A note on 'Analysis of gamma-ray burst duration distribution using mixtures of skewed distributions'

DOI:

[10.1093/mnras/stx2373](https://doi.org/10.1093/mnras/stx2373)

Document Version

Final published version

[Link to publication record in Manchester Research Explorer](#)

Citation for published version (APA):

Kwong, H. S., & Nadarajah, S. (2018). A note on 'Analysis of gamma-ray burst duration distribution using mixtures of skewed distributions'. *Monthly Notices of the Royal Astronomical Society*, 473(1), 625-646.
<https://doi.org/10.1093/mnras/stx2373>

Published in:

Monthly Notices of the Royal Astronomical Society

Citing this paper

Please note that where the full-text provided on Manchester Research Explorer is the Author Accepted Manuscript or Proof version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version.

General rights

Copyright and moral rights for the publications made accessible in the Research Explorer are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Takedown policy

If you believe that this document breaches copyright please refer to the University of Manchester's Takedown Procedures [<http://man.ac.uk/04Y6Bo>] or contact uml.scholarlycommunications@manchester.ac.uk providing relevant details, so we can investigate your claim.

A note on ‘Analysis of gamma-ray burst duration distribution using mixtures of skewed distributions’

Hok Shing Kwong and Saralees Nadarajah[★]

School of Mathematics, University of Manchester, Manchester M13 9PL, UK

Accepted 2017 September 11. Received 2017 September 11; in original form 2017 May 29

ABSTRACT

Tarnopolski [Monthly Notices of the Royal Astronomical Society, 458 (2016) 2024–2031] analysed data sets on gamma-ray burst durations using skew distributions. He showed that the best fits are provided by two skew normal and three Gaussian distributions. Here, we suggest other distributions, including some that are heavy tailed. At least one of these distributions is shown to provide better fits than those considered in Tarnopolski. Five criteria are used to assess best fits.

Key words: methods: statistical.

1 INTRODUCTION

There have been many studies in grouping of gamma-ray bursts. The gamma-ray bursts can be classified into two groups according to shorter and longer T_{90} durations. But is there a third group? Horvath found evidence of this by fitting a three Gaussian distribution to logarithms of T_{90} for the *BATSE* 3B data. Mukherjee et al. (1998) used multivariate clustering procedures to discriminate between distinct groups of gamma-ray bursts. The variables burst duration, fluence and spectral hardness were used. They found three groups. But there is no clear rationale for the existence of the third group. It may be influenced strongly by the satellites. Veres et al. (2010) argued that the third group might well be related to the so-called X-ray flashes, but Ripa & Meszaros (2016) questioned this relation.

Here, we focus on the analysis of data sets relating to gamma-ray burst durations: Savchenko, Neronov & Courvoisier (2012) found that the durations observed by *INTEGRAL* have a unimodal distribution; Tarnopolski (2015) modelled the logarithm of T_{90} durations by a mixture of skew distributions; Zitouni et al. (2015) showed that a mixture of three Gaussian distributions fitted the *Swift* data better than a two Gaussian distribution; for *BATSE* 4B and current and *Swift*, Zitouni et al. (2015) found that a three Gaussian distribution followed the observations better than a two Gaussian distribution; Tarnopolski (2016) suggested that mixtures of two non-Gaussian distributions are likely to provide better fits than a three Gaussian distribution if the assumption that data are Gaussian distributions is relaxed. This suggestion was motivated by non-symmetry in the distribution of the logarithm of T_{90} durations due to a non-symmetric distribution of the envelope masses of the progenitors. Tarnopolski (2016), in fact, showed that ‘mixtures of two skew normal or two sinh arcsinh distributions are more likely to describe the observed

duration distribution of *Fermi* than a mixture of three standard Gaussian distributions, and that mixtures of two sinh arcsinh or two skew normal distributions are models competing with the conventional three Gaussian distribution in the case of *BATSE* and *Swift*. For others, we refer the readers to Horvath et al. (2008), Zhang & Choi (2008), Huja, Meszaros & Ripa (2009), Horvath et al. (2010), Ripa et al. (2012) and Bhavé et al. (2017).

The aim of this note is to consider other distributions not considered in the related literature. The distributions considered include both heavy tailed and light tailed. We show that at least one of these distributions provides better fits than the best-fitting distributions noted by Tarnopolski (2016). The data sets from *Fermi*, *BATSE* and *Swift* will be studied.

Section 2 gives a description of the data. Section 3 discusses the distributions, their fitting and the criteria used to select the best-fitting distributions. The results of the fittings and their discussion are provided in Sections 4 and 5. Finally, some conclusions are noted in Section 6.

2 DATA SETS

Five data sets¹ were obtained from three different sources: *Fermi*,² *BATSE*³ and *Swift*.⁴ For the *Fermi* source, three different data sets were considered (i) data from the earliest – 2015 April 29, with outliers excluded; (ii) data from the earliest – 2015 April 29; and (iii) all available data at the time of access. (i) was used in order to make a sensible comparison with results in Tarnopolski (2015).

The following summary statistics of each data set are given in Table 1: minimum, first quartile, median, mean, third quartile,

* E-mail: mbbssn2@manchester.ac.uk

¹ All accessed on 2017 May 10.

² <http://heasarc.gsfc.nasa.gov/W3Browse/fermi/fermigbrst.html>

³ <http://gamma-ray.nsstc.nasa.gov/batse/grb/catalog/>

⁴ http://swift.gsfc.nasa.gov/archive/grbtext_table/

Table 1. Summary statistics of the data on $\ln T_{90}$.

Statistics	<i>Fermi</i> 2015 (outliers excluded)	<i>Fermi</i> 2015	<i>Fermi</i> 2017	<i>BATSE</i>	<i>Swift</i>
Number of data	1593	1596	2074	2041	914
Minimum	− 1.194	− 1.796	− 1.796	− 1.824	− 1.744
Maximum	2.78	2.918	2.918	3.116	3.322
Q1	0.735	0.730	0.751	0.350	1.021
Median	1.289	1.289	1.311	1.265	1.574
Q3	1.685	1.685	1.675	1.692	1.949
Mean	1.122	1.120	1.125	0.993	1.407
Skewness	− 0.797	− 0.810	− 0.837	− 0.691	− 1.098
Kurtosis	3.088	3.164	3.178	2.557	4.385
Variance	0.619	0.629	0.620	0.854	0.648
Range	3.975	4.714	4.714	4.940	5.067
IQR	0.949	0.954	0.925	1.341	0.928
CV	0.552	0.562	0.551	0.860	0.461

Table 2. Interpretation of AIC.

ΔAIC	Interpretation
$0 < \Delta AIC \leq 2$	Substantial support
$4 < \Delta AIC \leq 7$	Considerably less support
$10 < \Delta AIC$	Essentially no support

Table 3. Interpretation of BIC.

ΔAIC	Interpretation
$0 < \Delta BIC \leq 2.2$	Not worth more than a bare mention
$2.2 < \Delta BIC \leq 6$	Positive evidence against
$6 < \Delta BIC \leq 10$	Strong evidence against
$10 < \Delta BIC$	Very strong evidence against

maximum, skewness, kurtosis, variance, coefficient of variation, range and interquartile range.

All data sets are negatively skewed. The skewness is largest for *Swift* and smallest for *BATSE*. Four of the data sets have kurtosis greater than that of the Gaussian distribution. The kurtosis is largest for *Swift* and smallest for *BATSE*.

The variance is largest for *BATSE* and smallest for *Fermi* 2015 with outliers excluded. The range is largest for *Swift* and smallest for *Fermi* 2015 with outliers excluded. The interquartile range is largest for *BATSE* and smallest for *Fermi* 2017. The coefficient of variation is largest for *BATSE* and smallest for *Swift*.

For each data set, we use five different criteria to select the ‘best’-fitting distribution. These include the Akaike information criterion (AIC; Akaike 1974), the Bayesian information criterion (BIC; Schwarz 1978), the consistent Akaike information criterion

Table 4. Parameter estimates of the fitted distributions (number of independent parameters in parentheses) to the data set *Fermi* 2015 with outliers excluded; standard errors are in the scale of 10^{-3} .

Data: <i>Fermi</i> 2015 (outliers excluded)	$\hat{\omega}_k$ (s.e.)	$\hat{\mu}_k$ (s.e.)	$\hat{\sigma}_k$ (s.e.)	$\hat{\lambda}_k$ (s.e.)	$\hat{\tau}_k$ (s.e.)	$\ln L$
Three-Gaussian (8)	0.108 (0.222) 0.266 (0.466) 0.626 (0.547)	− 0.412 (0.926) 0.757 (1.387) 1.540 (0.397)	0.374 (0.503) 0.594 (1.293) 0.422 (0.250)			− 1709.4
Two-Skew normal (7)	0.205 (0.312) 0.795 (0.580)	− 0.760 (0.876) 1.869 (0.371)	0.977 (1.590) 0.676 (0.411)	3.070 (8.794) − 1.550 (1.972)		− 1708.75
Two-Generalized t (9)	0.404 (0.492) 0.596 (0.558)	0.557 (0.828) 1.504 (0.440)	1.485 (0.833) 0.656 (0.460)	16251.270 (13.442) 51.000 (81.077)	5.394 (12.881) 2.224 (3.080)	− 1707.95
Two-Exponential power (7)	0.399 (0.495) 0.601 (0.482)	0.538 (0.832) 1.508 (0.434)	1.072 (0.609) 0.460 (0.317)	5.343 (10.830) 2.203 (3.509)		− 1707.89
Two-Skew exponential power (9)	0.098 (0.212) 0.902 (0.603)	− 0.021 (0.066) 1.616 (0.379)	0.990 (0.139) 0.673 (0.198)	0.347 (0.364) 0.568 (0.330)	4.272 (13.329) 1.814 (2.310)	− 1707.17

Table 5. Information criteria of the distributions fitted (number of independent parameters in parentheses) to the data set *Fermi* 2015 with outliers excluded.

Data: <i>Fermi</i> 2015 (outliers excluded)	$\ln L$	AIC	BIC	CAIC	AICc	HQC
Three-Gaussian (8)	−1709.4	3434.8	3477.78	3485.78	3434.89	3450.76
Two-Skew normal (7)	−1708.75	3431.5	3469.12	3476.12	3431.58	3445.47
Two-Generalized t (9)	−1707.95	3433.9	3482.26	3491.26	3434.01	3451.85
Two-Exponential power (7)	−1707.89	3429.78	3467.39	3474.39	3429.85	3443.75
Two-Skew exponential power (9)	−1707.17	3432.34	3480.7	3489.7	3432.46	3450.3

Table 6. Parameter estimates of the fitted distributions (number of independent parameters in parentheses) to the data set *Fermi* 2015; standard errors are in the scale of 10^{-3} .

Data: <i>Fermi</i> 2015	$\widehat{\omega}_k$ (s.e.)	$\widehat{\mu}_k$ (s.e.)	$\widehat{\sigma}_k$ (s.e.)	$\widehat{\lambda}_k$ (s.e.)	$\widehat{\tau}_k$ (s.e.)	$\ln L$
Three-Gaussian (8)	0.107 (0.227) 0.259 (0.488) 0.635 (0.550)	−0.397 (1.053) 0.749 (1.571) 1.526 (0.402)	0.412 (0.648) 0.642 (1.561) 0.430 (0.260)			−1728.33
Two-Skew normal (7)	0.211 (0.309) 0.789 (0.576)	−0.680 (0.939) 1.840 (0.370)	0.900 (1.400) 0.644 (0.390)	2.113 (5.340) −1.355 (1.734)		−1728.21
Two-Generalized t (9)	0.418 (0.518) 0.582 (0.559)	0.608 (0.870) 1.486 (0.461)	1.485 (0.877) 0.642 (0.491)	1.793 (10.145) 19110.340 (221.435)	7.088 (20.232) 2.099 (3.053)	−1727.74
Two-Exponential power (7)	0.316 (0.408) 0.684 (0.368)	0.253 (0.965) 1.520 (0.376)	0.800 (0.663) 0.444 (0.278)	3.179 (7.003) 2.135 (3.056)		−1728.48
Two-Skew exponential power (9)	0.039 (0.170) 0.961 (0.620)	−0.717 (0.001) 1.684 (0.393)	0.443 (2.926) 0.735 (0.179)	0.245 (−) 0.630 (337)	0.184 (−) 1.634 (1.666)	−1710.16

Table 7. Information criteria of the distributions fitted (number of independent parameters in brackets) to the data set *Fermi* 2015.

Data: <i>Fermi</i> 2015	$\ln L$	AIC	BIC	CAIC	AICc	HQC
Three-Gaussian (8)	−1728.33	3472.67	3515.67	3523.67	3472.76	3488.64
Two-Skew normal (7)	−1728.21	3470.43	3508.05	3515.05	3470.5	3484.4
Two-Generalized t (9)	−1727.74	3473.48	3521.86	3530.86	3473.6	3491.45
Two-Exponential power (7)	−1728.48	3470.95	3508.58	3515.58	3471.02	3484.93
Two-Skew exponential power (9)	−1710.16	3438.33	3486.71	3495.71	3438.44	3456.3

(CAIC; Bozdogan 1987), the corrected Akaike information criterion (AICc; Hurvich & Tsai 1989) and the Hannan–Quinn criterion (Hannan & Quinn 1979). Tarnopolski (2016) used the AIC to discriminate among the fitted distributions.

3 DISTRIBUTIONS FITTED

Let X denote a continuous random variable representing the $\ln T_{90}$ duration of gamma-ray burst. Let $f(x)$ denote the probability density function of X , taken as a mixture of K distributions. Let $f_k(x)$ denote the probability density function of the k th component of $f(x)$. We suppose X can be described by the following five distributions:

(i) Mixture of K normal distributions specified by

$$f(x) = \sum_{k=1}^K \frac{\omega_k}{\sqrt{2\pi\sigma_k^2}} \exp \left[-\frac{(x - \mu_k)^2}{2\sigma_k^2} \right]$$

with $3K - 1$ independent parameters, where $\mu_k \in \mathbf{R}$ and $\sigma_k > 0$.

(ii) Mixture of K skew normal distributions (Azzalini 1985) specified by

$$f(x) = \sum_{k=1}^K \frac{2\omega_k}{\sigma_k} \phi \left(\frac{x - \mu_k}{\sigma_k} \right) \Phi \left(\lambda_k \frac{x - \mu_k}{\sigma_k} \right)$$

with $4K - 1$ independent parameters, where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function and the cumulative

Table 8. Parameter estimates of the fitted distributions (number of independent parameters in parentheses) to the data set *Fermi* 2017; standard errors are in the scale of 10^{-3} .

Data: <i>Fermi</i> 2017	$\hat{\omega}_k$ (s.e.)	$\hat{\mu}_k$ (s.e.)	$\hat{\sigma}_k$ (s.e.)	$\hat{\lambda}_k$ (s.e.)	$\hat{\tau}_k$ (s.e.)	$\ln L$
Three-Gaussian (8)	0.103 (0.175) 0.259 (0.370) 0.638 (0.423)	−0.388 (0.813) 0.739 (1.252) 1.527 (0.298)	0.396 (0.506) 0.668 (1.223) 0.416 (0.188)			−2208.41
Two-Skew normal (7)	0.214 (0.233) 0.786 (0.441)	−0.636 (0.696) 1.822 (0.274)	0.847 (0.989) 0.608 (0.282)	1.937 (3.689) −1.271 (1.261)		−2208.79
Two-Generalized t (9)	0.413 (0.388) 0.587 (0.383)	0.586 (0.667) 1.505 (0.334)	1.432 (0.671) 0.608 (0.355)	1.471 (5.759) 3027.160 (8140.490)	7.505 (16.648) 2.011 (2.101)	−2208.08
Two-Exponential power (7)	0.255 (0.242) 0.745 (0.430)	0.059 (0.703) 1.491 (0.267)	0.653 (0.489) 0.436 (0.201)	2.748 (5.032) 2.053 (2.154)		−2209.16
Two-Skew exponential power (9)	0.035 (0.128) 0.965 (0.478)	−0.717 (−) 1.677 (0.295)	0.440 (2.353) 0.736 (0.140)	0.250 (−) 0.609 (0.256)	0.180 (−) 1.528 (1.197)	−2206.39

Table 9. Information criteria of the distributions fitted (number of independent parameters in parentheses) to the data set *Fermi* 2017.

Data: <i>Fermi</i> 2017	$\ln L$	AIC	BIC	CAIC	AICc	HQC
Three-Gaussian (8)	−2208.41	4432.81	4477.91	4485.91	4432.88	4449.34
Two-Skew normal (7)	−2208.79	4431.58	4471.04	4478.04	4431.63	4446.04
Two-Generalized t (9)	−2208.08	4434.16	4484.89	4493.89	4434.25	4452.75
Two-Exponential power (7)	−2209.16	4432.33	4471.79	4478.79	4432.38	4446.79
Two-Skew exponential power (9)	−2206.39	4430.77	4481.51	4490.51	4430.86	4449.37

distribution function of the standard Gaussian distribution, respectively, $\mu_k, \lambda_k \in \mathbf{R}$ and $\sigma_k > 0$.

(iii) Mixture of K generalized- t distributions (McDonald & Newey 1988) specified by

$$f(x) = \sum_{k=1}^K \frac{\omega_k \tau_k}{2\sigma_k \lambda_k^{1/\tau_k} B(\lambda_k, 1/\tau_k)} \left[1 + \frac{1}{\lambda_k} \left| \frac{x - \mu_k}{\sigma_k} \right|^{\tau_k} \right]^{-(\lambda_k + 1/\tau_k)}$$

with $5K - 1$ independent parameters, where $\mu_k \in \mathbf{R}$ and $\tau_k, \lambda_k, \sigma_k > 0$.

(iv) Mixture of K exponential power distributions (Subbotin 1923) specified by

$$f(x) = \sum_{k=1}^K \frac{\omega_k \lambda_k}{2\sigma_k \Gamma(1/\lambda_k)} \exp \left[- \left(\frac{|x - \mu_k|}{\sigma_k} \right)^{\lambda_k} \right]$$

with $4K - 1$ independent parameters, where $\mu_k \in \mathbf{R}$ and $\sigma_k, \lambda_k > 0$.

(v) Mixture of K skew exponential power distributions (Zhu & Zinde-Walsh 2009) specified by

$$f(x) = \sum_{k=1}^K \omega_k f_k(x),$$

$$f_k(x) = \begin{cases} \frac{1}{\sigma_k} K_{EP}(\tau_k) \exp \left(-\frac{1}{\tau_k} \left| \frac{x - \mu_k}{2\lambda_k \sigma_k} \right|^{\tau_k} \right), & \text{if } x \leq \mu_k, \\ \frac{1}{\sigma_k} K_{EP}(\tau_k) \exp \left(-\frac{1}{\tau_k} \left| \frac{x - \mu_k}{2(1 - \lambda_k) \sigma_k} \right|^{\tau_k} \right), & \text{if } x > \mu_k \end{cases}$$

with $5K - 1$ independent parameters, where $K_{EP}(\tau) = \frac{1}{2\tau^{1/\tau} \Gamma(1+1/\tau)}$, $\mu_k \in \mathbf{R}$, $\tau_k, \sigma_k > 0$ and $0 < \lambda_k < 1$.

Several of these distributions are nested (Cox & Hinkley 1974): The generalized- t distribution for $\tau = 2$ and $\lambda \rightarrow \infty$ is a Gaussian distribution, the skew exponential power distribution for $\lambda = 0.5$ is an exponential power distribution, and the exponential power distribution for $\lambda = 2$ is a Gaussian distribution, and so on.

Each distribution was fitted by the method of maximum likelihood. That is, if $x_1, x_2, \dots, x_n$ are independent observations on X , then the parameters of each distribution are the values maximizing the likelihood

$$L(\Theta) = \prod_{i=1}^n f(x_i; \Theta)$$

or the log-likelihood

$$\ln L(\Theta) = \sum_{i=1}^n \ln f(x_i; \Theta),$$

Table 10. Parameter estimates of the fitted distributions (number of independent parameters in parentheses) to the data set *BATSE*; standard errors are in the scale of 10^{-3} .

Data: <i>BATSE</i> 2015	$\widehat{\omega}_k$ (s.e.)	$\widehat{\mu}_k$ (s.e.)	$\widehat{\sigma}_k$ (s.e.)	$\widehat{\lambda}_k$ (s.e.)	$\widehat{\tau}_k$ (s.e.)	$\ln L$
Three-Gaussian (8)	0.171 (0.208) 0.306 (0.365) 0.523 (0.397)	−0.454 (0.740) 0.752 (1.268) 1.606 (0.320)	0.474 (0.514) 0.736 (1.182) 0.382 (0.193)			−2443.47
Two-Skew normal (7)	0.376 (0.286) 0.624 (0.412)	−0.792 (0.654) 1.862 (0.300)	1.173 (0.949) 0.533 (0.308)	2.477 (3.811) −1.125 (1.415)		−2444.53
Two-Generalized t (9)	0.359 (0.256) 0.641 (0.410)	−0.032 (0.624) 1.569 (0.281)	1.057 (0.630) 0.564 (0.303)	29 022.660 (518.995) 13 705.780 (256.314)	2.912 (4.581) 1.870 (1.625)	−2444.03
Two-Exponential power (7)	0.611 (0.379) 0.389 (0.384)	0.621 (0.532) 1.577 (0.383)	1.227 (0.376) 0.341 (0.313)	4.499 (4.199) 2.328 (6.679)		−2443.43
Two-Skew exponential power (9)	0.287 (0.248) 0.713 (0.426)	−0.260 (0.627) 1.633 (0.289)	0.476 (0.323) 0.602 (0.198)	0.600 (0.465) 0.450 (0.251)	2.316 (4.416) 1.661 (1.944)	−2443.22

Table 11. Information criteria of the distributions fitted (number of independent parameters in brackets) to the data set *BATSE*.

Data: <i>BATSE</i>	$\ln L$	AIC	BIC	CAIC	AICc	HQC
Three-Gaussian (8)	−2443.47	4902.94	4947.91	4955.91	4903.01	4919.44
Two-Skew normal (7)	−2444.53	4903.07	4942.41	4949.41	4903.12	4917.5
Two-Generalized t (9)	−2443.63	4905.27	4955.86	4964.86	4905.35	4923.82
Two-Exponential power (7)	−2443.44	4900.87	4940.22	4947.22	4900.92	4915.3
Two-Skew exponential power (9)	−2443.22	4904.45	4955.04	4964.04	4904.54	4923

where $\Theta = (\theta_1, \theta_2, \dots, \theta_k)'$ is a vector of parameters specifying $f(\cdot)$. Let $\widehat{\Theta} = (\widehat{\theta}_1, \widehat{\theta}_2, \dots, \widehat{\theta}_k)'$ denote the maximum-likelihood estimate of Θ .

The expectation maximization (EM) algorithm (Dempster, Laird & Rubin 1977) was used for maximum-likelihood estimation. The following procedure was followed:

(i) E-step: set

$$\omega_{i,k} = \frac{f_k(x_i; \Theta)}{\sum_{m=1}^K f_m(x_i; \Theta)}$$

and

$$\omega_k = \frac{\sum_{i=1}^n \omega_{i,k}}{n}.$$

(ii) M-step: set

$$\widehat{\Theta} = \underset{\Theta}{\operatorname{argmax}} \sum_{k=1}^K \sum_{i=1}^n \omega_{i,k} \ln f_k(x_i; \Theta)$$

and

$$\widehat{\Theta}_k = \underset{\Theta_k}{\operatorname{argmax}} \sum_{i=1}^n \omega_{i,k} \ln f_k(x_i; \Theta).$$

(iii) Repeat E-step with $\Theta = \widehat{\Theta}$.

(iv) Keep repeating until $[\ln L(\widehat{\Theta}) - \ln L(\Theta)] \rightarrow 0$.

The M-step was performed using the routine `optim` in the R software package (R Development Core Team 2017). The standard errors of $\widehat{\Theta}$ were computed by approximating the covariance matrix of $\widehat{\Theta}$ by the inverse of the observed information matrix, i.e.

$$\operatorname{cov}(\widehat{\Theta}) \approx \left(\begin{array}{ccc} \frac{\partial^2 \ln L}{\partial \theta_1^2} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2} & \dots & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_k} \\ \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_2^2} & \dots & \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \ln L}{\partial \theta_k \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_k \partial \theta_2} & \dots & \frac{\partial^2 \ln L}{\partial \theta_k^2} \end{array} \right)^{-1} \Bigg|_{\Theta=\widehat{\Theta}}.$$

Many of the fitted distributions are not nested. Discrimination among them was performed using various criteria:

Table 12. Parameter estimates of the fitted distributions (number of independent parameters in parentheses) to the data set *Swift*; standard errors are in the scale of 10^{-3} .

Data: <i>Swift</i> 2015	$\hat{\omega}_k$ (s.e.)	$\hat{\mu}_k$ (s.e.)	$\hat{\sigma}_k$ (s.e.)	$\hat{\lambda}_k$ (s.e.)	$\hat{\tau}_k$ (s.e.)	$\ln L$
Three-Gaussian (8)	0.078 (0.330) 0.696 (0.319) 0.226 (0.305)	−0.531 (2.269) 1.444 (0.912) 1.960 (1.240)	0.479 (1.639) 0.597 (0.790) 0.290 (0.611)			−989.78
Two-Skew normal (7)	0.059 (0.281) 0.941 (1.066)	−0.199 (1.958) 2.208 (0.660)	0.623 (2.004) 0.915 (0.780)	−4.514 (112.633) −2.327 (4.925)		−991.112
Two-Generalized t (9)	0.224 (0.728) 0.776 (1.037)	0.579 (2.498) 1.635 (0.739)	1.782 (2.521) 0.793 (0.781)	4.624 (144.547) 22.920 (752.398)	5.787 (39.076) 2.427 (5.126)	−996.653
Two-Exponential power (7)	0.225 (0.725) 0.775 (1.036)	0.571 (2.487) 1.639 (0.738)	1.299 (1.801) 0.549 (0.537)	4.804 (21.597) 2.306 (5.227)		−996.675
Two-Skew exponential power (9)	0.056 (0.286) 0.944 (0.279)	−0.369 (1.861) 1.848 (0.656)	0.788 (1.864) 0.664 (0.337)	0.357 (1.317) 0.582 (0.559)	1.979 (16.118) 1.750 (3.823)	−989.527

Table 13. Information criteria of the distributions fitted (number of independent parameters in parentheses) to the data set *Swift*.

Data: <i>Swift</i>	$\ln L$	AIC	BIC	CAIC	AICc	HQC
Three-Gaussian (8)	−989.78	1995.56	2034.1	2042.1	1995.72	2010.27
Two-Skew normal (7)	−991.112	1996.22	2029.95	2036.95	1996.35	2009.1
Two-Generalized t (9)	−996.653	2011.31	2054.67	2063.67	2011.5	2027.86
Two-Exponential power (7)	−996.675	2007.35	2041.07	2048.07	2007.47	2020.22
Two-Skew exponential power (9)	−989.527	1997.05	2040.41	2049.41	1997.25	2013.61

(i) The AIC due to Akaike (1974) is defined by

$$\text{AIC} = 2k - 2 \ln L(\hat{\Theta}).$$

The relative probability of a fitted distribution is defined by

$$\text{Pr} = \exp\left(\frac{\text{AIC}_{\min} - \text{AIC}}{2}\right).$$

A general rule to interpret the AIC (Burnham & Anderson 2004) is shown in Table 2.

(ii) The BIC due to Schwarz (1978) is defined by

$$\text{BIC} = k \ln n - 2 \ln L(\hat{\Theta}).$$

A general rule to interpret the BIC (Raftery 1996) is shown in Table 3.

(iii) The CAIC due to Bozdogan (1987) is defined by

$$\text{CAIC} = -2 \ln L(\hat{\Theta}) + k(\ln n + 1).$$

The CAIC is similar to the BIC. It penalizes extra parameters harder than BIC does. The interpretation of ΔCAIC should be roughly the same as that described in Table 3.

(iv) The AICc (Hurvich & Tsai 1989) is defined by

$$\text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1}.$$

The AICc should be used when $n/k < 40$ (Burnham & Anderson 2004). The interpretation of ΔAICc should be the same as that described in Table 2.

(v) The Hannan–Quinn criterion (Hannan & Quinn 1979) is defined by

$$\text{HQC} = -2 \ln L(\hat{\Theta}) + 2k \ln \ln n.$$

The smaller the values of these criteria, the better the fit. For a more detailed discussion on these criteria, see Burnham & Anderson (2004) and Fang (2011).

4 RESULTS

The five distributions described in Section 3 were fitted to the data described in Section 2. The method of maximum likelihood was used. For all five data sets examined, there is strong evidence (in terms of the information criteria) that the optimal value of K for

Figure 1. Density of the distributions fitted to the data set *Fermi* 2015 with outliers excluded.

the skew normal mixture, generalized- t mixture, exponential power mixture and skew exponential power mixture distributions is equal to 2. The optimal value of K for the Gaussian mixture distribution is 3. The parameter estimates and information criteria for each mixture distribution are presented in Tables 4–13, where the smallest values of the information criteria are given in bold.

For *Fermi* 2015 with outliers excluded, the parameter estimates and information criteria are shown in Tables 4 and 5, respectively. Density plots, QQ-plots and PP-plots of the fitted distributions are shown in Figs 1, 6 and 11, respectively. For *Fermi* 2015, the parameter estimates and information criteria are shown in Tables 6 and 7, respectively. The density plot, QQ-plot and PP-plot are shown in Figs 2, 7 and 12, respectively. For *Fermi* 2017, the parameter estimates and information criteria are shown in Tables 8 and 9, respectively. The density plot, QQ-plot and PP-plot are shown in Figs 3, 8 and 13, respectively. For *BATSE*, the parameter estimates

and information criteria are shown in Tables 10 and 11, respectively. The density plot, QQ-plot and PP-plot are shown in Figs 4, 9 and 14, respectively. For *Swift*, the parameter estimates and information criteria are shown in Tables 12 and 13, respectively. The density plot, QQ-plot and PP-plot are shown in Figs 5, 10 and 15, respectively.

5 DISCUSSION

5.1 *Fermi*

5.1.1 *Fermi* 2015, outliers excluded

For this data set, a two exponential power distribution fits best in terms of all five information criteria. There is very strong evidence (ΔBIC , $\Delta\text{CAIC} > 10$) and strong evidence (ΔAIC , $\Delta\text{AICc} > 4$)

Figure 2. Density of the distributions fitted to the data set *Fermi* 2015.

that the two exponential power distribution fits better than the three Gaussian distribution. There is also very strong evidence (ΔBIC , $\Delta\text{CAIC} > 10$) that the former fits better than the two generalized- t distribution and the two skew exponential power distribution. The values of all of the information criteria for the two skew normal distribution are slightly higher than those for the two exponential power distribution. Although the two exponential power distribution is nested in the two skew exponential power distribution, the extra two parameters used in the two skew exponential power distribution increases the value of $\ln L$ by only 0.72. This suggests the possibility that the distribution of *Fermi* T_{90} can be modelled by a mixture of platykurtic distributions rather than a mixture of skew distributions. Each mixture component of the two exponential power distribution has a kurtosis less than that of the Gaussian distribution, meaning each component has heavier tails. The component having the larger

weight has the smaller scale and larger kurtosis. It is located to the right-hand side of the component having the smaller weight. In terms of QQ-plot and PP-plot, the three Gaussian, two skew normal, two generalized- t and two exponential power distributions fit the data very well. The two skew exponential power distribution fits the data slightly worse than the other four.

5.1.2 *Fermi* 2015

For this data set, the results obtained were quite different from those in Section 5.1.1. There is very strong evidence (ΔAIC , ΔBIC , ΔCAIC , ΔAICc , $\Delta\text{HQC} > 10$) that the two skew exponential power distribution describes the data better than the other four distributions. However, the density plot, QQ-plot and PP-plot show an unusual fit to the left half of the distribution. We argue that

Figure 3. Density of the distributions fitted to the data set *Fermi* 2017.

the unusual fitting is caused by a large amount of ties existing in the lower portion of the data set, which subsequently results in an unnatural high peak having a high impact on the maximum-likelihood estimates. If we exclude the results of the two skew exponential power distribution, we see that the two skew normal distribution fits best in terms of all of the five information criteria with very strong evidence in terms of BIC and CAIC. The two exponential power distribution arguably fits as well as the two skew normal distribution, with only a 0.5 difference in all five criteria. One mixture component of the two skew normal distribution has positive skewness. The other has negative skewness. The component with negative skewness has the larger weight and smaller scale. It is located to the right-hand side of the other component.

5.1.3 *Fermi* 2017

There is very strong evidence (ΔBIC , $\Delta\text{CAIC} > 10$) that both the two skew normal and two exponential power distributions are better models than the two skew exponential power distribution. Similar to the results in Section 5.1.2, unusual fits were recorded in the two skew exponential power distribution (see the density, QQ and PP plots). Excluding the two skew exponential power distribution, we have strong evidence that the two skew normal distribution fits best (ΔBIC , $\Delta\text{CAIC} > 6$). Again, the two exponential power distribution performs very closely to the two skew normal distribution, with a 0.7 difference in all five criteria. Once again, one mixture component of the two skew normal distribution has positive skewness, and the other has negative skewness with the larger weight and smaller scale.

Figure 4. Density of the distributions fitted to the data set *BATSE*.

5.2 *BATSE*

The two exponential power distribution performs best. With the lowest values in all of the five information criteria, there is strong evidence (ΔBIC , $\Delta\text{CAIC} > 7$) that it is a better model than the three Gaussian, two generalized- t and two skew exponential power distributions. There is also weak evidence ($\Delta\text{AIC} > 2$; ΔBIC , $\Delta\text{CAIC} > 2.2$) that the two exponential power distribution fits better than the two skew normal distribution. All five distributions appear to give adequate fits according to the density, PP and QQ plots. Considering both the information criteria and plots, there is arguably enough evidence that the two exponential power distribution is the best distribution to describe this data set. Each mixture component of the two exponential power distribution has a kurtosis less than that of the Gaussian distribution, meaning each component has heavier tails. The component having the larger weight has the larger

scale and smaller kurtosis. It is located to the left-hand side of the component having the smaller weight.

5.3 *Swift*

While both the two skew normal and two exponential power distributions performed the best in Sections 5.1.1, 5.1.2, 5.1.3 and 5.2, for this data set, there is very strong evidence ($\Delta\text{AIC} > 10$) and strong evidence (ΔBIC , $\Delta\text{CAIC} > 7$) to support the three Gaussian distribution against the two exponential power distribution. There is positive evidence that (ΔBIC , $\Delta\text{CAIC} > 4$) the two skew normal distribution fits better than the three Gaussian distribution. The density, PP and QQ plots show that the three Gaussian distribution fits the data better than the other four distributions. The component of the three Gaussian distribution having the largest weight also has the largest scale and it is located in the middle of the two other

Figure 5. Density of the distributions fitted to the data set *Swift*.

components. The component having the second largest weight has the smallest scale, and it is located to the right-hand side of the two other components. The component having the smallest weight has the second smallest scale, and it is located to the left-hand side of the two other components.

A possible reason for the better fit of the three Gaussian distribution could be that this data set has less significant bimodality. Figs 1–5 show that the bimodality of $\ln T_{90}$ from *Fermi* and *BATSE* is more significant than that from *Swift*. Tables 4–12 show that the best-fitting exponential power distributions are a mixture of a platykurtic distribution and a slightly platykurtic distribution (close to a Gaussian distribution). A possible reason that the two exponential power distribution fits all previous data sets better than the three Gaussian distribution is its better fits to the lower half of each data set. For the data set *Swift*, the less significant bimodality makes the

contribution of the fit of the exponential power distribution to the lower half of the data set less significant.

5.4 Comparison of the three data bases

For most data sets, the difference in log likelihoods between the best-fitting and three Gaussian distributions is small. The exception is the *Fermi* 2015 data set. The best-fitting distributions for *Fermi* 2015 without outliers, *Fermi* 2017 and *BATSE*, were chosen on the basis of the five information criteria: For the *Fermi* 2015 data without outliers, the differences between the best-fitting and three Gaussian distributions are substantial for all five criteria; for the *Fermi* 2017 data, the differences between the best-fitting and three Gaussian distributions are substantial for BIC, CAIC and HQC; and for the *BATSE* data, the differences between the best-fitting

Figure 6. QQ-plot of the distributions fitted to the data set *Fermi* 2015 with outliers excluded.

and three Gaussian distributions are substantial for BIC, CAIC and HQC.

6 CONCLUSIONS

Both two skew normal and two exponential power distributions provide significantly better fits than the three Gaussian distribution for

the gamma-ray burst durations recorded by *Fermi* and *BATSE*. The two distributions exhibit distinct properties: The extra parameter in the skew normal distribution adds different levels of skewness to the Gaussian distribution, and the extra parameter in the exponential power distribution adds different levels of kurtosis to the Gaussian distribution. The exponential power distribution has an elementary expression for its probability density function, which

Figure 7. QQ-plot of the distributions fitted to the data set *Fermi* 2015.

is not the case for the skew normal distribution. The exponential power distribution also has a closed-form expression for its cumulative distribution function. The skew normal distribution does not have a closed-form cumulative distribution function. Hence, the exponential power distribution should be preferred for practical use.

The number of different groups of gamma-ray burst durations is more likely to be two than three. However, we can-not find statistical

evidence to tell whether gamma-ray burst durations have a skew distribution or a platykurtic distribution as they both fit equally well. We also believe that the high amount of ties in *Fermi* could have affected the skewness and kurtosis, especially when the data are in logarithmic scale.

For the *Swift* data, although the two skew normal distribution fits better than the three Gaussian distribution in terms of BIC

Figure 8. QQ-plot of the distributions fitted to the data set *Fermi* 2017.

and CAIC, the three Gaussian distribution provides the best fit in terms of QQ-plot, PP-plot and other criteria. While we cannot formally find reasons why both two skew normal and two exponential power distributions perform worse, we believe that reasons could be due to differences in gamma-ray burst detector sensitivities (Band 2006) and the existence of ties. Moreover, the

fitted parameters of the three Gaussian distribution are significantly different from those for other data sets. For example, the third group of the three Gaussian distribution has a significantly higher mean and proportion compared to other data sets. This suggests that the third group in *Swift* may be closely related to the sensitivity of the *Swift* detector. An astrophysical meaning for the

Figure 9. QQ-plot of the distributions fitted to the data set *BATSE*.

presence of the third group is presented in Veres et al. (2010). They found evidence that the third group ‘is closely related to X-ray flashes’.

The best-fitting distributions in Section 5 compared to those in Tarnopolski (2016) are as follows: For the *Fermi* 2015 data with outliers excluded, the two exponential power distribution provides

better fits than the two skew normal distribution of Tarnopolski (2016) in terms of each of the five criteria, although the differences are small; and for the *BATSE* data, the two exponential power distribution provides better fits than the three Gaussian distribution of Tarnopolski (2016) in terms of each of the five criteria; most of the differences suggest either positive or strong evidence. Hence, the

Figure 10. QQ-plot of the distributions fitted to the data set *Swift*.

two exponential power distribution has gains over those discussed in Tarnopolski (2016).

The three Gaussian distribution giving the best fit can be attributed to the Central Limit Theorem (Durrett 2004). We also have two other distributions giving best fits for T_{90} durations. The two skew normal distribution giving the best fit can be attributed to varying

degrees of skewness in the data: One part of the data has positive skewness, while the other part has negative skewness, as shown in Section 5.1.2 for the *Fermi* 2015 data set and Section 5.1.3 for the *Fermi* 2017 data set. The two exponential power distribution giving the best fit can be attributed to the presence of heavier tails than the Gaussian distribution. It is also attributable to different degrees of

Figure 11. PP-plot of the distributions fitted to the data set *Fermi* 2015 with outliers excluded.

heaviness of the tails, one part of the data having heavy tails and the other part having even heavier tails, as shown in Section 5.1.1 for the *Fermi* 2015 data set with outliers excluded and Section 5.2 for the *BATSE* data set.

The message of this paper and Tarnopolski's (2015, 2016) is that often non-Gaussian distributions can provide better fits to gamma-ray burst durations. These include two skew normal, two sinh arc-

sinh and two exponential power distributions. These distributions have fewer parameters than the three Gaussian distribution and hence are more efficient. Are there astrophysical explanations for the presence of non-Gaussianity in gamma-ray burst durations? This is an important question for future research. The three Gaussian distribution, however, appears to provide the best fit for the *Swift* data. The presence of the third group in this case may be due to

Figure 12. PP-plot of the distributions fitted to the data set *Fermi* 2015.

X-ray flashes, as explained in Veres et al. (2010). It is also possible that the presence of a third group may be just a characteristic of the Gaussian distribution. The third group might not have appeared if the *Swift* data had been fitted adequately by a non-Gaussian distribution.

Future works will include: (i) fitting non-parametric and semi-parameter distributions to the data sets; ii) using gamma-ray burst data (for example, T_{50} durations) having more dimensions; and (iii) performing classification analysis between different groups of gamma-ray burst durations.

Figure 13. PP-plot of the distributions fitted to the data set *Fermi* 2017.

Figure 14. PP-plot of the distributions fitted to the data set *BATSE*.

Figure 15. PP-plot of the distributions fitted to the data set *Swift*.

ACKNOWLEDGEMENTS

The authors would like to thank the referee and the editor for careful reading and comments, which significantly improved this paper.

REFERENCES

- Akaike H., 1974, IEEE Trans. Autom. Control 19, 716
- Azzalini A., 1985, Scand. J. Stat., 12, 171
- Band D. L., 2006, ApJ, 644, 378
- Bhave A., Kulkarni S., Desai S., Srijith P. K., 2017, preprint ([arXiv:1708.05668](https://arxiv.org/abs/1708.05668))
- Bozdogan H., 1987, Psychometrika, 52, 345
- Burnham K. P., Anderson D. R., 2004, Sociological Methods and Research, 33, 261
- Cox D. R., Hinkley D. V., 1974, Theoretical Statistics. Chapman and Hall, London
- Dempster A. P., Laird N. M., Rubin D. B., 1977, J. R. Stat. Soc. B, 39, 1

- Durrett R., 2004, *Probability: Theory and Examples*, 3rd edn. Cambridge Univ. Press, New York
- Fang Y., 2011, *J. Data Sci.*, 9, 15
- Hannan E. J., Quinn B. G., 1979, *J. R. Stat. Soc. B*, 41, 190
- Horvath I., Balazs L. G., Bagoly Z., Veres P., 2008, *A&A*, 489, L1
- Horvath I., Bagoly Z., Balazs L. G., de Ugarte Postigo A., Veres P., Meszaros A., 2010, *ApJ*, 713, 552
- Huja D., Meszaros A., Ripa J., 2009, *A&A*, 504, 67
- Hurvich C. M., Tsai C.-L., 1989, *Biometrika*, 76, 297
- McDonald J. B., Newey W. K., 1988, *Econometric Theory*, 4, 428
- Mukherjee S., Feigelson E. D., Babu U. J., Murtagh F., Fraley C., Raftery A., 1998, *ApJ*, 508, 314
- R Development Core Team 2017, *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna
- Raftery A. E., 1996, *Biometrika*, 83, 251
- Ripa J., Meszaros A., 2016, *Ap&SS*, 361, 1
- Ripa J., Meszaros A., Veres P., Park I. H., 2012, *ApJ*, 756, 44
- Savchenko V., Neronov A., Courvoisier T. J.-L., 2012, *A&A*, 541, A122
- Schwarz G. E., 1978, *Ann. Stat.*, 6, 461
- Subbotin M. T., 1923, *Sbornik*, 31, 296
- Tarnopolski M., 2015, *A&A*, 581, A29
- Tarnopolski M., 2016, *MNRAS*, 458, 2024
- Veres P., Bagoly Z., Horvath I., Meszaros A., Balazs L. G., 2010, *ApJ*, 725, 1955
- Zhang Z.-B., Choi C.-S., 2008, *A&A*, 484, 293
- Zhu D., Zinde-Walsh V., 2009, *J. Econometrics*, 148, 86
- Zitouni H., Guessoum N., Azzam W. J., Mochkovitch R., 2015, *Ap&SS*, 357, 1

This paper has been typeset from a $\text{\LaTeX}$ file prepared by the author.