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The joint distribution of the sample minimum and maximum from a smooth distribution on $[w_1, w_2]$

by

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Abstract: We give expansions in inverse powers of the sample size n for the joint distribution function of the sample extremes when sampling from any smooth distribution on a finite interval when its density function is positive at its end points.

Keywords: Density function; Maximum; Moments

1 Introduction

Many variables in real life take values in finite intervals. The best known examples are proportions. Often interest is on extreme values of such variables, for example, lowest and highest proportions of people affected by a deadly disease.

Suppose $X_1, X_2, \dots, X_n$ is a random sample from a distribution on a finite interval. Suppose Z_{n1} denotes a suitably normalized $\min(X_1, X_2, \dots, X_n)$ and belongs to the Weibull domain of attraction such that

$$Prob.(Z_{n1} > z_1) \rightarrow e^{-z_1} \quad (1)$$

as $n \rightarrow \infty$. Suppose Z_{n2} denotes a suitably normalized $-\max(X_1, X_2, \dots, X_n)$ and belongs to the Weibull domain of attraction such that

$$Prob.(Z_{n2} > z_2) \rightarrow e^{-z_2} \quad (2)$$

as $n \rightarrow \infty$. Both (1) and (2) hold if for example the sample is from the uniform $[0, 1]$ distribution.

There is much work on the asymptotic joint distribution of sample minimum and sample maximum. For example, Leadbetter et al. (1983) showed under certain general conditions that sample minimum and sample maximum are asymptotically independent. If (1) and (2) hold then under certain general conditions

$$Prob.(Z_{n1} > z_1, Z_{n2} > z_2) \rightarrow e^{-z_1 - z_2}, \quad (3)$$

$$\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} Prob.(Z_{n1} \leq z_1, Z_{n2} \leq z_2) \rightarrow e^{-z_1 - z_2} \quad (4)$$

and

$$EZ_{n1}^r Z_{n2}^s \rightarrow r!s! \quad (5)$$

as $n \rightarrow \infty$. In practice, n is finite. It can never be infinity. So, models based on (3)-(5) cannot be realistic. Realistic models can be obtained by deriving expansions for the joint distribution of the minimum and maximum. Liao and Peng (2015) and Lu and Peng (2017) give the most recent work on the asymptotics of the joint distribution of the minimum and maximum.

The aim of this note is to derive expansions for the distributions of the minimum and maximum of a random sample of values in a finite interval. We derive expansions for the joint distribution function, joint density function and product moments. These expansions can be used for improved statistical modeling of the minimum and maximum. The expansions for the joint distribution function can be used for improved percentile estimation. For instance, by Corollary 3.1 below,

$$Prob. (Z_{n1} > z_1, Z_{n2} > z_2) = e^{-z_1 - z_2} (1 + Q_1 n^{-1})$$

which can be expected to perform better than (1). The expansions for the joint density function can be used for improved maximum likelihood estimation. For instance, by Corollary 3.1 below,

$$\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} Prob. (Z_{n1} \leq z_1, Z_{n2} \leq z_2) = e^{-z_1 - z_2} (1 + q'_1 n^{-1})$$

which can be expected to perform better than (2). The expansions for the moments can be used for improved moments estimation. For instance, by Corollary 3.1 below,

$$EZ_{n1}^r Z_{n2}^s = r!s! + M_{rs1} n^{-1}$$

which can be expected to perform better than (3).

The known work giving expansions for the joint distribution of sample minimum and sample maximum has focused on specific distributions or processes: a Brownian motion (Choi and Roh, 2013); the normal distribution (Shiryaeva, 2013); autoregressive process of order one (Withers and Nadarajah, 2015); and so on. We are aware of no work giving expansions for the joint distribution for general classes.

Let $X_1, \dots, X_n$ be a random sample with minimum m_n and maximum M_n from any distribution function $F(x)$ on a finite interval $[w_1, w_2]$ with finite derivatives $\{F_{\cdot r}(x), r \geq 0\}$ at its endpoints w_1, w_2 . However, $F_{\cdot r}(x)$ need not exist in (w_1, w_2) .

Section 2 gives the joint distribution function and product moments of

$$\mathbf{Y}_n = (Y_{n1}, Y_{n2}) = n(m_n - w_1, w_2 - M_n) \quad (6)$$

in terms of

$$A_r = F_{\cdot r}(w_1)/r!, \quad B_r = F_{\cdot r}(w_2)/r! \quad (7)$$

as expansions in powers of n^{-1} . Note that

$$Y_{n1} \geq 0, \quad Y_{n2} \geq 0, \quad Y_{n1} + Y_{n2} \leq e_n = n(w_2 - w_1), \quad (8)$$

and $B_0 = 1, A_1 \geq 0, B_1 \geq 0$. We assume that $A_0 = 0$ (that is F does not have an atom at w_1), and $A_1 > 0, B_1 > 0$. So, the density function of $F(x)$ is positive at its end points. For example, this rules out the beta distribution except for the uniform distribution.

Section 3 gives the joint distribution function and product moments of

$$\mathbf{Z}_n = (Z_{n1}, Z_{n2}) = (A_1 Y_{n1}, B_1 Y_{n2}) = n(A_1(m_n - w_1), B_1(w_2 - M_n)) \quad (9)$$

in terms of

$$a_r = \bar{a}_r = A_r/A_1^r, \quad b_r = B_r/B_1^r, \quad \bar{b}_r = (-1)^r b_r \quad (10)$$

as expansions in powers of n^{-1} under the same conditions. This considerably simplifies the results of Section 2. Sections 4 and 5 give the distribution function and moments of the marginals of $\mathbf{Y}_n$ and $\mathbf{Z}_n$ under the same conditions. Section 6 shows that $\text{corr}(Z_{n1}, Z_{n2}) = O(n^{-1})$ and outlines how to deal with the case $A_1 = 0$ or $B_1 = 0$.

We need some notations. For $n, r = 0, 1, \dots$, set $(n)_r = n!/(n-r)! = n(n-1)\cdots(n-r+1)$. Let $\delta_{i,j} = I(i=j)$ denote the Kronecker delta function. For a sequence $\mathbf{c} = (c_1, c_2, \dots)$ of real or complex numbers, define the partial ordinary Bell polynomial, $\tilde{B}_{r,k} = \tilde{B}_{r,k}(\mathbf{c})$, by

$$\left(\sum_{r=1}^{\infty} c_r t^r \right)^k = \sum_{r=k}^{\infty} \tilde{B}_{r,k} t^r$$

for $t \in C$, the plane of complex numbers, and $k = 0, 1, \dots$. Comtet (1974) tabled $\tilde{B}_{r,k}$ on page 309 for $1 \leq r \leq 10$. Bell polynomials are valuable tools for handling power series.

2 The joint distribution and moments of $\mathbf{Y}_n$

Here, we assume that (7) holds with $A_1 > 0$, $B_1 > 0$. For $w_1 \leq x \leq y \leq w_2$, the joint distribution function of (m_n, M_n) is

$$P_n = \text{Prob.}(x < m_n, M_n \leq y) = \Delta^n, \quad \text{Prob.}(m_n \leq x, M_n \leq y) = F(y)^n - \Delta^n$$

with marginals

$$P_{n1} = \text{Prob.}(m_n \leq x) = 1 - [1 - F(x)]^n, \quad P_{n2} = \text{Prob.}(M_n \leq y) = F(y)^n,$$

where $\Delta = F(y) - F(x)$. Now transform (x, y) to (a, b) , where $a = n(x - w_1)$, $b = n(w_2 - y)$, that is, $x = w_1 + a/n$, $y = w_2 - b/n$. So, $a \geq 0$, $b \geq 0$, $a + b \leq e_n$ and $\mathbf{Y}_n$ of (6) has joint distribution function

$$P_n = \text{Prob.}(Y_{n1} > a, Y_{n2} \geq b) = \Delta^n \quad (11)$$

with marginals

$$P_{n1} = \text{Prob.}(Y_{n1} > a), \quad P_{n2} = \text{Prob.}(Y_{n2} \geq b).$$

In terms of the end derivatives of (7), we have the Taylor expansions

$$\begin{aligned} F(x) &= \sum_{r=1}^{\infty} (a/n)^r A_r, \quad F(y) = 1 + \sum_{r=1}^{\infty} (-b/n)^r B_r, \quad \Delta = 1 - S, \\ S &= 1 - F(y) + F(x) = \sum_{r=1}^{\infty} c_r n^{-r}, \quad c_r = a^r A_r - (-b)^r B_r. \end{aligned} \quad (12)$$

To regain symmetry, set

$$\bar{A}_r = A_r, \quad \bar{B}_r = (-1)^r B_r, \quad (13)$$

so that

$$c_r = a^r \bar{A}_r + b^r \bar{B}_r. \quad (14)$$

Lemma 2.1 *If*

$$S = \sum_{r=1}^{\infty} c_r n^{-r}$$

then

$$n \ln(1 - S) = -n \sum_{r=1}^{\infty} C_r n^{-r} \quad (15)$$

for

$$C_r = C_r(\mathbf{c}) = \sum_{k=1}^r \tilde{B}_{r,k}(\mathbf{c})/k \quad (16)$$

and $\mathbf{c} = (c_1, c_2, \dots)$. *Furthermore, if*

$$T = \sum_{i=1}^{\infty} d_i n^{-i}$$

then

$$\exp(-T) = \sum_{i=0}^{\infty} Q_i n^{-i} \quad (17)$$

for

$$Q_i = \sum_{k=0}^i (-1)^k \tilde{B}_{i,k}(\mathbf{d})/k!$$

and $\mathbf{d} = (d_1, d_2, \dots)$.

Theorem 2.1 gives expansions for the joint distribution function of $\mathbf{Y}_n$. and the joint density function of $\mathbf{Y}_n$. An expansion for the product moments of $\mathbf{Y}_n$ is given in Theorem 2.2.

In Theorem 2.1, c_r , C_r , d_{r-1} are polynomials of degree r in (a, b) . Q_i is a polynomial of degree $2i$ in (a, b) . For f a function of (a, b) , set

$$\partial_a = \partial/\partial a, \quad f_{.a} = \partial_a f, \quad f_{.ab} = \partial_a \partial_b f, \quad J_1 = \partial_a - A_1, \quad J_2 = \partial_b - B_1. \quad (18)$$

If $J = J_1 J_2$ then

$$Jf = f_{.ab} - f_{.b}A_1 - f_{.a}B_1 + A_1B_1f \quad (19)$$

and

$$Jfg = f_{.ab}g + f_{.a}g_{.b} + g_{.ab}f + g_{.a}f_{.b} - B_1(f_{.a}g + fg_{.a}) - A_1(f_{.b}g + fg_{.b}) + A_1B_1fg.$$

So,

$$Ja^i = -ia^{i-1}B_1 + a^iA_1B_1, \quad Ja^ib^j = ija^{i-1}b^{j-1} - a^ib^{j-1}A_1 - b^ja^{j-1}B_1 + a^ib^jA_1B_1. \quad (20)$$

Theorem 2.1 For c_r of (14) and C_r of (16), set

$$d_i = C_{i+1}, \quad (21)$$

and $Q_i(a, b) = Q_i$ of (17). For $a \geq 0$, $b \geq 0$ and $a + b \leq e_n$, where e_n is given by (8), $\mathbf{Y}_n$ has joint distribution function

$$\text{Prob.}(Y_{n1} > a, Y_{n2} \geq b) = e^{-c_1} U_n, \quad (22)$$

where

$$U_n = U_n(a, b) = \sum_{i=0}^{\infty} Q_i n^{-i}.$$

For c_r of (14), Q_i of (17) and J of (19), $\mathbf{Y}_n$ of (6) has joint density function

$$p_n(a, b) = \partial_a \partial_b P_n = e^{-c_1} J U_n,$$

where

$$J U_n = \sum_{i=0}^{\infty} q_i(a, b) n^{-i}$$

and $q_i(a, b) = J Q_i$.

Proof: By (11), (15) and (17), P_n is equal to the left hand side of (22) for S of (12),

$$\ln P_n = n \ln(1 - S) = -C_1 - T$$

for

$$T = \sum_{r=2}^{\infty} C_r n^{1-r} = \sum_{i=1}^{\infty} d_i n^{-i}.$$

So, $P_n = \exp(-C_1) \exp(-T)$ for $C_1 = c_1$. Now apply the lemma. **The expansion for the joint density function follows by differentiation.** $\square$

Note that q_i is a polynomial of degree $2i$ in (a, b) , say

$$q_i(a, b) = \sum_{j+k \leq 2i} q_{ijk} a^j b^k. \quad (23)$$

For $q = q(a, b)$ any function of a, b , set

$$\mathcal{F}q = \int_0^{\infty} e^{-aA_1} da \int_0^{\infty} e^{-bB_1} q(a, b) db, \quad (24)$$

$$\mathcal{F}_{rs} q = \mathcal{F} a^r b^s q, \quad \mathcal{F}_{rs} 1 = \mathcal{F}_{rs} 1 = r! s! A_1^{-r-1} B_1^{-s-1}. \quad (25)$$

Lemma 2.2 With A_1, B_1 of (7) and J of (19),

$$\mathcal{F} J a^i b^j = 0,$$

$$\mathcal{F}_{r0}Ja^i = \begin{cases} r(r+i-1)!A_1^{-r-i}, & r+i \neq 0, \\ 1, & i=r=0, \end{cases}$$

and

$$\mathcal{F}_{rs}Ja^ib^j = \begin{cases} rs(r+i-1)!(s+j-1)!A_1^{-r-i}B_1^{-s-j}, & (r+i)(s+j) \neq 0, \\ r(r+i-1)!A_1^{-r-i}, & s=j=0, r+i \neq 0, \\ s(s+j-1)!B_1^{-s-j}, & r=i=0, s+j \neq 0, \\ 1, & r=s=i=j=0. \end{cases}$$

The proof of the next theorem can be used to give *exact* expansions for the moments of $\mathbf{Y}_n$, see Appendix A. We now introduce the notation $a_n \approx b_n$ if and only if $a_n = b_n + O(e^{-ng})$, where $g > 0$ does not depend on n .

Theorem 2.2 For p_n , q_i of [Theorem 2.1](#), q_{ijk} of (23) and $\mathcal{F}_{rs}$, F_{rs} of (24)-(25), $\mathbf{Y}_n$ of (6) has (r, s) moment

$$E Y_{n1}^r Y_{n2}^s \approx \mathcal{F}_{rs} p_n(a, b) = \sum_{i=0}^{\infty} m_{rsi} n^{-i} \quad (26)$$

for

$$m_{rsi} = \mathcal{F}_{rs} q_i(a, b) = \sum_{j+k \leq 2i} q_{ijk} F_{r+j, s+k}$$

by (23).

Proof: For e_n of (8),

$$E Y_{n1}^r Y_{n2}^s = \int_0^{e_n} a^r da \int_0^{e_n-a} e^{-c_1 b} p_n(a, b) db = \sum_{i=0}^{\infty} m_{nr si} n^{-i},$$

where

$$m_{nr si} = \int_0^{e_n} a^r da \int_0^{e_n-a} b^s e^{-c_1 b} q_i(a, b) db. \quad (27)$$

In Appendix A, we showed that

$$m_{nr si} \approx m_{rsi} = \int_0^{\infty} a^r da \int_0^{\infty} e^{-c_1 b} q_i(a, b) db = \mathcal{F}_{rs} q_i(a, b).$$

Now apply [Theorem 2.1](#). $\square$

3 The joint distribution and moments of $\mathbf{Z}_n$

The following corollary follows easily from [Theorems 2.1 and 2.2](#). They use the notation $\mathbf{Z}_n$ of (9) and $a_r, \bar{a}_r, b_r, \bar{b}_r$ of (10). Set $\mathbf{z} = (z_1, z_2) \in R^2$, $a = z_1/A_1$ and $b = z_2/B_1$. Then

$$\begin{aligned} c_r &= z_1^r \bar{a}_r + z_2^r \bar{b}_r, \quad a_{30} = C_3(\bar{\mathbf{a}}) = a_3 + \bar{a}_2 + 1/3, \quad a_{21} = \bar{a}_2 + 1, \\ a_{40} &= (\bar{a}_2 + 1/2)^2 / 2 = a_2^2/2 + \bar{a}_2/2 + 1/8, \quad a_{22} = -a_2 b_2 + \bar{a}_2/2 + \bar{b}_2/2 + 3/4. \end{aligned}$$

[Corollary 3.1](#) gives expansions for the joint distribution function of $\mathbf{Z}_n$, the joint density function of $\mathbf{Z}_n$ and the product moments of $\mathbf{Z}_n$.

Corollary 3.1 For d_i of (21) and Q_i of (17),

$$\text{Prob.}(\mathbf{Z}_n > \mathbf{z}) = \text{Prob.}(Z_{n1} > z_1, Z_{n2} \geq z_2) = e^{-z_1 - z_2} \sum_{i=0}^{\infty} Q_i n^{-i}.$$

For A_1, B_1 of (7), Q_i of (17) and $p_n(a, b), q_i(a, b)$ of Theorem 2.1, $\mathbf{Z}_n$ has joint density function

$$p'_n(\mathbf{z}) = p_n(a, b)/A_1 B_1 = e^{-z_1 - z_2} \sum_{i=0}^{\infty} q'_i n^{-i},$$

where $q'_i = q_i(a, b)/A_1 B_1 = J' Q_i$, $J' = J'_1 J'_2$ and $J'_i = \partial/\partial z_i$. For A_1, B_1 of (7) and m_{rsi} of Theorem 2.2,

$$E Z_{n1}^r Z_{n2}^s \approx \sum_{i=0}^{\infty} M_{rsi} n^{-i},$$

where $M_{rsi} = A_1^r B_1^s m_{rsi}$.

Example 3.1 For any continuous distribution function $F(x)$ on R , $(F(X_1), F(m_n), F(M_n))$ has the same distribution as (X_1, m_n, M_n) from $U[0, 1]$, the uniform distribution on $[0, 1]$. So, $(nF(m_n), n - nF(M_n))$ has the same distribution as $(nm_n, n - nM_n)$ from $U[0, 1]$.

4 Marginal expansions of $\mathbf{Y}_n$ for $A_1 > 0, B_1 > 0$

A dramatic simplification occurs when dealing with marginals. This is because $b = 0$ implies

$$c_r = a^r A_r, \tilde{B}_{r,k}(\mathbf{c}) = a^r \tilde{B}_{r,k}(\mathbf{A}), C_r(\mathbf{c}) = a^r C_r(\mathbf{A}), d_i = a^{i+1} D_i, \tilde{B}_{i,k}(\mathbf{d}) = a^{i+k} \tilde{B}_{i,k}(\mathbf{D})$$

for $D_i = C_{i+1}(\mathbf{A})$, $C_r(\mathbf{c})$ of Lemma 2.1, $\mathbf{c} = (c_1, c_2, \dots)$, $\mathbf{A} = (A_1, A_2, \dots)$, $\mathbf{d} = (d_1, d_2, \dots)$ and $\mathbf{D} = (D_1, D_2, \dots)$. Similarly, $a = 0$ implies

$$c_r = b^r \bar{B}_r, \tilde{B}_{r,k}(\mathbf{c}) = b^r \tilde{B}_{r,k}(\bar{\mathbf{B}}), C_r(\mathbf{c}) = b^r C_r(\bar{\mathbf{B}}), d_i = b^{i+1} D_i, \tilde{B}_{i,k}(\mathbf{d}) = b^{i+k} \tilde{B}_{i,k}(\mathbf{D})$$

for $D_i = C_{i+1}(\bar{\mathbf{B}})$, $\mathbf{c} = (c_1, c_2, \dots)$, $\bar{\mathbf{B}} = (\bar{b}_1, \bar{b}_2, \dots)$, $\mathbf{d} = (d_1, d_2, \dots)$ and $\mathbf{D} = (D_1, D_2, \dots)$.

Putting $b = 0$ or $a = 0$ in Theorem 2.1 gives the marginal distribution functions in terms of $C_r(\mathbf{A})$ and $C_r(\bar{\mathbf{B}})$. Corollary 4.1 gives the marginal distribution and density functions of $\mathbf{Y}_n$. The marginal moments of $\mathbf{Y}_n$ are given in Corollary 4.2.

Corollary 4.1 For A_i, B_i of (7), $\bar{B}_i$ of (13), $D_i = C_{i+1}(\mathbf{A})$, $C_r = C_r(\mathbf{A})$ of Lemma 2.1 and Q_i of Theorem 2.1,

$$P_{n1} = \text{Prob.}(Y_{n1} > a) = e^{-aA_1} \sum_{i=0}^{\infty} Q_i n^{-i},$$

where

$$Q_i = Q_i(a, 0) = a^i \sum_{k=0}^i (-a)^k \tilde{B}_{i,k}(\mathbf{D})/k!.$$

Similarly,

$$P_{n2} = \text{Prob.}(Y_{n2} \geq b) = e^{-bB_1} \sum_{i=0}^{\infty} Q_i n^{-i},$$

where $Q_i = Q_i(0, b)$, is given by replacing a and $C_r(\mathbf{A})$ above by b and $C_r(\overline{\mathbf{B}})$. For A_i of (7), J_1 of (18) and $C_r = C_r(\mathbf{A})$, Y_{n1} of (6) has density function

$$p_{n1}(a) = -e^{-aA_1} \sum_{i=0}^{\infty} q_{i1}(a) n^{-i},$$

where $q_{i1}(a) = J_1 Q_i(a, 0)$. For B_i of (7), $\overline{B}_i$ of (13), J_2 of (18) and $C_r = C_r(\overline{\mathbf{B}})$, Y_{n2} of (6) has density function

$$p_{n2}(b) = -e^{-bB_1} \sum_{i=0}^{\infty} q_{i2}(b) n^{-i},$$

where $q_{i2}(b) = J_2 Q_i(0, b)$, given by replacing $J_1 a^j$ by $J_2 b^j$ in $q_{i1}(a)$ above.

Proof: Note that for J_i of (18)

$$J_1 a^i = i a^{i-1} - A_1 a^i, \quad J_2 b^i = i b^{i-1} - B_1 b^i.$$

Using $P_n(a, b) = P_n$ and $U_n(a, b)$ of (22), we obtain

$$p_{n1}(a) = -\partial_a P_n(a, 0) = -e^{-aA_1} J_1 U_n(a, 0),$$

where

$$J_1 U_n(a, 0) = \sum_{i=0}^{\infty} q_{i1}(a) n^{-i}. \quad \square$$

$q_{i1}(a)$ is a polynomial of degree $2i$ in a and $q_{i2}(b)$ is a polynomial of degree $2i$ in b , say, then

$$q_{i1}(a) = \sum_{j=0}^{2i} q_{ij1} a^j, \quad q_{i2}(b) = \sum_{j=0}^{2i} q_{ij2} b^j. \quad (28)$$

For $q = q(a)$ any function, set

$$\begin{aligned} \mathcal{F}_1 q &= \int_0^{\infty} e^{-aA_1} q(a) da, \quad \mathcal{F}_{r1} q = \mathcal{F}_1 a^r q(a), \\ \mathcal{F}_2 q &= \int_0^{\infty} e^{-bB_1} q(b) db, \quad \mathcal{F}_{r2} q = \mathcal{F}_1 b^r q(b), \end{aligned}$$

when these integrals exist. Set

$$G_r(A) = \int_0^{\infty} a^r e^{-aA} da = r! A^{-r-1} \quad (29)$$

for $A > 0$. For J_i of (18), set

$$\begin{aligned} N_{rj} &= \mathcal{F}_{r1} J_1 a^j = \int_0^\infty (J_1 a^j) a^r e^{-aA_1} da = -r(r+j-1)! A_1^{-r-j}, \\ M_{sj} &= \mathcal{F}_{s2} J_2 b^j = \int_0^\infty (J_2 b^j) e^{-bB_1} db = -s(s+j-1)! B_1^{-s-j}. \end{aligned}$$

From (28) and [Corollary 4.1](#), we obtain

Corollary 4.2 For A_1, B_1 of (7), p_{n1}, p_{n2} of [Corollary 4.1](#), q_{ij1}, q_{ij2} of (28) and G_r of (29),

$$\begin{aligned} E Y_{n1}^r &\approx \mathcal{F}_{r1} p_{n1} = \sum_{i=0}^{\infty} m_{r0i} n^{-i}, \\ E Y_{n2}^s &\approx \mathcal{F}_{s2} p_{n2} = \sum_{i=0}^{\infty} m_{0si} n^{-i} \end{aligned}$$

for

$$\begin{aligned} m_{r0i} &= - \sum_{j=0}^{2i} q_{ij1} G_{r+j}(A_1), \\ m_{0si} &= - \sum_{j=0}^{2i} q_{ij2} G_{s+j}(B_1). \end{aligned}$$

5 Marginal expansions of $\mathbf{Z}_n$ for $A_1 > 0, B_1 > 0$

Theorems 2.1 and 3.1 of Withers and Nadarajah (2017) gave expansions for the marginal distribution functions, density functions and moments of Z_{n1} and Z_{n2} , respectively. Here, we take a somewhat different route giving more explicit results. Note that by (16), for $a_r, \bar{b}_r$ of (10), $\mathbf{a} = (a_1, a_2, \dots)$ and $\bar{\mathbf{b}} = (\bar{b}_1, \bar{b}_2, \dots)$,

$$C_r(\mathbf{A}) = A_1^r C_r \text{ for } C_r = C_r(\mathbf{a}) \quad (30)$$

and

$$C_r(\bar{\mathbf{B}}) = B_1^r C_r \text{ for } C_r = C_r(\bar{\mathbf{b}}). \quad (31)$$

So,

$$\begin{aligned} E Z_{n1}^r &= \sum_{i=0}^{\infty} M_{ir0} n^{-i}, \\ E Z_{n2}^s &\approx \sum_{i=0}^{\infty} M_{i0s} n^{-i}, \end{aligned}$$

where $M_{ir0} \approx A_1^r m_{ir0}$ and $M_{i0s} = B_1^s m_{i0s}$.

Putting $a = z_1/A_1$ and $b = z_2/B_1$ in Section 4 gives Theorem 5.1. The marginal density functions of $\mathbf{Z}_n$ are also given in Theorem 5.1. The marginal variance, the marginal standard deviation and the marginal reciprocal standard deviation of $\mathbf{Z}_n$ are given in Corollary 5.1.

Theorem 5.1 For $a_r, \bar{b}_r$ of (10), $C_r = C_r(\mathbf{a})$ of (30) and Q_i of Corollary 4.1,

$$\text{Prob.}(Z_{n1} > z_1) = e^{-z_1} \sum_{i=0}^{\infty} n^{-i} Q_i.$$

$\text{Prob.}(Z_{n2} > z_2)$ is given by replacing z_1 by z_2 and $C_r(\mathbf{a})$ by $C_r(\bar{\mathbf{b}})$ of (31), say

$$\text{Prob.}(Z_{n2} > z_2) = e^{-z_2} \sum_{i=0}^{\infty} n^{-i} Q_i.$$

For $a_r, \bar{b}_r$ of (10), $C_r = C_r(\mathbf{a})$ of (30) and Q_i of Corollary 4.1, Z_{n1} has density function

$$p_n(z_1) = e^{-z_1} \sum_{i=0}^{\infty} q_i n^{-i},$$

where $q_i = K_1 Q_i$ and $K_i = 1 - \partial_{z_i}$. For $C_r = C_r(\bar{\mathbf{b}})$ of (31) and Q_i of Corollary 4.1, Z_{n2} has density function $p_n(z_2)$ given by replacing z_1 by z_2 and $C_r(\mathbf{a})$ by $C_r(\bar{\mathbf{b}})$.

Corollary 5.1 For C_r of (30),

$$\sigma(Z_{n1})^2 = \text{var}(Z_{n1}) \approx \sum_{i=0}^{\infty} v_i n^{-i}$$

and

$$\sigma(Z_{n1})^{\pm 1} \approx \sum_{i=0}^{\infty} \sigma_i(\pm) n^{-i}$$

with v_i and $\sigma_i(\pm)$ determined by the proof. This also holds with M_{r0i} , Z_{n1} and C_r replaced by M_{0ri} , Z_{n2} and C_r of (31).

Proof: We use the convolution notation

$$\left(\sum_{i=0}^{\infty} f_i x^i \right) \left(\sum_{i=0}^{\infty} g_i x^i \right) = \sum_{i=0}^{\infty} e_i x^i,$$

where

$$e_i = f_i \otimes g_i = \sum_{j=0}^i f_j g_{i-j}.$$

If there is more than one subscript, the convolution is for the last. So, taking $f_i = M_{10i}$, $v_i = M_{20i} - e_i$ and $e_i = M_{i10} \otimes M_{i10}$ gives

$$[\text{var}(Z_{n1})]^{r/2} \approx \sum_{i=0}^{\infty} \sigma_i(\pm) n^{-i},$$

where

$$\sigma_i(\pm) = \sum_{k=0}^i \binom{r/2}{k} \tilde{B}_{i,k}, \quad \tilde{B}_{i,k} = \tilde{B}_{i,k}(\mathbf{v})$$

and $\mathbf{v} = (v_1, v_2, \dots)$. The proof is complete. $\square$

The results of Sections 4 and 5 for the marginal distribution functions and moments of Y_{n1} and Z_{n1} do not require any conditions on $\{B_i\}$. In fact, they also hold for $F(x)$ on $[w_1, \infty)$. Similarly, the results of Sections 4 and 5 for the marginal distribution functions and moments of Y_{n2} and Z_{n2} do not require any conditions on $\{A_i\}$; they also hold for $F(x)$ on $(-\infty, w_2]$.

6 Correlation and extensions

Corollary 6.1 gives expansions for the covariance and correlation between Z_{n1} and Z_{n2} .

Corollary 6.1 *We have*

$$\text{cov}(Z_{n1}, Z_{n2}) \approx \sum_{i=0}^{\infty} k_i n^{-i}$$

and

$$\text{corr}(Z_{n1}, Z_{n2}) \approx \sum_{i=0}^{\infty} r_i n^{-i}$$

with k_i and r_i determined by the proof.

Proof: Apply Theorem 5.1. Set $k_i = M_{11i} - M_{10i} \otimes M_{i01i}$ and $\sigma_i[a] = \sigma_i(-1)$ of Corollary 5.1. Then

$$\sum_{i=0}^{\infty} r_i n^{-i} = \left(\sum_{i=0}^{\infty} k_i n^{-i} \right) \left(\sum_{i=0}^{\infty} \sigma_i[a] n^{-i} \right) \left(\sum_{i=0}^{\infty} \sigma_i[b] n^{-i} \right)$$

for $r_i = k_i \otimes \sigma_i[a] \otimes \sigma_i[b]$. $\square$

Withers and Nadarajah (2017) also dealt with the case, where $w_1 = 0$ and for some g , $F(x)/x^g$ has finite derivatives at zero.

Finally, we consider the case $A_1 = 0$ or $B_1 = 0$. Let I, J be the lowest integers such that $A_I \neq 0, B_J \neq 0$. Then one can get similar expansions for $\mathbf{Z}_n$, where $Z_{n1} = n(m_n - w_1)(n/|A_I|)^{1/I}$ and $Z_{n2} = n(w_2 - M_n)(n/|B_J|)^{1/J}$. See also Theorems 2.2 and 3.2 in Withers and Nadarajah (2017).

Appendix A

Here, we give an exact formula for m_{nrs0} of (27). The method extends to general $m_{nr si}$.

Theorem A.1 *Set $v_n = e_n(A_1 - B_1)$ for e_n of (8) and A_1, B_1 of (7). If $A_1 > B_1$, then for m_{nrs0} of (27)*

$$m_{nrs0} A_1^r B_1^s = r!s! - s!e^{-A_1 e_n} \sum_{j=0}^r (r)_j (A_1 e_n)^{r-j} - \sum_{k=0}^s (s)_k L_{n,r,s-k}$$

for

$$\begin{aligned}
L_{nrs} &= e^{-B_1 e_n} (1 - B_1/a_1)^{-r-1} I_{nrs}, \quad I_{nrs} = S_{nrs} - e^{-v_n} T_{nrs}, \\
S_{nrs} &= \sum_{i=0}^s \binom{s}{i} (B_1 e_n)^{s-i} [-B_1/(A_1 - B_1)]^i (r+i)!, \\
T_{nrs} &= \sum_{i=0}^s \binom{s}{i} (B_1 e_n)^{s-i} [-B_1/(A_1 - B_1)]^i \sum_{k=0}^{r+i} (r+i)_k v_n^k = (B_1 e_n)^s v_n^r T_{rs}, \\
T_{rs} &= \sum_{i=0}^s \binom{s}{i} (r+i)! (-1)^i.
\end{aligned}$$

If $A_1 < B_1$, swap A_1, B_1, r, s with B_1, A_1, s, r . If $A_1 = B_1$, then

$$M_{nrs0} = r!s! - r!e^{-A_1 e_n} K_{nrs}$$

for

$$\begin{aligned}
K_{nrs} &= s! \sum_{j=0}^r (r)_j (A_1 e_n)^{r-j} + \sum_{k=0}^s (s)_k I_{nr,s-k}, \\
I_{nrs} &= (A_1 e_n)^{r+s+1} I_{rs}, \quad I_{rs} = \sum_{i=0}^s \binom{s}{i} (-1)^i / (r+i+1).
\end{aligned}$$

Furthermore, $M_{nrs0} \approx M_{rs0} = r!s!$ since

$$M_{nrs0} = r!s! - r!e^{-B_1 e_n} (B_1 e_n)^s [1 + O(n^{-1})]$$

if $A_1 > B_1$ and

$$M_{nrs0} = r!s! - e^{-A_1 e_n} (A_1 e_n)^{r+s+1} I_{rs} [1 + O(n^{-1})]$$

if $A_1 = B_1$.

Proof: Note that

$$\begin{aligned}
M_{nrs0} &= m_{nrs0}/A_1 B_1 = \int_0^{e_n} a^r e^{-a A_1} da \int_0^{e_n - a} b^s e^{-b B_1} db \\
&= A_1^{-r-1} B_1^{-s-1} \int_0^{A_1 e_n} u^r e^{-u} du \int_0^{y_n(u)} t^s e^{-t} dt
\end{aligned}$$

for $y_n(u) = (e_n - u/A_1) B_1$. By equations (6.5.3) and (6.5.32) in Abramowitz and Stegun (1968),

$$\int_x^\infty t^{g-1} e^{-t} dt = x^{g-1} e^{-x} \sum_{k=0}^\infty (g-1)_k x^{-k}. \quad (32)$$

This expansion diverges unless $g = 1, 2, \dots$; in that case we can replace

$$\sum_{k=0}^\infty$$

in (32) by

$$\sum_{k=0}^{g-1}.$$

So, for $y_n = y_n(u)$,

$$\int_0^{y_n} t^s e^{-t} dt = s! - e^{-y_n} \sum_{k=0}^s (s)_k y_n^{s-k}$$

implies

$$L_{nrs} = \int_0^{A_1 e_n} u^r e^{-u-y_n} y_n^s du$$

and

$$I_{nrs} = \int_0^{\alpha A_1 e_n} u^r e^{-\alpha u} y_n^s du$$

at $\alpha = 1 - B_1/A_1$. In the case $A_1 > B_1$,

$$\begin{aligned} I_{nrs} &= \sum_{i=0}^s \binom{s}{i} (B_1 e_n)^{s-i} [-B_1/(A_1 - B_1)]^i K_{n,r+i}, \\ K_{nr} &= \int_0^{v_n} x^r e^{-x} dx = r! - e^{-v_n} \sum_{k=0}^r (r)_k v_n^k = r! + O(e^{-v_n} v_n^r). \end{aligned}$$

In the case $A_1 = B_1$,

$$I_{nrs} = \int_0^{A_1 e_n} u^r (A_1 e_n - u)^s du = \sum_{i=0}^s \binom{s}{i} (A_1 e_n)^{s-i} (-1)^i \int_0^{A_1 e_n} u^{r+i} du.$$

The proof is complete. $\square$

This illustrates how replacing e_n by ∞ in the upper integral limits of (26) introduces errors that are exponentially small, that is, $m_{nr si} \approx m_{rs i}$.

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