functions whose poles and zeros are real and may have opposite sign; in particular we have illustrated classes that are phase-contained within $\mathcal{M}_{odd}$, but also shown by counterexample that some elements are not multipliers.

While our analysis shows that multipliers in $\mathcal{M}_+(0, 1)$ and $\mathcal{M}_-(0, 1)$ are only guaranteed to preserve the positivity of odd multipliers, it is straightforward to apply the analysis of (Heath et al., 2015a) at least first order multipliers to subclasses that preserve the positivity of quasi-odd nonlinearities. It would be interesting to see if such classification could be extended to higher order multipliers. Finally we note that an efficient search for FIR multipliers is presented in (Wang et al., 2014). It would be useful to find a similar classification for these multipliers.

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Appendix A. Proof of Lemma 2.14

Given $\delta_1 > 0$,

$$\operatorname{Re} \{M_a G\} \geq \delta_1, \forall \omega \quad (\text{A1})$$

it means

$$\operatorname{Re} \{M_a\} \operatorname{Re} \{G\} - \operatorname{Im} \{M_a\} \operatorname{Im} \{G\} \geq \delta_1, \forall \omega. \quad (\text{A2})$$

If $\angle M_a = \angle M_b$ at each frequency, i.e.

$$\frac{\operatorname{Im} \{M_a\}}{\operatorname{Re} \{M_a\}} = \frac{\operatorname{Im} \{M_b\}}{\operatorname{Re} \{M_b\}}, \forall \omega, \quad (\text{A3})$$

and

$$\operatorname{Re} \{M_a(z)\} > 0 \text{ and } \operatorname{Re} \{M_b(z)\} > 0 \text{ for all } |z| = 1. \quad (\text{A4})$$

then

$$\operatorname{Re} \{M_b\} \operatorname{Re} \{G\} - \operatorname{Im} \{M_b\} \operatorname{Im} \{G\} \geq \delta_1 \frac{\operatorname{Re} \{M_b\}}{\operatorname{Re} \{M_a\}}, \forall \omega. \quad (\text{A5})$$

Given

$$\delta_2 = \delta_1 \frac{\operatorname{Re} \{M_b\}}{\operatorname{Re} \{M_a\}} > 0, \quad (\text{A6})$$

$$\operatorname{Re} \{M_b\} \operatorname{Re} \{G\} - \operatorname{Im} \{M_b\} \operatorname{Im} \{G\} \geq \delta_2, \forall \omega. \quad (\text{A7})$$

So the multiplier M_b is a phase-substitute of the multiplier M_a .

Appendix B. Proof of Lemma 3.3

We start from two lemmas showing the subsets of first order transfer functions and their relationship with higher order transfer functions.

Let $\mathcal{M}_{RL1}$, $\mathcal{M}_{RC1}$ and $\mathcal{M}_{RL\cdot RC1}$ be the subsets of transfer functions consist of only first order transfer functions in the sets $\mathcal{M}_{RL}$, $\mathcal{M}_{RC}$ and $\mathcal{M}_{RL\cdot RC}$.

Definition B.1 ((Vidyasagar, 2002)). *Let H be a convolution operator with transfer function $H(s)$. The one-norm $\|H\|_1$ is*

$$\|H\|_1 = \int_{-\infty}^{\infty} |h(t)| dt \quad (\text{B1})$$

where $h(t)$ is the corresponding impulse response.

Lemma B.2. *Transfer functions in $\mathcal{M}_{RL1}$, $\mathcal{M}_{RC1}$ and $\mathcal{M}_{RL\cdot RC1}$ preserve the positivity of memoryless and monotone nonlinearities. Specifically*

- (a) $\mathcal{M}_{RL1} \subset \mathcal{M}_{ZF}$;
- (b) $\mathcal{M}_{RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$;
- (c) $\mathcal{M}_{RL\cdot RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$.

Proof. Let

$$M(s) = \frac{s+a}{s+b}. \quad (\text{B2})$$

We can write $M(s) = 1 - H(s)$ with

$$H(s) = \frac{b-a}{s+b}. \quad (\text{B3})$$

The norm of H is

$$\|H\|_1 = \frac{b-a}{b}. \quad (\text{B4})$$

Given a multiplier $M(s)$, there exists a phase-substitute multiplier $\tilde{M}(s)$ as shown in (4). It can be expressed as $\tilde{M}(s) = 1 - \tilde{H}(s)$ with

$$\tilde{H}(s) = \frac{b-a}{s-a}. \quad (\text{B5})$$

The norm of $\tilde{H}$ is

$$\|\tilde{H}\|_1 = \frac{b-a}{-a}. \quad (\text{B6})$$

- (a) $\mathcal{M}_{RL1} \subset \mathcal{M}_{ZF}$;

For $M(s) \in \mathcal{M}_{RL1}$ with $0 < a < b$, there exists $\|H\|_1 < 1$ with all terms in impulse response are non-negative. So $\mathcal{M}_{RL1} \subset \mathcal{M}_{ZF}$.

(b) $\mathcal{M}_{RC1} \stackrel{P}{\subset} \mathcal{M}_{ZF}$;

For $M(s) \in \mathcal{M}_{RC1}$ with $0 < b < a$, there exists a phase-substitute multiplier $\tilde{M}(s) = 1 - \tilde{H}(s)$ with $\|\tilde{H}\|_1 < 1$ with all terms in impulse response are non-negative. So

$\mathcal{M}_{RC1} \stackrel{P}{\subset} \mathcal{M}_{ZF}$.

(c) $\mathcal{M}_{RL-RC1} \stackrel{P}{\subset} \mathcal{M}_{ZF}$.

Let $M \in \mathcal{M}_{RL1}$ and $\tilde{M} \in \mathcal{M}_{RC1}$ be given as

$$M(s) = \frac{s+a}{s+b} \text{ and } \tilde{M}(s) = \frac{s-\tilde{b}}{s-\tilde{a}}, \quad (\text{B7})$$

with

$$0 < a < b \text{ and } 0 < \tilde{b} < \tilde{a}. \quad (\text{B8})$$

Then

$$M\tilde{M} = 1 - H - \tilde{H}, \quad (\text{B9})$$

with H being causal and given by

$$H(s) = \frac{(b-a)(b+\tilde{b})}{\tilde{a}+b} \frac{1}{s+b}, \quad (\text{B10})$$

and with $\tilde{H}$ being anticausal and given by

$$\tilde{H}(s) = \frac{(\tilde{b}-\tilde{a})(\tilde{a}+a)}{\tilde{a}+b} \frac{1}{s-\tilde{a}}. \quad (\text{B11})$$

It is straightforward to check that the impulse responses of $H(s)$ and $\tilde{H}(s)$ are both always non-negative, and

$$\|H\|_1 + \|\tilde{H}\|_1 = \frac{(b-a)(b+\tilde{b})}{(\tilde{a}+b)b} + \frac{(\tilde{b}-\tilde{a})(\tilde{a}+a)}{(\tilde{a}+b)(-a)} < 1. \quad (\text{B12})$$

To see this, first note that $-a\tilde{b}(\tilde{a}+b) < 0$. Rearranging

$$\tilde{a}(b-a)(b+\tilde{b}) - b(\tilde{b}-\tilde{a})(\tilde{a}+a) < (\tilde{a}+b)\tilde{a}b.$$

Dividing both sides by $(\tilde{a}+b)\tilde{a}b > 0$ gives the result. Hence $M\tilde{M} \in \mathcal{M}_{ZF}$.

□

Lemma B.3.

(a) Suppose $M \in \mathcal{M}_{RL}$ is given by (1) for some a_j, b_j with $j = 1, 2, \dots, n$. Then M can be written

$$M(s) = \sum_{j=1}^n \alpha_j \frac{s+a_1}{s+b_j}, \quad (\text{B13})$$

for some $\alpha_j > 0$, $j = 1, 2, \dots, n$.

- (b) Suppose $\tilde{M} \in \mathcal{M}_{RC}$ is given by (1) for some a_j, b_j with $j = 1, 2, \dots, n$. Then $\tilde{M}$ can be written

$$\tilde{M}(s) = \sum_{j=1}^n \alpha_j \frac{s - b_1}{s - a_j}, \quad (\text{B14})$$

for some $\alpha_j > 0$, $j = 1, 2, \dots, n$.

Proof. By induction.

- (a) Basis: the statement is true for $n = 1$ with $\alpha_1 = 1$.

Inductive step: assume the statement is true for multipliers of order n . Then if $M \in \mathcal{M}_{RL}$ has order $n + 1$ we can write

$$\begin{aligned} M(s) &= \prod_{j=1}^n \left(\frac{s + a_j}{s + b_j} \right) \left(\frac{s + a_{n+1}}{s + b_{n+1}} \right), \\ &= \left(\sum_{j=1}^n \alpha_j \frac{s + a_1}{s + b_j} \right) \left(\frac{s + a_{n+1}}{s + b_{n+1}} \right), \\ &= \sum_{j=1}^n \alpha_j \frac{(s + a_1)(s + a_{n+1})}{(s + b_j)(s + b_{n+1})}, \\ &= \sum_{j=1}^n \alpha_j \frac{s + a_1}{b_{n+1} - b_j} \left(\frac{a_{n+1} - b_j}{s + b_j} + \frac{b_{n+1} - a_{n+1}}{s + b_{n+1}} \right), \\ &= \sum_{j=1}^{n+1} \tilde{\alpha}_j \frac{s + a_1}{s + b_j}, \end{aligned} \quad (\text{B15})$$

with

$$\tilde{\alpha}_j = \alpha_j \frac{a_{n+1} - b_j}{b_{n+1} - b_j} > 0 \text{ for } j = 1, 2, \dots, n, \quad (\text{B16})$$

and

$$\tilde{\alpha}_{n+1} = \sum_{j=1}^n \alpha_j \frac{b_{n+1} - a_{n+1}}{b_{n+1} - b_j} > 0. \quad (\text{B17})$$

- (b) Similar.

□

Then we prove Lemma 3.3 based on Lemma B.2 and Lemma B.3.

- (a) $\mathcal{M}_{RL} \subset \mathcal{M}_{ZF}$;

By Lemma B.3(a) each element of $\mathcal{M}_{RL}$ can be expressed as a positively weighted sum of terms in $\mathcal{M}_{RL1} \subset \mathcal{M}_{ZF}$. Hence the result.

- (b) $\mathcal{M}_{RC} \overset{P}{\subset} \mathcal{M}_{ZF}$;

Suppose $M \in \mathcal{M}_{RC}$ is given by (1). Then $\tilde{M} \in \mathcal{M}_{RC}$ given by (4) is a phase substitute for

M . Hence $\mathcal{M}_{RC} \overset{P}{\subset} \tilde{\mathcal{M}}_{ZFC}$. By Lemma B.3(b) each element of $\tilde{\mathcal{M}}_{RC}$ can be expressed as a positively weighted sum of first order terms in $\tilde{\mathcal{M}}_{RC} \subset \mathcal{M}_{ZF}$. Hence the result.

(c) $\mathcal{M}_{RL-RC} \overset{P}{\subset} \mathcal{M}_{ZF}$

Following the analysis of parts (a) and (b) together with Lemma B.3, it suffices to show that the product of a first order term in $\mathcal{M}_{RL}$ (i.e. a multiplier in $\mathcal{M}_{RL}$) with a first order term in $\tilde{\mathcal{M}}_{RC}$ (i.e. a multiplier in $\tilde{\mathcal{M}}_{RC}$) is itself a Zames-Falb multiplier. This is given in Lemma B.2(c). Hence the result. ■

Appendix C. Proof of Theorem 3.4

We also start from the anticausal first order transfer functions.

Let $\tilde{\mathcal{M}}_{RL1}$, $\tilde{\mathcal{M}}_{RC1}$ and $\tilde{\mathcal{M}}_{RL-RC1}$ be the subsets of transfer functions consist of only first order transfer functions in the sets $\tilde{\mathcal{M}}_{RL}$, $\tilde{\mathcal{M}}_{RC}$ and $\tilde{\mathcal{M}}_{RL-RC}$.

Lemma C.1. *Transfer functions in $\tilde{\mathcal{M}}_{RL1}$, $\tilde{\mathcal{M}}_{RC1}$ and $\tilde{\mathcal{M}}_{RL-RC1}$ preserve the positivity of memoryless and monotone nonlinearities. Specifically*

(a) $\tilde{\mathcal{M}}_{RL1} \subset \mathcal{M}_{ZF}$;

(b) $\tilde{\mathcal{M}}_{RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$;

(c) $\tilde{\mathcal{M}}_{RL-RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$.

Proof. For any anticausal multiplier in the form of

$$\tilde{M}(s) = \frac{s-b}{s-a} \quad (C1)$$

it can be expressed as $\tilde{M}(s) = 1 - \tilde{H}(s)$ with $\tilde{H}(z)$ given in (B5). In addition, there exists a phase-substitute multiplier $M(s) = 1 - H(s)$ with $H(s)$ expressed in (B3).

(a) $\tilde{\mathcal{M}}_{RL1} \subset \mathcal{M}_{ZF}$

For $\tilde{M}(s) \in \tilde{\mathcal{M}}_{RL}$ with $0 < a < b$, there exists a phase-substitute multiplier $M(s) = 1 - H(s)$ with $H(s)$ in the form of (B3) such that $\|H\|_1 < 1$ and all terms in impulse response are non-negative. So we know $\tilde{\mathcal{M}}_{RL1} \overset{P}{\subset} \mathcal{M}_{ZF}$.

(b) $\tilde{\mathcal{M}}_{RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$

For $\tilde{M}(s) \in \tilde{\mathcal{M}}_{RC}$ with $0 < b < a$, there exists $\tilde{M}(s) = 1 - \tilde{H}(s)$ with $\tilde{H}(s)$ in the form of (B5) such that $\|\tilde{H}\|_1 < 1$ with all terms in impulse response are non-negative. So $\tilde{\mathcal{M}}_{RC1} \subset \mathcal{M}_{ZF}$.

(c) $\tilde{\mathcal{M}}_{RL-RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$

Similar to Lemma B.2(c). □

Then we consider the proof of Theorem 3.4

(a) $\tilde{\mathcal{M}}_{RL1} \subset \mathcal{M}_{ZF}$

Suppose $\tilde{M} \in \tilde{\mathcal{M}}_{RL}$ is given by (4). Then $M \in \mathcal{M}_{RL}$ given by (1) is a phase substitute for $\tilde{M}$. Hence $\tilde{\mathcal{M}}_{RL} \overset{P}{\subset} \mathcal{M}_{RL}$. By Lemma B.3(a) each element of $\mathcal{M}_{RL}$ can be expressed as a positively weighted sum of first order terms in $\mathcal{M}_{RL} \subset \mathcal{M}_{ZF}$. Hence the result.

$$(b) \tilde{\mathcal{M}}_{RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$$

By Lemma B.3(c) each element of $\tilde{\mathcal{M}}_{RC}$ can be expressed as a positively weighted sum of terms in $\tilde{\mathcal{M}}_{RC1} \subset \mathcal{M}_{ZF}$. Hence the result.

$$(c) \tilde{\mathcal{M}}_{RL-RC1} \overset{P}{\subset} \mathcal{M}_{ZF}$$

Following the analysis of parts (a) and (b) together with Lemma B.3, it suffices to show that the product of a first order term in $\tilde{\mathcal{M}}_{RC}$ (i.e. a multiplier in $\tilde{\mathcal{M}}_{RC}$) with a first order term in $\mathcal{M}_{RL}$ (i.e. a multiplier in $\mathcal{M}_{RL}$) is itself a Zames-Falb multiplier. This is given in Lemma C.1(c). Hence the result. ■

Appendix D. Proof of Theorem 4.2

Let

$$M(z) = \frac{z+a}{z+b}. \quad (D1)$$

We can write $M(z) = 1 - H(z)$ with

$$H(z) = \frac{b-a}{z+b}, \quad (D2)$$

we have

$$y_{k+1} = (b-a)u_k - by_k. \quad (D3)$$

Its impulse response is

$$h_k = \begin{cases} (-b)^{k-1}(b-a), & k \geq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (D4)$$

With $|b| < 1$, the response is causal.

The norm of H is

$$\|H\|_1 = \frac{|b-a|}{1-|b|}. \quad (D5)$$

Given a multiplier $M(z)$, there exists a phase-substitute multiplier $\tilde{M}(z)$ as shown in (33). It can be expressed as $\tilde{M}(z) = 1 - \tilde{H}(z)$ with

$$\tilde{H}(z) = \frac{a-b}{z^{-1}+a}. \quad (D6)$$

Then we have

$$y_{k-1} = (a-b)u_k - ay_k. \quad (D7)$$

Its impulse response is

$$\tilde{h}_k = \begin{cases} (-a)^{-k-1}(a-b), & k \leq -1 \\ 0, & \text{otherwise.} \end{cases} \quad (\text{D8})$$

With $|a| < 1$, the response is anticausal.

The norm of $\tilde{H}$ is

$$\|\tilde{H}\|_1 = \frac{|a-b|}{1-|a|}. \quad (\text{D9})$$

- (a) $\mathcal{M}_{1+} \subset \mathcal{M}$ and $\mathcal{M}_{1-} \overset{P}{\subset} \mathcal{M}$
- For $M(z) \in \mathcal{M}_{1+}$ with $-1 < a < b < 0$, there exists $\|H\|_1 < 1$ with all terms in impulse response (D4) are non-negative. So $\mathcal{M}_{1+} \subset \mathcal{M}$.
 - For $M(z) \in \mathcal{M}_{1-}$ with $-1 < b < a < 0$, there exists a phase-substitute multiplier $\tilde{M}(z) = 1 - \tilde{H}(z)$ with $\|\tilde{H}\|_1 < 1$ with all terms in impulse response (D8) are non-negative. So $\mathcal{M}_{1-} \overset{P}{\subset} \mathcal{M}$.
- (b) $\mathcal{M}_{2+} \subset \mathcal{M}_{\text{odd}}, \mathcal{M}_{2-} \overset{P}{\subset} \mathcal{M}_{\text{odd}}, (\mathcal{M}_{3+} \cup \mathcal{M}_{3-}) \overset{P}{\subset} \mathcal{M}_{\text{odd}}$
- For $M(z) \in \mathcal{M}_{2+}$ with $0 < a < b < 1$, there exists $\|H\|_1 < 1$ with terms in impulse response (D4) change signs. So $\mathcal{M}_{2+} \subset \mathcal{M}_{\text{odd}}$.
 - For $M(z) \in \mathcal{M}_{2-}$ with $0 < b < a < 1$, there exists a phase-substitute multiplier $\tilde{M}(z) = 1 - \tilde{H}(z)$ with $\|\tilde{H}\|_1 < 1$ with terms in impulse response (D8) change signs. So $\mathcal{M}_{2-} \overset{P}{\subset} \mathcal{M}_{\text{odd}}$.
 - For $M(z) \in \mathcal{M}_{3+}$ with $-1 < a < 0 < b < 1$ and either $2b < a+1$ or $b < 2a+1$, there exists a phase-substitute multiplier $\tilde{M}(z) = 1 - \tilde{H}(z)$ with $\|\tilde{H}\|_1 < 1$ with terms in impulse response (D8) change signs. So $\mathcal{M}_{3+} \overset{P}{\subset} \mathcal{M}_{\text{odd}}$. For $M(z) \in \mathcal{M}_{3-}$ with $-1 < b < 0 < a < 1$ and either $2b > a-1$ or $b > 2a-1$, there exists a phase-substitute multiplier $\tilde{M}(z) = 1 - \tilde{H}(z)$ with $\|\tilde{H}\|_1 < 1$ with terms in impulse response (D8) change signs. So $\mathcal{M}_{3-} \overset{P}{\subset} \mathcal{M}_{\text{odd}}$. Based on the above analysis, we have $(\mathcal{M}_{3+} \cup \mathcal{M}_{3-}) \overset{P}{\subset} \mathcal{M}_{\text{odd}}$.

Appendix E. Proof of Theorem 4.4

For any anticausal multiplier in the form of (33), it can be expressed as $\tilde{M}(z) = 1 - \tilde{H}(z)$ with $\tilde{H}(z)$ given in (D6). In addition, there exists a phase-substitute multiplier $M(z) = 1 - H(z)$ with $H(z)$ expressed in (D2).

- (a) $\tilde{\mathcal{M}}_{1+} \overset{P}{\subset} \mathcal{M}$ and $\tilde{\mathcal{M}}_{1-} \subset \mathcal{M}$
- For $\tilde{M}(z) \in \tilde{\mathcal{M}}_{1+}$ with $-1 < a < b < 0$, there exists a phase-substitute multiplier $M(z) = 1 - H(z)$ with $H(z)$ in the form of (D2) such that $\|H\|_1 < 1$ and all terms in impulse response (D4) are non-negative. So we know $M(z) \in \mathcal{M}_{1+}$ and $\tilde{\mathcal{M}}_{1+} \overset{P}{\subset} \mathcal{M}$.
 - For $\tilde{M}(z) \in \tilde{\mathcal{M}}_{1-}$ with $-1 < b < a < 0$, there exists $\tilde{M}(z) = 1 - \tilde{H}(z)$ with $\|\tilde{H}\|_1 < 1$ with all terms in impulse response (D8) are non-negative. So $\tilde{\mathcal{M}}_{1-} \subset \mathcal{M}$.
- (b) $\tilde{\mathcal{M}}_{2+} \overset{P}{\subset} \mathcal{M}_{\text{odd}}, \tilde{\mathcal{M}}_{2-} \subset \mathcal{M}_{\text{odd}}, (\tilde{\mathcal{M}}_{3+} \cup \tilde{\mathcal{M}}_{3-}) \overset{P}{\subset} \mathcal{M}_{\text{odd}}$
- For $\tilde{M}(z) \in \tilde{\mathcal{M}}_{2+}$ with $0 < a < b < 1$, there exists a phase-substitute multiplier $M(z) = 1 - H(z)$ with $H(z)$ in the form of (D2) such that $\|H\|_1 < 1$ with terms in impulse

- response (D4) change signs. So we know $M(z) \in \mathcal{M}_{2+}$ and $\tilde{\mathcal{M}}_{2+} \stackrel{P}{\subset} \mathcal{M}_{odd}$.
- For $\tilde{M}(z) \in \tilde{\mathcal{M}}_{1-}$ with $0 < b < a < 1$, there exists $\tilde{M}(z) = 1 - \tilde{H}(z)$ with $\|\tilde{H}\|_1 < 1$ and the terms in its impulse response (D8) change signs. So $\tilde{\mathcal{M}}_{2-} \subset \mathcal{M}_{odd}$.
 - For $\tilde{M}(z) \in \tilde{\mathcal{M}}_{3+}$ with $-1 < a < 0 < b < 1$ and either $2b < a + 1$ or $b < 2a + 1$, there exists a phase-substitute multiplier $M(z) = 1 - H(z)$ with $H(z)$ in the form of (D2) such that $\|H\|_1 < 1$ with terms in impulse response (D4) change signs. So we know $M(z) \in \mathcal{M}_{3+}$ and $\tilde{\mathcal{M}}_{3+} \stackrel{P}{\subset} \mathcal{M}_{odd}$. For $\tilde{M}(z) \in \tilde{\mathcal{M}}_{3+}$ with $-1 < b < 0 < a < 1$ and either $2b > a - 1$ or $b > 2a - 1$, there exists a phase-substitute multiplier $M(z) = 1 - H(z)$ with $H(z)$ in the form of (D2) such that $\|H\|_1 < 1$ with terms in impulse response (D4) change signs. So we know $M(z) \in \mathcal{M}_{3-}$ and $\tilde{\mathcal{M}}_{3-} \stackrel{P}{\subset} \mathcal{M}_{odd}$. Based on the above analysis, we have $(\tilde{\mathcal{M}}_{3+} \cup \tilde{\mathcal{M}}_{3-}) \stackrel{P}{\subset} \mathcal{M}_{odd}$.

Appendix F. Proof of Theorem 5.1

We first need a technical lemma.

Lemma F.1.

- (a) Suppose $M \in \mathcal{M}_+(\eta, \xi)$ is given by (5) for some η, ξ and for some a_j, b_j with $j = 1, 2, \dots, n$. Then M can be written

$$M(z) = \sum_{j=1}^n \alpha_j \frac{z + a_1}{z + b_j}, \quad (\text{F1})$$

for some $\alpha_j > 0, j = 1, 2, \dots, n$.

- (b) Suppose $\tilde{M} \in \tilde{\mathcal{M}}_-(\eta, \xi)$ is given by (14) for some η, ξ and for some a_j, b_j with $j = 1, 2, \dots, n$. Then $\tilde{M}$ can be written

$$\tilde{M}(z) = \sum_{j=1}^n \alpha_j \frac{z^{-1} + b_1}{z^{-1} + a_j}, \quad (\text{F2})$$

for some $\alpha_j > 0, j = 1, 2, \dots, n$.

Proof. By induction.

- (a) Basis: the statement is true for $n = 1$ with $\alpha_1 = 1$.

Inductive step: assume the statement is true for multipliers of order n . Then if $M \in$

$\mathcal{M}_+(\eta, \xi)$ has order $n+1$ we can write

$$\begin{aligned}
M(z) &= \prod_{j=1}^n \left(\frac{z+a_j}{z+b_j} \right) \left(\frac{z+a_{n+1}}{z+b_{n+1}} \right), \\
&= \left(\sum_{j=1}^n \alpha_j \frac{z+a_1}{z+b_j} \right) \left(\frac{z+a_{n+1}}{z+b_{n+1}} \right), \\
&= \sum_{j=1}^n \alpha_j \frac{(z+a_1)(z+a_{n+1})}{(z+b_j)(z+b_{n+1})}, \\
&= \sum_{j=1}^n \alpha_j \frac{z+a_1}{b_{n+1}-b_j} \left(\frac{a_{n+1}-b_j}{z+b_j} + \frac{b_{n+1}-a_{n+1}}{z+b_{n+1}} \right), \\
&= \sum_{j=1}^{n+1} \tilde{\alpha}_j \frac{z+a_1}{z+b_j}, \tag{F3}
\end{aligned}$$

with

$$\tilde{\alpha}_j = \alpha_j \frac{a_{n+1}-b_j}{b_{n+1}-b_j} > 0 \text{ for } j = 1, 2, \dots, n, \tag{F4}$$

and

$$\tilde{\alpha}_{n+1} = \sum_{j=1}^n \alpha_j \frac{b_{n+1}-a_{n+1}}{b_{n+1}-b_j} > 0. \tag{F5}$$

(b) Similar.

□

Now for the main proof:

(a) $\mathcal{M}_+(-1, 0) \subset \mathcal{M}$.

By Lemma F.1(a) each element of $\mathcal{M}_+(-1, 0)$ can be expressed as a positively weighted sum of terms in $\mathcal{M}_{1+} \subset \mathcal{M}$. Hence the result.

(b) $\mathcal{M}_-(-1, 0) \overset{P}{\subset} \mathcal{M}$.

Suppose $M \in \mathcal{M}_-(-1, 0)$ is given by (5). Then $\tilde{M} \in \tilde{\mathcal{M}}_-(-1, 0)$ given by (14) is a phase substitute for M . Hence $\mathcal{M}_-(-1, 0) \overset{P}{\subset} \tilde{\mathcal{M}}_-(-1, 0)$. By Lemma F.1(b) each element of $\tilde{\mathcal{M}}_-(-1, 0)$ can be expressed as a positively weighted sum of first order terms in $\tilde{\mathcal{M}}_{1-} \subset \mathcal{M}$. Hence the result.

(c) $\mathcal{M}_{+-}(-1, 0) \overset{P}{\subset} \mathcal{M}$.

Following the analysis of parts (a) and (b) together with Lemma F.1, it suffices to show that the product of a first order term in $\mathcal{M}_+(-1, 0)$ (i.e. a multiplier in $\mathcal{M}_{1+}$) with a first order term in $\tilde{\mathcal{M}}_-(-1, 0)$ (i.e. a multiplier in $\tilde{\mathcal{M}}_{1-}$) is itself a Zames-Falb multiplier. Let $M \in \mathcal{M}_{1+}$ and $\tilde{M} \in \tilde{\mathcal{M}}_{1-}$ be given as

$$M(z) = \frac{z+a}{z+b} \text{ and } \tilde{M}(z) = \frac{z^{-1}+\tilde{b}}{z^{-1}+\tilde{a}}, \tag{F6}$$

with

$$-1 < a < b < 0 \text{ and } -1 < \tilde{b} < \tilde{a} < 0. \quad (\text{F7})$$

Then

$$M\tilde{M} = m_0 - H - \tilde{H}, \quad (\text{F8})$$

with m_0 given by

$$m_0 = 1 + \frac{(b-a)(\tilde{a}-\tilde{b})}{1-\tilde{a}b}, \quad (\text{F9})$$

with H causal and given by

$$H(z) = \frac{(b-a)(1-b\tilde{b})}{1-\tilde{a}b} \frac{1}{z+b}, \quad (\text{F10})$$

and with $\tilde{H}$ anticausal and given by

$$\tilde{H}(z) = \frac{(\tilde{a}-\tilde{b})(1-a\tilde{a})}{1-\tilde{a}b} \frac{1}{z^{-1}+\tilde{a}}. \quad (\text{F11})$$

It is straightforward to check that $m_0 > 0$, the impulse responses of $H(z)$ and $\tilde{H}(z)$ are both always non-negative, and

$$\begin{aligned} \|H\|_1 + \|\tilde{H}\|_1 &= \frac{(b-a)(1-b\tilde{b})}{(1-\tilde{a}b)(1+b)} + \frac{(\tilde{a}-\tilde{b})(1-a\tilde{a})}{(1-\tilde{a}b)(1+\tilde{a})}, \\ &< \frac{(b-a)}{(1-\tilde{a}b)} + \frac{(\tilde{a}-\tilde{b})}{(1-\tilde{a}b)}, \\ &< m_0. \end{aligned} \quad (\text{F12})$$

To see this, first note that $(1+a)(1+\tilde{b}) > 0$, $-\tilde{a}(1+a) > 0$ and $-b(1+\tilde{b}) > 0$. Hence

$$(1+a)(1+\tilde{b}) - \tilde{a}(1+a) - b(1+\tilde{b}) > 0$$

Rearranging

$$1 - \tilde{a}b + (b-a)(\tilde{a}-\tilde{b}) > b-a + \tilde{a}-\tilde{b}$$

Dividing both sides by $1 - \tilde{a}b > 0$ gives the result. Hence $M\tilde{M} \in \mathcal{M}$. ■

Appendix G. Proof of Theorem 5.2

(a) If $M \in \mathcal{M}_+(-1, 0)$ then

$$0 < \angle M(e^{j\omega}) < (1 - \frac{\omega}{\pi})90^\circ \text{ on } \omega \in [0, \pi].$$

Every multiplier in $\mathcal{M}_{1+}$ has positive phase. Since each term in $\mathcal{M}_+(-1, 0)$ can be expressed as a product of terms in $\mathcal{M}_{1+}$, its phase is bounded below by zero. Similarly each term in $\mathcal{M}_-(-1, 0)$ has negative phase. But we can express each term in $\mathcal{M}_+(-1, 0)$ as a product of a term in $\mathcal{M}_{1+}$ and a term in $\mathcal{M}_-(-1, 0)$; hence it is bounded above by the phase of a term in $\mathcal{M}_{1+}$. This in turn is bound above by the limits given in the statement.

(b) If $M \in \mathcal{M}_-(-1, 0)$ then

$$-(1 - \frac{\omega}{\pi})90^\circ < \angle M(e^{j\omega}) < 0 \text{ on } \omega \in [0, \pi].$$

Similar.

(c) If $M \in \mathcal{M}_{+-}(-1, 0)$ then

$$-(1 - \frac{\omega}{\pi})90^\circ < \angle M(e^{j\omega}) < (1 - \frac{\omega}{\pi})90^\circ \text{ on } \omega \in [0, \pi].$$

Express each term as a product of terms in $\mathcal{M}_+(-1, 0)$ and $\mathcal{M}_-(-1, 0)$. ■

Appendix H. Proof of Theorem 5.3

Similar to the proof of Theorem 5.1, the proof of Theorem 5.3 is based on Lemma F.1.

(a) $\mathcal{M}_+(0, 1) \subset \mathcal{M}_{odd}$.

By Lemma F.1(a) each element of $\mathcal{M}_+(0, 1)$ can be expressed as a positively weighted sum of terms in $\mathcal{M}_{2+} \subset \mathcal{M}_{odd}$. Hence the result.

(b) $\mathcal{M}_-(0, 1) \stackrel{P}{\subset} \mathcal{M}_{odd}$.

Suppose $M \in \mathcal{M}_-(0, 1)$ is given by (5). Then $\tilde{M} \in \tilde{\mathcal{M}}_-(0, 1)$ given by (14) is a phase substitute for M . Hence $\mathcal{M}_-(0, 1) \stackrel{P}{\subset} \tilde{\mathcal{M}}_-(0, 1)$. By Lemma F.1(b) each element of $\tilde{\mathcal{M}}_-(0, 1)$ can be expressed as a positively weighted sum of first order terms in $\tilde{\mathcal{M}}_{2-} \subset \mathcal{M}_{odd}$. Hence the result.

(c) $\mathcal{M}_{+-}(0, 1) \stackrel{P}{\subset} \mathcal{M}_{odd}$.

Following the analysis of parts (a) and (b) together with Lemma F.1, it suffices to show that the product of a first order term in $\tilde{\mathcal{M}}_+(0, 1)$ (i.e. a multiplier in $\tilde{\mathcal{M}}_{2+}$) with a first order term in $\mathcal{M}_-(0, 1)$ (i.e. a multiplier in $\mathcal{M}_{2-}$) is itself a Zames-Falb multiplier. Let $\tilde{M} \in \mathcal{M}_{2+}$ and $M \in \mathcal{M}_{2-}$ be given as

$$M(z) = \frac{z^{-1} + b}{z^{-1} + a} \text{ and } \tilde{M}(z) = \frac{z + \tilde{a}}{z + \tilde{b}}, \quad (\text{H1})$$

with

$$0 < \tilde{b} < \tilde{a} < 1 \text{ and } 0 < a < b < 1. \quad (\text{H2})$$

Then

$$\tilde{M}M = m_0 - \tilde{H} - H, \quad (\text{H3})$$

with m_0 given by

$$m_0 = 1 + \frac{(\tilde{b} - \tilde{a})(a - b)}{1 - a\tilde{b}}, \quad (\text{H4})$$

with $\tilde{H}$ anticausal and given by

$$\tilde{H}(z) = \frac{(a - b)(1 - \tilde{a}a)}{1 - a\tilde{b}} \frac{1}{z^{-1} + a}, \quad (\text{H5})$$

and with H causal and given by

$$H(z) = \frac{(\tilde{b} - \tilde{a})(1 - \tilde{b}b)}{1 - a\tilde{b}} \frac{1}{z + \tilde{b}}. \quad (\text{H6})$$

It is straightforward to check that $m_0 > 0$, the impulse responses of $H(z)$ and $\tilde{H}(z)$ change signs, and

$$\|\tilde{H}\|_1 + \|H\|_1 = \frac{(b - a)(1 - \tilde{a}a)}{(1 - a\tilde{b})(1 - a)} + \frac{(\tilde{a} - \tilde{b})(1 - \tilde{b}b)}{(1 - a\tilde{b})(1 - \tilde{b})} < m_0. \quad (\text{H7})$$

To see this, first note that $(1 - \tilde{a})(1 - b)(1 - a\tilde{b}) > 0$. Rearranging

$$\begin{aligned} (b - a)(1 - a\tilde{a})(1 - \tilde{b}) + (\tilde{a} - \tilde{b})(1 - b\tilde{b})(1 - a) &< \\ &< [1 - a\tilde{b} + (\tilde{b} - \tilde{a})(a - b)](1 - a)(1 - \tilde{b}) \end{aligned} \quad (\text{H8})$$

Dividing both sides by $(1 - a)(1 - \tilde{b})(1 - a\tilde{b}) > 0$ gives the result. Hence $M\tilde{M} \in \mathcal{M}_{\text{odd}}$. ■

Appendix I. Proof of Theorem 5.4

(a) If $M \in \mathcal{M}_+(0, 1)$ then

$$0 < \angle M(e^{j\omega}) < \frac{\omega}{\pi} 90^\circ \text{ on } \omega \in [0, \pi].$$

Every multiplier in $\mathcal{M}_{1+}$ has positive phase. Since each term in $\mathcal{M}_+(0, 1)$ can be expressed as a product of terms in $\mathcal{M}_{1+}$, its phase is bounded below by zero. Similarly each term in $\mathcal{M}_-(0, 1)$ has negative phase. But we can express each term in $\mathcal{M}_+(0, 1)$ as a product of a term in $\mathcal{M}_{1+}$ and a term in $\mathcal{M}_-(0, 1)$; hence it is bounded above by the phase of a term in $\mathcal{M}_{1+}$. This in turn is bound above by the limits given in the statement.

(b) If $M \in \mathcal{M}_-(0, 1)$ then

$$-\frac{\omega}{\pi} 90^\circ < \angle M(e^{j\omega}) < 0 \text{ on } \omega \in [0, \pi].$$

Similar.

(c) If $M \in \mathcal{M}_{+-}(0, 1)$ then

$$-\frac{\omega}{\pi}90^\circ < \angle M(e^{j\omega}) < \frac{\omega}{\pi}90^\circ \text{ on } \omega \in [0, \pi].$$

Express each term as a product of terms in $\mathcal{M}_+(0, 1)$ and $\mathcal{M}_-(0, 1)$.

Figure 7. Phase of the multiplier $M_{3+} = \frac{z-1/3+\varepsilon}{z+1/3-\varepsilon}$ with $\varepsilon = 1/100$. The phase exceeds the bounds for both $\mathcal{M}_+(-1,0)$ and $\mathcal{M}_+(0,1)$.