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Effects of Edge on-Site Potential in a Honeycomb Topological Magnon Insulator

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While the deviation of the edge on-site potential from the bulk values in a magnonic topological honeycomb lattice leads to the formation of edge states in a bearded boundary, this is not the case for a zigzag termination, where no edge state is found. In a semi-infinite lattice, the intrinsic on-site interactions along the boundary sites generate an effective defect and this gives rise to Tamm-like edge states. If a nontrivial gap is induced, both Tamm-like and topologically protected edge states appear in the band structure. The effective defect can be strengthened by an external on-site potential, and the dispersion relation, velocity and magnon density of the edge states all become tunable.

I. INTRODUCTION

Many important phenomena in condensed matter physics are related to the formation of edge or surface states along the boundary of finite-size materials. Their existence has been commonly explained as the manifestation of Tamm’s or Shockley’s mechanisms. In recent years it has been revealed that the edge states in the so-called topological insulators are related to the bulk properties. One such property is characterized by an insulating bulk gap and conducting gapless topologically protected edge states that are robust against internal and external perturbations.

Edge states in topological magnon insulators have also attracted a lot of attention recently. Magnons are the quantized version of spin waves, which are the collective propagation of precessional motion of the magnetic moments in magnets. Since there is no particle movement, magnons can propagate over a long distance without dissipation by Joule heating. Similar to spintronics, the study of edge magnons will enrich the potential of magnonics, exploiting spin waves for information processing. For this purpose, the complete understanding of the edge magnon behavior in different lattice structures and the precise control of their properties are urgently required.

The magnon Hall effect was observed in the ferromagnetic insulator Lu2V2O7, in the Kagome ferromagnetic lattice, and in Y3Fe5O12 (YIG) ferromagnetic crystals, and has also been studied in the Lieb and honeycomb ferromagnetic lattices. Interestingly, it has been shown that a ferromagnetic Heisenberg model with a Dzyaloshinskii–Moriya interaction (DMI) on the honeycomb lattice realizes magnon edge states similar to those in the Haldane model for spinless fermions and the Kane–Mele model for electrons. By a topological approach, it has been shown that a nonzero DMI makes the band structure topologically nontrivial, and from the winding number of the bulk Hamiltonian, gapless edge states which cross the gap connecting the regions near the Dirac points have been predicted. The thermal Hall effect and spin Nernst effect have also been predicted for this magnetic system. Using a direct tight-binding formulation in a strip geometry, it was shown that the edge states in a lattice with a zigzag termination closely resemble their fermionic counterpart only if an external on-site potential is introduced at the outermost sites. Furthermore, a lattice with armchair termination has additional edge states to those predicted by a topological approach. Such edge states were found to be strongly dependent on the edge on-site potentials. On the other hand, in a semi-infinite ferromagnetic square lattice, a renormalization of the on-site contribution along the boundary gives rise to spin wave surface states, and most recent experiments on photonic lattices have observed unconventional edge states in a honeycomb lattice with bearded, zigzag and armchair boundaries, which are not present in fermionic graphene. In addition, Tamm-like edge states were also observed in a Kagome acoustic lattice. These unconventional edge states were found to be related to the bosonic nature of the quasi-particles in the lattice whose model Hamiltonians contain on-site interaction terms.

In this work, we explore in more detail the magnon edge states in a ferromagnetic honeycomb lattice with a DMI and an external on-site potential along the outermost sites. Extending our previous work, we develop a general approach applied to zigzag and bearded terminations and we derive analytical expressions for both the energy spectrum and wavefunctions. In a lattice with a boundary, the interaction terms along the outermost sites differ from the bulk values. Such a difference plays the role of an effective defect and gives rise to Tamm-like edge states – the type of edge states generated by a strong perturbation due to an asymmetric termination of a periodic potential. In a similar fashion, the on-site potential along the boundary plays an important role in the appearance of edge states in bosonic lattices. We found that the effective defect can be strengthened by an external on-site potential and this can be used to tune the dispersion relation, velocity and magnon density of the edge states present in the system. For both boundaries under consideration, we present a simple diagram with which the number of magnon edge states can be predicted. In addition, if a nontrivial gap is induced, the edge state band structure is found to be strongly dependent on the on-site interactions.
mulation implemented in this work facilitates the extraction of analytical solutions of both the energy spectrum and wavefunctions for better physical understanding. All our analytical results are in agreement with direct numerical calculations.

II. TIGHT-BINDING MODEL ON THE HONEYCOMB LATTICE

In this section, we briefly present the general approach for the study of the edge states with an arbitrary external on-site potential and with a DMI.

A. Harper’s equation

We consider the following Hamiltonian for a ferromagnetic honeycomb lattice:

\[
H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_{\langle\langle i,j \rangle\rangle} D_{ij} \cdot (\mathbf{S}_i \times \mathbf{S}_j),
\]

where the first summation runs over the nearest neighbors (NN) and the second runs over the next-nearest neighbors (NNN). \(J (> 0)\) is the isotropic ferromagnetic coupling, \(\mathbf{S}_i\) is the spin moment at site \(i\), and \(D_{ij}\) is the DMI vector between the sites. If the lattice is on the \(x\)-\(y\) plane, according to Moriya’s rules\(^{37}\), the DMI vector vanishes for the NN but has a nonzero component along the \(z\) direction for the NNN. Hence, we can assume \(D_{ij} = D_{0ij} \hat{z}\), where \(D_{0ij} = \pm 1\) is an orientation–dependent coefficient in analogy with the Kane–Mele model\(^{38}\). By the Holstein–Primakoff transformation in the linear spin wave approximation (LSWA), the Hamiltonian in Eq. (1) can be written as

\[
H = -t \sum_{\langle i,j \rangle} \left( a_i^\dagger b_j + a_i^\dagger b_j^\dagger - a_i^\dagger a_i - b_j^\dagger b_j \right) + H_D,
\]

where \(a_i\) and \(b_j\) are bosonic operators of the two sublattices, \(t = JS\) with \(S\) the spin quantum number and \(H_D = H_{D,A} + H_{D,B}\) is the DMI contribution. In particular,

\[
H_{D,A} = iDS \sum_{\langle i,j \rangle} \varrho_{ij} \left( a_i^\dagger a_j - a_i^\dagger a_j^\dagger \right),
\]

and a similar expression can be given for \(H_{D,B}\). The Hamiltonian in Eq. (2) is the bosonic equivalent to the Haldane model\(^{39}\), where the NNN complex hopping in Eq. (3) breaks the lattice inversion symmetry and makes the band structure topologically nontrivial. To analyze the edge states we consider a lattice with an open boundary along the \(x\) direction that is semi-infinite in the \(y\) direction as shown in Fig. 1. In the LSWA by denoting wavefunctions on the two sublattices of the honeycomb lattice as \(\psi_{A,n}\) and \(\psi_{B,n}\), respectively, the Harper’s equation provided by the Hamiltonian in Eq. (2) can be written as

\[
\begin{align*}
3\psi_{A,n} - J_1 \psi_{B,n} - J_2 \psi_{B,n-1} + f_{A,n} = \varepsilon \psi_{A,n}, \\
-3\psi_{A,n} - J_2 \psi_{A,n+1} + 3\psi_{B,n} - f_{B,n} = \varepsilon \psi_{B,n},
\end{align*}
\]

where \(n\) is the row index in the \(y\) direction, perpendicular to the boundary, and \(\varepsilon\) is the energy eigenvalue in the unit of \(t\). In the above equation, the DMI is given by \(f_{l,n} = J_3 \psi_{l,n} - J_4 (\psi_{l,n+1} + \psi_{l,n-1})\), with \(l (= A, B)\) the sublattice index. Furthermore, if \(k\) is the momentum in the \(x\) direction, the hopping amplitudes for the lattice with a zigzag boundary are given by: \(J_1 = 2 \cos (\sqrt{3}k/2)\), \(J_2 = 1\), \(J_3 = 2 D' \sin (\sqrt{3}k)\), and \(J_4 = 2 D' \sin (\sqrt{3}k/2)\), where \(D' = D/J\). In addition, the simple replacements of \(J_1 \rightarrow J_2\) and \(J_2 \rightarrow J_1\) in Eq. (4) provide the corresponding Harper’s equation for the lattice with a bearded boundary.

B. Effective Hamiltonian for the edge states

The Harper’s equation of Eq. (4) can be simplified if we assume a decaying Bloch wavefunction in the \(y\) direction of the form \(\psi_{l,n} = z^n \psi_l\), where \(l\) labels each sublattice and the Bloch phase factor \(z\) is a complex number\(^{40,41}\). The eigenequation for the effective Hamiltonian of the edge state can be written with the decaying wavefunction as \(H_{ef} \psi_{l,n} = \varepsilon \psi_{l,n}\), where

\[
H_{ef} = \begin{bmatrix}
3 + J_3 - J_4 \Delta & -w (J_1 + J_2 z^{-1}) \\
-w^{-1} (J_1 + J_2 z) & 3 - J_3 + J_4 \Delta
\end{bmatrix},
\]

and \(\Delta = z + z^{-1}\). In the above equation, the factor \(w\) takes into account the bearded (\(w = z\)) and zigzag (\(w = 1\)) boundaries. The nontrivial solution for the eigenstates of \(H_{ef}\) gives rise to the secular equation

\[
J_a^2 \Delta^2 - (2J_3J_4 - J_1J_2) \Delta - \varepsilon_r^2 + J_1^2 + J_2^2 + J_3^2 = 0,
\]

where \(\varepsilon_r = \varepsilon - 3\). Note that such a polynomial in \(\Delta\) is the same for both the considered boundaries. For a given momentum \(k\) and energy \(\varepsilon\), the solutions of Eq. (6) are
the Bloch phase factors $z_\nu$ with $\nu = 1, \ldots, 4$. In particular, for the infinite system, the Fourier transform in the $y$ direction is the solution $z = e^{\pm ik_y}$, which corresponds to Bloch extended states. In the case of a lattice with a boundary, the solutions of Eq. (6) satisfying $|z_\nu| = 1$ determine the bulk band structure (see Fig. 2). The states with $|z_\nu| \neq 1$ decay or grow exponentially in space, and they can be used to describe the edge states with the appropriate boundary conditions.

The factors $z_\nu$ and $z_\nu^{-1}$ in Eq. (6) always appear in pairs. Since we require a decaying (evanescent) wave from the boundary, setting the condition $|z_\nu| < 1$ implies that the general solution for the edge states can be written as a linear combination of the form,

$$\psi_{\nu,n} = c_1 z_\nu^n \psi^{(1)}_\nu + c_2 z_\nu^n \psi^{(2)}_\nu,$$

where the coefficients $c_i$, with $i = 1, 2$, are determined by the boundary conditions. In the above equation, $\psi^{(\nu)}_\nu$ with $\nu = 1, 2$ is the eigenvector of $H_{ef}$ corresponding to the $\nu$-th decaying solution of Eq. (6). To obtain the edge state energy spectrum, the wavefunctions given by Eq. (7) must satisfy the boundary conditions. This will be described in the following sections.

III. BOUNDARY CONDITIONS AND THE EDGE STATES

In this section, the boundary conditions for both zigzag and bearded boundaries are obtained. In particular, we introduce an on-site potential to the edges by adding the single-ion anisotropy term

$$H_A = \delta_1 J \sum_i (S_i^z)^2,$$

in the case of on-site potentials in the Fermionic models on a honeycomb lattice.\textsuperscript{42,43}

A. Zigzag boundary

In our previous work,\textsuperscript{29} we derived the equations for the energy and the wavefunctions considering a fixed on-site potential $\delta_1 = 1$, where the edge-state energy spectrum and the wavefunctions closely resemble those for fermionic graphene. Here, we first summarize, then extend the formalism to arbitrary values of the external on-site potential.

Due to the open boundary, the on-site potential along the boundary is different from that in the bulk. The Harper’s equation of Eq. (4) at $n = 1$ must be modified. Considering the missing bonds along the outermost $A$ site, the coupled Harper’s equation at $n = 1$ is written as

$$(2 - \delta_1) \psi_{A,1} - J_1 \psi_{B,1} + f_{A,1} = \varepsilon \psi_{A,1},$$

$$3\psi_{B,1} - (J_1 \psi_{A,1} + J_2 \psi_{A,2}) - f_{B,1} = \varepsilon \psi_{B,1},$$

where the external on-site potential $\delta_1$ given by Eq. (8) is introduced and $f_{1,1} = J_3 \psi_{1,1} - J_4 \psi_{1,2}$. In the above equation, the total energy at each sublattice is given by the on-site contribution (first term), the NN contribution (second term), and the DMI (third term). From Eqs. (4) and (9), we obtain the zigzag boundary conditions

$$(1 - \delta_1) \psi_{A,1} - J_4 \psi_{A,0} = 0,$$

$$\psi_{B,0} = 0,$$

for the edge-state wavefunctions in Eq. (7). Unlike the equivalent fermionic model,\textsuperscript{44} where the wavefunctions of both sublattices vanish at $n = 0$, Eq. (10) contains the on-site contribution at $n = 1$. As we will show in the following sections, such a contribution has important effects on the band structure of the edge states. From Eqs. (7) and (10) the nontrivial solution for the coefficients $c_i$ provides the following self-consistent equation for the edge-state energy spectrum:

$$\varepsilon = 3 + J_3 - J_4 \left[ \frac{f_1 + f_2}{\delta_0 J_1 z_1 z_2 - J_2 J_4} \right],$$

with

$$f_1 = [\delta_0 J_1 - J_2 J_4] (z_1 + z_2)$$

$$f_2 = [\delta_0 J_2 + J_1 J_4] (1 - z_1 z_2),$$

and $\delta_0 = \delta_1 - 1$. In Eqs. (11) and (12), $z_1$ and $z_2$ are two decaying solutions of Eq. (6). The corresponding edge-state wavefunctions are given by

$$\psi_{A,n} = c_1 (z_1^n - \alpha z_2^n) \psi^{(1)}_A,$$

$$\psi_{B,n} = c_1 (z_1^n - z_2^n) \psi^{(1)}_B,$$

FIG. 2. (Color online) (a) Edge-state dispersion relations induced by $\delta_1$ at the zigzag open boundary shown for $\delta_1 = 0.2, 0.6, 0.8$, and 1.0 (from the curved band to the flat band). For $\delta_1 = 0$ there are no edge states. The shaded (blue) region is the bulk continuum, where all the factors have $|z_\nu| = 1$ in Eq. (6). (b) Modulus of the decaying factors for the corresponding edge states, where $\pm 1$ is the sign of $z_1$.\textsuperscript{47}
where $c_1$ is a normalization term and
\begin{equation}
\alpha = \frac{(1 - \delta_1) z_1 - J_4}{(1 - \delta_1) z_2 - J_4}
\end{equation}
contains the contribution of the external on-site potential in the wavefunction. For a given momentum $k$, external potential $\delta_1$, and nonzero DMI, Eq. (11) is an implicit equation for the energy $\varepsilon$ and can be solved numerically. Equations (6), (11), and (13) provide a full description for the edge-state energy spectra and their corresponding wavefunctions.

### B. Bearded boundary

Similar to the zigzag case, by modifying the Harper’s equation at $n = 1$ to take into account the missing sites, the boundary conditions for the wavefunctions in Eq. (7) are given by
\begin{equation}
(2 - \delta_1) \psi_{B,1} + J_4 \psi_{B,0} = 0, \\
\psi_{A,0} = 0.
\end{equation}

From Eqs. (7) and (15) the nontrivial solution for the coefficients $c_i$ can also be obtained. We find that the simple replacements, $J_1 \rightarrow J_2$, $J_2 \rightarrow J_1$, $J_3 \rightarrow -J_3$, $J_4 \rightarrow -J_4$, and $\delta_1 \rightarrow \delta_1 + 1$ in Eq. (11) provide a self-consistent equation for the edge state-energy spectrum. The wavefunctions satisfying the boundary conditions of Eq. (15) are given by
\begin{equation}
\psi_{A,n} = c_1 (z_1^n - z_2^n) \psi_{A}^{(1)}, \\
\psi_{B,n} = c_1 (z_1^n - \alpha' z_2^n) \psi_{B}^{(1)},
\end{equation}
where $c_1$ is a normalization term and
\begin{equation}
\alpha' = \frac{(2 - \delta_1) z_1 + J_4}{(2 - \delta_1) z_2 + J_4}.
\end{equation}

Equations (6) and (15) together with Eq. (16) provide a full description for the edge-state energy spectrum and wavefunctions. For an arbitrary external on-site potential and zero DMI, the $k$-dependence of $\varepsilon(k)$ and the explicit solutions for the decaying $z$ factors are obtained in appendix A.

### IV. ENERGY SPECTRUM AND WAVEFUNCTIONS

#### A. Zero DMI

In a fermionic honeycomb lattice with a boundary, it is well known that there are flat edge states connecting the two Dirac points, $K \rightarrow K'$, in a lattice with a zigzag boundary, whereas in a lattice with a bearded boundary, the flat edge state is connecting the complementary region, $K' \rightarrow K$. In the equivalent bosonic models, we expect different situations due to the contribution of the on-site interactions along the boundary sites.
1. Zigzag boundary

For a zigzag boundary, in the absence of external on-site potentials and zero DMI, the solutions of Eqs. (6) and (9) provide bulk states with $z^2 = 1$ and energy $\varepsilon = 2 \pm |J_1|$, and no edge state is found. However, a Tamm-like edge state can be induced if the effective defect is strengthened by including an external on-site potential, Eq. (8). In Fig. 2 the energy spectra and the decaying factors of the induced edge states are shown for different values of $\delta_1$. As the external on-site potential increases ($\delta_1 \rightarrow 1$), the branch becomes flatter [Fig. 2(a)] and from the edge state wavefunctions

$$\begin{align*}
(\psi_{A,n}, \psi_{B,n}) &= z^n \left( \frac{z^{-1}}{1 - \delta_1 J_2} \right),
\end{align*}$$

the magnon density is found to be more localized in a single lattice (See Fig. 3). In the above equation, the decaying factor $z$ is a real number. For a wide ribbon\textsuperscript{4,5}, the edge-state energy spectrum is doubly degenerated and since the magnon velocity is the slope of the energy spectrum, the magnons are moving in the same direction at opposite edges, as illustrated in Fig. 4(a). As shown in Fig. 2(a), as $\delta_1$ is increased, the slope (and the edge magnon velocity) is reduced until $\delta_1 = 1$ where the edge state becomes nondispersive.

If the external on-site potential is increased, the number of edge states changes as well as their shape. Depending on the external on-site potential strength, a zigzag termination can have two edge states at each boundary. In the decaying factor diagram of Fig. 5(a), each edge state has a corresponding decaying factor $z_1$ or $z_1'$. For $0 < \delta_1 < 2$, there is a single decaying factor between the Dirac points (see also Fig. 2), and from Eq. (18) it is straightforward to show that the edge state in this region is mainly localized at the $A$ sublattice. As shown in Fig. 5(a), for $\delta_1 > 2$ there are two edge states. The first one, corresponding to $z_1'$, is defined over all the Brillouin zone with energy spectra over the bulk bands (due to the strong external on-site potential). The second edge state, corresponding to $z_1$, is defined between the $K'$ and $K$ points as in bearded graphene (note that $K'$ and $K$ have an $x$-component only). Such an edge state has a magnon density mainly localized at the $B$ sublattice with energy spectrum between the bulk bands. If the external on-site potential is even stronger, $\delta_1 > 2$, the system effectively shows the band structure of a bearded termination plus a high-energy Tamm-like edge state. Moreover, as we mentioned before, in the absence of external on-site potential ($\delta_1 = 0$) there are no edge states. At $\delta_1 = 2$, there are also no edge states. This can be observed in Fig. 5(a), where at such a value, $|z_1| = |z_1'| = 1$ for all values of $k$. At the transition lines (dashes) the modulus of the decaying factors reaches unity and the edge states are indistinguishable from the bulk bands.

The magnon excitations in a ferromagnetic lattice can be viewed as a synchronous precession of the spin vectors. The sign of the wavefunctions in Eq. (18) is related to the spin precession in successive rows and the wavefunction modulus is related to the radius of precession which decreases as $n$ increases. If we write the phase of the wavefunction as $e^{i\theta_l} = \text{sgn}(\psi_{l,n})$, then, for a given $k$ and $0 < \delta_1 < 2$, the synchronous precession of the spins in successive rows is in antiphase ($\theta_l = \pi$, optic-like) if $k < k_0(= \pi/\sqrt{3})$ and in-phase ($\theta_l = 0$, acoustic-like) if $k > k_0$. Furthermore, at the same $n$, the spins at different sublattices are precessing in antiphase for $k < k_0$ and in-phase for $k > k_0$ [See Fig. 3(a)]. At the transition point $k_0$, the edge-state energy is $\varepsilon_0 = 2 + \delta_1$ and the decaying factor is zero as shown in Fig. 2(b). Hence, for $\delta_1 \neq 0, 2$, and by Eq. (18), the magnon is completely localized at the edge site.

2. Bearded boundary

We now consider a bearded termination. As shown in Fig. 1(b), the outermost site has two missing bonds and the effective defect is stronger than that corresponding to a zigzag boundary. Contrary to the fermionic equivalent, the on-site terms provided by Eq. (2) substantially change the edge-state band structure. This is shown in Fig. 6(a), where for $\delta_1 = 0$ there are two edge-state energy bands as given by Eq. (A6), the first one between the Dirac points (dot-dashed black line) and the second one below the lower bulk bands (dashed black line). Such edge states are defined in a range of $k$ completely different from their fermionic equivalent\textsuperscript{46,47}. As shown in Fig. 6(b), the edge state below the bulk bands is defined over all the Brillouin zone, except at $k = 0, 2\pi/\sqrt{3}$, where the decaying factor reaches unity and the edge state is indistinguishable from the bulk bands. As is shown in Appendix A, the edge-state wavefunctions are given by

$$\begin{align*}
\left(\psi_{A,n}, \psi_{B,n}\right) &= z^n \left( \frac{2 - \delta_1}{z^{-1}} \right),
\end{align*}$$

where $z$ is a real number. In the above equation $z = z_1'$ for the edge state below the lower bulk bands and $z = z_1$ for the edge state between the Dirac points (see Fig. 5). In Figs. 6(c) and 6(d), we plot the magnon density $|\psi_{l,n}|^2$ for both edge states at different momentum. Note that the edge states are strongly localized in different sublattices.

Discussion of some interesting features of these edge states is in order here. From Fig. 6(a), for $\delta_1 = 0$ the slope of the edge-state energy spectra is positive if $k < k_0$ and negative if $k > k_0$. For a wide ribbon, each edge band is doubly degenerated; hence, the magnons are moving in the same direction (with different energy) at each boundary, as illustrated in Fig. 4(c). The fact that both edge states are strongly localized in different sublattices can be explained if we consider the edge itself as a defect. By closer inspection of the wavefunctions in Eq. (19), the edge state below the lower bulk bands is mainly localized along the boundary $B$ sites due to the strong attractive
The behavior of the phase in successive rows is different there are two decaying factors and their sign reveals that optic-like or acoustic-like. As described in Appendix A, the phase of the spin precession in successive rows. As an implicit form of the wavefunction in Eq. (19) is given by

\[ \psi = e^{i \Delta \theta} \]

the magnitude of the spin density are proportional to the radius of each circle with the phase given by \( e^{i \Delta \theta} = \pm 1 \).

FIG. 6. (Color online) (a) Bulk (shaded) and edge-state energy spectra for \( \delta_1 = 0 \) (black dashed and dot-dashed lines), \( \delta_1 = 1.8 \) (red continuous lines) and \( \delta_1 = 2 \) (green dotted lines). In (b) we show their corresponding decaying factors. The magnitude density profile is shown for the edge states with \( \delta_1 = 0 \) at (c) \( k = 1.40 \) and (d) \( k = 0.96 \). Here, the magnitudes of the spin density are proportional to the radius of each circle with the phase given by \( e^{i \theta} = \pm 1 \).

potential generated by the missing bonds. The edge state between the bulk bands is mainly localized along the A sublattice due to the presence of the outermost B site. In consequence, the outermost B site plays a double role, acting as an effective defect to host an edge state and contributing to the formation of the edge state between the Dirac points.

The number of edge states is determined by the number of solutions of Eq. (A3) with modulus lower than one and the edge-state dispersion can be tuned in all the Brillouin zone by small changes in the external on-site potential. This is shown in the decaying factor diagram in Fig. 5(b), where the dashed lines separate the regions in which each edge state is defined. In the region \( 0 \leq \delta_1 < 1 \), there are always two edge states (for \( z'_1 \) and \( z_1 \)). If \( \delta_1 = 0 \), the first edge state is defined over all the Brillouin zone with \( |z'| < 1 \), and the second one is defined between the Dirac points with \( |z| < 1 \). As \( \delta_1 \) is increased, both edge states gradually merge with the bulk bands. For \( \delta_1 = 2 \), there is a single edge state with a momentum between the \( K' \) and \( K \) points. This edge state is the flat band in Fig. 6(a) (dotted green line), where the energy spectrum closely resembles the fermionic graphene. If the external on-site potential is increased further, \( \delta_1 \gg 2 \), the hopping between sites at \( n = 1 \) is almost completely suppressed and the system effectively shows the band structure of a zigzag termination plus a high-energy Tamm-like edge state along the boundary sites.

Another important characteristic provided by the explicit form of the wavefunction in Eq. (19) is given by the phase of the spin precession in successive rows. As discussed in the previous section, the sign of the decaying factor determines whether the phase of the edge state is optic-like or acoustic-like. As described in Appendix A, there are two decaying factors and their sign reveals that the behavior of the phase in successive rows is different in both edge states. In particular, for \( \delta_1 = 0 \) the decaying factor of the edge state connecting the Dirac points is negative if \( k < k_0 \), and the spin precession in successive lattice sites is hence anti-phase (optic-like). However, the decaying factor of the edge state below the lower bulk bands is positive if \( k < k_0 \), and the spins in two successive rows are in-phase (acoustic-like). This provides us with two ways to distinguish these edge states, either by their energy or by their phase difference in successive rows.

Experimentally, the first observation of edge states in a honeycomb lattice with beaded boundaries was achieved in optical lattices\(^{34}\). Apart from the typical band structure, additional edge states have been observed near Van Hove singularities. As is shown in Fig. 6(a) for our model, similar edge states are obtained for an external on-site potential of \( \delta_1 = 1.8 \). Here a nearly flat band plus two highly dispersive edge states near the Van Hove singularities (continuous red lines) are obtained. As pointed out in the Ref.\(^{34}\), the origin of such edge states is also related to the effective defect generated by the on-site potential along the boundary sites.

B. Nonzero DMI

A nonzero DMI breaks the lattice inversion symmetry and a nontrivial gap is induced in the spin wave excitation spectra. By a topological approach using the wavefunctions for the infinite system, the Chern number predicts a pair of counterpropagating modes\(^{26}\) along the boundary of the finite system. However, the topological approach does not provide detailed properties of the edge states and also does not take into account the on-site potential along the boundary sites, which, as we will show in this section, has important effects on the band structure of the edge states.

1. Zigzag boundary

We first consider a zigzag boundary. The energy bands are obtained by solving the self-consistent Eq. (11) with the decaying factors provided by Eq. (6). In Fig. 7(a) we show the energy bands for a DMI strength of \( D = 0.1J \). The blue regions correspond to the bulk spectra where all the factors are \( |z_n| = 1 \). The bands which transverse the gap are the spectra of the edge states for different values of \( \delta_1 \). For completeness, we also include the energy spectrum for the edge state at the opposite edge (at large \( n \)) without external on-site potential. Contrary to the description obtained by a topological approach\(^{26,27}\), the edge state does not connect the regions near the Dirac points. As shown in Fig. 7(a), for \( \delta_1 = 0 \) (red continuous line) the intrinsic on-site potential along the boundary pulls the edge state within the bulk gap to a lower-energy region, slightly above the lower bulk band. Furthermore, a new edge state near the Van Hove singularities is re-
increased, the slope of the energy spectrum decreases. In particular, for $\delta_1 = 1$ (uniform case) the energy spectrum closely resembles that of fermionic graphene with merging points near the Dirac points and with the magnons moving in opposite directions at different boundaries, as illustrated in Fig. 4(d). In Fig. 7(c) the modulus of the decaying factors is shown for different values of the external on-site potential. Here, as $\delta_1$ increases, the merging points approach from the left to the $K$ and $K'$ points and the asymmetry around $k_0$ is reduced. In the finite region [Fig. 7(c)] around $k_0$, we have $|z_1| = |z_2|$, and from Eq. (6) it is evident that the decaying factors are complex conjugates to each other. At a certain value of $k$ both decaying factors become real and each $z$ is no longer identical, and as we mentioned before, while one factor increases, the other one decreases. The region around $k_0$, where the decaying factors are complex conjugates to each other, is defined for a nonzero DMI and is located within the bulk gap. Its boundaries in the $k$ space are given by the discriminant of Eq. (6) which is independent of the boundary conditions. In this region, the edge-state wavefunction is complex.

2. Bearded boundary

We now consider a bearded termination with a nonzero DMI and arbitrary external on-site potential. The solutions can be obtained by solving self-consistent equation provided by Eq. (15) and the wavefunctions provided by Eq. (16). As is shown in Fig. 8, for $\delta_1 = 0$ there is an edge state crossing the gap (red continuous line) and an edge state below the lower bulk bands (purple continuous line). Note that the nonzero DMI changes the shape of the edge magnon spectrum. In fact, the edge state within

![Graphical representation of energy spectra and wavefunctions](image-url)
the gap has a negative slope except near the $K$ point where it is almost flat. The edge-state energy spectrum below the lower bulk band has a maximum point where its slope changes. Before this point and away from the almost flat region, the propagation is similar to that in Fig. 4(b) where, for a fixed momentum and at the same boundary, the magnons move in opposite directions. On the other hand, as shown in Fig. 8(a), to the right of the $K'$ point, there are two edge bands with a negative slope (red and purple continuous lines) and a single edge band with a negative slope at the opposite boundary (green dot-dot-dashed line), hence the magnons are moving in the same direction at both edges.

The effective defect due to the missing bonds is strong in the bearded boundary, where the edge-state energy spectra are distinct from their fermionic equivalent. As shown in Fig. 8(b), the edge state within the bulk gap (red continuous line) is defined in the region to the right of the $K$ point. The edge state below the lower bulk bands is defined over the whole Brillouin zone and since its origin is due to the effective defect discussed in the previous section, it is not sensitive to small changes in the DMI strength. In Fig. 8(c) the decaying factors for this edge state are shown. The curves are almost symmetric around $k_0$ and since the decaying factors are real, the wavefunction decays exponentially to the inner bulk sites. As discussed in the previous section, as we move away from $k_0$, one decaying factor approaches to unity while the other one decreases. Note that Fig. 8(b) is similar to Fig. 7(b) except that the plots are tilted in the other direction. Here, as the external on-site potential increases, the merging points approach to $k_0$. In particular for $\delta_1 = 2$, the edge state has an energy spectrum connecting the Dirac points (black dotted line in Fig. 8(a)). However, for the same value of $\delta_1 = 2$ and around $k_0$ (Van Hove singularity), there is a small region in which a highly dispersive (and almost indistinguishable) edge state is also defined, (black-dotted line in Fig. 8(a) and 8(b)). If $\delta_1 \gg 2$, as in the case for $D = 0$, the system shows the band structure of a zigzag termination plus a high-energy Tamm-like edge state.

V. CONCLUSIONS

We have studied the on-site potential effects on the magnon edge states in a honeycomb ferromagnetic lattice with zigzag and bearded boundaries, extending our earlier work on the case of a fixed on-site potential. For zero DMI, the relation between the formation of the Tamm-like edge states and the effective defect due to the on-site potential along the outermost sites has been demonstrated. For nonzero DMI, we have found that the edge-state energy spectrum is modified by the missing bonds along the boundary sites and that their distribution in the momentum space is different from that predicted by a topological approach. For both zigzag and bearded boundaries and for zero and nonzero DMI, the edge-state properties have been discussed and Tamm-like edge states have been revealed. We have found that the Tamm-like and topologically protected edge states are tunable by modifying the external on-site potential and the DMI. Furthermore, the obtained analytical expressions for the edge-state energy spectrum and their corresponding wavefunctions have given us complete understanding of the edge state properties. Our results may explain the unconventional edge states recently found in optical and acoustic lattices and motivate new experiments on topological insulators.

The interesting properties of the honeycomb lattice may be experimentally accessible through engineered spin structures on metallic surfaces, by ultracold bosonic atoms trapped in optical lattices, and so forth. Therefore, the distribution of the edge magnons, the spin density, and their dependence on the DMI strength and external on-site potential as presented in this paper could be useful for experiments on small monolayers, thin-film magnets, and artificial lattices.

Finally, we would like to point out that a recent work on a system of two interacting bosons (doublon) in the Haldane model on the honeycomb lattice has derived an effective Hamiltonian similar to that of Eq. (2), and numerical solutions of an edge state similar to those in Fig. 8, have been found for a bearded boundary. Dispersive edge states similar to those in Fig. (2) for Dirac magnons in a honeycomb ferromagnet have also been produced. Both these works have confirmed the results of our general approach and the analytical solutions for both the energy spectra and wavefunctions presented here, as an extension of our earlier work.

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Appendix A: Analytical solutions for $D = 0$

In this appendix, we derive the edge-state energy spectrum and wavefunctions for a semi-infinite ferromagnetic honeycomb lattice with a bearded boundary in absence of the DMI and with an arbitrary external on-site potential $\delta_1$. From Eq. (6), for $D = 0$, the characteristic equation of the Hamiltonian in Eq. (5) is given by

$$ (3 - \varepsilon)^2 - J_1^2 - J_2^2 - J_1 J_2 \left( z + z^{-1} \right) = 0. $$

(A1)

For a fixed value of $k$, the above equation relates the decaying factor $z$ with the energy $\varepsilon$. From the Harper’s equation of Eq. (4) with the replacements $J_1 \rightarrow J_2$, $J_2 \rightarrow J_1$ and taking into account the missing bonds, the additional equation for the edge state at $n = 1$ is written
\[
(3 - \varepsilon) (1 + \delta_1 - \varepsilon) - J_1 J_2 z + J_2^2 = 0. \tag{A2}
\]

Here, both Eqs. (A1) and (A2) provide us with a complete set of equations for the decaying factor and the energy spectrum. Therefore, for an arbitrary external on-site potential \( \delta_1 \), the decaying factor satisfies,

\[
a z^2 + b z + c = 0, \tag{A3}
\]

where \( a = (-2 + \delta_1)^2 J_2 \), \( b = J_1 \left( (-2 + \delta_1)^2 - J_1^2 \right) \), and \( c = -J_1^2 J_2 \). Explicitly, we obtain

\[
z_1^{(1)} = \frac{-(\delta_0^2 - J_1^2) J_1 \pm |J_1| \sqrt{(\delta_0^2 - J_1^2)^2 + 4\delta_0^2}}{2\delta_0^2}, \tag{A4}
\]

where \( \delta_0 = -2 + \delta_1 \), \( J_2 = 1 \), and \( z_1 \), \( z_1' \) are the solutions corresponding to each sign. On the other hand, the edge-state energy spectrum satisfies,

\[
a_1 \varepsilon_r^2 + b_1 \varepsilon_r + c_1 = 0, \tag{A5}
\]

where \( \varepsilon_r = (\varepsilon - 3) - (-2 + \delta_1) \), \( a_1 = (-2 + \delta_1) J_1 \), \( b_1 = b \) and \( c_1 = -(-2 + \delta_1) J_1 J_2^2 \). For the edge-state energy spectra the two solutions are given by

\[
\varepsilon^{\pm} = \frac{6\delta_0 + \delta_0^2 + J_1^2 \pm \text{sgn}(J_1) \sqrt{(\delta_0^2 - J_1^2)^2 + 4\delta_0^2}}{2\delta_0}. \tag{A6}
\]

From the above equation and by closer inspection of the decaying factors given by Eq. (A4), two edge states can be defined. The wavefunction satisfying the boundary condition

\[
(2 - \delta_1) \psi_{B,1} - J_1 \psi_{A,0} = 0, \tag{A7}
\]

can be written as

\[
\psi_{l,n} = z_1^n \left( \frac{2\delta_1}{z_1^2} \right), \tag{A8}
\]

where the decaying factor \( z_1 \) is given by Eq. (A4). At \( k_0 = \pi/\sqrt{3} \), the edge states are completely localized at the boundary sites with energy

\[
\varepsilon^{\pm}_{k_0} = \frac{1}{2} (6 + \delta_0) \pm \sqrt{4 + \delta_0}. \tag{A9}
\]

In particular, as in graphene, for \( \delta_1 = 2 \) in Eq. (A3), a single decaying factor, \( z_1 = -J_2/J_1 \), with a corresponding flat energy band, \( \varepsilon = 3 \) in Eq. (A5), is obtained. Following the same procedure, the analytical form of the decaying factor and the edge state energy spectrum for a zigzag boundary can also be obtained.

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