ROBUST ANALYSIS AND SYNTHESIS FOR UNCERTAIN NEGATIVE-IMAGINARY SYSTEMS

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Abstract

Negative-imaginary systems are broadly speaking stable and square (equal number of inputs and outputs) systems whose Nyquist plot lies underneath (never touches for strictly negative-imaginary systems) the real axis when the frequency varies in the open interval 0 to ∞. This class of systems appear quite often in engineering applications, for example, in lightly damped flexible structures with collocated position sensors and force actuators, multi-link robots, DC machines, active filters, etc. In this thesis, robustness analysis and controller synthesis methods for uncertain negative-imaginary systems are explored.

Two new reformulation techniques are proposed that facilitate both the robustness analysis and controller synthesis for uncertain negative-imaginary systems. These reformulations are based on the transformation from negative-imaginary systems to a bounded-real framework via the positive-real property. In the presence of strictly negative-imaginary uncertainty, the robust stabilization problem is posed in an equivalent $\mathcal{H}_\infty$ control framework; similarly, a negative-imaginary robust performance analysis problem is cast into an equivalent μ-framework. The latter framework also allows robust stability analysis when the perturbations are a mixture of bounded-real and negative-imaginary uncertainties. The proposed two techniques pave the way for existing $\mathcal{H}_\infty$ control and μ theory to be applied to robustness analysis and controller synthesis for negative-imaginary systems.

In addition, a static state-feedback synthesis method is proposed to achieve robust stability of a system in the presence of strictly negative-imaginary uncertainties. The method is developed in the LMI framework, which can be solved efficiently using convex optimization techniques. The controller synthesis method is based on the negative-imaginary stability theorem: A positive feedback interconnection of two negative-imaginary systems is internally stable if and only if the DC loop gain is contractive and at least one of the systems in the interconnected loop is strictly negative-imaginary. Also, in order to handle non-strict negative-imaginary uncertainties, a strongly strictly negative-imaginary lemma is proposed that helps to ensure the strictly negative-imaginary property of the nominal closed-loop system for robustness. To this end, a state-space characterization for strictly negative-imaginary property is given for non-minimal systems where the conditions are convex and hence numerically attractive.

The results in this thesis hence facilitate both the robustness analysis and controller synthesis for negative-imaginary systems that quite often arise in practical scenarios. In addition, they can be applied to quantify the worse-case performance for this class of systems. Therefore, the proposed results have important implications in controller synthesis for uncertain negative-imaginary systems that achieve not only robust stabilization but also robust performance.

Keywords: negative-imaginary systems, lightly damped systems, bounded-real, positive-real, $\mathcal{H}_\infty$ control, robust stability, robust performance, μ analysis, LMI

Declaration

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Publications from this Thesis

Journal Papers

1. **Z. Song**, A. Lanzon, S. Patra, and I. R. Petersen, "Towards controller synthesis for systems with negative imaginary frequency response", *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1506-1511, June 2010.

2. **Z. Song**, A. Lanzon, S. Patra, and I. R. Petersen, "Robust performance analysis for uncertain negative-imaginary systems", *International Journal of Robust and Nonlinear Control*, In press.

3. A. Lanzon, **Z. Song**, S. Patra, and I. R. Petersen, "A strongly strict negative-imaginary lemma for non-minimal linear systems", *Communications in Information and Systems: Special Issue dedicated to the 70th Birthday of Brian Anderson*, In press.

4. **Z. Song**, A. Lanzon, S. Patra, and I. R. Petersen, "Robust stabilization in the presence of strictly negative-imaginary uncertainty via state-feedback", Submitted for Journal publication.

Conference Papers

1. **Z. Song**, A. Lanzon, S. Patra, and I. R. Petersen, "Analysis of robust performance for uncertain negative-imaginary systems using structured singular value", *in Proceedings of the 18th Mediterranean Conference on Control and Automation*, Marrakech, Morocco, June 2010, pp. 1025-1030.

2. A. Lanzon, **Z. Song**, and I. R. Petersen, "Reformulating negative imaginary frequency response systems to bounded-real systems", *in Proceedings of the 47th*

IEEE Conference on Decision and Control, Cancun, Mexico, December 2008, pp. 322–326.

3. I. R. Petersen, A. Lanzon, **Z. Song**, "Stabilization of uncertain negative-imaginary systems via state-feedback control", *in Proceedings of the 2009 European Control Conference*, Budapest, Hungary, August 2009, pp. 1605-1609.

Notation and Acronyms

Field of Numbers

$\mathbb{R}$	real numbers
$\mathbb{R}^n$	real column vectors with n entries
$\mathbb{R}^{m\times n}$	real matrices with n columns and m rows
j	the imaginary unit, i.e. $j = \sqrt{-1}$
$\mathbb{C}$	complex numbers
$\mathbb{C}_-$	open left-half plane
$\bar{\mathbb{C}}_-$	closed left-half plane
$\mathbb{C}_+$	open right-half plane
$\bar{\mathbb{C}}_+$	closed right-half plane
$\mathbb{C}^n$	complex column vectors with n entries
$\mathbb{C}^{m\times n}$	complex matrices with n columns and m rows

Relational Symbols

$\in$	belongs to
$\subset$	subset of
$\cup$	union
$:=$	defined by
$<$	less than
$\leq$	less than or equal to
$>$	greater than

$\geq$	greater than or equal to
$\neq$	not equal to
$\mapsto$	maps to
$\rightarrow$	tends to
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	is equivalent to

Miscellaneous

$\forall$	for all
$:$	such that
$\square$	end of proof
$\mathbf{Re}[s]$	real part of a complex number $s \in \mathbb{C}$
$\mathbf{Im}[s]$	imaginary part of a complex number $s \in \mathbb{C}$
$[M, N]$	denotes the positive-feedback interconnection of M and N
$\|x\|$	magnitude of $x \in \mathbb{C}$
$\angle x$	angle of $x \in \mathbb{C}$
$x \in (0, \infty)$	$0 < x < \infty$
$x \in [0, \infty)$	$0 \leq x < \infty$
$\lim_{x \rightarrow a} f(x)$	limit of $f(x)$ as x tends to a
$\inf(\cdot)$	infimum of a set
$\sup_{x \in \mathcal{X}} f(x)$	supremum of function $f(x)$ over $x \in \mathcal{X}$
$\text{ess} \sup_{x \in \mathcal{X}} f(x)$	supremum of function $f(x)$ over $x \in \mathcal{X}$ omitting isolating points

Matrix Operators

0	zero matrix of compatible dimensions
I	identity matrix of compatible dimensions
A^T	transpose of matrix A

A^*	complex conjugate transpose of matrix A
A^{-1}	inverse of matrix A
$A^\dagger$	pseudo-inverse of matrix A
A^{-T}	denotes $(A^{-1})^T$ or equivalently $(A^T)^{-1}$
A^{-*}	denotes $(A^{-1})^*$ or equivalently $(A^*)^{-1}$
$A > 0$	positive definite matrix, $x^*Ax > 0 \ \forall x \neq 0$
$A \geq 0$	positive semidefinite matrix, $x^*Ax \geq 0 \ \forall x \neq 0$
$A < 0$	negative definite matrix, $x^*Ax < 0 \ \forall x \neq 0$
$A \leq 0$	negative semidefinite matrix, $x^*Ax \leq 0 \ \forall x \neq 0$
$A > B$	denotes $(A - B) > 0$
$A \geq B$	denotes $(A - B) \geq 0$
$\det(A)$	determinant of matrix A
$\text{rank}(A)$	rank of matrix A
$\lambda_i(A)$	i-th eigenvalue of matrix A
$\bar{\lambda}(A)$	largest eigenvalue for a matrix A with only real eigenvalues
$\sigma_i(A)$	i-th singular value of A
$\bar{\sigma}(A)$	largest singular value of A
$\text{diag}(A, B)$	shorthand for block-diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$
$F_\ell(M, Q)$	lower linear fractional transformation of matrices M and Q
$F_u(M, Q)$	upper linear fractional transformation of matrices M and Q
$M \star N$	Redheffer Star-Product of two matrices M and N
$\mu_{\boldsymbol{\Delta}}(M)$	structured singular value of M with respect to a given set $\boldsymbol{\Delta}$

Function Spaces

$\mathcal{L}_\infty$	space of functions that are bounded on $j\mathbb{R}$ including ∞
$\mathcal{H}_\infty$	subspace of functions in $\mathcal{L}_\infty$ that are analytic and bounded in $\mathbb{C}_+$
$\mathcal{R}$	space of real rational functions

$\mathcal{R}^{m\times n}$	space of real rational functions with n inputs and m outputs
$\mathcal{RL}_\infty$	real rational subspace of $\mathcal{L}_\infty$
$\mathcal{RH}_\infty$	real rational subspace of $\mathcal{H}_\infty$

Systems Operations

$G(s)^T$	dual of real rational system $G(s)$
$G(-s)^T$	conjugate system of $G(s)$ also equivalent to $G^T(-s)$
$G(j\omega)^*$	complex conjugate transpose of frequency-response function $G(j\omega)$ at each frequency ω, that is $G(j\omega)^* = G(-j\omega)^T$
G^{-1}	inverse of real rational system G, that is $G^{-1}(s) = G(s)^{-1}$
G^{-T}	denotes transpose of G^{-1}, that is $(G^{-1})^T$ or equivalently $(G^T)^{-1}$
$G(j\omega)^{-*}$	denotes $(G(j\omega)^{-1})^*$ or equivalently $(G(j\omega)^*)^{-1}$
$\|G\|_\infty$	infinity-norm of $G \in \mathcal{RL}_\infty$
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	shorthand for state-space realization $C(sI - A)^{-1}B + D$

Sets of Systems

$\mathcal{I}$	negative-imaginary transfer function matrices
$\mathcal{I}_s$	strictly negative-imaginary transfer function matrices
$\mathcal{P}$	stable positive-real transfer function matrices

Acronyms

SISO	Single-Input Single-Output
MIMO	Multiple-Input Multiple-Output
NI	Negative-Imaginary
PR	Positive-Real
SNI	Strictly Negative-Imaginary
SPR	Strictly Positive-Real

WSNI	Weakly Strict Negative-Imaginary
WSPR	Weakly Strict Positive-Real
SSNI	Strongly Strict Negative-Imaginary
SSPR	Strongly Strict Positive-Real
BR	Bounded-Real
SBR	Strictly Bounded-Real
LFT	Linear Fractional Transformation
LTI	Linear Time-Invariant
LMI	Linear Matrix Inequality

Chapter 1

Introduction

1.1 Background and Motivation

Since the advent of the modern control theory, stability of interconnected systems is a topic of research interest among control systems community ([1–5] and references therein). The developed theories in this area are not only restricted to the nominal behaviors of the systems, but also capture the effects of uncertainty and unwanted exogenous signals in an interconnected loop. These results are specialized to different classes of systems, for example, the small-gain theorem for bounded-real (BR) systems [1, 6], the passivity theorem for positive-real (PR) systems [4, 5, 7–10] etc., and subsequently these results have widely been extended to robust control theory for achieving stability and performance of the uncertain closed-loop systems. In this thesis, the newly introduced class of systems, negative-imaginary (NI) systems is considered and both the analysis and synthesis frameworks are developed for robust control.

The concept of NI systems is similar to that of PR systems, where the phase of the systems is constrained. However, NI systems can have maximum relative degree of two, while PR systems cannot have more than unity. By definition, NI systems are Lyapunov stable systems (with no poles at the origin) with an equal number of inputs and outputs satisfying the frequency domain condition:

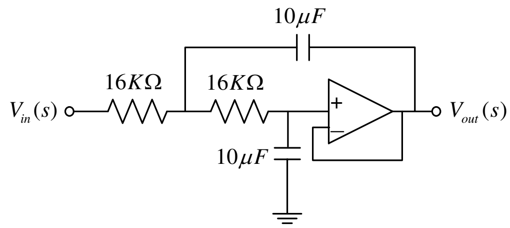

Figure 1.1: A Sallen-Key low-pass filter

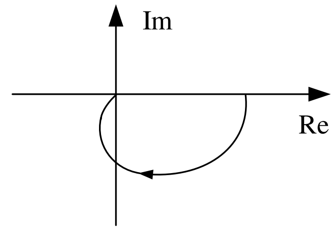

Figure 1.2: Nyquist plot for the transfer function $\frac{V_{out}(s)}{V_{in}(s)}$

$j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ [11–13], where $R(s)$ is the transfer function matrix of the system. Strictly negative-imaginary (SNI) systems are square stable systems that satisfy an SNI condition: $j[R(j\omega) - R^*(j\omega)] > 0$ for all $\omega \in (0, \infty)$ [11, 12]. An example of an NI system is shown in Figure 1.1 and the positive frequency branch of its Nyquist plot is shown in Figure 1.2. This figure illustrates (in a SISO setting) that the imaginary part of the frequency response in the positive frequency branch is negative. All real rational systems are real at zero frequency and at infinite frequency, so all NI systems must touch the real axis at zero and infinite frequency. NI systems can additionally touch the real axis at frequencies other than zero and infinite frequency. For SNI systems, the Nyquist plot cannot touch the real axis except at zero frequency and infinity. Note that the frequency domain condition for (strictly) NI systems is fulfilled on the punctured $j\omega$-axis (zero frequency and infinity are excluded), this fact limits the application of integral quadratic constraint (IQC) theory [14] for analysis and synthesis of NI systems. This is because IQC deals with frequency domain condition that is fulfilled on the entire $j\omega$-axis. The punctured $j\omega$-axis frequency condition introduces difficulties in both analysis and synthesis for NI systems. NI systems are closely connected to nonlinear/time-varying systems with counter-clockwise input-output dynamics [15–17].

NI systems are important in engineering applications, as this class of systems appear in many practical scenarios, for example in a DC machine [18], in electrical active filter circuits [19, 20], in multi-link robots [21], a lightly damped structure [11–13, 21–33] and in large vehicle platoons [34], etc. When force actuators and position sensors (such as piezoelectric sensors) are collocated on a flexible structure, the input/output map is NI. Those structures have been of great interest to the engineering community for some time (see [27–32] and references therein). They are in reality distributed parameter systems which are typically modeled with a sum of very high (or infinite) number of transfer functions $\frac{K_i}{s^2+2\zeta_i\omega_i s+\omega_i^2}$, where ζ_i and ω_i are the damping ratio and natural frequency associated with the i-th mode and K_i is determined by the boundary conditions on the partial differential equation which describes the distributed parameter system [27, 28]. Quite often, for controller synthesis, a truncated plant model is used and the unmodeled dynamics give rise to spill-over dynamics that make it difficult to control [27, 28]. Since the relative degree is more than unity, passivity theory is not applicable to this control problem [12]. In addition, as these dynamics are highly resonant, small-gain based design techniques would be rather conservative. However, these unmodeled spill-over dynamics satisfy the SNI property, as a consequence, NI theory is well suited to address the above mentioned flexible structure control problems.

On the other hand, it is well-known that flexible structures with collocated velocity sensors and force actuators give rise to PR systems (or passive systems in the nonlinear/time-varying case) [35–38]. It is PR theory or passivity theory that underpins the velocity feedback and robust controller synthesis methods for these structures [5, 10]. In this regard, one might suggest to apply PR theory to synthesis for flexible structures with collocated position sensors and force actuators by differentiating the output with respect to time for approximation of velocity feedback. However, it is not always true that NI systems and PR systems are related by a simple rotational transformation. From associated frequency conditions, it is apparent that an NI system can be transformed into a PR system by multiplying the transfer function matrix with $-\frac{1}{s}I$ or sI under some technical assumptions.

However, the former transformation raises instability issues and difficulties in invoking computational results that rely on asymptotically stable systems, and the latter may cause for improperness of the transformed system but more importantly also introduces a blocking zero at zero frequency. This blocking zero via the latter transformation results into an SNI system being always transformed into a non-strict PR system as opposed to a strict PR system as one would expect or hope for. Hence, the passivity theorem [5, 10] cannot capture the stability of the interconnection of NI systems because after the transformation both of the systems are always transformed into two PR systems instead of a PR system and a strictly positive-real (SPR) system. Furthermore, approaches based on SPR synthesis (see [39–41]) cannot be used for control of an NI system irrespective of whether it is strict or non-strict NI due to the aforementioned difficulties.

A robust stability analysis result for interconnected NI systems was proposed in [11–13]. A positive interconnection of NI systems is internally stable if and only if the DC loop gain is contractive provided that at least one of the systems is SNI and some technical assumptions hold [11–13]. Similar to the passivity theorem, the stability result of NI systems does not require the loop gain to be small at every frequency to establish stability, which is in contrast to the small-gain theorem [6]. However, the robust stability analysis result of NI systems is a conditional stability result on the DC loop gain and is hence different from unconditional small-gain and passivity theorems [5, 10]. This stability result unifies the classical graphical method of positive position feedback control [27, 28, 31], resonant control [29, 30] and integral resonant control [32] used in lightly damped structures with collocated position sensors and force actuators in a systematic and rigorous framework [22]. These control schemes rely on NI controllers to robust stabilize uncertain SNI systems. Recently, the NI theory has been used to extend the classical SISO integral controller synthesis method to the MIMO case in [33, 42]. These results have been successfully applied to a cantilever beam and high precision instrumentation, such as scanning tunneling microscopes and atomic force microscopes for damping vibrations [33, 42]. As a consequence, this stability analysis result indicates a direction for controller design of NI systems.

This thesis aims to provide systematic methods that facilitate engineers to design controller for the newly introduced NI systems with guaranteed robust stability and performance. In this thesis, controller synthesis methods and robust performance analysis for uncertain NI systems are explored. Here, an uncertain NI system is represented as a linear fractional transformation (LFT) of a linear time-invariant system and SNI uncertainties. SNI uncertainties arise for example, in the above mentioned unmodeled dynamics of lightly damped structures. Also, there are some uncertain systems that can equivalently be presented into systems with the uncertain part being SNI (such an example is given in Chapter 5). It is well-known that $\mathcal{H}_\infty$ theory and μ theory provide systematic robust analysis and synthesis framework to guarantee robust stability and performance of systems. The first part of this thesis is hence motivated by transforming the robust stabilization and performance problems of uncertain NI systems into the BR framework, so that the fruitful results from $\mathcal{H}_\infty$ theory and μ theory could enable robustness analysis and controller synthesis for NI systems. In the second part of this thesis, a novel robust state-feedback controller synthesis method is provided, primarily for systems with SNI uncertainty. In practical scenario, the uncertainties may not always satisfy the SNI property, however they satisfy (non-strict) NI property, say for example the uncertain mechanical plant in [12]. In this regard, the robust control problem for systems where non-strict NI uncertainties present is addressed in the second part of this thesis as well.

1.2 Organization of this Dissertation

This dissertation consists of six chapters. From Chapter 2 to Chapter 4, we focus on asymptotically stable NI systems for simplicity. In Chapter 5, however, the

NI systems considered are generalized to allow imaginary axis poles, which we refer to as "generalized NI systems" to discriminate from asymptotically stable NI systems. Main results of this dissertation are presented in Chapter 3 to Chapter 5 which proceed with an increasing order of complexity. The remainder of this dissertation is organized as follows:

Chapter 2: Preliminaries

In this chapter, background material is presented which lay the foundation of this dissertation. At first, useful matrix techniques are introduced and system spaces used in later chapters are defined. Then, robustness analysis results, namely, small-gain theorem, passivity theorem, NI stability theorem and μ theory are briefly reviewed. Finally, some basic and useful results on LMIs are collected.

Chapter 3: Reformulation from NI to BR Framework

This chapter reformulates closed-loop systems with NI property into closed-loop systems with bounded gain, so that theory and results from $\mathcal{H}_\infty$ control can be borrowed to enable controller synthesis for NI systems. This reformulation from NI systems into a BR framework is obtained via the PR property. This chapter also addresses a controller synthesis problem in an $\mathcal{H}_\infty$ optimal control framework for a generalized plant with an invariant zero at the origin in its (1,2) element which is due to the reformulation of the closed-loop system from an NI system to a BR framework. Here, the transformations between NI systems and PR systems are discussed in detail and the technical difficulties emerging due to the transformation are highlighted.

Chapter 4: Robust Performance Analysis for Uncertain Negative-imaginary Systems

Here, the reformulation technique in Chapter 3 is extended to assess robust performance of NI systems. This problem involves performance measurement via an $\mathcal{H}_\infty$ norm and physically motivated uncertainty that satisfies an SNI property.

It is thus cast into a structured singular value condition which gives a quantitative performance test for systems with SNI uncertainty. Unlike early works which involve uncertainties of the same type (size) that typically appear in robust performance problems, the framework for robust performance analysis in this chapter considers a mixture of SNI and SBR uncertainties. This proposed framework can be also applied to analyze robust stability when the perturbations are mixture of SBR and SNI uncertainties.

Chapter 5: Stabilization of Uncertain Negative-Imaginary Systems via State-Feedback

This chapter presents a systematic robust state-feedback synthesis for systems with SNI uncertainty assuming all states are available for feedback. This result is built on the concept of generalized NI systems and its robust stability analysis result. A relaxed version of the NI lemma without minimality assumption has been proposed which underpins the state-feedback synthesis method in this chapter. LMI conditions are developed to construct a state-feedback internally stabilizing controller such that the nominal closed-loop system satisfies the generalized NI property and a DC gain condition. As a result of this, the closed-loop system can then be guaranteed to be robustly stable against SNI uncertainties.

In order to handle non-strict NI uncertainties, it is desirable to enforce the nominal closed-loop system to be SNI, since the robust stability result of interconnected NI systems requires at least one of the system to be SNI. In this regard, an SNI lemma which gives a simple state-space characterization to ensure an SNI property, is also proposed in this chapter. This lemma facilitates robust analysis and synthesis methods to handle both the non-strict NI and NI uncertainty of the system. Numerical advantages are achieved by avoiding a non-convex rank constraint, a non-strict inequality condition and a minimality assumption present in previous literature.

Chapter 6: Conclusions

This chapter concludes the dissertation with a summary of the main contributions and an outline of several suggestions for future research in this area.

Several proofs can be found in the appendices.

Chapter 2

Preliminaries

This chapter briefly reviews basic mathematical tools and important background materials that are useful for the subsequent chapters.

In Section 2.1, some basic matrix techniques are introduced. This is followed by definitions of function spaces in Section 2.2 and state-space realizations of systems in Section 2.3. Then, several results on Lyapunov functions are collected in Section 2.4. In Section 2.5, robustness analysis results on feedback systems extensively used in the subsequent chapters are described. These results include stability analysis results on BR systems, passive systems and NI systems, and robust performance analysis machinery using the structured singular value. Finally, some basic results on LMIs are presented in Section 2.6.

2.1 Matrix Tools

In this section some basic matrix tools are briefly reviewed. More details can be found in [43–45].

2.1.1 Singular Value Decomposition

From systems point of view, the singular values of a matrix are good measures of input amplification in some specified directions and the corresponding singular vectors are good indicators of the strong/weak input or output directions. Hence, the concept of singular values is very useful in matrix analysis and applications.

By definition, for a matrix $A \in \mathbb{C}^{m \times n}$, the i-th singular value σ_i is a scalar that satisfies

$$Av_i = \sigma_i u_i,$$

where, $v_i \in \mathbb{C}^n$ with $v_i^T v_i = 1$, $u_i \in \mathbb{C}^m$ with $u_i^T u_i = 1$, and correspondingly, u_i and v_i are called the i-th left and the i-th right singular vector respectively. This definition implies that each right singular vector (indicating input direction) is mapped onto the corresponding left singular vector (indicating output direction), and the "magnification factor" is the corresponding singular value. Also, it is easy to see that σ_i^2 is an eigenvalue of AA^* or A^*A. Hence, it is often convenient to use the following alternative definition for the largest singular value $\bar{\sigma}$ as:

$$\bar{\sigma} := \sqrt{\bar{\lambda}(A^*A)}.$$

As will be seen in the sequel, the definitions of the infinity norm and structured singular value of a transfer function matrix are based on the concept of the largest singular value of a matrix.

2.1.2 Linear Fractional Transformation (LFT)

A linear fractional transformation (LFT) is a matrix function which provides a framework to standardize block diagrams for robust control analysis and design. The motivation for the terminologies of lower and upper LFTs can be seen clearly from the diagram shown in Figure 2.1.

Definition 2.1 *[44] For $M \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)}$ partitioned as* $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ *and complex matrices Δ_l and Δ_u of appropriate size, define a lower LFT with*

Figure 2.1: LFT diagram representations of lower and upper LFT

respect to Δ_l *as:*

$$\mathcal{F}_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21},$$

and an upper LFT with respect to Δ_u *as:*

$$\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12},$$

provided the inverse matrices exist.

The LFTs defined above are simply the closed-loop mapping from w_1 to z_1 and from w_2 to z_2, respectively, as shown in Figure 2.1.

LFTs are useful for representing an uncertain system in order to study perturbations. For example, the LFTs in Figure 2.1 can be used to represent an uncertain system, where M is the certain part of the system and Δ_u or Δ_l collects all the uncertainties that can appear at different points of the system block diagram. Such an LFT uncertain system separates the nominal model from the system uncertainty and hence facilitates analysis and synthesis.

An important property of LFTs is that any interconnection of LFTs is again an LFT [46]. This property is often used and is the heart of LFT machinery. The Redheffer Star-Product [44,46] is defined in the following to characterize the interconnection of LFTs in the uniform framework of Figure 2.2.

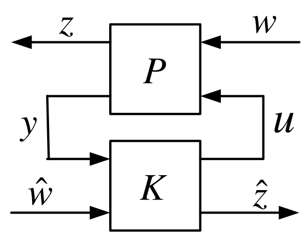

Figure 2.2: Interconnection of LFTs

Suppose that

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \quad \begin{bmatrix} u \\ \hat{z} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} y \\ \hat{w} \end{bmatrix},$$

and $(I - P_{22}K_{11})^{-1}$ exists. Then the Redheffer Star-Product of P and K, denoted by $P \star K$, is defined as

$$P \star K : \begin{bmatrix} w \\ \hat{w} \end{bmatrix} \mapsto \begin{bmatrix} z \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_l(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1}P_{21} & \mathcal{F}_u(K, P_{22}) \end{bmatrix}.$$

Suppose that the state-space realizations of P and K are given by

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad K = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{array}\right].$$

Then a state-space realization of $P \star K$ is given by

$$P \star K = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right],$$

where

$$
\begin{aligned}
\bar{A} &= \begin{bmatrix} A + B_2\tilde{R}^{-1}D_{K11}C_2 & B_2\tilde{R}^{-1}C_{K1} \\ B_{K1}R^{-1}C_2 & A_K + B_{K1}R^{-1}D_{22}C_{K1} \end{bmatrix}, \\
\bar{B} &= \begin{bmatrix} B_1 + B_2\tilde{R}^{-1}D_{K11}D_{21} & B_2\tilde{R}^{-1}D_{K12} \\ B_{K1}R^{-1}D_{21} & B_{K2} + B_{K1}R^{-1}D_{22}D_{K12} \end{bmatrix}, \\
\bar{C} &= \begin{bmatrix} C_1 + D_{12}D_{K11}R^{-1}C_2 & D_{12}\tilde{R}^{-1}C_{K1} \\ D_{K21}R^{-1}C_2 & C_{K2} + D_{K21}R^{-1}D_{22}C_{K1} \end{bmatrix}, \\
\bar{D} &= \begin{bmatrix} D_{11} + D_{12}D_{K11}R^{-1}D_{21} & D_{12}\tilde{R}^{-1}D_{K12} \\ D_{K21}R^{-1}D_{21} & D_{K22} + D_{K21}R^{-1}D_{22}D_{K12} \end{bmatrix}, \\
R &= I - D_{22}D_{K11}, \text{ and } \tilde{R} = I - D_{K11}D_{22}.
\end{aligned}
$$

Now suppose the loop between $\hat{w}$ and $\hat{z}$ is closed by $\hat{w} = Q\hat{z}$, then it is trivial that

$$\mathcal{F}_l(P, \mathcal{F}_l(K, Q)) = \mathcal{F}_l(P \star K, Q).$$

Thus the cascade of two LFTs can be easily characterized by another LFT involving the Redheffer Star-Product.

2.2 Function Spaces

This section defines the function spaces discussed in the following chapters. See [44, 46] for more details.

Definition 2.2 $\mathcal{R}$ *is the space of proper, real rational matrix-valued (or scalar-valued) functions.*

Definition 2.3 $\mathcal{L}_\infty$ *is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on $j\mathbb{R}$, with the norm defined by:*

$$\|G\|_\infty := ess \sup_{\omega \in \mathbb{R}} \bar{\sigma}\left[G(j\omega)\right].$$

Definition 2.4 *$\mathcal{RL}_\infty$ is a subspace of $\mathcal{L}_\infty$ which consists of all proper and real rational transfer function matrices with no poles on the imaginary axis.*

Definition 2.5 *$\mathcal{H}_\infty$ is a (closed) subspace of $\mathcal{L}_\infty$ that consists of all functions that are analytic and bounded in $\mathbb{C}_+$, with the norm defined by:*

$$\|G\|_\infty := \sup_{s\in\mathbb{C}_+} \bar{\sigma}[G(s)] = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[G(j\omega)].$$

Definition 2.6 *$\mathcal{RH}_\infty$ is a subspace of $\mathcal{H}_\infty$ which consists of all proper and real rational stable transfer function matrices.*

2.3 State-Space Systems

This section reviews some very important concepts in linear systems theory and design. Further details can be found in [47].

Suppose G(s) is a real rational system with the state-space realization:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.1}$$

Definition 2.7 *A is said to be stable or Hurwitz if all the eigenvalues of A are in $\mathbb{C}_-$.*

Definition 2.8 *(A, B) is said to be controllable if the matrix $[A-\lambda I \;\; B]$ has full row rank for all λ in $\mathbb{C}$, and (C, A) is said to be observable if the pair (A^T, C^T) is controllable.*

Definition 2.9 *(A, B) is said to be stabilizable if the matrix $[A-\lambda I \;\; B]$ has full row rank for all λ in $\bar{\mathbb{C}}_+$, and (C, A) is said to be detectable if the pair (A^T, C^T) is stabilizable.*

Definition 2.10 *A state-space realization (A, B, C, D) of $G(s)$ is said to be minimal if (A, B) is controllable and (C, A) is observable.*

Definition 2.11 *A complex number $z_0 \in \mathbb{C}$ is called an invariant zero of the system realization if it satisfies*

$$\text{rank}\begin{pmatrix} A - z_0 I & B \\ C & D \end{pmatrix} < \textit{normalrank}\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix},$$

where normalrank $(\cdot)$ denotes the normal rank, i.e the maximally possible rank of a polynomial matrix for at least one $s \in \mathbb{C}$.

Definition 2.12 *A complex number $z_0 \in \mathbb{C}$ is called a blocking zero of $G(s)$ if $G(z_0) = 0$.*

A linear system can be described by different state-space realizations. Two representations, $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ are said to be similar if there exists a nonsingular transformation matrix T such that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T\tilde{A}T^{-1} & T\tilde{B} \\ \hline \tilde{C}T^{-1} & \tilde{D} \end{array}\right].$$

The similarity transformation can be interpreted as a coordinate transformation, i.e. a mapping from one state base to another state base. The input/output dynamics of two similar systems are identical. Controllability (or stabilizability) and observability (or detectability) are invariant under similarity transformations. The following Kalman Canonical Decomposition is used to separate controllable, uncontrollable, observable and unobservable modes of the system realization and is hence useful in system analysis and synthesis.

Theorem 2.1 *[44] Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state-space realization for a real rational transfer function $G(s)$. Then there exists a nonsingular transformation T such* *that*

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cccc|c} \tilde{A}_{co} & 0 & \tilde{A}_{13} & 0 & \tilde{B}_{co} \\ \tilde{A}_{21} & \tilde{A}_{c\bar{o}} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{B}_{c\bar{o}} \\ 0 & 0 & \tilde{A}_{\bar{c}o} & 0 & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{\bar{c}\bar{o}} & 0 \\ \hline \tilde{C}_{co} & 0 & \tilde{C}_{\bar{c}o} & 0 & D \end{array}\right],$$

where the eigenvalues of $\tilde{A}_{co}$ *are controllable and observable modes, the eigenvalues of* $\tilde{A}_{c\bar{o}}$ *are controllable but unobservable modes, the eigenvalues of* $\tilde{A}_{\bar{c}o}$ *are observable but uncontrollable modes, and the eigenvalues of* $\tilde{A}_{\bar{c}\bar{o}}$ *are uncontrollable and unobservable modes of the sate-space realization.*

Some operations on systems are defined next.

Definition 2.13 *The transpose (or dual) of* G *is defined by*

$$G^T(s) = G(s)^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right].$$

Definition 2.14 *The conjugate system of* $G(s)$ *is defined as*

$$G^T(-s) = G(-s)^T = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right].$$

The representation of the cascade of two subsystems can be obtained in terms of the state-space realizations of these two subsystems as shown below.

Definition 2.15 *Suppose* $G_1(s)$ *and* $G_2(s)$ *are two subsystems with the realization*

$$G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right].$$

Then, the cascade (product) of $G_1(s)$ and $G_2(s)$ is given by

$$G_1G_2 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right].$$

2.4 Results on Lyapunov Equations

A continuous Lyapunov equation is an equation of this form:

$$A^TX + XA + Q = 0, \tag{2.2}$$

where A and Q are given square matrices, X is a square matrix variable and Q is symmetric.

In systems and control theory, Lyapunov equation plays a very important role. The following standard results, which will be used in the subsequent chapters, state the relationship between the stability of A and the solution of this equation.

Lemma 2.1 *[44] Suppose A is stable, then*

(i) $X > 0$ if $Q > 0$ and $X \geq 0$ if $Q \geq 0$.

(ii) if $Q \geq 0$, then (Q, A) is observable if and only if $X > 0$.

Lemma 2.2 *[44] Suppose X is the solution of the Lyapunov equation (2.2), then*

(i) $\mathrm{Re}\lambda_i(A) \leq 0$ if $X > 0$ and $Q \geq 0$.

(ii) A is stable if $X > 0$ and $Q > 0$.

(iii) A is stable if $X \geq 0$, $Q \geq 0$ and (Q, A) is detectable.

2.5 Robustness Analysis of Feedback Systems

Control systems are designed to make the output of a physical plant behave in a desired manner. Designing such a control system to deliver the required performance is usually done by employing certain nominal (usually linear) model of the plant to be controlled. This nominal model is chosen to capture the major dynamical features of the physical plant. However, it can never be an exact representation of the true plant. This discrepancy between the physical plant and its nominal model for controller design give rise to uncertainties. Uncertainties also arise for example from external disturbances and measurement noise in the physical system. These types of uncertainties are formidable adversaries and require tools to analyze a control system for robustness with respect to uncertainties. These analysis tools almost always lead to techniques to actually designing a control system with good robustness.

This section reviews several internal stability criteria. These criteria provide robust stability analysis tools for feedback systems in the face of several different classes of uncertainties of restricted information. Specifically, they are the small-gain theorem and μ analysis for bounded (in some sense) uncertainties, the passivity theorem for uncertain passive systems and the internal stability theorem for uncertain NI systems.

2.5.1 Small-Gain Theorem

The well-known small-gain theorem developed in [6] provides a stability criterion for feedback interconnections of systems with contractive gain. It conceptually generalizes the fact that the connection of two stable linear systems will be stable if the loop gain is less than unity. Prior to presenting this theorem, the following definitions of well-posedness and stability need to be given first.

Definition 2.16 *A feedback interconnection of real rational proper transfer function matrices is said to be "well-posed" if all closed-loop transfer function matrices exist, and are proper.*

Definition 2.17 *A feedback interconnection of real rational proper transfer function matrices is said to be "internally stable" if the feedback interconnection is well-posed and all closed-loop transfer function matrices belong to* $\mathcal{RH}_\infty$.

Subsequently, the small-gain theorem is stated as follows:

Theorem 2.2 *[6, 44] Consider the interconnected system shown in Figure 2.3. Suppose $M \in \mathcal{RH}_\infty$ and let $\gamma > 0$. Then this feedback interconnection is well-posed and internally stable for all $\Delta(s) \in \mathcal{RH}_\infty$ with*

(a) $\|\Delta\|_\infty \leq \gamma$ if and only if $\|M\|_\infty < 1/\gamma$;

(b) $\|\Delta\|_\infty < \gamma$ if and only if $\|M\|_\infty \leq 1/\gamma$.

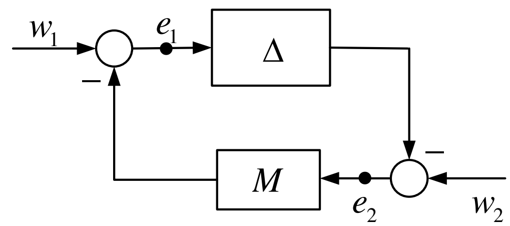

Figure 2.3: Standard feedback configuration

Given an appropriately defined stability property of Δ [1], stability in the above theorem can also be established for a nonlinear or time-varying Δ.

The small-gain theorem implies that the smaller the $\mathcal{H}_\infty$-norm of the transfer function matrix M is, the larger the $\mathcal{H}_\infty$-norm of the smallest stable perturbation Δ that destabilizes the interconnected system of M and Δ. This motivates the machinery of $\mathcal{H}_\infty$ control to design robust controllers. A typical $\mathcal{H}_\infty$ control problem reduces to the minimization of the $\mathcal{H}_\infty$-norm of a nominal closed-loop transfer function matrix (see Figure 2.4) over all stabilizing controllers [44], i.e.

$$\gamma_{opt} = \inf_{\text{stabilizing controllers } K(s)} \|\mathcal{F}_\ell(G, K)\|_\infty. \tag{2.3}$$

In Figure 2.4, w represents exogenous inputs (also the outputs of the uncertainty block); z represents controlled outputs (also the inputs of the uncertainty block); u represents controlled inputs and y represents measured outputs; G is a generalized plant which depends on the structure of how the uncertainties in the feedback control system block-diagram associate with the real nominal system,

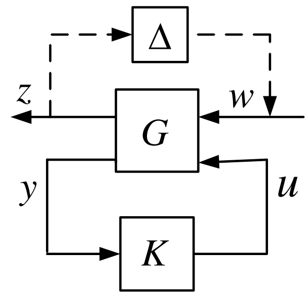

Figure 2.4: An LFT interconnection for robust control

and the performance specifications.

The problem posed in (2.3) is called an $\mathcal{H}_\infty$ optimal control problem. Knowing γ_{opt} is theoretically useful as it sets a limit on the achievable optimal $\mathcal{H}_\infty$-norm. However, finding an optimal $\mathcal{H}_\infty$ controller is often both numerically and theoretically complicated [44]. In practice, a suboptimal controller is usually designed that satisfies

$$\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma, \text{ where } \gamma > \gamma_{opt}, \tag{2.4}$$

and this problem is called the $\mathcal{H}_\infty$ suboptimal control problem. Then the optimization problem in (2.3) is usually solved by a bisection algorithm; i.e., optimizing γ in a sense that it approximates the optimal value γ_{opt}. Comprehensive solutions exist for this $\mathcal{H}_\infty$ suboptimal control problem, see for example the operator based solution in [48], the J-spectral factorization approach in [49], the Riccati equation based DGKF methods in [50,51] and LMI approaches in [52,53]. The DGKF methods typically require assumptions of 'no invariant zeros of (1,2) or (2,1) element of G on the $j\omega$-axis'. However, these assumptions can be lifted via methods in [52–54].

2.5.2 μ Analysis

When multiple sources of uncertainties appear in a system, robust stability and robust performance results based on direct use of the small-gain theorem may become very conservative. In this case, the terminology of "structured uncertainty" is defined to collect the independent uncertainty blocks that reflect the sources of the uncertainties; then the following definition of the structured singular value (denoted by μ) is thus introduced to address this issue (see [55] for a good review of the relevant literature). This concept provides a generalization of the $\mathcal{H}_\infty$-norm to permit small-gain type analysis of systems involving block-structured uncertainty.

Definition 2.18 *[44] (Structured Singular Value $\mu_{\mathbf{\Delta}}[M(j\omega)]$)*
Given $M \in \mathcal{RH}_\infty^{n\times m}$ and $\omega \in \mathbb{R} \cup \{\infty\}$, and let $\mathbf{\Delta} \in \mathbb{C}^{m\times n}$ be the uncertainty set which determines the uncertainty structure. Then,

$$\mu_{\mathbf{\Delta}}[M(j\omega)] := \frac{1}{min\{\overline{\sigma}[\Delta(j\omega)] : \Delta \text{ is structured}, \det(I - M(j\omega)\Delta(j\omega)) = 0\}},$$

unless no $\Delta(j\omega) \in \mathbf{\Delta}$ makes $I - M(j\omega)\Delta(j\omega)$ singular, in which case
$\mu_{\mathbf{\Delta}}(M(j\omega)) := 0$.

Hence, at every frequency ω, the structured singular value of $M(j\omega)$ is a measure of the inverse of the maximum allowable size of a structured uncertainty $\Delta(j\omega)$ such that $I - M(j\omega)\Delta(j\omega)$ is nonsingular for any $\Delta(j\omega)$ in the uncertainty set $\mathbf{\Delta}$. Then, similar arguments as in the stability analysis via the small-gain theorem can be applied to the structured uncertainty case using μ, as stated follows:

Theorem 2.3 *[44] (μ robust stability analysis) Consider the feedback interconnection depicted in Figure 2.3. Suppose $M \in \mathcal{RH}_\infty$ and let $\gamma > 0$. Then this feedback interconnection is internally stable for all structured $\Delta \in \mathcal{RH}_\infty$ with*

$\|\Delta\|_\infty < 1/\gamma$ *if and only if* $\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta}}[M(j\omega)] \leq \gamma$, *where the set* $\mathbf{\Delta}$ *determines the structure of* Δ.

Hence, the peak value of the μ-plot of $M(j\omega)$ determines the size of the perturbations that the loop is robustly stable against.

Robust stability is not the only property of a closed-loop system that is required to maintain in the presence of uncertainties. It is desirable that the system can also deliver the required performance when the designed controller based on a nominal model is implemented in the true physical system. Hence, from a practical point of view, the design paradigm is not only motivated along the direction of robust stability against uncertainties, but it is also very important to emphasise the robust performance perspective. The following theorem gives the required robust performance analysis test which indicates the worst-case performance associated with structured bounded uncertainties.

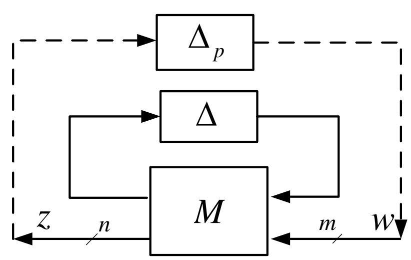

Figure 2.5: An LFT interconnection for μ robust performance analysis

Theorem 2.4 *[44] (μ robust performance analysis) Consider the feedback interconnection depicted in Figure 2.5. Suppose* $M \in \mathcal{RH}_\infty$ *and let* $\gamma > 0$. *Then this feedback interconnection is internally stable and satisfies* $\|F_u(M,\Delta)\|_\infty \leq \gamma$ *for all structured* $\Delta \in \mathcal{RH}_\infty$ *with* $\|\Delta\|_\infty < 1/\gamma$ *if and only if*

$$\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta}_{TOP}}[M(j\omega)] \leq \gamma,$$

where $\mathbf{\Delta}_{TOP} := \mathrm{diag}(\Delta, \Delta_P) : \Delta \in \mathbf{\Delta}, \Delta_P \in \mathbb{C}^{m\times n}$ *and the set* $\mathbf{\Delta}$ *determines the structure of* Δ.

This important theorem states that the robust performance problem posed in Figure 2.5 can be transformed equivalently into a robust stability problem with an augmented uncertainty by introducing an artificial full-block uncertainty operator Δ_P between the exogenous inputs w and the controlled outputs z.

2.5.3 Passivity Theorem

Passivity is an input-output property of a family of nonlinear physical systems which can only consume energy. The notion of passivity originated from electrical circuits and mechanical systems, and passive systems have received great attention in the last few decades (see [1, 2, 5, 9, 10] and references therein). When a passive system is connected with a strictly passive system in a negative feedback loop, energy is strictly dissipated as signals propagate around the loop, and hence the feedback interconnection is stable. This fact, also known as the passivity theorem, is considered as a natural companion of the small-gain theorem.

In the linear time-invariant setting, passive systems give rise to the well-known PR transfer function matrices. Here results of passivity theory for the LTI case are collected for an overview of similarities and differences between the notion of negative-imaginariness and positive-realness. Similar to PR systems, NI systems are phased constrained; however NI can have a maximum relative degree of two, while PR systems can not have a relative degree of more than unity. The results presented are useful for an understanding of the importance of passivity theorem and its link to NI systems. For more details of passivity theory, see [1, 5, 10].

Definition 2.19 *[5, 56] A proper rational transfer function matrix is said to be positive-real if*

1) $G(s)$ has no pole in $\mathbb{C}_+$,

2) for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega) \geq 0$,

3) any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the *residue matrix* $\lim_{s \to j\omega}(s - j\omega)G(s)$ *is positive definite Hermitian.*

The transfer function $G(s)$ *is said to be strictly positive-real if* $G(s-\varepsilon)$ *is positive-real for some* $\varepsilon > 0$.

The definition of PR systems implies that in the SISO case, PR systems have phase constraint between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This phase constrained property of PR systems and passive systems is in contrast to gain constrained property of BR systems.

The state-space characterizations of PR and SPR transfer function matrices are given in the following lemma.

Lemma 2.3 *[5, 56] Let* $G(s) = C(sI - A)^{-1}B + D$ *be a* $m \times m$ *transfer function matrix where* (A, B) *is controllable and* (C, A) *is observable. Then,* $G(s)$ *is strictly positive-real (positive-real) if and only if there exists matrices* $X = X^T > 0$, L, W *and a positive constant* ϵ ($\epsilon = 0$) *such that*

$$XA + A^*X = -L^*L - \epsilon X, \tag{2.5}$$

$$XB = C^* - L^*W, \tag{2.6}$$

$$W^*W = D + D^*. \tag{2.7}$$

Now we are ready to state the passivity theorem.

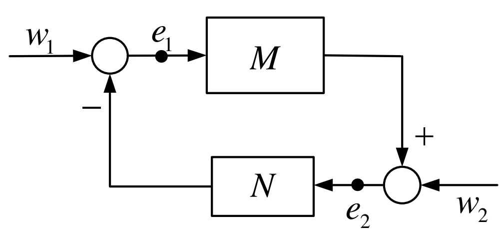

Figure 2.6: Feedback interconnection of passive systems

Theorem 2.5 *[5] Consider the feedback interconnection of Figure 2.6. Suppose* M *is positive-real and* N *is strictly positive-real. Then the feedback system is internally stable.*

Unlike the small-gain theorem which states that the connection of two stable linear systems will be stable if the loop gain is less than unity, the passivity theorem conceptually generalizes the fact that the connection of two stable linear systems will be stable if the loop phase is less than 180 degrees.

The passivity theorem can be used for analysis and synthesis of robust control systems in a uniform framework. For instance, an SPR controller can be designed to achieve stability for an uncertain passive system. This stability property is robust given that the passivity property of the system is retained. In an LFT framework, synthesizing a controller which renders a closed-loop transfer function SPR to guarantee robust stability of systems with PR uncertainties has been well-developed, see for example a frequency domain and operator theoretic approach in [57] and LMI approach in [39–41].

2.5.4 Stability Analysis for Negative-Imaginary Systems

As discussed in the last chapter, NI systems have desirable frequency domain interpretations. For instance, in the SISO case, the Nyquist plot of NI systems lies entirely on and below the real axis for all positive frequencies, for SNI systems, the Nyquist plot should not touch the real axis except at zero frequency and infinity. In the earlier chapters of this thesis, we will only consider asymptotically stable NI systems as defined below, the definition of NI systems will be relaxed to also allow imaginary axis poles (except at the origin) in Chapter 5.

Definition 2.20 *[11, 12] Let the set of negative-imaginary transfer function matrices be defined as*

$$\mathcal{I} := \{R(s) \in \mathcal{RH}_\infty^{n\times n} : j[R(j\omega) - R(j\omega)^*] \geq 0 \ \forall \omega \in (0, \infty)\}.$$

and the set of strictly negative-imaginary transfer function matrices be defined as

$$\mathcal{I}_s := \{R(s) \in \mathcal{RH}_\infty^{n\times n} : j[R(j\omega) - R(j\omega)^*] > 0 \ \forall \omega \in (0, \infty)\} \subset \mathcal{I}.$$

NI systems are stable systems with aforementioned frequency response in the open frequency interval between 0 and ∞, where only the imaginary part of the frequency response is considered. Definition 2.20 implies that in the SISO case, the system has a phase lag between 0 and $-\pi$ for all positive frequencies.

The phase constraint property of NI systems is similar to that of PR systems. However, NI systems can have a maximum relative degree of two, whereas positive-real systems cannot have more than unity. Most importantly, the frequency dependent condition for negative-imaginary systems is fulfilled on the punctured $j\omega$-axis; i.e., it excludes zero frequency whereas the positive-real condition is satisfied for all $\omega \in \mathbb{R}$.

The following lemma gives an algebraic criterion for a transfer function matrix to be NI in terms of the state-space realizations of the matrix. The lemma has a similar form to the well-known Positive Real Lemma [56] and hence, is referred to as the Negative Imaginary Lemma.

Lemma 2.4 *[12] (Negative-Imaginary Lemma) Let $R(s) = C(sI - A)^{-1}B + D$ be a $p \times p$ transfer function matrix where (A, B) is controllable and (A, C) is observable. Then $R(s) \in \mathcal{I}$ if and only if A has no poles on $j\omega$-axis, $D = D^*$ and there exists a real matrix $Y > 0$ such that*

$$AY + YA^* \leq 0 \quad \textit{and} \quad B = -AYC^*. \tag{2.8}$$

The state-space characterization for NI systems is useful in proving theoretical results for NI systems. The next lemma relates the gain at zero frequency and at infinity for NI systems. It is presented here for easy reference.

Lemma 2.5 *[12] Given $R(s) \in \mathcal{I}$ (respectively $\mathcal{I}_s$), then $R(0) - R(\infty) \geq 0$ (respectively > 0).*

Theorem 2.6, which was also given in [12], is the main analysis result that establishes the internal stability of positive feedback interconnections of NI systems.

The theorem is as follows:

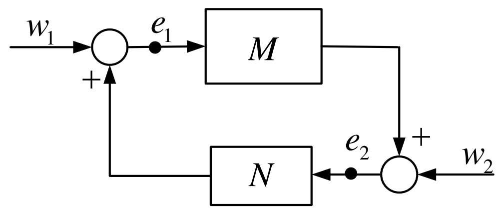

Figure 2.7: A positive feedback interconnection

Theorem 2.6 *[11, 12] Given $M(s) \in \mathcal{I}$ and $N(s) \in \mathcal{I}_s$ that also satisfy $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Let $[M, N]$ denote the positive-feedback interconnection of M and N illustrated in Figure 2.7. Then $[M, N]$ is internally stable if and only if*

$$\bar{\lambda}(M(0)N(0)) < 1. \tag{2.9}$$

Thus, in order to guarantee stability of a positive feedback interconnection of an NI system and an SNI system, the gain of the loop at zero frequency has to be strictly less than unity. Note that the eigenvalues of $M(0)N(0)$ are real as $M(0)$ is Hermitian and $N(0) > N(\infty) \geq 0$ via Lemma 2.5.

Similar to the passivity theorem, the stability result of NI systems does not require the loop gain to be small at every frequency to establish stability, which is in contrast to the small-gain theorem. However, the robust stability analysis result for NI systems is a conditional stability result on the DC loop gain and is hence different from the unconditional passivity theorem [5, 10]. This stability analysis result underpins the robustness analysis and controller synthesis for uncertain NI systems given in this thesis.

2.6 Linear Matrix Inequalities (LMIs)

Many problems arising in systems and control theory can be formulated as convex or quasiconvex optimization problems involving linear matrix inequalities (LMI) [45, 58, 59]. These LMI problems are numerically tractable and can be solved very efficiently using interior-point methods [60, 61]. In many cases, LMI-based design is less restrictive than conventional methods. For example, as mentioned in Section 2.5.1, LMI approach [52,53] to $\mathcal{H}_\infty$ suboptimal control problem doesn't require the assumptions of 'no invariant zeros on the imaginary axis' to be fulfilled, while the conventional DGKF methods [44,50] need such assumptions. Thus LMI is a valuable alternative when conventional methods fail [52].

Given these advantages of LMI, we use the LMI approach to address the robust stabilization problem of uncertain NI systems in Chapter 5. Henceforth, some basic and useful results on LMIs used in the following chapters are collected below.

A linear matrix inequality (LMI) is any constraint of the form

$$F(x) \ := F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{2.10}$$

where $x \ := [x_1 \ x_2 \cdots \ x_m] \in \mathbb{C}^m$ is the variable and the symmetric matrices $F_i = F_i^T \in \mathbb{C}^{n \times n}$ are given. The LMI in (2.10) is a convex constraint on x, so the solution set is also a convex set. A system of LMIs $F^1(x) > 0, \ldots, F^p(x) > 0$ can be expressed as a single LMI in the form $\operatorname{diag}[F^1(x), \ldots, F^p(x)] > 0$. Thus, we do not distinguish a single LMI from multiple LMIs. Note that there are many cases where the variable is a matrix, e.g, in the Lyapunov inequality $A^T X + XA < 0$, where A is given, and $X = X^T$ is the variable. Then instead of writing this equality as in the form of (2.10), we will declare the matrix X as the variable.

We also encounter non-strict LMIs, where the inequality in (2.10) is replaced by a non-strict one. For non-strict LMIs, YALMIP [62] and SeDuMi [63] can be used to obtain a solution efficiently.

Many matrix inequalities arising in control problems are nonlinear in the matrix variables. The following tricks help to transform a number of these nonlinear matrix inequalities into the LMI format so that they are readily solvable by LMI methods.

(a) Congruence Transformation [45]

Pre and post-multiplying a square matrix by a nonsingular matrix and its transpose is called congruence transformation. The definiteness of a matrix is unchanged under congruence transformation, i.e $Q < 0$ (or ≤ 0) if and only if $WQW^T < 0$ (or ≤ 0), where W is nonsingular. This technique, often together with the change of variables, can be used in many situations to remove bilinear terms in matrix inequalities and transform a bilinear matrix inequality into a linear one.

(b) Schur Complement

The following well-known result is usually used to transform quadratic matrix inequalities which are nonlinear in the matrix variable into linear matrix inequalities.

Lemma 2.6 *(Schur Complement Lemma) [45] Suppose P and Q are symmetric. The condition*

$$\begin{bmatrix} P & M \\ M^T & Q \end{bmatrix} < 0 \tag{2.11}$$

is equivalent to

$$P < 0,\ Q - M^T P^{-1} M < 0. \tag{2.12}$$

Suppose P, Q and M all depend on the vector variable x, then the above lemma reduces the nonlinear inequalities in (2.12) to the LMI in (2.11). The Schur Complement Lemma can be generalized into the non-strict version below.

Lemma 2.7 *(Non-strict Schur Complement Lemma) [45] Suppose P and Q are symmetric. The condition*

$$\begin{bmatrix} P & M \\ M^T & Q \end{bmatrix} \leq 0 \tag{2.13}$$

is equivalent to

$$P \leq 0,\ Q - M^T P^{\dagger} M \leq 0 \text{ and } (I - P^{\dagger} P)M = 0. \tag{2.14}$$

Chapter 3

Reformulation from NI to BR Framework

3.1 Introduction

This chapter will take a first step towards providing a systematic framework for controller synthesis for NI systems based on the negative-imaginary stability theorem (Theorem 2.6). More specifically, for uncertain systems where the perturbation belongs to the class of SNI systems, it is natural to seek to design a stabilizing controller such that the closed-loop system satisfies the NI condition. Via Theorem 2.6, it is then possible to quantify the largest family of perturbations that have SNI property in terms of the reciprocal of the DC gain of the nominal system. This chapter will reformulate the problem of finding such a controller into an equivalent problem of finding an internally stabilizing controller for a transformed system such that the closed-loop is BR. Parts of this chapter were also published in [23].

Although the reformulation technique posed in this chapter is based on a well-established bilinear transformation technique, which is normally used to reformulate the closed-loop system from PR systems to BR framework, the proposed work is not just a straightforward extension of previous results. We have to be mindful of significant difficulties that are introduced at zero frequency when an NI system is transformed into a PR system by differentiating the output with respect to time. In this regard, one may argue that the reformulation from an NI system to a BR system via an intermediate PR system may provide an easy solution to the proposed problem by multiplying the NI transfer function matrix with $-\frac{1}{s}I$ or sI. However, the former transformation raises instability issue and the latter may cause improperness of the system. Indeed, this chapter follows the latter transformation, however the transformation from an NI to a PR system results in significant technical difficulties relating to the preservation of stability due to the pole/zero cancellation at the origin resulting from the transformation. Furthermore, this transformation results in a strict NI system being reformulated as a non-strict BR system and hence, $\mathcal{H}_\infty$ optimal theory is required for controller synthesis instead of suboptimal $\mathcal{H}_\infty$ theory. In suboptimal $\mathcal{H}_\infty$ theory, the standard assumptions on the invariant zeros on the $j\omega$-axis can be lifted (see Section 2.5). However, the transformed system violates the assumption of 'no invariant zeros on the imaginary axis', which is a necessary assumption for optimal Riccati equation based $\mathcal{H}_\infty$ controller synthesis methods. Indeed, the reformulated (1,2) subsystem block of the generalized plant contains an invariant zero at the origin. To the best of our knowledge, there is no optimal $\mathcal{H}_\infty$ controller synthesis theory available in literature that can give a straightforward solution to this synthesis problem when such an invariant zero at the origin is present. In this context, [64, 65] may give some directions and this chapter flags an open research problem in $\mathcal{H}_\infty$ optimal control synthesis for a generalized plant with an invariant zero on the imaginary axis as this leads to a synthesis technique for NI systems.

3.2 Some Technical Results

In this section, some useful results are presented to describe the technical difficulties introduced by the transformations between NI and PR systems and also presents specific properties of NI systems. Some background material is also presented which is required to establish the main results of this chapter.

As pointed out in the previous chapter, the concept of NI systems is similar to that of PR systems where the frequency response is constrained in lie in half of the complex plane. However, it is very important to note that the frequency dependent condition for NI systems in Definition 2.20 excludes zero frequency and hence results in a punctured $j\omega$-axis, whereas the PR condition in Definition 2.19 is satisfied for all $\omega \in \mathbb{R}$. Under some technical assumptions, an NI system can be transformed into a PR system and vice versa. Before presenting the transformations between NI systems and PR systems, we consider the definition of stable PR systems as follows:

Definition 3.1 *Let the set of stable positive-real transfer function matrices be defined by*

$$\mathcal{P} := \{X(s) \in \mathcal{RH}_\infty^{n\times n} : [X(j\omega) + X(j\omega)^*] \geq 0 \; \forall \omega \in \mathbb{R}\}.$$

Note that Definition 2.19 implies that PR transfer function matrices can have poles on the imaginary axis, however Definition 3.1 considers the stable subset of PR transfer function matrices.

The following lemma gives an equivalent condition for the stability of a transfer function matrix that has a blocking zero at $s = 0$.

Lemma 3.1 *Given $X(s) \in \mathcal{R}$ satisfying $X(\infty) = 0$. Then, $Y(s) = sX(s) \in \mathcal{RH}_\infty$ and $Y(0) = 0$ if and only if $X(s) \in \mathcal{RH}_\infty$.*

Proof This is trivial by noting that $Y(s)$ has a blocking zero at $s = 0$, hence $X(s) = \frac{1}{s}Y(s)$ has the same poles as $Y(s)$. □

Now, we are ready to present the following two lemmas which show transformations between NI systems and PR systems.

Lemma 3.2 *Given $X(s) \in \mathcal{I}$. Then,*

(i) $s(X(s) - X(\infty)) \in \mathcal{P}$,

(ii) $-\frac{1}{s}(X(s) - X(0)) \in \mathcal{P}$.

Proof (i) Let $\hat{X}(s) = X(s) - X(\infty)$ and $X_1(s) = s(X(s) - X(\infty))$. First note that $\hat{X}(\infty) = 0$ and $\hat{X} \in \mathcal{RH}_\infty$, it follows from Lemma 3.1 that $X_1(s) \in \mathcal{RH}_\infty$. It is easy to verify that $[X_1(j\omega) + X_1(j\omega)^*] \geq 0 \ \forall \omega \in \mathbb{R}$ on noting that $X_1(j\omega) = j\omega(X(j\omega) - X(\infty))$ and $X(s) \in \mathcal{I}$.

(ii) Let $\tilde{X}(s) = (X(s) - X(0))$ and $X_2(s) = -\frac{1}{s}(X(s) - X(0))$. First note that $\tilde{X}(s)$ has a blocking zero at $s = 0$. Also, $\tilde{X}(s) \in \mathcal{RH}_\infty$ since $X(s) \in \mathcal{RH}_\infty$. Then it follows the proof of Lemma 3.1 that $X_2(s) \in \mathcal{RH}_\infty$. Now, let (A, B, C, D) be a minimal state-space realization of $X(s)$, then it follows from Lemma 2.4 that $D = D^*$, and there exist real matrix $Y = Y^T > 0$ such that

$$AY + YA^* \leq 0 \quad \text{and} \quad B + AYC^* = 0. \tag{3.1}$$

Hence, $X(0) = C(-A)^{-1}B + D = CYC^* + D = X(0)^T$. Consequently,

$$\begin{aligned} & X_2(j\omega) + X_2(j\omega)^* \\ = \; & -\frac{1}{j\omega}(X(j\omega) - X(0)) + \frac{1}{j\omega}(X(j\omega)^* - X(0)^T) \\ = \; & j\frac{1}{\omega}(X(j\omega) - X(j\omega)^*) \geq 0 \text{ for all } \omega \in (0, \infty). \end{aligned}$$

Also,

$$\begin{aligned} & X_2(0) + X_2(0)^T \\ = \; & \lim_{\omega \to 0} j\frac{X(j\omega) - X(j\omega)^*}{\omega} \\ = \; & \lim_{\omega \to 0} j\frac{-C(j\omega I - A)^{-1}AYC^* + (C(j\omega I - A)^{-1}AYC^*)^*}{\omega} \\ = \; & \lim_{\omega \to 0} jC[j(j\omega I - A)^{-2}AY + (j(j\omega I - A)^{-2}AY)^*]C^* \text{ (using L'Hospital's rule)} \\ = \; & -CA^{-1}[YA^* + AY]A^{-*}C^* \geq 0 \text{ via (3.1).} \end{aligned}$$

Hence, we have $[X_2(j\omega) + X_2(j\omega)^*] \geq 0 \ \forall \omega \in \mathbb{R}$. Thus, $X_2(s) \in \mathcal{P}$. □

Lemma 3.3 *Given* $X(s) \in \mathcal{P}$,

(i) if $X(\infty) = 0$, *then,* $-sX(s) \in \mathcal{I}$,

(ii) if $X(0) = 0$, *then,* $\frac{1}{s}X(s) \in \mathcal{I}$.

Proof The stability of $-sX(s)$ and $\frac{1}{s}X(s)$ can be proved similarly to the proof of the stability part of Lemma 3.2. The frequency response part can be readily verified using Definition 3.1 and Definition 2.20. □

If $X(s) \in \mathcal{I}_s$, then $X_1(s) = s(X(s) - X(\infty)) \in \mathcal{P}$, but $X_1(s)$ is not strictly positive-real. It can be easily proved by noting that $X_1(0) = 0$. Also, given $X(s) \in \mathcal{I}_s$, then $X_2(s) = -\frac{1}{s}(X(s) - X(0)) \in \mathcal{P}$, but $X_2(s)$ does not necessarily satisfy the SPR property. It is worth pointing out that the transformation from NI systems to PR systems always results into a non-strict PR system, irrespective of whether it starts from a strict or non-strict NI system. Hence, via this transformation, no strict problem can be transformed into an equivalent strict PR problem.

The results discussed so far for NI and PR systems highlight all the technical difficulties that appear due to the transformation. It also points out the restricted applicability of SPR theory for NI systems. This restriction introduces significant difficulty into the stability analysis for the interconnected NI system. In this chapter, a reformulation technique from closed-loop systems that satisfy the NI property into a BR framework via the PR property is developed by tackling all of these difficulties, especially the properness condition, the internal stability condition, and the violation of strictness due to the transformation.

The next three technical lemmas are given here to streamline presentation of the proof of the main result in the next section. The following set is introduced first for compactness of notation.

Definition 3.2 *Let the set of square stable contractive transfer function matrices whose Nyquist plot does not pass through* $-1 + j0$ *point be defined by*

$$\mathcal{B} := \{X \in \mathcal{RH}_\infty^{n\times n} : \|X\|_\infty \leq 1, \det(I + X(j\omega)) \neq 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}\}.$$

The following two lemmas state that if $X \in \mathcal{P}$ or $X \in \mathcal{B}$, then $(I + X)^{-1}$ is stable. These standard properties will be used to make a connection amongst different parts of the proof of the main result.

Lemma 3.4 *Given* $X \in \mathcal{P}$*, then* $(I + X)^{-1} \in \mathcal{RH}_\infty$*.*

Proof This is trivially established via simple application of Theorem 2.5. □

Lemma 3.5 *Given* $X \in \mathcal{B}$*. Then* $(I + X)^{-1} \in \mathcal{RH}_\infty$*.*

Proof First note that $(I + \alpha X)^{-1} \in \mathcal{RH}_\infty \ \forall \alpha \in (0, 1)$ and also $X \in \mathcal{B}$ gives $\det(I + X(j\omega)) \neq 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}$. As α increases continuously to unity, the transmission zeros of $(I + \alpha X(s))$ vary continuously and are in $\mathbb{C}_-$ for all $\alpha \in (0, 1)$, and they do not intersect the $j\omega$-axis at $\alpha = 1$. Therefore, at $\alpha = 1$, they must remain in $\mathbb{C}_-$. Thus $(I + X)^{-1} \in \mathcal{RH}_\infty$. □

The third technical lemma gives a simple necessary and sufficient condition for input-output stability of a particular Redheffer Star-Product. It will be used in the next section to make a connection between systems in $\mathcal{P}$ and $\mathcal{B}$.

Lemma 3.6 *Given* $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{RH}_\infty$*. Then,*

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T \in \mathcal{RH}_\infty \quad \Leftrightarrow \quad \left(I + T_{11}\right)^{-1} \in \mathcal{RH}_\infty.$$

Proof This equivalence can be seen directly from an expansion of the Redheffer

Star-Product (Section 2.1) of $\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T$. □

3.3 Reformulation Technique

The main result of this chapter is given in Theorem 3.1 below. Before this theorem is stated, the following definition of internal stability of an LFT interconnection is needed.

Definition 3.3 *Let $\langle G, K\rangle$ denote the feedback interconnection shown in Figure 3.1 and correspondingly let $T(G,K)$ denote the transfer function from* $\begin{pmatrix} w \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$ *to* $\begin{pmatrix} z \\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix}$. *We say $\langle G, K\rangle$ is internally stable when $T(G,K) \in \mathcal{RH}_\infty$.*

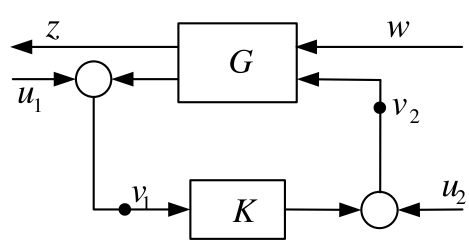

Figure 3.1: An LFT interconnection

The following main theorem broadly states that a controller internally stabilizes a generalized plant Σ and makes the input-output map satisfy an NI property if and only if the same controller internally stabilizes a different generalized plant G (constructed from Σ) and makes the input-output map contractive.

Theorem 3.1 *Given a controller $K \in \mathcal{R}^{q\times p}$ and a generalized plant*

$$\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{3.2}$$

with $A \in \mathbb{R}^{n\times n}$, $D_{11} = D_{11}^* \in \mathbb{R}^{m\times m}$, $D_{22} = 0 \in \mathbb{R}^{p\times q}$, $D_{12} = 0$, (A, B_2) stabilizable and (C_2, A) detectable. Let $U = I + C_1B_1$ and $V = I + B_1C_1$. Also, suppose U and V are invertible. Then, $\langle \Sigma, K\rangle$ is internally stable and $j\left[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*\right] \geq 0\ \forall\omega \in (0,\infty)$ if and only if $\langle G, K\rangle$ is internally stable, $\left\|F_\ell(G,K)\right\|_\infty \leq 1$ and $\det(I + F_\ell(G,K)(j\omega)) \neq 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}$, *where*

$$G = \left[\begin{array}{c|cc} V^{-1}A & B_1U^{-1} & V^{-1}B_2 \\ \hline -2U^{-1}C_1A & (I - C_1B_1)U^{-1} & -2U^{-1}C_1B_2 \\ C_2 - D_{21}U^{-1}C_1A & D_{21}U^{-1} & -D_{21}U^{-1}C_1B_2 \end{array}\right]. \tag{3.3}$$

Proof We will prove the result via a sequence of equivalent reformulations:

(a) $\langle \Sigma, K\rangle$ is internally stable and $j\left[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*\right] \geq 0\ \forall\omega \in (0,\infty)$.

(b) $\langle \bar\Sigma, K\rangle$ is internally stable and $j\left[F_\ell(\bar\Sigma, K)(j\omega) - F_\ell(\bar\Sigma, K)(j\omega)^*\right] \geq 0\ \forall\omega \in (0,\infty)$, where

$$\bar\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{3.4}$$

[The equivalence (a)⇔(b) follows on noting that $F_\ell(\Sigma, K)(\infty) = D_{11} = D_{11}^*$.]

(c) $\langle \hat\Sigma, K\rangle$ is internally stable and $\left[F_\ell(\hat\Sigma, K)(j\omega) + F_\ell(\hat\Sigma, K)(j\omega)^*\right] \geq 0\ \forall\omega \in \mathbb{R}$, where

$$\hat{\Sigma} \quad = \quad \begin{pmatrix} sI & 0 \\ 0 & I \end{pmatrix} \bar{\Sigma} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 A & C_1 B_1 & C_1 B_2 \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{3.5}$$

[The internal stability parts can be seen to be equivalent on noting that K is the same, $\hat{\Sigma}$ and $\bar{\Sigma}$ are both stabilizable and detectable, and that

$$\bar{\Sigma}_{22} = \hat{\Sigma}_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right], \tag{3.6}$$

thus allowing use of [46, Lemma A.4.1]. The frequency domain inequalities are also equivalent since $F_\ell(\hat{\Sigma}, K)(j\omega) = j\omega \cdot F_\ell(\bar{\Sigma}, K)(j\omega)$.]

(d) $\langle G, K\rangle$ is internally stable, $\bar{\sigma}\big[F_\ell(G, K)(j\omega)\big] \leq 1\ \forall\omega \in \mathbb{R}$, and $\det(I + F_\ell(G, K)(j\omega)) \neq 0\ \forall\omega \in \mathbb{R} \cup \{\infty\}$, where

$$G = \left[\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star \hat{\Sigma}\right] = \left[\begin{array}{c|cc} V^{-1}A & B_1 U^{-1} & V^{-1}B_2 \\ \hline -2U^{-1}C_1 A & (I - C_1 B_1)U^{-1} & -2U^{-1}C_1 B_2 \\ C_2 - D_{21}U^{-1}C_1 A & D_{21}U^{-1} & -D_{21}U^{-1}C_1 B_2 \end{array}\right].$$

[(c) $\Rightarrow$ (d): Since $F_\ell(\hat{\Sigma}, K) \in \mathcal{P}$, it follows that $(I + F_\ell(\hat{\Sigma}, K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 3.4. Then define $Y = (I - F_\ell(\hat{\Sigma}, K))(I + F_\ell(\hat{\Sigma}, K))^{-1}$ and note that $Y = F_\ell(G, K) \in \mathcal{RH}_\infty$. Also, since $(I+Y)^{-1} = \frac{1}{2}(I+F_\ell(\hat{\Sigma}, K)) \in \mathcal{RH}_\infty$, it follows that $\det(I + Y(j\omega)) \neq 0\ \forall\omega \in \mathbb{R} \cup \{\infty\}$. Also, $[F_\ell(\hat{\Sigma}, K)(j\omega) + F_\ell(\hat{\Sigma}, K)(j\omega)^*] \geq 0\ \forall\omega \in \mathbb{R}$ implies $\bar{\sigma}\,[Y(j\omega)] \leq 1\ \forall\omega \in \mathbb{R}$. Finally, since $\langle\hat{\Sigma}, K\rangle$ is internally stable, we have $T(\hat{\Sigma}, K) \in \mathcal{RH}_\infty$. Noting that $T_{11}(\hat{\Sigma}, K) = F_\ell(\hat{\Sigma}, K) \in \mathcal{P}$, we get $(I + T_{11}(\hat{\Sigma}, K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 3.4 and this in turn gives

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(\hat{\Sigma}, K) \in \mathcal{RH}_\infty$$

via Lemma 3.6, which implies $\langle G, K\rangle$ is internally stable as

$$T(G,K) = T\left(\begin{bmatrix} I & -2I \\ I & -I \end{bmatrix} \star \hat{\Sigma}, K\right) = \begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(\hat{\Sigma}, K). \tag{3.7}$$

(d) $\Rightarrow$ (c): Since $F_\ell(G,K) \in \mathcal{B}$, $(I + F_\ell(G,K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 3.5. Then define $X = (I - F_\ell(G,K))\,(I + F_\ell(G,K))^{-1} \in \mathcal{RH}_\infty$ and note that $X = F_\ell(\hat{\Sigma}, K)$. Then $\bar{\sigma}\,[F_\ell(G,K)(j\omega)] \leq 1\; \forall\omega \in \mathbb{R}$ implies $[X(j\omega) + X(j\omega)^*] \geq 0\; \forall\omega \in \mathbb{R}$. Finally, since $\langle G, K\rangle$ is internally stable, we have $T(G,K) \in \mathcal{RH}_\infty$. Noting that $T_{11}(G,K) = F_\ell(G,K) \in \mathcal{B}$, we get $(I + T_{11}(G,K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 3.5 and this in turn gives

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(G,K) \in \mathcal{RH}_\infty$$

via Lemma 3.6, which implies $\langle \hat{\Sigma}, K\rangle$ is internally stable as

$$\begin{pmatrix} 0 & 2I \\ \frac{1}{2}I & 0 \end{pmatrix} \star T(\hat{\Sigma}, K) = \begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(G,K) \text{ via (3.7).}]$$

$\square$

This theorem states that the original problem of synthesizing an internally stabilizing controller such that a closed-loop LFT satisfies the NI property can be transformed to an equivalent BR problem. This is a first step towards a controller synthesis method for closed-loop systems with NI property, allowing results to be borrowed from $\mathcal{H}_\infty$ control theory.

The following lemma shows that the restriction of $D_{12} = 0$ and $D_{22} = 0$ in the realization of the generalized plant for Σ in (3.2) can be easily circumvented.

Lemma 3.7 *Given a strictly proper controller K and a generalized plant*

$$\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{3.8}$$

with (A, B_2) stabilizable, (C_2, A) detectable, and $D_{11} = D_{11}^*$. Then, $\langle\Sigma, K\rangle$ is internally stable and $j\left[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*\right] \geq 0\ \forall\omega \in (0, \infty)$ if and only if $\langle\breve{\Sigma}, \breve{K}\rangle$ is internally stable and $j\left[F_\ell(\breve{\Sigma}, \breve{K})(j\omega) - F_\ell(\breve{\Sigma}, \breve{K})(j\omega)^*\right] \geq 0\ \forall\omega \in (0, \infty)$, where

$$\breve{\Sigma} = \left[\begin{array}{cc|cc} A & B_2 & B_1 & 0 \\ 0 & -\tau I & 0 & \tau I \\ \hline C_1 & D_{12} & D_{11} & 0 \\ C_2 & D_{22} & D_{21} & 0 \end{array}\right], \tag{3.9}$$

$$\breve{K}(s) = \left(\frac{s}{\tau} + 1\right) K(s), \text{ for any arbitrary } \tau > 0. \tag{3.10}$$

Proof Easily follows on noting that

$$T(\Sigma, K)\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{s/\tau+1}I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{s/\tau+1}I \end{pmatrix} T(\breve{\Sigma}, \breve{K})$$

because

$$\breve{\Sigma} = \Sigma \begin{pmatrix} I & 0 \\ 0 & \frac{1}{s/\tau+1}I \end{pmatrix}.$$

□

The following remarks are appropriate on Theorem 3.1 at this stage.

Remark 3.1 *The assumption on the invertibility of U and V in the theorem statement is imposed to ensure well-posedness of the transformed system G while* *the reformulation takes place from step* (c) *to* (d) *shown in the proof of the theorem.*

Remark 3.2 *Under the suppositions* (A, B_2) *stabilizable and* (C_2, A) *detectable, the state-space realization for* G *given in Theorem 3.1 is also stabilizable and detectable.*

Proof This can be easily shown via PBH test [44]. □

The following corollary is an immediate consequence of Theorem 3.1. It gives an equivalent BR condition where an internally stabilizing controller renders the closed-loop LFT structure to be SNI.

Corollary 3.2 *Given the suppositions of Theorem 3.1. Then,* $\langle \Sigma, K \rangle$ *is internally stable and* $j\left[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*\right] > 0 \ \forall \omega \in (0, \infty)$ *if and only if* $\langle G, K \rangle$ *is internally stable,* $F_\ell(G, K)(0) = I$, $\bar{\sigma}(F_\ell(G, K)(j\omega)) < 1 \ \forall \omega \in (0, \infty)$ *and* $\det(I + F_\ell(G, K)(j\infty)) \neq 0$, *where* G *is given in (3.3).*

Proof The result follows via a straightforward modification of the proof of Theorem 3.1 from steps (a) to (d) by replacing the frequency domain non-strict inequalities with strict inequalities for all $\omega \in (0, \infty)$. Note that $F_\ell(\hat{\Sigma}, K) = sF_\ell(\bar{\Sigma}, K)$ and $F_\ell(G, K) = (I - F_\ell(\hat{\Sigma}, K))(I + F_\ell(\hat{\Sigma}, K))^{-1}$, this implies $F_\ell(\hat{\Sigma}, K)(0) = 0$ and $F_\ell(G, K)(0) = I$. Hence the reformulated $\mathcal{H}_\infty$ constraint in step (d) can be restated as $F_\ell(G, K)(0) = I$, $\bar{\sigma}(F_\ell(G, K)(j\omega)) < 1 \ \forall \omega \in (0, \infty)$ and the determinant condition reduces to $\det(I + F_\ell(G, K)(j\infty)) \neq 0$. The proof of internal stability part remains the same. □

The reformulated closed-loop system $F_\ell(G, K)(s)$ in Theorem 3.1 and Corollary 3.2 is always identity at zero frequency i.e. $F_\ell(G, K)(0) = I$. This fact points out a significant technical difficulty in controller synthesis for NI systems. It indicates, the reformulated problem is always converted into a non-strict BR problem because $\bar{\sigma}(F_\ell(G,K)(0)) = 1$ independent of whether we started from a strict NI system or a non-strict NI system. This then necessarily invokes optimal $\mathcal{H}_\infty$ theory for controller synthesis. This problem occurs because the transformation from NI system to PR system, shown in step (b) to (c) in proof of the main theorem, involves multiplying the NI transfer function matrix with sI that always makes the transformed PR system as non-strict. Hence, controller synthesis methods for SPR systems [39–41] are not applicable in this context.

Remark 3.3 *The subsystem $G_{12}(s)$ of the transformed system G shown in (3.3) has an invariant zero at $s = j0$ and this violates a required assumption for $\mathcal{H}_\infty$ optimal controller synthesis, and hence Riccati equation based methods [44, 50] cannot be used.*

Proof Note that since (A, B_2) is stabilizable, it follows $\operatorname{rank}\begin{pmatrix} A & B_2 \end{pmatrix} = n$. Hence,

$$\begin{aligned} & \operatorname{rank}\begin{pmatrix} V^{-1}A & V^{-1}B_2 \\ -2U^{-1}C_1A & -2U^{-1}C_1B_2 \end{pmatrix} \\ = \; & \operatorname{rank}\left(\begin{pmatrix} V^{-1} & 0 \\ 0 & -2U^{-1} \end{pmatrix}\begin{pmatrix} I \\ C_1 \end{pmatrix}\begin{pmatrix} A & B_2 \end{pmatrix}\right) \\ = \; & n < n+m. \end{aligned}$$

□

The following remark states that under an additional assumption, the invariant zero in Remark 3 becomes a blocking zero, which may be a useful fact for controller synthesis for NI systems.

Remark 3.4 *When $\det(A) \neq 0$, $G_{12}(0) = 0$, (i.e. $s = 0$ becomes a blocking zero at zero frequency for $G_{12}(s)$), and hence $F_\ell(G,K)(0) = G_{11}(0) = I$.*

Proof This result is trivial on noting that

$$G_{12}(s) = -2U^{-1}C_1B_2 - 2U^{-1}C_1A(sI - V^{-1}A)^{-1}V^{-1}B_2,$$

$$G_{11}(s) = (I - C_1B_1)U^{-1} - 2U^{-1}C_1A(sI - V^{-1}A)^{-1}B_1U^{-1},$$

and hence, it follows that whenever $\det(A) \neq 0$, $G_{12}(0) = 0$, and $F_\ell(G, K)(0) = G_{11}(0) = I$. □

Remark 3.4 states when A is nonsingular, the reformulated closed-loop system is identity at zero frequency; i.e. $F_\ell(G, K)(0) = G_{11}(0) = I$ which is independent of the controller K. Also, it is shown via Corollary 3.2, that if we want to transform the controller synthesis problem for SNI systems, the reformulated system hits the boundary of $gain = 1$ only at zero frequency. In this regard, it might be possible to obtain a solution of the posed problem by extending the generalized Kalman-Yakubovich-Popov (KYP) lemma to tackle the frequency domain inequality at all frequencies excluding specific points (in this case, zero and infinity) as for example [66–68] have done on a given frequency interval. However, this is a nontrivial problem.

3.4 Illustrative Example

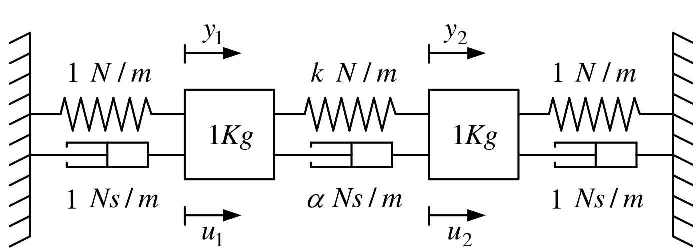

Figure 3.2: Lightly damped uncertain mechanical plant

Consider the lightly damped mechanical plant [12] depicted in Figure 3.2, which consists of two unit masses constrained to slide rectilinearly on a frictionless table. Each mass is attached to a fixed wall via a spring of known unit stiffness and via a damper of known unit viscous resistance. Furthermore, the two unit masses are coupled together via a spring of uncertain stiffness k (in Newton per meter, i.e., N/m) and via a damper of uncertain viscous resistance α (in Newton second per meter, i.e., Ns/m). A force is applied to each mass (denoted by u_1 and u_2, respectively) and the displacement of each mass is measured (denoted by y_1 and y_2 , respectively).

As mentioned in the introduction, lightly damped flexible structures with collocated position sensors and force actuators typically give rise to NI systems. Hence, the uncertain lightly damped mechanical system depicted in Figure 3.2 with collocated position sensors and force actuators is a typical example of an NI system. This physically motivated example was explored in [12] to illustrate the analysis results of [12]. As a consequence, the present work is a first step towards controller synthesis for this class of systems. For completeness, simplicity and also for ease of exposition, the same physically motivated example has been adopted in this section to illustrate the key reformulation of this chapter. Similar examples have been considered in the literature as benchmark problems, say for example [69, 70].

For shorthand, the following commonly appearing transfer functions and matrices are defined.

$$p(s) := \frac{1}{s^2+s+1}, \ \delta(s) := \frac{1}{s^2+(2\alpha+1)s+(2k+1)} \text{ and } \Psi := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Then, elementary mechanical modeling reveals that the transfer function matrix from the input

$$u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

to the output

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is described by $y = P_\triangle(s)u$, where

$$P_\triangle(s) := p(s)\delta(s) \times \begin{bmatrix} s^2 + (\alpha+1)s + (k+1) & (\alpha s + k) \\ (\alpha s + k) & s^2 + (\alpha+1)s + (k+1) \end{bmatrix}.$$

This plant is uncertain since α and k are unknown parameters.

For the purpose of robust controller synthesis, the controlled closed-loop system in Figure 3.3 is rearranged in a standard LFT interconnection shown in Figure 3.4. In these two figures, the generalized plant Σ, the nominal plant P

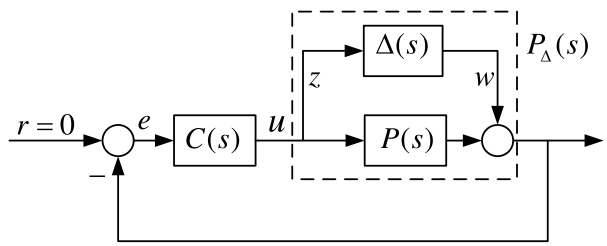

Figure 3.3: Controlled closed-loop system

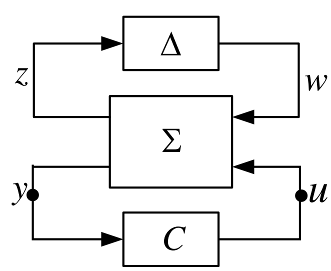

Figure 3.4: Rearranged LFT interconnection

and the uncertainty Δ are given respectively by

$$\Sigma = \begin{bmatrix} 0 & I \\ -I & -P \end{bmatrix}, \; P(s) = \Psi \mathrm{diag}(\frac{1}{2}p(s), 0)\Psi^*,$$

$$\text{and } \Delta(s) = \Psi^{-1}\mathrm{diag}(\frac{1}{2}\delta(s), 0)(\Psi^{-1})^*. \tag{3.11}$$

Since the uncertainty $\Delta(s)$ belongs to $\mathcal{I}$, a particular choice of controller $C(s)$ that internally stabilizes Σ and makes $F_\ell(\Sigma, C)$ belong to $\mathcal{I}_s$ was chosen in [12] as $C(s) = \Psi^{-*}\text{diag}(\frac{-2(s^2+s+1)}{2s^3+4s^2+4s+3}, \frac{-1}{s+1})\Psi^{-1}$. This guarantees robust stability for all perturbations in $\mathcal{I}$ as long as the DC loop gain condition is also satisfied [Theorem 2.6 Chapter 2].

Since $C(s)$ is strictly proper and the D_{12} matrix of Σ is nonzero, we first use Lemma 3.7 to give $\breve{C}(s) = (\frac{s}{\tau} + 1)C(s)$ and $\breve{\Sigma} = \begin{pmatrix} 0 & \frac{1}{s/\tau+1}I \\ -I & -\frac{1}{s/\tau+1}P \end{pmatrix}$ where we arbitrarily set $\tau = 1$. Then, using the construction in Theorem 3.1, we obtain the transformed generalized plant $G(s)$ as:

$$G = \left[\begin{array}{cccc|cccc} -0.9778 & -0.8526 & 0.6992 & 0.6992 & 0 & 0 & 0 & 0 \\ 1.1474 & -0.0222 & -0.1054 & -0.1054 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 2 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & -2 \\ -0.1054 & -0.6992 & -1 & 0 & -1 & 0 & 1 & 0 \\ -0.1054 & -0.6992 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right].$$

Since we know from [12] that the chosen $C(s)$ internally stabilizes $\Sigma(s)$ and makes $F_\ell(\Sigma, C) \in \mathcal{I}_s$, then via Lemma 3.7 and Corollary 3.2, we should get that $\breve{C}(s)$ internally stabilizes $G(s)$ and satisfies $F_\ell(G, \breve{C})(0) = I$, $\bar{\sigma}(F_\ell(G, \breve{C})(j\omega)) < 1\ \forall\omega \in (0, \infty)$ and $\det(I + F_\ell(G, \breve{C})(j\infty)) \neq 0$.

A simple computation gives $F_\ell(G, \breve{C})(0) = I$ and $\bar{\sigma}(F_\ell(G, \breve{C})(j\omega)) < 1\ \forall\omega \in (0, \infty)$ is satisfied as the plot of $\bar{\sigma}\left[F_\ell(G, \breve{C})(j\omega)\right]$ lies below the 0 dB line shown in Figure 3.5. Also, $\det(I + F_\ell(G, \breve{C})(j\infty)) = 0.8 \neq 0$. Finally, it is observed that $\breve{C}$ internally stabilizes G as the poles of $T(G, \breve{C})$ are at: $-0.5\ \pm j\ 0.8660$, $-0.5\ \pm j\ 0.8660$, -1, -0.7236, -0.2764.

This illustrative example demonstrates that the problem of finding an internally stabilizing controller such that the input-output map has SNI property can be

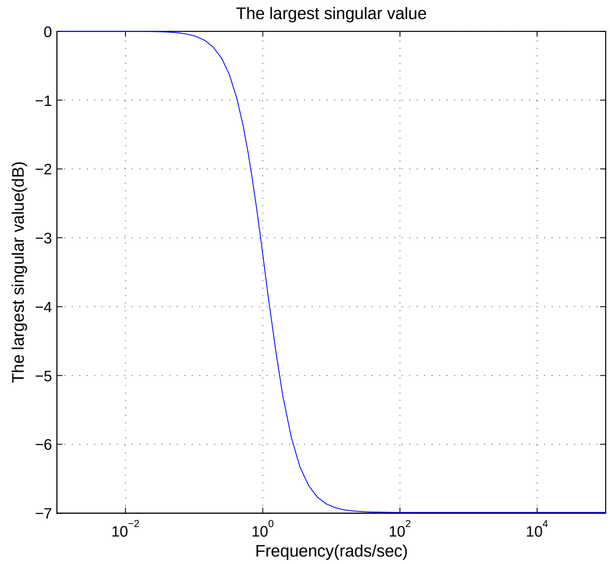

Figure 3.5: The largest singular value plot of $F_\ell(G, \breve{C})(j\omega)$

reformulated into a BR problem.

3.5 Discussions

Although this chapter doesn't give an explicit formulation for controller synthesis of an optimal $\mathcal{H}_\infty$ design problem with an invariant zero which is due to the reformulation of an NI synthesis problem into the BR framework, the following suboptimal $\mathcal{H}_\infty$ problem

$$\inf_{\breve{C} \text{ internally stabilizing } G} \gamma : \quad \|F_\ell(G, \breve{C})\|_\infty < \gamma, \tag{3.12}$$

will always give γ tending to unity. Hence, from a controller synthesis point of view, Corollary 3.2 is useful and more specifically, one can first solve the above suboptimal $\mathcal{H}_\infty$ problem and then, the conditions

$F_\ell(G, \breve{C})(0) = I$, $\bar{\sigma}(F_\ell(G, \breve{C})(j\omega)) < 1 \ \forall \omega \in (0, \infty)$ and $\det(I + F_\ell(G, \breve{C})(j\infty)) \neq 0$

can be checked to see if the conditions of Corollary 3.2 are fulfilled. Note that the A-matrix of $\breve{\Sigma}(s)$ is nonsingular, and hence via Remark 3.4, $F_\ell(G, \breve{C})(0) = I$ is always satisfied which is independent of the controller. In this case, we only need to check the remaining two conditions.

Note that the first element Σ_{11} of the generalized plant Σ rearranged in LFT structure as shown in Figure 3.4 is zero, and hence the well-posedness assumptions are satisfied, i.e $U = I + C_1B_1$ and $V = I + B_1C_1$ are invertible, and $D_{11} = D_{11}^*$ in Theorem 3.1 and Corollary 3.2. It is worth pointing out that the above conditions are also true if the uncertain plant of the controlled closed-loop system considered in Figure 3.3 is described as any of the following sets [71]:

$$
\begin{aligned}
P_\Delta &= P + W_1\Delta W_2 \text{ (Additive perturbed system)} \\
P_\Delta &= (I + W_1\Delta W_2)P \text{ (Output multiplicative perturbed system)} \\
P_\Delta &= P(I + W_1\Delta W_2) \text{ (Input multiplicative perturbed system)}
\end{aligned}
$$

3.6 Conclusions

Transformations between NI systems and PR systems are given in Section 3.2. It has been shown that an NI system is always transformed into a non-strict PR system, irrespective of whether it is a strict or non-strict NI system. Technical difficulties relating to preservation of stability and properness of the systems due to the transformations of NI systems are highlighted here that are helpful to understand the properties and particularity of NI systems. In Section 3.3, an LFT interconnection that has NI closed-loop properties is reformulated into a bounded-real LFT interconnection by tackling all these difficulties. In order to get rid of improperness issue, the simple assumptions $D_{11} = D_{11}^*$ and $D_{12} = 0$ are made and using a loop shifting technique in Lemma 3.7, it has been shown that the restriction $D_{12} = 0$ imposes no loss of generality. The other assumption $D_{11} = D_{11}^*$ does not restrict the applicability of the main theorem because it is a necessary assumption for NI systems and as is also demonstrated via a physically motivated numerical system in Section 3.4 (see also discussions in Section 3.5).

The reformulation result in this chapter is a first step towards a controller synthesis technique for NI systems. Although this chapter does not tackle the important step of explicit controller synthesis for such a class of systems, the main results in this chapter could constitute a key step in allowing results from $\mathcal{H}_\infty$ control synthesis to be borrowed for controller synthesis for closed-loop systems with NI property. However, the controller synthesis problem under consideration is not trivial and significant research effort is needed to solve it. To this end, important observations have been pointed out in the Corollary and Remarks of this chapter that sheds lights on controller synthesis for NI systems. It is hoped that this chapter highlights open technical problems in this area which need to be solved for effective controller synthesis for NI systems.

Chapter 4

Robust Performance Analysis For Uncertain Negative-Imaginary Systems

4.1 Introduction

In the previous chapter, the robust stabilization problem for systems with SNI uncertainty is reformulated into a BR framework. This result paves the way for allowing $\mathcal{H}_\infty$ synthesis method to enable controller synthesis for NI systems to guarantee robust stability. Robust stability is the minimum requirement of any practical control system. However, even if a closed-loop system is robustly stable, it is useless if it does not deliver the required performance. Hence, from a practical point of view, the design paradigm is not only to be motivated by the question of robust stability against uncertainties, but it is also very important to emphasize the robust performance perspective. It is well-known that the robust performance problem of uncertain linear time-invariant feedback systems can be transformed into a robust stability problem by introducing a fictional BR uncertainty; structured singular value theory is then usually used to assess the resulting robust stability problem which involves a structured uncertainty [44, 72, 73]. The standard definition of structured singular value assumes that the uncertainties are norm bounded [44, 72, 73] (see the μ stability and performance analysis results in Theorem 2.3 and Theorem 2.4). However, the uncertainties that arise in NI systems (for example, the spill-over dynamics in lightly damped structures) are typically highly resonant, hence direct μ analysis and synthesis would be rather conservative. These dynamics satisfy SNI properties, and also there are some uncertain systems that can equivalently be presented into systems with the uncertain part being SNI. The main purpose of this chapter is hence to utilize the properties of NI systems and extend the robust stabilization reformulation result in Chapter 3 to the robust performance problem that involves performance measured via an $\mathcal{H}_\infty$ norm and a physically motivated uncertainty that satisfies an SNI property.

The derivations in this chapter are based on algebraic operations on linear fractional transformations of feedback interconnected systems. The proposed structured singular value condition on a transformed closed-loop system equivalently gives a quantitative performance test for uncertain systems in the presence of SNI uncertainty. Note that this chapter gives an analysis framework for robust performance problems, where the framework considers a mixture of SNI and SBR uncertainties rather than uncertainties of the same type (size) that typically appear in robust performance problems. Moreover, the present work tackles significant difficulties that are introduced at zero frequency due to required DC loop condition in the NI stability criteria (see Theorem 2.6), and also because of the punctured $j\omega$-axis frequency property of the NI systems. The DC loop condition for stability of NI systems causes difficulty in the proposed robust performance problem, as the DC gain condition of the system has to be ensured for all arbitrary fictional SBR perturbations which will be shown in the main results. This proposed framework in this chapter will further underpin robust controller synthesis techniques for NI systems. Parts of this chapter were also published in [74] and a more detailed version of [74] will appear soon in [75].

The rest of the chapter is organized as follows: Section 4.2 gives the problem formulation. Section 4.3 contains some mathematical preliminary work which is useful to streamline the main results of the chapter. The main theorem, followed by four key corollaries, is presented in Section 4.4. It is important to understand that most of the practical significance of the chapter rests with the corollaries, rather than the main theorem, although the theorem is stated in the strongest sense. The usefulness of the main results is elucidated via a numerical example which is presented in Section 4.5. Section 4.6 concludes the chapter.

4.2 Problem Statement

Consider an LFT feedback interconnected system as shown in Figure 4.1, where Σ is a generalized plant, given by

$$\Sigma = \left[\begin{array}{c|cc:c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hdashline C_3 & D_{31} & D_{32} & D_{33} \end{array}\right] \tag{4.1}$$

with $A \in \mathbb{R}^{n\times n}$, $D_{11} \in \mathbb{R}^{p\times q}$, $D_{22} \in \mathbb{R}^{m\times m}$, $D_{33} \in \mathbb{R}^{l\times r}$, $K \in \mathcal{R}^{r\times l}$ is a given controller and $\Delta_2 \in \mathcal{RH}_\infty^{m\times m}$ is an arbitrary strictly NI uncertainty perturbing the system. The objective of this chapter is to quantify the worst-case (i.e. robust) performance for a given controller K when the model is perturbed by arbitrary SNI uncertainty Δ_2. This means that, subject to internal stability of the system, the infinity norm of the transfer function matrix from w_1 to z_1 remains smaller than some number γ (in particular, we choose $\gamma = 1$ for an appropriately weighted generalized plant Σ) for all possible SNI uncertainties Δ_2; i.e., $\|F_\ell(F_\ell(\Sigma, K), \Delta_2)\|_\infty \leq 1$ for all Δ_2, where the pre-specified weighting

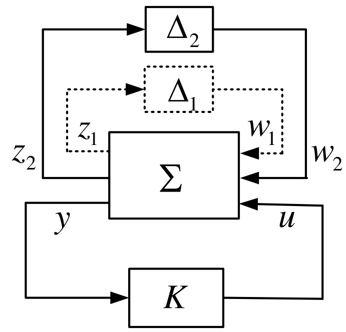

Figure 4.1: An LFT interconnection for performance analysis

functions are absorbed into Σ. For instance, w_1 represents the exogenous signals such as commands, disturbances, etc. whereas z_1 represents the error signals, control inputs, etc. in the feedback interconnection.

As is common in the literature, by introducing a fictional SBR uncertainty Δ_1, the above objective can be posed as a robust stability problem which involves a mix of SNI and SBR uncertainties; i.e., the internal stability of $\left[\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, F_\ell(\Sigma, K)\right]$ has to be ensured for all SBR uncertainties Δ_1 and SNI uncertainties Δ_2.

Note that the mixed uncertainty type makes the proposed problem a nontrivial extension of existing robust stability results.

4.3 Preliminaries

First we establish notational conventions that simplify the exposition in this chapter. Some uncertainty block-structure are defined as follows:

Definition 4.1 *Let $\mathcal{B}^\circ\boldsymbol{\Delta}$ be the stable SBR uncertainty set, i.e. $\mathcal{B}^\circ\boldsymbol{\Delta} = \{\Delta \in \mathcal{RH}_\infty : \|\Delta\|_\infty < 1\}$.*

Definition 4.2 *Let a complex block-structure $\boldsymbol{\Delta}_{\mathbf{TOT}}$ and a real block-structure*

$\mathbf{\Delta_{REAL}}$ *be defined respectively as:*

$$\mathbf{\Delta_{TOT}} = \left\{ \begin{bmatrix} \bar{\Delta}_1 & 0 \\ 0 & \bar{\Delta}_2 \end{bmatrix} : \ \bar{\Delta}_1 \in \mathbb{C}^{q\times p}, \ \bar{\Delta}_2 \in \mathbb{C}^{m\times m} \right\},$$

$$\mathbf{\Delta_{REAL}} = \left\{ \begin{bmatrix} \bar{\Delta}_1 & 0 \\ 0 & \bar{\Delta}_2 \end{bmatrix} : \ \bar{\Delta}_1 \in \mathbb{R}^{q\times p}, \ \bar{\Delta}_2 \in \mathbb{R}^{m\times m} \right\} \subset \mathbf{\Delta_{TOT}}.$$

Next, we present some technical lemmas which will streamline the proofs of the main results in the next section.

The first lemma gives an equivalent simpler condition for the input-output stability of a particular Redheffer Star-Product. This will be used in Section 4.4 to establish a stability equivalence result for two different star-product interconnections.

Lemma 4.1 *Given* $T = \left(\begin{array}{cc:c} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ \hdashline T_{31} & T_{32} & T_{33} \end{array}\right) \in \mathcal{RH}_\infty$. *Then,*

$$\left(\left[\begin{array}{cc:cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hdashline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T \right) \in \mathcal{RH}_\infty \Longleftrightarrow \ \left(I + T_{22}\right)^{-1} \in \mathcal{RH}_\infty. \quad (4.2)$$

Proof This equivalence can directly be seen by expanding the Redheffer Star-Product of $\left[\begin{array}{cc:cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hdashline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T$ (see Section 2.1). □

The next lemma is a simple restatement of the Main-Loop Theorem in μ analysis. It is given here for ease of reference in subsequent proofs.

Lemma 4.2 *(Main-Loop Theorem) [44] Let $M \in \mathcal{RH}_\infty^{(p+m)\times(q+m)}$ be partitioned as $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. Then, the following are equivalent:*

$$\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}(M(j\omega)) \leq 1 \Longleftrightarrow \begin{cases} \|M_{11}\|_\infty \leq 1 \quad \textit{and} \\ \sup\limits_{\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}} \|F_u(M, \Delta_1)\|_\infty \leq 1 \end{cases}$$

Proof This is precisely the Main-Loop Theorem with $\sup\limits_{\omega\in\mathbb{R}}$ included and specialized to a two full block case (see Theorem 11.7 of [44]). □

The following lemma gives an equivalent μ condition to estimate the least upper bound of the upper LFT of a constant real matrix with a contractive real matrix. This lemma will be used in the corollaries of the main results in the next section to quantify the largest family of SNI uncertainties for which the robust performance of the closed-loop system is guaranteed.

Lemma 4.3 *Given $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathbb{R}^{(p+m)\times(q+m)}$ such that $\bar{\sigma}(Q_{11}) \leq 1$. Suppose two real block sets $\mathbf{\Omega_1}$ and $\mathbf{\Omega_2}$ are defined as*

$$\mathbf{\Omega_1} = \left\{\bar{\Delta}_1 \in \mathbb{R}^{q\times p} : \ \bar{\sigma}(\bar{\Delta}_1) < 1\right\}, \quad \mathbf{\Omega_2} = \left\{\bar{\Delta}_2 \in \mathbb{R}^{m\times m} : \ \bar{\sigma}(\bar{\Delta}_2) < 1\right\}.$$

Then,

$$\inf\left\{\beta > 0 : \ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0 \\ 0 & \frac{1}{\beta}I \end{pmatrix} Q\right) \leq 1\right\} = \sup_{\bar{\Delta}_1 \in \mathbf{\Omega_1}} \bar{\sigma}(F_u(Q, \bar{\Delta}_1)).$$

Proof This is a consequence of applying Lemma 4.2 specialized to a real matrix and the frequency independent case (at zero frequency) with two real full-block structured uncertainty. □

The least upper bound of $\bar{\sigma}(F_u(Q, \bar{\Delta}_1))$ $\forall \bar{\Delta}_1 \in \mathbb{R}^{q\times p}$ satisfying $\bar{\sigma}(\bar{\Delta}_1) < 1$ can be estimated via numerical methods. However, the computational complexity increases as the dimension of $\bar{\Delta}_1$, i.e., $q \times p$ increases. The above lemma gives an analytical method so that real μ can be computed as a reasonably tight upper bound using real structured singular value techniques [76, 77].

The following lemma gives a necessary and sufficient condition for robust stability of a perturbed system with two full-block structured uncertainty. This lemma will be used in subsequent sections to derive robust performance analysis results for systems with SNI uncertainty by equivalently formulating the robust stability analysis results for systems with mixed perturbations of SBR and SNI uncertainties.

Lemma 4.4 *Given* $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \in \mathcal{RH}_\infty^{(p+m)\times(q+m)}$ *and* $\Delta_2 \in \mathcal{RH}_\infty^{m\times m}$. *Then,* $[\Delta_2, F_u(N, 0)]$ *is internally stable and* $\|F_\ell(N, \Delta_2)\|_\infty \le 1$ *if and only if*

$$\left[\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, N\right] \quad \textit{is internally stable } \forall \Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}.$$

Proof This is a transformation of the robust performance problem for system N with perturbation Δ_2 to a robust stability problem by introducing a fictional SBR uncertainty Δ_1 as in [44] without restricting the norm of the uncertainty Δ_2. □

Lemma 4.4, together with the stability analysis result of NI systems in Theorem 2.6, play a central role in establishing the robust performance analysis results for uncertain NI systems presented in the following section.

4.4 Robust Performance Analysis

In this section, the robust performance analysis problem for uncertain NI systems is equivalently cast into a specific μ analysis framework. Before the main results are stated, the following definition of internal stability for an LFT interconnection is needed.

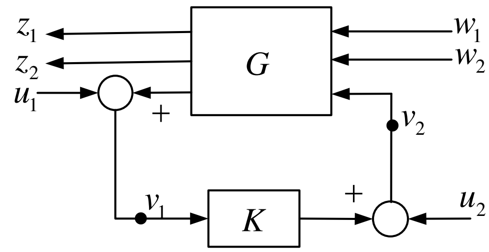

Figure 4.2: An LFT interconnection

Definition 4.3 *Let $\{G, K\}$ be the LFT feedback interconnection shown in Figure 4.2 of transfer function matrices $G(s)$ and $K(s)$, and correspondingly, let $T_f(G,K)$ be the transfer function matrix from $\begin{pmatrix} w_1^T & w_2^T & \begin{bmatrix} u_1^T & u_2^T \end{bmatrix} \end{pmatrix}^T$ to $\begin{pmatrix} z_1^T & z_2^T & \begin{bmatrix} v_1^T & v_2^T \end{bmatrix} \end{pmatrix}^T$. We say $\{G, K\}$ is internally stable when $T_f(G,K) \in \mathcal{RH}_\infty$.*

For a given controller that internally stabilizes an LFT closed-loop system in the presence of SNI uncertainty, the achieved performance can be quantified by solving the following proposed analysis problem. The following theorem gives an equivalent condition for the LFT interconnection shown in Figure 4.3 to be NI, from the signal vector w_2 to the output signal vector z_2, when the other two loops are closed with a given controller K and with a fictional SBR uncertainty Δ_1. It is emphasized that much of the engineering significance of the result lies with the corollaries to the following theorem statement, even though the theorem is written in the strongest form.

Theorem 4.1 *Given a controller $K \in \mathcal{R}^{r\times l}$ and a generalized plant Σ as in (4.1). Suppose $\det(A) \neq 0$, $D_{22} = 0$, $D_{33} = 0$, $D_{21} = 0$, $D_{23} = 0$, (A, B_3) is stabilizable and (C_3, A) is detectable. Then, $\{\Sigma, K\}$ is internally stable, $F_\ell(F_u(\Sigma, \Delta_1), K) \in \mathcal{I}$ for all $\Delta_1 \in \mathcal{B}^\circ\boldsymbol{\Delta}$ and $\|F_\ell(F_\ell(\Sigma, K), -sI)\|_\infty \leq 1$ if and only if $\{G, K\}$ is internally stable, $\sup_{\omega\in\mathbb{R}} \mu_{\boldsymbol{\Delta}_{\mathbf{TOT}}}[F_\ell(G,K)] \leq 1$ and $\det(I + F_u(F_\ell(G,K), \Delta_1)(j\omega)) \neq$*

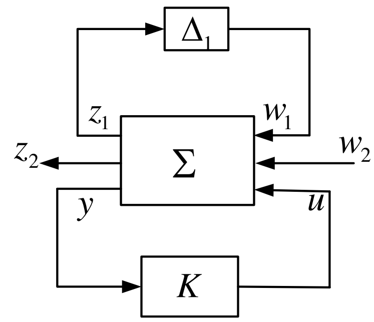

Figure 4.3: An LFT interconnection

0 *for all* $\omega \in \mathbb{R} \cup \{\infty\}$ *and* $\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}$, *where*

$$G = \left[\begin{array}{c|ccc} V^{-1}A & V^{-1}B_1 & \sqrt{2}B_2U^{-1} & V^{-1}B_3 \\ \hline C_1 - D_{12}U^{-1}C_2A & D_{11} - D_{12}U^{-1}C_2B_1 & \sqrt{2}D_{12}U^{-1} & D_{13} - D_{12}U^{-1}C_2B_3 \\ -\sqrt{2}U^{-1}C_2A & -\sqrt{2}U^{-1}C_2B_1 & (I - C_2B_2)U^{-1} & -\sqrt{2}U^{-1}C_2B_3 \\ C_3 - D_{32}U^{-1}C_2A & D_{31} - D_{32}U^{-1}C_2B_1 & \sqrt{2}D_{32}U^{-1} & -D_{32}U^{-1}C_2B_3 \end{array}\right],$$

$$U = I + C_2B_2 \quad and \quad V = I + B_2C_2. \tag{4.3}$$

Proof See Appendix A.1 for proof. □

Four remarks are appropriate on Theorem 4.1 to be given at this stage.

Remark 4.1 *The assumption that* $D_{33} = 0$ *is made without loss of generality as if it were non-zero, it could always be loopshifted to the controller* K*. Also, the condition* $D_{22} = 0$ *could easily be replaced by* $D_{22} = D_{22}^* \neq 0$ *with appropriate minor modifications in the theorem statement. Finally, the condition* $D_{21} = 0$ *and* $D_{23} = 0$ *could be replaced by* $D_{12} = 0$ *and* $D_{32} = 0$ *as this would be a dual generalized plant.*

Remark 4.2 *The assumption that* $\det(A) \neq 0$ *is imposed for mathematical convenience to prove the stability equivalence between* $F_\ell(F_u(\Sigma, \Delta_1), K)$ *and*

$F_\ell(F_u(\hat{\Sigma}, \Delta_1), K)$. *This assumption can be replaced by*

$$\left\| F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right], K) \right\|_\infty = \|F_\ell(F_\ell(\Sigma, K), 0)\|_\infty \leq 1,$$

which can be interpreted as nominal performance and the theorem statement still holds. This latter assumption is used instead of $\det(A) \neq 0$ *in the subsequent corollaries.*

Remark 4.3 *The zero D-term assumptions of the generalized plant in the suppositions of the theorem statement guarantee properness of* $F_\ell(F_\ell(\Sigma, K), -sI)$. *Noting that* $F_\ell(F_\ell(\Sigma, K), -sI) = F_\ell(F_\ell(G, K), 0)$, *it is easy to see that the assumption* $\|F_\ell(F_\ell(\Sigma, K), -sI)\|_\infty \leq 1$ *can be interpreted as a nominal performance property of the transformed interconnection* $\{G, K\}$. *This condition is part of the structured singular value condition* $\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] \leq 1$ *as seen in Lemma 4.2. It can be dropped if we are happy to replace* $\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] \leq 1$ *with* $\sup_{\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}} \|F_u(F_\ell(G, K), \Delta_1)\|_\infty \leq 1$.

Remark 4.4 *For all* $\omega \in \mathbb{R}$ *such that* $\mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] < 1$, *the condition* $\det(I + F_u(F_\ell(G, K), \Delta_1)(j\omega)) \neq 0$ *is automatically fulfilled for* $\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}$. *Consequently, this determinant condition needs to be checked only at the frequencies where* $\mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] = 1$.

More practically useful robust stability and robust performance results can be obtained as an immediate consequence of the main theorem as follows.

Corollary 4.2 *(Robust Stability I) Given the suppositions of Theorem 4.1 except* $\det(A) \neq 0$, $\gamma > 0$, *and* G *is as given in (4.3) such that*

$$\left\| F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right], K) \right\|_\infty \leq 1,$$ $\{G, K\}$ *is internally stable,* $\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] \leq 1$

and $\det(I+F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ *for all* $\omega \in \mathbb{R}\cup\{\infty\}$ *and* $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. *Then,* $\left[\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, F_\ell(\Sigma,K)\right]$ *is internally stable for all* $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ *and* $\Delta_2 \in \mathcal{I}_s$ *satisfying* $\Delta_2(\infty) \geq 0$ *and* $\bar{\lambda}(\Delta_2(0)) < \gamma (\leq \gamma)$ *if and only if*

$$\inf\left\{\beta > 0:\ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0 \\ 0 & \frac{1}{\beta}I \end{pmatrix} F_\ell(\Sigma,K)(0)\right) \leq 1\right\} \leq \frac{1}{\gamma}(< \frac{1}{\gamma}). \tag{4.4}$$

Proof $(\Longleftarrow)$ Let $F_\ell(\Sigma,K) = \left[\begin{array}{cc} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{array}\right]$, and note that

$$\|\bar{N}_{11}\|_\infty = \left\| F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right], K)\right\|_\infty \leq 1.$$

Then, from Theorem 2.2, we have $(I - \bar{N}_{11}\Delta_1)^{-1} \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Hence, $\det(I - \bar{N}_{11}(s_0)\Delta_1(s_0)) \neq 0$ for all $s_0 \in \bar{\mathbb{C}}_+$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Now, since $\{G,K\}$ is internally stable, $\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] \leq 1$ and for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R}\cup\{\infty\}$, it follows that $F_u(F_\ell(\Sigma,K),\Delta_1) \in \mathcal{I}$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ via Theorem 4.1 and Remark 4.2. Note that $\bar{N}_{21}(\infty) = 0$ and $\bar{N}_{22}(\infty) = 0$, it follows that $F_u(F_\ell(\Sigma,K),\Delta_1)(\infty) = 0$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Using Lemma 2.5, it follows that $F_u(F_\ell(\Sigma,K),\Delta_1)(0) \geq F_u(F_\ell(\Sigma,K),\Delta_1)(\infty) = 0$. Hence, $\bar{\lambda}(F_u(F_\ell(\Sigma,K),\Delta_1)(0)) = \bar{\sigma}(F_u(F_\ell(\Sigma,K),\Delta_1)(0))$ for all $\Delta_1(0) \in \mathbb{R}^{q\times p}$ satisfying $\bar{\sigma}(\Delta_1(0)) < 1$. Furthermore, since

$$\inf\left\{\beta > 0:\ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0 \\ 0 & \frac{1}{\beta}I \end{pmatrix} F_\ell(\Sigma,K)(0)\right) \leq 1\right\} \leq \frac{1}{\gamma}(< \frac{1}{\gamma}),$$

it follows from Lemma 4.3 that $\bar{\lambda}(F_u(F_\ell(\Sigma,K)(0),\Delta_1)(0)) = \bar{\sigma}(F_u(F_\ell(\Sigma,K),\Delta_1)(0)) \leq \frac{1}{\gamma}(< \frac{1}{\gamma})$ for all $\Delta_1(0) \in \mathbb{R}^{q\times p}$ satisfying $\bar{\sigma}(\Delta_1(0)) < 1$. Also, since $\Delta_2 \in \mathcal{I}_s$ satisfies

$\Delta_2(\infty) \geq 0$ and $\bar{\lambda}(\Delta_2(0)) < \gamma(\leq \gamma)$, it follows that $\bar{\lambda}(F_u(F_\ell(\Sigma, K), \Delta_1)(0)\Delta_2(0)) < 1$. Consequently, it follows from Theorem 2.6 that
$\det(I - F_u(F_\ell(\Sigma, K), \Delta_1)(s_0)\Delta_2(s_0)) \neq 0$ for all $s_0 \in \bar{\mathbb{C}}_+$, $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ and $\Delta_2 \in \mathcal{I}_s$ satisfying $\Delta_2(\infty) \geq 0$, and $\bar{\lambda}(\Delta_2(0)) < \gamma(\leq \gamma)$. Hence,

$$\begin{aligned} & \det\left[I - F_\ell(\Sigma, K)(s_0)\begin{pmatrix} \Delta_1(s_0) & 0 \\ 0 & \Delta_2(s_0) \end{pmatrix}\right] \\ = \; & \det(I - \bar{N}_{11}(s_0)\Delta_1(s_0)) \times \det(I - F_u(F_\ell(\Sigma, K), \Delta_1)(s_0)\Delta_2(s_0)) \\ \neq \; & 0 \;\forall s_0 \in \bar{\mathbb{C}}_+ \;\forall \Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta} \;\forall \Delta_2 \in \mathcal{I}_s \text{ satisfying } \Delta_2(\infty) \geq 0 \\ & \quad \text{and } \bar{\lambda}(\Delta_2(0)) < \gamma(\leq \gamma). \end{aligned}$$

Also, note that $F_\ell(\Sigma, K) \in \mathcal{RH}_\infty$ and $\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \in \mathcal{RH}_\infty$, it follows from Theorem 5.7 of [44] that the closed-loop system $\left[\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, F_\ell(\Sigma, K)\right]$ is internally stable.

($\Longrightarrow$) This can be proved via a contra-positive argument on choosing $\Delta_1 = 0$ and $\Delta_2 = \frac{1/\bar{\lambda}(\bar{N}_{22}(0))}{s+1} I$ as the destabilizing $\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}$. □

The following corollary is an immediate consequence of Corollary 4.2. It is not a restatement of Corollary 4.2, but a different version of the robust stability result, where the real μ condition is used to quantify the largest family of perturbations that are a mixture of BR and SNI uncertainties for which robust stability of the perturbed closed-loop system is guaranteed.

Corollary 4.3 *(Robust Stability II) Given the suppositions of Corollary 4.2 except $\gamma > 0$. Then, $\left[\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, F_\ell(\Sigma, K)\right]$ is internally stable for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$*

and $\Delta_2 \in \mathcal{I}_s$ *satisfying* $\Delta_2(\infty) \geq 0$ *if and only if*

$$\bar{\lambda}(\Delta_2(0)) < \frac{1}{\inf\left\{\beta > 0: \ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0 \\ 0 & \frac{1}{\beta}I \end{pmatrix} F_\ell(\Sigma, K)(0)\right) \leq 1\right\}}. \quad (4.5)$$

Proof This result is a straightforward consequence of Corollary 4.2 obtained by setting

$$\frac{1}{\gamma} = \inf\left\{\beta > 0: \ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0 \\ 0 & \frac{1}{\beta}I \end{pmatrix} F_\ell(\Sigma, K)(0)\right) \leq 1\right\}$$

for sufficiency and $\gamma = \bar{\lambda}(\Delta_2(0))$ for necessity. □

Corollary 4.4 *(Robust Performance I) Given the suppositions of Theorem 4.1 except* $\det(A) \neq 0$, $\gamma > 0$, *and* G *is as given in (4.3) such that* $\left\| F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right], K) \right\|_\infty \leq 1$, $\{G, K\}$ *is internally stable,* $\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] \leq 1$ *and* $\det(I + F_u(F_\ell(G, K), \Delta_1)(j\omega)) \neq 0$ *for all* $\omega \in \mathbb{R} \cup \{\infty\}$ *and* $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. *Then,* $[\Delta_2, F_u(F_\ell(\Sigma, K), 0)]$ *is internally stable and* $\|F_\ell(F_\ell(\Sigma, K), \Delta_2)\|_\infty \leq 1$ *for all* $\Delta_2 \in \mathcal{I}_s$ *satisfying* $\Delta_2(\infty) \geq 0$ *and* $\bar{\lambda}(\Delta_2(0)) < \gamma (\leq \gamma)$ *if and only if the condition in (4.4) is satisfied.*

Proof This result is straightforward to obtain by combining Corollary 4.2 and Lemma 4.4. □

The following corollary is an immediate consequence of Corollary 4.4. It can be used to quantify the largest family of SNI perturbations in terms of a DC loop gain condition for which robust performance of the perturbed closed-loop system is guaranteed.

Corollary 4.5 *(Robust Performance II) Given the suppositions of Corollary 4.4 except* $\gamma > 0$*. Then,* $[\Delta_2, F_u(F_\ell(\Sigma, K), 0)]$ *is internally stable and* $\|F_\ell(F_\ell(\Sigma, K), \Delta_2)\|_\infty \leq 1$ *for all* $\Delta_2 \in \mathcal{I}_s$ *satisfying* $\Delta_2(\infty) \geq 0$ *if and only if the condition in (4.5) is satisfied.*

Proof Similar to the proof of Corollary 4.3. □

Remark 4.5 *Note that,* $\left\| F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right], K) \right\|_\infty \leq 1$ *implies that the nominal performance of the system as structured in Figure 4.1 is satisfied; i.e., the infinity norm of the transfer function matrix from* w_1 *to* z_1 *is less than one when the physical SNI uncertainty* $\Delta_2 = 0$*.*

4.5 Numerical Example

The robust performance analysis results discussed in the previous section for uncertain NI systems will now be illustrated via a numerical example. Here, an uncertain lightly damped plant model is considered as follows:

$$
\begin{aligned}
P_\Delta(s) &:= p(s)\delta(s) \\
&\times \begin{bmatrix} 0.5(1-\alpha)s + 0.5(1-k) & -s^2 - 0.5(\alpha+1)s - 0.5(1+k) \\ -s^2 - 0.5(\alpha+1)s - 0.5(1+k) & (1-\alpha)s + (1-k) \end{bmatrix},
\end{aligned}
$$

where α and k are two unknown real parameters, $p(s) = \frac{1}{s^2+s+1}$ and $\delta(s) = \frac{1}{s^2+\alpha s+k}$.

The uncertain plant is expressed in additive uncertainty structure $P_\Delta(s) = P(s) + \Delta(s)$, the nominal plant $P(s)$ and the uncertainty $\Delta(s)$ are respectively given by:

$$
P(s) = \Psi \text{diag}(-0.5p(s), -0.5p(s))\Psi^*, \;\; \Delta(s) = \Psi^{-1}\text{diag}(\delta(s), 0)\Psi^{-*}, \tag{4.6}
$$

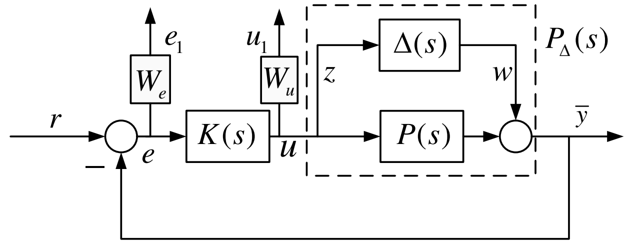

Figure 4.4: Controlled closed-loop system

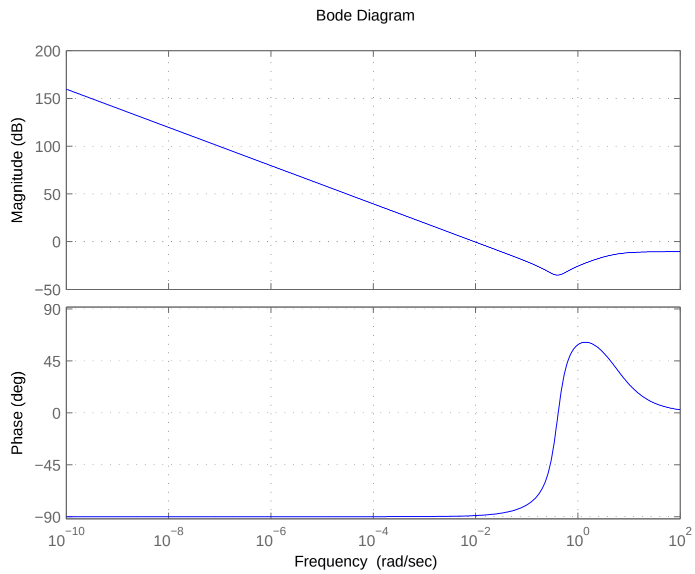

Figure 4.5: Bode plot of w_e

where $\Psi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Note that for all $\alpha > 0$ and $k > 0$, the uncertainty $\Delta(s) \in \mathcal{I}_s$. The associated closed-loop system is shown in Figure 4.4 along with the weighting functions. These two weights, $W_e(s)$ and $W_u(s)$ are, respectively, chosen to cost the error and control input signals to achieve a desired level of performance. To achieve tracking, as well as to attenuate the effects of disturbances at the output, the magnitude of the sensitivity function has to be small in a low frequency region. To achieve this goal, the weighting transfer function matrix $W_e(s) = w_e(s)I_2$ is chosen, where $w_e(s) = \frac{0.3(s^2+0.3s+0.16)}{s(s+5)}$, and the Bode plot of $w_e(s)$ is shown in Figure 4.5.

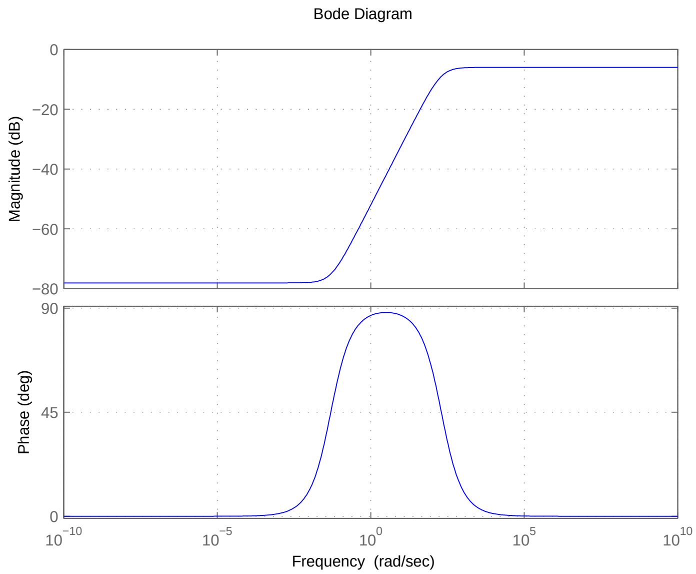

Figure 4.6: Bode plot of w_u

On the other hand, the control input is expected to be low at high frequency. To reduce the control action at high frequencies and to cost excessive control action at low frequencies, the weighting transfer function matrix $W_u(s) = w_u(s)I_2$ is selected, where $w_u(s) = \frac{0.5(s+0.05)}{s+200}$. The Bode plot of $w_u(s)$ is shown in Figure 4.6. The closed-loop control structure in Figure 4.4, has now been rearranged in a standard linear fractional form as depicted in Figure 4.1, where

$$\begin{aligned} \Sigma &= \begin{bmatrix} W_e & -W_e & -W_eP \\ 0 & 0 & W_u \\ 0 & 0 & I \\ I & -I & -P \end{bmatrix}, \ \Delta_2 = \Delta, \\ z_1 &= \begin{pmatrix} e_1 \\ u_1 \end{pmatrix}, \ w_1 = r, \ z_2 = z, \ w_2 = w, \ y = e, \end{aligned} \tag{4.7}$$

and $\Delta_1 \in \mathcal{B}^\circ\boldsymbol{\Delta}$ is a fictional BR contractive uncertainty considered for robust performance analysis.

The nominal plant $P(s) \in \mathcal{RH}_\infty$ and the following controller

$$K(s) = \Psi^{-*}\text{diag}(\frac{-2(s^2+s+1)}{s(s^2+2s+2)}, \frac{-2(s^2+s+1)}{s(s^2+2s+2)})\Psi^{-1}$$

achieves internal stability of the nominal closed-loop system since with this controller, $(I+P(s)K(s))^{-1} = \frac{s(s^2+2s+2)}{s^3+2s^2+2s+1}I$ and $K(s)(I+P(s)K(s))^{-1} = \frac{2}{s+1}\Psi^{-*}\Psi^{-1}$. Furthermore, note that $\|F_\ell(F_\ell(\Sigma, K), 0)\|_\infty = \left\| \begin{matrix} W_e(I+PK)^{-1} \\ W_uK(I+PK)^{-1} \end{matrix} \right\|_\infty = 0.3 \leq 1$ indicates the fulfillment of the nominal performance condition of the closed-loop system assumed in the main theorem and corollaries. Now, we analyze how much robust performance can be achieved by this controller in the presence of SNI uncertainty $\Delta(s)$ perturbing the feedback interconnection.

We know that the uncertainty in the system satisfies $\Delta(s) \in \mathcal{I}_s$ for all $\alpha > 0$ and $k > 0$, and also fulfills $\Delta(\infty) = 0$. Now, we use Corollary 4.4 given in Section 4.4 to analyze the robust performance problem for the closed-loop system with the selected weights $W_e(s)$ and $W_u(s)$. Transforming Σ as indicated in (4.7) into its state-space form and comparing it with (4.1), we notice that a non-zero D_{23} is obtained. This, however, does not satisfy the assumptions of Corollary 4.4. To fulfill these assumptions, Lemma 3.7 is applied that adopts a simple trick to transform the controller and the generalized plant, respectively, as $\breve{K}(s) = (\frac{s}{\tau}+1)K(s)$ and $\breve{\Sigma} = \begin{bmatrix} W_e & -W_e & -\frac{1}{s/\tau+1}W_eP \\ 0 & 0 & \frac{1}{s/\tau+1}W_u \\ 0 & 0 & \frac{1}{s/\tau+1}I \\ I & -I & -\frac{1}{s/\tau+1}P \end{bmatrix}$ exploiting the fact that $K(s)$ is strictly proper, where the value τ is arbitrarily set to 1. This then yields a generalized plant $\breve{\Sigma}$ and a controller $\breve{K}$ that satisfy the assumptions of Corollary 4.4. Subsequently, a transformed plant G is obtained from $\breve{\Sigma}$ as given in (4.3). Note that, $\breve{K}$ internally stabilizes G as the poles of $T(G, \breve{K})$ are at: $-0.5 \pm j\ 0.8660$ (2), -200 (2), -5 (2), -0.5669 (8), -0.1664 (8),

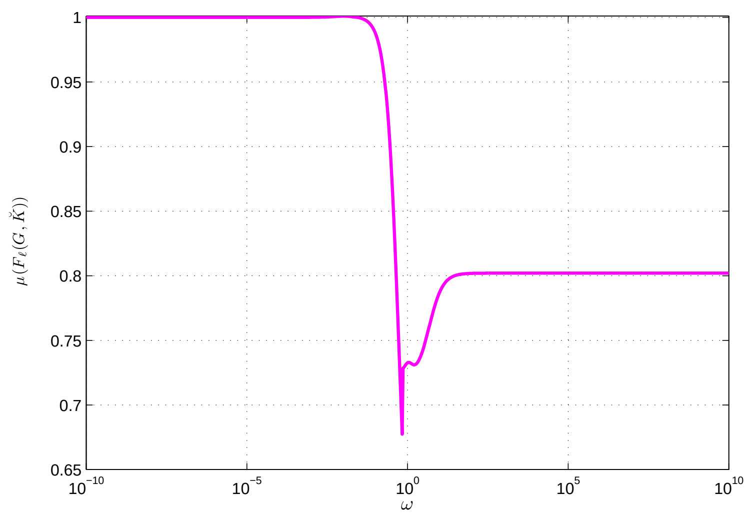

Figure 4.7: Structured singular value plot of $F_\ell(G, \breve{K})$

where the number mentioned in the bracket indicates the number of repeated poles. Then via computation we have $\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, \breve{K})(j\omega)] \leq 1$, where $\mathbf{\Delta_{TOT}} = \left\{ \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta \end{pmatrix} : \Delta_1 \in \mathbb{C}^{2\times 4},\ \Delta \in \mathbb{C}^{2\times 2} \right\}$, and the structured singular value plot of $F_\ell(G, \breve{K})$ is shown in Figure 4.7. Interestingly at $\omega = 0$, $\mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, \breve{K})] = 1$ and $\mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, \breve{K})(j\omega)] < 1$ for $\omega \neq 0$. Hence, to check whether $\det(I + F_u(F_\ell(G, \breve{K}), \Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, we only need to check the value of $F_u(F_\ell(G, \breve{K}), \Delta_1)(j\omega)$ for all $\Delta_1(0)$ satisfying $\bar{\sigma}(\Delta_1(0)) < 1$ at $\omega = 0$. Towards this end, for simplicity let us define $F_\ell(G, \breve{K}) = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}$. Now, we have $\bar{M}_{11}(0) = \begin{pmatrix} W_e(I + PK)^{-1} \\ W_u K(I + PK)^{-1} \end{pmatrix}(0)$, $\bar{M}_{21}(0) = 0$ as $\bar{M}_{21} = \sqrt{2}s\ K(I + PK - sK)^{-1}$, and $\bar{M}_{22}(0) = I$ as $\bar{M}_{22} = (I + PK + sK)(I + PK - sK)^{-1}$. After some calculations, we get $\det(I + F_u(F_\ell(G, \breve{K}), \Delta_1)(0)) = 4$ for all $\Delta_1(0) \in \mathbb{R}^{2\times 4}$ satisfying $\bar{\sigma}(\Delta_1(0)) < 1$. Hence, the assumption that $\det(I + F_u(F_\ell(G, \breve{K}), \Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ is satisfied.

Now, using Lemma 4.3, we get $\bar{\lambda}(F_u(F_\ell(\breve{\Sigma}, \breve{K}), \Delta_1)(0)) < 5.34$ for all $\Delta_1(0) \in \mathbb{R}^{2\times 4}$ satisfying $\bar{\sigma}(\Delta_1(0)) < 1$. It is easy to see that the condition $\bar{\lambda}(\Delta(0)) \leq$

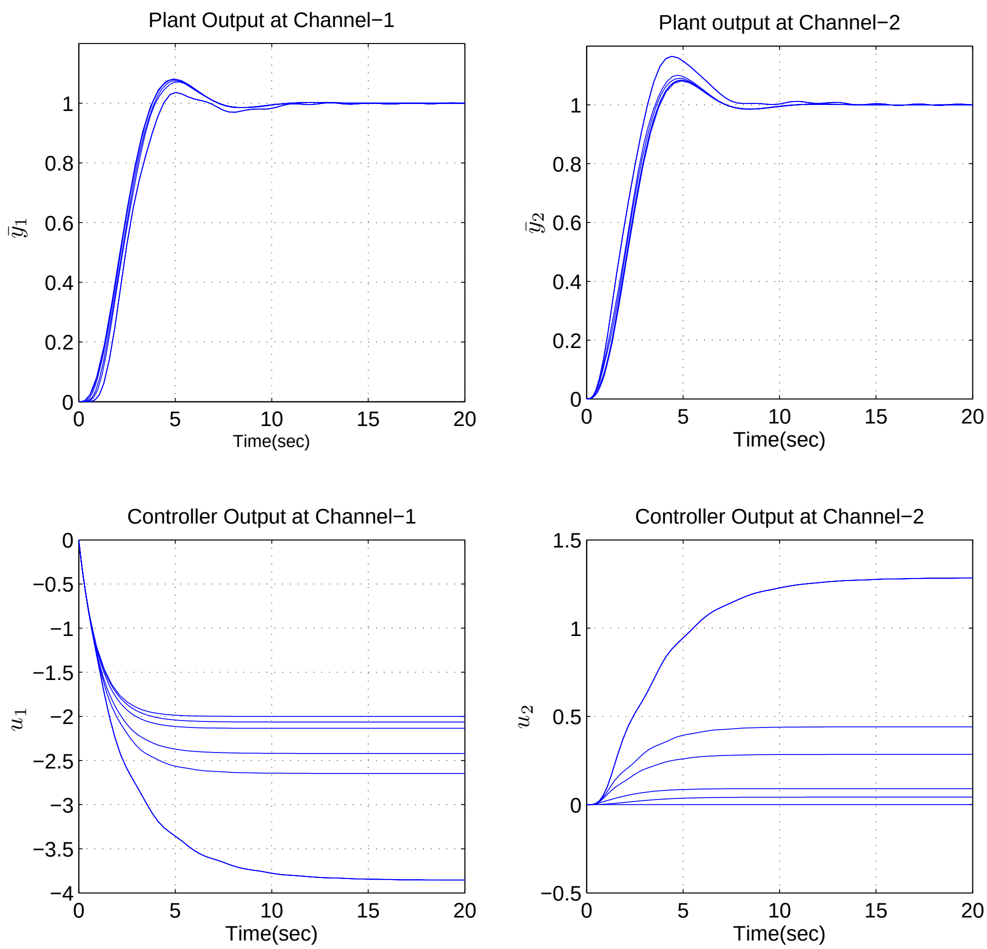

Figure 4.8: Simulation results for the closed-loop system with unit step reference command

$\frac{1}{5.34}$ is satisfied when $k \geq 6.99$ and $\alpha > 0$. Note that for this range of values, $\Delta(s) \in \mathcal{I}_s$ and $\Delta(\infty) \geq 0$. Hence, using Corollary 4.4, $\|F_\ell(F_\ell(\Sigma, K), \Delta)\|_\infty = \|F_\ell(F_\ell(\breve{\Sigma}, \breve{K}), \Delta)\|_\infty \leq 1$ for all $\alpha > 0$ and $k \geq 6.99$, which means that the given controller achieves a certain level of robust performance in the presence of SNI uncertainty in the form of (4.6) for any $\alpha > 0$ and $k \geq 6.99$. The step responses at the plant and controller outputs, respectively, $\bar{y} = [\bar{y}_1\ \bar{y}_2]^T$ and $u = [u_1\ u_2]^T$ are depicted in Figure 4.8, where the simulation results are shown for the nominal system along with five randomly chosen uncertainties.

Note that Corollary 4.4 and Corollary 4.5 indeed give a stronger result as they state that the robust performance demanded by the weighting functions

$W_e(s)$ and $W_u(s)$ in the feedback interconnection in Figure 4.4 is satisfied for all $\Delta \in \mathcal{I}_s$ (not just the form in (4.6)) satisfying $\Delta(\infty) \geq 0$ and $\bar{\lambda}(\Delta(0)) \leq \frac{1}{5.34}$. This powerful statement characterizes a large class of systems for which a level of robust performance of the closed-loop system is guaranteed.

4.6 Conclusions

This chapter considerably extends the robust stabilization reformulation technique in Chapter 3 to a generalized framework for analyzing the robust performance problem for uncertain NI systems. To characterize the robust performance, conditions are derived in the μ framework. It has been shown that a structured singular valued condition on a modified input-output map equivalently gives a quantitative performance test for systems with SNI uncertainty. This framework also quantifies the largest family of SNI perturbations for which robust performance of the perturbed closed-loop system is guaranteed in terms of a DC loop gain condition. This proposed result can be also applied to analyze robust stability when the perturbations are mixture of BR and NI uncertainties, this is different from the early works which involve uncertainties of the same type (size) that typically in robust performance and stability problems. A numerical example is demonstrated to show the usefulness of the proposed robust performance analytical method. This proposed framework will underpin future developments in controller synthesis to achieve a guaranteed robust performance level for uncertain NI systems.

Chapter 5

Stabilization of Uncertain Negative-Imaginary Systems via State-Feedback

5.1 Introduction

This chapter is concerned with the robust controller synthesis for uncertain NI systems when full state feedback is available. SNI uncertainties arise in many practical scenarios, for example, in unmodeled spill-over dynamics of lightly damped structures. There are also some uncertain systems that can equivalently be presented into systems with the uncertain part being SNI. Based on negative-imaginary stability theorem (Theorem 2.6), if a controller is synthesized such that a specified closed-loop transfer function is NI and satisfies the DC loop gain condition, then the interconnection of this closed-loop nominal system with SNI uncertainties is robustly stable. In Chapter 3, the problem of synthesizing a stabilizing controller that ensures an NI property of an LFT closed-loop system has been transformed into an equivalent $\mathcal{H}_\infty$ synthesis problem.

The original definition of NI systems in [11, 12] (Definition 2.20) considers asymptotically stable systems, and the previous chapters followed this definition.

In [13, 78], the definition of NI systems is further generalized to allow systems to have poles on the imaginary axis except at the origin. Subsequently, [13, 78] gives robust stability analysis results as Theorem 2.6 for the generalized NI systems. The generalized results for NI systems capture the largest known set of stabilizing systems in the presence of SNI uncertainty for which the robust stability holds. This concept is explored in the present chapter to design a controller such that a given LFT closed-loop system is generalized NI. The DC loop condition required for robust stability is equivalently cast into a simple LMI condition. Due to generalized definition of NI systems, note that the nominal closed-loop system may be marginally stable, however the uncertain system with SNI uncertainty is robustly asymptotically stable. Hence, the result in this chapter captures the largest known set of stabilizing state-feedback controllers in the presence of SNI uncertainty. Similar technique is employed in [24].

This chapter provides sufficient conditions for static state-feedback robust controller synthesis for systems in the presence of SNI uncertainty. The results are derived based on the Generalized Negative-Imaginary Lemma and generalized properties of NI systems. Note that the NI lemma in [12, 78] and [13] needs a minimality assumption for the state-space realization (A, B, C, D) of the system. In controller synthesis, this minimality assumption cannot be computed 'a priori' for the synthesized closed-loop system to be NI. However, without the minimality assumption, the conditions of the NI lemma are sufficient to guarantee the NI property of the system. In this chapter, a Generalized Negative-Imaginary Lemma is proposed with a relaxed assumption that (A, B, C, D) has no observable uncontrollable modes to retrieve the necessary condition. Although in literature, the controller synthesis methods with necessary and sufficient conditions for SPR system are available, these results cannot be used in the present synthesis framework as a strict or non-strict NI is always transformed into a non-strict PR system (as pointed out in Chapter 3 that there is always a blocking zero at the origin).

Furthermore, the non-strict KYP lemma [79], which seems to be useful in this regard, needs a controllability assumption to retrieve the necessary condition.

In practical scenario, the uncertainties in NI systems may not always satisfy the SNI property, but they belong to the NI class. An example can be seen in [12]. Since the robust stability result of interconnected NI systems requires at least one of the system to be SNI, in this situation, it is desirable to enforce the nominal closed-loop system to be SNI to handle NI uncertainties. Hence, in both analysis and synthesis frameworks related to NI systems, ensuring an SNI property is essential. Thus, this chapter also provides a state-space characterization for checking the SNI property of systems. An alternate characterization proposed in [13, 22] is referred to as the "Weakly Strict Negative-imaginary (WSNI) Lemma" as it is derived via an underpinning weakly strict positive-real (WSPR) property of the system [80]. The WSNI lemma in [13, 22] is difficult to apply for NI controller synthesis, for example, as it requires a minimality assumption and a non-convex rank condition to be fulfilled on a punctured $j\omega$-axis. However, in this chapter these difficulties have been circumvented. The proposed SNI lemma is referred to as the "Strongly Strict Negative-imaginary (SSNI) lemma" as it is developed via an underpinning strongly strict positive-real (SSPR) result [81]. This lemma facilitates controller synthesis for systems with non-strict NI uncertainty.

The rest of this chapter is organized as follows: Section 5.2 provides some background material which is required to establish main results of this chapter. This is followed by the Generalized Negative-Imaginary Lemma presented in Section 5.3 and by the LMI approach for static state-feedback controller synthesis method for systems with SNI uncertainties presented in Section 5.4. Then, the Strongly Strict Negative-Imaginary Lemma is proposed in Section 5.5. In Section 5.6, two illustrative examples are given to demonstrate the usefulness of the proposed results. Finally, Section 5.7 concludes the chapter.

5.2 Preliminaries

First, we recall the new generalized definition of NI systems.

Definition 5.1 *[13] The real rational proper transfer function matrix* $R(s) \in \mathcal{R}^{m\times m}$ *is said to be generalized NI if*

1) $R(\infty) = R^T(\infty)$*;*

2) $R(s)$ *has no poles at the origin and in* $\mathbf{Re}[s] > 0$*;*

3) $j[R(j\omega) - R(j\omega)^*] \geq 0$ *for all* $\omega \in (0, \infty)$ *except the values of* ω *where* $j\omega$ *is a pole of* $R(s)$*.*

4) If $j\omega_0$ *is a pole of* $R(s)$*, it is at most a simple pole and the residue matrix* $K_0 := \lim_{s\to j\omega_0}(s - j\omega_0)s(R(s) - R(\infty))$ *is positive-semidefinite Hermitian.*

The symmetry condition at $\omega = \infty$ is necessary for $R(s)$ to be generalized NI [12]. Furthermore, generalized NI transfer function matrices possess the aforementioned frequency response in the open frequency interval between 0 and ∞, where only the imaginary part of the frequency response is considered, and it also possesses additional properties for restricting poles in the closed-loop left half plane excluding the origin. Note that when the transfer function matrix is asymptotically stable, Definition 5.1 coincides with the definition of NI systems in Definition 2.20. As the above generalized definition allows for poles on the imaginary axis except at the origin, it is less restrictive than the definition previously given for NI systems in Definition 2.20. This is an important generalization because many real-life systems contain pure oscillatory poles which can be categorized as generalized NI systems using the above definition. We will follow this generalized definition throughout this chapter.

The next theorem, gives robust stability analysis results as in Theorem 2.6 for the generalized NI systems. This result underpins the controller synthesis method for uncertain NI systems given in this chapter. The theorem is restated as follows:

Theorem 5.1 *[13, 78] Given that $M(s)$ is generalized NI and $N(s)$ is SNI , and suppose that $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then, the positive-feedback interconnection $[M, N]$ is internally stable if and only if*

$$\overline{\lambda}(M(0)N(0)) < 1. \tag{5.1}$$

5.3 Generalized Negative-Imaginary Lemma

The original state-space characterization of NI transfer function matrices (namely the Negative-Imaginary Lemma) is established in Lemma 2.4 and later extended [13,78] to generalized NI systems. However, these results are built on the requirement that the state-space realization is minimal. Motivated by the generalized version of the PR lemma (see for example [82]), where the minimality condition is relaxed, a new version of the NI lemma is presented below in the absence of a minimality assumption. This lemma will be used in subsequent state-feedback design to fulfill the generalized NI property of the synthesized closed-loop system.

Lemma 5.1 *(Generalized Negative Imaginary Lemma) Let (A, B, C, D) be a realization of $R(s) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$ with $\det(A) \neq 0$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $m \leq n$.*

(i) If $D = D^T$ and there exist $X = X^T \geq 0$ such that the following LMI is satisfied:

$$\begin{bmatrix} XA + A^TX & A^TC^T - XB \\ B^TX - CA & -(CB + B^TC^T) \end{bmatrix} \leq 0, \tag{5.2}$$

then $R(s)$ is generalized NI.

(ii) If $R(s)$ is generalized NI, and its state-space realization (A, B, C, D) has no observable uncontrollable modes, then $D = D^T$ and there exist $X = X^T \geq 0$ such that the LMI condition in (5.2) is satisfied.

Proof First, we reformulate equivalently the generalized NI property of $R(s)$ as given in [13] as follows:

$R(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is generalized NI

$\Longleftrightarrow \hat{R}(s) = (R(s) - R(\infty)) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is is generalized NI, $D = D^T$,

$\Longleftrightarrow D = D^T$ and $F(s) = s\hat{R}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is PR (noting $\det(A) \neq 0$) via Definition 5.1 and Definition 2.19.

(i) If there exists a real matrix with $X = X^T \geq 0$ such that the LMI in (5.2) is satisfied, then there exists $N \in \mathbb{R}^{n\times(n+m)} = \begin{bmatrix} L & W \end{bmatrix}$ with $L \in \mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{n\times m}$ such that

$$\begin{bmatrix} XA + A^T X & A^T C^T - XB \\ B^T X - CA & -(CB + B^T C^T) \end{bmatrix} = -N^T N = \begin{bmatrix} -L^T L & -L^T W \\ -W^T L & -W^T W \end{bmatrix} \tag{5.3}$$

is satisfied. Hence, $F(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is PR via Corollary 1 and Corollary 3 of [82]. Also, since A is nonsingular and D is symmetric, it follows from the above equivalence that $R(s)$ is generalized NI.

(ii) Since $R(s)$ is generalized NI and $\det(A) \neq 0$, then we have $D = D^T$ and $F(s) = s\hat{R}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ being PR. Also, since the realization (A, B, C, D) for $R(s)$ has no observable uncontrollable modes, it follows that the realization (A, B, CA, CB) for $F(s)$ has no observable uncontrollable modes by noting that the controllability and observability of (A, B, C, D) and (A, B, CA, CB) are the same when A is nonsingular. Also note that since $F(s) = s\hat{R}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is PR, it follows from Corollary 2 and Corollary 3 of [82] that there exists a real matrix $X = X^T \geq 0$ such that the LMI condition in (5.2) is satisfied, which completes the proof. □

Corollary 5.2 *Let (A, B, C, D) be a state-space realization of $R(s) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $m \leq n$. If $\det(A) \neq 0$, $D = D^T$, and there exist real matrix $Y = Y^T > 0$, such that the following condition is satisfied,*

$$AY + YA^T \leq 0 \quad \text{and} \quad B + AYC^T = 0, \tag{5.4}$$

then $R(s)$ is generalized NI.

Proof This is a consequence of Lemma 5.1 (i) by restricting X to be positive definite, and also via equivalence between (5.3) and (5.4) by setting $Y = X^{-1}$ (see the proof of Lemma 1 in [12] for details). □

Corollary 5.2 states that we can relax the minimality condition in [12, 13, 78] so that the nonsingularity of A and symmetry of D together with the existence of positive definite solution to condition (5.4) are sufficient to guarantee the generalized NI property of $R(s)$. This corollary will be used in the next section to show the synthesized closed-loop system to be generalized NI.

5.4 State-Feedback Static Controller Synthesis

Based on the concept of generalized NI systems given in Section 5.2 and subsequent generalized NI lemma as given in Section 5.3, a robust static state-feedback controller synthesis method is presented for systems with SNI uncertainty.

Consider the uncertain system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x, \\ w &= \Delta(s) z, \end{aligned} \tag{5.5}$$

where x is state of the system, w is the disturbance acting on the system, z is the controlled output vector signal vector of the system and $\Delta(s)$ is the uncertainty transfer function matrix. Assume the full state vector x is available for feedback. Also suppose the plant uncertainty $\Delta(s)$ is SNI and satisfies $\Delta(\infty) \geq 0$, we want to construct a static state-feedback control law $u = Kx$ such that the closed-loop uncertain system is robustly stable. This problem can be formulated in the LFT framework as shown in Figure 5.1.

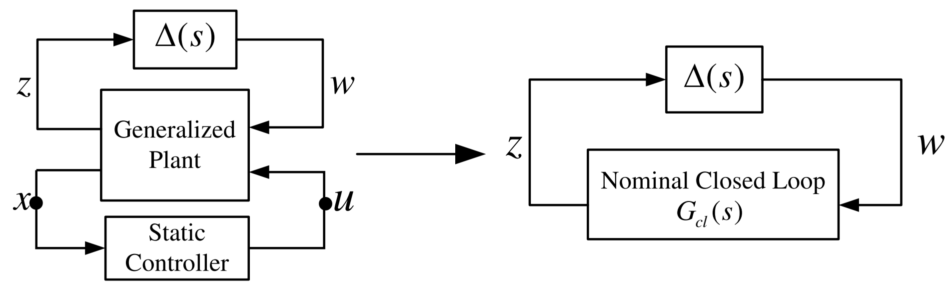

Figure 5.1: A feedback control system where the plant uncertainty $\Delta(s)$ is strictly negative-imaginary, and satisfies $\Delta(\infty) \geq 0$.

Since the uncertainty $\Delta(s)$ is SNI with $\Delta(\infty) \geq 0$, using Theorem 5.1, if the controller K is chosen such that the nominal closed-loop system $G_{cl}(s)$ is strictly proper, generalized NI and satisfies the DC loop condition $\overline{\lambda}(\Delta(0)G_{cl}(0)) < 1$, then the closed-loop is robustly stable. Keeping this idea in mind, we now present a synthesis method for K such that the conditions for robust stability are fulfilled.

If the control law $u = Kx$ is applied to this system, the resulting closed-loop uncertain system is described by the state equations

$$\begin{aligned} \dot{x} &= (A + B_2K)x + B_1w, \\ z &= C_1x, \\ w &= \Delta(s)z. \end{aligned} \tag{5.6}$$

The corresponding nominal closed-loop transfer function matrix is

$$G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1.$$

Note that $G_{cl}(s)$ is strictly proper. Before applying the DC loop condition for robust stability, the generalized NI property of $G_{cl}(s)$ has to be fulfilled; i.e., the transfer function matrix from w to z has to be NI. In this situation, either part (i) in Lemma 5.1 or Corollary 5.2 can be used to show the generalized NI property of the nominal closed-loop system $G_{cl}(s)$. Note that Lemma 5.1 (i) requires the existence of a positive semi-definite solution to the LMI condition in (5.2), while Corollary 5.2 requires the existence of a positive definite solution to the condition (5.4). Although Lemma 5.1 (i) is less conservative than Corollary 5.2, the latter result will be used in this chapter because the DC loop condition $\overline{\lambda}(\Delta(0)G_{cl}(0)) < 1$ can equivalently be formulated as an LMI condition which is shown later.

In order to construct the static state-feedback gain K so that $G_{cl}(s)$ is generalized NI according to Corollary 5.2, let

$$K = \bar{K}Y^{-1},$$

where the arbitrary matrix $Y > 0$ and a matrix $\bar{K}$ are to be determined. Then from Corollary 5.2, the system (5.6) is generalized NI if there exists matrices $\bar{K}$ and $Y > 0$ such that $A + B_2K = A + B_2\bar{K}Y^{-1}$ has no eigenvalues at the origin and

$$\begin{aligned} (A + B_2K)Y + Y(A + B_2K)^T &= AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T \leq 0, \\ B_1 + (A + B_2K)YC_1^T &= B_1 + AYC_1^T + B_2\bar{K}C_1^T = 0. \end{aligned}$$

Using the Schur complement, the above conditions can be equivalently reformulated as

$$\begin{bmatrix} AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T & B_1 + AYC_1^T + B_2\bar{K}C_1^T \\ B_1^T + C_1YA^T + C_1\bar{K}^T B_2^T & 0 \end{bmatrix} \leq 0. \qquad (5.7)$$

The requirement that $A + B_2\bar{K}Y^{-1}$ has no eigenvalues at the origin can be guaranteed if and only if the determinant condition

$$\det(A + B_2\bar{K}Y^{-1}) \neq 0$$

is satisfied. This determinant condition together with (5.7) lead to a set of sufficient conditions for the existence of a static state-feedback control law such that the closed-loop nominal system is generalized NI. The set of sufficient conditions is as follows:

$$\begin{bmatrix} AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T & B_1 + AYC_1^T + B_2\bar{K}C_1^T \\ B_1^T + C_1YA^T + C_1\bar{K}^T B_2^T & 0 \end{bmatrix} \leq 0, \; Y > 0, \quad (5.8)$$

$$\det(AY + B_2\bar{K}) \neq 0. \quad (5.9)$$

If there exists solution to these conditions, then the corresponding state-feedback control law is given by $u = \bar{K}Y^{-1}x$.

In order to use Theorem 5.1 to guarantee the robust stability of closed-loop uncertain system, it is also required that

$$\overline{\lambda}(\Delta(0)G_{cl}(0)) < 1.$$

The following lemma shows that the above DC loop condition can be equivalently reformulated as an LMI condition.

Lemma 5.2 *Suppose there exist matrices $\bar{K}$ and $Y > 0$ such that conditions in (5.8) and (5.9) are satisfied. Also, suppose $\Delta(s)$ is SNI and satisfies $\Delta(\infty) \geq 0$. Then*

$$\overline{\lambda}(\Delta(0)G_{cl}(0)) < 1 \Longleftrightarrow \begin{bmatrix} \Delta(0)^{-1} & C_1Y \\ YC_1^T & Y \end{bmatrix} > 0.$$

Proof

$$
\begin{aligned}
& \overline{\lambda}(\Delta(0)G_{cl}(0)) < 1, \\
\Leftrightarrow\ & G_{cl}(0) < \Delta(0)^{-1} \text{ noting } \Delta(0) > \Delta(\infty) \geq 0 \text{ via Lemma 2.5}, \\
\Leftrightarrow\ & C_1(-A_{cl})^{-1}B_1 < \Delta(0)^{-1}, \text{ where } A_{cl} = A + B_2\bar{K}Y^{-1}, \\
\Leftrightarrow\ & C_1(A_{cl})^{-1}A_{cl}YC_1^T < \Delta(0)^{-1} \text{ noting } B_1 = -A_{cl}YC_1^T, \\
\Leftrightarrow\ & C_1YC_1^T < \Delta(0)^{-1}, \\
\Leftrightarrow\ & \begin{bmatrix} \Delta(0)^{-1} & C_1Y \\ YC_1^T & Y \end{bmatrix} > 0.
\end{aligned}
$$

□

As a consequence of Lemma 5.2 together with the above results for $G_{cl}(s)$ to be generalized NI, the following sufficient results for the existence of a static state-feedback controller that guarantees closed-loop robust stability for the uncertain system are obtained.

Theorem 5.3 *Suppose there exist matrices $Y > 0$ and $\bar{K}$ such that the following LMIs and determinant condition are satisfied:*

$$
\begin{bmatrix} AY + YA^T + B_2\bar{K} + \bar{K}^TB_2^T & B_1 + AYC_1^T + B_2\bar{K}C_1^T \\ B_1^T + C_1YA^T + C_1\bar{K}^TB_2^T & 0 \end{bmatrix} \leq 0, \tag{5.10}
$$

$$
\begin{bmatrix} \Delta(0)^{-1} & C_1Y \\ YC_1^T & Y \end{bmatrix} > 0. \tag{5.11}
$$

$$
\det(AY + B_2\bar{K}) \neq 0 \tag{5.12}
$$

Then the static state-feedback control law $u = \bar{K}Y^{-1}x$ is robustly stabilizing for the uncertain system (5.5).

This theorem qualifies a set of robust static state-feedback stabilizing controllers for systems with SNI uncertainty in terms of the DC gain of the uncertainty. In practice, the DC gain of the uncertainty may not be known exactly. However, it may be bounded as $\overline{\lambda}(\Delta(0)) \leq \gamma$. Note that in this situation, the condition $\overline{\lambda}(G_{cl}(0)) < \frac{1}{\gamma}$ ensures $\overline{\lambda}(\Delta(0)G_{cl}(0)) < 1$ (see Corollary 6 in [12] for a small-gain type result on the DC loop gain). Hence, the condition $\overline{\lambda}(G_{cl}(0)) < \frac{1}{\gamma}$ is satisfied if and only if

$$G_{cl}(0) < \frac{1}{\gamma}I.$$

Again via the proof of Lemma 5.2, note that

$$G_{cl}(0) = C_1 Y C_1^T,$$

it follows that

$$C_1 Y C_1^T < \frac{1}{\gamma}I. \tag{5.13}$$

The following theorem is an immediate consequence of the above LMI condition in (5.13) and conditions in (5.10) and (5.12), which gives a sufficient static state-feedback synthesis result for system in the presence of strictly negative-imaginary uncertainty with a certain DC gain bound.

Theorem 5.4 *Given the uncertain system (5.5) with $\overline{\lambda}(\Delta(0)) \leq \gamma$. Suppose there exist matrices $Y > 0$ and $\bar{K}$ such that the LMI conditions in (5.10), (5.13) and the determinant condition in (5.12) are satisfied.*

Then the static state-feedback control law $u = \bar{K}Y^{-1}x$ is robustly stabilizing for the uncertain system (5.5).

Remark 5.1 *Note that the nominal closed-loop system $G_{cl}(s)$ may be marginally stable since in the definition of generalized NI systems, $G_{cl}(s)$ is allowed to have poles on the imaginary axis except at the origin. However, the robust stability of the uncertain system (5.5) in the presence of dynamical SNI uncertainty is guaranteed.*

Remark 5.2 *The LMI conditions in (5.10), (5.11) and (5.13) are convex while the determinant condition in (5.12) is not. However, this determinant condition* *can be easily checked after solving the LMI conditions.*

Remark 5.3 *In practice, a designer could use Theorem 5.4 to minimize* $(\frac{1}{\gamma})$ *in (5.13) to obtain the largest (in terms of the DC gain) family of SNI perturbations for which the feedback loop would still be internally stable.*

In this section, a state-feedback static controller synthesis method is developed to achieve robust stability in the presence of SNI uncertainties. This result is based on the Generalized Negative-Imaginary Lemma to render the nominal closed-loop system to be generalized NI with a proper DC gain. For systems with non-strict NI uncertainties, it is desirable to swap the system property in the loop, i.e., the nominal closed-loop system needs to satisfy the SNI property for robustness. This is because the stability of interconnected NI systems requires at least one of the systems to be SNI. In practice, one might also want to synthesize an SNI controller to stabilize an NI plant. For stability, the DC loop gain should be contractive. Hence, ensuring an SNI property is essential in analysis and synthesis for NI systems. In the following section, a lemma for characterizing such a property of a system is presented, which is in parallel to the Generalized Negative-Imaginary Lemma.

5.5 Strongly Strict Negative-Imaginary Lemma

In this section, a state-space characterization is given to check the SNI property of a system. This result relaxes the minimality assumption required in all previous versions of NI or WSNI lemmas in [12,13,22]. This relaxation facilitates controller synthesis as the minimality assumption cannot be computed ‘a priori’ in controller synthesis to satisfy the SNI property of the synthesized loop, which is necessary for robust stability of the closed-loop system. The proposed SSNI characterization also gives numerical advantages by avoiding the non-convex rank constraint and the non-strict inequality which are present in previous literature [12, 13, 22].

5.5.1 Some Useful Results for SSNI Lemma

This subsection presents some preliminary results that streamline the result of checking SNI property in the next subsection. First, we recall the definition of SSPR systems as follows:

Definition 5.2 *[83] A real rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ is SSPR if*

1) $G(s)$ has no poles in $\mathbf{Re}[s] \geq 0$,

2) $G(j\omega) + G(j\omega)^* > 0$ for all $\omega \in \mathbb{R}$,

3) $\lim_{\omega\to\infty} \omega^{2\rho}\det(G(j\omega) + G(j\omega)^*) > 0$ where ρ is the dimension of the null space of $G(\infty) + G(\infty)^T$.

Remark 5.4 *[83] For strictly proper transfer functions, condition 3) in Definition 5.2 reduces to $\lim_{\omega\to\infty} \omega^2(G(j\omega)+G(j\omega)^*) > 0$ which coincides with the condition previously presented in the literature (see [5, 81, 83–85] for details).*

Note that the frequency domain properties of NI systems are defined in the frequency interval $\omega \in (0, \infty)$, while for SSPR systems the frequency domain properties are fulfilled on the entire $j\omega$-axis.

The following lemma gives a state-space characterization for an SSPR property of a system. The standard Strictly Positive Real Lemma is given for minimal systems (for example in [9]), however, the following lemma is given for non-minimal systems. This lemma will be used in the next subsection to develop the SSNI Lemma.

Lemma 5.3 *Let $G(s) = C(sI - A)^{-1}B$ be a strictly proper $m \times m$ transfer function matrix and $G(s) + G(-s)^T$ has normal rank m.*

(i) If there exists a matrix $P = P^T > 0$ that satisfies

$$PA + A^T P < 0, \tag{5.14}$$

$$PB = C^T, \tag{5.15}$$

then A is Hurwitz and $G(s)$ is SSPR.

(ii) Suppose (A, B) is controllable. If A is Hurwitz and $G(s) = C(sI - A)^{-1}B$ is SSPR, then there exists a matrix $P = P^T > 0$ that satisfies conditions in (5.14) and (5.15).

(iii) Suppose (C, A) is observable. If A is Hurwitz and $G(s) = C(sI - A)^{-1}B$ is SSPR, then there exists a matrix $P = P^T > 0$ that satisfies conditions in (5.14) and (5.15).

(iv) Suppose the state-space realization (A, B, C) has no observable uncontrollable modes. If A is Hurwitz and $G(s) = C(sI - A)^{-1}B$ is SSPR, then there exists a matrix $P = P^T > 0$ that satisfies conditions in (5.14) and (5.15).

Proof See Appendix B.1 for proof. □

Remark 5.5 *The assumption that $G(s)+G(-s)^T$ has normal rank m is in order to avoid redundances in inputs and/or outputs [5].*

Now, a state-space realization of the reciprocal system will be given under the assumption that the state-space realization for the original system has no poles at the origin. This lemma will be invoked later to transform a system with a blocking zero at zero frequency into a strictly proper system.

Lemma 5.4 *Suppose a square transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ has a state-space realization (A, B, C, D) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Suppose $\det(A) \neq 0$. Then $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is a state-space realization of $G(\frac{1}{s})$, where*

$$\bar{A} = A^{-1},\ \bar{B} = -A^{-1}B,\ \bar{C} = CA^{-1},\ \bar{D} = D - CA^{-1}B. \tag{5.16}$$

Proof From (5.16), we can obtain that

$$\begin{aligned}
\bar{D} + \bar{C}(sI - \bar{A})^{-1}\bar{B} &= D - CA^{-1}B + CA^{-1}(sI - A^{-1})^{-1}(-A^{-1}B) \\
&= D - CA^{-1}\left[I + (sI - A^{-1})^{-1}A^{-1}\right]B \\
&= D - CA^{-1}(sA - I)^{-1}sAB \\
&= D + C\left(\frac{1}{s}I - A\right)^{-1}B.
\end{aligned}$$

This implies that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is a state-space realization of $G(\frac{1}{s})$ by noting that (A, B, C, D) is a state-space realization of $G(s)$. □

5.5.2 State-Space Characterization for SSNI Systems

A state-space characterization for the SNI property of a system is given in this subsection. The main theorem is derived via the SSPR property of a transformed system; and before stating the main result, two technical lemmas are presented to streamline the proof of the main theorem.

Lemma 5.5 *Given $R(s) \in \mathcal{R}^{m\times m}$ with $R(\infty) = R(\infty)^T$. The following statements are equivalent:*

(1) $R(s)$ is SNI;

(2) $\hat{R}(s) = R(s) - R(\infty)$ is SNI;

(3) $G(s) = s(R(s) - R(\infty)) \in \mathcal{RH}_\infty$, $G(0) = 0$ and $G(j\omega) + G(j\omega)^* > 0$ $\forall \omega \in (0, \infty)$.

Proof This trivially follows via definition of SNI transfer function matrix in *Definition* 2.20. □

The above lemma states that an SNI system $R(s)$ can be transformed into an equivalent system $G(s)$ with a blocking zero at the origin that satisfies the SPR frequency condition $G(j\omega) + G(j\omega)^* > 0$ in the frequency interval $\omega \in (0, \infty)$. Because of this blocking zero condition, the existing SPR Lemmas (strong [5, 85], extended [39], marginally stable [36], weakly [86]) cannot provide any useful solution for state-space characterizations of the SNI systems.

To this end, the following lemma, however, can provide a solution via the SSPR property of the reciprocal system $G(\frac{1}{s})$. The use of a reciprocal system is key to the result of SSNI Lemma proposed in this section. Using this concept, the blocking zero condition of $G(s)$ at zero frequency has been transformed into the strictly proper condition of its reciprocal system. Also, $G(j\frac{1}{\omega}) + G(j\frac{1}{\omega})^* > 0 \ \forall\omega \in (0,\infty)$ is equivalent to $G(j\omega) + G(j\omega)^* > 0 \ \forall\omega \in (0,\infty)$ [87]. These results are presented in the following lemma:

Lemma 5.6 *Let (A, B, C, D) be the state-space realization of an $m \times m$ transfer function matrix $G(s)$ and $G(s)+G(-s)^T$ has normal rank m. Suppose $\det(A) \neq 0$.*
(i) If $D - CA^{-1}B = 0$ and there exists a matrix $Y = Y^T > 0$ that satisfies

$$AY + YA^T < 0, \tag{5.17}$$

$$\text{and} \;\; B = -AYA^{-T}C^T, \tag{5.18}$$

then A is Hurwitz, and

$$\begin{aligned} G(s) \in \mathcal{RH}_\infty, \; G(0) = 0, \; G(j\omega) + G(j\omega)^* > 0 \ \forall\omega \in (0,\infty), \\ G(\infty) + G(\infty)^* > 0 \text{ and } \lim_{\omega\to 0}\frac{1}{\omega^2}(G(j\omega) + G(j\omega)^*) > 0. \end{aligned} \tag{5.19}$$

(ii) Suppose (C, A) is observable. If A is Hurwitz and the conditions in (5.19) are satisfied, then $D-CA^{-1}B = 0$ and there exists a matrix $Y = Y^T > 0$ that satisfies (5.17) and (5.18).
(iii) Suppose (A, B, C, D) has no observable uncontrollable modes. If A is Hurwitz and the conditions in (5.19) are satisfied, then $D - CA^{-1}B = 0$ and there exists a matrix $Y = Y^T > 0$ that satisfies (5.17) and (5.18).

Proof See Appendix B.2 for proof. □

As a consequence of Lemma 5.5 and Lemma 5.6, we obtain the following main theorem characterizing properties of SNI systems. In contrast to the Weakly Strict Negative-imaginary Lemma [13], we refer to this theorem as the Strongly Strict Negative-imaginary Lemma.

Theorem 5.5 *(SSNI Lemma) Given a square transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ with a state-space realization (A, B, C, D), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Suppose $R(s) + R(-s)^T$ has normal rank m and (C, A) is observable. Then, A is Hurwitz and $R(s)$ is SNI with*

$$\lim_{\omega\to\infty} j\omega(R(j\omega) - R(j\omega)^*) > 0 \quad \text{and} \quad \lim_{\omega\to 0} j\frac{1}{\omega}(R(j\omega) - R(j\omega)^*) > 0 \tag{5.20}$$

if and only if $D = D^T$ and there exists a matrix $Y = Y^T > 0$ such that

$$AY + YA^T < 0 \text{ and } B = -AYC^T. \tag{5.21}$$

Proof ($\Rightarrow$) Since $R(s)$ is SNI, we have $D = D^T$ via Lemma 1 of [12]. Then, it follows from Lemma 5.5 that $G(s) = s(R(s) - R(\infty))$ satisfying $G(s) \in \mathcal{RH}_\infty$, $G(0) = 0$ and $G(j\omega) + G(j\omega)^* > 0$ $\forall\omega \in (0, \infty)$. Also note

$$\begin{aligned}
& \lim_{\omega\to\infty} j\omega(R(j\omega) - R(j\omega)^*) > 0 \\
\Leftrightarrow\quad & \lim_{\omega\to\infty} [j\omega(R(j\omega) - R(\infty)) + (j\omega(R(j\omega) - R(\infty)))^*] > 0 \\
& (\text{noting } R(\infty) = R(\infty)^T) \\
\Leftrightarrow\quad & \lim_{\omega\to\infty} G(j\omega) + G(j\omega)^* > 0 \\
\Leftrightarrow\quad & G(\infty) + G(\infty)^T > 0.
\end{aligned} \tag{5.22}$$

Similarly,

$$\lim_{\omega\to 0} j\frac{1}{\omega}(R(j\omega) - R(j\omega)^*) > 0 \Leftrightarrow \lim_{\omega\to 0} \frac{1}{\omega^2}(G(j\omega) + G(j\omega)^*) > 0. \tag{5.23}$$

Hence $G(s) = s(R(s) - R(\infty))$ satisfies the conditions in (5.19). Also, since (A, B, CA, CB) is a state-space realization for $G(s)$, and (CA, A) is observable by noting (C, A) is observable and A is Hurwitz, then via Lemma 5.6 (ii), condition (5.19) implies that there exists a matrix $Y = Y^T > 0$ such that

$$AY + YA^T < 0 \text{ and } B = -AYA^{-T}(CA)^T = -AYC^T. \tag{5.24}$$

($\Leftarrow$) Since there exists a matrix $Y = Y^T > 0$ that satisfies conditions in (5.21), it follows that A is Hurwitz, which implies that A is nonsingular. Hence, (16) can be rewritten as $AY + YA^T < 0$ and $B = -AYA^{-T}(CA)^T$. Also, since (A, B, CA, CB) is a state-space realization for $G(s) = s(R(s) - R(\infty))$ and note that $CB - CA(A)^{-1}B = 0$, via Lemma 5.6 (i), it follows that $G(s) = s(R(s) - R(\infty))$ satisfies the conditions in (5.19). Also note $D = D^T$, hence, it follows from Lemma 5.5, (5.22) and (5.23) that $R(s)$ satisfies SNI property and conditions in (5.20), which completes the proof. □

Theorem 5.6 *(SSNI Lemma) Given a square transfer function matrix* $R(s) \in \mathcal{R}^{m\times m}$ *with a state-space realization* (A, B, C, D)*, where* $A \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^{n\times m}$*,* $C \in \mathbb{R}^{m\times n}$ *and* $D \in \mathbb{R}^{m\times m}$*. Suppose* $R(s) + R(-s)^T$ *has normal rank* m *and* (A, B, C, D) *has no observable uncontrollable modes. Then,* A *is Hurwitz and* $R(s)$ *is SNI with (5.20) satisfied if and only if* $D = D^T$ *and there exists a matrix* $Y = Y^T > 0$ *that satisfies conditions in (5.21).*

Proof First note that the statement that (A, B, C, D) has no observable uncontrollable modes implies that the state-space realization of $G(s) = s(R(s)-R(\infty))$: (A, B, CA, CB) has no observable uncontrollable modes when A is nonsingular. Then, the results follow from the same proof lines of Theorem 5.5 with only replacement of invoking Lemma 5.6 (i) and Lemma 5.6 (iii) instead of Lemma 5.6 (i) and Lemma 5.6 (ii). □

Remark 5.6 *The assumption that (C, A) is observable in Theorem 5.5 is only needed to prove necessity part of the theorem. Alternatively, the assumption that (A, B, C, D) has no observable uncontrollable modes is another necessary requirement to show the SNI property as posed in Theorem 5.6.*

Remark 5.7 *Theorem 5.5 and Theorem 5.6 can also be proven via the alternate transformation, $\bar{G}(s) = -\frac{1}{s}(R(s) - R(0))$ which relates the NI property of a system with the PR property. Here, a sketch of the alternative proof is given as this may be instructive in its own right: First note that $(A, A^{-1}B, -C, 0)$ is a state-space realization for $\bar{G}(s)$, where (A, B, C, D) is the state-space realization for $R(s)$. Also, note that the controllability and observability of $(A, A^{-1}B, -C)$ is the same as that of (A, B, C). Then, it follows that the fulfillment of the conditions in (5.21) is equivalently implying the strictly proper system $\bar{G}(s)$ to be an SSPR system via Lemma 5.3. Then, via the definitions of SNI and SSPR systems, it can be shown that $\bar{G}(s)$ being an SSPR system is equivalent to $R(s)$ satisfying the SNI property and the conditions in (5.20), which completes the sketch of this proof.*

Next, we give some physical interpretations of the mathematical conditions in (5.20).

Lemma 5.7 *Given $R(s)$ is a proper scalar SNI transfer function with $R(\infty) \geq 0$, then*

$$\lim_{\omega \to 0} j \frac{1}{\omega}(R(j\omega) - R(j\omega)^*) > 0 \quad \Leftrightarrow \quad \lim_{\omega \to 0} \frac{d\phi(\omega)}{d\omega} < 0,$$

where $\phi(\omega)$ denote the phase of $R(j\omega)$.

Proof See Appendix B.3 for proof. □

The above lemma states that for proper scalar transfer function $R(s)$ with SNI property and $R(\infty) \geq 0$, $\lim_{\omega \to 0} j\frac{1}{\omega}(R(j\omega) - R(j\omega)^*) > 0$ means that the phase of $R(j\omega)$ strictly decreases as frequency increases from $\omega = 0$.

Remark 5.8 *For strictly proper scalar transfer functions,*

$$\lim_{\omega \to \infty} j\omega(R(j\omega) - R(j\omega)^*) > 0$$

implies that $\mathbf{Im}[R(j\omega)]$ *cannot go to zero faster than* ω^{-1} *when* $|\omega| \to \infty$*. This implies that the relative degree of* $R(j\omega)$ *must be zero or one.*

As mentioned in Remark 5.8, if one uses the conditions in (5.21) to design an SNI controller, systems with relative degree two cannot be captured. Earlier (non-strict) NI lemmas [12,13,22] invoke a non-strict Lyapunov inequality in (5.21) and yield a complete state-space characterization of (non-strict) NI systems. When the Lyapunov inequality in (5.21) becomes strict as in Theorem 5.5 (Theorem 5.6), then we get a complete state-space characterization of SNI systems but we also enforce a departure condition from and an arrival condition to the real axis as described by the limiting condition in (5.20). For example, $\frac{1}{s^2+2s+2}$ and $\frac{2s+2}{s^2+2s+2}$ are two SNI systems, however, they violate the first and the second condition of (5.20), respectively.

5.5.3 Discussions

Theorem 5.5 and Theorem 5.6 will enable robust control synthesis for uncertain NI systems. Via this result, an SNI controller can be synthesized by considering the simple algebraic conditions shown in (5.21) to stabilize an NI plant interconnected via positive feedback in a closed-loop [12, 13, 22]; or we can design a controller such that an LFT closed-loop system satisfies (5.21) to ensure the SNI property that facilitates robust stability against NI uncertainties. For robust stability, the DC loop gain should be contractive [12]. Most importantly, the robust stability is retained for arbitrary plant variations as long as the plant satisfies NI property and the DC loop gain condition.

Note that existing results on robust control for uncertain NI systems typically only enforce a (non-strict) NI property on the closed-loop system comprising of the nominal plant and controller and thereby can only handle SNI uncertainty, say for example the state-feedback synthesis method in Section 5.4 and results in [22–24]. Using the proposed SSNI Lemma in this section, the synthesis methods can be extended for handle the NI and SNI uncertainty of the system. This lemma also helps solve numerical issues. For example, the determinant assumption in Lemma 5.1, which always results in a non-convex determinant condition to be fulfilled in the synthesis method (say for example in Theorem 5.3), is now lifted in the SSNI lemma.

5.6 Illustrative Examples

In this section, we present two illustrative examples that demonstrate the usefulness of the proposed state-feedback synthesis method for robust stability in the presence of SNI uncertainty. The first example comprises of a flexible structure with a collocated force actuator and position sensor which satisfies the SNI property, while in the second example, the uncertain system is equivalently transformed into a system representation where the uncertain part is SNI .

5.6.1 Example 1: Flexible Structure

Consider the uncertain system of [24] which is depicted in Figure 5.2. Such a system has a flexible structure which contains many lightly damped modes, the frequencies and damping ratio of the modes may not be known exactly. The force applied to this flexible structure is denoted by x_2 and the deflection of the structure at the same location is denoted by y. Let $F(s)$ denote the transfer function of this flexible structure. Since the position sensor and force actuator are collocated on this flexible structure, $F(s)$ satisfies SNI property [12]. If this flexible structure is replaced by a unity gain, which is an estimation of the DC gain of the structure, the resulting error is then considered as uncertainty $\Delta(s) = F(s) - 1$ in the system as shown in Figure 5.3. Using the state-feedback synthesis method proposed in this chapter, the stability of the closed-loop system is required to be robust against unmodeled dynamics of the flexible structure ($\Delta(s)$) that contains lightly damped modes. Note that $\Delta(s) = F(s) - 1$ satisfies the SNI

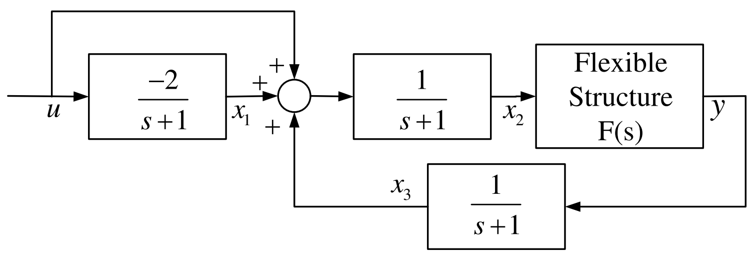

Figure 5.2: Block diagram of a system to be controlled using a state-feedback LMI approach to robust controller design.

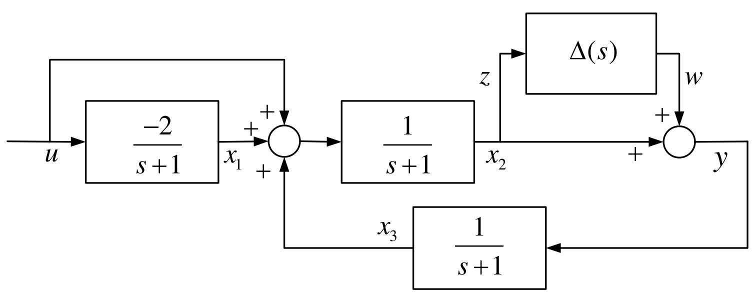

Figure 5.3: Block diagram of an uncertain system to be controlled by means of a state-feedback controller to be constructed using an LMI approach.

property since $F(s)$ is an SNI system. Assume $\Delta(0)$ is bounded by unity and also assume $\Delta(\infty) \geq 0$. In this situation, Theorem 5.4 can be invoked to synthesize a static state-feedback controller to robustly stabilize the uncertain system in

Figure 5.3. A state-space realization of this uncertain system is as follows

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} &= \begin{bmatrix}-1 & 0 & 0\\ 1 & -1 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} w + \begin{bmatrix}-2\\ 1\\ 0\end{bmatrix} u,\\ z &= \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},\\ w &= \Delta(s) z.\end{aligned}$$

YALMIP [62] and SeDuMi [63] are used to solve the LMI conditions in (5.13) and (5.10) with

$$A = \begin{bmatrix}-1 & 0 & 0\\ 1 & -1 & 1\\ 0 & 1 & -1\end{bmatrix}, \quad B_1 = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}, \quad B_2 = \begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}, \quad C_1 = \begin{bmatrix}0 & 1 & 0\end{bmatrix},$$

and $\gamma = 1$. The following solutions:

$$Y = \begin{bmatrix}6.8990 & -2.0000 & -6.1794\\ -2.0000 & 0.8791 & 1.8791\\ -6.1794 & 1.8791 & 6.1174\end{bmatrix} > 0, \ \bar{K} = \begin{bmatrix}-2.7196 & 1.0000 & 2.9411\end{bmatrix},$$

are obtained. The determinant condition in (5.12) is fulfilled since

$$\det(AY + B_2\bar{K}) = -1.2694 \neq 0.$$

Therefore, using Theorem 5.4, the required state feedback gain matrix K is obtained as follows

$$K = \bar{K}Y^{-1} = \begin{bmatrix}0.4558 & 0.4738 & 0.7957\end{bmatrix}.$$

A Bode plot of the corresponding closed-loop transfer function from w to z,

$$G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1,$$

is shown in Figure 5.4. It can be seen from this Bode plot that $G_{cl}(s)$ is indeed

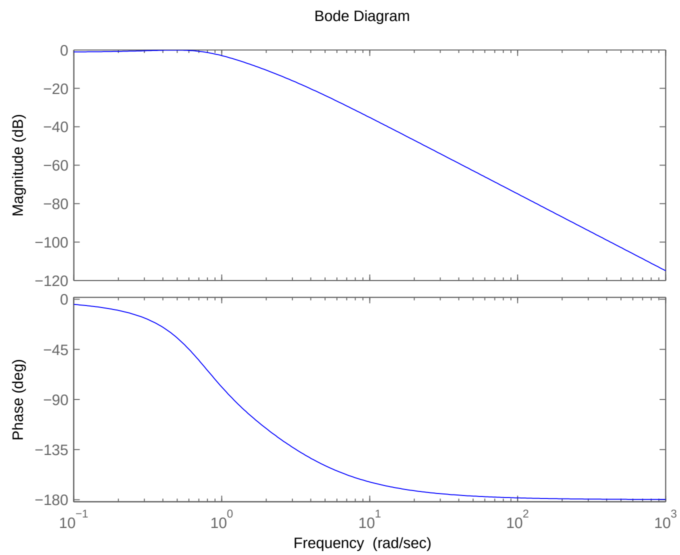

Figure 5.4: Bode plot of the closed-loop transfer function $G_{cl}(s)$ from the w to z.

NI since the phase lag is between 0 and $-\pi$ when the frequency varies from zero to infinity. Also, the gain of $G_{cl}(s)$ at zero frequency is 0.8701 which is less than unity. Since, the uncertainty transfer function $\Delta(s)$ in this example is SNI , then it follows from Theorem 5.1 that the true closed-loop system is stable provided that $\Delta(0) < 1$ and $\Delta(\infty) \geq 0$. The open-loop responses and closed-loop responses of the states for the nominal system under three different nonzero initial conditions are shown in Figure 5.5. As seen from this figure, all the states for the controlled closed-loop system go to zero as time tends to infinity while the states for the open-loop system do not.

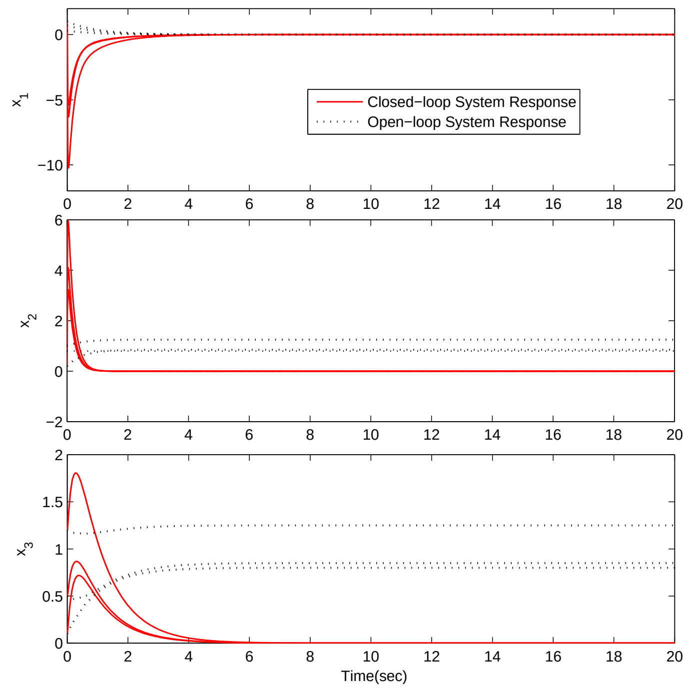

Figure 5.5: Open-loop responses and closed-loop responses of the nominal system under three randomly chosen nonzero initial states.

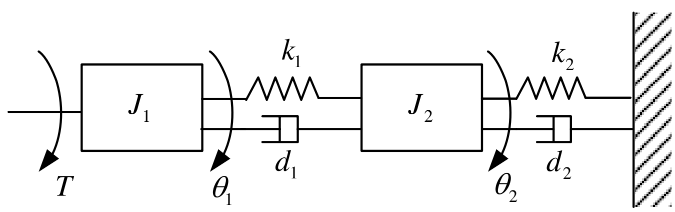

Figure 5.6: Two-mass model.

5.6.2 Example 2: Two-Mass Model

Consider the two-mass model shown in Figure 5.6, where the moment of inertia of the two masses are denoted by J_1 and J_2. The two masses are coupled via a spring with stiffness factor k_1 Nm/rad and a damper with damping factor d_1 $Nm.s/rad$. The second mass is attached to a fixed wall via a spring with stiffness factor k_2 Nm/rad and a damper with damping factor d_2 $Nm.s/rad$. We assume that all the springs and dampers are linear. T is the applied input torque (Nm), θ_1 and θ_2 are the angular position (rad) of the two masses, respectively. Here, we assume J_1, k_1 and d_1 are known exactly, while J_2, k_2 and d_2 are uncertain. The objective is to design a static state-feedback controller such that the system is robustly stable for all possible parameters $J_2 > 0$, $k_2 \geq 0$ and $d_2 \geq 0$.

The dynamics of the system can be described by the following equations:

$$T = J_1\ddot{\theta}_1 + d_1(\dot{\theta}_1 - \dot{\theta}_2) + k_1(\theta_1 - \theta_2), \tag{5.25}$$

$$k_1(\theta_1 - \theta_2) + d_1(\dot{\theta}_1 - \dot{\theta}_2) = J_2\ddot{\theta}_2 + d_2\dot{\theta}_2 + k_2\theta_2. \tag{5.26}$$

Then the transfer function from input T to output θ_2 can be obtained as

$$\begin{aligned} \frac{\theta_2(s)}{T(s)} &= \frac{d_1 s + k_1}{(J_2 s^2 + (d_1 + d_2)s + (k_1 + k_2))(J_1 s^2 + d_1 s + k_1) - (d_1 s + k_1)^2} \\ &= \frac{\frac{d_1 s + k_1}{(J_2 s^2 + (d_1 + d_2)s + (k_1 + k_2))(J_1 s^2 + d_1 s + k_1)}}{1 - \frac{(d_1 s + k_1)^2}{(J_2 s^2 + (d_1 + d_2)s + (k_1 + k_2))(J_1 s^2 + d_1 s + k_1)}}. \end{aligned} \tag{5.27}$$

Let

$$P_1 = \frac{(d_1 s + k_1)^2}{J_1 s^2 + d_1 s + k_1}, \quad P_2 = \frac{1}{d_1 s + k_1}, \text{ and } \Delta = \frac{1}{J_2 s^2 + (d_1 + d_2)s + (k_1 + k_2)}, \tag{5.28}$$

then we have

$$\frac{\theta_2(s)}{T(s)} = \frac{\Delta P_1 P_2}{1 - \Delta P_1}. \tag{5.29}$$

Hence the control problem can be equivalently casted into the block diagram shown in Figure 5.7, where K is a static state-feedback gain to be designed.

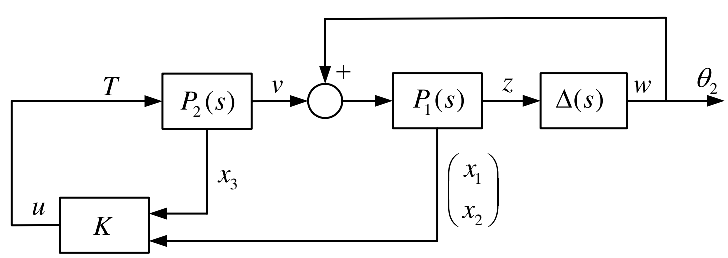

Figure 5.7: Block diagram of an uncertain two-mass model controlled via state-feedback.

It can be easily checked that P_1 is not NI, on the other hand, the uncertainty in this system Δ is SNI for all $J_2 > 0$, $d_1 > 0$, $k_1 > 0$, $d_2 \geq 0$ and $k_2 \geq 0$. A

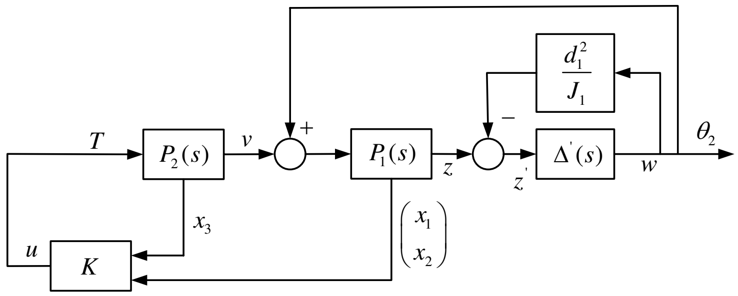

Figure 5.8: Block diagram of an uncertain tow-mass system by means of state-feedback control.

state-space realization of the uncertain system is as follows:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1}{J_1} & -\frac{d_1}{J_1} & 1 \\ 0 & 0 & -\frac{k_1}{d_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{d_1} \end{bmatrix} u, \\
z &= \begin{bmatrix} \frac{k_1^2 J_1 - d_1^2 k_1}{J_1^2} & \frac{2J_1 d_1 k_1 - d_1^3}{J_1^2} & \frac{d_1^2}{J_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{d_1^2}{J_1} w, \\
w &= \Delta(s) z. \qquad (5.30)
\end{aligned}$$

In order to apply Theorem 5.4, we transform the system in (5.30) into the form of (5.5). Now, let

$$z' = z - \frac{d_1^2}{J_1} w = \begin{bmatrix} \frac{k_1^2 J_1 - d_1^2 k_1}{J_1^2} & \frac{2J_1 d_1 k_1 - d_1^3}{J_1^2} & \frac{d_1^2}{J_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad w = \Delta'(s) z',$$

$$\text{where } \Delta'(s) = \frac{\Delta(s)}{1 - \frac{d_1^2}{J_1}\Delta(s)} = \frac{1}{J_2 s^2 + (d_1 + d_2)s + (k_1 + k_2 - \frac{d_1^2}{J_1})}. \qquad (5.31)$$

The controlled system in Figure 5.7 is now rearranged as shown in Figure 5.8. It can be verified that $\Delta'(s)$ is SNI for all $J_2 > 0$, $d_1 > 0$, $d_2 \geq 0$, $k_2 \geq 0$ and $k_1 > \frac{d_1^2}{J_1}$. Also, note that $\Delta'(\infty) = 0$ and $\Delta'(0) = \frac{1}{k_1 + k_2 - \frac{d_1^2}{J_1}} \leq \frac{1}{k_1 - \frac{d_1^2}{J_1}}$ for all $k_2 \geq 0$. Hence, Theorem 5.4 can be used to address the robust stabilization problem for the uncertain system in Figure 5.6.

Assume, in this example, $J_1 = 4kg.m^2$, $k_1 = 28Nm/rad$, $d_1 = 1Nm.s/rad$ and note that $k_1 - \frac{d_1^2}{J_1} = 27.75 > 0$. We now apply Theorem 5.4 where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1}{J_1} & -\frac{d_1}{J_1} & 1 \\ 0 & 0 & -\frac{k_1}{d_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -7 & -0.25 & 1 \\ 0 & 0 & -28 \end{bmatrix}, \tag{5.32}$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{d_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.33}$$

$$C_1 = \begin{bmatrix} \frac{k_1^2 J_1 - d_1^2 k_1}{J_1^2} & \frac{2J_1 d_1 k_1 - d_1^3}{J_1^2} & \frac{d_1^2}{J_1} \end{bmatrix} = \begin{bmatrix} 194.25 & 13.9375 & 0.25 \end{bmatrix}, \tag{5.34}$$

$$\text{and} \;\; \gamma = k_1 - \frac{d_1^2}{J_1} = 27.75. \tag{5.35}$$

This leads to the following solutions:

$$Y = \begin{bmatrix} 0.0009 & -0.0056 & 0.0440 \\ -0.0056 & 0.0937 & -0.8355 \\ 0.0440 & -0.8355 & 11.6412 \end{bmatrix} > 0,$$

$$\bar{K} = \begin{bmatrix} 2.0041 & -34.1436 & 325.3172 \end{bmatrix}.$$

The determinant condition in (5.12) is checked and it gives

$$\det(AY + B_2\bar{K}) = -198.9358 \neq 0.$$

Using Theorem 5.4, the designed state-feedback gain matrix K is then obtained as

$$K = \bar{K}Y^{-1} = \begin{bmatrix} -37.3102 & -322.9149 & 4.9106 \end{bmatrix}.$$

A Bode plot of the synthesized closed-loop transfer function from w to z',

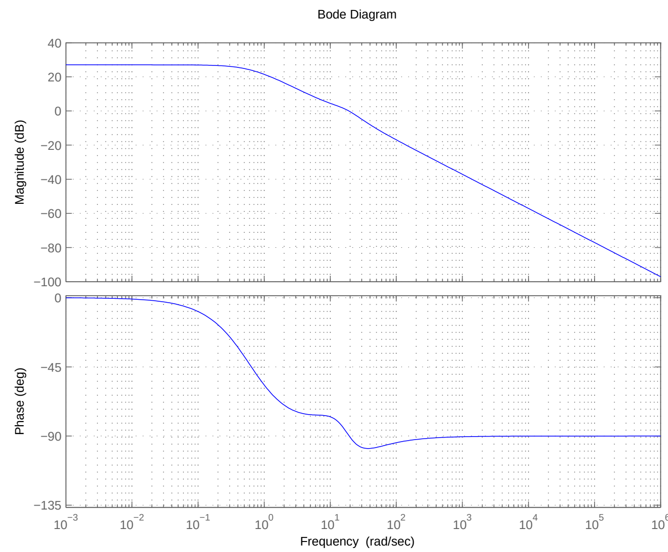

Figure 5.9: Bode plot of the closed-loop transfer function $G_{cl}(s)$ from the uncertainty output w to the uncertainty input $z^{'}$.

$$G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1,$$

is shown in Figure 5.9. As seen from this Bode plot,

$$\angle G_{cl}(j\omega) \in (-\pi, 0) \text{ for all } \omega \in (0, \infty),$$

hence $G_{cl}(s)$ indeed satisfies NI property. Also, the gain of $G_{cl}(s)$ at zero frequency is 22.5 which is less than $\gamma = 27.75$. The uncertainty transfer function $\Delta^{'}(s)$ in this example is SNI for all $J_2 > 0$, $d_2 \geq 0$ and $k_2 \geq 0$. Also, note that $\Delta^{'}(\infty) = 0$ and $\Delta^{'}(0) \leq \frac{1}{27.75}$, then it follows from Theorem 5.1 that the true uncertain closed-loop system in Figure 5.8 is robustly stable for all $J_2 > 0$, $d_2 \geq 0$ and $k_2 \geq 0$.

The time response of the states for uncertain open-loop system and closed-loop system for some randomly chosen parameters $J_2 > 0$, $d_2 \geq 0$ and $k_2 \geq 0$ under a nonzero initial condition are shown in Figure 5.10. As seen from this figure, all the states of the controlled closed-loop system go to zero as time tends to infinity while the states of the uncertain open-loop system do not.

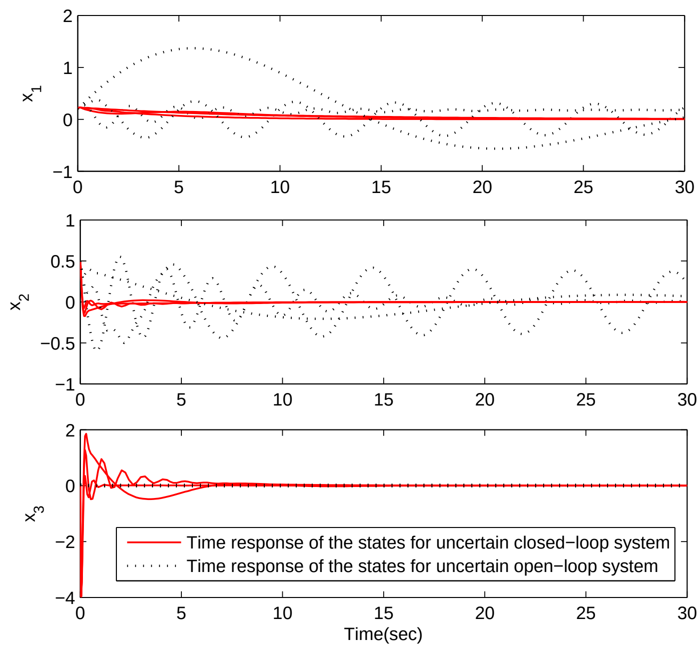

Figure 5.10: Open-loop responses and closed-loop responses of the uncertain system with three randomly chosen parameter of J_2, k_2 and d_2 under a nonzero initial condition.

5.7 Conclusions

This chapter proposes a synthesis framework for static state-feedback controllers to achieve robust stability of a system in the presence of SNI uncertainties. A relaxed version of the Generalized Negative-Imaginary Lemma where the minimality assumption is relaxed facilitates and underpins the proposed state-feedback synthesis method. LMI-based conditions are developed to design a robust state-feedback internally stabilizing controller with constant gain such that the closed-loop system satisfies generalized NI properties and a DC gain condition. In order to guarantee the property of generalized NI systems, no poles at the origin, an extra determinant condition needs to be fulfilled. Although the determinant condition is non-convex, this condition can easily be checked and it does not restrict the applicability of the proposed results which has been shown via two illustrative examples. This result facilitates engineers to design controllers that robustly stabilize systems against SNI uncertainties, those arise in many practical scenarios, for example, in unmodeled spill-over dynamics of lightly damped structures. Parts of this work have been submitted for publication in [88].

An SSNI Lemma has also been proposed to check the SNI property of a system. This lemma is derived using the SSPR property of a transformed system. The proposed characterization relaxes the minimality assumption, which is different from [12, 13, 22]; and this relaxation facilitates analysis and controller synthesis methods for uncertain NI systems. Using this result, the robust analysis and synthesis frameworks can be extended for both the NI and SNI uncertainty of the system. This result also clarifies the relationship between the strict Lyapunov inequality (see (5.21)) and the SNI property of the system. Parts of this work will appear soon in [89].

Chapter 6

Conclusions

This chapter summarises the contributions of this thesis and discusses possible future research directions. In this thesis, the robustness analysis and controller synthesis methods are explored for systems in the presence of SNI uncertainty. The robust control problem with non-strict NI uncertainty has also been addressed. The proposed methods are underpinning controller synthesis for systems with NI uncertainty to achieve not only robust stability but also robust performance.

6.1 Contributions

This thesis has made the following main contributions listed in this thesis in order of development and increasing complexity:

- The robust stabilization problem for uncertain NI systems is reformulated into an $\mathcal{H}_\infty$ synthesis problem. It has been shown that the problem of synthesising a stabilizing controller such that a closed-loop LFT satisfies NI property is equivalently transformed into a BR problem. This reformulation constitutes a key step towards applying $\mathcal{H}_\infty$ theory to enable controller synthesis for NI systems.
- An analytical framework for robust performance of systems with SNI uncertainty is proposed. To characterize robust performance, conditions are derived in

the μ framework. This framework also allows engineers to analyze robust stability in the presence of mixed NI and BR uncertainties. This framework underpins future developments in controller synthesis to achieve a guaranteed robust performance level for uncertain NI systems.

- An LMI approach to state-feedback controller synthesis method is proposed to achieve robust stability in the presence of SNI uncertainty. This result facilitates engineers to design controllers that robustly stabilize systems with SNI uncertainties. Two engineering motivated examples, a flexible structure where the unmodeled dynamics is SNI, and a two-mass model which can be transformed into a system with the uncertain part being SNI, are used to demonstrate the usefulness of the proposed synthesis method.

- A state-space characterization for SSNI property is proposed to facilitate robust controller synthesis and analysis for uncertain systems where non-strict NI uncertainties are present. This proposed characterization can handle non-minimal systems, and is convex and hence numerical attractive. Using this result, the robust analysis and synthesis frameworks can be extended to both the NI and SNI property of the system.

6.2 Future Research

A summary of related research directions which deserve future investigation is outlined below.

- $\mathcal{H}_\infty$ optimal control with invariant zeros on the $j\omega$-axis:

This problem is flagged in Chapter 3 and appears due to the particularity of the transformation of a closed-loop system with NI property into a BR framework. This not only leads to effective controller synthesis for practically motivated NI systems but also prefects the mature theory of $\mathcal{H}_\infty$ optimal control.

- Robust performance analysis for systems with a mixture of NI and PR uncertainties:

This is partly motivated by control systems of flexible structures which typically consist of multiple force actuators combined with collocated measurements of velocity and position. Flexible structures with collocated position sensors and force actuator give rise to NI systems, whereas flexible structures with collocated velocity sensors and force actuator give rise to PR systems. This necessities tools for robust analysis for mixed property of NI and PR uncertainties. It could be achieved by appropriately generalizing the transformation on the LFT presented in Chapter 4 by using the transformations between NI systems, PR systems and BR systems.

• Controller synthesis for uncertain NI systems to achieve robust performance:

To solve this problem, some techniques gleaning from passivity synthesis such as inner-outer loop design could be borrowed. More specifically, for a given lightly damped plant with NI property, an inner-loop controller satisfying SNI property and DC loop gain condition is chosen to increase damping of the system, then the outer loop controller is designed to improve performance.

Appendix A

Appendix to Chapter 4

A.1 Proof of Theorem 4.1

We will prove the theorem via a sequence of equivalent reformulations. In the foregoing reformulations, the equivalence between step (a) and step (b) is proved via transformations between NI systems and PR systems (using Lemma 3.2 and Lemma 3.3). Then the equivalence between step (b) and step (c) is shown in details in the proof. First, it is proved that step (b) implies step (c), then step (c) implies step (b).

(a) $\{\Sigma, K\}$ is internally stable, $F_\ell(F_u(\Sigma,\Delta_1),K) \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\boldsymbol{\Delta}$, $j\big[F_\ell(F_u(\Sigma,\Delta_1),K)(j\omega) - F_\ell(F_u(\Sigma,\Delta_1),K)(j\omega)^*\big] \geq 0$ for all $\omega \in (0,\infty)$, $\Delta_1 \in \mathcal{B}^\circ\boldsymbol{\Delta}$ and $\|F_\ell(F_\ell(\Sigma,K),-sI)\|_\infty \leq 1$.

(b) $\{\hat{\Sigma}, K\}$ is internally stable, $F_\ell(F_u(\hat{\Sigma},\Delta_1),K) \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\boldsymbol{\Delta}$, $\big[F_\ell(F_u(\hat{\Sigma},\Delta_1),K)(j\omega) + F_\ell(F_u(\hat{\Sigma},\Delta_1),K)(j\omega)^*\big] \geq 0$ $\forall\omega \in \mathbb{R}$, $\Delta_1 \in \mathcal{B}^\circ\boldsymbol{\Delta}$ and $\|F_\ell(F_\ell(\Sigma,K),-sI)\|_\infty \leq 1$ where

$$\hat{\Sigma} = \left[\begin{array}{c|cc:c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2A & C_2B_1 & C_2B_2 & C_2B_3 \\ \hdashline C_3 & D_{31} & D_{32} & 0 \end{array}\right]. \tag{A.1}$$

[Note that, K is the same for both the cases, $\hat{\Sigma}$ and Σ are both stabilizable and detectable, and $\Sigma_{33} = \hat{\Sigma}_{33} = \left[\begin{array}{c|c} A & B_3 \\ \hline C_3 & 0 \end{array}\right]$. Hence, the internal stability of $\{\Sigma, K\}$ and $\{\hat{\Sigma}, K\}$ are equivalent via Lemma A.4.1 of [46]. Also, note that $F_\ell(F_u(\hat{\Sigma}, \Delta_1), K) = s \cdot [F_\ell(F_u(\Sigma, \Delta_1), K)]\ \forall \Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}$, which shows the equivalence between the frequency dependent conditions of (a) and (b). Finally, note that for an arbitrary $\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}$, $F_\ell(F_u(\Sigma, \Delta_1), K)(\infty) = 0$ and $F_\ell(F_u(\hat{\Sigma}, \Delta_1), K)(0) = 0$ since $\det(A) \neq 0$. Hence, $F_\ell(F_u(\hat{\Sigma}, \Delta_1), K) \in \mathcal{RH}_\infty$ if and only if $F_\ell(F_u(\Sigma, \Delta_1), K) \in \mathcal{RH}_\infty$ using Lemma 3.1.]

(c) $\{G, K\}$ is internally stable, $\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] \leq 1$ and $\det(I + F_u(F_\ell(G, K), \Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}$, where

$$G = \left[\begin{array}{cc:cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hdashline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star \left[\begin{array}{c|cc:c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 A & C_2 B_1 & C_2 B_2 & C_2 B_3 \\ \hdashline C_3 & D_{31} & D_{32} & 0 \end{array}\right], \tag{A.2}$$

which is also equivalent to the formulation in (4.3). This can be proved as follows:

[(b) $\Rightarrow$ (c): Since $\{\hat{\Sigma}, K\}$ is internally stable, we have

$$T_f(\hat{\Sigma}, K) = \begin{bmatrix} T_{f_{11}} & T_{f_{12}} & T_{f_{13}} \\ T_{f_{21}} & T_{f_{22}} & T_{f_{23}} \\ T_{f_{31}} & T_{f_{32}} & T_{f_{33}} \end{bmatrix} \in \mathcal{RH}_\infty.$$

Note that, $T_{f_{22}}(\hat{\Sigma}, K) = F_\ell(F_u(\hat{\Sigma}, 0), K) \in \mathcal{RH}_\infty$ satisfies

$$T_{f_{22}}(\hat{\Sigma}, K)(j\omega) + T_{f_{22}}(\hat{\Sigma}, K)(j\omega)^* \geq 0\ \forall \omega \in \mathbb{R}.$$

Hence, $(I + T_{f_{22}}(\hat{\Sigma}, K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 3.4, and this implies

$$\left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T_f(\hat{\Sigma}, K) \in \mathcal{RH}_\infty \tag{A.3}$$

via Lemma 4.1, which shows that $\{G, K\}$ is internally stable as

$$\begin{aligned} T_f(G,K) &= T_f\left(\left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star \hat{\Sigma}, K\right) \\ &= \left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T_f(\hat{\Sigma}, K). \end{aligned} \tag{A.4}$$

Also, as for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, we have $F_\ell(F_u(\hat{\Sigma}, \Delta_1), K) \in \mathcal{RH}_\infty$ and $[F_\ell(F_u(\hat{\Sigma}, \Delta_1), K)(j\omega) + F_\ell(F_u(\hat{\Sigma}, \Delta_1), K)(j\omega)^*] \geq 0\ \forall \omega \in \mathbb{R}$, it follows that $(I + F_\ell(F_u(\hat{\Sigma}, \Delta_1), K))^{-1} \in \mathcal{RH}_\infty$ again via Lemma 3.4. Then defining $Y = (I - F_\ell(F_u(\hat{\Sigma}, \Delta_1), K))(I + F_\ell(F_u(\hat{\Sigma}, \Delta_1), K))^{-1}$, we have $Y = F_\ell(F_u(G, \Delta_1), K) \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Also, since $(I+Y)^{-1} = \frac{1}{2}(I + F_\ell(F_u(\hat{\Sigma}, \Delta_1), K)) \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, it follows that $\det(I + Y(j\omega)) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$; *i.e.*, $\det(I + F_u(F_\ell(G, K), \Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Furthermore, $[F_\ell(F_u(\hat{\Sigma}, \Delta_1), K)(j\omega) + F_\ell(F_u(\hat{\Sigma}, \Delta_1), K)(j\omega)^*] \geq 0$ for all $\omega \in \mathbb{R}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ implies that $\bar{\sigma}[Y(j\omega)] \leq 1$ for all $\omega \in \mathbb{R}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Hence, it follows that $\sup_{\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}} \|F_u(F_\ell(G, K), \Delta_1)\|_\infty \leq 1$. Now define $F_\ell(G, K) = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}$, noting that $\|\bar{M}_{11}\|_\infty = \|F_\ell(F_\ell(\Sigma, K), -sI)\|_\infty \leq 1$, it follows that $\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G, K)] \leq 1$ via Lemma 4.2.

(c) $\Rightarrow$ (b): Since $\{G, K\}$ is internally stable, we have $T_f(G,K) \in \mathcal{RH}_\infty$. Noting $T_{f_{22}}(G,K) = F_u(F_\ell(G,K),0) \in \mathcal{RH}_\infty$, we have $(I + T_{f_{22}}(G,K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 3.5 as $\det(I + F_u(F_\ell(G,K),0)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and $\|F_u(F_\ell(G,K),0)\|_\infty \leq 1$. This thus implies

$$\left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T_f(G,K) \in \mathcal{RH}_\infty \tag{A.5}$$

via Lemma 4.1, which shows $\{\hat{\Sigma}, K\}$ is internally stable as

$$T_f(\hat{\Sigma},K) = \left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T_f(G,K) \text{ via (A.4).}$$

Note that $F_\ell(G,K) = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix} \in \mathcal{RH}_\infty$. Since $\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] \leq 1$, then $\|F_\ell(F_\ell(\Sigma,K),-sI)\|_\infty = \|\bar{M}_{11}\|_\infty \leq 1$ and $\sup_{\Delta_1\in\mathcal{B}^\circ\mathbf{\Delta}} \|F_u(F_\ell(G,K),\Delta_1)\|_\infty \leq 1$ via Lemma 4.2, which implies $F_u(F_\ell(G,K),\Delta_1) \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ and $\bar{\sigma}(F_u(F_\ell(G,K),\Delta_1)(j\omega)) \leq 1$ for all $\omega \in \mathbb{R}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Also, since $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}\ \forall\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, it follows that $(I + F_u(F_\ell(G,K),\Delta_1))^{-1} \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ via Lemma 3.5. Defining $X = (I - F_u(F_\ell(G,K),\Delta_1)(I + F_u(F_\ell(G,K),\Delta_1)^{-1}$ for an arbitrary $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, we have $X = F_u(F_\ell(\hat{\Sigma},K),\Delta_1) \in \mathcal{RH}_\infty\ \forall\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Then, $\bar{\sigma}(F_u(F_\ell(G,K),\Delta_1)) \leq 1$ for all $\omega \in \mathbb{R}$ implies $[X(j\omega) + X(j\omega)^*] \geq 0$ for all $\omega \in \mathbb{R}$; *i.e.*,

$\left[F_\ell(F_u(\hat{\Sigma},\Delta_1),K)(j\omega) + F_\ell(F_u(\hat{\Sigma},\Delta_1),K)(j\omega)^*\right] \geq 0\ \forall\omega \in \mathbb{R}\ \forall\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}.]$

Appendix B

Appendix to Chapter 5

B.1 Proof of Lemma 5.3

(i) The proof is omitted as it is similar to the proof of the sufficiency part of Lemma 6.3 in [5].

(ii) This can be readily obtained via Theorem 3.1 of [84].

(iii) This can be readily obtained via Corollary 3.1 of [84].

(iv) Note that the state-space realization (A, B, C) has no observable uncontrollable modes, hence, without loss of generality, we suppose the state-space realization (A, B, C) is with the following Kalman canonical form:

$$G := \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} A_{11} & 0 & 0 & B_1 \\ A_{21} & A_{22} & A_{23} & B_2 \\ 0 & 0 & A_{33} & 0 \\ \hline C_1 & 0 & 0 & 0 \end{array}\right],$$

where the eigenvalues of A_{11} are controllable and observable modes, the eigenvalues of A_{22} are controllable but unobservable modes, and the eigenvalues of A_{33} are uncontrollable and unobservable modes of the sate-space realization (A, B, C).

Also, note that $G(s)$ is SSPR implies

$$\begin{aligned}
\hat{G}(s) : &= \hat{C}(sI - A)^{-1}\hat{B} + \hat{D} \\
&= \left[\begin{array}{ccc|ccc}
A_{11} & 0 & 0 & B_1 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & B_2 & I & 0 \\
0 & 0 & A_{33} & 0 & 0 & I \\
\hline
C_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{array}\right] \\
&= \begin{bmatrix} G(s) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\end{aligned}$$

is SSPR. Also, note that A is Hurwitz and $(A, \hat{B})$ is controllable, it follows from Theorem 3.1 of [84] that there exist $\hat{P} = \hat{P}^T > 0$ and $L = L^T > 0$, $\varepsilon > 0$ and real matrices Q and W such that

$$\hat{P}A + A^T\hat{P} = -QQ^T - \varepsilon L, \tag{B.1}$$

$$\hat{C} - \hat{B}^T\hat{P} = W^TQ, \tag{B.2}$$

$$\hat{D} + \hat{D}^T = W^TW. \tag{B.3}$$

Partitioning W as $W = \begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix}$ with compatible dimension, we have $W_1 = 0$ as the (1,1) block of $\hat{D}$ is zero. Considering the part of (B.1) and (B.2) corresponding to (1,1) block of $\hat{G}(s)$, namely $G(s)$, we obtain that there exist $\hat{P} = \hat{P}^T > 0$, $L = L^T > 0$, $\varepsilon > 0$ and real matrix Q such that

$$\begin{aligned}
\hat{P}A + A^T\hat{P} &= -QQ^T - \varepsilon L, \\
C - B^T\hat{P} &= 0,
\end{aligned}$$

which implies that there exists $P = \hat{P} = P^T > 0$ that satisfies conditions in (5.14) and (5.15).

B.2 Proof of Lemma 5.6

(i) Since there exists a matrix $Y = Y^T > 0$ such that conditions in (5.17) and (5.18) are satisfied, it follows that there exists $P = Y^{-1} > 0$ such that

$$PA^{-1} + (A^{-1})^T P < 0, \tag{B.4}$$

$$P(-A^{-1}B) = (CA^{-1})^T. \tag{B.5}$$

Also, since $D - CA^{-1}B = 0$, it follows from Lemma 5.3 (i) and Lemma 5.4 that the strictly proper transfer function $\tilde{G}(s) = G(\frac{1}{s})$ is SSPR. Then, via Definition 5.2, we have $\tilde{G}(j\omega) + \tilde{G}(j\omega)^* > 0 \ \forall \omega \in \mathbb{R}$ and $\lim_{\omega\to\infty} \omega^2(\tilde{G}(j\omega) + \tilde{G}(j\omega)^*) > 0$ which implies $G(\frac{1}{j\omega}) + G(\frac{1}{j\omega})^* > 0 \ \forall \omega \in [0, \infty)$ and $\lim_{\omega\to\infty} \omega^2(G(\dfrac{1}{j\omega}) + G(\dfrac{1}{j\omega})^*) > 0$. Also, note that

$$\begin{aligned}
& G(\frac{1}{j\omega}) + G(\frac{1}{j\omega})^* > 0 \ \forall \omega \in (0, \infty) \\
\Leftrightarrow \quad & G(-j\frac{1}{\omega}) + G(-j\frac{1}{\omega})^* > 0 \ \forall \omega \in (0, \infty) \\
\Leftrightarrow \quad & G(-j\frac{1}{\omega}) + G(j\frac{1}{\omega})^T > 0 \ \forall \omega \in (0, \infty) \\
\Leftrightarrow \quad & G(j\frac{1}{\omega}) + G(-j\frac{1}{\omega})^T > 0 \ \forall \omega \in (0, \infty) \\
\Leftrightarrow \quad & G(j\omega) + G(j\omega)^* > 0 \ \forall \omega \in (0, \infty).
\end{aligned} \tag{B.6}$$

Similarly,

$$\lim_{\omega\to 0}(G(\frac{1}{j\omega}) + G(\frac{1}{j\omega})^*) = \left[G(\frac{1}{s}) + G(-\frac{1}{s})^T\right](0) > 0 \Leftrightarrow G(\infty) + G(\infty)^* > 0, \tag{B.7}$$

$$\lim_{\omega\to\infty} \omega^2(G(\frac{1}{j\omega}) + G(\frac{1}{j\omega})^*) > 0 \Leftrightarrow \lim_{\omega\to 0}\frac{1}{\omega^2}(G(j\omega) + G(j\omega)^*) > 0. \tag{B.8}$$

Hence, $G(j\omega)+G(j\omega)^* > 0\ \forall\omega \in (0,\infty)$, $G(\infty)+G(\infty)^* > 0$ and $\lim_{\omega\to 0}\frac{1}{\omega^2}(G(j\omega)+G(j\omega)^*) > 0$. Also, $D - CA^{-1}B = 0$ implies $G(0) = 0$. Finally, condition (5.17) implies that A is Hurwitz and $G(s) \in \mathcal{RH}_\infty$.

(ii) First note that the conditions in (5.19) are satisfied, it follows from (B.6), (B.7) and (B.8) that $G(\frac{1}{j\omega}) + G(\frac{1}{j\omega})^* > 0\ \forall\omega \in [0,\infty)$ and $\lim_{\omega\to\infty}\omega^2(G(\frac{1}{j\omega}) + G(\frac{1}{j\omega})^*) > 0$. Also, since $G(0) = 0$, it follows from Lemma 5.4 that $\tilde{G}(s) = G(\frac{1}{s})$ satisfies $\tilde{G}(\infty) = D - CA^{-1}B = G(0) = 0$. Furthermore, since A is Hurwitz, it follows that A^{-1} is Hurwitz which implies $G(\frac{1}{s}) \in \mathcal{RH}_\infty$ via Lemma 5.4. Consequently, $G(\frac{1}{s})$ is SSPR via Definition 5.2. Using Lemma 5.4, $(\bar{A},\bar{B},\bar{C},\bar{D})$ is a state-space realization of $G(\frac{1}{s})$, where $(\bar{A},\bar{B},\bar{C},\bar{D})$ is given in (5.16). Also, note that (C,A) being observable implies that $(\bar{C},\bar{A})$ is observable by noting that A is nonsingular. Then, it follows from the SSPR Lemma (Lemma 5.3 (iii)) that there exists a matrix $P = P^T > 0$ that satisfies conditions in (B.4) and (B.5). Let $Y = P^{-1}$, then the condition in (B.5) is equivalent to (5.18). Finally, it follows via a simple algebraic computation that condition (B.4) is equivalent to (5.17).

(iii) First note A is nonsingular as A is Hurwitz. Also, note that the statement that (A,B,C,D) has no observable uncontrollable modes implies that $(\bar{A},\bar{B},\bar{C},\bar{D})$ has no observable uncontrollable modes when A is nonsingular, where $(\bar{A},\bar{B},\bar{C},\bar{D})$ is given in (5.16). Then, it follows the same lines of the proof of (ii) with only replacement of invoking Lemma 5.3 (iv) instead of Lemma 5.3 (iii).

B.3 Proof of Lemma 5.7

$$
\begin{aligned}
& \lim_{\omega\to 0} j\frac{1}{\omega}(R(j\omega) - R(j\omega)^*) > 0 \\
\Leftrightarrow\quad & \lim_{\omega\to 0} j\frac{1}{\omega}\left[r(\omega)e^{j\phi(w)} - r(\omega)e^{-j\phi(w)}\right] > 0 \\
& \text{where } r(\omega) \text{ denotes the magnitude of } R(j\omega) \\
\Leftrightarrow\quad & \lim_{\omega\to 0} j\frac{1}{\omega}(2j\sin\phi(\omega)) > 0 \text{ noting } r(\omega) > 0 \text{ near } \omega = 0 \\
& \text{since } R(0) > R(\infty) \geq 0 \text{ via Lemma 2.5} \\
\Leftrightarrow\quad & \lim_{\omega\to 0} \frac{\sin\phi(\omega)}{\omega} < 0 \\
\Leftrightarrow\quad & \lim_{\omega\to 0} \cos\phi(\omega)\frac{d\phi(\omega)}{d\omega} < 0 \text{ since } R(0) > R(\infty) \geq 0 \text{ via Lemma 2.5,} \\
& \text{hence } \phi(0) = 0 \text{, we can use L'Hospital's rule} \\
\Leftrightarrow\quad & \lim_{\omega\to 0} \frac{d\phi(\omega)}{d\omega} < 0 \text{ noting } \cos\phi(\omega) > 0 \text{ in the neighborhood of } \omega = 0.
\end{aligned}
$$

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