MULTI-MODE RECEIVER SYSTEMS FOR COSMIC MICROWAVE BACKGROUND B-MODE POLARISATION EXPERIMENTS

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<tr>
<td><strong>BTB horn</strong></td>
<td>Back-to-Back horn</td>
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<tr>
<td><strong>CDM</strong></td>
<td>Cold Dark Matter</td>
</tr>
<tr>
<td><strong>CEM</strong></td>
<td>Computational ElectroMagnetics</td>
</tr>
<tr>
<td><strong>CMB</strong></td>
<td>Cosmic Microwave Background</td>
</tr>
<tr>
<td><strong>EFIE</strong></td>
<td>Electric Field Integral Equation</td>
</tr>
<tr>
<td><strong>FDTD</strong></td>
<td>Finite-Difference Time-Domain</td>
</tr>
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<td><strong>FEM</strong></td>
<td>Finite Element Method</td>
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<tr>
<td><strong>FWHM</strong></td>
<td>Full Width at Half maximum</td>
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<tr>
<td><strong>GUTs</strong></td>
<td>Grand Unified Theories</td>
</tr>
<tr>
<td><strong>HDPE</strong></td>
<td>High-Density PolyEthylene</td>
</tr>
<tr>
<td><strong>HEMT</strong></td>
<td>High-Electron-Mobility-Transistor</td>
</tr>
<tr>
<td><strong>HFI</strong></td>
<td>High Frequency Instrument</td>
</tr>
<tr>
<td><strong>HPBW</strong></td>
<td>Half Power Beam Width</td>
</tr>
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<td><strong>HWP</strong></td>
<td>Half-Wave Plate</td>
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<tr>
<td><strong>LE</strong></td>
<td>Lab Excitation method</td>
</tr>
<tr>
<td><strong>LFI</strong></td>
<td>Low Frequency Instrument</td>
</tr>
<tr>
<td><strong>LSPE</strong></td>
<td>Large Scale Polarisation Explorer</td>
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<tr>
<td><strong>MF</strong></td>
<td>Merit Function</td>
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<tr>
<td><strong>MLFMM</strong></td>
<td>MuLti Fast Multipole Method</td>
</tr>
<tr>
<td><strong>MoM</strong></td>
<td>Method of Moments</td>
</tr>
<tr>
<td><strong>NEP</strong></td>
<td>Noise-Equivalent Power</td>
</tr>
<tr>
<td><strong>NSM</strong></td>
<td>Normalised Scattering Matrix</td>
</tr>
<tr>
<td><strong>OMT</strong></td>
<td>Orthogonal Mode Transducer</td>
</tr>
<tr>
<td><strong>PO</strong></td>
<td>Physical Optics</td>
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<tr>
<td><strong>QWP</strong></td>
<td>Quarter-Wave Plate</td>
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<tr>
<td><strong>RL-angle</strong></td>
<td>Ray Launching angle</td>
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<tr>
<td><strong>RL-GO</strong></td>
<td>Ray Launching – Geometrical Optics</td>
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<tr>
<td><strong>RMS</strong></td>
<td>Root mean square</td>
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<tr>
<td><strong>SQUID</strong></td>
<td>Superconducting Quantum Interference Device</td>
</tr>
<tr>
<td><strong>STRIP</strong></td>
<td>Survey TeneRIfe Polarimeter</td>
</tr>
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**SWE:** Spherical Wave Expansion  
**SWIPE:** Short-Wavelength Instrument for the Polarisation Explorer  
**TE:** Transverse Electric  
**TES:** Transition Edge Sensor  
**TM:** Transverse Magnetic  
**UTD:** Uniform Theory of Diffraction  
**VNA:** Vector Network Analyser  
**WPE:** Waveguide Port Excitation
Abstract

MULTI-MODE RECEIVER SYSTEMS FOR COSMIC MICROWAVE BACKGROUND B-MODE POLARISATION EXPERIMENTS

Stephen Legg

A thesis submitted to The University of Manchester for the degree of Doctor of Philosophy, September 2017

A measurement of the primordial B-mode polarisation of the Cosmic Microwave Background would provide direct evidence of inflation in the early universe. The extremely weak nature of the B-mode signal necessitates an instrument with a high sensitivity and precise control over systematic effects. Multi-mode antenna feed horns offer higher sensitivity than their single-mode counterparts, however their behaviour is much more complex. The Short Wavelength Instrument for the Polarisation Explorer (SWIPE) onboard the Large Scale Polarisation Explorer (LSPE) is one instrument planning to implement multi-mode feed horns. SWIPE will attempt to detect the primordial B-mode at large angular scales, measuring the sky in three bands at 140, 220 and 240 GHz. A single on-axis High-Density PolyEthylene (HDPE) lens and polarisation-splitting wire grid combine to focus the radiation from the sky onto two focal planes of multi-mode horns feeding bolometric detectors. A large diameter rotating metal-mesh half-wave plate allows both polarisations to be measured by the same pixel, therefore bypassing many detector systematics. Simulations are performed to predict the sky beam for two key pixels: closest to and furthest from the centre of the focal plane. For the 140 GHz channel the cross-polarisation is predicted, and the optimum location at which to place the telescope’s focus behind the horn aperture to maximise gain and optimise beam shape is investigated. A measurement of the multi-mode horn is performed using a room-temperature bolometer. An investigation is also conducted to assess to what extent the same measurements can be performed using a coherent measurement system such as a vector network analyser. A working coherent measurement technique is devised, however it is limited to horns carrying only the first 3 modes.
Declaration

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Proceedings papers


Conference posters

S. Legg, B. Maffei and P. Schemmel, “Multi-Mode Receiver Systems For Cosmic Microwave Background B-mode Experiments”, URSI Festival of Radio Science (FRSci), 2014 (Winner of best poster)
1. Introduction

1.1. Introduction

The Cosmic Microwave Background (CMB) allows us to look at the Universe when it was just 400,000 years old. The information contained within its tiny fluctuations across the sky has allowed cosmologists to constrain theories about the birth and evolution of the Universe. One component of cosmology which is yet to be fully constrained is the theory of cosmic inflation. This states that the very early Universe underwent a nearly exponential period of expansion, creating the initial conditions for the Big Bang. The key to constraining theories of inflation actually lies within anisotropies in the polarisation of the CMB. The CMB polarisation can be split into E-mode and B-mode components, so called because their patterns resemble those of electric and magnetic fields respectively. Of these components, it is the B-mode that could only have been generated if inflation occurred. Detection of the B-mode is therefore a primary goal in modern cosmology, with many experiments being developed specifically for this purpose.

To measure the B-mode requires an instrument with extremely high sensitivity, superb control of systematic effects and an efficient method of removing dominant foreground signals coming from the rest of the Universe. Historically, CMB instruments have been based on the use of antenna feed horns to couple radiation from the telescope on to the detector. These horns were single-moded, in the sense that the waveguide diameter at the base of the horn is restricted so that only the fundamental electromagnetic waveguide mode can propagate through. Single-mode horns have a well characterised behaviour, but limit the amount of power being coupled to the detector. In a multi-mode horn the waveguide diameter is opened up to allow higher order modes to propagate. When used in conjunction with an incoherent detector such as a bolometer, each mode independently couples power from the sky, resulting in an increased throughput and therefore increased sensitivity. The drawback of the multi-mode horns is that they are much more difficult to characterise. This is because each mode acts independently and the overall behaviour
depends on the final balance of power between modes after individual propagation through the horn and detector assembly.

Ground based instruments have limited sky coverage and limited frequency coverage due to the atmosphere. This makes component separation of the foregrounds and B-mode signal much more difficult at large angular scales. A balloon-borne or satellite experiment provides a solution. The Large Scale Polarisation Explorer (LSPE) is an Italian based balloon-borne experiment which aims to measure the B-mode polarisation at large angular scales. Onboard LSPE, the Short Wavelength Instrument for the Polarisation Explorer (SWIPE) is a Stokes polarimeter and will measure the sky in 3 bands at 140, 220 and 240 GHz. A single on-axis High-Density PolyEthylene (HDPE) lens and polarisation-splitting wire grid focuses radiation from the sky onto two focal planes of multi-mode horns feeding Transition Edge Sensor (TES) bolometric detectors. A large diameter rotating metal-mesh Half-Wave Plate (HWP) modulates the incoming polarisation, allowing the same detector to measure both polarisations. Combining SWIPE with a partner ground based instrument (the Survey TeneRife Polarimeter) (STRIP) covering low frequencies, LSPE is able to measure over a large frequency range. This aids in the separation of the B-mode and foreground signals through the exploitation of their different frequency spectra. This thesis concerns the development of the multi-mode horn-lens configuration of the SWIPE instrument.

1.2. The Standard Cosmology

1.2.1. The Standard Model

The theoretical and observational motivations underpinning the standard model of Big Bang cosmology are well described in any modern cosmology textbook. Here, an approach similar to that of (Liddle 2015) is taken. The cosmological principle states that the Universe is homogeneous and isotropic on large scales. Homogeneity means that the Universe is the same at all points and isotropy means that the Universe looks the same in all directions. On small scales the Universe departs from this principle; when we look around our solar system there is clearly significant structure. This remains true as we look around the Milky Way and even the local
1 Introduction

It is not until we look on scales of a few 100 Mpc that the Universe begins to look very smooth (isotropic). Of course, we cannot directly test homogeneity on such scales because this would involve travelling this vast distance and observing the Universe from the new location. However, homogeneity is inferred from the isotropy since we do not occupy a ‘special’ place in the Universe.

In 1929 Edwin Hubble examined the relationship between the distance and redshift of galaxies and found that for most galaxies, the further away the galaxy is, the faster its recessional velocity is (Hubble 1929). The proportionality constant between recessional velocity and distance is known as Hubble’s constant. The implication of Hubble’s discovery is that the Universe must be expanding, and the cosmological principle means that this expansion must be occurring everywhere throughout the Universe. Therefore, if time is traced backwards far enough, everything must have been much closer together at some point in the past, perhaps to a single point. Hence, it was predicted that the universe began in an extremely dense state in a process which later became known as the ‘Big Bang’ (Lemaître 1931; Gamow 1946).

General relativity provides a rigorous treatment of gravity with which the expansion of the Universe can be described. Space-time has 1 time dimension $x^0$ and three spatial coordinates $x^1$, $x^2$ and $x^3$. These coordinates are chosen to be comoving coordinates meaning that, as space-time expands, the coordinates move along with the expansion. Thus, the real distance, $r$, between two points, as would be measured if one could place a cosmic tape measure between them, is related to the comoving coordinate distance, $x$, as

$$r = a(t)x,$$

where $a(t)$ is the scale factor of the Universe which describes the rate at which the Universe expands. The cosmological principle means that the scale factor must be the same everywhere, therefore it is only a function of time. The distance between points in space-time, $ds$, is given by

$$ds^2 = \sum_{\mu,\nu} g_{\mu \nu} dx^\mu dx^\nu,$$

where $g_{\mu \nu}$ is the metric which accounts for the curvature of the space and $dx^i$ is the separation between points in the $i^{th}$ coordinate. The cosmological principle states that the Universe does not have any special locations. This provides us with a
simplification because it means that the spatial part of the metric must have a constant curvature. The most general spatial metric with constant curvature (incorporated into space-time) is the Robertson-Walker metric (Liddle 2015)

\[ ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \right] , \]

where \( k \) is a constant which tells us about the curvature of the space and a transition has been made from Cartesian to spherical polar coordinates.

The geometry of a Universe which obeys the cosmological principle can either be flat \((k = 0)\), spherical \((k > 0)\) or hyperbolic \((k < 0)\). A flat geometry obeys Euclidean geometry whereby, amongst other axioms, the angles of a triangle sum to 180°. The other types of geometry are non-Euclidean and can be visualised by projecting onto a lower dimensional space within our 3 dimensional space. A spherical geometry would relate to the surface of a sphere, whereby the angles of a triangle sum to greater than 180°. This is also known as a closed geometry since parallel lines always converge. A hyperbolic geometry would relate to the surface of a saddle, whereby the angles of a triangle sum to less than 180°. This is also known as an open geometry since parallel lines always diverge.

Next we need to know how the presence of matter curves space-time as this determines how the metric evolves. This is described by the Einstein equation

\[ R^\mu_\nu - \frac{1}{2} g^\mu_\nu R = \frac{8\pi G}{c^4} T^\mu_\nu , \]

where \( R^\mu_\nu \) is the Ricci Tensor, \( R \) is the Ricci scalar and \( T^\mu_\nu \) is the energy-momentum tensor of the matter. The Ricci tensor and scalar give the curvature of space-time. Assuming the matter is a perfect fluid with no viscosity or heat flow, the matter then has an energy-momentum tensor given by

\[ T^\mu_\nu = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} , \]

where \( \rho \) is the mass density and \( p \) is the pressure. The Friedmann equation can be derived as the time-time Einstein equation to be

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho , \]
where the dot notation always denotes the time derivative. This is a very important equation which describes how the Universe expands with time by showing how the scale factor evolves with time. The space-space Einstein equation gives the acceleration equation as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2}\right). \quad \text{(1.7)}$$

As its name suggests, the acceleration equation describes the acceleration of the scale factor with time. From the Friedmann and acceleration equations, the fluid equation can be derived to be

$$\dot{\rho} + 3\frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2}\right) = 0. \quad \text{(1.8)}$$

The fluid equation tells us how the density of matter in the Universe changes with time.

We now have two independent equations (one of the equations can be derived from the other two) and three unknowns: the scale factor $a(t)$; the mass density $\rho(t)$ and the pressure $p(t)$. Therefore another equation is needed; the equation of state relates the mass density $\rho$ to the pressure $p$. The equation of state differs depending on which type of particle is being considered. For non-relativistic matter (referred to as just ‘matter’ from this point) effectively no pressure is exerted and the equation of state is

$$p = 0. \quad \text{(1.9)}$$

For relativistic matter, including particles of light (referred to as ‘radiation’ from this point) and particles moving close to the speed of light, a radiation pressure is exerted and the equation of state is

$$p = \frac{\rho c^2}{3}. \quad \text{(1.10)}$$

Assuming a flat Universe ($k = 0$), it turns out that for a Universe comprised entirely of matter, the density and scale factor evolve as

$$\rho(t) = \frac{\rho_0}{a^3} \quad \text{(1.11)}$$

and

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}, \quad \text{(1.12)}$$
where the 0 subscript denotes the present day value. The Universe expands forever at an ever decreasing rate, asymptotically tending to a static Universe. For the case of a flat Universe comprised entirely of radiation, the density and scale factor can be shown to evolve as

$$\rho(t) = \rho_0 \frac{t}{a^4}$$

and

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}.$$  

The radiation Universe expands more slowly than the matter Universe. The Universe was initially dominated by radiation, however, the equations show that the density of radiation falls off faster than the density of matter as the Universe expands. Therefore at some point in time matter must come to dominate, leading to a change in the expansion rate of the Universe.

Cases can also be considered where the curvature is non-zero. Assuming a matter dominated universe, consider the Friedman equation in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2}.$$  

If $k$ is negative then the right hand side must be positive meaning that the universe expands forever. If $k$ is positive then the right hand side eventually becomes zero as the two terms become equal. This leads to a universe which collapses in on itself and is often referred to as the ‘Big Crunch’.

Looking back at the Hubble constant which relates the recessional velocity of galaxies, $v$, to their distance, we see that in fact it is not a constant at all because the expansion of the Universe varies with time. Therefore is it better to consider it generally as the Hubble parameter, $H$, and to consider only its present day value as the Hubble constant, $H_0$. The Hubble parameter defines the relation: $v = Hr$. Using the fact that $r = a(t)x$ and that the comoving distance is constant with time then the Hubble parameter can be written as

$$H = \frac{v}{r} = \frac{\dot{r}}{r} = \frac{\dot{a}}{a}.$$  

Thus the Friedmann equation can be written as
Another useful parameter to introduce is the density parameter, $\Omega$. Looking at the Friedmann equation above, there is a certain value of density which gives a flat Universe ($k = 0$). This is called the critical density and is given by

$$\rho_c = \frac{3H^2}{8\pi G}.$$  

Introducing the density parameter

$$\Omega(t) = \frac{\rho}{\rho_c},$$

the Friedmann equation can then be written as

$$[1 - \Omega(t)] = -\frac{kc^2}{a^2H^2},$$

where a density parameter equal to 1 gives a flat Universe.

There is one element still missing from the above equations which is known as the cosmological constant, $\Lambda$. When Einstein originally proposed his equations he believed that the universe was static. To achieve this solution he added an extra term associated with a repulsive force, which is now know as the cosmological constant. However Einstein was never happy with this since he realised that such a universe would be highly unstable and a small perturbation would lead to run-away expansion or contraction. Therefore he was relieved to learn of Hubble’s discovery of an expanding universe since he could now remove the term from his equations. However, recent observations of type Ia supernovae have actually shown that the expansion of the universe appears to be accelerating. To explain these observations the cosmological constant has now been reintroduced into the equations.

The cosmological constant fits into the Friedmann and acceleration equations as

$$\left(\frac{\dot{a}}{a}\right) = \frac{8\pi G}{3} \rho - \frac{k c^2}{a^2} + \frac{\Lambda}{3},$$  

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}.$$

The cosmological constant in terms of a critical density is then expressed as

$$\Omega_\Lambda(t) = \frac{\Lambda}{\Lambda_c} = \frac{\Lambda}{3H^2}.$$
The overall density parameter is then redefined as

\[ \Omega = \Omega_{\text{matter}} + \Omega_{\text{radiation}} + \Omega_{\Lambda}. \]  \hspace{1cm} 1.24

\( \Omega_{\Lambda} \) is referred to more generally as the dark energy density parameter. Dark energy is the energy which permeates all space and causes the accelerated expansion of the Universe. The cosmological constant is just one proposed form of dark energy. Another proposed form of the dark energy are scalar fields, whose energy density vary in time and space.

The cosmological constant can be considered as the energy of ‘empty’ space. An explanation of this so-called ‘zero-point’ energy comes from quantum mechanics. Due to the uncertainty principle, even empty space is not entirely empty; virtual particle pairs are constantly being created and annihilated. This means that the ground state energy of a vacuum cannot be zero. One problem with this theory, however, is that the predicted value of the cosmological constant is much larger than suggested by observations.

By looking at the rotation curves of galaxies, it becomes immediately apparent that there is not enough visible matter to hold the galaxy together. Therefore, in addition to the baryonic matter that we can see, there must also be an invisible component. This invisible matter is called ‘dark matter’, and actually contributes the majority of the overall matter density. Furthermore, it is likely that this matter is actually Cold Dark Matter (CDM) meaning that its velocity was much less than the speed of light in the early universe.

Combining the ideas within this section, the overall standard cosmological model is therefore usually referred to as the \( \Lambda \)CDM model. Current best estimates from the Planck satellite reveal that the mass-energy density of the universe is very close to the critical density, and is made up of 5% baryonic matter, 27% dark matter and 68% dark energy (Planck Collaboration, Ade et al. 2016).
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1.2.2. Inflation

The standard cosmological model cannot explain a number of aspects of our current Universe.

The flatness problem

The density parameter varies with time for a radiation dominated universe as

$$|1 - \Omega(t)| \propto t,$$

and for a matter dominated universe as

$$|1 - \Omega(t)| \propto t^{2/3},$$

where $|1 - \Omega(t)|$ represents the deviation from a flat universe. Thus, a flat universe is unstable meaning that a small deviation from flatness in the early universe will grow into a universe with significant curvature. However, measurements of our present day Universe show it to be very flat with a density which is very close to the critical density, $|1 - \Omega_0| \leq 0.005$ (Planck Collaboration, Ade et al. 2016). Extrapolating this value back to the earliest time scales (the Planck time) means that the initial deviation from flatness must have been around $|1 - \Omega| \leq 10^{-60}$. This extremely small value seems statistically far too small to be coincidental and there is no reason or preference why the density should be close to the critical density.

The horizon problem

Within the Universe information can only be transferred between two regions for which light has had time to traverse the space between. This is known as causal contact. For instance, we can only see a portion of the Universe (the observable Universe); light from beyond the observable Universe has not reached us yet. For two regions of the Universe to be in equilibrium they must be in causal contact or have been in causal contact at some point in the past. When we look at the Universe across all of the sky, the light (CMB) appears to be roughly the same temperature across the whole sky (2.7 K). This implies that opposite sides of the observable Universe must have been in equilibrium. However, how could they of been in causal contact if light from two opposite sides has only just had chance to reach us in the middle. Furthermore, the regions would have had to reach equilibrium by the time the CMB radiation was released at the time of last scattering, a very short time after
the Big Bang when the observable Universe was much smaller. In fact it turns out that any points separated by >2° on the sky would not have been in causal contact and therefore should have drastically different temperatures.

**The monopole problem**

Grand unified theories (GUTs) predicts the generation of a high amount of magnetic monopoles in the early Universe. These would be non-relativistic particles and would therefore become dominant over the radiation whose density reduces faster with expansion. If this is the case we would expect the Universe today to be dominated by magnetic monopoles, however they are yet to be observed.

In the 1980's the theory of inflation was first developed by (Guth 1981) as an extension to the standard model in an attempt to alleviate the flatness, horizon and monopole problems, whilst keeping the successes of the standard model of Big Bang cosmology intact. Inflationary theory proposes that the Universe underwent a short period of nearly exponential expansion which set the conditions for the Big Bang. During inflation the scale factor would therefore have been accelerating ($\ddot{a} > 0$). Looking at the Friedmann equation (Eq. 1.20), a scale factor with positive acceleration means that the time derivative of ($aH$) must be positive and therefore ($\Omega - 1$) is driven towards zero. This provides a solution to the flatness problem. If the ($\Omega - 1$) is driven close enough to zero then even with the subsequent expansion of the Universe ($\Omega - 1$) still remains very close to zero as we measure in the present Universe. The horizon problem can also be solved because a small part of the Universe, which was small enough to be in thermal equilibrium, rapidly expanded to be greater than the size of our current observable Universe. Thus, looking at opposite sides of the observable Universe, this would explain why they are the same temperature. The monopole problem is also able to be solved simply by realising the fact that inflation will rapidly reduce the concentration of magnetic monopoles throughout the Universe. Given the expected amount of inflation the dilution would be more than enough to explain why we do not see magnetic monopoles today.

The origin of the large scale structure of the Universe can also be explained by inflation. Beginning with an initial singularity (or possibly an eternally inflating universe with no origin), inflation drove the exponential expansion of the universe.
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At this point the inhomogeneities, anisotropies and the curvature of space were all smoothed out, fitting with our observations of the Universe today. The resulting universe is filled with an inflaton field. The inflaton field, however, was not completely homogeneous due to quantum fluctuations amplified during the inflationary process. At the end of the inflation era the inflaton field decayed into the familiar particles from the Standard Model of particle physics. At this point the initial conditions for Big Bang cosmology are created. Models of the fluctuations in the inflaton field and its subsequent decay have been shown to give rise to the current large scale structure in the Universe that we observe today.

1.3. The Cosmic Microwave Background

In the 1940’s George Gamow and others predicted that the Big Bang was actually a Hot Big Bang in which the early universe was an ionised plasma containing a radiation field of photons. Consequently it was predicted that there should be a residual radiation signature with a blackbody distribution in the microwave regime (Gamow 1946; Alpher et al. 1948). In the 1960’s Arno Penzias and Robert Wilson stumbled across microwave radiation coming from all directions on the sky at a temperature of 2.7 K (Penzias & Wilson 1965). Penzias and Wilson could not find a source for this radiation, either from their instrument or from known astronomical sources. By collaborating with Robert Dicke and Jim Peebles they realised that what they had detected was in fact the remnant radiation from the early universe; the Cosmic Microwave Background (CMB).

When the universe was 3 minutes old the Universe consisted of an ionised plasma containing nuclei and free electrons in a sea of photons. At this point the temperature of the Universe was far too hot for neutral hydrogen atoms to form because the atoms would be immediately ionised by the high energy photons. The high abundance of free electrons meant that the photons could only travel very short distances before interacting with an electron via Thomson scattering, meaning that the Universe was essentially opaque to the photons. As the Universe expanded and its temperature reduced, eventually when the universe was around 400,000 years old the temperature was reached where the photons did not have enough energy to ionise the forming atoms and neutral hydrogen was formed. At this point photons could travel freely
through the Universe without encountering an electron. These freely travelling photons are the CMB photons which we observe today. They allow us to observe a snapshot of how the Universe was at the time when they were released, known as the surface of last scattering (in effect, this event happened over a short period of time and is therefore really a ‘shell of last scattering’).

The CMB radiation has a distribution which is associated with a black-body with a temperature of 2.7 K. Originally the temperature was much hotter than this however since the photons were released the universe has expanded. This expansion caused the photons to be diluted and stretched to longer wavelengths, placing them in the microwave part of the electromagnetic spectrum. Precise examination of the CMB reveals that it is not perfectly smooth. Within it actually exist tiny temperature variations at the level of 0.001%. These anisotropies were first measured by COBE (Smoot et al. 1992) then subsequently measured to a higher precision by WMAP (Bennett et al. 2013) and Planck (Planck Collaboration, Adam et al. 2016a). The current best measurement of the CMB temperature map over the whole sky is shown in Figure 1.1.

![CMB Temperature Map](image)

Figure 1.1: A full-sky map of the CMB temperature as measured by the Planck telescope. (Planck Collaboration, Adam et al. 2016a)
The anisotropies in the CMB reflect the fluctuations in the matter density (which came from quantum fluctuations amplified by inflation) at the time of last scattering. In the primordial ionised plasma that existed just before the time of last scattering, the areas of higher matter density want to contract under gravity. This attraction is opposed by a radiation pressure exerted by the interaction of photons with electrons. This cosmic fluid acts like a driven oscillator creating acoustic waves which oscillate in time. Peaks and troughs of the acoustic waves represent areas undergoing compression and rarefaction. The compressions heat the plasma while the rarefactions cause it to cool. At the time of last scattering the hotter regions release more energetic photons whilst the cooler regions release less energetic photons. Therefore the spectrum of these acoustic waves is concreted into the CMB. Furthermore, at the time of last scattering the repulsive radiation pressure was removed and matter fluctuations have grown under the effect of gravity to form the large scale structure we see today.

Measurements of the CMB anisotropies are used to constrain cosmological models. The exact pattern of temperature fluctuation on the sky is not the most useful result, rather it is the statistical properties that are of most concern. Therefore a statistical treatment of the anisotropies is required as outlined by (Ryden 2002). It is useful to expand the fractional temperature anisotropies in terms of spherical harmonics

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi),$$

where $\theta$ and $\phi$ denote the direction on the sky in spherical polar coordinates and $Y_{lm}(\theta, \phi)$ are the spherical harmonics with coefficients $a_{lm}$. The index $l$ relates to the considerations of anisotropies on an angular scale of approximately $180^\circ/l$. A correlation function, $C(\theta)$, is defined which compares the fluctuations for two points on the surface of last scattering

$$C(\theta) = \langle \frac{\Delta T}{T}(\hat{n}) \frac{\Delta T}{T}(\hat{n}') \rangle_{\hat{n} \cdot \hat{n}' = \cos(\theta)},$$

where the two points are in the directions $\hat{n}$ and $\hat{n}'$, and separated by an angle $\theta$, given by $\cos(\theta) = \hat{n} \cdot \hat{n}'$. The angular brackets denotes an ensemble average over all possible observers in our Universe (averaging over index $m$). Expanding into spherical harmonics, the correlation function can be written as
where \( P_l \) are the Legendre polynomials. Thus the correlation function can be broken down into its multipole moments, \( C_l \). It is usual to plot the function

\[
\Delta T = \left( \frac{l(l+1)}{2\pi} C_l \right)^{\frac{1}{2}} \langle T \rangle,
\]

where \( \Delta T \) is plotted against \( \log(l) \) to give the radiation angular power spectrum.

For a specific model of the Universe, \( C_l \) can be calculated and compared with observation in order to evaluate the model. Because we are comparing one fixed observation against the ensemble average, \( C_l \) will differ from the measured multipole on the order of the RMS deviation of \( |a_{lm}|^2 \) from \( C_l \). This is known as cosmic variance. The cosmic variance is a limiting factor at low multipoles but becomes negligible at high multipoles.

![Figure 1.2: Measurement of the angular power spectrum overlaid with the ΛCDM model prediction. (Planck Collaboration, Adam et al. 2016a)](image)

The monopole \( (l = 0) \) corresponds to the background radiation field and vanishes if the mean \( T \) is correct. We cannot measure the monopole since we cannot measure the
CMB from a different position in the universe. The dipole \((l = 1)\) corresponds to the Doppler shift due to the motion of the observer relative to the CMB photons. It is the multipoles with \(l \geq 2\) that are of the most interest to cosmologists because they tell us about the fluctuations at the time of last scattering. Figure 1.2 shows the incredible agreement between the data from Planck and the prediction from the \(\Lambda\)CDM model.

By thinking of the acoustic waves you can understand where the peaks in the angular power spectrum come from. The acoustic waves can be decomposed into a series of modes with different wavelengths. Modes which were at the extrema of oscillation at the time of last scattering show the greatest contrast. The first peak in the angular power spectrum represents a mode that compressed just once. The second peak represents a mode that compressed then rarefied. The third peak represents a mode which compressed, then rarefied, then compressed again. This continues to form a harmonic series. The amplitude of the peaks decreases as you go to higher order. This is because the recombination process did not happen instantaneously. During the short time of recombination photons travelled randomly covering a small scale. The scale of the fluctuations of the higher order peaks is small enough that the random motion of the photons has mixed hot and cold photons on this scale, causing exponential damping of the peaks.

From the CMB angular power spectrum the values for cosmological parameters can be extracted. However, some parameters have similar effects on the power spectrum and therefore degeneracies are present. For example, the presence of gravitational waves and the process of reionisation both suppress the power spectrum at high multipoles. Fortunately, the intensity alone does not reveal all of the information contained within the CMB since the CMB is actually polarised. It is the extra information contained within the polarisation anisotropies of the CMB that can alleviate some of these degeneracies. Furthermore, the polarisation also has the potential to tell us if inflation really did occur.

### 1.3.1. Polarisation

When unpolarised radiation Thomson scatters off a free electron, the resulting radiation is linearly polarised in the plane perpendicular to the direction of the incident radiation (see Figure 1.3 (a)). If the overall incident radiation is isotropic or
has a dipole anisotropy then the combination of polarisation caused by individual events averages to zero. However, if the overall incident radiation has a quadrupole anisotropy then a net polarisation is observed (see Figure 1.3 (b)). Such quadrupole anisotropies were present in the radiation field at the time of last scattering, producing the polarisation in the CMB that we see today. These quadrupole anisotropies have 3 geometrically distinct sources (scalar, vector and tensor), each producing different patterns in the polarisation. Scalar perturbations arise due to density fluctuations in the cosmological fluid, vector perturbations represent vortical motions of the matter and tensor perturbations are generated by the presence of gravitational waves generated during inflation. (Hu & White 1997)

Figure 1.3: The net polarisation generated after Thomson scattering of unpolarised radiation: (a) incident from a single direction; and (b) with a quadrupole anisotropy. The orthogonal lines represent the strength of each polarisation component. See main text for further explanation. This figure was adapted from one created by (Hu & White 1997).

The linearly polarised CMB can of course be described using the Q and U Stokes parameters that represent the intensity differences between orthogonal Cartesian orientations of the polarisation. However, it is more convenient from a theoretical standpoint to use a coordinate free description, which distinguishes two types of linear polarisation patterns in the CMB using their different parities. The two components are the curl-free "E-mode" and the divergence-free "B-mode" (see Figure 1.4). Furthermore, these two components are directly linked to different physical processes in the early Universe (Zaldarriaga & Seljak 1997). At the surface
of last scattering the E-mode was produced by scalar and tensor perturbations whereas the B-mode was only produced by tensor perturbations. As stated, these tensor perturbations are caused by gravitational waves generated by inflation. Therefore, a detection of the B-mode would provide direct evidence of inflation, and a measurement of its strength would also set the energy scale of inflation. The strength of the B-mode is usually parameterised by tensor-to-scalar ratio, $r$, the ratio of the tensor fluctuations to the scalar fluctuations.

![Pattern of the E-mode and B-mode polarisation patterns surrounding an intensity extremum.](image)

Figure 1.4: Pattern of the E-mode and B-mode polarisation patterns surrounding an intensity extremum. The B-mode is orientated at 45° relative to the E-mode and also possesses handedness unlike the E-mode. Also shown are the Q and U Stokes parameters.

In addition to the B-mode which originates from the surface of last scattering (the "primordial B-mode"), the E-mode at the surface of last scattering can be transformed into an apparent B-mode due to gravitational lensing of the light as it travels towards us (the "lensing B-mode"). This gravitational lensing is caused by intervening large-scale structure which distorts the fabric of space-time, causing light to be deflected as it traverses this path. Fortunately the two B-mode components peak at different angular scales. However, at the intervening angular scales at which the two components may have comparable strengths, it is critical that the two components are not confused. This may be an issue for certain experiments which cannot view a large enough portion of the sky in order to target the large angular scales on which the primordial B-mode dominates. This is explored further in the proceeding section.
1.3.2. Polarisation Measurements

Measurements of the CMB are the primary evidence supporting the current standard model of cosmology and provide us with tight constraints on cosmological parameters. The CMB temperature anisotropy has been measured by many experiments over the past three decades. From these measurements the angular power spectrum has been constructed to a high precision and over a large range of scales from arcminutes to tens of degrees. The highest precision measurements to date come from the Planck satellite (shown previously in Figure 1.2). More specifically, what is plotted is actually the self-correlation of the temperature anisotropies $C_{l}^{TT}$ (TT). Self-correlations can also be plotted for the E-mode (EE) and B-mode (BB), as well as cross-correlations for (TE).

The first detection of the E-mode polarisation was made by DASI (Degree Angular Scale Interferometer) in 2002. Since then it has been extensively measured by a multitude of experiments, the results of which are summarised in Figure 1.5.

![Figure 1.5: Summary taken from (QUIET Collaboration et al. 2012) of published measurements of the pure E-mode power spectrum (EE) measured by a range of experiments: DASI (Leitch et al. 2005); BOOMERanG (Montroy et al. 2006); CBI (Sievers et al. 2007); MAXIPOL (Wu et al. 2007); CAPMAP (Bischoff et al. 2008); QUaD (QUaD Collaboration et al. 2009); BICEP (Chiang et al. 2010); WMAP (Larson et al. 2011); and QUIET (QUIET Collaboration et al. 2011; QUIET Collaboration et al. 2012). The solid line shows the predicted ΛCDM EE power spectrum.](image-url)
The B-mode is predicted to be even weaker, at a level of two orders of magnitude less than the E-mode. The first detection of the lensing B-mode was made in 2012 using data from SPTpol (Hanson et al. 2013) cross-correlated with data from Hershel-SPIRE (Griffin et al. 2010). In 2014, the BICEP2 team (BICEP2 Collaboration et al. 2014b) reported a detection of the B-mode including an apparent detection of the elusive primordial B-mode at $l \sim 80$ at a level of $r = 0.2$ (see Figure 1.6). However, subsequent analysis using the data from Planck revealed that the signal actually came mainly from polarised light emitted from dust within the Milky Way (BICEP2 and Keck Collaborations et al. 2015).

Figure 1.6: Summary taken from (BICEP2 Collaboration et al. 2014b) showing an apparent detection of the primordial B-mode (black circles). Also shown are 95% confidence level upper limits placed by previous experiments (see the caption of Figure 1.5 for references). The lower dashed line and the solid line show the predicted $\Lambda$CDM primordial BB spectrum for a tensor-to-scalar ratio of $r = 0.2$ and the gravitational lensing BB spectrum respectively. The upper dashed line shows the combined BB spectrum.

The B-mode signal remains undetected at degree scales where it is predicted to be by the inflationary model of the universe. The latest measurements are summarised in
Figure 1.7. The measurements follow the predicted lensing B-mode spectrum, with no extra signal which could be interpreted as the primordial B-mode. Thus, results from BICEP2 and the Keck array, combined with data from Planck, provide an current upper limit on the strength of the primordial B-mode, given as $r < 0.11$ at 95% confidence (BICEP2 Collaboration et al. 2016).

![Figure 1.7: Summary taken from (The POLARBEAR Collaboration et al. 2017) showing measurements of the B-mode polarisation power spectrum from POLARBEAR; SPTpol (Keisler et al. 2015); ACTPOL (Louis et al. 2017); Keck Array (BICEP2 and Keck Array Collaborations et al. 2015). Error bars correspond to 68.3% confidence levels. The triangular data point is an upper limit quoted at 95.4% confidence level. The black curve is a theoretical Planck 2015 lensed $\Lambda$CDM spectrum.]

1.3.3. Polarised Foregrounds

The B-mode signal is far from being the only source of polarised radiation in the sky. In fact it is swamped by much stronger polarised radiation coming from the rest of the Universe. These foregrounds must be carefully characterised and removed from the data so that they are not mistaken for the B-mode signal. Hence, foreground discrimination is likely to be the limiting factor in any B-mode experiment.
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The dominant foregrounds at angular scales emanate from within the Milky Way's interstellar medium. There are two main components of these diffuse foregrounds: synchrotron radiation generated from relativistic cosmic-ray electrons and positrons accelerated around galactic magnetic field lines (Sazhin et al. 2002); and thermal emission from dust grains, which is poorly understood at CMB frequencies. Other foreground sources include free-free emission, anomalous microwave emission and point sources. At low frequencies (<60 GHz) the synchrotron emission is dominant and at high frequencies (>60 GHz) dust emission dominates the foreground. The predicted level of diffuse foregrounds compared to the level of the B-modes for different tensor-to-scalar ratios is shown in Figure 1.8. Several data analysis methods have been proposed to separate these foregrounds from the B-mode signal (Leach et al. 2008). A primary technique is to use multi-frequency data to exploit the fact that the foregrounds and B-mode signal have different frequency spectra. Therefore the ability of an instrument to be able to measure the sky over a large range of frequencies is extremely important.

![Figure 1.8: Plot taken from (Remazeilles et al. 2017). Shown are the spectra of the synchrotron and dust foregrounds based on (Planck Collaboration, Adam et al. 2016b) computed on 40' angular scales. The striped lines indicate the foreground levels in the quietest 10% of the sky. Also shown are the E-mode and B-mode spectra for different tensor-to-scalar ratios, $r$. The grey bars indicate the frequency bands of the CORE experiment and can be ignored.](image-url)
1.4. **Multi-mode Technology for the Next Generation of B-mode Instruments**

Creating an instrument capable of measuring the B-mode signal is extremely challenging. The instrument must have a very high sensitivity and exquisite control of systematic effects, as well as being capable of observing across a range of frequencies. The rich scientific gains from measurements of the CMB has led to the development of new technology and detection techniques specifically for this purpose. Historically the detector assemblies, which collect light from the telescope and guide it onto detectors, have been based around the use of antenna feed horns. The general structure of a conical horn is demonstrated in Figure 1.9. Several of these horns are then arranged side-by-side to form the focal plane of the instrument, where each horn corresponds to a different pixel on the sky.

![Diagram of a conical antenna feed horn](image)

**Figure 1.9:** Basic representation of a conical antenna feed horn. Radiation from the telescope enters the horn from the left and is coupled to the detector on the right.

Conventionally, antenna feed horns have generally operated in a single-moded fashion. In a single-mode horn the waveguide filter is restricted in diameter so that it only permits the fundamental electromagnetic waveguide mode to propagate. Single-mode horns have well defined characteristics in terms of their beam pattern and the efficiency with which they couple radiation from the telescope onto the detectors. The beam pattern describes the acceptance of power from different directions (as a function of angle). This makes them relatively easy to integrate into the overall design of the instrument.

The sensitivity of an instrument is simply defined as the weakest signal that it can measure. This typically will depend on how the signal strength compares with the
total noise levels, hence it is important to identify the largest source of noise. When measuring electromagnetic radiation there is always a fundamental uncertainty due to the discrete particle-like nature of photons. This is called the ‘photon noise limit’. Historically, the intrinsic noise of the detectors outweighed the photon noise and therefore the emphasis was on improving the detectors. However, the current generation of bolometric detectors have advanced to the point where their intrinsic noise is close to the photon noise limit. Thus no significant gains in sensitivity can be achieved by improving individual detectors and emphasis is instead placed on reducing photon noise. The photon noise can be modelled as a Poisson distribution and therefore scales as $N^{-1/2}$, where $N$ is the total number of photons detected. Hence the only way to reduce photon noise is by collecting more photons.

One way to collect more photons is to simply increase the number of detectors. However, since each detector is accompanied by a horn, this leads to large heavy focal planes which are expensive and difficult to manufacture. This is a particular problem for space based or balloon-borne experiments because of their stringent size and weight requirements. Traditionally the horns have been made by direct machining. New manufacturing techniques have been developed to ease the manufacture of these large focal planes. One technique used for SPTpol (Austermann et al. 2012) involves etching holes of varying sizes into silicon sheets. These sheets are then aligned and stacked together to create an array of horns of the desired shape. The whole structure is then coated with a thin layer of metal to create the conductive surface. Part of the SPTpol focal plane is shown in the left of Figure 1.10.

Another solution is to replace the horn entirely and have a large focal plane of ‘detector assemblies’ based on planar technology. This design has been implemented for BICEP2 (see the right of Figure 1.10) and SPIDER (Filippini et al. 2011). By making use of photolithographic techniques the planar equivalents of the detector assembly can be mass produced to create thinner and lighter, large focal planes featuring hundreds or even thousands of detectors. The focal plane of BICEP2, for instance, is based on a new planar detector technology consisting of antenna-coupled transition-edge sensor (TES) arrays fabricated at the Jet Propulsion Laboratory (Kuo et al. 2008). The whole detection assembly shares a single, photolithographically fabricated, monolithic silicon wafer with the detector itself. The focal plane contains
a total of 500 photon-noise limited sub-Kelvin polarisation-sensitive bolometer detectors to achieve a high sensitivity.

Figure 1.10: *Left:* Part of the focal plane developed for SPTpol constructed from stacked silicon plates (Hubmayr et al. 2012). *Right:* BICEP2 planar focal plane (BICEP2 Collaboration et al. 2014a).

Another solution, which still uses the familiar horn technology, is also possible. The number of photons collected for individual horn receivers is increased by opening up the waveguide filter of the horn to allow higher order waveguide modes to propagate. The effect on the overall signal-to-noise ratio is explained by (Kogut et al. 2011). Noise-equivalent power (NEP) is the signal power that provides a signal-to-noise ratio of 1 in 0.5 seconds of integration time. The NEP of photon noise in a single linear polarisation is given by

\[
\text{NEP}^2 = \frac{4A\Omega(kT)^5}{c^2h^3} \int \frac{\alpha \epsilon \Phi \lambda^4}{e^x - 1} \left(1 + \frac{\alpha \epsilon f}{e^x - 1}\right) dx, \tag{1.31}
\]

where \(A\) is the effective aperture area, \(\Omega\) is the antenna beam solid angle, \(x = h\nu/kt\), \(\nu\) is the observing frequency, \(\alpha\) is the detector absorptivity, \(T\) is the physical temperature of the source, \(\epsilon\) is the emissivity of the source, and \(f\) is the power transmission through the optics (Mather 1982). A key characteristic of a light collecting system is its throughput, \((A\Omega)\). The throughput gives a measure of the total amount of radiation that an optical system handles. For a multi-mode system transmitting \(n\) modes at an observing wavelength \(\lambda\), the throughput is given by \(A\Omega = n\lambda^2\). Thus, since the desired signal increases linearly with throughput, \((A\Omega)\), but
the photon noise also increases with throughput as, $\text{NEP} \propto \sqrt{A \Omega}$, the overall signal-to-noise ratio improves as $\sqrt{A \Omega}$ (and therefore as $\sqrt{n}$).

Multi-mode horns therefore allow a sensitivity, equivalent to hundreds of single-mode horns, to be achieved using only a limited number of pixels. The drawback of the multi-mode operation is a loss of angular resolution, and the fact that the overall efficiency with which radiation is coupled from the telescope onto the detector is much more difficult to predict. This is because the beam pattern of the horn is affected by the propagation and detector coupling of each allowed mode; and the efficiency with which each mode is transmitted through the system, and with which each mode couples to the detector is likely to differ between modes. This point will be become clearer in Chapter 2. Such a multi-mode solution is planned for the LSPE-SWIPE instrument. The integration and development of the multi-moded technology is the subject of this thesis.

1.5. The Planck Satellite

The Planck telescope (Tauber, J. A. 2010) (see Figure 1.11) was a space based satellite which performed five full-sky surveys measuring the CMB temperature and polarisation anisotropies with unprecedented sensitivity and angular resolution. Planck also mapped foreground sources and achieved many other scientific objectives. The satellite was launched on 14 May 2009 under the control of the European Space Agency (ESA) and was deactivated on 23 October 2013.

Planck measured the sky in nine frequency bands between 27 - 900 GHz. The wide frequency range allowed efficient separation of foreground sources from the CMB signal based on their different frequency spectra. The frequency range was divided between two onboard instruments: the Low Frequency Instrument (LFI) (Bersanelli et al. 2010); and the High Frequency Instrument (HFI) (Lamarre, J.-M. et al. 2010). A single offset Gregorian-like telescope with an effective diameter of 1.5 m illuminated a focal plane shared by both instruments. The LFI had bands at 30, 44 and 70 GHz and was based on the use of a radiometer array cooled to 20 K. The HFI had bands at 100, 143, 217, 353, 545 and 857 GHz and used highly sensitive bolometric detectors cooled to 0.1 K. The four lowest bands, which were focused
around the peak of the CMB intensity, were polarisation-sensitive and performed direct measurements of the CMB. The two highest frequency bands were unpolarised and designed for observations of foreground sources.

Figure 1.11: Image of the fully constructed Planck satellite just before launch (Tauber, J. A. 2010).

The HFI aimed to have sensitivity limited by the photon-noise limit. A schematic of the HFI detector assembly chain (Maffei et al. 2010) is shown in Figure 1.12. In this design multiple horns were placed together to create a back-to-back horn (BTB horn). The BTB horn configuration for HFI was based on a previous design for a 90 GHz radiometer prototype (Church et al. 1996). It has previously been demonstrated that placing optical components between the horn and telescope will impact on the beam characteristics (Maffei et al. 2008). Thus, the advantage of the HFI BTB horn configuration was that the filters were not placed beyond the front horn, thereby limiting their effect on the final beam to a small loss in transmission. Furthermore the performance of the filters is reduced when they are not placed at the beamwaist with rays at normal incidence (Ade et al. 2010), which was not the case in the HFI pixel.

The walls of the horns were corrugated to produce a beam with higher polarisation purity and lower side-lobes in comparison to the equivalent smooth-walled
1 Introduction

single-mode horn. Side-lobes are peaks in the beam pattern outside of the main on-axis lobe and normally represent the collection of unwanted radiation from beyond the telescope’s main optical element. Further enhancements were made to the beam shape by giving the front horn a custom profile (shape). This profile also prevented shadowing of adjacent horns in the focal plane. The waveguide filter cut low frequencies and the mesh filters cut high frequencies. The mesh filters were positioned between the back and detector horn, which is where thermal filters were also located to prevent radiative transfer between heat stages.

![Diagram of a typical layout of a Planck-HFI detector assembly](image)

Figure 1.12: Schematic of a typical layout of a Planck-HFI detector assembly used for single- and multi-mode channels. Radiation from the telescope enters from the left and is coupled to the bolometer on the right. The front horn is responsible for the overall beam pattern definition and the cut-off of low frequencies. The back horn and detector horn are responsible for efficiently coupling the radiation onto the detector and filtering out high frequencies. A lens aids the efficient coupling of radiation between the back and detector horns. Horn corrugations not shown. Cooling is achieved in three thermal stages at 4 K, 1.6 K and 0.1 K. (Ade et al. 2010).

For the 545 and 857 GHz bands a resolution of the order of a few arcminutes was achieved. This was problematic because point source contamination was too high and observation strategy requirements were unfulfilled (Maffei et al. 2010). Therefore, whereas the first four HFI bands worked under the conventional single-mode operation, the 545 and 857 GHz bands were multi-moded. This made the beams more ‘top-hat’ shaped as opposed to Gaussian, widening the beamwidth to be closer to that of the lower frequency CMB channels. Of course the secondary benefit was an increased throughput and thus better sensitivity. A disadvantage was that the inclusion of higher modes drastically reduces polarisation purity, hence why these channels were not polarisation sensitive.

The design, simulation and testing of these multi-mode pixel assemblies is detailed in (Murphy et al. 2001; Colgan 2001; Gleeson 2004; Murphy et al. 2010).
The Planck Satellite

simulation and measurement of the multi-mode pixels was found to be drastically harder than for their single-mode counterparts. In the simulations each mode must be propagated individually in separate simulations, leading to very long simulation times. Furthermore, for the single-mode horn, the beam shape wholly depends on the simulation of the front horn. The effect of components behind the waveguide filter is to reduce the overall amplitude of the beam but not affect the beam shape. However, the final beam pattern of a multi-mode horn is a combination of the beam patterns from each mode which exists in the waveguide filter. Therefore the beam pattern depends on the balance of power between modes after the individual propagation of each mode through the horn assembly, and taking into account the coupling efficiency of each mode with the detector. This is difficult to simulate given the complex nature of some components and the free space coupling section between the horns, giving the possibility for multiple reflections and scattering between modes. Overall this makes the beam much less predictable. Simulation techniques available through commercially available software were far too time consuming therefore pre-existing modelling techniques were adapted to perform the simulations. A mode-matching technique was tailored to simulate the propagation of radiation through the detector assembly and telescope optics, onto the sky (see Murphy et al. 2010 for references).

After some modifications due to mechanical limitations, the final focal plane in Figure 1.13 was manufactured. All 545 and 857 GHz horns were tested and qualified at single component level (Murphy et al. 2010). Measurements of a horn under single-mode operation are relatively straightforward. Measuring the horn while it is operating in a multi-moded fashion is much more difficult. Single-mode measurements showed very little deviation from the simulated single-mode beam. This measurement was useful in ruling out any obvious fundamental problems with the horn such as manufacturing defects. Measurements of the multi-mode pixel, however, were much more difficult due to the same principles listed in the previous paragraph (modal filtering and coupling efficiency with the detector directly affects the beam shape). An exact measurement of the in-flight operation was not possible. Instead, measurements were conducted by placing the flight model multi-mode horns outside of the cryostat and using an extra horn assembly inside the cryostat to couple
with the bolometer (Murphy et al. 2010). Understandably this set-up led to some disagreement with simulation due to extra modal filtering.

Figure 1.13: Planck-HFI focal plane. The multi-mode 545 and 857 GHz horns are 3rd from the left.

The work performed on the multi-moded Planck pixels forms an outstanding knowledge base on which this thesis aims to extend.

1.6. The Large Scale Polarisation Explorer

The Large Scale Polarisation Explorer (LSPE) (The LSPE collaboration et al. 2012) is an experiment aiming to measure the CMB B-mode polarisation at large angular scales. Secondly, LSPE will map foreground polarisation from synchrotron and interstellar dust emission within the Milky Way. LSPE will survey the sky over the frequency range 40 - 240 GHz using two instruments: a ground based instrument called the Survey Tenerife Polarimeter (STRIP) (Bersanelli et al. 2012), covering the low frequencies; and a balloon-borne instrument called the Short Wavelength Instrument for the Polarisation Explorer (SWIPE) (de Bernardis et al. 2012), covering the high frequencies. A wide spectral range is covered in order to achieve component separation of the B-mode signal and the foreground by exploiting their different frequency spectra.
1.6 The Large Scale Polarisation Explorer

1.6.1. The STRIP Instrument

STRIP was originally designed to fly on the LSPE gondola with SWIPE, however it has now been converted into a ground based instrument to be deployed at the Izana site in Tenerife during Spring 2018. The design of STRIP is based on that of the QUIET experiment (Bischoff et al. 2013), which achieves a high level of suppression and redundancy of systematics. A schematic of STRIP is shown in Figure 1.14. A side-fed crossed-Dragone dual reflector telescope, with a projected diameter of 1.5 m, feeds a focal plane consisting of corrugated feed-horn antennas. This optical set-up offers excellent performance in terms of low cross polarisation and low beam asymmetry. Polarisation discrimination is achieved using a grooved polariser and OMT (Orthogonal Mode Transducer), which feed an array of coherent polarimeters using cryogenic High-Electron-Mobility-Transistor (HEMT) low noise amplifiers.

![Figure 1.14: The STRIP instrument. (http://planck.roma1.infn.it/lspe/strip.html accessed on 04/08/2017)](http://planck.roma1.infn.it/lspe/strip.html)

The focal plane is structured to have 49 Q-band horns (44 GHz) located in the centre, surrounded by 7 W-band horns (90 GHz). The Q-band horns are arranged into 7
hexagonal elements as shown in Figure 1.15. The horns have a custom optimised profile to help meet the optical requirements of STRIP. The low frequency Q-band pixels will map the polarised Galactic synchrotron foregrounds, and will also contribute to the CMB polarised sensitivity in foreground-clean regions.

Figure 1.15: Q-band focal plane of STRIP. Each hexagonal element is made up of many layers stacked together. Each layer has holes of different radii which come together to form the shape of the horns when stacked. This manufacturing technique allows for mass production of the horns, which would be difficult and expensive to achieve using direct machining or electroforming. (http://planck.roma1.infn.it/lspe/strip.html accessed on 04/08/2017)

1.6.2. The SWIPE Instrument

SWIPE will be the sole payload on a circumpolar stratospheric balloon flight to be launched from Longyearbyen (Svalbard islands). The scale of the SWIPE gondola is shown in Figure 1.16. An 800000 m$^3$ balloon will used to carry the payload to an altitude of around 37 km. The balloon will then travel in a circumpolar path at a latitude close to 78°N during the polar night. The polar night flight is necessary to avoid contamination from sunlight. The payload will be airborne for around 2 weeks and will then be recovered from Greenland. A flight test to verify a possible trajectory for the payload has been performed by The Italian Space Agency (ASI), the results of which are shown in Figure 1.17.
1.6 The Large Scale Polarisation Explorer

Figure 1.16: The LSPE-SWIPE gondola. (http://planck.roma1.infn.it/lspe/index.html accessed 07/08/2017)

Figure 1.17: A flight test for a balloon launched from Svalbard. (The LSPE collaboration et al. 2012)
1 Introduction

The scan strategy is aimed to maximise sky coverage. The gondola will spin at a rate of 2 rpm around the azimuthal direction, and the instrument is able to move in elevation to observe calibration sources. Combined with the circumnavigation of the North Pole, this enables SWIPE to scan around 1/4 of the sky. A degree scale angular resolution is targeted to match where the primordial B-mode signal is most prominent.

SWIPE will measure the sky in three bands centred at 140, 220 and 240 GHz. The two high frequency bands are for removal of the dust foreground, which dominates the overall foreground above 60 GHz. SWIPE is built to prioritise collection efficiency and polarisation purity, thus angular resolution is sacrificed for a higher sensitivity through the use of BTB multi-mode feed horns. The loss of angular resolution is not an issue since the primordial B-mode resides at large angular scales as stated. The key components and subsystems of SWIPE are outlined in Figure 1.18.

![Figure 1.18: The layout of the SWIPE instrument (de Bernardis et al. 2012). Geometrical optics rays show how light is focussed onto two curved focal planes. See main text for detail on the individual components.](image)

SWIPE is a Stokes polarimeter. Radiation from the sky enters the instrument through a vacuum window consisting of a thin polypropylene (PP) film covering a thick foam layer. A single on-axis 480 mm diameter high-density polyethylene (HDPE) lens with a cold aperture stop focuses the radiation onto two curved focal planes.
Polarisation discrimination is achieved using a rotating 500 mm diameter metal-mesh rotating Half-Wave Plate (HWP) and wire-grid polariser. The wire grid is orientated at 45° to the optical axis so that the one polarisation is transmitted onto one focal plane, and the orthogonal polarisation is reflected onto a second focal plane, thereby doubling the number of detectors available. The focal planes consist of a curved array of large-throughput multi-moded smooth-walled conical BTB horns feeding cryogenically cooled bolometric detectors. The $^4\text{He}$ cryostat operates through 4 thermal stages, from the 250 K in-flight ambient atmospheric temperature to 2 K. The thermal filters prevent radiative heat transfer between thermal stages. A $^3\text{He}$ sorption fridge then cools the focal plane and detectors down to 300 mK. The cryogen tank contains 290 L of $^4\text{He}$ resulting in around 2 weeks of operation. The size of the instrument is limited by the maximum manufacturable diameter of the band-pass filters, thermal filters, and the HWP.

The large rotating HWP is based on the metal-mesh technology (Pisano et al. 2012a) which has been used to construct bandpass filters, Quarter-Wave plates (QWP) and lenses (Ade et al. 2006; Pisano et al. 2012b; Pisano et al. 2013). The HWP is constructed by stacking metal-mesh grids, composed of orthogonal capacitive and inductive elements as shown in Figure 1.19. The capacitive and inductive elements interact with orthogonal incoming polarisations, shifting them oppositely in phase. After six grids the differential phase shift cumulates to 180° to create a HWP.

![Figure 1.19: A metal-mesh HWP made from layers of capacitive and inductive elements. (Pisano et al. 2012a)](image)

The manufacture of the grids is achieved using photolithographic techniques. A thin plastic mask is created with the required geometry printed onto it. A 2 $\mu\text{m}$ thick layer
of copper is evaporated onto a polypropylene (PP) substrate, to which a photoresist coating is then applied. A UV light is shone on the copper through the mask. The UV light reacts with the photoresist making the exposed regions soluble to a developer solution. The developer solution is used to remove the UV-exposed photoresist, leaving unprotected copper in these parts. Finally, the sample is placed in an etchant which removes the unprotected copper to give a device fitting the desired geometry. The grids are embedded inside a dielectric to set the spacing between grids and increase the overall robustness of the device. The HWP is rotated using a stepper motor positioned outside of the cryostat. The stepper motor is connected to the HWP by a set of thermally insulated shafts and gears. The rotation angle of the HWP is known to a precision of $< 0.1^\circ$ using pairs of optical fibres.

Figure 1.20: Geometrical optics simulation performed using ZEMAX (http://www.zemax.com/) by Prof. Marco de Petris. Shown are the half-wave plate (HWP), lens (L1), aperture stop (AS), polarisation-splitting wire grid (WG) and the two focal planes (CFP). The ray tracing paths for the centre and edge focal plane pixels are shown corresponding to the different frequency bands. Each focal plane consists of 165 pixels distributed between the different frequency bands. The black arrows show the scanning direction. (Lamagna et al. 2015)
The lens and detector assembly design of SWIPE are discussed in full detail in Chapter 3, however some brief points on their design are made here. The choice of having a single plano-convex lens creates a curved focal plane as demonstrated in Figure 1.20.

The layout of the SWIPE BTB horn assembly is shown in Figure 1.21. The high frequency cut-off is provided by a bandpass filter placed in the filter cap at the front of the front horn. The low frequency cut-off is provided by the waveguide filter at the base of the front horn. The detector is placed in a resonance cavity at the back of the BTB horn, a quarter of a wavelength from each wall. Coupling efficiency would be reduced if the cavity was attached directly to the waveguide filter, since each mode has a different effective wavelength in the guide. Therefore, a transition horn is included which expands the guide causing the range of effective wavelengths to narrow.

Figure 1.21: LSPE-SWIPE BTB horn detector assembly. Dimensions are shown in mm.

The detectors are optimised Transition Edge Sensors (TES) (see (Gualtieri et al. 2016). TES bolometers take advantage of the steep variation of resistance against temperature within the superconducting phase transition. High energy cosmic ray particles can be detrimental to the results of B-mode experiments if they are detected in large numbers. To combat this, a SiN spider-web mesh is used to absorb the incoming radiation. The structure of the mesh is made smaller than the wavelength of the desired radiation. Therefore the desired radiation is absorbed efficiently whilst most of the high energy cosmic rays will pass through the gaps in the mesh and not be detected. The TESs are linked to superconducting quantum interference devices (SQUIDs) amplifiers to perform the read-out electronics.
Figure 1.22: Projected SWIPE sensitivity compared to Planck HFI, SPIDER and the strength of B-modes for different values of the tensor-to-scalar ratio. The vertical dotted lines indicate the lowest multipoles (largest angular scales) which can be targeted by each experiment.

The projected sensitivity of SWIPE is shown in Figure 1.22. Assuming photon noise limited detectors, the sensitivity is calculated by considering the photon noise in the radiative background incident on the detectors. The radiative background includes sources of noise such as the thermal emission by the lens and the atmosphere, as well as the signal from the CMB itself.

1.6.3. Main Beam Polarisation Systematic Effects

Systematic effects arise due to imperfections in an instrument and cause a consistent error in the measurement. Each systematic must be well controlled, so not to become the limiting factor of the experiment. Some systematics can be eliminated by scan strategy, polarisation modulation or optical design. Other systematics can be limited through clever design. Remaining systematics must be understood to a high degree so that they can be accounted for in post-processing of the data. Understanding of
these systematics comes from simulations and measurements, as well as ground and in-flight calibration using well-known celestial objects.

The amount of systematic effects are numerous. Here, only the systematic effects associated with the polarisation of the main beam are focussed upon. Each polarisation systematic can be broadly categorised as either cross-polarisation or instrumental polarisation. Consider a general CMB polarimeter in which the polarisation of each pixel is measured by differencing the signal from two matched detector pairs, each sensitive to orthogonal polarisations. Cross-polarisation is the presence of the polarisation orthogonal to that which is desired for each polarised detector. This causes the E-mode signal to be converted into an apparent B-mode signal (E→B). Instrumental polarisation is an apparent measured polarisation for an unpolarised patch of sky after differencing the signal from the two detectors. This arises due to differences in the beam patterns of each detector and causes the unpolarised temperature anisotropy to be converted into an apparent E-mode and B-mode signal (T→E; T→B). B-mode measurements are highly susceptible to systematics effects due to the sheer weakness of the B-mode signal compared with strength of the temperature anisotropy and the E-mode. A further systematic which is important is the presence of far-sidelobes in the sky beam. These cause pickup of unwanted radiation from sources outside of the main beam, such as the Sun, Earth and Moon. A list of systematics relating to the sky beam shape is given in Table 1.1.

The symmetry of each systematic effect can be different. These symmetries are categorised in Table 1.1 and demonstrated pictorially in Figure 1.23. This is important since the temperature anisotropy coupling to the B-mode directly relates to the symmetry of the systematic (i.e. T, ∇T, ∇²T). If the two beams have the same azimuthal profile but with different beams sizes then a monopole effect is produced. If the beams have a different pointing then a dipole effect is produced. Finally, beams with a different ellipticity will produce a quadrupole effect. Monopole and dipole effects can be averaged by rotating the instrument azimuthally, however quadrupole effects have the same symmetry as the B-mode and therefore cannot be distinguished from it.
Table 1.1: Beam systematic effects as detailed in (Bock et al. 2008). ‘Differential’ refers to the differences between two polarised matched detector pairs used to measure the sky polarisation.

<table>
<thead>
<tr>
<th>Systematic effect</th>
<th>Azimuthal symmetry</th>
<th>Type</th>
<th>Conversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differential beam size</td>
<td>Monopole</td>
<td>Instrumental polarisation</td>
<td>T→B</td>
</tr>
<tr>
<td>Differential gain</td>
<td>Monopole</td>
<td>Instrumental polarisation</td>
<td>T→B</td>
</tr>
<tr>
<td>Differential pointing</td>
<td>Dipole</td>
<td>Instrumental polarisation</td>
<td>VT→B</td>
</tr>
<tr>
<td>Differential ellipticity</td>
<td>Quadrupole</td>
<td>Instrumental polarisation</td>
<td>V²T→B</td>
</tr>
<tr>
<td>Non-orthogonality of the polarisation vectors</td>
<td>Quadrupole</td>
<td>Cross-polarisation</td>
<td>E→B</td>
</tr>
<tr>
<td>Differential orientation</td>
<td>Quadrupole</td>
<td>Cross-polarisation</td>
<td>E→B</td>
</tr>
<tr>
<td>Optical cross polarisation</td>
<td>Quadrupole</td>
<td>Cross-polarisation</td>
<td>E→B</td>
</tr>
<tr>
<td>Far-sidelobe</td>
<td>-</td>
<td>-</td>
<td>T,E→B</td>
</tr>
</tbody>
</table>

(from bright celestial sources)

Figure 1.23: *From left to right:* the shape of beam systematics for differential beam width (monopole), differential pointing (dipole) and differential ellipticity (quadrupole). Figure adapted from one in (Bock et al. 2008).
Previous experiments (Archeops, SPIDER, BOOMERANG, QUAD, BICEP, PLANCK HFI) have discriminated between orthogonal polarisations using two polarisation sensitive detectors for each pixel. This has the disadvantage that any uncorrelated drifts between the two detectors measuring orthogonal polarisations will result in a systematic error. In SWIPE this is avoided by the use of the rotating HWP which modulates the polarisation. Combined with a polarising wire-grid in front of the focal plane, this allows the same detector to measure both polarisations. Furthermore, the HWP is the first element in the optical chain therefore any beam asymmetries become irrelevant. The HWP, however, introduces many systematic effects of its own. A full list of SWIPE systematics and their mitigation strategies can be found in (de Bernardis et al. 2012). A prediction of the level of main beam systematics for SWIPE using a horn-lens simulations is the final result of Chapter 3.

1.7. Outline of Thesis

Chapter 2 introduces the modal description of electromagnetic waves in circular waveguides. Important ideas are explained and key equations are derived which are required for later chapters. Chapter 3 describes the simulations performed on the SWIPE multi-mode horns and optics. The sky beam pattern is predicted for key pixels in the focal plane and the level of systematics is investigated. An analysis on the optimum location of the telescope focus in relation to the horn aperture is also conducted. Chapter 4 presents the measurement techniques undertaken to characterise the SWIPE multi-mode horns. The first part of the chapter deals with purely incoherent measurements using a bolometric detector. The second half of the chapter details an investigative study into the feasibility of using coherent measurement techniques to characterise the horns. An understanding of the simulation techniques used throughout this thesis is presented in Appendix A. This is provided outside of the main text since the simulation techniques have been implemented through the use of simulation software suites, and thus no direct manipulation of the equations describing the techniques has occurred.

LSPE is being developed primarily at Sapienza Università di Roma, Italy. The author of this thesis worked at Sapienza for a period of 6 months, funded by an STFC Long Term Attachment grant. During this time the author collaborated directly on the
1 Introduction

development of LSPE by developing the simulations of the horn in Chapter 3, and discussing the overall design and measurement of the horn pixel assembly.
2. The Electromagnetic Properties of Multi-Moded Horns

2.1. Introduction

At GHz frequencies, the traditional type of waveguides are hollow metallic structures, usually made from brass, copper, silver or aluminium. The general shape of these waveguides is usually either cuboidal or cylindrical. The propagating electric field within the waveguides can be conveniently described as a combination of electric field patterns which represent modal solutions to the wave equation. The natural set of electromagnetic modes are referred to as Transverse-Electric (TE) and Transverse-Magnetic (TM) modes. The equations used to describe these modes are derived in § 2.2.

In general in a CMB instrument, radiation passing through the telescope element must be collected and directed onto the detector with high efficiency. A conical horn antenna is the conventional choice of device to perform this task. Conical and BTB conical horns have been introduced briefly in Chapter 1 (§ 1.4 onwards). In modelling the behaviour of a horn it is equally valid, and often more convenient, to treat the system in reverse due to its reciprocal nature. The detector is treated as an emitter and the radiation is propagated backwards through the horn to reveal the field at the aperture, and thus the far-field beam pattern which determines the coupling with the telescope. This reciprocal treatment is used in the modelling of the multi-mode BTB horn antenna for SWIPE in Chapter 3. The simulation software used to perform the modelling uses the modal description of the fields to describe the excitation in the waveguide filter at the throat of the horn. To give the correct result for the case of a bolometric detector, it is appropriate to provide each modal excitation with equal power, weighted by the coupling efficiency of each mode with the detector. A problem occurs since the excitation of each mode in the simulation software is actually specified in terms of an excitation magnitude instead of an excitation power. An excitation magnitude of unity gives each mode its fundamental
power, and this fundamental power is different for each mode. Therefore, to achieve equal excitation power between modes, equations which describe a unity power for each mode in terms of the excitation magnitude are derived in § 2.3. For electrically large waveguides, many modes can propagate, leading to long simulation times and high computational requirements. Fortunately, the complexity of the simulation can be vastly reduced by taking advantage of the inherent symmetry of the modes. This symmetry is described in § 2.4.

In Chapter 4, incoherent and coherent measurement techniques are used to determine the far-field beam pattern of a prototype SWIPE horn. The incoherent technique attempts to directly measure the horn beam by replicating in-flight conditions using a room-temperature bolometer. The coherent detection technique makes use of a vector network analyser and associated techniques to indirectly measure the far-field beam. The difference between coherent and incoherent excitation of modes within the horn is discussed in detail in § 2.5.1. As modes propagate along the horn, the increase in radius causes scattering between modes. The horn can therefore be characterised in terms of a modal scattering matrix. In the coherent measurement technique the field is measured in a plane in front of the horn and propagated backwards to infer the field at the aperture. The Fourier optics techniques used to propagate the field are described in § 2.5.3. The modal content of the aperture field is then calculated and used to deduce the scattering matrix of the horn. Using the scattering matrix, the aperture field can be reconstructed and used to calculate the far-field beam by means of a Fourier transform. The technique of calculating the far-field is given in § 2.5.2. Modes are partially reflected at the aperture of the horn creating standing waves. The equation used to describe the standing waves envelope is derived in § 2.6.

2.2. Electric and Magnetic Field Equations of Cylindrical Waveguide Modes

A cylindrical waveguide of radius, \( a \), is described by cylindrical polar coordinates as shown in Figure 2.1. The guided electromagnetic wave propagates along the \( z \) direction and is confined in the transverse \( r\phi \) plane. It is useful to describe the overall electric field in terms of an infinite natural set of cylindrical waveguide modes. This modes set consists of two types of modes: those with \( E_z = 0 \) (TE
2.2 Electric and Magnetic Field Equations of Cylindrical Waveguide Modes

modes); and those with $H_z = 0$ (TM modes), where $E_z$ and $H_z$ represent the electric and magnetic field components along the $z$ direction respectively. Each of these modes represents a solution to the two dimensional wave equation and appropriate boundary conditions. In addition to this infinite mode set, a second mode set is also allowed, associated with the orthogonal polarisation vector; i.e. each mode has an orthogonal counterpart with an orthogonally orientated electric field. From this doubly infinite set, only a finite amount of modes are allowed to propagate freely in the waveguide, the exact number of which is determined by the waveguide radius. The remainder of the modes are evanescent in the waveguide and decay in power as they propagate. Throughout this section all theoretical treatment and simulations assume the walls of such a waveguide to be made from a perfect electric conductor, thus completely confining the field to the interior of the guide.

Figure 2.1: Coordinate system used to describe cylindrical waveguide modes. The radius of the waveguide is parameterised by $a$.

The modal analysis equations are derived here using the same method as that which is presented by (Ramo 1993). Considered are time-harmonic waves propagating down the waveguide with variations in time and along the propagation direction described by $e^{i\omega t - \gamma z}$. The propagation constant $\gamma$ determines the extent of attenuation of each mode. If $\gamma$ is imaginary then the mode propagates freely along the guide however if $\gamma$ is real then the mode is evanescent and will decay in power along the guide.

Maxwell’s equations in differential form are
2 The Electromagnetic Properties of Multi-Moded Horns

\[ \nabla \cdot D = \rho \]  \hspace{1cm} 2.1
\[ \nabla \cdot B = 0 \]  \hspace{1cm} 2.2
\[ \nabla \times E = -\dot{B} \]  \hspace{1cm} 2.3
\[ \nabla \times H = J + \dot{D}, \]  \hspace{1cm} 2.4

where all symbols have their usual meanings and the dot denotes a time derivative.

From Maxwell’s equations the wave equations can be derived

\[ \nabla^2 E = \frac{k^2}{\omega^2} \dot{E} \]  \hspace{1cm} 2.5
\[ \nabla^2 H = \frac{k^2}{\omega^2} \dot{H}. \]  \hspace{1cm} 2.6

These equations relate to the instantaneous field values however it is often more convenient to work with time invariant phasor forms of the fields. For a single frequency, a phasor represents the entire sinusoidal waveform (over all time) as a single complex number. The magnitude of the phasor gives the amplitude of the sine wave and the argument of the phasor gives the phase shift relative to the zero point. The instantaneous field value can be found from the phasor form by multiplying by \( e^{i\omega t} \) and taking the real part. From this point forward the fields correspond to the phasor forms (unless stated otherwise) however the notation is kept the same.

The time derivative of the instantaneous field becomes the phasor value multiplied by \( i\omega \), thus the wave equations for an electric and magnetic phasor field reduce to Helmholtz equations

\[ \nabla^2 E = -k^2 E \]  \hspace{1cm} 2.7
\[ \nabla^2 H = -k^2 H, \]  \hspace{1cm} 2.8

where \( k = 2\pi/\lambda \) is the wavenumber in the guide. By breaking down the \( \nabla^2 \) into longitudinal and transverse parts these equations reduce to

\[ \nabla_t^2 E = -(k_t^2)E \]  \hspace{1cm} 2.9
\[ \nabla_t^2 H = -(k_t^2)H, \]  \hspace{1cm} 2.10

where \( \nabla_t \) is the two-dimensional Laplacian in the transverse plane; where

\[ \frac{\partial^2 E}{\partial z^2} = \gamma^2 E \]  \hspace{1cm} 2.11

has been used; and where

\[ k_t^2 \equiv \gamma^2 + k^2 = k^2 - \beta^2 \]  \hspace{1cm} 2.12
2.2 Electric and Magnetic Field Equations of Cylindrical Waveguide Modes

has been defined. An efficient method is to find the z components of \( E \) and \( H \) that satisfy the boundary conditions and Eq. 2.9 and 2.10 respectively, then use Maxwell’s equations to find the remaining components of the field. Thus, taking only the z components, Eq. 2.9 and 2.10 become

\[
\nabla_z^2 E_z = -(k_z^2) E_z
\]

\[
\nabla_z^2 H_z = -(k_z^2) H_z.
\]

Inserting the transverse Laplacian in cylindrical coordinates these equations become

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} = -k_z^2 E_z
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial H_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \phi^2} = -k_z^2 H_z.
\]

2.2.1. \( TM \) Modes

These waves have \( H_z = 0 \) therefore Eq. 2.15 becomes the appropriate equation to solve. Using a separation of variables technique (Ramo 1993) the solution is found to be

\[
E_{z,mn} = [A_{mn}J_m(k_z r)] \left( \frac{\cos(m\phi)}{\sin(m\phi)} \right)
\]

where \( A_{mn} \) determines the magnitude of the mode, \( J_m \) is an \( m^{th} \) order Bessel function of the first kind. The solution also includes a second order Bessel function \( N_m(k_z r) \) but this becomes infinite at \( r = 0 \) for any value of \( n \) and therefore is not included in the solution. The sinusoidal terms refer to the azimuthal variation of the field. Two solutions are shown (shown stacked in the parentheses), associated with the two orthogonal versions of the same mode. If a single polarisation is used to excite the system then only one of the orthogonal mode sets is excited.

The equations relating the z components of the fields to the \( r \) and \( \phi \) components are given by (Ramo 1993) as

\[
E_r = -\frac{i}{k_z^2} \left[ \beta \frac{\partial E_z}{\partial r} + \omega \mu \frac{\partial H_z}{\partial \phi} \right]
\]

\[
E_\phi = \frac{i}{k_z^2} \left[ -\beta \frac{\partial E_z}{\partial \phi} + \omega \mu \frac{\partial H_z}{\partial r} \right]
\]

\[
H_r = -\frac{i}{k_z^2} \left[ \omega \epsilon \frac{\partial E_z}{\partial \phi} - \beta \frac{\partial H_z}{\partial r} \right]
\]
The Electromagnetic Properties of Multi-Mode Horns

\[ H_\phi = -\frac{i}{k_c} \left[ \omega \frac{\partial E_z}{\partial r} + \frac{\beta}{r} \frac{\partial H_z}{\partial \phi} \right]. \tag{2.21} \]

Inserting Eq. 2.17 into Eq. 2.18-2.21 with \( H_z = 0 \) gives the transverse components as

\[ E_r, mn = -\frac{i \beta}{k_c} A_{mn} J'_m(k_c r) \begin{pmatrix} \cos(m \phi) \\ \sin(m \phi) \end{pmatrix} \tag{2.22} \]

\[ E_\phi, mn = -\frac{i \beta m}{k_c^2 r} A_{mn} J_m(k_c r) \begin{pmatrix} -\sin(m \phi) \\ \cos(m \phi) \end{pmatrix} \tag{2.23} \]

\[ H_r = -\frac{E_\phi}{Z_{TM}} \tag{2.24} \]

\[ H_\phi = \frac{E_r}{Z_{TM}} \tag{2.25} \]

where the prime denotes the derivative with respect to \( r \) and where the impedance, \( Z_{TM} \), for each mode is given by

\[ Z_{TM} = \frac{\beta}{\omega \epsilon} = \frac{1}{\omega \epsilon} \sqrt{k_r^2 - k_z^2} = \frac{\omega \sqrt{\mu \epsilon}}{\omega \epsilon} \sqrt{1 - \left( \frac{k_z}{k_c} \right)^2} = Z_0 \sqrt{1 - \left( \frac{k_z}{k_c} \right)^2}, \tag{2.26} \]

where \( Z_0 = \sqrt{\mu / \epsilon} \) is the impedance of the waveguide medium. The boundary condition is such that \( E_z \) and \( E_\phi \) equal zero at the edge of the guide \((r = a)\). Imposing this condition on Eq. 2.17 reveals that \( k_c a \) is a zero of the Bessel function

\[ k_c a = p_{mn}, \tag{2.27} \]

where \( J_m(p_{mn}) = 0 \).

Eq. 2.23 shows that \( E_\phi(r = a) = 0 \) also. The cut-on frequency of any order TM mode can be found easily using Eq. 2.27. There are an infinite number of modes since there are an infinite number of Bessel functions of increasing order \( (J_m) \), each with an infinite number of roots \( n \). Each TM mode is therefore denoted \( TM_{mn} \) where \( m \) and \( n \) are associated with angular and radial variations in the field pattern respectively.

### 2.2.2. TE Modes

A similar derivation can be done for TE modes where \( E_z = 0 \). The solutions for the field components are given by

\[ H_{z, mn} = [A_{mn} J_m(k_c r)] \begin{pmatrix} \sin(m \phi) \\ \cos(m \phi) \end{pmatrix} \tag{2.28} \]
2.2 Electric and Magnetic Field Equations of Cylindrical Waveguide Modes

\[ E_{r,mn} = -\frac{i\omega \mu m}{k_e^2 r} A_{mn} J_m(k_c r) \left( \begin{array}{l} \cos(m\phi) \\ -\sin(m\phi) \end{array} \right) \]  
2.29

\[ E_{\phi,mn} = \frac{i\omega \mu}{k_e} A_{mn} J'_m(k_c r) \left( \begin{array}{l} \sin(m\phi) \\ \cos(m\phi) \end{array} \right) \]  
2.30

\[ H_r = -\frac{E_\phi}{Z_{TE}} \]  
2.31

\[ H_\phi = \frac{E_r}{Z_{TE}} \]  
2.32

where the prime denotes the derivative with respect to \( r \) and where the impedance, \( Z_{TE} \), for each mode is given by

\[ Z_{TE} = \frac{\omega \mu}{\beta} = \frac{Z_0}{\sqrt{1 - \left(\frac{k_e}{k}\right)^2}} \]  
2.33

The boundary condition gives that \( k_e a \) is a zero of the derivative of a Bessel function

\[ k_e a = p'_{mn}, \]  
2.34

where \( J'_m(p'_{mn}) = 0 \), and the modes are denoted as \( TE_{mn} \).

The modal cut-ons can also be expressed in terms of guide radius at a fixed frequency. Table 2.1 shows the cut-on value of each mode in terms of the radius of the waveguide expressed in terms of wavelengths.

Table 2.1: The cut-on radius in terms of wavelength for \( TM_{mn} \) and \( TE_{mn} \) modes up to \( m = 4, n = 4 \). The fundamental mode (\( TE_{11} \)) has been emboldened.

<table>
<thead>
<tr>
<th></th>
<th>( TM_{0n} )</th>
<th>( TM_{1n} )</th>
<th>( TM_{2n} )</th>
<th>( TM_{3n} )</th>
<th>( TM_{4n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( TM_{m1} )</td>
<td>0.3827</td>
<td>0.6098</td>
<td>0.8174</td>
<td>1.0154</td>
<td>1.2077</td>
</tr>
<tr>
<td>( TM_{m2} )</td>
<td>0.8785</td>
<td>1.1166</td>
<td>1.3396</td>
<td>1.5535</td>
<td>1.7610</td>
</tr>
<tr>
<td>( TM_{m3} )</td>
<td>1.3773</td>
<td>1.6192</td>
<td>1.8494</td>
<td>2.0714</td>
<td>2.287</td>
</tr>
<tr>
<td>( TM_{m4} )</td>
<td>1.8767</td>
<td>2.1205</td>
<td>2.3548</td>
<td>2.5820</td>
<td>2.8037</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( TE_{0n} )</th>
<th>( TE_{1n} )</th>
<th>( TE_{2n} )</th>
<th>( TE_{3n} )</th>
<th>( TE_{4n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( TE_{m1} )</td>
<td>0.60983</td>
<td><strong>0.29303</strong></td>
<td>0.4861</td>
<td>0.66864</td>
<td>0.84632</td>
</tr>
<tr>
<td>( TE_{m2} )</td>
<td>1.1167</td>
<td>0.843853</td>
<td>1.06731</td>
<td>1.27567</td>
<td>1.47734</td>
</tr>
<tr>
<td>( TE_{m3} )</td>
<td>1.61916</td>
<td>1.3586</td>
<td>1.58669</td>
<td>1.80576</td>
<td>2.01839</td>
</tr>
<tr>
<td>( TE_{m4} )</td>
<td>2.12053</td>
<td>1.86307</td>
<td>2.09613</td>
<td>2.32141</td>
<td>2.54077</td>
</tr>
</tbody>
</table>
The fundamental mode is the $TE_{11}$ mode followed by the $TM_{01}$ mode. The field patterns of the first 30 modes are shown in Figure 2.2. Note that only one orthogonal variation of each mode is shown, the orthogonal modes are found by rotating the field pattern by $90^\circ/m$. A longitudinal cut of the field patterns for the first two modes is shown in Figure 2.3.
2.2 Electric and Magnetic Field Equations of Cylindrical Waveguide Modes

Figure 2.2: A transverse cut of the electric and magnetic field patterns of the first 30 modes in a cylindrical waveguide. (Lee et al. 1985)
The Electromagnetic Properties of Multi-Moded Horns

Figure 2.3: A longitudinal cut of the electric and magnetic field patterns of the first two modes demonstrating the zero electric and magnetic field components in the z direction for the TE and TM modes respectively. (Terman 1943)

In summary, on combining Eq. 2.22 and 2.23; and Eq. 2.29 and 2.30, and substituting in Eq. 2.27 and 2.34 respectively, the electric field for TM and TE ordered modes is given by

\[ E_{mn}^{TM} = \left[ c_{mn}^{TM} j m \left( \frac{r}{\alpha} \right) \left( \cos(m\phi) \right) \hat{\Phi} + c_{mn}^{TM} \frac{m}{k_c} j m \left( \frac{r}{\alpha} \right) \left( -\sin(m\phi) \right) \cos(m\phi) \right] \hat{\Phi} \]

\[ E_{mn}^{TE} = \left[ c_{mn}^{TE} B \frac{m}{k_c} j m \left( \frac{r}{\alpha} \right) \left( \cos(m\phi) \right) \hat{\Phi} - c_{mn}^{TE} j' m \left( \frac{r}{\alpha} \right) \left( -\sin(m\phi) \right) \hat{\Phi} \right] \]

where

\[ c_{mn}^{TM} = \frac{-i \beta A_{mn}^{TM}}{k_c} \]

\[ c_{mn}^{TE} = \frac{-i \omega \mu A_{mn}^{TE}}{k_c} \]

and where the impedance of the modes is given by

\[ Z_{mn}^{TM} = Z_0 \sqrt{1 - \left( \frac{r_{mn}}{\omega \sqrt{\mu \epsilon a}} \right)^2} \]

\[ Z_{mn}^{TE} = \frac{Z_0}{\sqrt{1 - \left( \frac{r'_{mn}}{\omega \sqrt{\mu \epsilon a}} \right)^2}} \]

The effective wavelength of modes within the waveguide is given by


2.2 Electric and Magnetic Field Equations of Cylindrical Waveguide Modes

\[ \lambda_{TM} = \frac{2\pi}{\beta} = 2\pi / \sqrt{k^2 - \left( \frac{p_{mn}}{a} \right)^2} \]  \hspace{1cm} 2.41

\[ \lambda_{TE} = \frac{2\pi}{\beta} = 2\pi / \sqrt{k^2 - \left( \frac{p'_{mn}}{a} \right)^2} \]  \hspace{1cm} 2.42

For convenience, the electric field can also be expressed in Cartesian coordinates \((E_x = E_r \cos(\phi) - E_\phi \sin(\phi) ; E_y = E_r \sin(\phi) + E_\phi \cos(\phi))\) as

\[ E_{mn}^{TM} = \frac{C_{mn}^{TM}}{2} \left[ J_{m-1} \left( \frac{p_{mn}}{a} \right) \left( \cos(m-1)\phi \right) - J_{m+1} \left( \frac{p_{mn}}{a} \right) \left( -\sin(m+1)\phi \right) \right] \hat{x} \]

\[ - \left( J_{m-1} \left( \frac{p_{mn}}{a} \right) \left( \sin(m-1)\phi \right) + J_{m+1} \left( \frac{p_{mn}}{a} \right) \left( \cos(m+1)\phi \right) \right) \hat{y} \]  \hspace{1cm} 2.43

\[ E_{mn}^{TE} = \frac{C_{mn}^{TE}}{2} \left[ J_{m-1} \left( \frac{p'_{mn}}{a} \right) \left( \cos(m-1)\phi \right) + J_{m+1} \left( \frac{p'_{mn}}{a} \right) \left( -\sin(m+1)\phi \right) \right] \hat{x} \]

\[ - \left( J_{m-1} \left( \frac{p'_{mn}}{a} \right) \left( \sin(m-1)\phi \right) - J_{m+1} \left( \frac{p'_{mn}}{a} \right) \left( \cos(m+1)\phi \right) \right) \hat{y} \]  \hspace{1cm} 2.44

\[ H_{mn}^{TM} = \frac{C_{mn}^{TM}}{2Z_{mn}} \left[ J_{m-1} \left( \frac{p_{mn}}{a} \right) \left( \sin(m-1)\phi \right) + J_{m+1} \left( \frac{p_{mn}}{a} \right) \left( \cos(m+1)\phi \right) \right] \hat{x} \]

\[ + \left( J_{m-1} \left( \frac{p_{mn}}{a} \right) \left( -\sin(m-1)\phi \right) - J_{m+1} \left( \frac{p_{mn}}{a} \right) \left( \cos(m+1)\phi \right) \right) \hat{y} \]  \hspace{1cm} 2.45

\[ H_{mn}^{TE} = \frac{C_{mn}^{TE}}{2Z_{mn}} \left[ J_{m-1} \left( \frac{p'_{mn}}{a} \right) \left( \sin(m-1)\phi \right) - J_{m+1} \left( \frac{p'_{mn}}{a} \right) \left( \cos(m+1)\phi \right) \right] \hat{x} \]

\[ + \left( J_{m-1} \left( \frac{p'_{mn}}{a} \right) \left( -\sin(m-1)\phi \right) + J_{m+1} \left( \frac{p'_{mn}}{a} \right) \left( \cos(m+1)\phi \right) \right) \hat{y} \]  \hspace{1cm} 2.46

where the recurrence relations for a general Bessel function \(J_m(x)\)

\[ J_{m+1}(x) + J_{m-1}(x) = \frac{2m}{x} J_m(x) \]  \hspace{1cm} 2.47

\[ J_{m+1}(x) - J_{m-1}(x) = -2J'_m(x) \]  \hspace{1cm} 2.48

have been used.

Modes decay in amplitude as

\[ e^{-\gamma z} \]  \hspace{1cm} 2.49

where

\[ \gamma = i\beta = (k_c^2 - k^2)^{\frac{1}{2}} \]  \hspace{1cm} 2.50

and \(k_c\) is given from Eq. 2.27/2.34 for TM and TE modes respectively.
2.3. Modal Power

The power per unit area of an electromagnetic wave passing through a surface is described by the Poynting vector

\[ S(t) = E(t) \times H(t), \]

where the fields represent the instantaneous electric and magnetic fields. Therefore the Poynting vector gives the instantaneous power flow. This power flow fluctuates as the electromagnetic wave oscillates. For our purposes it is more useful to know the time averaged power flow which does not fluctuate. The time-averaged instantaneous Poynting vector is thus given by

\[ \langle S \rangle = \frac{1}{T} \int_0^T S(t) \, dt = \frac{1}{T} \int_0^T E(t) \times H(t) \, dt, \]

where \( T \) is the time period of a full cycle of the sinusoidal wave. As with the derivation of the electric field equations for modes, for a single frequency, a more convenient definition is found by expressing the electric and magnetic field using phasor notation. Again, subbing in the instantaneous field in terms of the phasor field \( E(t) = \text{Re}[E e^{i\omega t}] \) but keeping the notation the same, Eq. 2.52 becomes

\[ \langle S \rangle = \frac{1}{T} \int_0^T \text{Re}(E e^{i\omega t}) \times \text{Re}(H e^{i\omega t}) \, dt \]

\[ = \frac{1}{T} \int_0^T \frac{1}{2} (E e^{i\omega t} + E^* e^{-i\omega t}) \times \frac{1}{2} (H e^{i\omega t} + H^* e^{-i\omega t}) \, dt \]

\[ = \frac{1}{T} \int_0^T \frac{1}{4} (E \times H^* + E^* \times H + E \times H e^{2i\omega t} + E^* \times H^* e^{-2i\omega t}) \, dt \]

\[ = \frac{1}{T} \int_0^T \frac{1}{2} \text{Re}(E \times H^*) + \frac{1}{2} \text{Re}(E \times H e^{2i\omega t}) \, dt \]

\[ \langle S \rangle = \frac{1}{2} \text{Re}(E \times H^*) + 0. \]

A new Poynting vector expressed in terms of phasors is thus defined as

\[ S = \frac{1}{2} E \times H^*, \]

with the real part representing the time-averaged power flow as seen from Eq. 2.53. Additionally, the imaginary part represent the reactive power, which is the power that is returned to the source due to the interference from standing waves for instance.
2.3 Modal Power

For the field in the waveguide the power flow in the z direction is given by integrating over the surface of the transverse plane

\[ P = \int \text{Re}(S) \cdot dS = \int \frac{1}{2} \text{Re}(E \times H^*) \cdot dS \]

\[ P = \int \frac{1}{2} \text{Re}(E_r H^*_\phi + E_\phi H^*_r) \cdot dS. \]

Subbing in Eq. 2.44 and 2.25

\[ P = \int \frac{1}{2} \text{Re}\left(\frac{|E|^2 + |E|^2}{z}\right) \cdot dS \]

where \(|E|\) is the complex magnitude of \(E\) and \(Z\) is either \(Z_{TM}\) or \(Z_{TM}\) depending on the mode. Substituting in Eq. 2.22 and 2.23 (TM modes) or Eq. 2.29 and 2.30 (TE modes) into Eq. 2.57 and expanding the integral gives

\[ P = \frac{C_{mn}^2}{2Z} \int_0^a \int_0^{2\pi} \left\{ J_{m}^2(k_c r) \left( \cos^2(m\phi) \right) + \frac{m^2}{k_c^2 r^2} J_{m}^2(k_c r) \left( \sin^2(m\phi) \right) \right\} r dr d\phi, \]

where

\[ C_{mn} = C_{mn}^{TM} \quad \text{and} \quad Z = Z_{mn}^{TM} \quad (\text{TM modes}) \]

\[ C_{mn} = C_{mn}^{TE} \quad \text{and} \quad Z = Z_{mn}^{TE} \quad (\text{TE modes}). \]

Performing the azimuthal integral gives

\[ P = \frac{C_{mn}^2}{2Z} \int_0^a \int_0^{2\pi} \left\{ J_{m}^2(k_c r) \left( 1 + \delta_{m0}\pi \right) + \frac{m^2}{k_c^2 r^2} J_{m}^2(k_c r) \left( 1 + \delta_{m0}\pi \right) \right\} r dr \]

\[ = \frac{C_{mn}^2 (1 + \delta_{m0})\pi}{2Z} \int_0^a \left\{ J_{m}^2(k_c r) + \frac{m^2}{k_c^2 r^2} J_{m}^2(k_c r) \right\} r dr, \]

where \(\delta_{m0}\) is a Kronecker delta function (zero when \(m = 0\)), representing the fact that \(m = 0\) modes do not have an orthogonal counterpart.

Using the recurrence relations for Bessel functions (Eq. 2.47 and 2.48) the integral reduces to

\[ P = \frac{C_{mn}^2 (1 + \delta_{m0})\pi}{2Z} \int_0^a \left\{ J_{m+1}^2(k_c r) + J_{m-1}^2(k_c r) \right\} r dr, \]

which can be evaluated using the integral identity (Ramo 1993)

\[ \int \mu J_\phi^2(\alpha \mu) d\mu = \left[ \frac{\mu^2}{2} \left( J_\phi^2(\alpha \mu) + \left( 1 - \frac{\nu^2}{\alpha^2 \mu^2} \right) J_\phi^2(\alpha \mu) \right) \right], \]

giving
The Electromagnetic Properties of Multi-Mode Horns

\[ P = \frac{C^2_m (1 + \delta_m \alpha^2)}{4Z} \left( J^2_{m+1}(k_c a) + \left( 1 - \frac{(m+1)^2}{k_c^2 a^2} \right) J^2_{m+1}(k_c a) + J^2_{m-1}(k_c a) + \left( 1 - \frac{(m-1)^2}{k_c^2 a^2} \right) J^2_{m-1}(k_c a) \right) \]

2.3.1. **TM Modes**

Using the recurrence relation Eq. 2.47 and substituting in the fact that \( J_m(k_c a) = J_m(p_{mn}) = 0 \) for TM modes means that

\[ J_{m+1}(p_{mn}) = -J_{m-1}(p_{mn}). \]

Therefore, using the identities (Ramo 1993)

\[ \mu J'_{v+1}(\mu) = -(v+1)J_{v+1}(\mu) + \mu J_v(\mu) \]
\[ \mu J'_{v-1}(\mu) = (v-1)J_{v-1}(\mu) - \mu J_v(\mu) \]

for TM modes means that

\[ J_{m+1}(p_{mn}) = -J_{m-1}(p_{mn}). \]

Therefore, using Eq. 2.65 and 2.66 to substitute for the \( J' \) terms in Eq. 2.63 and using Eq. 2.64 gives

\[ P_{TM} = \frac{|C^TM_{mn}|^2 (1 + \delta_m \alpha^2) \pi a^2}{4Z^TM_{mn}} J^2_{m+1}(p_{mn}). \]

2.3.2. **TE Modes**

Using the recurrence relations and substituting in the fact that \( J'_m(k_c a) = J'_m(p'_{mn}) = 0 \) for TE modes means that

\[ J_{m+1}(p'_{mn}) = J_{m-1}(p'_{mn}) = \frac{m}{p_{mn}} J_m(p'_{mn}). \]

Therefore, using Eq. 2.65 and 2.66 to substitute for the \( J' \) terms in Eq. 2.63 and using Eq. 2.64 gives

\[ P_{TE} = \frac{|C^{TE}_{mn}|^2 (1 + \delta_m \alpha^2) \pi a^2}{4Z^{TE}_{mn}} \left( 1 - \frac{m^2}{p_{mn}^2} \right) J^2_{m}(p'_{mn}). \]

2.3.3. **Power Normalised Modes**

If the modes are normalised so that they carry equal power, \( P \), then

\[ C^{TM}_{mn} = \left[ \frac{P (1 + \delta_m \alpha^2) \pi a^2}{4Z^TM_{mn}} J^2_{m+1}(p_{mn}) \right]^{\frac{1}{2}} \]

and

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2.4 Modal Symmetry

\[ C_{mn}^{TE} = \left[ \frac{P(1 + \delta_{m0}) \pi a^2}{4 Z_{mn}^{TE}} \left( 1 - \frac{m^2}{p_{mn}^2} \right) j_m^2(p_{mn}) \right]^{\frac{1}{2}}. \]  

Alternatively, if the modes are normalised so that

\[ \int (E_m \cdot E_m^*) dS = 1, \]

evaluating the integral similarly to Eq. 2.57 yields the constants as

\[ C_{mn}^{TM} = \left[ \frac{(1 + \delta_{m0}) \pi a^2}{2} j_{m+1}^2(p_{mn}) \right]^{\frac{1}{2}}, \]

\[ C_{mn}^{TE} = \left[ \frac{(1 + \delta_{m0}) \pi a^2}{2} \left( 1 - \frac{m^2}{p_{mn}^2} \right) j_m^2(p_{mn}) \right]^{\frac{1}{2}}. \]

2.4. Modal Symmetry

Each of the \( TE \) and \( TM \) modes have an inherent symmetry plane along the \( x \)- and \( y \)-axis which can be exploited in order to reduce computational requirements and simulation time. There are two types of symmetry planes: perfect electric (PE) and perfect magnetic (PH) as illustrated in Figure 2.4. PE planes are defined such that they could be replaced by an ideal electrically conducting wall without changing the field structure, hence the electric field is purely perpendicular and the magnetic field is purely tangential. Similarly, a PH plane could be replaced by an ideal magnetically conducting wall without changing the field structure, hence the magnetic field is purely perpendicular and the electric field is purely tangential. (FEKO user manual)

![Figure 2.4: Perfect electric and magnetic planes of symmetry, showing the only component of the electric and magnetic fields which is present. (FEKO user manual)](image)

The symmetry of the \( TE \) and \( TM \) modes is shown in Table 2.2. The symmetry of a mode is dependent on whether it is a \( TE \) or \( TM \) mode and whether its azimuthal
index number \((m)\) is odd or even. By definition the symmetry of the orthogonal version of the mode is orthogonal \((\text{PE} \rightarrow \text{PH}; \text{PH} \rightarrow \text{PE})\).

Table 2.2: Categorisation of the four possible combinations of perfect electric (PE) and perfect magnetic (PH) symmetry planes of the cylindrical waveguide modes based on mode type \((TE\) or \(TM\)) and the parity of the azimuthal index number \(m\).

<table>
<thead>
<tr>
<th>Mode type</th>
<th>Parity of (m)</th>
<th>x-axis symmetry</th>
<th>y-axis symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>(TM)</td>
<td>Even</td>
<td>PE</td>
<td>PE</td>
</tr>
<tr>
<td>(TE)</td>
<td>Odd</td>
<td>PE</td>
<td>PH</td>
</tr>
<tr>
<td>(TM)</td>
<td>Odd</td>
<td>PH</td>
<td>PE</td>
</tr>
<tr>
<td>(TE)</td>
<td>Even</td>
<td>PH</td>
<td>PH</td>
</tr>
</tbody>
</table>

2.5. Multi-mode Smooth-walled Conical Horn

The BTB horn of SWIPE is shown again in Figure 2.5 in order to illustrate the points made within this section, ignoring the effects of the filter cap. A full description of the horn simulation is given in Chapter 3, however some fundamental points about the behaviour of modes within the horn are stated here. In the simulation of the BTB horn, the detector is instead treated as an emitter, taking advantage of the reciprocal nature of the system. Modes are excited individually in the waveguide filter, with their excitation power weighted by the efficiency with which they couple to the detector through the transition horn. As these modes pass through the front horn the change in waveguide radius causes power to be scattered between modes. In the case of an azimuthally symmetric component such as a conical horn, the scattering is restricted to take place between modes of the same azimuthal order (same azimuthal index \(m\)) only. The exact scattering relation depends on the profile of the horn. The resultant electric field at the aperture of the horn can be found, and thus the far-field beam pattern of the horn can be calculated.
2.5 Multi-mode Smooth-walled Conical Horn

Figure 2.5: LSPE-SWIPE BTB horn design. The transition horn and detector cavity are designed to efficiently couple radiation from the waveguide filter onto the detector. The waveguide in the centre acts as a modal filter, determining how many modes the horn can support and providing a high frequency cut-off. The filter cap provides the corresponding low frequency cut-off.

2.5.1. Incoherent and Coherent Operation

Throughout later chapters in this thesis, techniques and results are referred to as either ‘coherent’ or ‘incoherent’, and the term ‘modal field’ is used. Each of these terminologies are defined within this section. Figure 2.6 provides an illustration to aid the text.

In an incoherent system a detector, such as a bolometer, absorbs only the total power of each incident mode and ignores any phase information. Thus, each mode in the waveguide filter couples independently to the detector through the transition horn, or using the reciprocity of the system, each mode is excited by the detector independently with equal power (assuming perfect coupling of each mode with the detector through the transition horn). These waveguide filter modes can therefore be described as being partially coherent because they are spatially coherent but have no fixed phase relationship (Withington & Murphy 1998). Such a system is referred to in this thesis as ‘incoherent’.

Each partially coherent field is referred to as a ‘modal field’. The terminology is introduced to distinguish from talking about individual cylindrical waveguide modes. Each modal field is associated with a different mode in the waveguide filter. The waveguide mode scatters into different modes as it travels through the front horn, however the scattered modes are still fully coherent between themselves and part of the same modal field associated with the original waveguide mode. The horn is
described as being single-moded when there is only one mode present in the waveguide filter (1 modal field), and multi-moded when there are multiple modes present in the waveguide filter (multiple modal fields).

For an incoherent system, the overall incoherent multi-mode electric field at any point along the horn is found by propagating independently each modal field to that point, then summing in quadrature the electric fields of each modal field. The same principal applies to calculating the incoherent multi-mode far-field beam pattern. For a coherent system the detector does not ignore phase information. Thus each of the modal fields present in the waveguide filter are now fully coherent (but it is still allowable to treat them separately). Therefore the overall coherent multi-mode electric field at any point along the horn is found by propagating each modal field to that point, then performing a complex sum of the electric fields of each modal field.

Figure 2.6: Illustration of the incoherent and coherent behaviour of modes in a conical horn antenna. See main text for explanation.
2.5 Multi-mode Smooth-walled Conical Horn

2.5.2. Far-field Calculation

For smooth-walled conical horns (specified by the parameters defined in Figure 2.7) the field at the aperture can be well approximated as the theoretical $TE_{mn}$ and $TM_{mn}$ waveguide modal equations (Eq. 2.43 and 2.44) in the waveguide filter, plus a phase error term given by

$$\exp\left(-\frac{ik(r')^2}{2L}\right)$$  \hspace{1cm} (2.75)

to account for the horn flare, where $L$ is the horn slant length (Olver et al. 1994).

The far-field can be calculated by Fourier transforming the aperture field. The full treatment of the aperture field to far-field calculation is found in (Balanis 2005) and is only valid for large values of $kR$. The Fourier transform expressed in Cartesian components is given by

$$f_x(\theta, \phi) = \int_{-a}^{a} \int_{-a}^{a} E_{xa}(x', y') \exp(ik(x'\sin(\theta)\cos(\phi) + y'\sin(\theta)\sin(\phi))dx'dy'$$  \hspace{1cm} (2.76)

$$f_y(\theta, \phi) = \int_{-a}^{a} \int_{-a}^{a} E_{ya}(x', y') \exp(ik(x'\sin(\theta)\cos(\phi) + y'\sin(\theta)\sin(\phi))dx'dy'$$  \hspace{1cm} (2.77)

where $E_a$ is the field at the aperture. For a discretely sampled field the integration is replaced by a summation over all measurement points giving

$$f_x(\theta, \phi) = \sum \sum E_{xa}(x', y') \exp(ik(x'\sin(\theta)\cos(\phi) + y'\sin(\theta)\sin(\phi))dx'dy'$$  \hspace{1cm} (2.78)

$$f_y(\theta, \phi) = \sum \sum E_{ya}(x', y') \exp(ik(x'\sin(\theta)\cos(\phi) + y'\sin(\theta)\sin(\phi))dx'dy'$$  \hspace{1cm} (2.79)

where $dx'$ and $dy'$ are the distances between measurement points.

The far-field is then given by

$$E(R, \theta, \phi) \equiv \frac{k\exp(-ikR)}{2\pi R} [ (f_x \cos(\phi) + f_y \sin(\phi))\hat{\theta}$$

$$+ \cos(\theta)(-f_x \sin(\phi) + f_y \cos(\phi))\hat{\Phi}].$$  \hspace{1cm} (2.80)

From a measurement standpoint, it is often more convenient to express the far-field in terms of Ludwig’s III polarisation (Ludwig 1973) which defines the polarisation as coordinates on a spherical surface

$$E_{y-pol} = E_\theta \sin(\phi) + E_\Phi \cos(\phi)$$  \hspace{1cm} (2.81)

$$E_{x-pol} = E_\theta \cos(\phi) - E_\Phi \sin(\phi).$$  \hspace{1cm} (2.82)
Figure 2.7: Parameter definitions for far-field calculations showing the polar angle $\theta$ and the azimuthal angle $\phi$. 
We will see in Chapter 3 that, although this approximate method of finding the aperture field is useful as a quick check of the horn beam pattern, in order to achieve a higher accuracy the horn must be simulated using computational electromagnetic techniques.

### 2.5.3. Fourier Optics

An electric field measured in front of the horn can be propagated backwards in order to infer the electric field at the horn aperture. If the aperture is many wavelengths across ($2a \gg \lambda$), the propagation can be done using scalar diffraction theory, where the light is approximated by a complex scalar potential. The full method is explained in (Goodman 1996). The parameters used are the same as those defined in Figure 2.7. For distances of $z > \lambda/2$ the diffraction at the aperture is well described by the Rayleigh-Sommerfeld diffraction formula (complete diffraction integral)

$$u(x',y') = \frac{1}{2\pi} \int \int u(x,y,z) \frac{\partial}{\partial z} \left( \frac{e^{ikR}}{R} \right) dxdy,$$

where $u(x',y')$ is the field at the aperture, $u(x,y,z)$ is the field in the measurement plane, $R$ is the distance between points in each plane given by

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2},$$

and the integral is performed over the extent of the measurement plane. Performing the differential, Eq. 2.83 becomes

$$u(x',y') = \frac{1}{2\pi} \int \int u(x,y,z) \frac{e^{ikR}}{R^2} \left( ik - \frac{1}{R} \right) z dxdy.$$

For a discretely sampled field the integral is replaced by a summation over all sample points giving

$$u(x',y') = \frac{1}{2\pi} \sum \sum u(x,y,z) \frac{e^{ikR}}{R^2} \left( ik - \frac{1}{R} \right) z dxdy,$$

where $dx$ and $dy$ are the distances between sample points.

### 2.6. Standing Waves

On encountering a change in impedance, such as a boundary with free space or a reflector, a guided wave is reflected and a standing wave is formed. The forward travelling wave can be written as
2 The Electromagnetic Properties of Multi-Moded Horns

\[ e^{i(\omega t - kx)}, \]

where \( x \) now refers to the propagation direction of the wave. At the boundary the wave is reflected with a magnitude related to the reflection coefficient, \( r \), and is phase shifted by an amount, \( \phi \). Thus the reflected wave can be written as

\[ re^{i(\omega t + kx + \phi)} = \Gamma e^{i(\omega t + kx)}, \]

where the sign of \( kx \) has been reversed since the wave is travelling backwards and \( \Gamma = re^{i\phi} \) is the complex reflection coefficient. The resulting standing wave which forms is then the sum of the forward and backwards travelling waves

\[ e^{i(\omega t - kx)} + r e^{i(\omega t + kx + \phi)} = e^{i\omega t} \left( e^{-ikx} + re^{-i(kx+\phi)} \right). \]

A more useful result is the envelope of the standing wave, which is the maximum amplitude to which the wave is confined. The right hand side of Eq. 2.89 can be rewritten as

\[ e^{i\omega t} e^{i\left(\frac{\phi}{2}\right)} \left( e^{-i(kx+\frac{\phi}{2})} + r e^{i(kx+\frac{\phi}{2})} \right) \]

\[ = e^{i\omega t} e^{i\left(\frac{\phi}{2}\right)} \left( 1 - r \right) e^{-i(kx+\frac{\phi}{2})} + r \left( e^{-i(kx+\frac{\phi}{2})} + e^{i(kx+\frac{\phi}{2})} \right) \]

\[ = e^{i\omega t} e^{i\left(\frac{\phi}{2}\right)} \left( 1 - r \right) e^{-i(kx+\frac{\phi}{2})} + 2r \cos \left( kx + \frac{\phi}{2} \right). \]

To get the equation of the envelope we wish to work out the complex magnitude of this, which is given by

\[ = \left[ \left( 1 - r \right) \cos \left( kx + \frac{\phi}{2} \right) + 2r \cos \left( kx + \frac{\phi}{2} \right) \right]^2 + \left[ -\left( 1 - r \right) \sin \left( kx + \frac{\phi}{2} \right) \right]^2 \]

\[ = (1 + 2r - r) \cos^2 \left( kx - \frac{\phi}{2} \right) + (1 - 2r + r^2) \sin^2 \left( kx - \frac{\phi}{2} \right) \]

\[ = (1 + r^2) \cos^2 \left( kx - \frac{\phi}{2} \right) + \sin^2 \left( kx - \frac{\phi}{2} \right) + 2r \cos^2 \left( kx - \frac{\phi}{2} \right) \]

\[ - \sin^2 \left( kx - \frac{\phi}{2} \right) \]

Using the identity

\[ \cos^2(u) + \sin^2(u) = 1, \]

and the double angle formula

\[ \cos^2(u) - \sin^2(u) = \cos(2u), \]
we get

$$\sqrt{1 + r^2 + 2rcos(2kx - \phi)}.$$ 

Figure 2.8: Incident wave (blue), reflected wave (red) and the standing wave resulting from the combination of the two waves (black). The waves are shown at three phases of $\omega t = 0^\circ$, $45^\circ$ and $90^\circ$. 

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For the case where a perfectly conducting reflector is placed at the end of the waveguide, the wave is reflected with equal and opposite amplitude ($r=1$, $\phi=180^\circ$). The resulting standing wave and standing wave envelope are shown in Figure 2.8 and Figure 2.9 respectively. The standing wave oscillates in time but the envelope remains stationary.

![Standing wave envelope for fully reflected wave.](image)

**Figure 2.9: Standing wave envelope for fully reflected wave.**

### 2.7. **Summary**

The treatment of an electromagnetic wave inside a cylindrical waveguide in terms of cylindrical waveguide modes is a fundamental idea throughout this thesis. These modes have been introduced and the equations describing their electric field within the waveguide have been derived from first principles. Equations describing the power flow within each mode have been worked out in order to normalise the power carried by each mode. Furthermore, the symmetry of each mode has been described and categorised. Possible detection schemes can be categorised as either coherent or incoherent; the behaviour of the modes in each case has been explained. An approximate method of calculating the field at the aperture of the horn in terms of modes at the base of the horn has been presented. Equations relating such an aperture field to the far-field of the horn in terms of a Fourier transform have also been given. It is difficult to measure the horn aperture field directly in the lab, therefore equations used to propagate a field measured in front of the horn in order to infer the aperture field have been presented. Finally, the equations governing standing waves within a...
waveguide have been presented. Overall this chapter has introduced the key equations and ideas used throughout Chapter 3 and Chapter 4.
3. Modelling of the Multi-Mode Horn-Lens Configuration for LSPE-SWIPE

3.1. Introduction

The measured signal for each pixel in the focal plane at a single frequency can be expressed as (Dodelson 2003)

\[ s_i = \int \theta(n)B_i(n)dn, \]

where \( B_i(n) \) is the beam pattern of the \( i \)th pixel, \( \theta(n) \) is the underlying temperature distribution of the astrophysical source and \( n \) is a unit vector directed towards the sky. The beam pattern must therefore be known and accounted for in order to recover the correct CMB polarisation signal. Furthermore, beam systematics can cause conversion of the unpolarised temperature anisotropy and the E-mode into an apparent B-mode signal. These systematics must also be known and accounted for. It is difficult to accurately measure the beam of the SWIPE pixels on the ground whilst replicating in-flight conditions. Therefore a prediction of the beams from simulation is likely to be the main indicator before the beams can be accurately measured in-flight using well-known celestial sources. A good model of the instrument is therefore highly important.

The large electrical size of the SWIPE multi-mode BTB horn and telescope, combined with the fact that a separate simulation is required for each allowed mode in the waveguide filter, makes simulation extremely difficult. Fortunately, modern computing power has now made it possible for these simulations to be carried out on standard desktop PCs, whilst still achieving highly accurate results. There are many simulation techniques available, each with varying accuracy and computational requirements. An important part of constructing the simulation is choosing the most appropriate technique for the model, and optimising the parameters of the simulation to strike the correct balance between simulations time and accuracy. The main simulation techniques used within this chapter are described in detail in Appendix A.
The principles of single and multi-moded BTB horns have been discussed in general in Chapter 1 and Chapter 2. In this chapter the design of SWIPE is considered specifically. A simulation is performed for the multi-mode BTB horn feeding the lens. Pixels which are closest to and furthest from the centre of the focal plane are considered. The simulations are used to predict the shape of the main beam on the sky and the levels of main beam polarisation systematics. The simulated beams for the horn without the lens are compared against measured beams in Chapter 4.

Another important factor in the design of a CMB instrument is the location of phase centre of the horn. This is the point at which the spherical wavefront of the horn’s emission appears to emanate from. Optimum sensitivity and beam shape can be achieved by placing the telescope focus at the phase centre, hence it is important to know its exact location. Building on from past work on the phase-centre of the Planck-HFI multi-mode horns (Gleeson et al. 2002), the phase-centre of the SWIPE horn is investigated in § 3.4. A discussion of the results from each section is provided at the end of the chapter.

### 3.2. SWIPE Pixel Assembly

#### 3.2.1. Design

The purpose of the SWIPE pixel assembly (or BTB horn) is to couple radiation from the telescope onto the detectors with a high efficiency, to define the beam on the sky and to define the frequency band of the pixel. A Schematic showing the dimensions of the full pixel assembly is shown again in Figure 3.1.

![Figure 3.1: SWIPE BTB horn pixel. Dimensions shown are in mm. This figure is the same as Figure 1.21.](image-url)

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3 Modelling of the Multi-Mode Horn-Lens Configuration for LSPE-SWIPE

The low frequency cut-off is provided by the waveguide filter; the high frequency cut-off is provided by a filter cap containing a ~5 mm thick bandpass filter. The 10 mm long waveguide filter has sufficient length to ensure that all evanescent modes decay to negligible power upon passing through it. The number of allowed modes across each band is shown in Table 3.1. The high frequency channels have a smaller bandwidth since these are primarily for removal of the dust foreground.

Table 3.1: The number of modes which are allowed to propagate in the SWIPE BTB horn waveguide filter across each frequency band. The size of the band is specified as the half-power bandwidth. The number of modes are shown in the format of ‘regular modes + orthogonal modes’. The number of orthogonal modes is fewer since modes with an azimuthal index of $m = 0$ are considered not to have an orthogonal counterpart.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>Half-power bandwidth</th>
<th>Number of modes at lower frequency</th>
<th>Number of modes at centre frequency</th>
<th>Number of modes at upper frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>140</td>
<td>30 %</td>
<td>10+7</td>
<td>12+9</td>
<td>17+13</td>
</tr>
<tr>
<td>220</td>
<td>5 %</td>
<td>28+23</td>
<td>30+24</td>
<td>31+25</td>
</tr>
<tr>
<td>240</td>
<td>5 %</td>
<td>32+26</td>
<td>34+28</td>
<td>35+29</td>
</tr>
</tbody>
</table>

The transition horn and detector cavity are optimised to maximise coupling of the radiation to the detector. The absorber of the detector is an 8 mm diameter SiN spider web structure, which is placed in the centre of the resonance cavity, a quarter of a wavelength from each wall. There is a problem, however, since the effective wavelength of modes can vary significantly (according to Eq. 2.41 and 2.42). The transition horn alleviates this problem since the increased radius causes the second term in the equations to reduce, thereby narrowing the range of effective wavelengths. Furthermore, radiation is therefore expanded to exploit the full area of the absorber. The exact shape of the transition horn has been optimised by (Lamagna et al. 2015). Polarisation separation is performed prior to the radiation entering the pixel assembly by the polarisation-splitting wire grid, therefore the detector is not polarisation-sensitive. The horns have a conical profile in order to ease mechanical fabrication and simulation, although more complex profiles may further optimise the beam pattern and coupling efficiency.
3.2 SWIPE Pixel Assembly

3.2.2. Simulation Set-up

Several approximations are made to reduce the complexity of the initial simulation in order to achieve reasonable computational demands and runtimes. Firstly, each allowed mode in the waveguide filter (or ‘modal field’) is assumed to couple equally to the bolometer through the transition horn. ‘Modal field’ is used to refer to all radiation associated with a particular mode in the waveguide filter (see § 2.5.1). Thus, anything behind the front horn is excluded from the simulation and each modal field is excited with equal power in the waveguide filter to reveal the horn’s beam pattern. In reality the coupling efficiency will vary for different modal fields. In addition to affecting the overall throughput, this will directly affect the shape of the beam pattern, therefore this should be taken into consideration once an accurate model or measurement of the modal coupling within the detector cavity is available. This can be done by a simple weighting of the excitation power of each modal field.

Furthermore, the beam for each pixel should be defined by the average beam across the whole band, weighted by the transmission profile of the bandpass filter. For now the beam has been approximated by a monochromatic simulation at the centre of the frequency band. For the horn alone this should be a good approximation since the narrowing of the beam due to higher frequencies is expected to be cancelled out by the presence of extra modes in the waveguide filter. Thus, the main difference will lie in the effect of the bandpass filter transmission profile. Lastly, the filter cap itself has not been included in the simulation as doing so causes a severe increase in simulation time. As well as having a frequency dependent transmission profile, the effects of the band pass filter may vary for different modal fields, which will directly affect the shape of the beam.

Simulation are performed initially at 140 GHz and simulation techniques and parameters are investigated. The result for the 220 GHz band is examined once an optimum simulation has been constructed. All simulations are performed using a standard desktop PC with 192 GB of RAM and 2 Intel Core i7 processors, each with 6 cores.
3.2.3. Single-mode Simulation

Before performing the full multi-mode simulation, the result of exciting only the fundamental mode ($TE_{11}$) is investigated. This is useful to ensure the simulation is behaving as expected and to compare the results from several simulation techniques, without incurring the long runtime of the multi-mode simulation (since each mode must be simulated individually). It is important that the amount of memory required must stay below the 192 GB available on the PC, otherwise the memory is borrowed from the hard drive which drastically increases runtimes.

The front horn is simulated firstly using the full-wave Method of Moments (MoM) simulation technique in the FEKO simulation suite (https://www.feko.info/). MoM is described in detail in § A.1. A representation of the horn in the FEKO simulation software is shown in the top of Figure 3.2.

![Figure 3.2: Illustration of the SWIPE front horn simulation in: (a) FEKO; and (b) HFSS, for the case of perfect electric (PE) symmetry in the x-axis plane and perfect magnetic (PH) symmetry in the y-axis plane (matching the symmetry of the fundamental mode). For the FEKO simulation the simulation mesh is overlaid on the structure and the red ring shows the waveguide port. In HFSS only a quarter of the geometry is drawn and the flat walls of the quarter horn are selected to be symmetry planes.](image-url)
3.2 SWIPE Pixel Assembly

The geometry is represented in the calculation by a mesh. The MoM solves currents only on the surface of the geometry and therefore only requires a surface mesh. A default ‘standard’ RMS (root mean square) mesh size of $\lambda/12$ is used. This is reduced to $\lambda/15$ for the waveguide port because the mesh must be fine enough to resolve the more complex transverse electric field pattern of the modal excitations. $\lambda/15$ is deemed by FEKO to be the mesh size at which this condition is satisfied. In total the mesh contains 200,312 triangular elements. The runtime and computational requirements of the simulation are reduced by taking advantage of the inherent symmetry of the modes in an azimuthally symmetric waveguide (as described in § 2.4).

The fundamental mode is excited at the waveguide port. The orientation of the modal field pattern adheres to the symmetry shown in Figure 3.2. The far-field beam pattern is extracted and written to a .ffe file. The origin of the far-field calculation is placed at the aperture of the horn. The choice of origin affects far-field phase but not amplitude. Post processing and plotting are performed using a custom MATLAB (https://www.mathworks.com/) script. The data is extracted from the .ffe and converted into Ludwig’s III definition of polarisation (as defined at the end of § 2.5.2). The far-field beam intensity and phase are shown in Figure 3.3 and Figure 3.4 respectively, and the runtime and memory requirements are shown in Table 3.2. Far-field coordinates are defined previously in Figure 2.7. The intensity is the square of the electric field amplitude.

Table 3.2: Comparison of run time per mode and memory requirement for a 140 GHz simulation of the SWIPE horn using MoM (FEKO), MLFMM (FEKO) and FEM (HFSS). Mesh information is also included.

<table>
<thead>
<tr>
<th>Simulation technique</th>
<th>Simulation time per mode (minutes)</th>
<th>Memory requirement (GB)</th>
<th>RMS mesh size</th>
<th>Total number of mesh elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>MoM</td>
<td>485</td>
<td>3</td>
<td>$\lambda/12$ (on port)</td>
<td>200,312 triangular</td>
</tr>
<tr>
<td>MLFMM</td>
<td>219</td>
<td>3</td>
<td>$\lambda/12$ (on port)</td>
<td>200,312 triangular</td>
</tr>
<tr>
<td>FEM</td>
<td>491</td>
<td>45</td>
<td>$\lambda/6$</td>
<td>706,652 tetrahedral</td>
</tr>
</tbody>
</table>
Figure 3.3: SWIPE horn fundamental mode ($TE_{11}$) 140 GHz normalised far-field beam pattern intensity showing azimuthal cuts of each polarisation at $\phi = 0^\circ$, $45^\circ$ and $90^\circ$ for MoM, MLFMM and FEM simulations. The MLFMM result is almost entirely overlaid with the MoM result.
Figure 3.4: SWIPE horn fundamental mode ($TE_{11}$) 140 GHz far-field beam pattern phase for the y-polarisation at $\phi=0^\circ$ and x-polarisation at $\phi=45^\circ$ for MoM, MLFMM and FEM simulations.
The field is sampled every 1° in $\theta$ and the MATLAB line plot of the beam has used a simple linear interpolation between points. All intensity plots in this section are normalised with respect to the overall maximum from both polarisations. The $TE_{11}$ mode is highly polarised with most of the power in the polarisation component matching the excitation polarisation (co-polarisation). There is some power in the orthogonal polarisation (cross-polarisation) which peaks at $\phi=45^\circ$. This is explained by looking at the modal field pattern of the $TE_{11}$ mode (see Figure 2.2 previously). The phase information is unimportant as an end result for the beam on the sky, however, the phase information is important for the modelling of the beam through intermediate components between the horn and the sky, such as the lens.

An alternative simulation technique is MLFMM (described in § A.2). This technique is an extension of MoM, and can reduce simulation time in certain situations. The mesh size is kept the same as the previous model. MLFMM does not benefit from exploiting the symmetry of the modes. MLFMM uses an iterative solver and a residuum of $3e^{-3}$ (default) was achieved. The result is compared with MoM in Figure 3.3, Figure 3.4 and Table 3.2. It is clear that the difference between the beams is very small whilst the runtime is more than halved.

Finally, the simulation is also performed using a completely different simulation technique in a different software; the Finite Element Method (FEM) in HFSS (http://www.ansys.com/en-GB). A representation of the horn in the HFSS simulation software is shown in the bottom of Figure 3.2. Again, the modal symmetry is exploited to reduce the simulation time. In contrast to MoM, the FEM requires a volumetric mesh. Furthermore, the mesh is adaptive, meaning that the simulation repeats iteratively with a decreasing mesh size until a solution with a certain accuracy is reached. The solution accuracy is set by choosing the value of a residuum which measures the convergence of the iterative solver to the solution of the matrix equation (HFSS manual). A very small residual of $0.00035$ is achieved for this simulation giving an RMS mesh size of $\lambda/6$ containing a total of 706652 tetrahedral elements. The result is compared with MOM in Figure 3.3, Figure 3.4 and Table 3.2. The beam pattern agrees strongly with the previous results in terms of intensity, giving confidence in the simulation. The phase also shows reasonable agreement,
although it is shifted by 90° due to a different convention for the initial phase of the excitation between the two software. The simulation time is comparable to MoM.

As discussed in § 2.5.2, an analytical approach to deduce the beam pattern of the horn can be done by approximating the aperture field as being equal to the theoretical modal field pattern at the throat of the horn multiplied by a spherical phase factor to account for the horn flare. The far-field is then proportional to the Fourier Transform of the aperture field. The method requires virtually no run time and has been shown to provide good results for smooth walled conical horns with low flare angle in the past (Olver et al. 1994). Using a slant length of 77 mm for the horn, the result is compared to the full MOM simulation of the fundamental mode in Figure 3.5.

![Figure 3.5: SWIPE horn fundamental $TE_{11}$ mode 140 GHz normalised far-field beam pattern intensity showing cuts of each polarisation at $\phi = 0^\circ$, 45° and 90° for the MoM simulation and for the approximate method.](image)
The approximate method predicts the location of the peaks and troughs of the pattern well, but there is substantial disagreement in the depth of the troughs. This shows that the approximate method is good for quickly designing horns but full simulations are required for final characterisation. Overall, the best technique is clearly MLFMM due to its low run-time and high accuracy. The beam pattern intensity from the MLFMM simulation is plotted over all azimuthal angles as a beam map in Figure 3.6.

![Figure 3.6: SWIPE horn fundamental TE_{11} mode 140 GHz normalised far-field beam pattern intensity uv-plane beam map extending to 20° in θ.](image)
The simulation time is virtually independent of the number of far-field points requested, therefore no overall increase in runtime is incurred by plotting the full beam map. The beam map has been created using a custom code in MATLAB which projects the spherical data onto the uv-plane, where \( u = \sin(\theta) \cos(\phi) \) and \( v = \sin(\theta) \sin(\phi) \). The far-field has been sampled every 0.1° in \( \theta \) and every 0.1° in \( \phi \). Due to the nature of the projection, an equal increment between points in spherical polar coordinates becomes larger in Cartesian coordinates the further it is from the centre. If the projected resolution was such that all points in the spherical far-field were included, this would lead to significant gaps in the data at large \( u \) and \( v \) values. Therefore the resolution of the grid is limited so that there are no missing data points in the projection, and higher resolution points close to the centre overlap. The resolution should be high enough to resolve the features of the beam, for which it is in this case.

### 3.2.4. Multi-mode Simulation

In the multi-mode simulation each modal field is excited independently with equal power using the waveguide port at the throat of the horn. The far-field beams are then extracted corresponding to each modal field. The whole process is automated using a custom EDITFEKO script (.pre file) created in MATLAB. The final incoherent multi-mode beam is calculated by summing in quadrature the electric far-fields (see § 2.5.1 for an explanation of coherent and incoherent operation).

Simulation time is almost halved by realising that the orthogonal mode set does not need to be simulated. This is because, for azimuthally symmetric models, the far-field relating to the orthogonal mode excitation can be generated from the result of the regular mode excitation by a simple rotation of the field pattern and the polarisation vector. For a mode with azimuthal index, \( m \), and an electric far-field, \( E \), specified in spherical polar coordinates by a polar angle, \( \theta \), and an azimuthal angle, \( \phi \), the electric far-field of the orthogonal mode is given by

\[
E_{\alpha}^{\text{orth}}(\phi = \beta) = E_{\alpha+90^\circ/m}(\phi = \beta + 90^\circ/m),
\]

where \( \alpha \) denotes either the x or y-polarisation (in the Ludwig III definition). For example, to generate the orthogonal version of the \( TE_{11} \) mode, the field pattern is
rotated by 90° and the polarisation is rotated by 90° (the x-polarisation becomes the y-polarisation in this case).

Care has to be taken to sample the far-field with a fine enough increment in $\phi$ to allow the rotation to be calculated without any error. For example, at 140 GHz the highest value of $m$ encountered is 4 for the $TE_{41}$ mode. Therefore the sampling of the field must be at most in increments of $0.5^\circ$ in $\phi$ to incur no error when rotating the field (must be a factor of $22.5^\circ$). At 220 GHz it must have increments of $0.05$ in $\phi$ (must be a factor of $11.25^\circ$ for the $TE_{91}$ mode). This only applies to modes with even $m$, since modes with odd $m$ can simply be rotated by 90° for all values of $m$.

The total simulation time at 140 GHz is 46 hours (12 modes). The fields are calculated with a $0.1^\circ$ resolution in $\theta$ and $\phi$. The multi-mode beam cuts and beam map are shown respectively in: Figure 3.7 and Figure 3.8 (excluding the orthogonal modes); and Figure 3.9 and Figure 3.10 (including the orthogonal modes). The unpolarised plot refers to the case where the electric field polarisation components have been added in quadrature. The case where the orthogonal modes are included corresponds to how the SWIPE horn actually operates since the bolometer is not polarisation sensitive. The unpolarised beam including orthogonal modes is azimuthally symmetric as you would expect. The multi-mode beam has a more ‘Top-hat’ like shape compared with a single-mode horn beam which is quasi-Gaussian. This is due to the higher order modes contributing more power off-axis. Also shown on the plots is the angle at which the cold aperture stop cuts the horn beam.
3.2 SWIPE Pixel Assembly

Figure 3.7: SWIPE horn 140 GHz multi-mode (excluding orthogonal modes) normalised far-field beam pattern intensity showing cuts of each polarisation at $\phi = 0^\circ$, $45^\circ$ and $90^\circ$. 
Figure 3.8: SWIPE horn 140 GHz multi-mode (excluding orthogonal modes) normalised far-field intensity uv-plane beam map extending to 20° in θ.
Figure 3.9: SWIPE horn 140 GHz multi-mode (including orthogonal modes) normalised far-field beam pattern intensity showing cuts of each polarisation at $\phi = 0^\circ$, $45^\circ$ and $90^\circ$. The angle at which the beam is designed to be cut by the cold aperture stop of the telescope is indicated by the vertical dashed black line.
Figure 3.10: SWIPE horn 140 GHz multi-mode (including orthogonal modes) normalised far-field intensity uv-plane beam map extending to 20° in θ.
3.2.5. High Frequency Pixel

The beam is also calculated for the 220 GHz operation (30 modes) of the SWIPE horn, and compared to the 140 GHz result. Only considered, is the case where the horn is multi-moded and the orthogonal modes are included. The field sampling is kept at 0.1° in θ and ϕ therefore an error will occur when generating the orthogonal version of the $TE_{\beta_1}$ mode. However, given the large number of modes, this error is extremely small. Beam cuts are shown in Figure 3.11 and beam maps are shown in Figure 3.12. As expected, the beam changes very little with frequency since the narrowing at higher frequencies is counteracted by the presence of more modes with off axis power. Simulation time and parameters are compared in Table 3.3.

![Figure 3.11: An azimuthal cut of the unpolarised multi-mode beam (including orthogonal modes) of the SWIPE horn including the 220 GHz band. The black dashed vertical line indicates the angle at which the aperture stop cuts the beam.](image)

Table 3.3: Comparison of horn simulation parameters at 140 and 220 GHz.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>Total simulation time (hours)</th>
<th>Average simulation time per mode (hours)</th>
<th>Mesh size</th>
<th>Iterative solver residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>140</td>
<td>46</td>
<td>3.8</td>
<td>$\lambda/12$ ($\lambda/15$ on port)</td>
<td>3e-3</td>
</tr>
<tr>
<td>220</td>
<td>462</td>
<td>15.4</td>
<td>$\lambda/12$ ($\lambda/15$ on port)</td>
<td>3e-3</td>
</tr>
</tbody>
</table>
Figure 3.12: SWIPE horn multi-mode (including orthogonal modes) 220 GHz normalised far-field beam pattern intensity uv-plane beam map extending to 20° in $\theta$. 
3.3. **Telescope**

3.3.1. **Thick Lens Design Equations**

A general biconvex thick lens is shown in Figure 3.13. The lens is fully defined by its centre thickness, \( t_c \), diameter, \( D \), focal length, \( f \), and refractive index, \( n \). The remaining parameters shown in the figure are useful in the construction of the lens and can be expressed in terms of the defining parameters. Parallel rays of light coming from the left at different elevations will refract upon entering the first surface then refract again upon exiting the second surface before converging at the focal point. Equivalently, a principal plane, \( H_B \), can be defined whereby a ray entering from the left will undergo a single refraction at the principal plane before converging at the focal point.

![Figure 3.13: A representation of a biconvex thick lens with spherical surfaces. Shown are the design parameters of the lens including: the diameter, \( D \); centre thickness, \( t_c \); and focal length, \( f \). Other useful parameters in the construction of the lens are: the front and back principal planes, \( H_F \) and \( H_B \); surface radii, \( R_F \) and \( R_B \); focal points, \( F_F \) and \( F_B \); and focal distances, \( FFD \) and \( BFD \).](image)

The focal length can be expressed as (Hecht 2002)
3 Modelling of the Multi-Mode Horn-Lens Configuration for LSPE-SWIPE

\[
\frac{1}{f} = (n - 1) \left( \frac{1}{R_B} - \frac{1}{R_F} + \frac{(n - 1)t_c}{nR_BR_F} \right),
\]

3.3

and the front and back focal distances are given by

\[
FFD = f \left( 1 + \frac{(n - 1)t_c}{nR_B} \right)
\]

3.4

\[
BFD = f \left( 1 - \frac{(n - 1)t_c}{nR_F} \right).
\]

3.5

If \( R_B \) becomes infinite then the lens becomes Plano-Convex as shown in Figure 3.14.

![Diagram of a Plano-Convex Thick Lens](image)

Figure 3.14: A representation of a plano-convex thick lens with a single spherical surface. Symbols have been previously defined in Figure 3.13. The lens is orientated so that light from the sky enters the spherical surface from the left and is focused onto the focal plane on the right. Although the reverse orientation would have the same focusing power, the spherical aberrations would be higher therefore this orientation is preferred (Hecht 2002).

Neglecting all terms where \( R_B \) is in the denominator, Eq. 3.3-3.5 reduce to

\[
\frac{1}{f} = (n - 1) \left( \frac{1}{R_F} \right)
\]

3.6

\[
FFD = f
\]

3.7

\[
BFD = f \left( 1 - \frac{(n - 1)t_c}{nR_F} \right).
\]

3.8
3.3.2. Initial Optimisation of the Lens Using Zemax

An initial design for the SWIPE lens was generated and optimised by Prof. Marco de Petris and Gabriele Coppi using the Zemax-EE 2003 software (http://www.zemax.com/). Zemax is designed to model, analyse and optimise imaging systems. Problems are solved using the ray tracing geometrical optics technique described in § A.3. A representation of the SWIPE optical system in Zemax is shown in Figure 3.15. The model considers simultaneously the 140 and 220 GHz frequencies. The fields (angle of incoming rays) which the model considers are: an on-axis ray; and off-axis rays at polar angles of 10° and 10.5° at azimuthal angles of 0°, 90°, 180° and 270°. Each field is weighted equally in the optimisation. The highest aberrations result from radiation entering the system from the furthest off-axis angles, therefore this is where the majority of the rays are concentrated.

Figure 3.15: A Zemax model of the SWIPE optical configuration. The components shown (from left to right) are: the thermal filters (TF), rotating half-wave plate (HWP), lens (L1), aperture stop (AS), polarisation-splitting wire grid (WG) and the two curved focal planes (CFP). The wire grid splits the incident polarisations, reflecting one and transmitting the other onto separate focal planes. The fields shown are: the on-axis field (blue); and the fields at ±10° off axis (red and green). The fields at ±10.5° off axis are not shown. The curved focal plane is a consequence of the single lens design.
The model is represented in Zemax on a spreadsheet as a series of surfaces with various parameters representing the properties each surface. The sequential mode of Zemax is used, meaning that the rays intersect the surfaces in the order that they are presented in the spreadsheet. Each parameter can be made fixed or variable; only variable parameters are considered during the optimisation. Prior to the optimisation, the geometry of several of the components had already been fixed. The thermal filters and HWP are difficult to manufacture for large diameters therefore they constrain the diameter of the whole system. The radius of the filters in this case is 242 mm, thus the radius of the lens is set to 240 mm. This radius relates to the curved portion of the lens; there is an extra lip around the lens to hold it in place. The radius of the aperture stop is constrained to 212 mm. Previous work on the optics and mechanical constraints had fixed the focal plane to have a radius of curvature of -333 mm and a radius of 156 mm. The remaining parameters which were left free to vary in the optimisation were the radius of curvature and the conic constant of the curved lens surface.

The optimisation goal is selected as RMS wavefront centroid. This minimises the spherical aberration (measured in waves), where the RMS computation is referenced to the centroid of all the data coming from that field point. The goal of the optimisation is expressed by Zemax as a merit function, \( MF \), which is composed of a number of weighted operands (Zemax Manual)

\[
MF^2 = \frac{\sum W_i (V_i - T_i)^2}{\sum W_i},
\]

where \( W_i \) is the weight of the \( i^{th} \) operand, \( V_i \) is its computed value and \( T_i \) is its target value, and the summation is over all the operands in the merit function. The merit function is expressed as a single value, whereby the closer it is to zero, the better the design matches the desired optimisation goal. Using several algorithms, each of the variable parameters are automatically adjusted to bring the merit function as close to zero as possible. The completed optimisation gave a final merit function of 0.095, fixing the radius of curvature and conic constant of the lens as 477.5 mm and -0.54 respectively.

The performance of the lens can be analysed using a spot diagram (Figure 3.16). The spot diagram shows where rays from each field fall on the image plane (focal plane).
3.3 Telescope

OBJ shows the off-axis polar angle of the field listed as two numbers representing angles along azimuthal angular directions of 0° and 90°. Not all azimuthal beams are shown due to the azimuthal symmetry of the model. IMA shows the image plane intercept coordinates of the centroid relative to the centroid of the central field, listed as two numbers representing distances along the x and y axes. It is evident that the off-axis pixels suffer from a high degree of the aberration coma, which would be expected for such a configuration. The parameters of the final optimised lens are summarised in Table 3.4.

Table 3.4: Parameters to describe the optimised lens.

<table>
<thead>
<tr>
<th>Lens parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lens diameter (mm)</td>
<td>480</td>
</tr>
<tr>
<td>Centre thickness (mm)</td>
<td>55</td>
</tr>
<tr>
<td>Focal length (mm)</td>
<td>847.22</td>
</tr>
<tr>
<td>Refractive index</td>
<td>1.57</td>
</tr>
<tr>
<td>Conic constant</td>
<td>-0.54</td>
</tr>
<tr>
<td>Radius of front surface (mm)</td>
<td>477.5</td>
</tr>
<tr>
<td>BFD (mm)</td>
<td>805</td>
</tr>
<tr>
<td>Lens-aperture stop distance (mm)</td>
<td>5</td>
</tr>
<tr>
<td>Aperture stop diameter (mm)</td>
<td>424</td>
</tr>
<tr>
<td>Lens lip thickness (mm)</td>
<td>2.7</td>
</tr>
</tbody>
</table>

The Zemax optimisation deduced the best result for the lens shape based on minimising the final aberrations at the focal plane when rays with a parallel wavefront are traced from the sky through the system. The simulation time for each iteration is extremely low, therefore the optimisation can vary multiple parameters in the same optimisation procedure whilst keeping the total simulation time reasonable. However, there are some limitations of the Zemax simulation: the beam pattern of the horn in the focal plane is not considered; and the final beam pattern on the sky of the whole system is not determined. Therefore the analysis of the horn-lens system is extended to resolve these limitations by simulating the system in FEKO.
3 Modelling of the Multi-Mode Horn-Lens Configuration for LSPE-SWIPE

Figure 3.16: Spot diagram at 140 GHz (blue) and 220 GHz (green) referenced to the chief ray. The colours do not correspond to the colours in Figure 3.15. See main text for details.
3.3.3. FEKO Simulation Technique

The large electrical size of the lens (~200 λ diameter at 140 GHz) makes the simulation extremely demanding on computational resources. Furthermore, as for the horn, a separate simulation is required for each mode permitted by the horn waveguide filter, leading to very long overall run times. Therefore anyway in which the simulation time can be reduced without compromising too much on accuracy is very important, and the technique used to simulate the lens should be chosen carefully.

Simulation time is vastly reduced by avoiding having to re-simulate the horn when simulating the lens. Instead, the horn is represented by an equivalent source, which is exported from the horn simulations in § 3.2. By doing this, multiple reflections between the lens and the horn geometry are not taken into account because the actual horn geometry is excluded from the simulation. However, this effect is expected to be small, given the large distance between the two components. The lens is constructed using the lens design equations from § 3.3.1. To simplify the lens initially, the conic constant is approximated to be 0 (a spherical surface). After the initial investigation of simulation parameters, the non-zero conic constant is reintroduced later (§ 3.3.6). To construct the lens a sphere of radius 477.5 mm is split and extended cylindrically to give the lens the correct centre thickness, \( t_c \). The lens is then positioned relative to the horn-equivalent source according to the BFD. In this initial simulation no aperture stop is included. Furthermore, the horn-equivalent source is calculated from § 3.2 with its origin at the aperture of the horn. Thus the telescope focus coincides directly with the horn aperture, however, as we will see later in § 3.4, this may not be the ideal configuration.

The full wave simulation techniques MOM, MLFMM and FEM are not viable for simulation of the lens since they require too much memory. Instead, more approximate techniques such as Physical Optics (PO) and Ray Launching-Geometrical Optics (RL-GO) are appropriate. RL-GO is described in § A.3. The accuracy and run-times for PO and RL-GO are compared with that of the more accurate MLFMM for a lens which has been scaled down by a factor of 10. Only the beam of the fundamental mode of the horn is used to illuminate the lens. The results are compared in Figure 3.17 and the simulation times are compared in Table 3.5. PO
Modelling of the Multi-Mode Horn-Lens Configuration for LSPE-SWIPE shows poor agreement with the more accurate MLFMM technique, even with a very fine mesh, therefore RL-GO is deemed to be the best choice to solve the full lens model.

A comparison of the accuracy of these simulation techniques against measured data can be found in the PhD thesis of Fahri Ozturk (Ozturk 2013). Ozturk compared simulated (MLFMM) and measured far-field beams for a single-mode horn feeding a small dielectric lens (16 λ diameter) and medium dielectric lens (30 λ diameter) at 97 GHz. The co-polarisation beams showed strong agreement: for the main lobe down to ~ -30 dB for the small lens; and for the main lobe and first two sidelobe peaks down to ~ -30 dB for the medium lens.

![Figure 3.17: Comparison of far-field beams for a 1/10 scale SWIPE lens fed by the fundamental mode beam of the SWIPE horn at 140 GHz.](image)

<table>
<thead>
<tr>
<th>Simulation technique</th>
<th>Simulation time</th>
</tr>
</thead>
<tbody>
<tr>
<td>MLFMM</td>
<td>30 minutes</td>
</tr>
<tr>
<td>PO</td>
<td>8.5 seconds</td>
</tr>
<tr>
<td>RL-GO</td>
<td>11.5 seconds</td>
</tr>
</tbody>
</table>

Table 3.5: Comparison of simulation times for a 1/10 scale SWIPE lens fed by the fundamental mode beam of the SWIPE horn.
For the full-wave techniques the mesh is required to accurately model the electric current distribution over the surface, and is frequency dependent. In this case the size of the mesh elements are usually a small fraction of the wavelength. The large number of mesh elements is the primary reason why the computational memory requirements and simulation time are so high when simulating the lens. An advantage of RL-GO over the full-wave techniques is that the mesh is only required to resolve the geometry relative to the increment between launched rays, independent of frequency. Therefore mesh elements can be multiple wavelengths in size, which helps reduce computational requirements. A model of the RL-GO horn-lens simulation is shown in Figure 3.18.

Figure 3.18: A model of the SWIPE horn-lens RL-GO simulation in FEKO. The blue lines represent the launched rays, only a portion which have been drawn. Most of each ray’s power is transmitted through the lens however the ray is not shown to continue along its path through the lens in the simulation. This is because FEKO calculates the far-field from the ray distribution and transmission efficiency over the front surface of the lens. The aperture stop is shown in this image but not included in the initial simulation of the lens.

Within each simulation technique are further options to refine the desired accuracy of the model. For RL-GO the most important are: the angular increment between adjacently launched rays from the horn-equivalent source (RL angle); and the maximum number of interactions (bouts of transmission, reflection and refraction at a surface) which each ray is allowed to undergo. A warning is issued if the RL angle is not small enough so that the distance between the points where adjacent rays intersect the geometry fails to satisfy the Nyquist sampling criteria (\(\lambda/2\)). The size of the mesh is still important in determining the accuracy of the solution, but is expected to have much less impact on simulation time for RL-GO. Using symmetry
does not reduce the runtime when RL-GO is being used. Another simulation parameter which will have a strong effect on the result is the number of points which are used to represent the beam pattern of the horn-equivalent source which illuminates the lens, and the type of equivalent source used (see next § 3.3.4). The initial choices of simulation parameter are listed in Table 3.6.

Table 3.6: Initial values for simulation parameters in the horn-lens simulation. The mesh size is the RMS size of all mesh elements on the lens.

<table>
<thead>
<tr>
<th>Simulation parameter</th>
<th>Initial value</th>
</tr>
</thead>
<tbody>
<tr>
<td>RL angle</td>
<td>0.3° (default)</td>
</tr>
<tr>
<td>Maximum number of ray interactions</td>
<td>2 (default)</td>
</tr>
<tr>
<td>Mesh size</td>
<td>1 λ</td>
</tr>
<tr>
<td>Number of points in horn-source far-field /Resolution of horn source far-field points</td>
<td>7560 / 1°</td>
</tr>
</tbody>
</table>

### 3.3.4. Horn-equivalent Source

An attempt is made to reduce the simulation time by avoiding having to re-simulate the horn in the horn-lens simulation. A standard technique is to replace the horn with an equivalent source which represents the field produced by the horn, which has already been simulated in § 3.2. The lens is 805 mm from the horn which places it in the far-field. Therefore the most appropriate choices of equivalent source for illumination of a lens in the far-field are a FEKO far-field file (.ffe) or a spherical wave expansion (SWE). The .ffe file is a direct output of the far-field (electric field strength versus angle). The SWE file is a approximation of the .ffe file into an expansion in terms of spherical modes. The coefficients of the SWE are calculated based on the values of the original far-field which it represents. The accuracy and calculation time of the SWE itself depends on the number of points in the original far-field and on the maximum number of SWE modes included in the SWE expansion. The number of SWE modes included should be high enough to accurately represent the far-field i.e. finely sampled far-field points require higher order SWE modes.

The .ffe file and the SWE are calculated for the far-field of the SWIPE horn carrying the \( TE_{11} \) mode only. The far-field request only extends up to 20° off-axis since the remainder of the field will not interact with the lens or aperture stop. The calculation
3.3 Telescope

time for the SWE from the far-field is negligibly short. Thus an overestimation of the number of SWE modes required could be used if an accurate representation of the horn far-field is the only concern. However, it turns out that the simulation time of the horn-lens simulation increases rapidly as the number of SWE modes are increased, therefore the fewest modes possible, which still give an accurate representation of the horn far-field, should be used. To find the optimum number of SWE modes, the SWE is compared against the original far-field, for variations in the number of SWE modes ranging from 15-30 (Figure 3.19) where the usual convention for far-field coordinates has been used. The far-field is sampled in 1° increments along the polar angle $\theta$.

![Image of Figure 3.19: Comparison of the SWIPE horn y-polarisation normalised far-field intensity (top) and phase (bottom) of the SWE against the original far-field, where the number of SWE modes (SWE) used in the SWE is increased. The horn has been excited with the $TE_{11}$ mode only. Only a cut at $\phi=0^\circ$ is shown. The ‘Original’ and ‘30 SWE’ plots are identical.]

Clearly, the SWE starts to represent the far-field to an appropriate accuracy when 30 SWE modes are used. The lens is simulated with RL-GO using the initial simulation parameters (Table 3.6) for both the original far-field (.ffe) and SWE source. It is
found that, even with this low number of SWE modes, the simulation runtime becomes longer when using the SWE than when using the direct far-field data. Therefore the .ffe field data is used directly as the horn-equivalent source.

### 3.3.5. Simulation Parameter Optimisation

The simulation parameters for the horn-lens simulation: RL angle, maximum number of ray interactions; mesh size and number of points in the horn-equivalent source, are investigated to understand their effect on simulation accuracy and run-time. The maximum number of ray interactions is kept at 2 (no multiple reflections within the lens) for now, since this is a more complicated issue for which Anti-Reflective Coating (ARC) on the lens should be taken into account. A major difference between the MLFMM horn simulation and the RL-GO lens simulation, is that for the lens simulation the simulation time largely depends on the number of points requested in the final far-field beam. For the current case where the final multi-mode beam is azimuthally symmetric, the far-field only needs to be calculated for a single azimuthal cut, leading to a largely reduced run-time. However, for cases where the beam is being calculated for an off-axis pixel and where the beam is polarised, the beam will not be azimuthally symmetric and therefore needs to be calculated over all azimuthal angles. In order to keep the simulation parameter study relevant for these cases as well, at least with regards to simulation time, the beam is therefore calculated over all azimuthal angles (sampled every 1° in φ, and every 0.1° in θ up to 3.5°). In the investigation of the simulation parameters the full multi-mode beam is not considered. Instead, to reduce run-time, it is approximated that the \( TE_{11} \) mode is representative of the rest of the modes. It is known that the run-time is almost equivalent between modes for the lens simulation, and it is expected that the effect on accuracy is also similar.

In the case where the simulation parameter is specified by a numerical value, the simulation parameter is varied and the trend in accuracy and simulation time is examined. To evaluate the accuracy of each solution, a criteria must be decided in order to quantify it in some way. If a result is available which is known to be more accurate (for example an MLFMM simulation), then the best way to assess the accuracy of the RL-GO simulation is to compare the solution directly with the
MLFMM result. However, the lens is too large to be simulated by MLFMM, therefore another method needs to be used to assess the accuracy of the beam. A standard technique is to perform a convergence study. This is used in most commercial software, for instance, when the size of the mesh is iteratively deduced for a target accuracy. In this case the number of mesh elements is gradually increased and the effect on the final beam is quantified in some way, usually as a residual. This process is repeated until the residual falls below a certain threshold which corresponds to meeting the desired accuracy. Overall this tells you that the beam is not sensitive (according to the desired accuracy) to changes in the mesh at the resultant level. This was the case for the FEM simulation of the horn in HFSS.

A similar technique is adopted for the simulation parameters. Each simulation parameter is varied and the results from consecutive iterations are compared to quantify the effect on the final result. The effect on the beam is quantified as the mean intensity difference over the whole electric far-field:

\[ M = \frac{\sum_{\alpha=1}^{A} \left[ \frac{l_{n+1}(\alpha) - l_{n}(\alpha)}{l_{n}(\alpha)} \right]}{A}, \]

where \( l_n \) is the intensity of the \( n^{th} \) iteration at the \( \alpha \) point in the far-field containing a total of \( A \) points. The phase difference is also noted but is not used to define the stopping criteria. The stopping difference is ultimately set by the maximum reasonable simulation time. The convergence at this simulation time is then used as a guide for the accuracy of the simulated beam. A nominal value of around 1 hour per mode is aimed for. If the time limit is not reached then a level of convergence of -30 dB is used instead as the stopping criteria. This means that the far-field intensity beam difference of consecutive iterations should average out to be 0.1% of the beam from the former iteration.

The horn-lens far-field beam for a \( TE_{11} \) mode excitation with simulation parameters set at their initial values (Table 3.6) is shown in Figure 3.20. The average beam difference is actually calculated for an unpolarised beam from \( \phi = 0^\circ \) to \( 90^\circ \) (azimuthally) and up to a \( \theta \) angle of 3.5\(^\circ\) off-axis (polar angle). Only a quarter of the field is required due to the symmetry of the model. The field is sampled in angular increments of 0.5\(^\circ\) in \( \phi \) and 0.1\(^\circ\) in \( \theta \). The beams for each solution are normalised before the beam differences are calculated. For average phase difference the
difference is taken between unwrapped phase (the function is continuous and extends beyond $2\pi$) in order to avoid large values where the phase wraps at different $\theta$ angles.

![Graphs showing y- and x-polarisation intensity versus angle](image)

Figure 3.20: SWIPE horn-lens normalised far-field beam cuts for a $TE_{11}$ mode excitation at 140 GHz with simulation parameters set to their initial values. The beams are normalised to the maximum electric field intensity from both polarisations.

**Mesh size**

As stated previously, the mesh size only has to be sufficiently fine enough to give a good representation of the geometry shape. The default RL angle of 0.3° means that the rays intersect the geometry every 4.2 mm ($2\lambda$). The mesh size is adjusted above and below the default size in the convergence study. Fundamentally, the solution accuracy is dependent on the number of mesh elements rather than the mesh size,
therefore this is the parameter which is varied. The fraction by which the number of mesh elements is increased between consecutive solutions needs to be thought about carefully. The overall purpose of the convergence study is to find the point at which the result (to a predefined precision) becomes independent of the number of mesh elements i.e. adding more elements does not affect the result at that precision (-30 dB in this case). If the fractional change is too small then consecutive solutions will give near identical models and thus a high degree of convergence, but the significance of the convergence is also very small. Oppositely, a very large fractional change will skip intermediate levels where the solution has reached convergence. Taking this into account, a fractional change of 1.5 is used.

Since FEKO only allows input of RMS mesh size, a conversion from the number of mesh elements must be known. The power-law relationship between the two parameters is found by plotting a graph for several data points (Figure 3.21) and fitting a power-law trend line to reveal that the parameters have the relationship

\[ RMS \text{ mesh size} = 415 \times (\text{number of mesh elements})^{-0.498}. \]

![Graph showing the relationship between mesh size and number of mesh elements](image)

Figure 3.21: The relationship between the number of mesh elements and the RMS mesh size for the lens.

Figure 3.22 shows how the intensity and phase converge in comparison to the increase in simulation runtime when the number of mesh elements is increased.
Figure 3.22: Average far-field beam difference (blue dashed line) between successive iterations for intensity (top) and unwrapped phase (bottom) plotted against the number of mesh elements used to represent the lens geometry. The increase in simulation runtime is shown in comparison (green dotted line).

The solution converges steadily for intensity with some small fluctuations at large mesh sizes (small number of elements), levelling off just below -45 dB. The -30 dB criteria is met when $0.45 \times 10^5$ mesh elements are used, hence the previous iteration to which the agreement has been considered, pertaining to $0.3 \times 10^5$ mesh elements, is selected to be used going forward. This corresponds to a mesh size of $2.45 \lambda$. From the graph it is evident that the local fluctuations in convergence are small compared to the global trend in convergence, indicating that this is a truly representative solution. The phase converges rapidly, reaching $0.022^\circ$ for $0.3 \times 10^5$ mesh elements. The simulation runtime increases steadily at a very slow rate, with an
abnormality when a very small number of mesh elements is used. Although at a slow rate, the increase is still important because it will be magnified when other simulation parameters, which cause simulation runtime to increase at a faster rate, are varied. At the desired convergence the runtime is around 137 seconds, way below the limit of 1 hour.

**Number of far-field points in the horn-equivalent source**

The next simulation parameter to be investigated is the number of points in the equivalent far-field source used to represent the horn. The number of points is increased in multiples of 1.5 and the convergence is shown in Figure 3.23.

![Graphs showing the relationship between number of far-field points and runtime, intensity difference, and phase difference.](image)

Figure 3.23: Average far-field beam difference (blue dashed line) between successive iterations for intensity (top) and unwrapped phase (bottom) plotted against the **number of far-field points used to represent the horn source**. The increase in simulation runtime is shown in comparison (green dotted line).
The solution converges steadily for intensity with some fluctuations when a small number of points are used. The -30 dB criteria is met when $6.2 \times 10^5$ points are used, hence the previous iteration to which the agreement has been considered, pertaining to $4.1 \times 10^5$ points, is selected and used going forward. This corresponds to an angular spacing between far-field points of $0.13^\circ$, which is rounded to $0.1^\circ$. Again, from the graph it is evident that the local fluctuations in convergence are small compared to the global trend in convergence, indicating that this is a truly representative solution. The phase converges rapidly, reaching $0.086^\circ$ for $4.1 \times 10^5$ far-field points. The runtime shows a general increase, with some abnormal results when a low number of points is used. The runtime at convergence is around 134 seconds, which remains much lower than the 1 hour limit.

**Ray Launching angle (RL angle)**

The RL angle has the biggest effect on simulation accuracy and runtime. The number of rays is increased in multiples of 1.5 and the convergence is shown in Figure 3.24. The results are slow to converge for far-field intensity and the runtime increases dramatically as the number of rays is increased. A -30 dB convergence cannot be achieved if the simulation time is to remain under 1 hour. A -20 dB convergence is achieved when 3.5 million rays are used and the solution seems to converge steadily after this point. Hence the previous iteration to which the agreement has been considered, pertaining to 2.3 million rays, is selected and used going forward. This corresponds to a RL angle of $0.0322^\circ$ with a simulation runtime of 32 minutes.* The convergence on the phase is around $0.3^\circ$.

The final values of the simulation parameters are summarised in Table 3.7. The $TE_{11}$ mode horn-lens beam is compared for simulations using the initial and final values of the simulation parameters in Figure 3.25. Since the simulation parameters are interdependent to some extent, their associated levels of convergence may have changed as other simulation parameters have changed throughout the convergence study. For instance, the result for the convergence of the mesh may have changed somewhat when RL angle was decreased since the geometry is intersected at a higher

---

* Note that in the final version of the simulation, the number of far-field points requested is double that which has been used for the convergence study. Therefore, times quoted within this section should be doubled to be representative of times for the final simulation.
resolution. Therefore, a final convergence measurement is performed for each parameter to quantify the final accuracy of the simulation. The result is shown in Table 3.7. Clearly the level of convergence is dominated by the effect of RL angle. Hence, combining the convergences of each parameter in quadrature gives a final convergence on the beam of -21.8 dB, corresponding to a 0.0066 fractional uncertainty.

![Figure 3.24](image-url)

Figure 3.24: Average far-field beam difference (blue dashed line) between successive iterations for intensity (*top*) and unwrapped phase (*bottom*) plotted against the **number of rays launched from the source**. The increase in simulation runtime is shown in comparison (green dotted line). The time is now expressed in minutes.
Table 3.7: Simulation parameter values used in the final version of the horn-lens simulation. The final level of convergence for each parameter is also shown.

<table>
<thead>
<tr>
<th>Simulation parameter</th>
<th>Final value</th>
<th>Final intensity convergence (dB)</th>
<th>Final phase convergence (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh size</td>
<td>2.45 λ</td>
<td>-31.98</td>
<td>0.017</td>
</tr>
<tr>
<td>Angular increment of source far-field</td>
<td>0.1°</td>
<td>-35.2</td>
<td>0.0077</td>
</tr>
<tr>
<td>Maximum number of ray interactions</td>
<td>2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>RL angle</td>
<td>0.0322°</td>
<td>-21.8</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Figure 3.25: Horn-lens normalised far-field beam cuts for a $TE_{11}$ mode excitation at 140 GHz with simulation parameters set to their final values compared with the result using the initial values.
3.3.6. Inclusion of the Optimised Conic Constant of the Lens

The lens has been assumed to be spherical, when in fact the Zemax optimisation gave a prolate ellipsoid (conic constant of -0.54) for the optimised lens surface. The lens is constructed in exactly the same way except an ellipsoid is constructed first instead of a sphere. An ellipsoid with a radius, \( R \), and conic constant, \( K \), which is prolate along the \( z \)-direction has semi-principal axes given by

\[
\begin{align*}
a_x &= a_y = \frac{R}{\sqrt{K + 1}} \\
a_z &= \frac{a_x}{\sqrt{K + 1}}.
\end{align*}
\]

The effect on the beam due to the optimised conic constant is shown in Figure 3.26.

![Graphs showing beam intensity for different conic constants](image)

Figure 3.26: Horn-lens \( TE_{11} \) mode far-field beam for the lens with a spherical surface compared to one with a **conic constant of -0.54**.
Although the true conic constant was not used in investigating the simulation parameters, it is unlikely to make a significant difference to the overall convergence result, meaning that the results of § 3.3.5 are still valid as a guide to the accuracy of the simulation. The new lens geometry is used in all simulations from this point onward.

3.3.7. Inclusion of the Aperture Stop

Any part of the horn beam, which is not intersected by the telescope, contributes to unwanted radiative loading on the detectors due to the thermal emission from the rest of the instrument. The cold aperture stop is introduced in front of the lens to reduce this effect. The effect on the horn-lens beam is shown in Figure 3.27.

![Figure 3.27: Horn-lens TE_{11} mode far-field beam with and without the aperture stop present.](image)
The aperture stop has a diameter of 424 mm and is placed 5 mm from the flat surface of the lens. A model of the lens including the aperture stop has been shown previously in Figure 3.18. The aperture stop is modelled as PEC (Perfect Electric Conductor), whereby rays are perfectly reflected from the surface.

Inclusion of the aperture stop causes minor changes to the beam for the $TE_{11}$ mode, however, we will see later (§ 3.3.9; Figure 3.32) that the aperture stop has the effect of reducing the far-sidelobe (at around 15°) in the multi-mode horn-lens beam. The aperture stop is included in all simulations from this point onward.

3.3.8. Single-mode Beam Map

The horn-lens far-field beam map corresponding to the fundamental $TE_{11}$ mode is shown in Figure 3.28. The full simulation took 1.4 hours. To allow a high resolution plot to be generated without having missing data points at large angles, a single bout of cubic spline interpolation has been applied to the data before the uv projection is made. The cubic spline interpolation works by fitting a piecewise curve going through each data point. There is a separate curve with its own coefficients fitted for each interval. Furthermore, adjacent fitted curves are constrained to have matching gradients at connecting data points, making the whole curve continuous. The interpolation is carried out on a 2D array of points, arranged so that the rows relate to increasing $\theta$ and the columns relate to increasing $\phi$. The interpolation returns the interpolated values on a refined grid format by repeatedly dividing the coordinate intervals $N$ times in each dimension (MATLAB online documentation). The interpolation is also made to be carried out between the first and last columns of the array (first and last $\phi$ values).
3 Modelling of the Multi-Mode Horn-Lens Configuration for LSPE-SWIPE

3.3.9. Multi-mode Beam Map

The simulation is extended to model the full multi-mode horn-lens beam. As with the horn simulation, a separate simulation is required for each mode which can exist in the horn waveguide filter (each modal field). For each modal field, the far-field horn beam is propagated through the lens and the resultant far-field is calculated. The final

Figure 3.28: Horn-lens 3.5° angular radius normalised far-field beam map for a $TE_{11}$ mode excitation at 140 GHz.
horn-lens multi-moded result is then given by summing in quadrature the resultant electric far-fields associated with each modal field.

For an on-axis azimuthally symmetric system, it is possible to generate the result of the orthogonal modes from the result of the regular modes, as was done for the horn simulation. However, as mentioned previously, the field has to be sampled finely enough in \( \phi \) so not to lead to an error when rotating the field pattern for modes with a high value of \( m \) (see § 3.2.4 previously). This is problematic since, unlike with the horn simulation, the simulation time depends on the number far-field points requested. Overall, it is more time efficient to directly include the orthogonal modes in the simulation as horn-equivalent sources rather than to calculate the far-field with a high enough resolution in \( \phi \) to generate the orthogonal mode results without significant error. Furthermore, for off-axis pixels and for when the horn beam is polarised, it will no longer be possible to generate the orthogonal mode result due to the breakdown of azimuthal symmetry. Thus, the horn-equivalent sources relating to all 21 modes at 140 GHz (including orthogonal modes) are included in the horn-lens simulation. The multi-mode far-field beam cuts are shown in Figure 3.29 for the 140 and 220 GHz pixels. Also shown are the equivalent single-mode beam cuts relative to the multi-mode beam cut of the same frequency, thereby demonstrating the large increase in collected power under multi-mode operation. The multi-mode beam maps are shown in Figure 3.30 - Figure 3.31.

![Intensity (dB) vs. \( \theta \) (°) for horn-lens normalised far-field beam cuts at 140 and 220 GHz.](image)

Figure 3.29: Horn-lens normalised far-field beam cuts for a multi-mode (MM) excitation at 140 and 220 GHz. The relative single mode (SM) beams for each frequency are shown in comparison.
Figure 3.30: Horn-lens 3.5° angular radius normalised far-field beam map for a multi-mode excitation at 140 GHz. (1/2)
3.3 Telescope

Figure 3.30: Horn-lens 3.5° angular radius normalised far-field beam map for a multi-mode excitation at 140 GHz. (2/2)

Figure 3.31: Horn-lens 3.5° angular radius normalised far-field beam map for a multi-mode excitation at 220 GHz. (1/2)
Figure 3.31: Horn-lens 3.5° angular radius normalised far-field beam map for a multi-mode excitation at 220 GHz. (2/2)
The beam changes very little with frequency. Simulation parameters and run-time are compared in Table 3.8. The mesh size has not been changed for the 220 GHz simulation since mesh size is independent of frequency.

Table 3.8: Simulation lens mesh size and run-time. $\lambda_{140}$ is the wavelength at 140 GHz.

<table>
<thead>
<tr>
<th>Freq (GHz)</th>
<th>Mesh size</th>
<th>Number of modes</th>
<th>Simulation time (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>140</td>
<td>2.45*$\lambda_{140}$</td>
<td>12+9</td>
<td>42.7</td>
</tr>
<tr>
<td>220</td>
<td>2.45*$\lambda_{140}$</td>
<td>30+24</td>
<td>72.8</td>
</tr>
</tbody>
</table>

So far the horn-lens simulation has only considered the beam up to 3.5° in polar angle. Figure 3.32 shows an extended beam cut. There is an obvious additional feature of multiple far-sidelobes beginning at around 10°. These are thought to originate due to the flat geometry of the lip which surrounds the lens (used for mounting the lens in place). The addition of the aperture stop between the horn and the lens reduces the far-sidelobe by a large margin, however a remnant of the sidelobe still remains at around 20°. The level of the far-sidelobe will be mitigated further through the use of large shields surrounding the aperture of the instrument (visible in Figure 1.16 previously).

Figure 3.32: Horn-lens multi-mode 140 GHz normalised unpolarised far-field extended beam cuts for simulations with and without the aperture stop present.
3.3.10. Accounting for the Layout of the Focal Plane

The focal plane of SWIPE has a hexagonal shape as shown in Figure 3.33. All of the horns are orientated to face the centre of the flat surface of the lens according to the Zemax optimisation. This gives a radius of curvature across the focal plane of -333 mm. The pixels for each of the three bands are contained within the single focal plane so that the same area of sky is measured at each frequency during each scan.

![Diagram of SWIPE focal plane](image)

Figure 3.33: The layout of the horns in the SWIPE focal plane. The focal plane is made up of hexants as shown. Also highlighted are the pixels in each band which are closest to and further from the focal plane centre.

The positions of the horns with respect to the centre of the flat surface of the lens (the BFD) are given in Table 3.9 in terms of the spherical polar coordinate system illustrated in Figure 3.34.

Previous simulations within this thesis have assumed an on-axis pixel, however it is important to understand how the result changes for off-axis pixels across the focal plane. Furthermore, it is only the 220 GHz band which actually has an on-axis pixel.
Due to the number of pixels in the focal plane and the difficulty of the simulation, it is impractical to simulate every pixel. Therefore only the most extreme pixels are considered. The positions of the pixels which are closest to and furthest from the centre of the focal plane for each frequency are given in Table 3.10. These pixels are also highlighted in Figure 3.33.

Table 3.9: Horn positions with respect to the centre of flat surface of the lens for one sector out of the six sectors of the hexagonal focal plane, where $\theta$ is the polar angle and $\phi$ is the azimuthal angle.

<table>
<thead>
<tr>
<th>Horn number</th>
<th>Radial distance (mm)</th>
<th>$\theta$ (°)</th>
<th>$\phi$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>805</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>805</td>
<td>1.65</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>803</td>
<td>3.29</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>801</td>
<td>4.95</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>797</td>
<td>6.59</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>793</td>
<td>8.23</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>788</td>
<td>9.87</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>804</td>
<td>2.86</td>
<td>30.08</td>
</tr>
<tr>
<td>8</td>
<td>802</td>
<td>4.37</td>
<td>19.19</td>
</tr>
<tr>
<td>9</td>
<td>802</td>
<td>4.35</td>
<td>41.19</td>
</tr>
<tr>
<td>10</td>
<td>799</td>
<td>5.96</td>
<td>14.13</td>
</tr>
<tr>
<td>11</td>
<td>799</td>
<td>5.75</td>
<td>30.17</td>
</tr>
<tr>
<td>12</td>
<td>799</td>
<td>5.97</td>
<td>46.19</td>
</tr>
<tr>
<td>13</td>
<td>795</td>
<td>7.58</td>
<td>11.13</td>
</tr>
<tr>
<td>14</td>
<td>796</td>
<td>7.25</td>
<td>23.61</td>
</tr>
<tr>
<td>15</td>
<td>796</td>
<td>7.25</td>
<td>36.66</td>
</tr>
<tr>
<td>16</td>
<td>795</td>
<td>7.59</td>
<td>49.12</td>
</tr>
<tr>
<td>17</td>
<td>790</td>
<td>9.21</td>
<td>9.2</td>
</tr>
<tr>
<td>18</td>
<td>791</td>
<td>8.81</td>
<td>19.35</td>
</tr>
<tr>
<td>19</td>
<td>792</td>
<td>8.68</td>
<td>30.11</td>
</tr>
<tr>
<td>20</td>
<td>791</td>
<td>8.82</td>
<td>40.87</td>
</tr>
<tr>
<td>21</td>
<td>790</td>
<td>9.23</td>
<td>50.99</td>
</tr>
<tr>
<td>22</td>
<td>784</td>
<td>10.86</td>
<td>7.86</td>
</tr>
<tr>
<td>23</td>
<td>786</td>
<td>10.42</td>
<td>16.39</td>
</tr>
<tr>
<td>24</td>
<td>787</td>
<td>10.19</td>
<td>25.46</td>
</tr>
<tr>
<td>25</td>
<td>787</td>
<td>10.19</td>
<td>34.72</td>
</tr>
<tr>
<td>26</td>
<td>786</td>
<td>10.43</td>
<td>43.78</td>
</tr>
<tr>
<td>27</td>
<td>784</td>
<td>10.87</td>
<td>52.31</td>
</tr>
</tbody>
</table>
Figure 3.34: Spherical polar coordinate system used to specify the locations of horns in the focal plane, where the lens lies in the xy-plane with the centre of the flat surface coincident with the origin. Also shown in red is the final definition of the workplane of the off-axis source orientation (U'V' plane).

Table 3.10: Pixel positions for pixels closest to and furthest from the centre of the focal plane.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>Pixel w.r.t. focal plane centre</th>
<th>Radial distance (mm)</th>
<th>θ (°)</th>
<th>Φ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>140</td>
<td>Closest</td>
<td>801</td>
<td>4.95</td>
<td>300</td>
</tr>
<tr>
<td>140</td>
<td>Furthest</td>
<td>784</td>
<td>10.87</td>
<td>352.31</td>
</tr>
<tr>
<td>220</td>
<td>Closest</td>
<td>805</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>220</td>
<td>Furthest</td>
<td>787</td>
<td>10.19</td>
<td>25.46</td>
</tr>
<tr>
<td>240</td>
<td>Closest</td>
<td>801</td>
<td>4.95</td>
<td>120</td>
</tr>
<tr>
<td>240</td>
<td>Furthest</td>
<td>784</td>
<td>10.87</td>
<td>112.31</td>
</tr>
</tbody>
</table>
The simplest way to change the simulation to represent the off-axis pixel is by translating and rotating the horn-equivalent source position. In FEKO the source workplane is defined by two Cartesian vectors defined with respect to the global Cartesian coordinate system

\[
U = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{3.14}
\]

\[
V = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{3.15}
\]

The correct orientation is achieved by applying a series of rotation matrices to these vectors. The rotation matrices which represent a 3D rotation in spherical polar coordinates are given by

\[
R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \quad \text{3.16}
\]

\[
R_y(\alpha) = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \quad \text{3.17}
\]

\[
R_z(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{3.18}
\]

where \( R_i(\alpha) \) is a rotation of \( \alpha \) around the \( i \) axis. As can be seen in Figure 3.34, the orientation of the off-axis pixel is defined by a rotation of \( \theta \) around the x-axis, followed by a rotation of \( \phi \) around the z-axis. Applying this rotation to the source workplane gives

\[
U' = R_z(\phi)R_x(\theta)U = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{bmatrix} \quad \text{3.19}
\]

\[
V' = R_z(\phi)R_x(\theta)V = \begin{bmatrix} -\sin(\phi)\cos(\theta) \\ \cos(\phi)\cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad \text{3.20}
\]

A final rotation of the workplane is required because the polarisation of the source is defined along \( U' \) and \( V' \), but these vectors now correspond to the \( \hat{\theta} \) and \( \hat{\phi} \) axes because of the rotation, and we want the polarisation to relate to the projection of the original x and y axes. Therefore an additional rotation of \( -\phi \) around the \( \hat{z} \) axis is performed. Finally, the source is translated along the \( \hat{z} \) direction by \( r \) to give the correct horn-lens distance. The final off-axis source location and polarisation axes are shown in Figure 3.34 in red.
Figure 3.35: Horn-lens multi-mode 3.5° angular radius normalised **unpolarised** far-field beam map at 140 GHz for pixels closest to *(top)* and furthest from *(bottom)* the centre of the focal plane. The far-field calculation has been centred at the maximum of the beam.
For an off-axis pixel, keeping the far-field horn-lens beam calculation with its origin along the central axis ($z$-axis) of the coordinate system is inadequate since the sampling resolution of the beam becomes too low at the new off-axis location of the beam centre. Therefore the far-field origin is relocated to be at the maximum of the off-axis beam by using the same transformation equations used to relocate the off-axis horn-equivalent source. Note that the maximum of the far-field has the same value for $\phi$ as the off-axis horn position however the value for $\theta$ is less due to the effect of the lens. The centred multi-mode far-field beam maps are shown in Figure 3.35.

**3.3.11. Polarised Horn Beam**

In the SWIPE instrument the beam is polarised by a polarisation-splitting wire grid. This is placed after the lens in the optical chain, therefore polarisation effects caused by the lens are an important factor. For now, the wire grid is assumed to perfectly polarise the beam without causing any other effects. Thus the horn-equivalent source beam is simply polarised by removing one polarisation component. The polarised horn beams are then used to illuminate the lens. The result is calculated for the 140 GHz pixels closest to and furthest from the centre of the focal plane. The polarised beam maps are shown in Figure 3.36-Figure 3.39. These beam maps form the final prediction of the beam on the sky and are used to extract the main beam systematics in the following section.
Figure 3.36: Horn-lens multi-mode 3.5° angular radius normalised far-field beam map at 140 GHz for the pixel closest to the centre of the focal plane for an x-polarised horn source. The maximum y-polarisation is at -43 dB.
Figure 3.37: Horn-lens multi-mode 3.5° angular radius normalised far-field beam map at 140 GHz for the pixel closest to the centre of the focal plane for an y-polarised horn source. The maximum x-polarisation is at -43 dB.
Figure 3.38: Horn-lens multi-mode 3.5° angular radius normalised far-field beam map at 140 GHz for the **pixel furthest** from the centre of the focal plane for an **x-polarised** horn source. The maximum y-polarisation is at -44 dB.
Figure 3.39: Horn-lens multi-mode 3.5° angular radius normalised far-field beam map at 140 GHz for the pixel furthest from the centre of the focal plane for an y-polarised horn source. The maximum x-polarisation is at -45 dB.
3.3.12. Beam Systematics

Beam systematics relating to polarisation have been discussed previously in § 1.6.3. The 140 GHz horn-lens simulation is now used to quantify the size of the main beam and predict the level of these polarisation systematics. This is done for the pixels closest to and furthest from the centre of the focal plane. Furthermore, the level of edge taper and spillover at the point where the cold aperture stop intersects the horn beam is also extracted. The results are summarized in Table 3.11, and a discussion of how they are obtained is given in the subsequent sections.

Table 3.11: SWIPE horn-lens 140 GHz simulated main beam systematics. The results are categorised in terms of x-polarised or y-polarised horn beams feeding the telescope.

<table>
<thead>
<tr>
<th>Horn beam polarisation</th>
<th>Pixel closest to the focal plane centre</th>
<th>Pixel furthest from the focal plane centre</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x-polarised</td>
<td>y-polarised</td>
</tr>
<tr>
<td>Edge taper (maximum)</td>
<td>- 5 dB</td>
<td>-5 dB</td>
</tr>
<tr>
<td>Spillover</td>
<td>0.049</td>
<td>0.050</td>
</tr>
<tr>
<td>HPBW (°)</td>
<td>0.34 x 0.30</td>
<td>0.31 x 0.34</td>
</tr>
<tr>
<td>Cross-polarisation (maximum)</td>
<td>-43 dB</td>
<td>-43 dB</td>
</tr>
<tr>
<td>Cross-polarisation (integrated)</td>
<td>-42 dB</td>
<td>-42 dB</td>
</tr>
<tr>
<td>Instrumental polarisation</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>Far sidelobe</td>
<td>~ -40 dB</td>
<td></td>
</tr>
</tbody>
</table>

**Edge taper**

The edge taper is specified as the intensity of the horn beam at the point where it is cut by the aperture stop, relative to the intensity at the beam centre. Since the pixels are off-axis, the angle at which the aperture stop cuts the beam traces out an oblique cone as demonstrated in Figure 3.40.
Figure 3.40: Parameters to define the angular size of the aperture stop as seen by an off-axis pixel in the focal plane.

The two vector directions are given by

\[
\hat{s} = \begin{pmatrix} r \cos(t) \\ r \sin(t) - d \\ -h \end{pmatrix} \quad \text{(3.21)}
\]

\[
\hat{c} = \begin{pmatrix} 0 \\ -d \\ -h \end{pmatrix} \quad \text{(3.22)}
\]

Thus, \( \alpha \) can be found using the fact that

\[
\cos(\alpha) = \frac{\hat{s} \cdot \hat{c}}{|\hat{s}| |\hat{c}|} \quad \text{(3.23)}
\]

hence

\[
\alpha = \cos^{-1} \left( \frac{(-d + r \sin(t))(-d) + h^2}{\left(r^2 \cos^2(t) + (-d + r \sin(t))^2 + h^2\right)^{\frac{1}{2}} \left(d^2 + h^2\right)^{\frac{1}{2}}} \right) \quad \text{(3.24)}
\]

The final step is to rotate the angle, \( t \), in the equations to account for the, \( \phi \), rotation of the pixel location in the focal plane (as shown in Figure 3.34 previously), i.e. the largest angle of the oblique cone should be when \( t = \phi \). Given the way the angles are defined, this means that \( t \) is rotated by \( \phi - 90 \).
Figure 3.41: SWIPE horn 140 GHz multi-mode (including orthogonal modes) normalised far-field intensity uv-plane beam map extending to 20° in θ. The black and blue dashed lines represent where the aperture stop cuts the beam for pixels closest to and furthest from the centre of the focal plane respectively.

Figure 3.41 shows the 140 GHz horn beam overlaid with the position where it is cut by the aperture stop. Note that the aperture stop angle has also been projected onto the uv-plane. The angle at which the aperture stop cuts the beam varies between 15.2° and 14.5° for the pixel closest to the centre of the focal plane, and between 15.4° and 14.0° for the pixel furthest from the centre of the focal plane. The variation
is small due to the large distance between the horn and the lens relative to other parameters.

**Spillover**

The spillover is the power in the horn beam which falls outside of the telescope. The ratio of the sum of the intensity inside the edge taper ellipse to the sum of intensity over the whole beam (up to 20°) is calculated. 1 minus this value gives the spillover.

**Telescope beam width**

For single-mode horns the beam is quasi-Gaussian therefore an appropriate measure of the beam width is given by the Full Width at Half Maximum (FWHM). For multi-mode beams, the beam is no longer Gaussian therefore the FWHM becomes an inappropriate measure to use. This is because the beam width should tell you information about how much power there is within the enclosed portion of the beam. For a Gaussian beam it is proportional to the FWHM, however for a multi-mode beam the enclosed power becomes very sensitive to on-axis gain. The point is illustrated in Figure 3.42.

![Figure 3.42](image)

Figure 3.42: A representation of two multi-mode beams with the same FWHM but with a vastly different amount of power within the enclosed portion of the beam.

A better measurement of the beam width is sought. A more appropriate measure is the Half-Power Beam Width (HPBW). Along any particular azimuthal cut of the beam, the integrated power within the HPBW amounts to half of the total power along that cut. Due to the ellipticity in the beam, the HPBW is specified along the widest and narrowest cuts of the beam as shown in Figure 3.43 and Figure 3.45. To increase precision, the two cuts are re-calculated from the simulation, with an increased $\theta$ resolution of 0.01°. The beam cuts are plotted in Figure 3.44 and Figure 3.46 with the HPBW indicated.
Figure 3.43: Horn-lens multi-mode 3.5° angular radius normalised far-field beam map at 140 GHz for the pixel closest to the centre of the focal plane. Overlaid are the directions of the widest and narrowest beam cuts.
Figure 3.44: Widest and narrowest cuts of the beam highlighted in Figure 3.43. The vertical dashed lines show the respective HPBW.
Figure 3.45: Horn-lens multi-mode 3.5° angular radius normalised far-field beam map at 140 GHz for the pixel furthest from the centre of the focal plane. Overlaid are the directions of the widest and narrowest beam cuts.
Figure 3.46: Widest and narrowest cuts of the beam highlighted in Figure 3.45. The vertical dashed lines show the respective HPBW.
Cross-polarisation

There is a component of cross-polarisation due to the off-axis horn beam intersecting the refractive optic at an angle. The maximum of the cross-polarisation is extracted from the polarised horn beam simulation in § 3.3.11 (Figure 3.36 - Figure 3.39). Also specified is the integrated cross-polarisation, expressed relative to the integrated co-polarisation.

Instrumental polarisation

Modulation of the polarisation is carried out by the HWP as the first element in the optical chain. This means that the same identical polarised beam from the horn-lens system is used to measure both orthogonal sky polarisations. Therefore there is no instrumental polarisation affects due to horn-lens beam asymmetry.

Far-sidelobe

The peak of the far-sidelobe is estimated from the extended beam cut for an on-axis pixel in Figure 3.32. The value may change slightly if off-axis pixels in the focal plane are considered.

3.4. Horn Phase Centre

It is important to know where to place the focus of the telescope in relation to the horn in order to achieve an optimal final beam in terms of gain, angular resolution and beam shape. For a single-mode horn the radiation from the aperture forms a spherical wavefront. The spherical wave can be traced back to a virtual point source at which the wave appears to emanate from. This is known as the phase centre and usually resides a small distance behind the horn aperture for conical horns. When the focus of the telescope is made coincident with the phase centre the gain, beam shape and angular resolution of the final beam are optimised. For a multi-mode horn the concept of a phase centre is more complex because each of the incoherent modal fields can have their own location of phase centre, each of which may differ by a significant distance (as before, modal field refers to the field associated with each existent mode in the horn waveguide filter). In consequence, the position where the on-axis gain is optimised may differ from the position where the angular resolution
or beam shape is optimised. Thus, for a multi-mode system, the term ‘phase centre’ is used to mean the position where the telescope focus should be located in order to optimise the particular beam parameter of interest (Gleeson et al., 2002).

Since the angular resolution is of less importance for B-mode searches, and optimisation of the beam shape requires the full horn-lens simulation (see later), the optimisation is first investigated for the maximisation of on-axis gain. In this definition the ‘phase centre’ is defined as the position behind the horn aperture where the telescope focus should be placed in order to maximise on-axis gain in the horn-lens far-field beam. The position and significance of the phase centre is calculated in the following sections using several different techniques.

### 3.4.1. Optimising On-axis Gain by Locating the Virtual Beam Waist

In the first technique the field at the aperture of the horn is propagated backwards as though in free space using a Fresnel transformation. This technique has been successfully implemented previously in the development of the Planck-HFI multi-mode horn antennas (Gleeson et al., 2002). The Fresnel transform for any component of the field has the form (Ramo et al., 1994)

\[
\Psi(x, y, z) = \frac{i e^{-ik\Delta z}}{\lambda \Delta z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(x', y', z') \exp\left(-\frac{ik}{2\Delta z} [(x - x')^2 + (y - y')^2]\right) dx' dy',
\]

where \( \Psi(x', y', z') \) is a scalar field at the aperture and \( \Psi(x, y, z) \) is the scalar field in a plane at a distance \( \Delta z = |z - z'| \) behind the aperture. When the resulting field is plotted for increasing values of \( \Delta z \), a virtual field is constructed behind the horn aperture from which the position of maximum on-axis gain can be determined. Again, each modal field at the aperture must be propagated back separately with the multi-mode result being generated by summing the resultant electric fields in quadrature (summing intensity). To reduce simulation time only modes with on-axis power (\( m=1 \)) are considered.

The technique is performed twice using two different methods to obtain the field at the aperture of the horn. In the first method the aperture fields are obtained using the approximate method (see the first paragraph of § 2.5.2). Following the approach in (Gleeson et al., 2002), the Fresnel transform in cylindrical polar coordinates is
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\[ \psi(r, \phi, z) = \frac{i e^{-i k \Delta z}}{\lambda \Delta z} \int_0^{2\pi} \int_0^a \psi(r', \phi', z') \exp \left( -\frac{ik}{2\Delta z} [r'^2 + r'^2 - 2rr' \cos(\phi - \phi')] \right) r'dr'd\phi'. \]

Upon inserting the waveguide modal equations (Eq. 2.43 and 2.44) multiplied by the spherical phase factor (Eq. 2.75) into the Fresnel integral, the integration over \( \phi' \) can be done analytically by rewriting the \( \cos(\phi') \) term as an exponential and using the integral representation of a Bessel function (Born & Wolf 1970)

\[ J_n(x) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} e^{i \alpha x} e^{i n \alpha} d\alpha. \]

The final result is

\[ E_{mn}(r, \phi, z) = l_{m-1,n}(r, \phi, z) \pm l_{m+1,n}(r, \phi, z) \hat{i} \pm l_{m-1,n}(r, \phi, z) \hat{j}, \]

where

\[ l_{m \pm 1,n}(r, \phi, z) = \frac{i}{\lambda \Delta z} \int_{r'=0}^{r'=a} J_{m \pm 1} \left( \frac{x_{mn} r'}{a} \right) J_{m \pm 1} \left( \frac{kr'}{\Delta z} \right) \exp \left( -\frac{ik(r')^2}{2L} \right) \]

\[ \exp \left( -ik \left( \frac{(r')^2 + r^2}{2\Delta z} \right) \right) 2\pi i^{m \pm 1} \left[ \cos(m \pm 1) \phi \right] \sin(m \pm 1) \phi r'dr'. \]

At 140 GHz the first 12 modes (excluding orthogonal modes) are allowed to propagate in the SWIPE horn waveguide modal filter, of which only the \( TE_{11}, TM_{11} \) and \( TE_{12} \) have on-axis power. Each mode is given equal power and the aperture fields are propagated backwards. Figure 3.47 shows the virtual field behind the horn aperture resulting from the combination of the backwards propagation of each of these modal aperture fields.

An on-axis cut of the propagated field is plotted against distance behind the horn aperture: for individual modes in Figure 3.48; and for the combination of modes (as a fraction of the value at the phase centre) in Figure 3.49 (black line). The peak of the graph clearly shows the phase centre to be located at around 24 mm behind the horn aperture.
3.4 Horn Phase Centre

Figure 3.47: The intensity of the virtual electric field behind the horn aperture generated by backwards propagation of the on-axis modes calculated using the approximate method at 140 GHz. The horn aperture is at the top of the plot. Both polarisation components have been summed in quadrature.

Figure 3.48: On-axis E-field intensity plotted against distance behind the horn aperture for individual modes, where the aperture fields have been obtained using the approximate method.

A more accurate result is given if the aperture fields are instead obtained from the horn simulation performed in § 3.2. In this case the integral is evaluated numerically by performing the equivalent summation over all points in the aperture field. The on-axis intensity is shown in comparison in Figure 3.49 (red dashed line). This shows the phase centre to be located at around 21 mm behind the aperture, close to the previous result. There is good agreement in the shape of the decay of the field after the phase centre and just before the phase centre. The large disagreement in the
remaining region close to the aperture can be attributed to a breakdown of the Fresnel approximation at small distances for a finitely sampled field.

Figure 3.49: Fractional on-axis E-field intensity plotted against distance behind the horn aperture for the combination of modes. The aperture fields have been obtained using two different methods. ‘Aperture fields from modal field equations’ is described in the text as ‘the approximate method’.

3.4.2. Optimising On-axis Gain by Translation of a Lens-equivalent Reflector

In the second technique, a simulation is constructed of the SWIPE horn feeding a parabolic reflector with the same focal length as the lens. The reflector approximates the effect of the lens whilst drastically reducing the simulation time thereby allowing many simulations to be performed on a reasonable timescale. The reflector focus is translated incrementally further behind the horn aperture and the on-axis gain of the far-field is extracted. The model is created in GRASP (www.ticra.com) which uses physical optics to perform the simulation.

The parabola (Figure 3.50) is constructed by its radius, \( r \), and depth, \( t \), which are related to its focal length, \( f \), and angular radius, \( \theta_E \), by

\[
\begin{align*}
  r &= -\frac{4f}{\tan(\theta_E)} \pm \left(\frac{16f^2}{\tan(\theta_E)^2} + 16f^2\right)^{\frac{1}{2}} \\
  t &= \frac{r^2}{4f}
\end{align*}
\]

3.30 3.31
3.4 Horn Phase Centre

Both the SWIPE horn and the reflector are constructed in GRASP and the simulation is run using the GRASP batch mode. MATLAB is used to generate a GRASP command file (.tci) which separately excites each propagating mode at the throat of the horn and propagates this through the system using PO before calculating the resultant far-field. The .tci file is placed in the directory of the GRASP program. The command file is run by opening up the windows cmd prompt, navigating to the GRASP directory, then typing grasp9 “filename.tci”. As usual, the final beam is given by summing the far-field intensities resulting from each modal excitation.

Firstly, the reflector is given a overly large angular radius of 65° in order to examine the horn without including, to a high degree, effects related to the spatial truncation of the horn beam. Again, only modes with on-axis power are included. The variation in on-axis far-field intensity is compared with the previous results in Figure 3.51 (dark blue dotted line). Note that it is acceptable to directly compare this to the result of the previous section (propagation of horn aperture field backwards), since the far-field is essentially an image of the field at the focus of the telescope. The x-axis figure label (‘Distance behind horn aperture’) now refers explicitly to the translation of the telescope focus behind the horn aperture. The result shows good agreement with the previous technique for phase centre location (20-21 mm) and variation in the on-axis intensity across the translation (at large distances from the aperture). Since

Figure 3.50: Representation of a horn feeding a parabolic reflector.
no numerical Fresnel transformation has been used, there is no error at small distances therefore the result agrees better with the first technique which also did not suffer from this error. The small remaining differences may be due to remaining beam truncation effects or the approximations made in the Physical optics simulation technique employed by GRASP.

In reality, there is an aperture stop in SWIPE which has an angular radius of around 15°. This causes the beam to be highly spatially truncated. The situation is modelled by reducing the angular radius of the reflector to 15° whilst keeping the focal length the same so as to still match the SWIPE lens. The result is compared in Figure 3.51 (light blue line). The beam truncation causes the variation in on-axis intensity to be much less severe. The phase centre is fairly flat over the region 20-25 mm, remaining in agreement with the previous results.

Figure 3.51: Fractional on-axis E-field intensity plotted against distance behind the horn aperture for different cases.

### 3.4.3. Optimising On-axis Gain by Translation of the Lens

In the final technique, the full horn-lens simulation in § 3.3 (including the aperture stop) is used to deduce the phase centre. For an on-axis pixel, the lens is translated incrementally and the result is compared to the previous techniques in Figure 3.52 (pink line). A magnified plot is also shown. The variation in intensity is similar to the previous result, however the phase centre is predicted to be slightly further back, in the region of 26-30 mm behind the aperture.
Figure 3.52: Fractional on-axis E-field intensity plotted against distance behind the horn aperture for different cases. The bottom plot has a restricted y-axis scale in order to show the difference between pink and blue lines.

Figure 3.53: On-axis E-field intensity plotted against distance of the telescope focus behind the horn aperture for the case where the full horn-lens simulation is used.
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The translation of the lens is repeated for the 220 GHz pixel and the result is compared in Figure 3.53. The 220 GHz result closely matches the 140 GHz result, both in terms of phase centre location (24-26 mm) and variation of intensity across the whole displacement.

3.4.1. Optimising Integrated Gain and Beam Shape

For a single-mode system ($TE_{11}$ mode) the horn beam has maximum power on-axis and forms a quasi-Gaussian beam shape. Therefore, on-axis gain is an appropriate measure in determining the phase centre of the horn when trying to maximise overall throughput. However, for a multi-mode system, higher order modes with an azimuthal index not equal to one have zero power on-axis and instead contribute the majority of their power off-axis. This gives the beam its flat-top shape. The off-axis power is substantial and thus needs to be taken into account. This is done by considering integrated power over the whole beam instead. All modes (including orthogonal modes) are included in the simulation and a single cut of the azimuthally symmetric unpolarised beam is calculated for different translations of the lens for the 140 GHz pixel. The result is compared against the on-axis gain result in Figure 3.54. Again, a magnified plot is also shown. The result is now almost the same for both the 140 and 220 GHz pixels. The phase centre is also shifted forwards slightly and is now located at 21-23 mm behind the aperture at both frequencies. The integrated intensity at the aperture is 0.93 times what it is at the phase centre.
Figure 3.54: Fractional on-axis (dashed line) and integrated (solid line) E-field intensity plotted against distance of the telescope focus behind the horn aperture. In the bottom plot the resolution of the translation has been increased and both axes have been restricted to show clearly the differences between results.

In addition to the overall throughput, the shape of the beam must also be considered. The variation of the beam for different translations of the lens focus is demonstrated in Figure 3.55. The beam is considered most optimal when most of its power is concentrated within a specific angle (falling off sharply at the edge). For both frequencies the beam is thus optimised when the telescope focus is placed 10 mm behind the horn aperture. This differs somewhat with the result for optimal integrated gain (21-23 mm).
3.4.2. Accounting for the Layout of the Focal Plane

The phase centre investigation is repeated at 140 GHz for the true locations of the pixels in the focal plane. The layout of the focal plane has previously been discussed in § 3.3.10. Pixels which are closest to and furthest from the centre of the focal plane are considered. All modes are included in each case. The full horn-lens beam maps when the telescope focus is located at the horn aperture were shown previously in Figure 3.35. Since the unpolarised beam is no longer azimuthally symmetric, the whole beam should be considered when calculating the integrated gain. However, the simulation time for a full beam calculation at multiple horn-lens distances is too long. Therefore, the calculation of integrated gain is instead approximated using the
combined integrated gain along 2 beam cuts taken at the narrowest and widest parts of the beam. Figure 3.56 shows the effect of the translation on the integrated gain. As the pixel is moved further off-axis the phase centre is shifted towards the aperture. This is believed to be due to a disagreement between the Zemax and FEKO simulations regarding the predicted shape of the focal plane, although further investigation is required. For the most central pixel the phase centre is at 19-21 mm and the integrated intensity at the aperture is 0.945 times the value at the phase centre. This changes to 9-14 mm and 0.965 respectively for the least central pixel.

![Graph](image)

Figure 3.56: Fractional integrated E-field intensity plotted against distance of the telescope focus behind the horn aperture taking into account the true location of pixels in the focal plane.

The beams used in the calculation of integrated gain are shown in Figure 3.57 and Figure 3.58 for the most and least central pixels respectively. The lack of azimuthal symmetry in the off-axis beam makes determination of the optimal beam less obvious. Furthermore, there appears to be some defocusing effects as the horn-lens distance is changed. These defocusing effects may be due to the fact that the horn faces the centre of the flat surface of the lens and not the principal plane. For the most central pixel the beam is optimised at 10 mm, agreeing with the result for the fictitious on-axis pixel in the previous section. For the least central pixel the beam is difficult to analyse given how an optimised beam has been defined. The first cut would suggest the beam is optimised at the aperture whereas the second cut may suggest the beam is optimised >30 mm. Overall the results are inconclusive for this pixel. The full results for the phase centre are summarised in Table 3.12.
Figure 3.57: Widest (top) and narrowest (bottom) cuts of the far-field beam at 140 GHz for the pixel closest to the focal plane centre for different translation of the telescope focus relative to the horn aperture.
Figure 3.58: Widest (top) and narrowest (bottom) cuts of the far-field beam at 140 GHz for the pixel furthest from the focal plane centre for different translation of the telescope focus relative to the horn aperture.
Table 3.12: SWIPE horn-lens location of phase centre specified as distance behind the horn aperture. In brackets is the fractional value of the tabulated parameter at the aperture of the horn relative to the value at the phase centre.

<table>
<thead>
<tr>
<th></th>
<th>140 GHz</th>
<th>220 GHz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>On-axis pixel</td>
<td>Pixel closest to focal plane centre</td>
</tr>
<tr>
<td>On-axis gain</td>
<td>26-30 mm (0.87)</td>
<td>-</td>
</tr>
<tr>
<td>Integrated gain</td>
<td>21-23 mm (0.93)</td>
<td>19-21 mm (0.945)</td>
</tr>
<tr>
<td>Beam shape</td>
<td>10 mm</td>
<td>10 mm</td>
</tr>
</tbody>
</table>

3.5. Discussion

3.5.1. Beam Systematics

The SWIPE horn-lens simulation has been used to predict the beams for the horn and the horn-lens set-up. These beams have been used to make a prediction of the edge taper and spillover for the horn beam intersecting the cold aperture stop, and for the cross-polarisation and far-sidelobe level of the horn-lens beam. The instrumental polarisation due to differential beam shape of the horn-lens beam for each polarisation normally contributes a large source instrumental polarisation in a CMB experiment. However, many systematic errors, including this one, are mitigated through the introduction of the rotating HWP as the first element in the optical chain, since this allows the same beam to measure both polarisations.

A report by (Bock et al. 2006) estimates the level at which the systematics must be suppressed for a B-mode detection at the level of $r = 0.01$ (~30 nK rms signals). The goal is to keep the level of each systematic effect a factor 10 below this level. The goal for cross-polarisation is set at $<-25$ dB. The predicted cross-polarisation for the horn-lens set-up is $<-40$ dB for the 140 GHz pixels. This suggests that the optical horn-lens cross-polarisation should not be an issue for SWIPE. The spillover and far-sidelobe level cannot be evaluated using the report due to the differences in instrument design. Instead, further investigation is required to fully understand their effect on the experiment. The spillover is around 5% for the 140 GHz pixels due to
3.5 Discussion

the top-hat shape of the multi-mode beam. This is counteracted in the design of SWIPE through the use of a cold aperture stop in front of the lens. The value of the spillover looks acceptable, however a full analysis should be performed to be certain. This would be done by converting the spillover into a noise on the detector by using a black-body at the temperature of the cold aperture stop. The spillover noise would then be combined with other sources of noise to give the total noise levels on the detector. The far-sidelobe is around -40 dB for the 140 GHz pixel. This is negated further by forebaffles which surround the aperture of SWIPE. A full evaluation of the far-sidelobe would require full experiment level simulations where the scan strategy of SWIPE is taken into account.

It must be noted that these systematics are due to the horn-lens set-up only, with the polarisation-splitting wire grid assumed to act perfectly. Beyond the current simulation, the effects of other components in the optical chain should be taken into account. A true model of the polarisation-splitting wire grid should be included, taking into account the differential effects of reflection and transmission of different polarisations. This may induce some instrumental polarisation into the final beam. Furthermore, although the HWP mitigates many systematics, it is also a source of many systematics itself, some of which may be quite severe. Thus the output from the horn-lens simulation should be propagated through a model of the HWP in order to predict these. Current models of such a large diameter HWP, however, rely on transmission line modelling, and are thus incapable of doing this.

3.5.2. Phase Centre

Ideally, each horn in the focal plane should be individually shifted so that the telescope focus coincides with the horn phase centre according to the results in § 3.4. However, the phase centres for optimal gain and beam shape do not coincide, therefore a compromise must be made. Maximising the sensitivity of the experiment is more important than angular resolution and beam shape, therefore maximising gain should take precedence, providing that the beams remain reasonable. The phase centre for optimal gain is located 19-21 mm and 9-14 mm behind the aperture for the 140 GHz pixels closest to and furthest from the centre of the focal plane respectively. The integrated electric field intensities at the aperture relative to the value at the phase centre are 0.945 and 0.965 respectively. The beams are not optimised but
remain reasonable at these locations (see Figure 3.57 and Figure 3.58). The same analysis should be repeated for each pixel in the focal plane to determine the phase centre so that each pixel can be positioned accordingly.

In practice, in positioning each horn according to the phase centre, the mechanical considerations and shadowing of other horns in the focal plane must also be considered. Consequently, it may not be overall beneficial to achieve optimal positioning of the horns according to phase centre. Nevertheless, even if the positions of the horns are restricted so that the telescope focus remains at the aperture, it has been demonstrated that the effect on the gain is not severe, and the beams are acceptable. The integrated electric field at the aperture only drops to around 95% of the value at the phase centre.

**3.5.3. Horn-lens Simulation**

The horn-lens simulation is successful in determining the beams and their associated systematics, and finding the optimal phase centre location for the SWIPE horn-lens configuration. The full simulation can be performed on a reasonable timescale, taking about 3 days for the 140 GHz pixel and 1 week for the 220 GHz pixel. To achieve a high accuracy on these timescales, several assumptions and approximations have been made. Going forward, these approximations should be directly removed from the simulation, or, for cases where this is not possible, the effect of the approximation should be quantified in some way.

The full frequency band has been approximated by a monochromatic simulation at the centre of band. The simulation should instead be performed at several frequencies across the band, with the final result given as an average across the band, weighted by the transmission profile of the band-pass filter in the horn filter cap. The effect of the filter cap, itself, on the beam should also be examined by including a realistic model of the filter cap and bandpass filter in the simulation. This has been attempted, however an accurate model, which could be run on the available resources, could not be achieved.

Another important consideration which should be taken into account is the behaviour of the detector cavity and the detector itself. It is not practical to directly add a model
of the detector to the current simulation of the horn. Rather, the coupling of each modal field onto the detector through the transition horn should be modelled separately. The results should then be used to weight the power associated with each modal field in the horn simulation, before combining them to get the multi-mode result. Initial simulations have been performed to optimise the transition horn and detector cavity by minimising the return loss as a function of absorber surface impedance and distance to the backshort (Lamagna et al. 2015). These show an average return loss for modes excited in the waveguide filter of -20 dB, -24 dB and -21 dB at 140, 220 and 240 GHz respectively. Once these simulations have been developed to include a more accurate model of the detector, and the data are available for individual modes, the results should be used as the weighting parameters.

There are several further improvements that should be made to the simulation. A consideration should be made of the effect of multiple reflections within the lens after anti-reflective coating is taken into account. A better model of the aperture stop should be implemented (currently modelled as PEC). Finally, a separate simulation should be performed for the 240 GHz pixel. Furthermore, the inclusion of other components, as detailed at the end of § 3.5.1, should be addressed.

3.6. Conclusion

A simulation has been constructed of the SWIPE multi-mode horn-lens configuration for 140 GHz and 220 GHz pixels closest to and furthest from the centre of the focal plane. The large electrical size of the lens and the fact that a separate simulation is required for each mode, makes this a challenging problem to simulate. A variety of simulation techniques are investigated in order to determine the most appropriate in terms of accuracy and run-time. Within the specific technique, simulation parameters are investigated through a convergence study to choose the optimal values. The final horn-lens simulation is used to extract the beam pattern on the sky and determine the level of polarisation systematic effects in the main beam for the 140 GHz band. The optical cross-polarisation is predicted to be < -40 dB, well below the performance goals for a B-mode detection at the level of $r = 0.01$. Furthermore, the spillover of
the horn beam outside the telescope is around 5% and the far-sidelobe of the horn-lens beam is around -40 dB.

The optimum location at which the telescope focus should be placed in relation to the horn aperture has also been investigated. This is referred to as the ‘phase centre’. The phase centre with respect to maximising gain is found using several different techniques, all of which predict consistent results. The most accurate technique involves translation of the horn in the full horn-lens simulation. The phase centre is found to be 19-21 mm and 9-14 mm behind the horn aperture for 140 GHz pixels closest to and furthest from the centre of the focal plane respectively. The integrated electric field intensity at the aperture is 0.945 and 0.965 respectively, as a fraction of the value at the phase centre. The phase centre with respect to optimising beam shape is found to be ~10 mm behind the horn aperture for the most central pixel, however the result is unclear for the pixel furthest from the centre. Practical limitations have shown that it may only be possible to position the horns with the telescope focus at the horn aperture. In this case, it has been shown that the loss of gain is small and the telescope beams are not adversely affected.
4. Measurements of the Multi-mode Horn for LSPE-SWIPE

4.1. Introduction

The simulations for the SWIPE multi-mode horns have been dealt with in Chapter 3. This section focuses on measurement techniques used to validate these simulations and to check for manufacturing defects in the horn and the detector. In this work two measurement set-ups are investigated. An incoherent set-up is presented in § 4.2; and a coherent set-up is presented in § 4.3. The difference between coherent and incoherent operation is previously explained in § 2.5.1. The overall aim of each measurement set-up is to retrieve the incoherent multi-mode far-field beam pattern of the horn.

The measurement using the incoherent set-up is straightforward in the sense that it attempts to directly mimic the in-flight operation of the full BTB horn using an incoherent bolometric detector. The main difference from in-flight operation is the omission of the cryostat through the utilisation of a room-temperature version of the bolometer. The coherent set-up, on the other hand, is the result of an investigation into the feasibility of using a coherent detector to measure and infer the incoherent beam of the front horn. This is done by measuring how individual modes scatter as they pass through the horn. From this knowledge a scattering matrix is constructed and used to infer the incoherent far-field beam. The coherent technique is useful in assessing the performance of the front horn, however no direct information is gained on the coupling efficiency of modes onto the detector in the detector cavity. Some information is gained indirectly however, since, upon attempting to excite individual modes within the horn waveguide filter, information is learnt about the alignment sensitivity of modal excitations which proves useful in understanding the detector cavity performance.
4 Measurements of the Multi-mode Horn for LSPE-SWIPE

There are two prototype versions of the SWIPE BTB horn which have been manufactured. One is an older prototype with a waveguide filter of radius 2.05 mm; designated the P1 horn. The other is a prototype of the flight model design with a waveguide filter of radius 2.25 mm; designated the P2 horn. For the simulations in Chapter 3 and the measurements using the incoherent set-up, it is the P2 horn which is used. However, for the measurements using the coherent set-up, it is the P1 horn which is measured since the P2 horn was unavailable. The simulations within the coherent measurement section are adjusted accordingly to match this. This is not a problem since the primary aim for the coherent set-up is to validate the measurement technique against simulation.

4.2 Incoherent Measurements

The incoherent set-up is designed to measure the full SWIPE BTB horn (described previously in § 3.2.1) including the detector cavity but not the filter cap. The measurements are compared against the simulations performed in Chapter 3 using the P2 prototype. The incoherent set-up has been developed at Sapienza Università di Roma, primarily by Luca Lamagna, Grazia Giuliani, Riccardo Gualtieri and Fabio Columbro. The author of this thesis has played only a small part in the development of the incoherent set-up and the performance of the measurements, but has contributed to ongoing discussions regarding the analysis and interpretation of the results. A diagram of the full incoherent set-up is shown in Figure 4.1. This is accompanied by photographs shown in Figure 4.2 and Figure 4.3.

![Diagram of the incoherent set-up](image)

Figure 4.1: A diagram of the incoherent set-up. See the main text for explanation.
Ideally the horn would be measured under exact in-flight conditions with cryogenically cooled components, however this is difficult to achieve in the lab. Instead, an approximate measurement is made by placing a room-temperature bolometer in the detector cavity. The room-temperature bolometer has a Pt thermistor which achieves good sensitivity under low vacuum conditions at room temperature. In the set-up the radiation source and horn under test are placed at opposite ends of an Eccosorb cage to reduce reflections and limit contamination from sources outside of the set-up.

Figure 4.2: Incoherent test set-up showing the radiation source (near-side) and horn under test (far-side) surrounded by an Eccosorb cage.
Measurements of the Multi-mode Horn for LSPE-SWIPE

The radiation source consists of a mounted pyramidal feed horn fed by a 116 GHz Gunn diode oscillator, producing a polarised beam. The horn under test is placed in the far-field of the source thereby making the source effectively a plane wave. A beam splitter is angled at 45° in front of the source so that a percentage of the power is reflected onto a diode detector. This allows the drifts in the output power of the Gunn diode to be measured and corrected for. An Eccorsorb circular aperture stop is placed in front of the source to remove radiation which is scattered off the metallic ring which holds the beam splitter.

![Figure 4.3: Incoherent test set-up showing a side view of the horn under test mounted to the rotary scanner.](image)

A closer view of the end of the set-up housing the receiver is shown in Figure 4.3 and the BTB horn assembly is shown fully in Figure 4.4. The room temperature bolometer requires a high and stable vacuum to operate correctly. A sharp drop in responsivity can be seen if the vacuum is removed. Therefore the horn under test is encased by an air-tight metallic casing. The gap between the casing and the outside of the horn is made air-tight using a series of alternating rubber o-rings and metallic rings which slot in between. The seal is improved by coating the rings in vacuum grease. The air-tight seal is completed inside the horn by placing a thin piece of polypropylene inside the waveguide filter. A vacuum pump is attached to the back of the casing to create the vacuum. The vacuum remains pumping throughout the measurements and a barometer is attached to check that the vacuum is stable. The horn is rotated using an automated 2-axis rotary scanner which is controlled using...
4.2 Incoherent Measurements

Labview (http://www.ni.com/labview/). The room temperature bolometer sits in the detector cavity at the back of the horn.

Figure 4.4: The P2 BTB horn consists of a 59.12 mm long front horn of aperture radius 10 mm, and a 37.67 mm long transition horn of aperture radius 8.5 mm which guides radiation from the back of the front horn onto the full area of the bolometer cavity. A waveguide filter of length 10 mm and radius of 2.25 mm sits between the two horns; half of the filter is attached to the back of the front horn and the other half is attached to the front of the transition horn. The detector cavity has a depth of λ/2 and the bolometer is placed at midway thereby creating a resonant cavity to maximise absorption. The bolometer is not in place in the image.

The bolometer signal is extracted and separated from background noise using a lock-in-amplifier which modulates the signal at 2.0 Hz. Even under vacuum conditions the time constant of the bolometer is relatively high, therefore each point is given an integration time of 50 seconds. Measurements of the horn beam have been performed at 116 GHz by Luca Lamagna and Fabio Columbro. At 116 GHz only the first 9 modes are non-evanescent in the waveguide filter, 3 fewer than at 140 GHz. Simulations from Chapter 3 have be recalculated at 116 GHz to compare with the measured data.

4.2.1. Far-field Beam Pattern

The measured horn beam is compared with simulation in Figure 4.5. The beam falls off much faster than predicted. This is indicative of modes with off-axis power being filtered out, which is likely due to poor coupling of these modes with the bolometer in the detector cavity. The most probably causes are manufacturing defects in the bolometer and poor alignment of the bolometer in the cavity. The exact reasons are
Measurements of the Multi-mode Horn for LSPE-SWIPE

explored further in § 4.4.2, taking into account the results of the coherent measurements in § 4.3.

Figure 4.5: Measured P2 horn normalised far-field beam at 116 GHz compared with simulation.

4.3. Coherent Measurements

As we have seen in § 4.2, performing an accurate direct incoherent measurement of the multi-moded SWIPE pixel assembly is difficult due to the incoherent detector. Therefore, an investigation is carried out to understand to what extent these measurements can be done if a more robust, coherent measurement system, such as a Vector Network Analyser (VNA), is used instead. The VNA interfaces directly with the waveguide filter at the base of the front horn, taking the place of the transition horn and detector cavity. The advantage being that, the waveguide filter and front horn performance can be characterised separately without the complications of the bolometric detector. This is done by attempting to infer the incoherent far-field beam from the coherent measurements, as would be given if the modes coupled perfectly to the bolometer in the incoherent set-up. Furthermore, the method also gives information on how misalignments in the excitation of modes in the waveguide filter affects the modal content at the horn aperture. This information could be used.

The coherent measurement technique does not directly measure the performance of the bolometer in the detector cavity. This cannot be neglected in the overall characterisation of the pixel assembly and still must be understood using the incoherent set-up.
4.3 Coherent Measurements

to indirectly gain understanding of how misalignment in the bolometer in the
detector cavity may affect the modal content at the horn aperture and thus the
far-field beam. This information could contribute to the understanding of the poor
match between the simulated and measured beams in the incoherent set-up.

A VNA has the advantage that the measurements can be performed simultaneously
across the whole frequency band of operation. Furthermore, the coherent nature of
the detection scheme means that cryogenics are not required to cool the components.
The available VNA is a Rohde & Schwarz ZVA40 with W-band (75-110 GHz)
converter ports. The output from the converter ports is in the form of a rectangular
waveguide which remains single moded over the entire W-band. A
rectangular-to-circular waveguide transition can be attached to the end of the
rectangular waveguide in order to interface with circular waveguides and horns. The
gradual taper of the transition means that the circular waveguide is predominantly
single moded (TE_{11} mode) over the entire frequency range. Using this set-up, the
beam patterns of single-mode conical horns can be directly measured by attaching a
horn of matching throat radius to the circular waveguide. However, simply applying
this same technique to the multi-mode horn does not yield the desired incoherent
multi-mode beam. This is because, firstly, only the first mode (TE_{11}) is excited at the
horn throat, therefore the measured beam pattern would be the just the single mode
operation of the horn. Secondly, even if all modes could be excited at the horn throat,
the modes would have different powers and the beam you would measure is the
coherent sum of the electric modal fields rather than the incoherent sum of the
intensities. Incoherent and coherent operation, and the ‘modal field’ terminology has
been explained previously in § 2.5.1.

A new technique is developed with the overall aim of using coherent measurements
to indirectly infer the incoherent far-field beam. A brief overview of the technique is
given here, however this will become clearer as the sections within this chapter are
progressed. Individual modes are excited within the horn waveguide filter and are
allowed to propagate through the front horn. As each mode propagates it scatters into
other modes. The scattered modes and their associated waveguide mode are referred
to as a single modal field. The modal content of each modal field at the aperture of
the horn is measured to determine the exact scattering relation, and an overall
4 Measurements of the Multi-mode Horn for LSPE-SWIPE

scattering matrix is constructed which describes the scattering of modes within the horn. After normalising modal field power within the scattering matrix so that each modal field has an equal excitation power in the waveguide filter, the electric far-field for each modal field is constructed and summed in quadrature to give the multi-mode incoherent far-field beam.

The technique is first developed using purely simulated waveguides and horns. The jump from simulation to measurement raises further technical challenges which must be overcome. Currently the validity of the technique as a whole is limited to only the first 3 modes being excited at the horn throat. A discussion as to why this is and of how the technique could be extended beyond 3 modes is given at the end of the chapter.

An overview of the process by which the coherent measurement technique is developed is provided for clarity:

- **Simulation**
  - Modal content testing
    - Purely theoretical
    - Cylindrical waveguide – internal field measurement
      - 2 waveguide ports so that no radiation is reflected
      - 1 reflective end so that radiation is fully reflected
      - 1 open end so that radiation is partially reflected
    - Cylindrical waveguide – external field measurement
    - Conical horn – external field measurement
  - Reconstruction of the incoherent multi-mode far-field beam
    - Waveguide and horn

- **Measurement**
  - Modal content test and beam reconstruction
    - Cylindrical waveguide (single-mode)
      - Probe study
    - SWIPE P1 horn
      - Using a field cut at 300 mm from the horn aperture
      - Using a field cut at 150 mm from the horn aperture
4.3.1. Theoretical Overview of the Modal Content Calculation for a Simulated Horn

The first three circular waveguide modes \((TE_{11}, \, TM_{01} \text{ and } TE_{21})\) are excited together at the throat of a horn. In the FEKO simulation software the excitation power in each mode is specified by inputting the excitation magnitude. Initially, a value of unity is given for the excitation magnitude of each mode, resulting in each mode being excited with its fundamental power (Eq. 2.67 and 2.69), where the magnitude term corresponds to the amplitude term, \(A\), which resides in the coefficient term (Eq. 2.37 and 2.38). Note that the fundamental power changes depending on the mode.

The horn is simulated using the usual Method of Moments techniques as was used in Chapter 3, and the complex electric field at the aperture of the horn is extracted from the simulation. Following a similar technique to (Shimozuma 2008; Jawla et al. 2012), the modal content of the aperture field is calculated by performing overlap integrals with the theoretical electric field patterns of each mode (Eq. 2.43 and 2.44). As seen previously in Chapter 2, the complex electric field at the aperture of a horn can be decomposed into waveguide modes and their associated coefficients. Therefore the aperture electric field is given as

\[
u = \sum_{m}^{M} \sum_{n}^{N} (E_{mn}^{TM} + E_{mn}^{TE}) = \sum_{m}^{M} \sum_{n}^{N} (C_{mn}^{TM} \Phi_{mn}^{TM} + C_{mn}^{TE} \Phi_{mn}^{TE}) = \sum_{l=1}^{L} C_{l} \Phi_{l}, \tag{4.1}
\]

where \(\Phi\) represents the eigenfunction part of the mode and the representation has been simplified by categorising both \(M \times N \, TE\) and \(TM\) modes into a single set of \(L\) modes, where \(l\) represent all combinations of indices \(mn\). Performing the dot product with \(\Phi_{l}^{*}\) and integrating over the aperture surface, \(S\), gives an equation for the coefficients

\[
C_{l} = \frac{\int \nu \cdot \Phi_{l}^{*} dS}{\int |\Phi_{l}|^2 dS}. \tag{4.2}
\]

More usefully, similarly to Eq. 2.57, it follows that the power in each mode can be calculated by

\[
P_{l} = \frac{1}{2Z_{l}} |C_{l}|^2 \cdot \int |\Phi_{l}(x, y)|^2 dS, \tag{4.3}
\]
where $Z_l$ is the theoretical impedance of the $l$ mode in the waveguide at the aperture. The form of the above equations relates to a continuous field distribution, which is not appropriate for the simulation or lab measurements, for which the output is a discretely sampled numerical field. Thus, for a field comprised of a finite amount of sample points, the equations become

$$C_l = \frac{\sum u(x,y) \cdot \Phi_l^*(x,y)}{\sum |\Phi_l(x,y)|^2}$$  \hspace{1cm} 4.4$$

$$p_l = \frac{1}{2} \frac{1}{Z_l} |C_l|^2 \cdot \sum |\Phi_l(x,y)|^2,$$  \hspace{1cm} 4.5$$

where the summation is performed over all sample points in the measurement plane. The fractional modal power content can be found simply by

$$FP_l = \frac{p_l}{\sum p_i}$$  \hspace{1cm} 4.6$$

In the simulation the modes have been excited with their fundamental power. To be representative of the case where the modes are instead excited with equal power, the measured modal coefficients are multiplied by the ratio of the fundamental modal coefficient (Eq. 2.37 and 2.38) to the power normalised modal coefficient (Eq. 2.70 and 2.71). The modal power content is multiplied by the square of this ratio. The value is referred to as the boosted detected modal power. This provides an instant check to see how much of a mode has been scattered in the horn since, if no scattering took place, the boosted modal power content is equal to unity.

### 4.3.2. Modal Content Calculation for a Simulated Cylindrical Waveguide

A custom MATLAB code reads in the electric field exported from the simulation and performs the modal content calculation. The technique is developed by starting with the simplest case possible and then gradually adding more complexity, whilst ensuring that the result remains as expected. The modal content is first tested for the simplest case possible, a circular waveguide of radius 1.5 mm and length 3 mm. The constant radius means that no scattering takes place between the modes, therefore the measured modal content should match exactly the modal content which is excited. At 75 GHz only the fundamental $TE_{11}$ mode can propagate, whereas at 110 GHz the first 3 modes can propagate. For now, the orthogonal modes are not included in the excitation since their scattering behaviour is identical to their orthogonal counterpart.
The orthogonal modes however are always tested for during the modal content calculation. The fundamental coefficient and power of the first three modes in this waveguide geometry at 110 GHz are shown in Table 4.1.

Table 4.1: Fundamental coefficients and power of the first three modes in a cylindrical waveguide of radius 1.5 mm at 110 GHz.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fundamental coefficient (absolute magnitude)</th>
<th>Fundamental power (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>707.6</td>
<td>4.741e-4</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0.999</td>
<td>3.516e-09</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>426.6</td>
<td>5.405e-05</td>
</tr>
</tbody>
</table>

**Purely theoretical field input**

Before using data from the simulation, an even simpler case is tested to check that the modal content code is working as intended. The theoretical modal field equations, which are used to analyse the modal content, are tested against themselves.

![Theoretical electric field for a 3-mode coherent excitation in a circular waveguide.](image)

Figure 4.6: **Theoretical electric field** for a 3-mode coherent excitation in a circular waveguide. The two columns corresponds to each polarisation and the two rows show amplitude (*top*) and phase (*bottom*). This is the standard layout of how the fields are displayed for the remainder of the chapter.
The 3-mode theoretical field is constructed at 110 GHz with each mode given its fundamental power. This acts as the input field and is shown in Figure 4.6. The modes in the input field are always summed coherently to match what would be the case in the lab using the coherent detection scheme.

Figure 4.7: Theoretical modal eigenfunction electric fields for the first three circular waveguide modes. (1/2)
4.3 Coherent Measurements

Figure 4.7: Theoretical modal eigenfunction electric fields for the first three circular waveguide modes. (2/2)

The modal content is calculated by performing an overlap integral of the input field with each modal field eigenfunction (Eq. 4.4 and 4.5) up to a maximum azimuthal index, $m$, of 4 and radial index, $n$, of 10. The modal field eigenfunctions of the first three modes are plotted in Figure 4.7 for reference. Any minor deviations from the obvious symmetry are purely an artefact of the Matlab plot and do not impact on the calculation of the modal content.

The detected modal content is shown in Table 4.2. As expected, the detected coefficients and power match the fundamental coefficient and power (Table 4.1) given to modes in the input field, with a very small error introduced which is discussed below. The total fractional power sums to unity for these three modes, showing that there is no power in modes which were not excited. Finally, the boosted power is unity for each mode as expected (boosted power is explained previously at the end of § 4.3.1).

The very small error in the modal content calculation is thought to originate from the fact that the resolution of the field is limited. It fundamentally stems from the approximation of an integral as a summation. Table 4.3 shows how this error increases as the resolution is changed from the resolution of $201 \times 201$ field points which is used in the above case. The simple solution would be to use an extremely
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high resolution, however, this increases the overall runtime of the process. Therefore a resolution of $201 \times 201$ is used and the error is taken as a guideline for the precision at which the modal power can be detected. Note that this error does not originate from the problem of distinguishing between the three modes in the combined 3-mode input field, since measuring the modal content of each field individually provides the same result. Rather, the error comes directly from the overlap integral used to calculate the modal content. Furthermore, when exciting modes individually, zero modal content is measured in modes which have not been excited.

Table 4.2: The detected modal content for a 3-mode **theoretical input field**. The fractional power is a fraction of the total power measured for all modes up to an azimuthal index of 4 and radial index of 10. The last column shows what the detected power is after boosting to represent the case where all modes are excited with unity power.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Detected coefficient</th>
<th>Detected power (W)</th>
<th>Total fractional power</th>
<th>Detected power (boosted) (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Complex</td>
<td>Absolute magnitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>2.054e-19 – 707.3i</td>
<td>707.5</td>
<td>4.739e-4</td>
<td>0.829</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>1.000- 0.000i</td>
<td>1.000</td>
<td>3.513e-09</td>
<td>3.737e-06</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>-1.237-18 – 426.4i</td>
<td>426.4</td>
<td>5.402e-05</td>
<td>0.171</td>
</tr>
</tbody>
</table>

Table 4.3: The error in the detected power for different resolutions of the input electric field.

<table>
<thead>
<tr>
<th>Resolution</th>
<th>Fractional error in detected power</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$TE_{11}$</td>
</tr>
<tr>
<td>401x401</td>
<td>2.222e-4</td>
</tr>
<tr>
<td>201x201</td>
<td>4.123e-4</td>
</tr>
<tr>
<td>151x151</td>
<td>9.121e-4</td>
</tr>
<tr>
<td>101x101</td>
<td>2.185e-3</td>
</tr>
<tr>
<td>51x51</td>
<td>7.367e-3</td>
</tr>
</tbody>
</table>
4.3 Coherent Measurements

Simulated waveguide with 2 ports

The next simplest case is an actual simulation of a section of a waveguide of the same dimensions using the simulation software. The simulation is performed using FEKO and a representation of the model is shown in Figure 4.8. One waveguide port is added at the left end of the waveguide and the first three modes are excited with their fundamental power. A second passive waveguide port is added on the other end of the waveguide to ensure that no power is reflected at the end of the guide.

Figure 4.8: Model of a circular waveguide simulated in FEKO with 2 waveguide ports.

The electric field, extracted at the midpoint of the guide, is shown in Figure 4.9 and the detected modal content result is shown in Table 4.4. The electric field resembles that of the theoretical input case in Figure 4.6, but is slightly different. This is due to the fact that the relative phase difference between the 3 modes has been made to be zero in the theoretical input case, however in the simulated waveguide the phase relationship is non-zero. This is because each mode has its own different effective wavelength within the waveguide, thus the phase relationship between modes changes along the length of the waveguide. This is also evident in the phase of the complex detected modal coefficients, however, the absolute magnitude of the coefficient, which does not depend on the phase relationship, shows good agreement with the previous results of Table 4.2.
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Figure 4.9: Simulated electric field for a 3-mode coherent excitation in a 2-port circular waveguide.

Table 4.4: The detected modal content for a 3-mode excitation of a 2-port simulated waveguide.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Detected coefficient</th>
<th>Detected power (W)</th>
<th>Total fractional power</th>
<th>Detected power (boosted) (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Complex</td>
<td>Absolute magnitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>-151.1 + 691.3i</td>
<td>707.6</td>
<td>4.741e-4</td>
<td>0.897</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>-0.632 + 0.845i</td>
<td>1.056</td>
<td>3.922e-09</td>
<td>7.423e-6</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>-426.6+ 19.98i</td>
<td>427.1</td>
<td>5.420e-05</td>
<td>0.103</td>
</tr>
</tbody>
</table>

A significant error is introduced for the $TM_{01}$ mode detection, compared with the purely theoretical input field. The source of this error is investigated by exciting each mode individually in a separate simulation and measuring the modal content. The results are shown in Table 4.5. The detected boosted power in the $TM_{01}$ mode is now very close to unity again, when only this mode is excited. Therefore the erroneous power must come from interference with the other modes. There is a significant amount of erroneous detected power in the $TM_{01}$ mode for $TE_{11}$ and $TE_{21}$ excitations, however, the sum of the modal content for the $TM_{01}$ mode still does not accumulate to the 1.044 W detected in the 3-mode excitation. Therefore another error must originate from the modal fields interfering when excited in the same simulation.
This point is further supported by comparing the reflection coefficients for individual and 3-mode excitations (Table 4.6 columns 1 and 2). It appears that the presence of the other modes are causing the $TM_{01}$ mode to appear to be reflected, thus contributing the extra power in the modal content measurement.

Table 4.5: The detected modal content for a 3-mode excitation of a 2-port simulated waveguide where the modes are excited individually in separate simulations.

<table>
<thead>
<tr>
<th>Modal excitation</th>
<th>Mode</th>
<th>Detected coefficient</th>
<th>Detected power (W)</th>
<th>Detected power (boosted) (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>$TE_{11}$</td>
<td>-151.1+691.3i</td>
<td>707.6</td>
<td>4.741e-4</td>
</tr>
<tr>
<td></td>
<td>$TM_{01}$</td>
<td>-</td>
<td>-</td>
<td>2.380e-4</td>
</tr>
<tr>
<td></td>
<td>$TE_{21}$</td>
<td>-</td>
<td>-</td>
<td>7.980e-8</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>$TE_{11}$</td>
<td>-</td>
<td>-</td>
<td>9.014e-15</td>
</tr>
<tr>
<td></td>
<td>$TM_{01}$</td>
<td>-0.633+0.816i</td>
<td>1.032</td>
<td>3.750e-9</td>
</tr>
<tr>
<td></td>
<td>$TE_{21}$</td>
<td>-</td>
<td>-</td>
<td>2.598e-12</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>$TE_{11}$</td>
<td>-</td>
<td>-</td>
<td>7.923e-10</td>
</tr>
<tr>
<td></td>
<td>$TM_{01}$</td>
<td>-</td>
<td>-</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>$TE_{21}$</td>
<td>-426.7+20.07i</td>
<td>427.2</td>
<td>5.422e-5</td>
</tr>
</tbody>
</table>

Table 4.6: Reflection coefficients (absolute magnitude) for the waveguide with 2 ports.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Individual excitation</th>
<th>3-mode excitation</th>
<th>3-mode excitation (equal power)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0.002</td>
<td>0.0242</td>
<td>0.002</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
</tr>
</tbody>
</table>

The error in the 3-mode excitation is likely caused by an error in the MoM calculations when there is such a large power difference between modes. Therefore, instead of exciting the modes with the fundamental power and boosting the modal content afterwards, the magnitude of the excitation in the simulation is instead boosted to give each mode unity power in the excitation. The new reflection coefficients are shown as the last column in Table 4.6 and the modal content for a
Measurements of the Multi-mode Horn for LSPE-SWIPE

3-mode excitation is shown in Table 4.7. The large error is no longer present, thus the modes are excited in this way in all proceeding simulations.

Table 4.7: The detected modal content for a 3-mode excitation of a 2-port simulated waveguide where the modes have been excited with equal power.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Detected coefficient</th>
<th>Detected power (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Complex</td>
<td>Absolute magnitude</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>-6.937e3+3.175e4i</td>
<td>3.250e4</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>-1.033e4+1.332e4i</td>
<td>1.686e4</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>-5.805e4+2.734e3i</td>
<td>5.811e4</td>
</tr>
</tbody>
</table>

**Simulated waveguide closed at one end**

If the passive second waveguide port is removed, each mode is partially reflected at the end of the guide. The partially reflected wave interferes with the incoming excitation and a standing wave is formed. A full description of the mathematics and terminology used to describe standing waves is given in § 2.6. To understand what effect the standing wave has on the modal content calculation in the simplest case possible, the end of the waveguide is sealed with a perfect electric conductor (PEC) so that 100% of the power is reflected.

Each mode actually forms a different standing wave with a wavelength equal to the effective wavelength of that mode within the guide. The maxima of the standing wave pattern of the $TE_{11}$ mode is shown for example at the top of Figure 4.10. The boundary conditions dictate that the standing wave must form a node at the end of the closed waveguide since the electric field is zero inside the conductor and must be continuous. In this particular waveguide at 110 GHz, the $TE_{11}$ mode has an effective wavelength of 3.2 mm, which is just longer than the 3 mm length of the waveguide. With regards to measuring the modal content, it is not the instantaneous electric field of the standing wave which is of most use, rather it is the time-independent standing wave envelope which is needed. The standing wave envelope is given by Eq. 2.98 with a reflection coefficient ($r=1$) and a phase shift ($\phi=180^\circ$), and its modulus is plotted at the bottom of Figure 4.10 for the $TE_{11}$ mode. The modal power content
can be corrected for the presence of the standing wave by dividing by the value of the square of the standing wave envelope at that position in the guide.

![Graph showing theoretical standing wave pattern and standing wave envelope for TE11 mode in a circular waveguide closed at one end.](image)

Figure 4.10: Theoretical standing wave pattern (top) and standing wave envelope (bottom) of the $TE_{11}$ mode in a circular waveguide which is closed at one end. ‘amplitude’ refers to the amplitude of the single $TE_{11}$ mode travelling along the waveguide before it is reflected.

A simulation is performed with all 3 modes excited with equal power. The simulation measures the reflection coefficient for each mode to be 1.000 as expected. Field cuts are taken incrementally along the length of the guide and the detected and standing wave corrected modal content is shown in Table 4.8. Furthermore, a plot of the detected power is plotted against the theoretical standing wave for each mode in Figure 4.11. Errors start to occur in the corrected modal content (bold in the table) when the field is taken close to the node of the standing wave, however this is
expected because theoretically the field goes to zero at this point. Therefore the results show that the standing wave correction works as expected.

Table 4.8: The modal power content with a standing wave correction at different positions along the waveguide which is closed at one end.

<table>
<thead>
<tr>
<th>Field cut position (mm)</th>
<th>Detected power (W)</th>
<th>Detected power (W)</th>
<th>Detected power (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$TE_{11}$</td>
<td>$TM_{01}$</td>
<td>$TE_{21}$</td>
</tr>
<tr>
<td>Aperture</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.8</td>
<td>0.579</td>
<td>0.423</td>
<td>0.184</td>
</tr>
<tr>
<td>2.6</td>
<td>1.979</td>
<td>1.512</td>
<td>0.702</td>
</tr>
<tr>
<td>2.4</td>
<td>3.392</td>
<td>2.808</td>
<td>1.460</td>
</tr>
<tr>
<td>2.2</td>
<td>3.997</td>
<td>3.760</td>
<td>2.317</td>
</tr>
<tr>
<td>2.0</td>
<td>3.447</td>
<td>3.972</td>
<td>3.119</td>
</tr>
<tr>
<td>1.8</td>
<td>2.058</td>
<td>3.346</td>
<td>3.716</td>
</tr>
<tr>
<td>1.6</td>
<td>0.635</td>
<td>2.152</td>
<td>4.001</td>
</tr>
<tr>
<td>1.4</td>
<td>0.002</td>
<td>0.895</td>
<td>3.920</td>
</tr>
<tr>
<td>1.2</td>
<td>0.524</td>
<td>0.104</td>
<td>3.488</td>
</tr>
<tr>
<td>1.0</td>
<td>1.900</td>
<td>0.114</td>
<td>2.787</td>
</tr>
<tr>
<td>0.8</td>
<td>3.332</td>
<td>0.920</td>
<td>1.941</td>
</tr>
<tr>
<td>0.6</td>
<td>3.993</td>
<td>2.184</td>
<td>1.108</td>
</tr>
</tbody>
</table>

Figure 4.11: Detected modal content (● markers) plotted against theoretical standing wave envelope (dotted line) for a waveguide closed at one end.
Simulated waveguide open at one end

If the waveguide is left open at one end, the modes are partially reflected by different amounts at the waveguide aperture, forming a partial standing wave. In this case the fields close to the aperture become much more complicated and the boundary condition of there being a node at the end of the waveguide is removed. The partial reflection of the field means that there are no points along the guide where the field is zero, hence, the modal content should be able to be detected and corrected at any point along the waveguide without error. The reflection coefficients for each mode can be extracted directly from the simulation, and are 0.0216, 0.322 and 0.168 respectively. The modal content result is shown in Table 4.9 and the plot against the theoretical standing wave is shown in Figure 4.12.

Table 4.9: The modal power content with a standing wave correction at different positions along the waveguide which is open at one end.

<table>
<thead>
<tr>
<th>Field cut position</th>
<th>Detected power (W)</th>
<th>Detected power (W) (standing wave corrected)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$TE_{11}$</td>
<td>$TM_{01}$</td>
</tr>
<tr>
<td>Aperture</td>
<td>0.958</td>
<td>0.963</td>
</tr>
<tr>
<td>2.8</td>
<td>0.963</td>
<td>0.758</td>
</tr>
<tr>
<td>2.6</td>
<td>0.958</td>
<td>0.497</td>
</tr>
<tr>
<td>2.4</td>
<td>0.977</td>
<td>0.491</td>
</tr>
<tr>
<td>2.2</td>
<td>1.009</td>
<td>0.749</td>
</tr>
<tr>
<td>2.0</td>
<td>1.037</td>
<td>1.147</td>
</tr>
<tr>
<td>1.8</td>
<td>1.042</td>
<td>1.530</td>
</tr>
<tr>
<td>1.6</td>
<td>1.024</td>
<td>1.734</td>
</tr>
<tr>
<td>1.4</td>
<td>0.992</td>
<td>1.670</td>
</tr>
<tr>
<td>1.2</td>
<td>0.964</td>
<td>1.363</td>
</tr>
<tr>
<td>1.0</td>
<td>0.958</td>
<td>0.948</td>
</tr>
<tr>
<td>0.8</td>
<td>0.975</td>
<td>0.596</td>
</tr>
<tr>
<td>0.6</td>
<td>1.007</td>
<td>0.458</td>
</tr>
<tr>
<td>0.4</td>
<td>1.035</td>
<td>0.592</td>
</tr>
<tr>
<td>0.2</td>
<td>1.043</td>
<td>0.940</td>
</tr>
</tbody>
</table>
The phase of the standing wave at the aperture for each mode is found by shifting the theoretical standing wave to find the best match with the detected modal content. The phase of the standing wave at the aperture is 38.5°, 48° and 3.5° for each mode respectively. It is not understood what the physical reason is why each modal standing wave has this particular phase however a different simulation software CST (https://www.cst.com/) using a different simulation technique (FEM) gives the same result. The standing wave correction works as expected everywhere except at the aperture where the modal content drops below the expected value. The reason for this is not known but is expected to be due to the complex nature of the boundary condition at the aperture. Nevertheless, the detected modal content at the aperture (without standing wave correction) is taken to be the correct value since this produces the correct far-fields to match the FEKO simulation.

**Measuring the field in front of the aperture**

Of course, in the lab it is impractical to measure inside or at the aperture of the waveguide. Therefore, the field must be measured at a distance in front of the waveguide and propagated backwards to infer the field at the aperture. Before being used on measured data, the method is first tested on the simulation to check the consistency of results with the aperture field which is directly extracted from the simulation. A 2D cut of the field measuring 100 mm × 100 mm and at a distance of 40 mm from the aperture is extracted from the simulation and propagated backwards.
using scalar diffraction theory as outlined in § 2.5.3. The field cut is well into the far-field regime of the waveguide (6.60 mm at 110 GHz). The size of the field cut relative to the waveguide is shown in Figure 4.13. This is provided in order to put the scale of the field cut in context.

Figure 4.13: The circular waveguide and the position of the field cut as displayed in the simulation software. Dimensions shown are in mm.

The field at 40 mm from the aperture and the inferred aperture field are shown in comparison to the directly extracted aperture field in Figure 4.14. The modal content of the inferred aperture field is calculated and is compared to that of the directly extracted aperture field in Table 4.10. Correction for standing waves is not necessary since the field is measured outside of the waveguide.
Figure 4.14: Simulated electric field for a 3-mode coherent excitation of an open ended circular waveguide. (1/2)
4.3 Coherent Measurements

Figure 4.14: Simulated electric field for a 3-mode coherent excitation of an open ended circular waveguide. (2/2)

Table 4.10: Modal content for a 3-mode coherent excitation of a circular waveguide. The modes are listed in order of power. Modes contributing less than 1% of the power are not shown.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Detected power (W)</th>
<th>Inferred aperture field</th>
<th>Directly extracted aperture field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>0.741</td>
<td>0.958</td>
<td></td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0.320</td>
<td>0.963</td>
<td></td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0.179</td>
<td>0.633</td>
<td></td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>0.059</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$TE_{22}$</td>
<td>0.030</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$TE_{23}$</td>
<td>0.020</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

The inferred aperture field resembles the directly extracted aperture field, however there are significant differences. Furthermore, the detected modal content is much lower and also there are some modes detected in error. One reason is that the approximations inherent in the scalar diffraction theory used to propagate back the field have been broken, since the diameter of the waveguide is not many times the wavelength. Therefore, to propagate this field would require a more rigorous
4 Measurements of the Multi-mode Horn for LSPE-SWIPE

treatment such as vector diffraction theory. Another explanation is developed on noticing that the measured modal content is particularly low for the $TM_{01}$ and $TE_{21}$ modes. As stated, for this particular waveguide the aperture is small (relative to the wavelength), therefore a very broad beam is produced. Power not passing through the field cut taken at 40 mm will not contribute to the detected modal content. This effect would be particularly strong for modes with mainly off-axis power, such as the $TM_{01}$ and $TE_{21}$, as is the case. Consequently, this means that the current implementation of the modal content calculation is not expected to give the correct result for any small diameter waveguides. However, with regards to the SWIPE horn, the aperture size is far greater than the wavelength, hence the scalar diffraction theory should hold and the narrower beam will mean that modes with off axis power are not discriminated against. This conjecture is explored in the proceeding sections.

**Scattering matrix**

The modal content result can be put in the form of a scattering matrix which describes how each input mode scatters. In the case of a waveguide it is simplistic since no scattering takes place, however it is included here in its full form for completeness. The scattering matrix is described in terms of the complex coefficients since both the relative phase and amplitude are important. The scattering matrix* is given in Table 4.11 for the case where the inferred aperture field is used, and in Table 4.12 for the case where the aperture field is extracted directly. The scattering matrices are used in § 4.3.4 to reconstruct the far-field beam.

---

* The scattering matrices are not a convenient form of the result to evaluate directly since the coefficients are complex. Instead, the modal content (Table 4.10) or the reconstructed beam (§ 4.3.4 later) should be examined. The scattering matrices have been included here (and in the following section) in order to illustrate the full process of the coherent measurement technique.
Table 4.11: **Inferred** aperture field **circular waveguide** scattering matrix for the first three modes. Only scattered modes up to an azimuthal index of $m = 2$ and a radial index of $n = 3$ are shown.

<table>
<thead>
<tr>
<th></th>
<th>$TE_{11}$</th>
<th>$TM_{01}$</th>
<th>$TE_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{01}$</td>
<td>0</td>
<td>16.313+4.984i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{02}$</td>
<td>0</td>
<td>-7.84-2.418i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{03}$</td>
<td>0</td>
<td>6.095+1.883i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0</td>
<td>9236.265-2398.766i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{02}$</td>
<td>0</td>
<td>19.386-44.123i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{03}$</td>
<td>0</td>
<td>31.178+6.492i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>11870.069-25333.903i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{12}$</td>
<td>-1183.784+2430.237i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{13}$</td>
<td>1151.298-2394.775i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>2822.05-5401.639i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{12}$</td>
<td>-784.219+1533.825i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{13}$</td>
<td>425.911-836.332i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0</td>
<td>0</td>
<td>1042.641+24530.183i</td>
</tr>
<tr>
<td>$TE_{22}$</td>
<td>0</td>
<td>0</td>
<td>-183.346-6584.766i</td>
</tr>
<tr>
<td>$TE_{23}$</td>
<td>0</td>
<td>0</td>
<td>142.248+5072.647i</td>
</tr>
<tr>
<td>$TM_{21}$</td>
<td>0</td>
<td>0</td>
<td>112.706+3312.504i</td>
</tr>
<tr>
<td>$TM_{22}$</td>
<td>0</td>
<td>0</td>
<td>-39.233-1244.427i</td>
</tr>
<tr>
<td>$TM_{23}$</td>
<td>0</td>
<td>0</td>
<td>22.4+722.368i</td>
</tr>
</tbody>
</table>
Table 4.12: **Directly extracted** aperture field circular waveguide scattering matrix for the first three modes. Only scattered modes up to an azimuthal index of $m = 2$ and a radial index of $n = 3$ are shown.

<table>
<thead>
<tr>
<th></th>
<th>$TE_{11}$</th>
<th>$TM_{01}$</th>
<th>$TE_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{01}$</td>
<td>0</td>
<td>17.073+1.377i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{02}$</td>
<td>0</td>
<td>-9.006-5.548i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{03}$</td>
<td>0</td>
<td>8.518+6.438i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0</td>
<td>14474.052-8018.274i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{02}$</td>
<td>0</td>
<td>-1817.572+3870.518i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{03}$</td>
<td>0</td>
<td>1543.314-2830.9i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>13250.354-28904.649i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{12}$</td>
<td>2685.925+986.152i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{13}$</td>
<td>-1072.555-874.088i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>3425.693-154.536i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{12}$</td>
<td>-2111.601-1250.538i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{13}$</td>
<td>1563.155+1245.86i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0</td>
<td>0</td>
<td>3141.391+46034.412i</td>
</tr>
<tr>
<td>$TE_{22}$</td>
<td>0</td>
<td>0</td>
<td>-3949.711-2270.475i</td>
</tr>
<tr>
<td>$TE_{23}$</td>
<td>0</td>
<td>0</td>
<td>2047.216+1557.569i</td>
</tr>
<tr>
<td>$TM_{21}$</td>
<td>0</td>
<td>0</td>
<td>-6606.077-1687.193i</td>
</tr>
<tr>
<td>$TM_{22}$</td>
<td>0</td>
<td>0</td>
<td>4619.423+1910.736i</td>
</tr>
<tr>
<td>$TM_{23}$</td>
<td>0</td>
<td>0</td>
<td>-3715.033-1752.752i</td>
</tr>
</tbody>
</table>
4.3.3. Modal Content Calculation for a Simulated Conical Horn

A conical horn with a throat radius of 1.5 mm, an aperture radius of 5 mm and a length of 20 mm is simulated and the modal content is calculated. The size of the field cut is kept the same as for the waveguide case so that the results are directly comparable. Hence, a 100 mm × 100 mm field cut is taken at 40 mm in front of the horn aperture. This is towards the far-field regime of the horn (73mm at 110 GHz). The horn and the size of the field cut are shown in Figure 4.15. It is expected that the larger electrical size of the aperture will vastly improve the modal content measurement in comparison to the waveguide in the previous section.

Figure 4.15: The horn and the position of the field cut as displayed in the simulation software. Dimensions shown are in mm.

The horn is simulated at 110 GHz and the first 3 modes ($TE_{11}$, $TM_{01}$ and $TE_{21}$) are excited with equal power at the throat of the horn. The shape of the horn now causes scattering of the modes into modes of the same azimuthal order, $m$, as the modes propagate through the horn. Furthermore, the number of modes which are allowed to propagate increases to 26 at the horn aperture. The reflection coefficients for each
modal excitation are 0.0165, 0.070 and 0.0218 respectively. Figure 4.16 shows the field at 40 mm in front of the aperture and the inferred aperture field in comparison to the directly extracted aperture field. The detected modal content is shown in Table 4.13.

Figure 4.16: Simulated electric field for a 3-mode coherent excitation of a **conical horn**. (1/2)
4.3 Coherent Measurements

Figure 4.16: Simulated electric field for a 3-mode coherent excitation of a conical horn. (2/2)

Table 4.13: Modal content for a 3-mode coherent excitation of a conical horn. The modes are listed in order of power. Modes contributing less than 1% of the power are not shown.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Detected power (W)</th>
<th>Inferred aperture field</th>
<th>Direct aperture field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>0.885</td>
<td>0.900</td>
<td></td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0.856</td>
<td>0.902</td>
<td></td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0.810</td>
<td>0.867</td>
<td></td>
</tr>
<tr>
<td>$TM_{02}$</td>
<td>0.061</td>
<td>0.084</td>
<td></td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>0.043</td>
<td>0.059</td>
<td></td>
</tr>
<tr>
<td>$TM_{21}$</td>
<td>0.034</td>
<td>0.051</td>
<td></td>
</tr>
<tr>
<td>$TM_{22}$</td>
<td>0.017</td>
<td>0.0161</td>
<td></td>
</tr>
<tr>
<td>$TM_{13}$</td>
<td>0.017</td>
<td>0.013</td>
<td></td>
</tr>
<tr>
<td>$TE_{12}$</td>
<td>0.013</td>
<td>0.0140</td>
<td></td>
</tr>
<tr>
<td>$TM_{12}$</td>
<td>0.010</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>$TE_{22}$</td>
<td>-</td>
<td>0.0116</td>
<td></td>
</tr>
</tbody>
</table>
The inferred aperture field is a much better match than for the waveguide case as expected. The difference in modal content for the main modes is also now only a few percent. The scattering matrix is given in Table 4.14 for the case where the inferred aperture field has been used, and in Table 4.15 for the case where the aperture field is extracted directly. The scattering matrices are used in § 4.3.4 to reconstruct the far-field beam.

Table 4.14: **Inferred** aperture field **conical horn** scattering matrix for the first three modes. Only scattered modes up to an azimuthal index of \( m = 2 \) and a radial index of \( n = 3 \) are shown.

<table>
<thead>
<tr>
<th></th>
<th>( TE_{11} )</th>
<th>( TM_{01} )</th>
<th>( TE_{21} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( TE_{01} )</td>
<td>0</td>
<td>-1.342+1.284i</td>
<td>0</td>
</tr>
<tr>
<td>( TE_{02} )</td>
<td>0</td>
<td>-0.849-1.368i</td>
<td>0</td>
</tr>
<tr>
<td>( TE_{03} )</td>
<td>0</td>
<td>0.458+1.437i</td>
<td>0</td>
</tr>
<tr>
<td>( TM_{01} )</td>
<td>0</td>
<td>4849.37-2169.418i</td>
<td>0</td>
</tr>
<tr>
<td>( TM_{02} )</td>
<td>0</td>
<td>1478.92+1506.924i</td>
<td>0</td>
</tr>
<tr>
<td>( TM_{03} )</td>
<td>0</td>
<td>-783.503-22.63i</td>
<td>0</td>
</tr>
<tr>
<td>( TE_{11} )</td>
<td>-2523.41-8108.898i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( TE_{12} )</td>
<td>1426.408-564.49i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( TE_{13} )</td>
<td>-169.021+382.559i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( TM_{11} )</td>
<td>1948.188-1002.917i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( TM_{12} )</td>
<td>490.848+1228.791i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( TM_{13} )</td>
<td>-741.337-1392.515i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( TE_{21} )</td>
<td>0</td>
<td>0</td>
<td>-6070.872+9439.189i</td>
</tr>
<tr>
<td>( TE_{22} )</td>
<td>0</td>
<td>0</td>
<td>-1508.541-500.21i</td>
</tr>
<tr>
<td>( TE_{23} )</td>
<td>0</td>
<td>0</td>
<td>588.496-502.962i</td>
</tr>
<tr>
<td>( TM_{21} )</td>
<td>0</td>
<td>0</td>
<td>-2071.162-857.802i</td>
</tr>
<tr>
<td>( TM_{22} )</td>
<td>0</td>
<td>0</td>
<td>1307.432-1177.95i</td>
</tr>
<tr>
<td>( TM_{23} )</td>
<td>0</td>
<td>0</td>
<td>-366.835+1730.925i</td>
</tr>
</tbody>
</table>
Table 4.15: **Directly** extracted aperture field **conical horn** scattering matrix for the first three modes. Only scattered modes up to an azimuthal index of $m = 2$ and a radial index of $n = 3$ are shown.

<table>
<thead>
<tr>
<th></th>
<th>$TE_{11}$</th>
<th>$TM_{01}$</th>
<th>$TE_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{01}$</td>
<td>0</td>
<td>-1.331+1.242i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{02}$</td>
<td>0</td>
<td>-0.875-1.245i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{03}$</td>
<td>0</td>
<td>0.512+1.002i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0</td>
<td>4892.999-2501.614i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{02}$</td>
<td>0</td>
<td>1399.285+2043.445i</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{03}$</td>
<td>0</td>
<td>-626.577-802.693i</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>-2738.986-8111.538i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{12}$</td>
<td>1527.191-547.378i</td>
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<td>0</td>
</tr>
<tr>
<td>$TE_{13}$</td>
<td>-268.444+328.094i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>2361.793-1003.862i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{12}$</td>
<td>-125.121+1220.591i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{13}$</td>
<td>330.938-1333.434i</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0</td>
<td>0</td>
<td>-5880.538+9911.855i</td>
</tr>
<tr>
<td>$TE_{22}$</td>
<td>0</td>
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<td>$TE_{23}$</td>
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<td>$TM_{21}$</td>
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<tr>
<td>$TM_{22}$</td>
<td>0</td>
<td>0</td>
<td>1656.958-333.485i</td>
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<tr>
<td>$TM_{23}$</td>
<td>0</td>
<td>0</td>
<td>-1048.69+428.819i</td>
</tr>
</tbody>
</table>

**4.3.4. Incoherent Beam Reconstruction for the Simulated Waveguide and Horn**

The scattering matrix for the horn and waveguide has been deduced from simulations which resembles a coherent detection technique. However, the scattering matrix is the same regardless of whether an incoherent or a coherent detection scheme is used. Therefore the incoherent far-field beam can be reconstructed. First the fields at the aperture for each modal field (each column of the scattering matrix) are reconstructed separately using the theoretical waveguide modal field equations (Eq. 2.43 and 2.44) for each mode with coefficients according to the values in the scattering matrix. For each of these aperture fields, the resulting far-field is then calculated using Eq. 2.80 – 2.82 with $R = 1$. The multi-mode coherent beam is found
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by taking the complex sum of the far-fields and the multi-mode incoherent beam is found by adding the complex fields in quadrature.

Simulated waveguide

Both the coherent and incoherent far-fields are generated for the simulated waveguide in § 4.3.2 according to the scattering matrix and are compared against the far-field which has been simulated directly in FEKO. The small waveguide diameter caused the measured modal content to deviate quite far from the expected result, therefore the match between the directly simulated and constructed beams is expected to be equally deviated. However, putting the modal content result into the form of the far-field beams puts into context how large the effect of this error actually is.

Before constructing and comparing the beams, a check is made to see how well the aperture field to far-field calculation performed using a custom MATLAB code matches the undocumented one used by the FEKO software. The aperture field is directly exported from the simulation of the waveguide and the far-field is calculated and compared with a far-field requested from FEKO directly, for the coherent case. The results are compared in Figure 4.17 (solid and dashed lines). The far-fields agree on-axis, but start to deviate at larger angles. As with the previous errors caused when propagating the field back, this error is caused because of the small electrical size of the waveguide aperture and therefore is also expected to be reduced for a large diameter horn.

The aperture field is reconstructed according to the scattering matrix (Table 4.11) which was found by propagating a field cut back from 40 mm in front of the waveguide. The coherent (Figure 4.17, ● markers) and incoherent (Figure 4.18, ● markers) far-field is calculated and compared with the result which is directly exported from FEKO (solid line). As expected, the above mentioned errors cause poor agreement in both cases. The far-field is also significantly lower in power overall since the broadness of the beam means that a lot of the power is not captured in the field cut at 40 mm.
Figure 4.17: Coherent 3-mode far-field (excluding orthogonal modes) of the circular waveguide: directly exported from FEKO (solid line); calculated using the directly exported aperture field (dashed line); and calculated using the reconstructed aperture field (● markers).
Figure 4.18: **Incoherent** 3-mode far-field (excluding orthogonal modes) of the circular waveguide: directly exported from FEKO (solid line); and calculated using the reconstructed aperture field (● markers).
The aperture and far-fields can also be generated for the orthogonal modes, assuming that they scatter in exactly the same way. The incoherent far-field including both orthogonal modes sets is shown in Figure 4.19. The unpolarised beam is azimuthally symmetric as expected.

![Graphs showing x-polarisation, y-polarisation, and unpolarised electric field vs. angle]

Figure 4.19: Incoherent 3-mode far-field (including orthogonal modes) of the circular waveguide: directly exported from FEKO (solid line); and calculated using the reconstructed aperture field (● markers).
Simulated horn

The far-field for the simulated horn in § 4.3.3 is calculated from the scattering matrix (Table 4.14) and compared with the direct calculation from FEKO. The coherent and incoherent far-fields are shown in Figure 4.20 and Figure 4.21 respectively. As expected, the agreement is far superior in comparison to the result for the waveguide. Furthermore, the agreement is good enough to be able to clearly distinguish between the different beam shapes of the coherent and incoherent far-fields. The incoherent field with the orthogonal modes included is shown in Figure 4.22. The addition of the extra modes further increases the agreement.

![Graphs showing comparison of far-field measurements](image)

Figure 4.20: **Coherent** 3-mode far-field (**excluding orthogonal modes**) of the **conical horn**: directly exported from FEKO (solid line); calculated using the directly exported aperture field (dashed line), and calculated using the reconstructed aperture field (● markers).
Incoherent 3-mode far-field (excluding orthogonal modes) of the conical horn: directly exported from FEKO (solid line); and calculated using the reconstructed aperture field (● markers).
Figure 4.22: Incoherent 3-mode far-field (including orthogonal modes) of the conical horn: directly exported from FEKO (solid line); and calculated using the reconstructed aperture field (● markers).
4.3.5. Coherent Measurement Set-up

The coherent measurement technique has been shown to work in simulation for a conical horn in the previous sections. In this section the technique is developed for actual measurements in the lab. The coherent test set-up is shown in Figure 4.23.

Figure 4.23: The test set-up used to measure coherently the modal content of horns and waveguides. The automated scanner moves the probe to scan the field radiating from the device under test (DUT). Axes relating to the measured data are shown.

The radiation is both transmitted and received from the system using a VNA (Rohde & Schwarz ZVA40 with 75-110 GHz converter ports). The converter port which is acting as the probe (Port 2) is attached to a 3-axis scanner that was developed by Peter Schemmel (Schemmel et al. 2013). The scanner is automated using a LABVIEW script. The device under test (DUT) is attached to the second extension head (Port 1) which rests on a test bench in front of the scanner. Alignment is
achieved by levelling the scanner and test bench using the adjustable feet, then ensuring each vertical strut at the front of the scanner is equidistant from the test bench. Eccosorb is added around the DUT and probe to prevent unwanted reflections. The cables are positioned to minimise distortion of their shape as they inevitably move with the motion of the scanner.

The converter port output is a WR-10 rectangular waveguide. A rectangular-to-circular waveguide transition can be used in order to interface with a circularly symmetric waveguide. For both rectangular and circular waveguide outputs the radiation from the converter port is polarised. Therefore, to perform a scan of the cross-polarisation of the DUT, a 31 mm long 90° rectangular waveguide twist is introduced onto the end of the rectangular waveguide of Port 2. To account for this extra length of waveguide, the port is moved backwards by 31 mm and the measured phase for the cross-polarisation is shifted by an amount corresponding to 31 mm at the measurement frequency.

The choice of probe is very important since what is being measured is actually a convolution of the probe and the DUT. Therefore, either the probe should be insignificant in the final result (probe beam pattern varies insignificantly over measured angle), or the beam of the probe should be taken into account by performing a probe correction. For near-field measurements the probe correction is heavily complicated since a full deconvolution must be performed. However, for far-field measurements, probe correction can be achieved with high accuracy by simply dividing the measured intensity by the beam of the probe at the corresponding measurement angle. For this reason, efforts are made to keep measurements in the far-field region at all times.

In the lab the probe is centred with the DUT by eye. This is not critical however since the beam is centralised again in the post processing. A separate centralisation has to be performed for both measured polarisations since the rectangular waveguide 90° twist slightly moves the location of the probe aperture. Originally the post process centralisation was attempted by shifting the data to place the maximum or minima of the beam at the centre. This is fine for the $TE_{11}$ mode which has a defined maximum in the co-polarisation and minima in the cross-polarisation. However, this is not the case for the other modes. Overall, it is found that the most accurate
centralisation is achieved through centring the beam patterns by eye, by considering the overall shape of both the amplitude and phase. It is found that the best result is achieved when two bouts of centralisation are applied, before and after the beam has been propagated to infer the aperture field. The full measurement process is summarised in Table 4.16.

Table 4.16: Steps in the coherent set-up measurement procedure.

<table>
<thead>
<tr>
<th>Step</th>
<th>Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Field of DUT scanned</td>
</tr>
<tr>
<td>2.</td>
<td>1st centralisation of the data by eye</td>
</tr>
<tr>
<td>3.</td>
<td>Probe correction</td>
</tr>
<tr>
<td>4.</td>
<td>Normalisation</td>
</tr>
<tr>
<td>5.</td>
<td>Backwards propagation to infer the aperture field of the DUT</td>
</tr>
<tr>
<td>6.</td>
<td>2nd centralisation of the data by eye</td>
</tr>
<tr>
<td>7.</td>
<td>Cut of any remaining field beyond aperture bounds</td>
</tr>
<tr>
<td>8.</td>
<td>Modal content calculation</td>
</tr>
<tr>
<td>9.</td>
<td>Deduction of the scattering matrix of the DUT</td>
</tr>
<tr>
<td>10.</td>
<td>Reconstruction of the incoherent beam of the DUT</td>
</tr>
</tbody>
</table>

Throughout this chapter the number of field points has been 201 × 201 for a 100 mm × 100 mm field, giving a resolution of 0.5 mm. The average scanner speed is around 60 points per minute, meaning that this scan would take around 11 hours per polarisation. This is far too long considering the many scans that have to be performed, therefore a compromise is made on both the resolution and scan area in order to give a reasonable total scan time. As well as the errors from lowering the resolution as mentioned in § 4.3.2, this will give further errors from the centralisation of the beam, since this cannot be performed as precisely for a lower resolution scan.

4.3.6. Basic Circular Waveguide Measurement

As an initial test, the modal content of a circular waveguide is measured at 90 GHz and compared with simulation. This simplicity of the geometry, with no modal scattering, means that the expected modal content is already well understood at the aperture, therefore this is a good test to check that the measurement procedure works as expected. The opportunity is also taken to test a variety of probes in order to
determine which gives the best result. The circular waveguide is actually the circular end of the rectangular-to-circular waveguide transition. A model of the transition (originally constructed by Giampaolo Pisano) is shown in Figure 4.24. The circular waveguide has a radius of 1.55 mm and supports only the first two modes at 90 GHz. The rectangular waveguide only supports the $TE_{10}$ rectangular waveguide mode over the whole of the W-band. Upon entering the rectangular-to-circular transition the $TE_{10}$ is primarily converted into the $TE_{11}$ circular waveguide mode since these two modal field patterns have a high resemblance. As seen for the waveguide in § 4.3.2, the small electrical size of the aperture is expected to cause problems since several of the techniques used to manipulate the electric field rely on the aperture being many times the wavelength. Nevertheless, these errors apply to both simulation and measurement, therefore still allowing a comparison to be done for a very simple DUT.

![Diagram](image)

Figure 4.24: A cut-plane of the rectangular-to-circular waveguide transition (with waveguide choke) used to interface the rectangular waveguide output of the VNA converter port with circularly symmetric waveguides and horns. Dimensions shown are in mm.

**Simulation**

A simulation is firstly performed which replicates the lab measurement, disregarding the effects of the probe. The input excitation is a single $TE_{10}$ mode at the rectangular waveguide end, excited with 1 W of power. A 30 mm × 30 mm field cut is taken at 40 mm in front of the aperture. This is well into the far-field regime for the guide (around 1.5 mm at 90 GHz). The resolution of the field cut is 1 mm, matching what will be the case for the lab measurements. The simulated fields are shown in Figure
4.3 Coherent Measurements

4.25 and the detected modal content is given in the first data column of Table 4.17. Since the excitation power of the VNA is not known precisely, the 40 mm input field is normalised to the overall maximum (from both polarisations) in order to be directly comparable with the measurement.

Figure 4.25: Simulated electric field for the rectangular-to-circular waveguide transition.
4 Measurements of the Multi-mode Horn for LSPE-SWPE

Table 4.17: $TE_{11}$ modal content for the simulated and measured rectangular-to-circular waveguide transition at 90 GHz for a variety of probes.

<table>
<thead>
<tr>
<th>$TE_{11}$ mode</th>
<th>Simulation</th>
<th>Circular waveguide without flange</th>
<th>WR-10 waveguide with flange</th>
<th>Corrugated horn</th>
<th>WR-10 waveguide without flange</th>
</tr>
</thead>
<tbody>
<tr>
<td>Detected power (W)</td>
<td>7.115e-07</td>
<td>8.059e-07</td>
<td>5.537e-07</td>
<td>1.956e-10</td>
<td>9.552e-07</td>
</tr>
</tbody>
</table>

**Measurement**

The field is scanned in the lab with several probes in order to decide on the best choice. The probes available are shown in Figure 4.26. The advantage of the waveguide probes over the horn is that they have a broader beam. This allows them to receive more power at large angles, meaning that the scan area can be made larger before the noise floor is reached. Furthermore, their effect on the overall measurement is less severe, thus making the probe correction easier.

![Figure 4.26: Probes available to scan the field include: (a) circular waveguide without flange; (b) rectangular waveguide with flange; (c) corrugated horn; and (d) rectangular waveguide without flange. Images not to the same scale. The corrugated horn was designed for a previous experiment called Clover (Maffei et al. 2005).](image)

The results are shown in Figure 4.27 to Figure 4.30, and the modal content results are compared with the simulation in Table 4.17. Since the measurement is in the far-field, the probe correction is performed simply by dividing the measured field by the simulated beam of the probe.
Circular waveguide without flange

40 mm field

Simulated probe beam

Figure 4.27: Measured electric field of the rectangular-to-circular transition using the circular waveguide without flange as a probe. (1/2)
Figure 4.27: Measured electric field of the rectangular-to-circular transition using the circular waveguide without flange as a probe. (2/2)
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Rectangular waveguide with flange

Figure 4.28: Measured electric field of the rectangular-to-circular transition using the rectangular waveguide with flange as a probe. (1/2)
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Figure 4.28: Measured electric field of the rectangular-to-circular transition using the rectangular waveguide with flange as a probe. (2/2)
Corrugated horn

40 mm field

Simulated probe beam

Figure 4.29: Measured electric field of the rectangular-to-circular transition using the corrugated horn as a probe. (1/2)
Figure 4.29: **Measured** electric field of the **rectangular-to-circular** transition using the **corrugated horn** as a probe. (2/2)
4.3 Coherent Measurements

Rectangular waveguide without flange

![Measurements](image)

40 mm field

Simulated probe beam

Figure 4.30: Measured electric field of the rectangular-to-circular transition using the rectangular waveguide without flange as a probe. (1/2)
Figure 4.30: **Measured** electric field of the **rectangular-to-circular** transition using the **rectangular waveguide without flange** as a probe. (2/2)
4.3 Coherent Measurements

The corrugated horn (Figure 4.29) gives by far the worst result. The beam of the probe drops off quickly off-axis and the first null in the beam pattern also comes into view. The result is therefore dominated by the beam of the corrugated horn itself, thus the desired result is almost destroyed during the probe correction. This rules out the corrugated horn as a probe. The waveguide probes all have much broader beams and all give modal content results in the region of what was predicted by the simulation (Table 4.17). The differences in the inferred aperture field can be subtle therefore, to determine the best probe, it is the normalised probe-corrected fields which are compared. It is important to consider both the amplitude and phase when making judgment, since it is the complex field as a whole which is considered in the modal content calculation. Firstly, the y-pol beam is considered. The addition of the flange on the rectangular waveguide causes a large ellipticity in the beam of the probe (Figure 4.28). This magnifies any residual alignment errors, resulting in a poor beam and thus ruling out this probe. Hence the choice is made between the circular and rectangular waveguides probes, both with no flange. The rectangular waveguide (Figure 4.30) gives a superior y-pol beam but a far-worse x-pol beam compared to the circular waveguide (Figure 4.27). In fact, the poor x-pol beam means that, on-balance, the detected modal content is closer to the simulation for the circular waveguide probe. However, the poor x-pol beam of the rectangular waveguide is most likely due to poor alignment for this particular scan since the discontinuities in the measured phase do not form the cross shape as found in the other scans. Overall both probes are similar and are both adequate choices, however the decision is made to go with the rectangular waveguide probe. This is due to its superior y-pol beam measurement and also the fact that the beam of the probe itself has a lower cross-polarisation component, which could become important later on.

4.3.7. SWIPE P1 Horn Measurement at 300 mm

The modal content is found and the incoherent beam is reconstructed for an older prototype of the SWIPE horn, designated the P1 horn. It has the same design as the P2 horn (shown previously in Figure 4.4) but with a smaller waveguide filter radius of 2.05 mm. The availability of equipment in the lab limits measurements to the W-band (75-110 GHz). Originally a 90 GHz channel was planned for SWIPE however this has been removed because the horns were too large in the focal plane
design. Therefore the remaining bands are at 140, 220 and 240 GHz, outside of the available measurement range. Nevertheless, the W-band measurements serve as a proof of concept to see how well the incoherent beam can be deduced using the coherent detection technique. The number of modes which are allowed to propagate in the waveguide filter and at the aperture of the horn across the W-band are shown in Table 4.18. The measurements are performed at 75 GHz since this is limited to 3-modes (as is the restriction of this technique), and will have the highest attenuation of the remaining evanescent modes within the waveguide filter.

Table 4.18: Allowed modes for the P1 horn.

<table>
<thead>
<tr>
<th>Frequency (GHz)</th>
<th>Number of allowed modes</th>
<th>Waveguide filter</th>
<th>Aperture</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>3</td>
<td>3</td>
<td>39</td>
</tr>
<tr>
<td>85</td>
<td>3</td>
<td>3</td>
<td>46</td>
</tr>
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<td>95</td>
<td>5</td>
<td>5</td>
<td>54</td>
</tr>
<tr>
<td>105</td>
<td>6</td>
<td>6</td>
<td>59</td>
</tr>
<tr>
<td>110</td>
<td>6</td>
<td>6</td>
<td>64</td>
</tr>
</tbody>
</table>

**Simulation with waveguide port excitation**

Before performing measurements, a simulation is performed in order to confirm the method is working as expected. All 3 modes are excited with equal power at the beginning of the waveguide filter using a direct waveguide port excitation, and the modal content is measured. A 200 mm × 200 mm field cut of 5 mm resolution is taken at 300 mm in front of the horn aperture to match what will be the case for the measurements. This distance from the horn aperture is the maximum distance which is allowed by the set-up of the 3D scanner and is well into the far-field regime of the horn (200 mm at 75 GHz). The horn model and the size of the field cut relative to the horn are shown in Figure 4.31.
In the usual way, the field is propagated back to infer the aperture field, the modal content is calculated and the scattering matrix is deduced. The aperture modal content is shown in Table 4.19. The coherent and incoherent far-fields are reconstructed according to the coefficients in the scattering matrix, and compared to those directly output by FEKO and those calculated directly from the aperture field. As before, this is used to assess any problems in various stages of the process (mainly the aperture field to far-field conversion and the modal content calculation). The coherent and incoherent beam reconstruction is shown in Figure 4.32 - Figure 4.34.

Table 4.19: Aperture modal content for a simulated 3-mode coherent waveguide port excitation of the simulated P1 horn. The modes are listed in order of power. Modes contributing less than 1% of the power are not shown.
Figure 4.32: **Coherent 3-mode far-field (excluding orthogonal modes)** of the simulated P1 horn: directly exported from FEKO (solid line); calculated using the directly exported aperture field (dashed line); and calculated using the scattering matrix deduced from the field cut at 300 mm (● markers).
Figure 4.33: **Incoherent** 3-mode far-field (excluding orthogonal modes) of the simulated P1 horn: directly exported from FEKO (solid line); and calculated using the scattering matrix deduced from the field cut at 300 mm (● markers).
Figure 4.34: Incoherent 3-mode far-field (including orthogonal modes) of the simulated P1 horn: directly exported from FEKO (solid line); and calculated using the scattering matrix deduced from the field cut at 300 mm (● markers).
Figure 4.32 demonstrates that the aperture field to far-field conversion is working as expected (solid and dashed lines match). The far-field calculated using the scattering matrix deduced from the inferred aperture field (● markers), however, shows poor agreement. This is also true for the incoherent far-fields (Figure 4.33 - Figure 4.34). The reason for this is that the angular size of the field cut in front of the horn is too small, thus the modal power in modes with predominantly off-axis power is not captured sufficiently. This is demonstrated by increasing the scan area in the simulation beyond what would be the limits of the scanner in the lab. If the dimensions of the field-cut are quadrupled (800 mm x 800 mm), keeping the same resolution, the beam reconstruction is much better, as shown in Figure 4.35.

![Diagram showing simulated 3-mode far-field of the P1 horn](image.png)

Figure 4.35: Simulated 3-mode far-field of the P1 horn: directly exported from FEKO (solid line); calculated using the directly exported aperture field (dashed line); and calculated using the scattering matrix deduced from the extended field cut (800 mm x 800 mm) at 300 mm (● markers). The top plot is the coherent field y-pol and the bottom plot is the unpolarised incoherent field including the orthogonal modes.
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An initial set of measurements have been conducted using the 200 mm × 200 mm field cut, therefore the error caused by the field cut being too small persists into the final result. Nevertheless, these measurements are still useful in validating the technique and comparing with the simulation. Later, in § 4.3.8, it is explored if measuring closer to the horn, in order to increase the angular size of the field cut, at the expense of measuring in the far-field regime, can improve the result.

Lab measurement discussion and modal excitations

In the simulations each mode has simply been excited with equal power by a wave port placed at the throat of the horn. In the lab the excitation of the desired modes is not a straightforward problem to solve. On the original formulation of the ideas within this section, it was thought that a certain generalised field distribution, such as a plane-wave, could excite all mode which were allowed in a specific waveguide. However this is not the case. Rather, to excite any particular mode, the field pattern of that mode must be generated specifically or matched by an incoming excitation field, and there are many ways of doing this. For example, the above idea of a plane wave excitation upon a waveguide which is able to support the first 3 modes would give virtually all excitation power to the $TE_{11}$ mode because of its likeness to the plane wave field pattern relative to the other modes (field patterns of modes are shown previously in Figure 2.2).

It is unlikely that all three modes can be excited at the same time with sufficient power using a single excitation, therefore the technique for the lab measurements is adapted somewhat. The single measurement is instead split into three separate ones, each with the aim of exciting sufficient power in one of the three modes and then measuring how it scatters through the horn. This is achieved by designing specific input excitation techniques at the horn throat for each mode. Modes can only scatter into modes with the same azimuthal index for an azimuthally symmetric geometry such as a horn. Therefore it does not matter if some of the power goes into exciting the other modes with a different azimuthal index since their result can be separated at the horn aperture. Rather, the overall goal is simply to give sufficient power to the desired mode so that it can be measured above the noise level of the measurement set-up.
The excitation of evanescent modes must also be taken into consideration. The specific field pattern of each desired mode cannot be produced exactly. Therefore, rather than exciting only a specific mode, realistically it is all modes that have field components in a favourable direction for the particular excitation source that are excited. In other words, a single mode is not sufficient to describe the excitation field, rather, many higher order modes are also required to give a better match. Fortunately, if these higher order modes are evanescent they remain localised to the source field. When thinking of impedance matching, the non-evanescent modes represent a resistive load on the source and the evanescent modes represent purely reactive loads on the source (Ramo 1993). The effectiveness of the half-length of waveguide filter at the throat of the P1 front horn is tested by examining the simulated insertion loss for each mode at 75 GHz. The results in Table 4.20 show that the length of the filter is sufficiently effective at removing any evanescent modes.

Table 4.20: Insertion loss for modes passing through the 5 mm half-length of wavelength filter at 75 GHz. The value continues to decrease for higher order modes.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Insertion loss (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>0</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0</td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>-43.976</td>
</tr>
<tr>
<td>$TM_{21}$</td>
<td>-57.155</td>
</tr>
<tr>
<td>$TM_{02}$</td>
<td>-84.760</td>
</tr>
</tbody>
</table>

Throughout this section only the first three modes have been considered by making sure that the geometry in question is long and narrow enough that all higher order modes are evanescent and filtered out. This is actually due to the limitations imposed by the technique of exciting the modes. As discussed, the first three modes have different azimuthal indices, therefore their associated modal content at the aperture can be separated, assuming the geometry is azimuthally symmetric. However, the 4th mode ($TM_{11}$) has the same azimuthal index as the first mode ($TE_{11}$). Given that both of these modes are likely to be excited because of the similarity of their fields, modes at the aperture with an azimuthal index of 1 can no longer be distinguished as having
come from the $TE_{11}$ or $TM_{11}$ mode. Therefore the scattering matrix can no longer be deduced. This is the main limitation of this technique in its current form. The possible extension of this technique beyond 3 modes is discussed later in § 4.4.3.

**$TE_{11}$ excitation**

To excite the $TE_{11}$ mode the rectangular-to-circular transition is attached to the converter head and the 1.55 mm radius circular waveguide is place flush and concentric with the 2.05 mm radius waveguide filter at the base of the P1 front horn, as shown in Figure 4.36. The 1.55 mm waveguide carries almost only the $TE_{11}$ mode, therefore it is expected that almost all of the power will go into exciting the $TE_{11}$ mode in the P1 horn. The two waveguides are aligned by eye by looking down the P1 horn and ensuring they are concentric as shown in Figure 4.37.

Figure 4.36: Set-up for exciting the $TE_{11}$ mode in the P1 horn.
4.3 Coherent Measurements

Figure 4.37: Concentric alignment of circular waveguide and P1 horn.

An exact replica of the lab set-up (with the same excitation mechanism) is also simulated for comparison. The simulated and measured fields are shown in Figure 4.38 and Figure 4.39 respectively, and the modal content is shown in Table 4.21. Due to the unknown excitation energy and phase in the lab, the measured field is normalised (after probe correction) to the maximum of the simulated field, and the phase of the measured field is shifted to match the simulation. Also plotted at the end of Figure 4.39 is the residual field. The residual field shows the difference between the simulated and measured inferred aperture fields, plotted in dB, as a fraction of the maximum of the simulated inferred aperture field. Finally, the results are also compared with the aperture field which is directly extracted from the simulation (directly extracted aperture field). The same plotting methodology used here is also used for the $TM_{01}$ and $TE_{21}$ modal excitation plots, and for the repeat case in the following section.
Figure 4.38: Simulation of the P1 front horn excited by a circular waveguide placed flush against the waveguide filter to excite the $TE_{11}$ mode. (1/2)
4.3 Coherent Measurements

Figure 4.38: **Simulation** of the P1 front horn excited by a circular waveguide placed flush against the waveguide filter to excite the $TE_{11}$ mode. (2/2)

Figure 4.39: **Measurement** of the P1 front horn excited by a circular waveguide place flush against waveguide filter to excite the $TE_{11}$ mode. (1/2)
Figure 4.39: Measurement of the P1 front horn excited by a circular waveguide placed flush against waveguide filter to excite the TE11 mode. (2/2)
4.3 Coherent Measurements

Table 4.21: Fractional modal content for a targeted $TE_{11}$ excitation of the P1 horn.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fractional modal content</th>
<th>Simulated: inferred aperture field</th>
<th>Simulated: direct aperture field</th>
<th>Measured: inferred aperture field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td></td>
<td>0.945</td>
<td>0.924</td>
<td>0.916</td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td></td>
<td>0.039</td>
<td>0.051</td>
<td>0.045</td>
</tr>
<tr>
<td>$TE_{12}$</td>
<td></td>
<td>0.009</td>
<td>0.015</td>
<td>0.012</td>
</tr>
<tr>
<td>$TM_{12}$</td>
<td></td>
<td>0.006</td>
<td>0.007</td>
<td>0.008</td>
</tr>
</tbody>
</table>

$TM_{01}$ excitation

A coaxial cable consists of an inner conductor and outer conductor concentric along the same axis. The electric field lines point between the two conductors as shown in Figure 4.40 (a). This strongly resembles the field of the $TM_{01}$ mode (Figure 4.40 (b)). Hence the coaxial cable is used as a method to excite the $TM_{01}$ mode.

(a)

(b)

Figure 4.40: Electric field lines: (a) within a coaxial cable; (b) of the $TM_{01}$ mode. (http://physwiki.apps01.yorku.ca/index.php?title=Main_Page/PHYS_4210/Coaxial_Cable) (Lee et al. 1985; Terman 1943).
4 Measurements of the Multi-mode Horn for LSPE-SWIPE

Simply placing the coaxial cable at the throat of the P1 horn, will not excite the $TM_{01}$ mode sufficiently well since the power radiated from an open ended coaxial cable is very low. However, if the central conductor is extended beyond the coaxial cable and into the P1 horn waveguide filter (Figure 4.41 (a)), the electric field begins to loop around and a large amount of power is transferred into a $TM_{01}$ mode excitation within the guide.

![Figure 4.41: Coaxial cable with central conductor extended into waveguide: (a) depiction (Ramo 1993); and (b) actual.](image)

A rectangular-to-coaxial converter is attached to the converter head and a piece of wire is then inserted into the hollow central conductor as shown in Figure 4.41 (b). The end of the coaxial connector is then placed in line with the end of the P1 waveguide filter as shown in Figure 4.42 (a). Alignment is achieved by eye by ensuring the central pin is concentric with the P1 horn (Figure 4.42 (b)).

Again a simulation is also performed for comparison. However, in this case the coaxial excitation could not be modelled therefore the simulation is instead for a direct excitation of the $TM_{01}$ mode using a waveguide port. The simulated and measured fields are shown in Figure 4.43 and Figure 4.44 respectively and the modal content is shown in Table 4.22.
4.3 Coherent Measurements

Figure 4.42: (a) Overall set-up for exciting the $TM_{01}$ mode in the horn showing (b) the concentric alignment.

Figure 4.43: Simulation of the P1 front horn excited by a coaxial cable to excite the $TM_{01}$ mode. (1/2)
Figure 4.43: **Simulation** of the P1 front horn excited by a coaxial cable to excite the $TM_{01}$ mode. (2/2)
4.3 Coherent Measurements

Figure 4.44: Measurement of the P1 front horn excited by a coaxial cable to excite the $TM_{01}$ mode. (1/2)
Figure 4.44: Measurement of the P1 front horn excited by a coaxial cable to excite the $TM_{01}$ mode. (2/2)

Table 4.22: Fractional modal content for a targeted $TM_{01}$ excitation of the P1 horn.

<table>
<thead>
<tr>
<th>mode</th>
<th>Simulated: inferred aperture field</th>
<th>Simulated: direct aperture field</th>
<th>Measured: inferred aperture field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TM_{01}$</td>
<td>0.926</td>
<td>0.922</td>
<td>0.926</td>
</tr>
<tr>
<td>$TM_{02}$</td>
<td>0.071</td>
<td>0.056</td>
<td>0.069</td>
</tr>
<tr>
<td>$TM_{03}$</td>
<td>0.004</td>
<td>0.023</td>
<td>0.004</td>
</tr>
</tbody>
</table>

**$TE_{21}$ excitation**

The $TE_{21}$ mode is the most difficult to excite because the electric field points in opposite directions in each half of the waveguide as shown in Figure 4.45. The mode could be excited in a similar manner to the $TM_{01}$ excitation by using a field probe (coaxial cable with extended pin). To excite the $TE_{21}$ mode it would require two probes out of phase by $180^\circ$ and placed in opposite halves of the guide. However, the size of the available coaxial cable relative to the waveguide is too large. Furthermore, the output of two out of phase coaxial cables would require a custom coaxial or
waveguide assembly which is not available in the lab. Therefore a different solution is explored.

Figure 4.45: $TE_{21}$ mode electric field vectors.

The circular waveguide is rotated by 45° about the centre of the P1 waveguide as shown in Figure 4.46. Although this is not an ideal way of exciting the mode since most of the power will still go into the $TE_{11}$ mode excitation, it is expected that a significant of power will be put into the $TE_{21}$ mode due to the asymmetry in the set-up.

Figure 4.46: Set-up for excitement of the $TE_{21}$ mode.

Again a replica of the lab set-up is also simulated for comparison. The simulated and measured fields are shown in Figure 4.47 and Figure 4.48 respectively and the modal content is shown in Table 4.23. The $TE_{11}$ mode is mostly excited therefore, although the $TE_{21}$ exists equally in both polarisations, in the y-pol it is dominated by the co-polarisation of the $TE_{11}$ mode. In the x-pol, however, the cross polarisation of $TE_{11}$ is low therefore the pattern is dominated by the $TE_{21}$ mode. The pattern
symmetry is moved away from the centre due to the interference between the coherent $TE_{11}$ and $TE_{21}$ modes. Therefore the previous method of centralisation, where both the phase and amplitude of the beam were made symmetric about the origin, could not be repeated exactly. Instead the phase is made symmetric about the origin whilst ensuring that the amplitude pattern matches what is seen in the simulation.

Figure 4.47: Simulation of the P1 front horn excited by a circular waveguide placed at 45° to the waveguide filter. (1/2)
4.3 Coherent Measurements

Directly extracted aperture field

Figure 4.47: Simulation of the P1 front horn excited by a circular waveguide placed at 45° to the waveguide filter. (2/2)

300 mm field

Figure 4.48: Measurement of the P1 front horn excited by a circular waveguide placed at 45° to the waveguide filter. The inferred aperture field has been flipped horizontally to match the simulation. (1/2)
Figure 4.48: Measurement of the P1 front horn excited by a circular waveguide placed at 45° to the waveguide filter. The inferred aperture field has been flipped horizontally to match the simulation. (2/2)
Table 4.23: Fractional modal content for a targeted $TE_{21}$ excitation of the P1 horn.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fractional modal content</th>
<th>Simulated: inferred aperture field</th>
<th>Simulated: direct aperture field</th>
<th>Measured: inferred aperture field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TE_{11}$</td>
<td>0.625</td>
<td>0.572</td>
<td>0.620</td>
<td></td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0.319</td>
<td>0.347</td>
<td>0.297</td>
<td></td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>0.027</td>
<td>0.031</td>
<td>0.031</td>
<td></td>
</tr>
<tr>
<td>$TM_{21}$</td>
<td>0.017</td>
<td>0.017</td>
<td>0.021</td>
<td></td>
</tr>
<tr>
<td>$TE_{01}$</td>
<td>-</td>
<td>-</td>
<td>0.018</td>
<td></td>
</tr>
</tbody>
</table>

**Normalised scattering matrix and beam reconstruction**

At the start of this section the 3-mode scattering matrix of the P1 horn was deduced based on a simulation where each mode is excited with equal power at the horn throat. The coherent and incoherent beams were then reconstructed using the scattering matrix and compared with the direct output from FEKO. The aim in this section is to deduce the scattering matrix of the horn and reconstruct the beams based on the data from the lab measurements. The main difference of the lab measurements is that the excitation power of each waveguide mode is unequal and unknown, hence, a raw export of the scattering matrix would yield the incorrect result. Instead, the scattering matrix should be normalised to represent the case where each mode has been excited in the waveguide filter with equal power.

As a reminder, the term ‘modal field’ is used to refer to all modes associated with a particular waveguide filter mode (a single column of the scattering matrix). Ideally, the normalised scattering matrix would be produced by dividing each modal field by the excitation power in the associated waveguide filter mode. However, the excitation power is unknown therefore a different solution is sought. One approximate solution is to normalise the scattering matrix internally by dividing by the sum of the power in all modes in the modal field. However, this excludes power escaping beyond the edges of the field cut in front of the horn, and power in each mode which is reflected at the aperture and travels backwards towards the excitation (the return loss information of each mode is lost). Due to the complex nature of the excitation methods, this return loss cannot be measured for each mode by simply
looking at the reflection S-parameter because the returned power cannot be
distinguished as coming from a specific mode. Furthermore, there are also likely to
be further losses when travelling back through the excitation mechanism.

The $TE_{11}$ excitation measurement is first considered. The complex scattering matrix
is extracted featuring only modes of azimuthal order $m = 1$. The matrix is then
ormalised by dividing by the quadrature sum of the coefficients for modes of $m = 1$.
This result becomes the first column in the overall P1 horn scattering matrix. This is
repeated for the $TM_{01}$ and $TE_{21}$ excitations which then form columns 2 and 3 of the
scattering matrix respectively.

Figure 4.49: Simulated **incoherent** 3-mode far-field (**including orthogonal modes**) of the P1 horn: directly exported from FEKO (solid line); and calculated using the scattering matrix deduced from an **extended field cut** (800 mm $\times$ 800 mm) at 300
mm (dashed lines). The green dashed line demonstrates the effect of **internally normalising** the scattering matrix. WPE: waveguide port excitation; NSM: normalised scattering matrix.
Before considering the measured data, the effect of the approximation in the internal normalisation of the scattering matrix is examined by considering an already understood result. At the start of this section a simulation was performed where the modes were directly excited with a waveguide port at the throat of the horn and an extended field cut ($800 \text{ mm} \times 800 \text{ mm}$) was used to find the modal content (Figure 4.35). The reconstructed incoherent beam (including orthogonal modes) in this case gave a very good match with the direct beam output from FEKO. The same scattering matrix used to construct this beam is normalised internally and the reconstructed beam is recalculated. The result is demonstrated in Figure 4.49. The results are now normalised so that results with and without the normalised scattering matrix can be compared. The normalisation causes an upwards shift of the beam which is particularly worse at large angles. Ultimately, the new result represents the best match that can now be achieved with the simulated beam, even if the remainder of the measurement is perfect. The normalisation error forms another large limitation in the coherent measurement technique.

Focus is now returned to the $200 \text{ mm} \times 200 \text{ mm}$ field cut measurement and corresponding simulations. The scattering matrix is deduced from the measurements and is normalised internally. The scattering matrix is also deduced and normalised using the simulations in which the lab modal excitation techniques have been replicated (i.e. not a direct waveguide port excitation). This is done for both the directly extracted aperture field and the inferred aperture field. The incoherent beam is reconstructed and compared in Figure 4.50. The directly extracted aperture field case shows good agreement with the direct FEKO output, and only disagrees at larger angles due to the error caused by normalisation of the scattering matrix as expected. The simulation and measurement for the field cut taken at $300 \text{ mm}$ both produce beams which fall below what is expected at large angles. This is expected due to the error caused by the angular size of the field cut being too small as was discussed previously (Figure 4.34 and Figure 4.35). In the proceeding section an attempt is made to increase the angular size of the field cut in order to achieve a better match.
4.3.8. SWIPE P1 Horn Measurement at 150 mm

The angular size of the scan area is increased by making some adjustments to the scanner to extend the maximum scan area, and by scanning the field closer to the horn. A 340 mm × 340 mm field cut at 5 mm resolution is taken at 150 mm from the horn aperture instead of at 300 mm. Thus the angular size of the field cut has been increased at the expense of the measurement no longer being performed in the far-field. Firstly a simulation is performed where the modes are directly excited with a waveguide port excitation at the throat of the horn with equal power and the incoherent beam is reconstructed. The result is shown in Figure 4.51. Using the inferred aperture field now gives a very good match with the correct result as anticipated (compared to Figure 4.34 for the smaller field cut). The effect of normalising the scattering matrix internally is also shown. Again it is seen that this slightly raises the beam at high angles.
4.3 Coherent Measurements

Figure 4.51: **Incoherent** P1 horn 3-mode far-field (**including orthogonal modes**) for a **simulated direct excitation** of modes and where a larger 340 mm × 340 mm field cut at 150 mm is used to infer the aperture field. WPE: waveguide port excitation; NSM: normalised scattering matrix.

The modes are excited the same way in the lab as in the previous section, and the electric field and modal content for each excitation is compared against simulation in Figure 4.52 - Figure 4.54 and Table 4.24 - Table 4.26.
**TE_{11} excitation**

Simulated

150 mm field

Simulated inferred aperture field

Figure 4.52: Electric fields for a targeted $TE_{11}$ mode lab excitation with a larger field cut. (1/3)
Figure 4.52: Electric fields for a targeted $TE_{11}$ mode lab excitation with a larger field cut. (2/3)
Measurements of the Multi-mode Horn for LSPE-SWIPE

Simulated directly extracted aperture field

Simulation - measurement residual

Figure 4.52: Electric fields for a targeted $TE_{11}$ mode lab excitation with a larger field cut. (3/3)
**TM\textsubscript{01} excitation**

Simulated

150 mm field

Simulated inferred aperture field

---

Figure 4.53: Electric fields for a targeted $\text{TM}_{01}$ mode lab excitation with a larger field cut. (1/3)
Figure 4.53: Electric fields for a targeted $TM_{01}$ mode lab excitation with a larger field cut. (2/3)
4.3 Coherent Measurements

Figure 4.53: Electric fields for a targeted $TM_{01}$ mode lab excitation with a larger field cut. (3/3)
Measurements of the Multi-mode Horn for LSPE-SWIPE

**TE$_{21}$ excitation**

Figure 4.54: Electric fields for a targeted $TE_{21}$ mode lab excitation with a larger field cut. (1/3)
4.3 Coherent Measurements

Figure 4.54: Electric fields for a targeted $TE_{21}$ mode lab excitation with a larger field cut. (2/3)
Figure 4.54: Electric fields for a targeted $TE_{21}$ mode lab excitation with a larger field cut. (3/3)
Table 4.24: Modal content for a targeted $TE_{11}$ mode lab excitation with a larger field cut.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fractional modal content</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Simulated: inferred aperture field</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>0.934</td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>0.041</td>
</tr>
<tr>
<td>$TE_{12}$</td>
<td>0.015</td>
</tr>
</tbody>
</table>

Table 4.25: Modal content for a targeted $TM_{01}$ mode lab excitation with a larger field cut.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fractional modal content</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Simulated: inferred aperture field</td>
</tr>
<tr>
<td>$TM_{01}$</td>
<td>0.931</td>
</tr>
<tr>
<td>$TM_{02}$</td>
<td>0.045</td>
</tr>
<tr>
<td>$TM_{03}$</td>
<td>0.025</td>
</tr>
</tbody>
</table>

Table 4.26: Modal content for a targeted $TE_{21}$ mode lab excitation with a larger field cut.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fractional modal content</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Simulated: inferred aperture field</td>
</tr>
<tr>
<td>$TE_{11}$</td>
<td>0.725</td>
</tr>
<tr>
<td>$TE_{21}$</td>
<td>0.208</td>
</tr>
<tr>
<td>$TM_{11}$</td>
<td>0.033</td>
</tr>
<tr>
<td>$TE_{12}$</td>
<td>0.011</td>
</tr>
<tr>
<td>$TM_{21}$</td>
<td>-</td>
</tr>
</tbody>
</table>
The reconstructed incoherent beam for the measurements and simulated equivalent of the measurements (including the normalised scattering matrix), for the 340 mm × 340 mm field cut at 150 mm are shown in Figure 4.55. The results are compared to the results from Figure 4.51 where the modes have been excited directly and various cases have been considered. The far-field from the measurement itself (black dotted line) shows good agreement, with only a slight disagreement at large angles. A detailed discussion of the final result is given in § 4.4.

Figure 4.55: Incoherent P1 horn 3-mode far-field (including orthogonal modes) for several cases. WPE: waveguide port excitation; LE: lab excitation method; NSM: normalised scattering matrix. See main text for explanation. Pink and blue dashed lines almost entirely overlap.
4.4. Discussion

4.4.1. Sources of Error in the Coherent Measurements

There are two categories of error associated with the deduction of the incoherent beam from coherent measurements in § 4.3: those associated with the overall technique itself and those associated with the measurement of the data. The main errors are summarised in Table 4.27. The effect of these errors on the measurement of the SWIPE P1 horn is discussed in this section.

Table 4.27: The main sources of error when deducing the incoherent far-field beam from coherent measurements.

<table>
<thead>
<tr>
<th>Sources of error in the measurement set-up:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Alignment of the 3D scanner and DUT</td>
</tr>
<tr>
<td>• Alignment of the modal excitation mechanism</td>
</tr>
<tr>
<td>• Centralisation of the measured beam</td>
</tr>
<tr>
<td>• Use of a simulated probe beam instead of a measured probe beam for the probe correction</td>
</tr>
<tr>
<td>• Movement of VNA cables during the scan.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sources of error in the overall technique:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Power escaping outside of the field cut in front of the horn</td>
</tr>
<tr>
<td>• Method of propagating the field back to infer the aperture field</td>
</tr>
<tr>
<td>• Method of generating the far-field from the aperture field</td>
</tr>
<tr>
<td>• Scattering matrix normalisation error - due to unknown modal excitation power in the waveguide filter</td>
</tr>
<tr>
<td>• Field scan resolution.</td>
</tr>
</tbody>
</table>

Sources of error in the measurement set-up

Sources of error in the measurement set-up cause the disagreement between the beams in Figure 4.55 related to the measurements (black dotted) and the simulated equivalent of the measurement set-up (pink dashed). To understand these errors, the measured and simulated electric fields and detected modal content in § 4.3.8 are compared. It is important to note that, when comparing the phase the absolute value
is not important. What is important is the relative change in phase across the beam, and between each polarisation, for the particular case being considered.

By far the most difficult part of the measurement is achieving a good alignment between the horn and the automated scanner, and between the modal excitation mechanism and the horn waveguide filter. Achieving a good alignment between the horn and the scanning plane is made difficult by the fact that the converter port, which is attached to the probe, tilts down with respect to the automated scanner struts. This is compensated for by raising the adjustable feet at the front of the scanner, however some residual alignment error remains. Achieving a good alignment for the modal excitation mechanism is difficult purely due to the alignment sensitivity and the small dimensions of the components involved.

A simulation is performed to investigate the effects of different misalignments within the measurement set-up. The magnitude of the effect on the amplitude pattern relative to the phase pattern is examined for various misalignments, and contrasted against the measured results in order to deduce which misalignments are most prominent within the set-up. The $TE_{11}$ mode excitation, with the field taken at 150 mm in front of the horn, is looked at as an example. The result with no misalignment is shown again for convenience in Figure 4.56. Note that in the x-pol there is a central phase discontinuity boundary along both axes. Rotational misalignment between the horn and scan plane are first investigated, followed by rotational and translational misalignment in the excitation mechanism. Translational misalignment are not investigated for the horn and scan plane because the field is centralised in post-processing. Misalignments are described in terms of the coordinates defined in Figure 4.23.

First to be considered is a misalignment of the horn (DUT) with respect to the scan plane. The effect of a 5° rotational misalignment about the x-axis is shown in Figure 4.57. The amplitude pattern suffers a small upwards displacement, however the overall shape changes very little. The effect on the phase is to shift upwards the phase boundary at the centre of the x-pol so that the four quadrants no longer symmetrically cut the circular phase front.
Figure 4.56: Field cut at 150 mm for the simulated equivalent of the measurement to excite the $TE_{11}$ mode, where **no misalignment** is present. This figure is the same as the first plot at the top of Figure 4.52.

Figure 4.57: Simulated effect of a $5^\circ$ rotational misalignment (about the x-axis) between the P1 horn and the scanning plane.
Second, misalignments in the excitation mechanism are considered. The excitation mechanism of the $TE_{11}$ mode consists of placing a smaller diameter circular waveguide, carrying the $TE_{11}$ mode, against the larger diameter P1 horn waveguide filter. The effect of a 0.1 mm translational misalignment along the y-axis is shown in Figure 4.58. The misalignment causes a slight asymmetry in the amplitude pattern and causes the horizontal phase boundary to be degraded in the x-pol. This is particularly bad further away from the centre. The vertical phase boundary appears to be unaffected.

![Figure 4.58: Simulated effect of 0.1 mm translational misalignment (along the y-axis) between the circular waveguide excitation and the P1 horn waveguide filter.](image)

Finally a rotational misalignment in the excitation mechanism is looked at. The circular waveguide is pivoted about its furthest off-axis contact point with the P1 horn waveguide filter. The effect of a 3° misalignment around the x-axis is shown in Figure 4.59. The effect is the same as the translational case; there is an asymmetry in the amplitude and the horizontal phase boundary is degraded away from the centre in the x-pol.

In light of the knowledge of the effects of the misalignments, the 150 mm distance measured field scans of § 4.3.8 are examined. We have seen that the main effect of a misalignment between the scanner and the horn is to shift the whole amplitude and
4.4 Discussion

phase pattern. Misalignments in the excitation mechanism do not cause this effect. For the $TE_{11}$ excitement (Figure 4.52: measured 150 mm field), the centre of the phase boundary cross in the x-pol is at the centre. Therefore the alignment between the horn and the scanner must be small, and is not enough to account for the misalignment effects in the amplitude pattern, for which the misalignment in the excitation mechanism must then be responsible. This is further backed up by the fact that there is clear degradation of the phase boundaries along both axes; a characteristic of misalignments within the excitation misalignment. The horn-scanner alignment does not change between scans, hence this misalignment can be assumed not to be the critical factor for the $TM_{01}$ and $TE_{21}$ mode excitations also.

![Figure 4.59: Simulated effect of 3° rotational misalignment (about the x-axis) between the circular waveguide excitation and the P1 horn waveguide filter.](image)

For the $TE_{11}$ mode excitation (Figure 4.52), the fractional detected modal content (Table 4.24) matches the simulation in terms of detecting only significant power in the same modes, however, the $TE_{11}$ content is 0.893, lower than the 0.934 expected. Comparing the simulated and measured inferred aperture fields, it is evident that this disagreement is primarily due to the poor match in the phase of the x-pol caused by the misalignment errors. In the $TM_{01}$ excitation (Figure 4.53: measured 150 mm field) there is also the potential for the extended pin of the coaxial cable connector to be tilted with respect to the coaxial cable connector itself. The alignment in the field
however looks remarkably good. There is a clear phase boundary in both polarisations over the majority of the scan, and the amplitude patterns are symmetric. There is a slight asymmetry in the y-pol amplitude which is likely due to a small tilt in the coaxial pin or coaxial connector itself. The fractional modal content (Table 4.25) features the same modes as predicted by the simulation, however the $TM_{01}$ content is 0.907, lower than the expected 0.931 (from the equivalent simulation). This is primarily due to the alignment error affecting the y-pol. The $TE_{21}$ excitation (Figure 4.54: measured 150 mm field) is more difficult to assess due to its off-axis nature and the fact that the $TE_{11}$ mode, which is excited with a higher power, dominates the y-pol scan. Comparing with the simulation, the alignment looks very good. The amplitude patterns are symmetric about the horizontal and there is a clear phase boundary in the x-pol. The fractional modal content (Table 4.26) shows by far the worst agreement out of the three modal excitations. The $TE_{11}$ and $TE_{21}$ modes are measured at 0.632 and 0.197 respectively compared with the equivalent simulation which predicted 0.725 and 0.208. The only real difference in the simulated and measured beams is the absence of the ‘tails’ in the x-pol amplitude pattern. This is due to the circular waveguide not facing the centre of the guide exactly. To demonstrate this, the effect of a 2 mm translational off-set along the x-axis of the $TE_{21}$ mode excitation mechanism (circular waveguide tilted at 45°) in the simulation is shown in Figure 4.60. It is clear that this misalignment causes the ‘tails’ to disappear.

There are several more errors which are applicable to the measurements. The movement of the cables as the scanner moves is inevitable given the type of scanning system. This will primarily affect the phase. Also, the probe correction has been performed using a simulated beam whereas the beam of the actual probe may differ. This will affect both amplitude and phase. Finally, the centralisation is limited by the resolution of the scan. All these errors, however, are much smaller than the error due to misalignment of the excitation mechanism. To improve the measurements therefore, a test bench mount should be manufactured which holds both the P1 horn and the excitation waveguide in precise but adjustable alignment. With a better alignment it should be possible to achieve a very strong agreement between the measured and measurement-equivalent simulated results (black dotted and pink lines in Figure 4.55).
Figure 4.60: Simulated effect of the waveguide excitation being off-set along the x-axis by 2 mm in the TE$_{21}$ excitation.

**Sources of error in the overall technique**

The errors which are fundamental to the technique itself are now discussed. Each error is isolated by looking at different results in Figure 4.55.

- There is almost no error in moving from the direct excitation of modes in the waveguide (Figure 4.55: pink dashed line) to the method of exciting modes in the lab (Figure 4.55: blue dashed line).

- There is a large error caused by the way in which the scattering matrix has been normalised internally. This happened because the excitation power in each mode in the P1 horn waveguide filter in the lab is unknown. This can be seen as the disagreement between the results in Figure 4.55: including the normalisation (blue dashed line [under the pink line]); and without the normalisation (green dashed line).

- There is a very small error caused by having to infer the aperture field from a field cut in-front of the horn. This causes disagreement in Figure 4.55 between the reconstructed beams for the simulations: where a field cut is
taken in-front of the horn and propagated back (green dashed line); and where the aperture field is extracted directly (red dashed line). The two main contributing errors are caused by the method of propagating back the field and by the escape of power outside of the field cut. The propagation of the field caused problems when trying to enact the method on a waveguide (§ 4.3.2) because of the small electrical size of the aperture. However this error became very small when dealing with large aperture horns (§ 4.3.3 and § 4.3.7). The error caused by power leakage outside of the field cut also caused problems for the waveguide, and for the P1 horn when the field cut was taken at 300 mm distance. However, this error also became very small when the angular size of the field cut was extended by taking measuring at 150 mm distance instead of 300 mm. The small nature of the error caused by both of these errors is evident given the close agreement between the two mentioned results in Figure 4.55.

- There is a small error caused by the difference in the method in which the far-field is calculated using the custom code (Figure 4.55: red dashed line) and using FEKO directly (Figure 4.55: black solid line).

Thus, to improve the overall result, the way in which the scattering matrix is normalised must be improved. However it is not obvious in how this can be achieved. Power is not excited equally for each mode in the waveguide filter due to the differences in the three excitation mechanisms. To solve this, it is assumed that modes have only been scattered into modes of the same azimuthal index, and the scattering is matrix is normalised so that the total power in each azimuthal index for each excitation mechanism is equal to unity. However, this introduces the problem that information on any power not passing through the field cut is lost. This includes power escaping beyond the field cut and the power in each mode which is reflected at the aperture. As demonstrated in § 4.3.8, the field cut is large enough that the power escaping beyond the field is small. Therefore the majority of the problem is predicted to be due to the loss of the return loss information at the aperture.

One solution could be to try to determine how each modes is reflected at the aperture, then weight the modes accordingly in the construction of the incoherent
beam. However, it is not currently understood how this could be achieved, either through simulation or measurement.

Figure 4.61: FEKO model of a device used to excite the first 3 circular waveguide modes. (Sharma & Thyagarajan 2012)

An alternative solution is to understand how much power each mode has in the waveguide filter before entering the horn. In this way, the scattering matrix could be normalised exactly to account for this excitation power. This seems like a conceivable solution however the specifics of how this can be accomplished are not currently understood. A further solution may be possible by improving the way in
which the modes are excited in the waveguide filter. Similarly to the previous solution, the idea would be to know the excitation power of each mode in the waveguide filter. Potentially, a device similar to the one designed by (Sharma & Thyagarajan 2012) and shown in Figure 4.61, may be a capable of doing this. Each mode is excited by applying power to the corresponding port. The top of the device would be interfaced with the P1 horn waveguide directly. If all ports were covered except the $TE_{11}$ port, subtracting the power returned through the $TE_{11}$ port from the power excited through the $TE_{11}$ port should give the power escaping through the top. Hence, this would be the power with which the $TE_{11}$ mode is excited in the guide. The same technique should be possible for the two other modes also, but it depends on the purity with which they are excited.

4.4.2. Systematics in the Incoherent Set-up

The incoherent measurements in § 4.2 have shown that the measured beam is much narrower than predicted by simulation (Figure 4.5). This is likely caused by modes with mainly off-axis power not coupling to the detector efficiently. There are two main systematics which are put forward to explain this behaviour: misalignment of the bolometer within the detector cavity; and non-uniformity of the bolometer absorber across its surface.

Since the system is reciprocal (detector can be treated as an emitter), the results from the coherent set-up are somewhat applicable and may provide some insight in the analysis of the incoherent set-up. The coherent measurements have demonstrated that the modal content at the aperture of the P1 horn is very sensitive to misalignments in the excitation of the modes in the waveguide filter. Even with the best alignment of the coherent excitations in the lab, the best fractional modal content for the $TE_{11}$ and $TM_{01}$ modes were 0.893 and 0.907 respectively, compared to the values predicted by the simulation of 0.934 and 0.931 respectively. The value for the $TE_{21}$ mode is not considered since this is a special case where the excitation is at 45°. This demonstrates that, in principle, bolometer misalignment may be responsible for the measured incoherent beam disagreeing with the simulation. However, for a more conclusive result, the effect of the transition horn and detector cavity must be taken into account. To do this a simulation should be performed which investigates directly
how misalignment of the bolometer in the cavity affect the beam. Furthermore, the bolometer should be purposely misaligned in the incoherent set-up to examine the effect on the measured beam.

If a simulation is performed of the full horn pixel assembly, whereby incident plane-waves are excited from different directions onto the front horn, the power distribution across the absorber changes drastically depending on the direction of the plane wave. If the power distribution is more central for on-axis directions, and less central for off-axis plane waves, then it becomes apparent that decreased coupling efficiency at the edges of the absorber can lead to the beam which is measured. Furthermore, is the absorber also varies azimuthally, this can also explain the apparent asymmetry in the beam.

A similar explanation is also available if the simulation is thought of in the usual reciprocal sense (absorber as an emitter). As normal, each mode is excited in the waveguide filter and the far-field beam of the front horn is calculated. In order to model the non-uniformity of the absorber surface, an impedance sheet is added within the waveguide filter, whose impedance varies randomly across its surface. The loss of azimuthal symmetry means that within a single modal field, modes will now scatter into modes of different azimuthal orders, and between the two orthogonal mode sets. Since the modes within each modal field are coherent, their interference is now able to explain the narrowness and loss of azimuthal symmetry of the measured beams.

Further investigation is ongoing in order to understand the exact nature of these effects.

4.4.3. Extension of the Coherent Measurements Technique Beyond 3 modes

The mechanisms used to target the excitation of specific modes in the lab are not exact. This means that all modes with a generalised field pattern similar to that of the excitation field are excited. For example, the mechanism used to excite the $TE_{11}$ mode would also likely excite the higher order $TE_{1n}$ and $TM_{1n}$ modes. To combat this, the frequency is restricted so that only the first three modes are non-evanescent
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($TE_{11}$, $TM_{01}$ and $TE_{21}$). The remaining modes are all filtered out in the waveguide filter. Any power which happens to be excited in the $TM_{01}$ or $TE_{21}$ modes is not an issue because, assuming the horn is azimuthally symmetric, modes only scatter into modes of the same azimuthal order. Thus the modal content corresponding to each of the first three modes can be separated at the aperture. However, if the frequency is increased to allow the 4th mode to propagate ($TM_{11}$), this is problematic because you cannot distinguish the modal content at the aperture as having come from the $TE_{11}$ or $TM_{11}$ mode. Hence you cannot construct the scattering matrix and thus the incoherent far-field beam. Therefore the overall technique is limited to the first 3 modes only. This is an issue since the lowest frequency band in SWIPE supports the first 12 modes (excluding orthogonal modes).

Extension of the technique beyond 3 modes is far from straightforward. One solution could be to excite specifically each individual mode in the waveguide, without exciting any other modes. This requires replicating the exact electric field shape of the targeted mode. A device such as the modal exciter discussed at the end of § 4.4.1 (Figure 4.61) may be capable of this if its principles are extended to higher order modes, however it is unlikely that it will be precise enough. Another solution which may be capable of specifically exciting modes is a planar grid of dual polarisation slot antennas. With each slot antenna acting as a pixel, the power to each pixel could be adjusted to combine to form the electric field pattern a specific mode. This would require a very high pixel density, however, which would be difficult to achieve.

A different solution may be to remove the horn and measure the modal content which has been excited in the waveguide filter itself by the particular excitation mechanism. This way you would know exactly which modes enter into the horn and could therefore normalise the scattering matrix accordingly. The problem is that the small electrical size of the aperture of the waveguide causes many problems as has been demonstrated in § 4.3.2. This is due to the approximations used to manipulate the fields and the escape of power at large angles not passing through the field cut in front of the horn. It may be possible to fix these errors by using more rigorous field propagation techniques and by increasing the angular size of the field cut.

Another idea which could be explored is that the scattering matrices for different modes are related. This could especially be true if they are modes of the same
azimuthal order. For instance, the way in which the $TE_{12}$ mode scatters could be related to the way in which the $TE_{11}$ scatters. This way the scattering matrices of all higher order modes of the same azimuthal index could be extrapolated from the $TE_{11}$ measurement. Furthermore, if the $TE$ and $TM$ mode scattering matrices are related also, and exciting higher azimuthal orders ($>2$) is possible, then in theory the full scattering matrix for all modes could be found.

4.5. Conclusion

The far-field beam of the SWIPE horn has been measured at 116 GHz using an incoherent set-up with a room-temperature bolometer placed in the detector cavity. The beam shows a poor match with the simulation, being much narrower than predicted. This is attributed to some modes not coupling to the detector due to misalignment of the detector in the cavity and defects in the detector absorber. Modes with mainly off-axis power seem to be particularly affected, giving the beam its narrow shape. Later work using a coherent measurement set-up demonstrated the highly sensitivity nature of the modal content at the aperture of the horn to misalignments in the excitation of the modes in the waveguide filter.

Due to the difficult nature of performing the incoherent measurements directly, an investigation is performed to assess if a coherent VNA can be used to measure and infer useful information about the incoherent multi-mode operation of the SWIPE horn. The overall aim is to deduce the multi-mode incoherent beam (as would be measured if an incoherent detector was used) from the coherent measurements for the SWIPE P1 front horn. The first three modes are excited separately in the horn waveguide filter at 75 GHz using different excitation mechanisms. The field is scanned at a distance in front of the horn aperture and propagated back to infer the aperture field. The modal content of the inferred aperture field is measured and used to construct a scattering matrix to describe how modes are scattered within the horn. The scattering matrix is normalised to give equal power to the fields associated with each waveguide mode. For each of these fields the far-field is calculated and the multi-mode incoherent far-field is generated by summing the electric far-fields in quadrature. The technique is currently limited to only the first three modes due to the imprecise nature of the mechanisms used to excite the modes in the waveguide filter.
The reconstructed incoherent beam (from the coherent measurements) shows good agreement with the simulated beam, however there are some obvious deviations at high angles. Two main errors cause the disagreement. Each modes is excited in the waveguide with unequal power. The scattering matrix therefore needs to be normalised by dividing by the excitation power of each mode in the waveguide filter, however this excitation power is unknown. Instead an approximation is made by normalising the scattering matrix internally. This however removes the information about how much power is reflected at the aperture for each mode, leading to an error in the final beam. The second error is due to the misalignments in the excitation mechanism used to excite the specific modes in the horn waveguide filter.
5. Conclusions and Future Work

This thesis concerns the development of receiver systems employing multi-mode feed horns to increase sensitivity in order to measure the B-mode polarisation component of the CMB. Particular focus is placed on simulations and measurements of the horn pixel assembly and telescope of the SWIPE instrument, which forms part of the LSPE experiment. The goal is to predict the optical performance, and to understand how measurements can be performed for a multi-moded system.

In Chapter 3 a simulation of the SWIPE horn-lens set-up is performed for pixels which are closest to and furthest from the centre of the focal plane. The far-field beam on the sky is predicted for the 140 GHz and 220 GHz bands. The 140 GHz beam is used to assess the level of optical cross-polarisation, which is found to be at an acceptable level for a targeted B-mode measurement at a level of $r = 0.01$. The strength of the far-side lobe and the level of spillover of the horn beam outside of the telescope are also predicted. Furthermore, the horn-lens simulation is also used to find the optimal location to place the telescope focus relative to the horn aperture, with regards to maximising gain and optimising beam shape. This is referred to as the ‘phase centre’. For multi-mode horns, it is found that the drop in gain at the horn aperture compared to at the phase centre is small, relative to what is usually the case for single-mode horns.

In Chapter 4 a measurement of the full SWIPE multi-mode horn pixel assembly is performed at 116 GHz using a room-temperature bolometer (incoherent detector). The measured beam is narrower than predicted due to modes with mainly off-axis power not coupling onto the bolometer efficiently. This is theorised to be due to misalignments of the bolometer in the detector cavity and defects in the bolometer absorber. Due to the difficulty in using an incoherent detector, a separate study is undertaken to investigate if useful information about the incoherent behaviour of the SWIPE horn can be inferred from measurements using a coherent detector, such as a VNA. Modes are excited in the waveguide filter, and the scattering behaviour of modes within the horn is measured by analysing the modal content at the horn aperture. The incoherent 3-mode far-field beam of the SWIPE front horn is deduced,
and shows good agreement with simulation, with some discrepancies at large angles. The main errors come from misalignments in the mechanism used to excite the modes in the horn waveguide filter, and from the normalisation problem caused by the unequal and unknown excitation energy of the modes. The modal content at the aperture is found to be highly sensitive to misalignments of the excitation of modes in the waveguide filter. This strengthens the claim that misalignments of the bolometer may contribute to the poor beam in the incoherent measurements. The main limitation of the coherent measurement technique is that it is currently limited to only the first 3 waveguide modes due to the imprecise nature of the mechanisms used to excite the modes.

Initially, the goal of Chapter 3 was to produce an accurate simulation of the full SWIPE optical chain. This would include accurate models of the horn (including the filter cap), polarisation-splitting wire grid, lens, thermal filters and the rotating HWP. A simulation of the coupling efficiency of modes onto the detector would also be included by weighting the modal excitations in the waveguide filter of the horn. Furthermore, this simulation would be performed across the full frequency band for the 140, 220 and 240 GHz pixels, and for a variety of different pixels in the focal plane. The result of the model would be used to deduce the beam on the sky and to extract the levels of main beam polarisation systematics. Furthermore, the optimum position to place the telescope focus relative to the horn aperture would be determined. Some of these goals were only partially achieved for several reasons. The large electrical size of the horn and lens, combined with the fact that a separate simulation is required for each mode, gave a very long simulation time and high computational requirements, even for the lowest accuracy. Efforts to include the other components gave simulations which could not be run on a reasonable timescale or which exceeded the capabilities of the available computer. Ideally the HWP would have been included in the simulation, since this generates many important systematic effects, being the first component in the optical chain. However, the metal-mesh HWP is a relatively new technology and has a very complex structure. As such, no existing simulation techniques are capable of simulating it to this accuracy, and developing one is beyond the scope of this thesis. Simulating the horn-lens system over the whole frequency band (instead of monochromatically) would give a long simulation time, but is feasible. However, this was not done since no accurate
information on the transmission profile of the bandpass filter was available at the
time of the work, and the beam itself did not appear to change much going from
140 GHz to 220 GHz. Finally, the simulations do not include information about the
modal coupling to the detector. Simulations to assess modal coupling of individual
modes were attempted but proved highly difficult. Efforts were instead focussed on
ensuring the horn-lens simulation was correct, and on simulating off-centre pixels in
the focal plane. The investigation of the optimum location at which to place the
telescope focus with respect to the horn aperture has been carried out successfully. It
would have been desirable to confirm the result with a measurement however this
was not achieved since the lens has not yet been manufactured.

The initial goal of the Chapter 4 was to measure the SWIPE horn incoherent far-field
beam using an incoherent and coherent detection scheme. These two results would
then be directly compared to each other and against the simulations of Chapter 3.
Some of these goals were only partially achieved due to difficulties in developing the
coherent detection technique. The coherent detection technique in its entirety is
something which has not been attempted before, therefore its capabilities were
unknown at the start. Part way through its development it was realised that it is very
difficult to excite specific modes in a waveguide. This means that is difficult to
extend the coherent technique beyond the first 3 modes. This issue is not straight
forward to solve and holds the technique back from being directly comparable to the
measurements using the incoherent set-up. Nevertheless, the development of the
 technique has been a useful exercise in determining the feasibility of such a
measurement, and useful information has been extracted about how sensitive the
modal content at the horn aperture is to misalignments in the modal excitation in the
horn waveguide filter.

Regarding future work, there are several objectives which are identified as the most
important. The simulation should be extended to take into account the whole
frequency band and the coupling efficiency of modes onto the detector. Furthermore,
the systematics of the HWP should be assessed by introducing it into the simulation.
For the incoherent measurements, the systematics effecting the measured beam must
be identified and resolved. This is an ongoing area of investigation. For the coherent
measurements, misalignment in the modal excitation mechanism should be removed,
5 Conclusions and Future Work

the error from normalising the scattering matrix must be overcome, and a way in which the method can be extended beyond 3 modes must be identified. This will most likely be achieved by improving the way in which the modes are excited in the waveguide filter.

Design and testing of the LSPE-SWIPE gondola, cryostat, optics, focal plane and readout electronics are all thoroughly underway. Regarding the focal plane, the horns have been manufactured and tested using a room-temperature model of the bolometer. Once the systematics affecting the beam are understood and rectified, a flight-model of the bolometer must be manufactured and tested. The filter cap must also be introduced and tested. Pending these tests, the final designs will be mass produced and integrated into the focal plane. The focal plane housing, which secures the horns in place, has already been manufactured. The whole focal plane will then undergo a further test before being integrated into the instrument. The instrument will then undergo final system level tests before being shipped to the launch site. A 1 month long launch window for SWIPE has been scheduled for the end of 2018.

Although the techniques developed within this thesis have been specifically tailored to the development of SWIPE, they also have an extended scope of application beyond multi-moded horn experiments. Regarding Chapter 3, the insight gained into the simulation of electrically large horn-lens systems remains applicable to future experiments employing a similar single lens design. This is particularly the case if FEKO is used to perform the simulations, since a large understanding has been acquired regarding the specific implementation of the simulations within the FEKO software, and of how the choice of each simulation parameter effects the overall accuracy of the end result. Looking outside of the field of experimental cosmology, the knowledge gained in Chapter 4 of how modes behave within physical waveguide structures remains relevant in many areas of physics. For example, many experimental set-ups use waveguides to transmit radiation between components. Modal purity is often important in these systems since the scattering of modes due to the structure of the waveguides and misalignments in the system directly leads to a loss of transmitted power. A specific example of where this is the case is during the electron cyclotron resonance heating of plasma in fusion reactors (Shimozuma 2008).
Appendix A

A Simulation Techniques

When performing simulations using commercially available software, it can be very useful to have an insight into the fundamental principles on which the software is based. The advantage being that, with knowledge of the approximations and limitations of the simulation method, an intuition is gained into how simulation efficiency (time and computational resources) can be increased without effecting the results significantly. Furthermore, it becomes easier to quickly determine the cause of any unexpected results or errors.

Finding a full solution to an electromagnetic problem such as the scattering of light off a conducting surface, or the refraction of light by a dielectric lens requires finding solutions to Maxwell's equations. However, to find actual solutions is complex and therefore some form of approximation is usually required. The numerical approximation of Maxwell's equations is called computational electromagnetics (CEM). Within the branch of CEM, many different formulations of solution methods have been developed. The choice of method generally involves a trade-off between the required accuracy of the result and the computational requirements. Furthermore, the appropriateness of individual methods depends on the complexity and size of the geometry as well as the type of materials involved.

The two main categories of CEM methods are “full-wave” methods, which approximate the Maxwell equations numerically without any initial physical approximations being made; and “asymptotic” methods, which require fundamental approximations in the Maxwell equations that become asymptotically increasingly valid as the frequency is increased (Davidson 2005). For the analysis of quasi-optical systems, which are being studied in this work, the basic set of full-wave solution methods are: the Finite Difference Time Domain (FDTD) method; the Method Of Moments (MOM); and the Finite Element Method (FEM). Additionally, the main asymptotic methods include: Physical Optics (PO); Geometrical Optics (GO); and the Uniform Theory of Diffraction (UTD). The full-wave methods tend to be more
accurate, but also more computationally intensive. The majority of the simulations in this thesis are performed using MOM or GO therefore the principles of these methods are described in detail.

### A.1 The Method of Moments

The Method of Moments (MoM) is extensively described in many CEM books (Davidson 2005; Gibson 2007; Bondeson et al. 2005). An electric field incident on a conducting surface will excite surface currents, which themselves in turn produce a scattered electric field. Large scale problems have many conducting surfaces of various geometries forming an overall scattering structure. The idea of the MoM is to replace the scattering structure by the equivalent surface currents. This surface current is then discretised into triangular elements in a process known as “meshing”. A finer mesh will lead to a result with higher accuracy however the computational requirements increase rapidly. Triangular elements are selected since tessellation of this shape produces the most efficient representation of a general geometry.

To replace the scattering structure by the equivalent surface currents an equation must be derived from Maxwell's equations for the incident and scattered electric field in terms of surface currents. This is an integral equation called the Electric Field Integral Equation (EFIE). The EFIE cannot be solved analytically therefore the MoM is used to convert this integral equation into a linear system of equations (represented as a matrix equation) that can be solved numerically by a computer. The matrix equation represents the interaction of all mesh elements with all other mesh elements.

#### General Method of Moments

Consider a generalised problem

\[ L(f) = g \]

where L is a linear operator, g is a known forcing function, and f is unknown. In an electromagnetic problem L is typically an integro-differential operator, f is an unknown function (e.g. current) and g is a known excitation source (e.g. incident field) (Gibson 2007). The unknown function f is expanded as a series of N known basis functions
\[ f = \sum_{n=1}^{N} \alpha_n f_n \]  

(\text{A.2})

with unknown weighting coefficients \( \alpha_n \). A set of \( N \) testing functions \( \omega_m \) are then defined and the inner product is taken of Eq. \( \text{A.1} \) with \( \omega_m \) to compare the two functions. After substituting in \( \text{A.2} \) this leads to the equation

\[ \sum_{n=1}^{N} \alpha_n <\omega_m, L f_n> = <\omega_m, g>. \]  

(\text{A.3})

Since \( <\omega_m, L f_n> \) has one index that runs up to \( N \) and has another index that is summed up to \( N \), it can be represented as a matrix \([A_{mn}]\). This leads to the matrix equation

\[ [A_{mn}][\alpha_n] = [g_m] \]  

(\text{A.4})

where \([A_{mn}] = <\omega_m, L f_n>\) and \([g_m] = <\omega_m, g>\). If \([A_{mn}]\) is not singular, the unknowns can be found by inverting Eq. \( \text{A.4} \) to give

\[ [\alpha_n] = [A_{mn}]^{-1}[g_m]. \]  

(\text{A.5})

**Simulating Electromagnetic Scattering Using the Method of Moments**

As mentioned above, the MoM finds solutions to the integral form of Maxwell’s equations, which must first be derived starting from the individual Maxwell equations in their usual form. The electric field \( E \) and magnetic field \( B \) are assumed to have an \( \exp(j\omega t) \) time dependence, and are associated with a current density \( J \).

Starting from Ampère’s law

\[ \nabla \times B = \mu_0 J + \epsilon_0 \mu_0 \dot{E} \]  

(\text{A.6})

and substituting in the scalar and vector potential descriptions of the field

\[ E = -\nabla \phi - \dot{A} \]  

(\text{A.7})

\[ B = \nabla \times A \]  

(\text{A.8})

gives\(^1\)

\[ \nabla (\nabla \cdot A) - \nabla^2 A = \mu_0 J - j\omega \epsilon_0 \mu_0 (\nabla \phi + j\omega A). \]  

(\text{A.9})

\(^1\) Using the vector identity: \( \nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A. \)

Choosing to work in the Lorentz gauge

\[ \nabla \cdot A = -j\omega \epsilon_0 \mu_0 \phi \]  

(\text{A.10})

further reduces Eq. \( \text{A.9} \) to the vector Helmholtz equation

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\[-(\nabla^2 + k^2) A = \mu_0 J\]  \hspace{1cm} (A.11)

where \( k = \frac{\omega}{c} \). In the Cartesian coordinate system the scalar Helmholtz equations are then

\[-(\nabla^2 + k^2) A_i = \mu_0 J_i \]  \hspace{1cm} (A.12)

where the sub index \( i \) represents the three Cartesian axes individually. Assuming all currents flow only on the surfaces of conductors, (Bondeson et al. 2005) shows that the solution at a point \( r \) to this equation is given by

\[ A_i(r) = \int_{\partial \Omega_c} G(r, r') \hat{\mathbf{s}}_i \cdot \mathbf{J}_s(r') dS' \]  \hspace{1cm} (A.13)

with a Green's function for a point current at \( r' \) flowing in the \( i \)th direction given by

\[ G = \frac{\mu_0}{4\pi} \frac{\exp(-jk|r - r'|)}{|r - r'|} \]  \hspace{1cm} (A.14)

and where \( \hat{\mathbf{s}}_i \cdot \mathbf{J}_s \) is the \( i \)th components of the of the surface current \( \mathbf{J}_s \) flowing along the conductor surface \( \partial \Omega_c \). The full vector potential is thus given by

\[ A(r) = \frac{\mu_0}{4\pi} \int_{\partial \Omega_c} \frac{\exp(-jk|r - r'|)}{|r - r'|} \mathbf{J}_s(r') dS'. \]  \hspace{1cm} (A.15)

Similarly the scalar potential can also be found in terms of surface currents

\[ \phi(r) = \frac{1}{4\pi\epsilon_0} \int_{\partial \Omega_c} \frac{\exp(-jk|r - r'|)}{|r - r'|} \rho(r') dS'. \]  \hspace{1cm} (A.16)

where \( \rho \) is the surface charge density.

The scattered electric field is given by

\[ \mathbf{E}^s = -j\omega A - \nabla \phi . \]  \hspace{1cm} (A.17)

Applying the boundary condition that the sum of the incident and scattered tangential electric field must be zero on the surface of the conductor:

\[ \mathbf{E}^i_{\parallel} + \mathbf{E}^s_{\parallel} = 0, \]  \hspace{1cm} (A.18)

and subbing in for the vector and scalar potential gives the EFIE:

\[ \mathbf{E}^i_{\parallel} = \frac{j\omega \mu_0}{4\pi} \int_{\partial \Omega_c} \frac{\exp(-jk|r - r'|)}{|r - r'|} \mathbf{J}_s(r') dS' \]  \hspace{1cm} (A.19)

\[ \mathbf{E}^s_{\parallel} = \frac{j}{4\pi\epsilon_0 \omega} \nabla \int_{\partial \Omega_c} \frac{\exp(-jk|r - r'|)}{|r - r'|} \nabla' \cdot \mathbf{J}_s(r') dS'. \]
As stated above, the EFIE is solved numerically using the MoM. In solving the EFIE, issues regarding singularities can occur and therefore careful treatment is required. The full treatment is presented in (Gibson 2007).

The choice of basis and testing function is crucial to achieving an efficient and good solution. In the MoM simulations in this report, the surface current is expanded in terms of Rao-Wilton-Glisson (RWG) triangular basis functions (Rao et al. 1982)

\[
J_s = \sum_{n=1}^{N} a_n s_n(r).
\]

Linear combinations of RWG basis functions can give a description of the surface currents with high accuracy. These also match the triangular mesh elements into which the geometry is discretised. For the testing functions “Galerkin's method” is used, where the basis functions are also used as testing functions (Davidson 2005).

### A.2 Multilevel Fast Multipole Method

The computational requirements of the MoM scale rapidly with increasing problem size. This is because the N basis functions which are used to describe the surface currents are treated individually. Therefore the interaction has to be calculated between each and every basis function with every other basis function. The Multilevel Fast Multipole Method (MLFMM) is an alternative formulation of the MoM. As with the MoM, the MLFMM models the interaction between all triangular mesh elements. What differentiates the MLFMM is that it groups basis functions and models the interaction between groups of basis functions rather than between individual basis functions. The grouping is based on treating far away elements as if they were a single element. This reduces the computational time scaling from \(N^2\) to \(N[\log(N)]^2\). (FEKO 2014)

### A.3 Geometrical Optics

If the wavelength of radiation is small compared to the optical components, electromagnetic waves can be well approximated as rays which are perpendicular to the wavefronts. The behaviour of the light is predicted using the basic principles of reflection and refraction of light. Effects such as interference are not taken into account in this treatment. Reflections are modelled using the simple property that the
angle of incidence equals the angle of reflection. Refractions are calculated using Snell’s law

\[ n_1 \sin(\theta_1) = n_2 \sin(\theta_2), \]

where \( n_1 \) and \( n_2 \) are the refractive indices of successive media and \( \theta_1 \) and \( \theta_2 \) are the angles of incidence and refraction at the interface between the two media.

**Ray Launching and Ray Tracing for Dielectric Lenses**

Consider a focal plane of antenna feed horns illuminating a dielectric lens; there are two main approaches for the propagation of rays: ray tracing and ray launching. A ray tracing approach is implemented in Zemax. Rays are traced between each source (sky) and each observation point (focal plane), through the lens. An algorithm will calculate these rays by looking for valid ray paths between the two points using the principles of GO. A ray launching approach is implemented in FEKO. Rays are launched from the horn source at fixed angular increments independent of the number of observation points. Upon the ray intersecting a dielectric surface mesh element the reflected and transmitted rays are calculated and propagated onwards. Refraction is also taken into account at this point. If the ray encounters a wedge or edge mesh element, rays pertaining to the diffraction cone are computed and propagated further. The whole process occurs iteratively with each ray being propagated on an individual basis. On the final surface from which the far-field beam is calculated, the accumulation of rays hitting the surface combine to predict the final electric field strength over the surface from which the far-field is calculated. The disadvantage of ray launching is that the repeated splitting of rays can lead to an enormous number of rays causing the simulation runtime to increase rapidly for many ray interactions.


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