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A unified and integrated numerical-analytical approach for evaluation of J_k -integrals of an interfacial crack in orthotropic and isotropic bimetals**Azam Tafreshi**School of Mechanical, Aerospace and Civil Engineering, University of Manchester, M13 9PL,atafreshi@manchester.ac.uk

A new unified and integrated method for the numerical-analytical calculation of J_k -integrals of an in-plane traction free interfacial crack in homogeneous orthotropic and isotropic bimetals is presented. The numerical algorithm, based on the boundary element crack shape sensitivities, is generic and flexible. It applies to both straight and curved interfacial cracks in anisotropic and/or isotropic bimetals. The shape functions of semi-discontinuous quadratic quarter point crack tip elements are correctly scaled to adapt the singular oscillatory near tip field of tractions. The length of crack is designated as the design variable to compute the strain energy release rate precisely. Although an analytical equation relating J_1 and stress intensity factors (SIFs) exists, a similar relation for J_2 in debonded anisotropic solids for decoupling SIFs is not available. An analytical expression was recently derived by this author for J_2 in aligned orthotropic/orthotropic bimetals with a straight interface crack. Using this new relation and the present computed J_k values, the SIFs can be decoupled without the need for an auxiliary equation. Here, the aforementioned analytical relation is reconstructed for cubic symmetry/isotropic bimetals and used with the present numerical algorithm. An example with known analytical SIFs is presented. The numerical and analytical magnitudes of J_k for an interface crack in orthotropic/orthotropic and cubic symmetry/isotropic bimetals show an excellent agreement.

Keywords

Boundary element, interface crack in bimetals, J_k -integrals, anisotropic materials, SIF, shape sensitivities

NOMENCLATURE

a	=	half crack length
a_k	=	local coordinates
$C_{jk}(P_B)$	=	limiting value of $T_{jk}(P_B, Q_B)$
E_j	=	moduli of elasticity ($j=1,2$)
G_{12}	=	modulus of elasticity in shear
I_{sc}	=	integral related to strain discontinuities and strain energy density
K_1, K_2	=	stress intensity factors

N_i, N_{c_i} = geometrical and collocation nodes, respectively ($i=1,3$)

n_k = unit normal vector

S_{mn} = compliance matrix ($m,n=1,2,6$)

t_j = traction vector ($j=1,2$)

u_j = displacement vector ($j=1,2$)

x_j = rectangular Cartesian coordinates

z_j = complex coordinates ($z = x_1 + ix_2$)

β_0 = bimaterial constant

ε = bimaterial constant

ε_{jk} = strain tensor ($j,k=1,2$)

$\eta_{jk,1}, \eta_{1,jk}$ = mutual influence coefficients

ν_{jk} = Poisson's ratio

σ_{jk} = stress tensor ($j,k=1,2$)

ζ_j = coordinates of load point ($j=1,2$)

ζ = natural coordinate

INTRODUCTION

Layered dissimilar anisotropic materials are extensively used in a variety of industries where it is vital to employ materials with high specific stiffness and strength. A typical example is the application of fibre reinforced laminated composites. However, delamination or cracking is the most common form of failure in these materials during manufacturing or in service applications.

Williams¹ was the pioneer in discovering that the stress field along an interfacial crack has an oscillatory behavior. His analytical approach mainly focused on linear elastic interfacial fracture mechanics (LEIFM) of isotropic materials and followed by Rice and Sih², Erdogan and England³⁻⁵. Clements⁶, Willis⁷, Wu⁸, Qu and Li⁹, Qu and Bassani¹⁰, Suo¹¹, Ting¹²⁻¹³, Beom and Alturi¹⁴ derived the related theories of LEIFM for debonded anisotropic bimaterials.

Numerical methods such as the boundary element (BE) and finite element (FE) methods are popular techniques for the solution of problems related to LEIFM. As discussed in references¹⁵⁻¹⁶, a variety of numerical algorithms are available for analysis of debonded isotropic bimaterials and fracture in homogeneous solids. However, in anisotropic bimaterials with interfacial cracks, the traction and displacement data close to the crack tip are commonly

employed to compute SIFs¹⁷⁻²³. Therefore, a very fine mesh in these regions is usually employed.

In the recent studies of this author¹⁵⁻¹⁶, a new algorithm for the computation of J_k -integrals for flaws in isotropic and homogeneous bimetals was introduced. This method was based on the boundary element crack shape sensitivities (BECSS) and is applicable to both straight and curved interfacial cracks. In contrast to the available methods, this algorithm does not need the stress and strain values at internal points or an auxiliary equation to separate SIFs. Here that algorithm is expanded to cover the analysis of anisotropic bimetals. Once numerical J_k -values are obtained, analytical relations are required to decouple SIFs. Although an analytical equation relating J_1 -integral and SIFs exists, a similar relation for the J_2 -integral in debonded anisotropic solids is not available. It should be noted that for a straight crack J_1 -integral and strain energy release rate (SERR) are identical.

In a recent study by this author²⁴, an analytical relation between J_2 -integral of a straight interfacial crack in aligned orthotropic/orthotropic bimetals using the complex function method (CFM) was derived. In this study numerical J_k values in conjunction with this analytical expression can be used to estimate SIFs of these materials.

It is well-known that an analytical relation for an anisotropic material is not always applicable to an isotropic material by merely inserting its engineering constants. The other objective of this study is to reconstruct this new analytical relation for the J_2 -integral of cubic symmetry/isotropic bimetals and then use it in conjunction with the present numerical algorithm.

In LEIFM, the majority of numerical and experimental methods rely on the analytical equations relating J_k -integrals and SIFs. As mentioned before, an analytical expression exists relating J_1 and SIFs for debonded anisotropic solids¹¹. For a mixed mode flaw in a homogeneous and isotropic material, the SIFs and J_2 are related by an analytical equation. This relation mainly depends on SIFs and modulus of elasticity (E). Gosz et al²⁵ and Khandelwal et al²⁶ derived a relation for J_2 -integral of an interface flaw in dissimilar isotropic and homogeneous bimetals. This equation depends on SIFs, engineering constants of the solids and a bimaterial constant (ε). This relation becomes numerically indefinite when applied to homogenous materials where $\varepsilon = 0$. Wang et al²⁷ and Chu and Hong²⁸ showed that the J_2 -integral of a crack in a homogeneous anisotropic structure can be directly related to the corresponding SIFs. This relation depends on SIFs, engineering constants and characteristic roots of the material. This relation becomes numerically infinite when applied

to isotropic materials where their characteristic roots are the same and equal to 1. Therefore, it would be ideal to have a unique relation between J_2 -integral and SIFs which can be applied to an interfacial crack in dissimilar solids with any combinations of material properties. This is provided, in this unique relation, the multipliers of the SIFs are defined in terms of the material properties and bimaterial constants.

Here a new numerical algorithm for computation of J_k -integrals of in-plane anisotropic and/or isotropic bimaterials with a straight or curved interfacial crack using the BECSS is presented. The shape functions of the crack tip elements are accurately scaled to model the singular oscillatory near tip field of tractions. Since the crack length, straight or curved, is designated as the design variable, the SERR is precisely computed. Having the computed J_k -values and using the recently derived analytical relation for J_2 , it is possible to estimate the SIFs of aligned orthotropic/orthotropic bimaterials with a straight interface crack. The aforementioned analytical expression is also reconstructed for cubic symmetry/isotropic bimaterials and used with the present numerical algorithm. Generally, this study presents a unified and integrated framework for LEIFM using BECSS of an in-plane crack in orthotropic/orthotropic, cubic symmetry/isotropic and isotropic/isotropic bimaterials. It is also perfectly applicable to a mixed mode flaw in a homogeneous solid, anisotropic or isotropic. An example with known analytical SIFs and a variety of material properties is presented to verify this study.

STRESS-STRAIN RELATIONS FOR PLANE ELASTICITY: ANISOTROPIC VS ISOTROPIC

The constitutive equations for a plane stress, homogeneous and anisotropic material is

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \quad (1)$$

where ε_{jk} , σ_{jk} and S_{mn} are the strains, stresses and compliances, respectively. The compliances and engineering constants are related as follows²⁹:

$$S_{11} = \frac{1}{E_1}, S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}, S_{16} = \frac{\eta_{12,1}}{E_1} = \frac{\eta_{1,12}}{G_{12}}, S_{22} = \frac{1}{E_2}, S_{26} = \frac{\eta_{12,2}}{E_2} = \frac{\eta_{2,12}}{G_{12}}, S_{66} = \frac{1}{G_{12}} \quad (2)$$

The above equation can also be used for isotropic materials where

$$E_1 = E_2 \equiv E, \nu_{12} \equiv \nu, G_{12} \equiv G = \frac{E}{2(1+\nu)}$$

The constitutive, compatibility and equilibrium equations for 2D plane (stress or strain) homogeneous and anisotropic materials can be combined to attain the governing equations. For the stress function to be solved the following equation must be satisfied.

$$S_{11}\mu^4 - 2S_{16}\mu^3 + (2S_{12} + S_{66})\mu^2 - 2S_{26}\mu + S_{22} = 0. \quad (3)$$

The roots of equation (3) are either complex or purely imaginary²⁹. For specially orthotropic materials where $S_{16} = S_{26} = 0$, these roots are purely imaginary and can be written as: $\mu_1 = ip_1, \mu_2 = ip_2, \mu_3 = \bar{\mu}_1, \mu_4 = \bar{\mu}_2$ where p_1 and p_2 are real numbers. For a material with cubic symmetry, equation (3) reduces to

$$\mu^4 + \frac{2S_{12} + S_{66}}{S_{11}}\mu^2 + 1 = 0. \quad (4)$$

If $\rho = \frac{2S_{12} + S_{66}}{2S_{11}} > 1$, the roots will be purely imaginary. For an isotropic material which is a degenerate orthotropic material $\rho = 1$ and $p_1 = p_2 = 1$.

For 2D in-plane linear elastic, homogeneous and anisotropic materials, the stress (σ_{jk}) and displacement fields (u_j) can be formulated^{11,30-31} using two complex functions $f_j(z_j)$ which are analytical and $z_j = x_1 + \mu_j x_2$ as

$$u_j = 2\text{Re}\left\{A\begin{bmatrix} f_1(z_1) \\ f_2(z_2) \end{bmatrix}\right\}, [\sigma_{j2}] = 2\text{Re}\left[B\begin{bmatrix} f_1'(z_1) \\ f_2'(z_2) \end{bmatrix}\right], [\sigma_{j1}] = -2\text{Re}\left[C\begin{bmatrix} f_1'(z_1) \\ f_2'(z_2) \end{bmatrix}\right] \quad (j=1,2) \quad (5)$$

The imaginary and real parts of a complex function are shown by (Im) and (Re), respectively. The prime sign is for the derivative with respect to the complex variable. In equations (5), A, B and C are complex matrices which for an aligned orthotropic material are defined as

$$A_{1k} = S_{11}\mu_k^2 + S_{12}, A_{2k} = S_{12}\mu_k + \frac{S_{22}}{\mu_k}, B_{1k} = -\mu_k, B_{2k} = 1, P_{kk} = \mu_k, P_{jk} = 0. (j \neq k), C = BP \quad (6)$$

Beom el al¹⁷ introduced the function G(z) as

$$G(z) = Bf'(z) = B\begin{bmatrix} f_1'(z) \\ f_2'(z) \end{bmatrix} \quad (7)$$

where equation (7) is a modified version of the Stroh formalism and is also applicable to a degenerate orthotropic material. Muskhelishvili³² showed that for 2D in-plane linear elastic, homogeneous and isotropic materials, the displacements and stresses can also be defined using two complex functions $\gamma_j(z)$, as shown below,

$$\begin{aligned}\sigma_{11} &= \text{Re}[3\gamma_1(z) - \gamma_2(z) - (\bar{z} - z)\gamma_1'(z)] , \quad \sigma_{12} = \text{Im}[\gamma_2(z) - \gamma_1(z) + (\bar{z} - z)\gamma_1'(z)] \\ \sigma_{22} &= \text{Re}[\gamma_1(z) + \gamma_2(z) + (\bar{z} - z)\gamma_1'(z)] , \quad -2iG \frac{d}{dx_1}(u_2 + iu_1) = \left[\chi \overline{\gamma_1(z)} - \gamma_2(z) - (\bar{z} - z)\gamma_1'(z) \right] \quad (8)\end{aligned}$$

where χ for plane stress and plane strain cases are: $\chi = \frac{3-\nu}{1+\nu}$ and $\chi = 3-4\nu$, respectively.

Choi et al³³ re-wrote equations (8) in a similar way to equations (5) and designated A^*, B^*, C^* and $g'(z)$ instead of A, B, C and $f'(z)$,

$$u_{j,1} = 2\text{Re}[A^* g'(z)] , \quad [\sigma_{2j}] = 2\text{Re}[B^* g'(z)] , \quad [\sigma_{1j}] = 2\text{Re}[C^* g'(z)] \quad (9)$$

$$\text{where } A^* = \frac{1}{4Gi} \begin{bmatrix} \chi i & -i \\ \chi & 1 \end{bmatrix} , \quad B^* = \frac{1}{2} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} , \quad C^* = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ i & -i \end{bmatrix} , \quad g'(z) = \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix}$$

For an orthotropic material, the matrix Δ is defined as

$$\Delta = iAB^{-1} = H + iQ = \begin{bmatrix} S_{11}(p_1 + p_2) & 0 \\ 0 & \frac{S_{22}(p_1 + p_2)}{p_1 p_2} \end{bmatrix} + i \begin{bmatrix} 0 & (\sqrt{S_{11}S_{22}} + S_{12}) \\ -(\sqrt{S_{11}S_{22}} + S_{12}) & 0 \end{bmatrix} \quad (10)$$

H and Q are real matrices while Δ is a Hermitian, positive definite matrix. Similarly for an isotropic material the Hermitian matrix Δ^* is

$$\Delta^* = iA^*B^{*-1} = H^* + iQ^* = \frac{1}{4G} \begin{bmatrix} \chi + 1 & 0 \\ 0 & \chi + 1 \end{bmatrix} + \frac{i}{4G} \begin{bmatrix} 0 & (\chi - 1) \\ -(\chi - 1) & 0 \end{bmatrix} \quad (11)$$

For a bimaterial comprising of two aligned orthotropic solids whose principal axes coincide with the Cartesian coordinates axes, as displayed in Fig. 1a, two matrices (α, β) are needed

for definition of the stresses at their interface^{12,14}. Assuming $\bar{H}_{11}^T = \bar{H}_{11}^{(1)} + \bar{H}_{11}^{(2)}$ and $\bar{H}_{22}^T = \bar{H}_{22}^{(1)} + \bar{H}_{22}^{(2)}$ we have,

$$\alpha = \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \frac{\bar{H}_{11}^{(2)} - \bar{H}_{11}^{(1)}}{\bar{H}_{11}^T} & 0 \\ 0 & \frac{\bar{H}_{22}^{(2)} - \bar{H}_{22}^{(1)}}{\bar{H}_{22}^T} \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\bar{Q}_{12}^{(2)} - \bar{Q}_{12}^{(1)}}{\bar{H}_{11}^T} \\ \frac{\bar{Q}_{21}^{(2)} - \bar{Q}_{21}^{(1)}}{\bar{H}_{22}^T} & 0 \end{bmatrix} \quad (12)$$

The superscript indicates the material number (n=1,2) and $\beta_{12}\beta_{21} = -\beta_0^2$. β_0 is a dimensionless bimaterial parameter. For an isotropic bimaterial $\alpha_{11} = \alpha_{22} = \alpha_0$ and $\beta_{12} = -\beta_{21} = \beta_0$ where

$$\alpha_0 = \frac{G^{(1)}(\chi+1) - G^{(2)}(\chi+1)}{G^{(1)}(\chi+1) + G^{(2)}(\chi+1)} \text{ and } \beta_0 = \frac{G^{(1)}(\chi-1) - G^{(2)}(\chi-1)}{G^{(1)}(\chi+1) + G^{(2)}(\chi+1)} \quad (13)$$

α_0 and β_0 are the well-known Dundor parameters^{12,34}. For an isotropic/cubic symmetry bimaterial the matrices α and β reduce to the same form as an isotropic/isotropic bimaterial. According to equations (10) and (11), for an isotropic/cubic symmetry bimaterial, the elements of matrices α and β reduce to

$$\alpha_{11} = \alpha_{22} = \frac{\frac{\chi+1}{G^{(2)}} - S_{11} \begin{pmatrix} (1) & (1) \\ P_1 & P_2 \end{pmatrix}}{\frac{\chi+1}{G^{(2)}} + S_{11} \begin{pmatrix} (1) & (1) \\ P_1 & P_2 \end{pmatrix}} \text{ and } \beta_{12} = -\beta_{21} = \frac{\frac{\chi-1}{G^{(2)}} - S_{11} \begin{pmatrix} (1) & (1) \\ P_1 & P_2 \end{pmatrix}}{\frac{\chi+1}{G^{(2)}} + S_{11} \begin{pmatrix} (1) & (1) \\ P_1 & P_2 \end{pmatrix}} \quad (14)$$

where materials 1 and 2 are isotropic and cubic symmetry, respectively.

INTERFACIAL CRACK AND INTERFACE SIFS

Fig. 1b shows a straight crack between materials (1) and (2). The centre of the Cartesian coordinate system, x_j ($j=1,2$), is located at the tip of the crack. The crack and the intact interface both are aligned along the x_1 -axis and r is a small distance from the crack tip. If both materials are orthotropic whose principal axes are aligned in the x_j directions, the complex function $G(z)$ in equation (7) can be written as^{8-9,12,14,17}

$$G^{(n)}(z) = \frac{1}{2\sqrt{2\pi}} (I + \Omega i \beta) Y(z^{i\varepsilon}) K \quad (15)$$

Ω is equal to +1 and -1 for materials 1 and 2 (n=1,2), respectively. I is a unit matrix and ε is defined as

$$\varepsilon = \frac{1}{2\pi} \text{Ln} \frac{1 + |\beta_0|}{1 - |\beta_0|} \quad (16)$$

K is the vector of interface SIFs and Y is a complex matrix defining the oscillatory field,

$$K = \begin{bmatrix} K_2 \\ K_1 \end{bmatrix} = \lim_{x_1 \rightarrow 0} \sqrt{2\pi x_1} Y(x_1^{-i\varepsilon}) \begin{bmatrix} \sigma_{21}(x_1, 0) \\ \sigma_{22}(x_1, 0) \end{bmatrix}, \quad Y(z^{i\varepsilon}) = \frac{1}{2} (z^{i\varepsilon} + z^{-i\varepsilon}) I + \frac{i}{2|\beta_0|} (z^{i\varepsilon} - z^{-i\varepsilon}) \beta \quad (17)$$

Designating the complex functions $D_1, D_2, \psi^{(n)}, \phi^{(n)}$ as

$$D_1(z) = z^{\frac{i\varepsilon-1}{2}} + z^{\frac{-i\varepsilon-1}{2}}, D_2(z) = z^{\frac{i\varepsilon-1}{2}} - z^{\frac{-i\varepsilon-1}{2}}, \psi^{(n)}(z) = D_1(z) + \Omega|\beta_0|D_2(z),$$

$$\phi^{(n)}(z) = \Omega D_1(z) + \frac{D_2(z)}{|\beta_0|} \quad (18)$$

Equation (15) can be written as²⁴

$$G^{(n)}(z) = \frac{1}{4\sqrt{2\pi}} \begin{bmatrix} \psi^{(n)}(z) & i\beta_{12}^{(n)}\phi^{(n)}(z) \\ i\beta_{21}^{(n)}\phi^{(n)}(z) & \psi^{(n)}(z) \end{bmatrix} \begin{bmatrix} K_2 \\ K_1 \end{bmatrix} = \frac{1}{4\sqrt{2\pi}} L K \quad (19)$$

Combining equations (7) and (19) we have²⁴

$$[f'(z)]^{(n)} = \frac{1}{4\sqrt{2\pi}} \times \frac{1}{\mu_2 - \mu_1} \begin{bmatrix} \left[\psi^{(n)}(z) + i\mu_2\beta_{21}^{(n)}\phi^{(n)}(z) \right] K_2 + \left[\mu_2\psi^{(n)}(z) + i\beta_{12}^{(n)}\phi^{(n)}(z) \right] K_1 \\ - \left[\psi^{(n)}(z) + i\mu_1\beta_{21}^{(n)}\phi^{(n)}(z) \right] K_2 - \left[\mu_1\psi^{(n)}(z) + i\beta_{12}^{(n)}\phi^{(n)}(z) \right] K_1 \end{bmatrix} \quad (20)$$

For an isotropic/isotropic interface^{25-26,33},

$$[g'(z)]^{(n)} = \begin{bmatrix} \gamma_1(z) \\ \gamma_2(z) + (\bar{z} - z)\gamma_1'(z) \end{bmatrix}^{(n)} \quad (21)$$

$$\text{where } \gamma_1^{(n)}(z) = \frac{(K_1 - iK_2)(1 - \Omega\beta_0)}{2\sqrt{\pi}} z^{\frac{1}{2} - i\varepsilon}, \gamma_2^{(n)}(z) = \frac{(K_1 + iK_2)(1 + \Omega\beta_0)}{2\sqrt{\pi}} z^{\frac{1}{2} + i\varepsilon}$$

Ω is equal to +1 and -1 for materials 1 and 2 ($n=1,2$), respectively. For a cubic symmetry /isotropic bimaterial, a combination of relations (20) and (21) can be applied. Assuming materials 1 and 2 are cubic symmetry and isotropic, respectively, we have

$$\begin{bmatrix} f'(z) \\ g'(z) \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2\pi}} B^{-1}(I + i\beta)Y(z^{i\varepsilon})K \\ \begin{bmatrix} \gamma_1^{(2)}(z) \\ \gamma_2^{(2)}(z) + (\bar{z} - z)\gamma_1'^{(2)}(z) \end{bmatrix} \end{bmatrix} \quad (22)$$

REVIEW OF THE ANALYTICAL EVALUATION OF THE J_K -INTEGRALS FOR AN INTERFACIAL CRACK IN ORTHOTROPIC BIMATERIALS USING THE COMPLEX FUNCTION METHOD

The following briefly demonstrates the previous study²⁴ of this author for the theoretical derivation of J_K -integrals for a straight interfacial flaw between two aligned orthotropic materials using the CFM. The same procedure will be employed for the evaluation of the J_2 -integral of a cubic symmetry /isotropic bimaterial.

For a 2D in-plane linear or non-linear elastic material of unit thickness, as shown in Fig. 1c, J_k are the line integrals calculated along an anticlockwise contour(s) surrounding the crack tip where

$$J_k \equiv \lim_{r \rightarrow 0} \oint_S (W n_k - t_j u_{j,k}) ds \quad j, k = 1, 2 \quad (23)$$

t_j, u_j and n_k are the traction, displacement and unit normal (to the contour path) vectors, respectively. W is the strain energy density (SED). Figs. 1d-e show two narrow rectangular contours, s_1 and s_2 , surrounding two straight cracks for evaluation of J_1 and J_2 , respectively. Using these contours, equation (23) can be reduced to

$$J_1 = - \lim_{\substack{R \rightarrow 0 \\ \delta/R \rightarrow 0}} \oint_{s_1} (\sigma_{j2}^T u_{j,1}) (n_2) ds \quad (24a)$$

$$J_2 = - \lim_{\substack{R \rightarrow 0 \\ \delta/R \rightarrow 0}} \oint_{s_2} (\sigma_{j1}^T u_{j,2}) (n_1) ds \quad (24b)$$

For contour s_1 , $z_1 = z_2 = z$, $(n_2) ds = -dz$ and for contour s_2 , $z_1 = -i\mu_1 z$, $z_2 = -i\mu_2 z$, $(n_1) ds = -idz$. Based on equations (5), the displacement derivative vectors for each material are,

$$[u_{j,1}]^{(n)} = 2 \operatorname{Re} \left\{ A \begin{bmatrix} f_1'(z) \\ f_2'(z) \end{bmatrix} \right\}^{(n)}, [u_{j,2}]^{(n)} = 2 \operatorname{Re} \left\{ AP \begin{bmatrix} f_1'(z_1) \\ f_2'(z_2) \end{bmatrix} \right\}^{(n)} \quad (25)$$

where²⁴

$$\begin{aligned} \begin{bmatrix} f_1'(z_1) \\ f_2'(z_2) \end{bmatrix}^{(n)} &= \frac{1}{4\sqrt{2\pi}} \times \frac{1}{\sqrt{-i}} \begin{bmatrix} \frac{1}{\sqrt{\mu_1(\mu_2 - \mu_1)}} & 0 \\ 0 & \frac{1}{\sqrt{\mu_2(\mu_2 - \mu_1)}} \end{bmatrix}^{(n)} \begin{bmatrix} \begin{bmatrix} \psi_1^{(n)}(z) + i\mu_2 \beta_{21} \phi_1^{(n)}(z) \\ \mu_2 \psi_1^{(n)}(z) + i\beta_{12} \phi_1^{(n)}(z) \end{bmatrix} & \begin{bmatrix} \mu_2 \psi_1^{(n)}(z) + i\beta_{12} \phi_1^{(n)}(z) \\ \mu_1 \psi_2^{(n)}(z) + i\beta_{12} \phi_2^{(n)}(z) \end{bmatrix} \\ -\begin{bmatrix} \psi_2^{(n)}(z) + i\mu_1 \beta_{21} \phi_2^{(n)}(z) \\ \mu_1 \psi_2^{(n)}(z) + i\beta_{12} \phi_2^{(n)}(z) \end{bmatrix} & -\begin{bmatrix} \mu_2 \psi_1^{(n)}(z) + i\beta_{12} \phi_1^{(n)}(z) \\ \mu_1 \psi_2^{(n)}(z) + i\beta_{12} \phi_2^{(n)}(z) \end{bmatrix} \end{bmatrix}^K \\ &= \frac{1}{4\sqrt{2\pi}} \times \frac{1}{\sqrt{-i}} \times q^{(n)} M^{(n)} K \end{aligned} \quad (26)$$

$$\begin{aligned} \psi_j^{(n)}(z) &= \left[d_{1j}^{(n)}(z) + \Omega |\beta_0| d_{2j}^{(n)}(z) \right], \phi_j^{(n)}(z) = \left[\Omega d_{j1}^{(n)}(z) + \frac{d_{2j}^{(n)}(z)}{|\beta_0|} \right] \\ d_{1j}^{(n)}(z) &= \left[p_j^{i\varepsilon} z^{i\varepsilon - \frac{1}{2}} + p_j^{-i\varepsilon} z^{-i\varepsilon - \frac{1}{2}} \right], d_{2j}^{(n)}(z) = \left[p_j^{i\varepsilon} z^{i\varepsilon - \frac{1}{2}} - p_j^{-i\varepsilon} z^{-i\varepsilon - \frac{1}{2}} \right] \end{aligned} \quad (27)$$

Substituting σ_{j2} , u_{j1} , σ_{j1} and u_{j2} (equations 5 and 25) in equations (24a-b), J_k -integrals can be written as,

$$J_1 = -\frac{1}{16\pi} \text{Im} \left[\oint \left(K^T L^T \Delta L K \right) dz \right] \quad (28)$$

$$J_2 = -\frac{1}{16\pi} \text{Im} \left[\frac{1}{i} \oint \left(K^T M^T q^T P^T B^T A P q M K \right) dz \right] \quad (29)$$

Integrating equations (28-29), where for material-1 $0 \leq \theta \leq \pi$ and for material-2 $-\pi \leq \theta \leq 0$, the J_k values in terms of SIFs can be obtained.

$$J_1 = \frac{1}{4} (1 - \beta_0^2) \left\{ H_{11}^T K_2^2 + H_{22}^T K_1^2 \right\} \quad (30)$$

$$J_2 = \Lambda_{\varepsilon_{11}} K_2^2 + \Lambda_{\varepsilon_{22}} K_1^2 + 2\Lambda_{\varepsilon_{12}} K_1 K_2 \quad (31)$$

where

$$\begin{aligned} \Lambda_{\varepsilon_{11}} &= \frac{1}{8\pi\varepsilon} \sum_{n=1}^2 \left\{ \omega_{\varepsilon_{11}} \left[\Omega \beta_0^2 + \Omega p_2^2 \beta_{21}^2 - 2 p_2 \beta_{21} \right] \sin 2\varepsilon \ln p_1 r + \omega_{\varepsilon_{22}} \left[\Omega \beta_0^2 + \Omega p_1^2 \beta_{21}^2 - 2 p_1 \beta_{21} \right] \sin 2\varepsilon \ln p_2 r \right\} \\ \Lambda_{\varepsilon_{22}} &= \frac{-1}{8\pi\varepsilon} \sum_{n=1}^2 \left\{ \omega_{\varepsilon_{11}} \left[\Omega \beta_{12}^2 + \Omega p_2^2 \beta_0^2 + 2 p_2 \beta_{12} \right] \sin 2\varepsilon \ln p_1 r + \omega_{\varepsilon_{22}} \left[\Omega \beta_{12}^2 + \Omega p_1^2 \beta_0^2 + 2 p_1 \beta_{12} \right] \sin 2\varepsilon \ln p_2 r \right\} \\ \Lambda_{\varepsilon_{12}} &= \frac{|\beta_0|}{8\pi\varepsilon} \sum_{n=1}^2 \left\{ \omega_{\varepsilon_{11}} \left[\Omega \beta_{12} - \Omega p_2^2 \beta_{21} + 2 p_2 \right] \cos 2\varepsilon \ln p_1 r + \omega_{\varepsilon_{22}} \left[\Omega \beta_{12} - \Omega p_1^2 \beta_{21} + 2 p_1 \right] \cos 2\varepsilon \ln p_2 r \right\} \end{aligned} \quad (32)$$

$$\left(\omega_{\varepsilon_{11}} \right)^{(n)} = \frac{S_{22} - S_{11} \mu_1^4}{(\mu_2 - \mu_1)^2}, \quad \left(\omega_{\varepsilon_{22}} \right)^{(n)} = \frac{S_{22} - S_{11} \mu_2^4}{(\mu_2 - \mu_1)^2} \quad (33)$$

A UNIFIED APPROACH FOR ANALYTICAL EVALUATION OF THE J_2 -INTEGRAL IN DISSIMILAR AND HOMOGENEOUS ORTHOTROPIC, CUBIC SYMMETRY AND ISOTROPIC MATERIALS

J_1 in dissimilar or homogeneous solids with anisotropic and/or isotropic material properties can be directly evaluated¹¹. However, for calculation J_2 , coefficients (32) for a homogeneous material ($\varepsilon = 0$) and for an isotropic material ($\mu_1 = \mu_2 = 1$) become numerically infinite and indeterminate, respectively. For an isotropic material, based on equations (9) and contour s_2 in Fig. 1e, the displacement derivative vectors are,

$$[u_{j,2}]^{(n)} = 2 \text{Re} \{ \eta^* g'(z) \}^{(n)} \quad (34)$$

where $\eta^* = \frac{1}{4G} \begin{bmatrix} i(\chi + 2) & -i \\ (\chi - 2) & 1 \end{bmatrix}$.

For an isotropic/isotropic bimaterial, $[u_{j,2}]^{(n)}$ and $[\sigma_{j1}]^{(n)}$ (equations 9, 34) are substituted in equation (24b) and integrated. Considering that $\oint d(z - \bar{z})\gamma_1'^2(z) = 0$, J_2 can be determined.

$$J_2 = \frac{|\beta_0|}{2\pi\varepsilon} \sum_{n=1}^2 \left[\omega_\varepsilon^{*(n)} (\Omega\beta_0 + 1) \sin 2\varepsilon \ln r \right] (K_2^2 - K_1^2) - 2 \frac{|\beta_0|}{2\pi\varepsilon} \sum_{n=1}^2 \left[\omega_\varepsilon^{*(n)} (\Omega\beta_0 + 1) \cos 2\varepsilon \ln r \right] K_1 K_2 \quad (35)$$

where $\omega_\varepsilon^{*} = \frac{1 + \chi^{(n)}}{8G^{(n)}}$. Equation (35) is also reported elsewhere²⁵⁻²⁶ but in a different form.

Comparing equations (31-32) with equation (35), it can be concluded that for an isotropic/isotropic bimaterial the coefficients $\Lambda_{\varepsilon_{11}}^*$, $\Lambda_{\varepsilon_{22}}^*$ and $\Lambda_{\varepsilon_{12}}^*$ can be defined as

$$\Lambda_{\varepsilon_{22}}^* = -\Lambda_{\varepsilon_{11}}^* = \frac{|\beta_0|}{2\pi\varepsilon} \sum_{n=1}^2 \left[\omega_\varepsilon^{*(n)} (\Omega\beta_0 + 1) \sin 2\varepsilon \ln r \right], \Lambda_{\varepsilon_{12}}^* = \frac{|\beta_0|}{2\pi\varepsilon} \sum_{n=1}^2 \left[\omega_\varepsilon^{*(n)} (\Omega\beta_0 + 1) \cos 2\varepsilon \ln r \right] \quad (36)$$

$$\omega_{\varepsilon_{11}}^{(n)} = \omega_{\varepsilon_{22}}^{(n)} = \omega_\varepsilon^{*(n)}$$

In a homogeneous orthotropic material, $\varepsilon = \beta_{12} = \beta_{21} = \beta_0 = 0$., the coefficients $\omega_{\varepsilon_{11}}$ and $\omega_{\varepsilon_{22}}$ are the same at both sides of the crack, and Ω equals to + 1 and -1 for n=1 and 2, respectively. Now by applying the L'Hospital rule to $\frac{\beta_0}{\varepsilon}$ and knowing that $|\beta_0| = \frac{e^{2\pi\varepsilon} - 1}{e^{2\pi\varepsilon} + 1}$, it

can be concluded that $\Lambda_{\varepsilon_{12}} = \frac{1}{4} (p_2 \omega_{\varepsilon_{11}} + p_1 \omega_{\varepsilon_{22}})$, $\Lambda_{\varepsilon_{11}} = \Lambda_{\varepsilon_{22}} = 0$.. Therefore, equation (31)

for a homogeneous and orthotropic material can now be written as

$$J_2 = 2\Lambda_{\varepsilon_{12}} K_1 K_2 \quad (37)$$

This agrees with the relation derived in Refs.²⁷⁻²⁸ for these materials. For a flaw in an isotropic and homogeneous material ($\varepsilon \rightarrow 0$), the L'Hospital rule can be applied to equation (35), therefore,

$$J_2 = -\frac{2K_1 K_2}{E} \quad (38)$$

Comparing equations (31) and (38), it can be said that for a homogeneous and isotropic material,

$$\Lambda_{\varepsilon_{11}}^* = \Lambda_{\varepsilon_{22}}^* = 0, -\Lambda_{\varepsilon_{12}}^* = \omega_\varepsilon^{*(1)} = \omega_\varepsilon^{*(2)} = \frac{1 + \chi}{8G} = \frac{1}{E} \quad (39)$$

It can be concluded that equations (35), (37) and (38), which are commonly used by researchers worldwide, are specific forms of equation (31). For a cubic symmetry /isotropic bimaterial, equations (14) and (22) represent the bimaterial matrices (α, β) and related complex functions, respectively. Using these relations and integrating equation (24b), an

analytical relation for J_2 can be obtained which is identical to equation (31). However, the coefficients ($\Lambda_{\varepsilon_{11}}, \Lambda_{\varepsilon_{22}}, \Lambda_{\varepsilon_{12}}$) are defined as follows:

$$\begin{aligned}\Lambda_{\varepsilon_{11}} &= \Lambda_{\varepsilon_{11}}^{(1)} + \Lambda_{\varepsilon_{11}}^{*(2)} = \frac{1}{8\pi\varepsilon} \left\{ \omega_{\varepsilon_{11}}^{(1)} \left[\beta_0^2 + p_2^{(1)} \beta_{21}^2 - 2 p_2^{(1)} \beta_{21} \right] \sin 2\varepsilon \ln p_1 r + \omega_{\varepsilon_{22}}^{(1)} \left[\beta_0^2 + p_1^{(1)} \beta_{21}^2 - 2 p_1^{(1)} \beta_{21} \right] \sin 2\varepsilon \ln p_2 r \right\} \\ &+ \frac{\beta_0 [1 - \beta_0]}{2\pi\varepsilon} \left\{ \omega_{\varepsilon}^{*(2)} \sin 2\varepsilon \ln r \right\} \\ \Lambda_{\varepsilon_{22}} &= \Lambda_{\varepsilon_{22}}^{(1)} + \Lambda_{\varepsilon_{22}}^{*(2)} = \frac{-1}{8\pi\varepsilon} \left\{ \omega_{\varepsilon_{11}}^{(1)} \left[\beta_{12}^2 + p_2^{(1)} \beta_0^2 + 2 p_2^{(1)} \beta_{12} \right] \sin 2\varepsilon \ln p_1 r + \omega_{\varepsilon_{22}}^{(1)} \left[\beta_{12}^2 + p_1^{(1)} \beta_0^2 + 2 p_1^{(1)} \beta_{12} \right] \sin 2\varepsilon \ln p_2 r \right\} \\ &- \frac{\beta_0 [1 - \beta_0]}{2\pi\varepsilon} \left\{ \omega_{\varepsilon}^{*(2)} \sin 2\varepsilon \ln r \right\} \\ \Lambda_{\varepsilon_{12}} &= \Lambda_{\varepsilon_{12}}^{(1)} + \Lambda_{\varepsilon_{12}}^{*(2)} = \frac{|\beta_0|}{8\pi\varepsilon} \left\{ \omega_{\varepsilon_{11}}^{(1)} \left[\beta_{12} - p_2^{(1)} \beta_{21} + 2 p_2^{(1)} \right] \cos 2\varepsilon \ln p_1 r + \omega_{\varepsilon_{22}}^{(1)} \left[\beta_{12} - p_1^{(1)} \beta_{21} + 2 p_1^{(1)} \right] \cos 2\varepsilon \ln p_2 r \right\} \\ &+ \frac{|\beta_0| [1 - \beta_0]}{2\pi\varepsilon} \left\{ \omega_{\varepsilon}^{*(2)} \cos 2\varepsilon \ln r \right\}\end{aligned}\tag{40}$$

For the above coefficients, it is assumed that materials 1 and 2 are cubic symmetry ($\rho > 1$) and isotropic, respectively. For the sake of brevity, the complete integration is not included. It can be observed that the orthotropic properties of material-1 influence the asymptotic field only by the bimaterial constants ε and β_0 .

CALCULATION OF STRESS INTENSITY FACTORS

SIFs can be written in terms of J_1 and J_2 , by combining equations (30) and (31) and then rearranging them as follows:

$$K_1 = \pm \sqrt{\frac{c_{\varepsilon_2} \pm \sqrt{(c_{\varepsilon_2})^2 - c_{\varepsilon_1} c_{\varepsilon_3}}}{c_{\varepsilon_1}}}, K_2 = \pm \sqrt{\frac{\dot{c}_{\varepsilon_2} \pm \sqrt{(\dot{c}_{\varepsilon_2})^2 - \dot{c}_{\varepsilon_1} \dot{c}_{\varepsilon_3}}}{\dot{c}_{\varepsilon_1}}}\tag{41}$$

where,

$$\begin{aligned}C_{\varepsilon_1} &= \left[\frac{\Lambda_{\varepsilon_{11}} \beta_{12}^2}{\beta_0^2} - \Lambda_{\varepsilon_{22}} \right]^2 + \frac{4\Lambda_{\varepsilon_{12}}^2 \beta_{12}^2}{\beta_0^2} \text{ and } C'_{\varepsilon_1} = \left[\frac{\Lambda_{\varepsilon_{22}} \beta_{21}^2}{\beta_0^2} - \Lambda_{\varepsilon_{11}} \right]^2 + \frac{4\Lambda_{\varepsilon_{12}}^2 \beta_{21}^2}{\beta_0^2} \\ C_{\varepsilon_3} &= J_{2r} \left[1 - \frac{4J_1 \Lambda_{\varepsilon_{11}}}{(1 - \beta_0^2) H_{11}^T} \right]^2 \text{ and } C'_{\varepsilon_3} = J_{2r} \left[1 - \frac{4J_1 \Lambda_{\varepsilon_{22}}}{(1 - \beta_0^2) H_{22}^T} \right]^2 \\ C_{\varepsilon_2} &= J_{2r} \left[\Lambda_{\varepsilon_{22}} - \frac{\Lambda_{\varepsilon_{11}} \beta_{12}^2}{\beta_0^2} \right]^2 + \frac{4J_1}{(1 - \beta_0^2) H_{11}^T} \left[\frac{\beta_{12}^2}{\beta_0^2} \Lambda_{\varepsilon_{11}}^2 - \Lambda_{\varepsilon_{11}} \Lambda_{\varepsilon_{22}} + 2\Lambda_{\varepsilon_{12}}^2 \right] \text{ and}\end{aligned}$$

$$C'_{\varepsilon_2} = J_{2r} \left[\Lambda_{\varepsilon_{11}} - \frac{\Lambda_{\varepsilon_{22}} \beta_{21}^2}{\beta_0^2} \right]^2 + \frac{4J_1}{(1 - \beta_0^2) H_{22}^T} \left[\frac{\beta_{21}^2}{\beta_0^2} \Lambda_{\varepsilon_{22}}^2 - \Lambda_{\varepsilon_{11}} \Lambda_{\varepsilon_{22}} + 2\Lambda_{\varepsilon_{12}}^2 \right]$$

The crack opening displacements calculated by the BEM analysis can be used to determine the signs of K_1 and K_2 .

For the numerical analysis, the convergence test is carried out according to the J_1 quantity which is independent of r . Using the corresponding value of J_2 and for different values of

$10^{-6} \leq \frac{r}{a} \leq 10^{-4}$ and in conjunction with equations (41), a series of K_1 and K_2 values can be

obtained. The average values of K_1 and K_2 , in turn, provide a very good estimation of the SIFs. An example for an isotropic/isotropic bimaterial is demonstrated in Ref.¹⁵

COMPUTATION OF THE J_K -INTEGRALS USING BOUNDARY ELEMENT CRACK SHAPE SENSITIVITIES

The BEM applied to in-plane anisotropic materials can be represented by an integral equation involving boundary displacements (u_j) and tractions (t_j)³⁵. If $Q_B(x_1, x_2)$ and $P_B(\zeta_1, \zeta_2)$ are the field and load points, respectively, on the complex plane we have

$$z_k = (x_1 - \zeta_1) + \mu_k(x_2 - \zeta_2) \quad k = 1, 2 \quad (42)$$

and the boundary integral equation (BIE) can be shown as

$$C_{jk} u_j(P_B) + \int_S T_{jk}(P_B, Q_B) u_j(Q_B) ds(Q_B) = \int_S U_{jk}(P_B, Q_B) t_j(Q_B) ds(Q_B) \quad j, k = 1, 2 \quad (43)$$

$T_{jk}(P_B, Q_B)$ and $U_{jk}(P_B, Q_B)$ are the fundamental solutions for the tractions and displacements, respectively. For a bimaterial with an interfacial crack, the solid may be divided into two separate regions where the crack is located on the interface boundary. The BIE (Equation 43) can be used individually for each sub-region in turn. The equilibrium and continuity equations at the intact interface are applied before the final set of linear equations are solved to compute the unknown displacements or tractions.

For calculation of the crack shape sensitivities (CSS), the crack shape variables are defined in terms of the boundary nodes using the design velocity fields. The BIE can be differentiated with respect to the associated boundary node coordinates to determine the sensitivities of the multi-region domain. For more details the reader can refer to references³⁶⁻⁴¹.

Fig. 1c shows a curved interfacial flaw between two homogeneous elastic materials, anisotropic and/or isotropic. a_k is a local coordinate system placed at the tip of the crack. a_1

and a_2 are in the tangent and transverse directions of the crack, respectively. x_j ($j=1,2$) are the fixed global axes. The contours S_{1a} and S_{2a} in materials 1 and 2, respectively, surround the crack tip but exclude it. S_r is the boundary of a small circle with radius r centred at the crack tip. S_{P_c} and S_{U_c} include the crack surfaces and interface regions near the crack tip, respectively. In Ref.¹⁵ it was proved that

$$\frac{\partial(E_p)}{\partial a_k} = \int_{S_p=S_p^++S_p^-} \left(W n_k - t_i \frac{\partial u_i}{\partial a_k} \right) ds \quad (44)$$

where E_p is the overall potential energy of the solid shown in Fig. 1c. For a straight crack which is traction free ($n_1 = 0$ and $t_j = 0$.)

$$J_1 = \frac{\partial(E_p)}{\partial a_1} \quad (45)$$

For a traction free curved crack ($n_1 \neq 0$ and $t_j = 0$.), the jump of the SED over crack faces in equation (44) must be considered¹⁵. However, for computation of J_1 -integral⁴² or the SERR in this study, the design variable is the crack length. Therefore, for the curved cracks the integration over the crack surfaces will not be necessary. This method was used in Ref.⁴¹ for estimation of the crack propagation path in homogeneous and anisotropic materials where the SERR at each new crack tip was determined. It was also proved that¹⁵

$$J_2 = \frac{\partial(E_p)}{\partial a_2} + I_{S_c} \quad (46)$$

where

$$I_{S_c} = \int_{S_{P_c}+S_{U_c}} \left(W n_k - t_i \frac{\partial u_i}{\partial a_2} \right) ds \quad (47)$$

It can be said that for $k=2$, equation (44) is not J_2 -integral but by addition of I_{S_c} to it, J_2 can be determined. The crack faces are traction free and across the interface, t_j and u_j are continuous. On the other hand, the strains across the interface are discontinuous. Therefore¹⁵,

$$I_{S_c} = \int_{S_{P_c}+S_{U_c}} \langle\langle W \rangle\rangle n_2 ds - \int_{S_{U_c}} t_i \langle\langle \frac{\partial u_i}{\partial a_2} \rangle\rangle ds \quad (48)$$

Each $\langle\langle \rangle\rangle$ term represents the change of the function across the interface and the crack surfaces.

In order to obtain J_2 , it is necessary to compute $\frac{\partial(E_p)}{\partial a_2}$ and $I_{S_c} \cdot \frac{\partial(E_p)}{\partial a_2}$ can be determined using the BECSS while I_{S_c} contains two line integrals with crack tip singularity. Here these line integrals are only evaluated for the crack tip elements.

CRACK TIP ELEMENTS WITH SCALED SHAPE FUNCTIONS

In the present study, for modelling of each crack tip, straight semi-discontinuous quadratic quarter point (SDQQP) element is employed where its shape functions are appropriately modified to accommodate the oscillatory singularity of the tractions. Standard quadratic (SQ) elements are used elsewhere. Fig. 1f shows an SQ element. Figs. 1g-h show two SDQQP elements where their collocation nodes are located at $\left[-1, 0, \frac{3}{4}\right]$ and $\left[-\frac{3}{4}, 0, 1\right]$, respectively.

L is the length of each element. N_{c_i} and N_i are the collocation and geometrical nodes, respectively, where $i = 1, 3$. Here the crack tip element's geometry is defined using the SQ shape functions (SF_i) where

$$SF_1 = \frac{1}{2}\zeta(\zeta - 1) \quad , \quad SF_2 = (1 - \zeta^2) \quad , \quad SF_3 = \frac{1}{2}\zeta(\zeta + 1) \quad (49)$$

The displacements and tractions are represented using the following shape functions (sf_i),

$$sf_1 = \frac{(\zeta - \zeta_2)(\zeta - \zeta_3)}{(\zeta_1 - \zeta_2)(\zeta_1 - \zeta_3)} \quad , \quad sf_2 = \frac{(\zeta - \zeta_1)(\zeta - \zeta_3)}{(\zeta_2 - \zeta_1)(\zeta_2 - \zeta_3)} \quad , \quad sf_3 = \frac{(\zeta - \zeta_1)(\zeta - \zeta_2)}{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)} \quad (50)$$

For the crack tip element, the variations of u_j and t_j are proportional to $r^{\frac{1}{2}}$ and $r^{-\frac{1}{2}}$, respectively. However, with SDQQP, both u_j and t_j have $r^{\frac{1}{2}}$ variations. Here, for simulation of the oscillatory singularity of the tractions in anisotropic bimetals, the traction vector of each crack tip element is multiplied by $\sqrt{\frac{L}{r}}Y\left(\frac{r}{L}\right)^{i\epsilon}$ and the associated shape functions are modified accordingly. Tan et al²⁰ used a similar approach for a standard quadratic quarter point crack tip element. If t_j and $\tilde{t}_j$ are the actual and computed traction values of a crack tip element then

$$t_j = \sqrt{\frac{L}{r}}Y\left(\frac{r}{L}\right)^{i\epsilon} \tilde{t}_j \quad (j = 1, 2) \quad (51)$$

For a SDQQP crack tip element, r and ζ are related as

$$\zeta = 2\sqrt{\frac{r}{L}} - 1 \text{ or } \zeta = 1 - 2\sqrt{\frac{r}{L}} \quad (52)$$

Therefore,

$$r = \begin{bmatrix} L, \frac{L}{4}, \frac{L}{64} \end{bmatrix} \text{ or } r = \begin{bmatrix} \frac{L}{64}, \frac{L}{4}, L \end{bmatrix} \quad (53)$$

Combining equations (51) and (53), the following relations must apply between the computed and actual traction coefficients at node c ($c=1,3$) of the crack tip element,

$$\begin{aligned} \tilde{t}_j^{(1)} &= t_j^{(1)} & \tilde{t}_j^{(3)} &= t_j^{(3)} \\ \tilde{t}_j^{(2)} &= \frac{1}{2} Y \left[\left(\frac{1}{4} \right)^{-i\varepsilon} \right] t_j^{(2)} & \text{or} & \tilde{t}_j^{(2)} = \frac{1}{2} Y \left[\left(\frac{1}{4} \right)^{-i\varepsilon} \right] t_j^{(2)} \\ \tilde{t}_j^{(3)} &= \frac{1}{8} Y \left[\left(\frac{1}{64} \right)^{-i\varepsilon} \right] t_j^{(3)} & \tilde{t}_j^{(1)} &= \frac{1}{8} Y \left[\left(\frac{1}{64} \right)^{-i\varepsilon} \right] t_j^{(1)} \end{aligned} \quad (54)$$

With these modified shape functions, the oscillatory variant of the tractions near the crack tip will be accommodated.

EXAMPLE: INTERFACIAL CRACK IN A PLATE

Fig. 1i shows a central crack, with the length $2a=20\text{mm}$, in a thin bimaterial plate which is made up of Material-1 (top) and Material-2 (bottom) under a tensile stress of $\sigma_{y0} = 100\text{MPa}$.

As shown, the plate is discretized to quadratic boundary elements. Each material includes 37 elements with 12 elements for the full crack length. A close-up view of two SDQQP elements is also shown. It is assumed that a/W and H/W are small enough for the plate to be considered an infinite plate where H and W are the height and width of the plate, respectively. The plane stress condition is considered. To maintain continuity of the strain along the bimaterial interface, in the infinite plate solution, $(\sigma_x)_0$ must be prescribed on the side of material-2, as shown in Fig.1i where¹⁰

$$(\sigma_{x0})_2 = \frac{E_1^{(2)}}{E_1^{(1)}} (\sigma_{x0})_1 + \left[\nu_{12}^{(2)} - \frac{E_1^{(2)}}{E_1^{(1)}} \nu_{12}^{(1)} \right] (\sigma_{y0}) \quad (55)$$

The analytical SIFs of an anisotropic bimaterial plate which is infinite and has a central interface crack is derived by Qu and Bassani¹⁰ as,

$$K = \sqrt{\pi a} Y \left[(1 + 2i\varepsilon)(2a)^{-i\varepsilon} \right] \left[\sigma_{j2} \right]_0 \quad (56)$$

The SIFs obtained using equation (56) are inserted in equation (31) to evaluate the analytical values of J_1 . The computed J_2 magnitudes are obtained using equations (31) and (56) for $10^{-5} \leq \frac{r}{a} \leq 5 \times 10^{-4}$. Table 1 displays the properties of the materials selected for this study.

As shown, the characteristic roots of a number of fibre reinforced composite materials change in the following ranges $2.7 < p_1 < 6$, $0.5 < p_2 < 1.0$ where⁴³,

$$\frac{E_1}{E_2} = \mu_1^2 \mu_2^2 = p_1^2 p_2^2, \quad \frac{E_1}{G_{12}} = \mu_1^2 + \mu_2^2 + 2\nu_{12} = p_1^2 + p_2^2 + 2\nu_{12} \quad (57)$$

An orthotropic rescaling method⁴⁴⁻⁴⁵ for a homogeneous orthotropic material has been employed to produce a new material which has cubic symmetry. The engineering constants of materials H and I in Table 1 are obtained by scaling the properties of T300/934 graphite/epoxy and Boron/5505 boron/epoxy, respectively. The following relations apply between S_{jk} and $\tilde{S}_{jk}$ of the original and scaled materials, respectively

$$\tilde{S}_{11} = \lambda^{-\frac{1}{2}} S_{11}, \tilde{S}_{22} = \lambda^{\frac{1}{2}} S_{22}, \tilde{S}_{12} = S_{12}, \tilde{S}_{66} = S_{66}, \lambda = \frac{S_{11}}{S_{22}} \quad (58)$$

Two categories of bimaterial interface are considered: orthotropic/orthotropic and cubic symmetry /isotropic. In the first category (orthotropic/orthotropic), material-1 is designated as T300/934 graphite epoxy and 3 sets of properties are considered for material-2. Table 2 shows different bimaterial combinations studied in this category. For set number-1 in this category, the properties of material-2 are designated as $G_{12}^{(2)} = 6.9GPa$, $\nu_{12}^{(2)} = 0.3$, $p_2^{(2)} = 0.9$ and $2.5 < p_1^{(2)} < 4$. Figs. 2a-b show J_k -integrals, numerical and analytical, vs the characteristic root $(p_1^{(2)})$ and $(E_1^{(2)}/E_2^{(2)})$ of material-2 for this set, respectively. For set number-2, $E_2^{(2)} = 10.3GPa$, $\nu_{12}^{(2)} = 0.3$, $p_2^{(2)} = 0.9$ and $2.5 < p_1^{(2)} < 4$. Figs. 3a-b show the numerical and analytical values of J_k vs $(p_1^{(2)})$ and $(E_1^{(2)}/G_{12}^{(2)})$ of material-2 for this set, respectively. For set number-3, materials B, C, D, F or G from Table 1 are designated as material-2. Their corresponding bimaterial constants (ε) are shown in Table 2. Fig. 4 shows the numerical and analytical values of J_k vs (ε) for these bimaterials.

In the second category (cubic symmetry/isotropic), material-2 is isotropic where $10GPa < E^{(2)} < 100GPa$, $\nu^{(2)} = 0.3$. Material H or I from Table 1, in turn, is designated as

material-1. Figs. 5a-b show the variation of the numerical and analytical values of J_k vs $\left(\frac{r}{E}\right)^{(2)}$, where material-1 is designated as material H and I, respectively.

It can be observed that for all the presented results in Figs 2-5, the numerical and analytical magnitudes of J_1 or SERR are almost identical. The numerical data for J_2 are close to the corresponding analytical data when $10^{-5} < \frac{r}{a} < 5 \times 10^{-4}$. Tables 2-3 show the dE_p/da_2 and I_{sc} magnitudes for each data point of the J_2 curves presented in Figs 2-5. The analytical and estimated SIFs for each of the bimetals analysed in this study were in very good agreement. The full crack length of each model was discretised to 12-20 elements and relatively coarser mesh was used elsewhere.

Here the BEM results for J_k are compared with the corresponding data of an infinite bimaterial cracked plate with known analytical solution. To the author's knowledge, similar numerical data in relation to orthotropic or cubic symmetric bimetals are not available elsewhere. A recent study⁴⁶⁻⁴⁸ shows the analytical and numerical evaluations of brittle interfacial cracking between two dissimilar elastic layers. The numerical validation⁴⁷ of their proposed analytical formulation was carried out using FEM in conjunction with MSC/NASTRAN. Since their study was mainly based on isotropic bimetals, their results cannot be used for comparison with the present data.

CONCLUSIONS

First, a brief review of linear elastic interfacial fracture mechanics for in-plane anisotropic and isotropic bimetals has been made. The highlights of boundary integral equation and boundary element crack shape sensitivities for evaluation of J_k -integrals for an interfacial crack in dissimilar anisotropic materials are included. The shape functions of the semi discontinuous quadratic quarter point elements are scaled to accommodate the nonlinear and oscillatory behavior of tractions in these regions. Because of the flexibility of the formulation, the algorithm is capable of analysis of both straight and curved cracks and any combination of isotropic/anisotropic bimetals. In order to compute the strain energy release rate, the crack span is designated as the design variable which gives an excellent result without any integration over the curved crack faces. With the computed J_k values and using an analytical relation for J_2 which was recently obtained by this author, the SIFs for an aligned orthotropic bimaterial with straight interface crack can be obtained without the need

for an auxiliary equation. It is shown that using J_1 and J_2 with small variation of $\frac{r}{a}$ near the crack tip, the SIFs can be well estimated. It is well-known that an analytical relation for an anisotropic material is not always applicable to an isotropic material by merely inserting its engineering constants. Here the aforementioned analytical expression for J_2 is reconstructed for cubic symmetry /isotropic bimetals and used with the present numerical algorithm. Therefore, this paper presents a unified and integrated numerical-analytical framework for linear elastic interfacial fracture mechanics using boundary element crack shape sensitivities of an in-plane interface crack in orthotropic/orthotropic, cubic symmetry/isotropic and isotropic/isotropic bimetals. It is also perfectly applicable to a homogeneous anisotropic or isotropic solid with a mixed mode flaw. An example with known analytical SIFs and different combinations of material properties is presented to validate this study. The results show that the analytical and numerical J_1 values are almost identical. The numerical J_2 magnitudes are similar to their associated analytical values when $10^{-5} < \frac{r}{a} < 5 \times 10^{-4}$.

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Table 1 Typical values of lamina engineering constants for several composites

Material	ID		E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12}	$\mu_1 = p_1 i$	$\mu_2 = p_2 i$
Orthotropic	A	T300/934 Graphite/epoxy	131	10.3	6.9	0.22	4.222i	0.844i
	B	AS/3501 Graphite/epoxy	138	9	6.9	0.3	4.309i	0.908i
	C	Kevlar 49/934 aramid/epoxy	75.8	5.5	2.3	0.34	5.643i	0.657i
	D	Scotchply 1002 E-glass/epoxy	38.6	8.27	4.14	0.26	2.870i	0.752i
	E	Boron/5505 boron/epoxy	204	18.5	5.59	0.23	5.977i	0.555i
	F	Spectra 900/826 Polyethylene/epoxy	30.7	3.52	1.45	0.32	4.483i	0.658i
	G	E-glass/470-36 E-glass/vinylester	24.4	6.87	2.89	0.32	2.705i	0.696i
cubic symmetry	H	Scaled material A	36.73	36.73	6.9	0.061	2.236i	0.447i
	I	Scaled material E	61.43	61.43	5.59	0.069	3.279i	0.304i

Table 2 Numerical values of dE/da_2 and I_{Sc} for the orthotropic/orthotropic category

	Set No.	Fig. No.	Properties of material-2			Components of J_2	
						dEp/da_2 (mJ/mm ²)	I_{Sc} (mJ/mm ²)
Material-1 (T300/934, graphite/epoxy)	1	(Figs. 2a-b)	$G_{12}^{(2)} = 6.9GPa$ $\nu_{12}^{(2)} = 0.3$ $p_2^{(2)} = 0.9$	$p_1^{(2)}$	2.5	-2.785	-25.158
					2.75	-2.353	-21.611
					3	-1.917	-17.962
					3.25	-1.470	-14.117
					3.5	-1.037	-10.274
					3.75	-0.610	-6.382
					4	-0.216	-2.725
	2	(Figs. 3a-b)	$E_2^{(2)} = 10.3GPa$ $\nu_{12}^{(2)} = 0.3$ $p_2^{(2)} = 0.9$	$p_1^{(2)}$	2.5	-2.876	-25.675
					2.75	-2.117	-20.828
					3	-1.594	-16.055
					3.25	-1.023	-11.432
					3.5	-0.501	-7.014
					3.75	-0.016	-2.786
					4	0.428	1.260
	3	(Fig. 4)	AS/3501 Graphite/epoxy	(ϵ)	0.0014	-0.229	-1.764
			Kevlar 49/934aramid/epoxy		0.0259	-3.809	-21.692
			Scotchply 1002 E-glass/epoxy		0.0356	-4.845	-35.761
			Spectra 900/826 Polyethylene/epoxy		0.0538	-9.156	-42.705
			E-glass/470-36 E-glass/vinylester		0.0463	-6.749	-42.378

Table 3 Numerical values of dE/da_2 and I_{Sc} for the cubic symmetry/isotropic category:

material-2 is isotropic where $10 < E^{(2)} < 100$ and $\nu^{(2)} = 0.3$

Material-1	Material-2 (Isotropic)	Components of J_2	
	$E^{(2)}$	$\frac{dE_p}{da_2}$ (mJ / mm ²)	I_{Sc} (mJ / mm ²)
Material H (Fig. 5a)	10	-3.529	-11.776
	20	-0.459	-3.814
	25	0.156	-1.305
	33.33	0.715	1.281
	50	1.120	3.300
	75	1.203	3.516
	100	1.565	3.925
Material I (Fig. 5b)	10	-4.294	-12.503
	20	-1.412	-7.309
	25	-0.791	-5.146
	33.33	-0.182	-2.547
	50	0.347	0.206
	75	0.573	1.593
	100	0.608	1.825

Figure 1 a) an intact interface b) a straight interface crack c) a curved interface crack d) a straight crack and a narrow rectangular contour for calculation of the J_1 -integral e) a straight crack and a narrow rectangular contour for calculation of the J_2 -integral f) a continuous quadratic element g,h) semi-discontinuous quadratic quarter point elements i) a bimaterial plate with an interface crack discretised to boundary elements with a close-up view of two SDQQP elements

Figure 2 Variation of the numerical and analytical values of J_k for the orthotropic/orthotropic bimaterial category for set number-1 vs a) $p_1^{(2)}$ b) $E_1/E_2^{(2)}$

Figure 3 Variation of the numerical and analytical values of J_k for the orthotropic/orthotropic bimaterial category for set number-2 vs a) $p_1^{(2)}$ b) $E_1/G_{12}^{(2)}$

Figure 4 Variation of the numerical and analytical values of J_k for the orthotropic/orthotropic bimaterial category for set number-3 vs (ϵ)

Figure 5 Variation of the numerical and analytical values of J_k for the cubic symmetry /isotropic bimaterial category with respect to the modulus of elasticity of material-2 when material-1 is a) material H, b) material I