RADON TRANSFORMS AND MICROLOCAL ANALYSIS IN COMPTON SCATTERING TOMOGRAPHY

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Abstract

Radon transforms and microlocal analysis in Compton scattering tomography
James W Webber
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In this thesis we present new ideas and mathematical insights in the field of Compton Scattering Tomography (CST), an X-ray and gamma ray imaging technique which uses Compton scattered data to reconstruct an electron density of the target. This is an area not considered extensively in the literature, with only two dimensional gamma ray (monochromatic source) CST problems being analysed thus far [50, 48, 46]. The analytic treatment of the polychromatic source case is left untouched and while there are three dimensional acquisition geometries in CST which consider the reconstruction of gamma ray source intensities [41, 47, 64, 37], an explicit three dimensional electron density reconstruction from Compton scatter data is yet to be obtained. Noting this gap in the literature, we aim to make new and significant advancements in CST, in particular in answering the questions of the three dimensional density reconstruction and polychromatic source problem. Specifically we provide novel and conclusive results on the stability and uniqueness properties of two and three dimensional inverse problems in CST through an analysis of a disc transform and a generalized spindle torus transform. In the final chapter of the thesis we give a novel analysis of the stability of a spindle torus transform from a microlocal perspective. The practical application of our inversion methods to fields in X-ray and gamma ray imaging are also assessed through simulation work.
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Author contributions

The contributions by the authors to the papers presented in this thesis are stated below:

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Authors: James Webber
All content by James Webber with helpful suggestions and insight provided by William Lionheart.

Three dimensional Compton scattering tomography

Authors: James Webber and William Lionheart
Idea and analysis of the problem by JW and WL. Written by JW. All proofs and simulations by JW.

Micolocal analysis of a spindle transform

Authors: James Webber and Sean Holman
Idea by JW, SH and WL. Written by JW and SH. Proofs and analyses by JW and SH. Simulations by JW.
Chapter 1

Introduction

In geometrical inverse problems in tomography we aim to reconstruct an image of a physical quantity (e.g. density or absorption coefficient) represented by a real valued function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) from its integrals over a set of curves or hypersurfaces \((n - 1)\) manifolds embedded in \( \mathbb{R}^n \), described by a Radon transform

\[
Rf(y) = \int_{H_y} f dS,
\]

where \( y \) parameterises a set of hypersurfaces \( H_y \) in \( \mathbb{R}^n \) and \( dS \) is the surface measure on \( H_y \). For the moment \( f \) is a function for which the above integral makes sense. We will visit the types of functions for which various Radon transforms are defined throughout this thesis. The fundamental questions we ask when we perform an analysis of \( R \) are listed below:

(i) On what classes of functions is \( R \) injective? If \( R \) is injective, is there an explicit left inverse of \( R \)?

(ii) What is the range of \( R \)? Can we give conditions to determine if a given dataset is in the range of \( R \)? Can we fully characterize the range?

(iii) How stable is \( R \) and if \( R \) has a left inverse, how unstable is its left inverse? In what Sobolev spaces is our solution stable, if any?

(iv) How are the support of the function \( f \) and the support of \( Rf \) related?

(v) Can we identify the singularities of a function \( f \) (image edges) in all directions from \( Rf \) and if not, in which directions are the singularities the hardest to identify?
With recent developments in energy sensitive detectors in imaging applications [11, 21], Compton Scattering Tomography (CST) is gaining more relevance, and there are various Radon transforms and imaging ideas already introduced in the literature [48, 46, 41]. In this thesis we introduce a new set of Radon transforms which describe two and three dimensional problems in CST and we provide an analysis of their injectivity, stability and microlocal properties.

1.1 The hyperplane Radon transform

The most classical, well studied and well understood example of a Radon transform is the hyperplane Radon transform, which gives the integrals of a function over hyperplanes [45, page 9]. For \( s \in \mathbb{R} \) and \( \theta \in S^{n-1} \) we define the hyperplane \( H_{s,\theta} = \{ x \in \mathbb{R}^n : x \cdot \theta = s \} \). Then for \( f \in S(\mathbb{R}^n) \) in the Schwartz space on \( \mathbb{R}^n \) (see Appendix A), we define the hyperplane Radon transform

\[
Rf(s,\theta) = \int_{H_{s,\theta}} f \, dS = \int_{\mathbb{R}^n} \delta(s - x \cdot \theta) f(x) \, dx,
\]

(1.2)

where \( \delta \) denotes the Dirac delta function and \( dS \) is the surface measure on \( H_{s,\theta} \).

Here \( Rf \) is defined as a function on the unit cylinder \( Z^n = \mathbb{R} \times S^{n-1} \) in \( \mathbb{R}^{n+1} \).

The hyperplane Radon transform is widely applied in medicine in X-ray CT, emission CT and ultrasound CT, and also has applications in other fields such as astronomy, electron microscopy and nuclear magnetic resonance [19, 34, 16, 10, 62, 17]. The application of the hyperplane Radon transform in two dimensional X-ray CT arises from the Beer-Lambert law [39, 8], which describes the loss in intensity of an incident radiation through a medium

\[
I = I_0 e^{-\int_L f \, dl},
\]

(1.3)

where \( I_0 \) is the initial intensity of a beam travelling along a straight line \( L \), \( I \) is the resulting intensity and \( f \) is the attenuation coefficient of the medium. Here \( dl \) denotes the arc length measure on \( L \). The integral in (1.3) is a value of the X-ray transform [44] of \( f \), which is defined for attenuation coefficients \( f \in S(\mathbb{R}^n) \). In the plane (we are imaging a two dimensional slice of the density which we wish to reconstruct), if we consider a radiation source and a detector which are connected via a line \( L = H_{s,\theta} \) in the plane, then the logarithm of the ratio of
the measured and incident intensities of the beam is a Radon transform of the attenuation coefficient
\[ Rf(s, \theta) = \log \frac{I_0}{T}. \]  
(1.4)

See figure 1.1. This is the basis of X-ray transmission CT. The first generation of acquisition geometries in X-ray CT consist of a single source and detector pair which are translated and rotated about the scanned object. With a full set of translations and rotations of the source and detector about the object, we can determine a full dataset for the two dimensional Radon transform of the attenuation coefficient function \( f \) from the Beer-Lambert law. So we know \( Rf(s, \theta) \) for all \( s \in \mathbb{R} \) and \( \theta \in S^1 \), where \( s \) corresponds to a source detector translation and \( \theta \) corresponds to a rotation. Hence the problem in two dimensional X-ray CT results in reconstructing the attenuation coefficient function from its Radon transform. Later modalities in X-ray CT use multiple sources and detectors, or a single fan beam source and an array of detectors [34, page 77] which serves to reduce the acquisition time.

The backprojection operator of the Radon transform is defined as
\[ R^* g(x) = \int_{S^{n-1}} g(x \cdot \theta, \theta) d\theta, \]  
(1.5)

which takes functions on \( \mathbb{Z}^n \) to functions on \( \mathbb{R}^n \). Given a set of projections \( g = Rf \), \( R^* g(x) \) averages (or backprojects) \( g \) out over all planes passing through \( x \). The backprojection operator \( R^* \) is the formal adjoint of \( R \) [45]. For \( f \in S(\mathbb{R}^n) \) and for \( n \) even, we can reconstruct \( f \) from the data \( g = Rf \) via the formula [44, 45]:
\[ f = \frac{(-1)^{(n-2)/2}(2\pi)^{1-n}}{2} R^* H g^{n-1}, \]  
(1.6)

where \( H \) is the Hilbert transform (see Appendix A) and \( g^{n-1} \) is the derivative of \( g \) with respect to the \( s \) variable. Equation (1.6) can be written alternatively as
\[ f(x) = (-1)^{n/2}(2\pi)^{-n} \int_{S^{n-1}} \text{P.V.} \int_{\mathbb{R}} \left\{ \frac{\partial^{n-1}}{\partial s^{n-1}} Rf(s, \theta) \right\}_{s=x \cdot \theta + t} \frac{dtd\theta}{t}. \]  
(1.7)

where P.V. denotes the Cauchy principle value of the inner integral. If \( n \) is odd,
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Figure 1.1: First generation two dimensional X-ray CT acquisition geometry,

then $f$ can be reconstructed by

$$f = \frac{(-1)^{(n-1)/2}(2\pi)^{1-n}}{2} R^* g^{n-1}$$

(1.8)

or

$$f(x) = \frac{(-1)^{(n-1)/2}(2\pi)^{1-n}}{2} \int_{S^{n-1}} \left\{ \frac{\partial^{n-1}}{\partial s^{n-1}} R f(s, \theta) \right\}_{s=x, \theta} \ d\theta,$$

(1.9)

For the case when $n = 2$, the formula (1.7) becomes

$$f(x) = \frac{1}{4\pi^2} \int_{S^1} \text{P.V.} \int_{\mathbb{R}} \left\{ \frac{\partial}{\partial s} R f(s, \theta) \right\}_{s=x, \theta+t} \ \frac{d\theta dt}{t},$$

(1.10)

which was first derived by Johann Radon in his classical paper [55].

So after we apply the intermediate Hilbert transform and derivative operator
(a filter) to the data \( g \), our image \( f \) can then be reconstructed by backprojecting the filtered data over \( \mathbb{R}^n \). This is the basis of the filtered back-projection algorithm [34, page 60] used extensively in medical and industrial imaging and security screening [60, 58, 43].

In [29] Helgason provides a full characterization of the range of the Radon transform on Schwartz functions in \( \mathbb{R}^n \).

**Definition 1.** Let \( S(Z^n) \) denote the set of Schwartz functions on the cylinder (see Appendix A). Then we denote by \( S_H(Z^n) \) the vector space of even functions in \( \varphi \in S(Z^n) \) which for any \( k \in \mathbb{Z}^+ \) there exists a homogeneous polynomial \( P_k \) of degree \( k \) such that

\[
\int_{-\infty}^{\infty} \varphi(s, \theta) s^k ds = P_k(\theta), \quad \theta \in S^{n-1}.
\]

Helgason proves the equality \( R(S(\mathbb{R}^n)) = S_H(Z^n) \), which is to say that, given some data \( g \) on \( Z^n \), there exists a function \( f \in S(\mathbb{R}^n) \) such that \( g = Rf \) if and only if the \( k^{th} \) moments of \( g \) in \( s \) are polynomials of degree \( k \) in \( \theta \). Further characterization of the range of the Radon transform is given by Helgason’s support theorem [29], where he explains the relation between the support of a function \( f \in S(\mathbb{R}^n) \) and the support of its Radon transform. We will visit this result later in the thesis in chapter 2. The stability and microlocal properties of the hyperplane Radon transform will be discussed later in the introduction in sections 1.6 and 1.7.

### 1.2 The Funk–Radon transform

Another classical example of a Radon transform is the Funk–Radon transform, which can be defined for \( f \in C(S^2) \) and \( v \in S^2 \) as

\[
R_M f(v) = \int_{C_v} f ds,
\]

where \( ds \) is the arc length measure on the circle \( C_v = \{ x \in S^2 : x \cdot v = 0 \} \).

The Funk–Radon transform defines the integral of a function on the sphere over great circles (i.e. the intersection of a plane through the origin with \( S^2 \)) and was first considered by Paul Funk in [24]. The Funk–Radon transform on even functions on the sphere and the hyperplane Radon transform in two dimensions share an equivalence through the gnomonic projection (extensively used in map projections) which maps great semicircles on the hemisphere to straight lines on
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a plane. This equivalence is formalised in [51, page 47]. So we could say that Funk was the first to consider the two dimensional hyperplane Radon transform, just in a different coordinate system to Radon.

Let \( Y_l^m \) be a spherical harmonic (see Appendix A). Funk [24], for functions \( f \in L^2(S^2) \), showed that

\[
R_M f(v) = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} P_l(0) f_{lm} Y_l^m(v),
\]

where \( P_l \) is a Legendre polynomial of degree \( l \) (see Appendix A) and the \( f_{lm} \) are the harmonic coefficients of \( f \). As \( P_l(0) = 0 \) for odd \( l \), the odd coefficients of \( R_M f \) vanish for odd \( l \) and hence the odd functions are contained in the null space of \( R_M \). For even \( l \), \( f_{lm} = \frac{(R_M f)_l}{P_l(0)} \), where the \((R_M f)_l\) denote the harmonic coefficients of \( R_M f \). This is possible since \( P_l(0) \neq 0 \) for all even \( l \). Hence any even function \( f \in L^2(S^2) \) can be reconstructed explicitly from its Funk–Radon transform by the formula

\[
f(x) = \sum_{l \text{ even}} \sum_{|m| \leq l} f_{lm} Y_l^m(x)
\]

\[
= \sum_{l \text{ even}} \sum_{|m| \leq l} \left( \frac{(R_M f)_l}{P_l(0)} \right) Y_l^m(x)
= \sum_{l \text{ even}} \sum_{|m| \leq l} \frac{1}{P_l(0)} (-1)^m \left[ \int_{S^2} (R_M f) Y_l^{m-m} d\Omega \right] Y_l^m(x),
\]

where \( d\Omega \) is the surface measure on \( S^2 \).

We define the shifted backprojection operator of the Funk–Radon transform as

\[
\tilde{R}_M g(p, x) = \frac{1}{2\pi \cos p} \int_{\|v\|=1, v \cdot u = \sin p} g(v) dv.
\]

This gives the average value of \( g \) over all great circles at an arc distance \( p \) from \( x \). The backprojection (or dual) operator of \( R_M \) is defined by \( R_M^* g(x) = \tilde{R}_M g(0, x) \) (this gives the average value of \( g \) over great circles which intersect \( x \)). In [30, page 66] Helgason gives the alternate inversion formula for the Funk–Radon transform

\[
f(x) = \frac{1}{2\pi} \left\{ \frac{d}{du} \int_0^u \tilde{R}_M(R_M f)(\cos^{-1} v, x) v(u^2 - v^2)^{-\frac{1}{2}} dv \right\}_{u=1}.
\]

So far we have reviewed inversion formulae of two types, inversion by some kind of
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filtered backprojection (Helgason’s inversion formula for the Funk–Radon transform and Radon’s inversion formula for the hyperplane Radon transform) and inversion by decomposition into some orthogonal basis of special functions (Funk’s inversion formula for the Funk–Radon transform). Later in this thesis in chapter 3, using some of the same ideas as Funk, we provide an explicit inversion formula for a spindle torus transform. Our inversion formula is a hybrid of the two types considered so far. That is, our inversion is obtained by first taking the components of the spindle torus transform in a basis of spherical harmonics, and then solving a set of one dimensional integral equations by applying an inverse integral operator.

1.3 Radon transforms over a general class of hypersurfaces in \( \mathbb{R}^n \)

Inversion formulae for Radon transforms which define the integrals over more general classes of curves and surfaces are considered in the literature [52, 49, 13, 46]. We review the geometry considered by Palamodov in [52].

Let \( X \) and \( \Sigma \) be smooth \( n \)-dimensional manifolds with \( n > 1 \), let \( Y \) be a smooth hypersurface in \( X \times \Sigma \) and \( p : Y \rightarrow X \) and \( \pi : Y \rightarrow \Sigma \) be natural projections. Let us suppose there exists a \( \Phi \in C^\infty(X \times \Sigma) \) such that \( Y = \{ (x, \sigma) : \Phi(x, \sigma) = 0 \} \), where \( \sigma = (s, \theta) \) and we use the semicolon notation to distinguish this, \( s \in \mathbb{R}, \theta \in S^{n-1} \), where \( \omega \) is a smooth function on \( X \times S^{n-1} \).

Palamodov considers in more detail the generating functions which are resolved and regular.

Definition 2. Let \( T^*(X) \) and \( N^*(Y) \) denote the cotangent and conormal bundles (see Appendix A) of \( X \) and \( Y \) respectively. A generating function \( \Phi \) is said to be resolved and regular if the following conditions are satisfied:

1. \( \Sigma = Z^n \) and \( \Phi \) is of the form \( \Phi(x; s, \theta) = \omega(x, \theta) - s \) (here \( \sigma = (s, \theta) \) and we use the semicolon notation to distinguish this), \( s \in \mathbb{R}, \theta \in S^{n-1} \), where \( \omega \) is a smooth function on \( X \times S^{n-1} \).

2. The map \( \pi \) has rank \( n \) and the mapping \( P : N^*(Y) \rightarrow T^*(X) \) defined by \( P(x, \sigma, v_x, v_\sigma) = (x, v_x) \) is a local diffeomorphism.

3. The equations \( \Phi(x, \sigma) = \Phi(y, \sigma) \) and \( d_\sigma \Phi(x, \sigma) = d_\sigma \Phi(y, \sigma) \) are satisfied for no \( x \neq y \in X, \sigma \in \Sigma \).
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Condition 2. in Definition 2 implies that for any $\sigma \in \Sigma$, $Y(\sigma) = \pi^{-1}(\sigma) = \{ x : \Phi(x, \sigma) = 0 \}$ is a smooth hypersurface in $X$ and that for any point $x \in X$ and tangent hyperplane $h \subset T_x(X)$ there exists a locally unique hypersurface $Y(\sigma)$ through $x$ which is tangent to $h$. So Palamodov considers sets of hypersurfaces which intersect every $x \in X$ orthogonal to every direction $\xi \in T^*_xX$, where $(x, \xi) \in T^*X$.

Let $X$ be a submanifold of $\mathbb{R}^n$. Then the Funk–Radon transform generated by a resolved and regular $\Phi$ for $f \in C_0(X)$ is defined as

$$M_\Phi f(\sigma) = \int_{Y(\sigma)} f \frac{dS}{|\nabla_x \Phi(x, \sigma)|},$$

(1.16)

where $dS$ denotes the surface measure on $Y(\sigma)$. If the gradient modulus is separable, that is, $|\nabla_x \Phi(x, \sigma)| = m(x)\mu(\sigma)$ for some positive continuous functions $m$ in $X$ and $\mu$ in $\mathbb{Z}^n$, then the Funk–Radon transform data is equivalent to the Radon transform

$$R_\Phi f(\sigma) = \int_{Y(\sigma)} f dS = \mu(\sigma) M_\Phi (mf)(\sigma).$$

(1.17)

So in this case the reconstruction from $R_\Phi f$ data can be obtained through an inversion of $M_\Phi$.

For a resolved, regular generating function $\Phi = \omega - s$ let

$$\Theta_n(x, y) = \lim_{\epsilon \to 0} \int_{S^{n-1}} \frac{d\theta}{(\omega(x, \theta) - \omega(y, \theta) - i\epsilon)^n}.$$  

(1.18)

Then for $f \in L^2(X)$ of compact support the reconstruction of $f$ from the data $M_\Phi f$ is given by the formula

$$f(x) = -\frac{1}{(2\pi)^{n-1}D_n(x)} \int_{S^{n-1}} \text{P.V.} \int_{\mathbb{R}} \frac{d\theta}{\Phi(x; s, \theta)} M_\Phi f(s, \theta) \frac{ds d\theta}{\Phi(x; s, \theta)}$$

(1.19)

if $n$ is even and $\Re \Theta_n(x, y) = 0$ for any $x \neq y \in X$. Here P.V. denotes the Cauchy principal value of the inner integral. If $n$ is odd and $\Im \Theta_n(x, y) = 0$ for $x \neq y$ then $f$ can be reconstructed by

$$f(x) = \frac{1}{2(2\pi)^{n-1}D_n(x)} \int_{S^{n-1}} \left\{ \frac{\partial^{n-1}}{\partial s^{n-1}} M_\Phi f(s, \theta) \right\}_{s = \omega(x, \theta)} d\theta,$$

(1.20)
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where

\[ D_n(x) = \frac{1}{|S^n|} \int_{S^{n-1}} \frac{d\theta}{|\nabla_x \omega(x, \theta)|^n}. \]  

(1.21)

The above inversion result amounts to a generalization of the inversion formulae (1.7) and (1.8) and a further generalization of the inversion results given by Radon in [55]. We can see this as if we set \( \Phi(x; s, \theta) = x \cdot \theta - s \), then the conditions of Definition 2 hold trivially and the inversion formulae (1.19) and (1.20) reduce to (1.7) and (1.8). The inversion formulae given by Palamodov are shown to hold for any \( n \)-dimensional manifold \( X \). We have reviewed the Euclidean case when \( X \) is a submanifold of \( \mathbb{R}^n \).

In some sense the Funk–Radon transform must fit into Palamodov’s framework given the equivalence of the Funk–Radon transform and the hyperplane Radon transform in two dimensions. To apply the general inversion formulae of Palamodov to Funk–Radon data we just have to be careful about how our transforms are defined. Condition 1. of Definition 2 requires that \( \Sigma = \mathbb{Z}^n \). So although it is more natural to parameterise great circles by some \( v \in S^2 \), we can define the Funk–Radon transform of a function as a function on \( \mathbb{Z}^2 \) through the standard projection \( S^2 \to \mathbb{Z}^2 \) (again used in map projections).

An example of a generating function which does not satisfy Palamodov’s conditions for invertibility is considered later in this thesis in chapter 4, when we provide an analysis of the microlocal properties of a spindle torus transform. Specifically we show that the left projection of the spindle torus transform drops rank and hence its generating function fails to satisfy condition 2. of Definition 2.

### 1.4 The spherical Radon transform

A Radon transform of great interest in CST, which is the field of focus of this thesis, as well as other fields such as Photo Acoustic Tomography (PAT), Thermo Acoustic Tomography (TAT) and Ultra Sound Tomography (UST) [46, 48, 71, 70, 36, 5], is the spherical Radon transform, which defines the integrals of a function over spheres in \( \mathbb{R}^n \).

The spherical Radon transform can be defined for \( f \in C_0(\mathbb{R}^n) \) as

\[ R_S f(r, y) = \int_{S(r, y)} f \, dA, \]  

(1.22)
where $y \in \mathbb{R}^n$, $r \in \mathbb{R}^+$ and $dA$ is the surface measure on $S(r, y) = \{x \in \mathbb{R}^n : |y - x| = r\}$. For the two dimensional case of the spherical Radon transform, which gives the integrals over circles in the plane, a full classification of its injectivity on the set of compact supported continuous functions is given by Agranovsky and Quinto in [2]. They give the definition of a set of injectivity of $R_S$ for $f \in C_0(\mathbb{R}^n)$:

**Definition 3.** The Radon transform $R_S$ is said to be injective on a set $A \subseteq \mathbb{R}^n$ if for any $f \in C_0(\mathbb{R}^n)$

$$R_S f(r, y) = 0, \forall r \in \mathbb{R}^+, y \in A \implies f \equiv 0. \quad (1.23)$$

One can see how we could generalize definition (3) to define injectivity sets for any Radon transform.

In [2] the injectivity sets for the spherical Radon transform in two dimensions are shown to be those which do not lie in a finite union of lines unioned with a finite set. Let $\Sigma_N = \bigcup_{k=0}^{N-1} \{ t \left( \cos \frac{\pi k}{N}, \sin \frac{\pi k}{N} \right) : t \in \mathbb{R} \}$ and let $M(n)$ be the group of rigid motions in $\mathbb{R}^n$. Let $\varphi : M(n) \times \mathbb{R}^n \to \mathbb{R}^n$ define a left action of $M(n)$ on $\mathbb{R}^n$ in the natural way. Then we have:

**Theorem 1.** $A \subseteq \mathbb{R}^2$ is a set of injectivity of $R_S$ in two dimensions if $A$ is not contained in any set of the form $\varphi(\omega, \Sigma_N) \cup F$, where $\omega \in M(2)$ and $F \subseteq \mathbb{R}^2$ is a finite set.

So if we know the integrals of a function $f \in C_0(\mathbb{R}^2)$ over circles of all radii with centres on a set of points not in $\varphi(\omega, \Sigma_N) \cup F$ (a rigid motion $\omega \in M(n)$ can be written as the composition of a translation and an orthogonal transformation), then $f$ is uniquely determined by the data $R_S f$.

The reconstruction problem from the full spherical Radon transform data is formally overdetermined (an $n+1$ dimensional dataset), but we typically consider $n$ dimensional subsets of the full dataset for the spherical Radon transform in application. For example, in TAT a ring of transducers placed around the patient record acoustic signals (circular waves) travelling through the body after the patient is exposed to a short pulse of electromagnetic radiation. See figure 1.2. Let $\Omega_n$ denote the open unit ball in $\mathbb{R}^n$ centred at the origin. Here the problem reduces to recovering an image of the patient $f \in C_0(\Omega_2)$ from $R_S f$ known for all radii and $y \in S^1$ [5]. Since $S^1 \cap (\varphi(\omega, \Sigma_N) \cup F) = F_1 \neq S^1$, where $F_1$ is a finite set, Theorem 1 holds and the image is uniquely determined by the TAT data.
In [5] an explicit inversion formula is provided for the TAT acquisition geometry displayed in figure 1.2, where it is shown that in fact only a limited number of circle radii are needed to reconstruct a function $f$ supported on an annulus.

An $n$ dimensional TAT problem is considered in [38], where it is shown that a function $f \in C^1_0(\Omega_n)$ can be reconstructed explicitly from its integrals over spheres with centres on $S^{n-1}$ (an $n$ dimensional generalization of the modality considered in [5]), and in [15] Cormack and Quinto show that any $f \in C^\infty(\mathbb{R}^n)$ can be reconstructed explicitly from its integrals over spheres containing the origin (we shall consider the two dimensional case of this transform in more detail later in the thesis in chapter 2).

We now review a limited data case of the spherical Radon transform in two dimensions of particular interest in Compton scattering tomography.
1.5 The circular arc Radon transform

An example of a Radon transform in CST is the circular arc Radon transform, which was first introduced by Nguyen and Truong in [46] and can be defined as

\[
R_C f(r, \phi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho \sqrt{\frac{1 + r^2}{1 + r^2 \cos^2 \varphi}} F(\rho, \varphi + \phi) \mid_{\rho=\sqrt{r^2 \cos^2 \varphi + 1 - r \cos \phi}} \, d\varphi, \tag{1.24}
\]

where \( F(\rho, \theta) = f(\rho \cos \theta, \rho \sin \theta) \) is the polar form of \( f \in C_0^\infty(B_{\epsilon_1,\epsilon_2}^2) \) for some \( 0 < \epsilon_1 < \epsilon_2 < 1 \), where \( B_{\epsilon_1,\epsilon_2}^2 = \{ x \in \mathbb{R}^2 : \epsilon_1 < |x| < \epsilon_2 \} \). This gives the integral of \( f \) over the circular arc which is the intersection of the circle \( \{(x, y) \in \mathbb{R}^2 : (x - r \cos \phi)^2 + (y - r \sin \phi)^2 = 1 + r^2\} \) and the unit disc. See figure 1.3. There has been recent interest in circular arc transforms due to their application in CST [46, 48, 50].

The Compton effect, first observed by Arthur Compton [12], describes the loss in energy of a photon when scattering from charged particles (usually electrons). The equation for the loss in energy is given by

\[
E_s = \frac{E}{1 + (E/E_0) (1 - \cos \omega)}, \tag{1.25}
\]

where \( E_s \) is the scattered energy, \( E \) is the initial energy, \( \omega \) is the scattering angle and \( E_0 \approx 511 \text{keV} \) is the electron rest energy. So if the imaging source is monochromatic (\( E \) is fixed) and the detector is energy sensitive (we can measure intensity at a given energy \( E_s \)), then for each scattered energy we measure, the scattering angle \( \omega \) is fixed and hence the locus of scattering points is a circular arc with its end points at the source and detector points.

The scattering distribution of a photon scattering from a free electron at rest is given by the Klein–Nishina formula [35]

\[
\frac{d\sigma}{d\Omega}(E_s, \omega) = \frac{r_0^2}{2} \left( \frac{E_s}{E} \right)^2 \left( \frac{E_s}{E} + \frac{E}{E_s} - 1 + \cos^2 \omega \right). \tag{1.26}
\]

Neglecting the attenuation of the incoming and scattered radiation, we can model the scattered intensity with energy \( E_s \) measured at the detector as the integral of the electron density over a circular arc

\[
I = I_0 \frac{d\sigma}{d\Omega} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S d\Omega R_C f, \tag{1.27}
\]
Figure 1.3: Nguyen and Truong’s CST acquisition geometry. $s$ and $d$ denote the source and detector points. The circular arc is the intersection of the scanning region (the unit disc) and the circle with centre $(-r,0)$ and radius $R = \sqrt{1 + r^2}$.

where $I_0$ is the initial intensity, $S$ is the incoherent scattering function [32] (this corrects for the vibrations of the atomic electrons), $d\Omega$ is the solid angle subtended by the source and detector and $f$ is the electron density. The above model (1.27) also assumes that only single scattering events occur. So with a full set of rotation angles $\phi$ and scattered energies $E_s$ we can determine a full data set for $R_C f$ (here $r$ is determined by $E_s$), and so in two dimensional CST, we aim to reconstruct the electron density function from its circular arc transform.

Given the rotational invariance of the circular arc transform we can reduce equation (1.24) to a set of one dimensional integral equations to solve for the
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Fourier coefficients of \( f \):

\[
R_C f_l(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho \sqrt{\frac{1 + r^2}{1 + r^2 \cos^2 \varphi}} F_l(\rho) e^{-il\varphi} \bigg|_{\rho = \sqrt{r^2 \sin^2 \varphi + 1 - r \cos \varphi}} \ d\varphi, \tag{1.28}
\]

where \( l \in \mathbb{Z} \) and \( F_l \) and \( R_f_l \) are Fourier coefficients of \( F \) and \( Rf \). This idea is seen in other publications [16, 5, 6] and in higher dimensions in [15]. In fact, for any rotationally invariant (about the origin) Radon transform acting on the domain of smooth functions on \( \mathbb{R}^n \) such a decomposition exists using a basis of higher dimensional spherical harmonics [15, 59].

Nguyen and Truong give the explicit solution to equation (1.28) using the properties of a Chebyshev transform studied by Cormack [16, 13] and Li [40]:

\[
F_l(\rho) = \frac{1 + \rho^2}{2\pi \rho^2} \left\{ \int_{t}^{\infty} \frac{T_l(q/t)}{\sqrt{q^2 - t^2}} dq \left( \frac{R_C f_l(1/q)}{\sqrt{1 + q^2}} \right) d\varphi \right\}_{t = \frac{2\rho}{1 - \rho^2}}, \tag{1.29}
\]

where \( T_l \) is a Chebychev polynomial of the first kind order \( l \) (see Appendix A).

Let us define:

\[
\frac{1}{|x|^2} \hat{f} \left( \frac{x}{|x|^2} \right) = f \left( \sqrt{|x|^2 + 1} - |x| \cdot \frac{x}{|x|} \right) \frac{|x|}{\sqrt{1 + |x|^2}}. \tag{1.30}
\]

Then it follows that

\[
R_C f(r, \varphi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho \sqrt{\frac{1 + r^2}{1 + r^2 \cos^2 \varphi}} F_l(\rho, \varphi + \phi) \bigg|_{\rho = \sqrt{r^2 \sin^2 \varphi + 1 - r \cos \varphi}} \ d\varphi
\]

\[
= \sqrt{1 + r^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{r^2 \cos^2 \varphi} \hat{F} \left( \frac{1}{r \cos \varphi}, \varphi + \phi \right) d\varphi
\]

\[
= \sqrt{1 + r^2} \frac{1}{r} R \hat{f} \left( \frac{1}{r}, \phi \right), \tag{1.31}
\]

where \( \hat{F} \) is the polar form of \( \hat{f} \) and \( R \hat{f} \) is the polar form of the two dimensional hyperplane Radon transform of \( \hat{f} \) [13, 14]. So the hyperplane Radon transform in two dimensions and the circular arc transform are equivalent via the diffeomorphism \( x \rightarrow \left( \sqrt{\frac{1}{|x|^2} + 1} - \frac{1}{|x|} \right) \cdot \frac{x}{|x|} \) and hence a reconstruction of the function \( f \) from \( R_C \) can be obtained through an inversion of \( R \). This equivalence also has implications on the stability of \( R_C \) which we will discuss later in section 1.6.
1.6 Sobolev space estimates

The use of Sobolev space theory as a tool in stability analysis of Radon transforms is well known in the literature \[44, 50, 26\]. The intuitive idea “compares” the stability of the inversion of a Radon transform to the stability of taking a number of derivatives.

For example, let us consider the Abel transform

$$ A f(y) = \int_0^y \frac{f(x)}{\sqrt{y^2 - x^2}} \, dx, \quad y \in [0, b] $$

defined for \( f \in C([0, b]) \), where \( b > 0 \). Then we can show that

$$ A^2 f(z) = \int_0^z \int_0^y \frac{f(x)}{\sqrt{z^2 - y^2} \sqrt{y^2 - x^2}} \, dx \, dy $$

$$ = \int_0^z \int_x^z \frac{1}{\sqrt{z^2 - y^2} \sqrt{y^2 - x^2}} \, dy \, f(x) \, dx $$

$$ = \frac{\pi}{2} \int_0^z f(x) \, dx. $$

So we can invert the operator \( A^2 \) by taking a derivative with respect to \( z \), and hence the inversion of \( A \) is as stable as taking \( 1/2 \) a derivative (its singular values decay at the same rate as a square root).

We now give the definitions of the Sobolev spaces \[1, \text{page 44}\].

**Definition 4.** For integers \( k \geq 0 \) and \( 1 \leq p \leq \infty \) the Sobolev space \( W^{k,p}(\mathbb{R}^n) \) is defined as

$$ W^{k,p}(\mathbb{R}^n) = \{ \text{tempered distributions } f : D^\beta f \in L^p(\mathbb{R}^n), |\beta| \leq k \}, $$

where the derivative \( D^\beta \) is taken in the weak sense and \( \beta \) is a multi–index.

The spaces \( W^{k,p} \) are Banach spaces with the norms

$$ \| f \|_{W^{k,p}(\mathbb{R}^n)}^p = \sum_{|\beta| \leq k} \| D^\beta f \|_{L^p(\mathbb{R}^n)}^p, \quad 1 \leq p < \infty $$

and

$$ \| f \|_{W^{k,\infty}(\mathbb{R}^n)} = \sum_{|\beta| \leq k} \| D^\beta f \|_{L^\infty(\mathbb{R}^n)}. $$
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We now state part of the Sobolev embedding theorem [1, page 97], which gives important embeddings between the Sobolev spaces $W^{k,p}$:

**Theorem 2** (Sobolev embedding theorem). If $n > kp$, then for every $q$ satisfying $p \leq q \leq np/(n-kp)$, then $W^{j+k,p}(\mathbb{R}^n) \hookrightarrow W^{j,q}(\mathbb{R}^n)$ for all $j \in \mathbb{Z}^+$. That is, under the given conditions, there exists a constant $C(n,k)$ such that

$$\forall f \in W^{j+k,p}(\mathbb{R}^n), \quad \|f\|_{W^{j,q}(\mathbb{R}^n)} \leq C(n,k) \|f\|_{W^{j+k,p}(\mathbb{R}^n)}. \quad (1.37)$$

Suppose that $kp > n$. Then $W^{j+k,p}(\mathbb{R}^n) \hookrightarrow C^j_B(\mathbb{R}^n)$ (see appendix A for the definition of $C^j_B$) for all $j \in \mathbb{Z}^+$.

The Sobolev embedding theorem asserts the existence of embeddings $W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{j,p}(\mathbb{R}^n)$ and $W^{k,p}(\mathbb{R}^n) \hookrightarrow C^j_B(\mathbb{R}^n), \forall j \leq k$. The particular case of $j = 0$ gives embeddings into $L^q$ space; $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$. Embedding theorems are an essential tool in the study of Radon transforms, as when we analyse the stability of Radon transforms, we aim to find bounds on our solution $f$ in terms of bounds on our data $Rf$ in some suitable norms. Since Radon transforms are smoothing operators (i.e. the Radon transform of a function in one Sobolev space belongs to a Sobolev space of higher order), bounds on our solution in $L^p$ are not possible and we aim to find a bound in some Sobolev space $W^{k,p}$.

The Sobolev spaces of particular use in this thesis are those for which $p = 2$, denoted by $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$. The definition of the Sobolev spaces $H^k$ can be extended to all $k = \alpha \in \mathbb{R}$:

**Definition 5.** We define the Sobolev spaces of order $\alpha \in \mathbb{R}$:

$$H^\alpha(\mathbb{R}^n) = \{\text{tempered distributions } f : \langle \xi \rangle^\alpha \hat{f} \in L^2(\mathbb{R}^n)\}, \quad (1.38)$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ and $\hat{f}$ denotes the Fourier transform of $f$ (see Appendix A).

The spaces $H^\alpha$ are Hilbert spaces [1, page 47] with inner product

$$\langle f, g \rangle_{H^\alpha(\mathbb{R}^n)} = \langle \langle \xi \rangle^\alpha \hat{f}, \langle \xi \rangle^\alpha \hat{g} \rangle_{L^2(\mathbb{R}^n)} \quad (1.39)$$

and norm

$$\|f\|_{H^\alpha(\mathbb{R}^n)}^2 = \langle f, f \rangle_{H^\alpha(\mathbb{R}^n)}. \quad (1.40)$$
For $\alpha = k \in \mathbb{Z}$, the norm (1.40) is equivalent to the norm (1.35). This follows from a simple application of the Plancherel theorem and from the properties of derivatives of Fourier transforms.

As Radon transforms are smoothing operators, inversions from Radon transform data are ill–posed (the error in our solution does not depend continuously on the error in our data). There are varying degrees to which a Radon transform is ill–posed, which we shall now explain. For a Radon transform $R$, if for some $\alpha \in \mathbb{R}$, $\beta \geq 0$ we can bound our solution $f$ in $H^\alpha$:

$$\|f\|_{H^\alpha} \leq \|Rf\|_{H^{\alpha+\beta}}$$

(1.41)

in terms of bounds on $Rf$ in $H^{\alpha+\beta}$ (the singular values of $R$ decay at the rate of a polynomial degree $\beta$), then we say that the inverse problem described by $R$ is mildly ill–posed. If no such bound exists (the singular values of $R$ decay at rate faster than any polynomial, e.g. exponential), then we say that the problem is severely ill–posed.

An example of a Radon transform whose stability is well understood is the hyperplane Radon transform $R$. We define Sobolev norms for functions $g$ on the cylinder $\mathbb{Z}^n$ as

$$\|g\|^2_{H^\alpha(\mathbb{Z}^n)} = \int_{S^{n-1}} \int_{\mathbb{R}} (1 + \sigma^2)^\alpha |\hat{g}(\sigma,\phi)|^2 d\sigma d\phi,$$

(1.42)

where the Fourier transform of $g$ is taken in the first variable. Then we have the Sobolev space estimates for $R$ [44, page 42]:

$$c(\alpha,n) \|f\|_{H^\alpha(\mathbb{R}^n)} \leq \|Rf\|_{H^{\alpha+(n-1)/2}(\mathbb{Z}^n)} \leq C(\alpha,n) \|f\|_{H^\alpha(\mathbb{R}^n)}$$

(1.43)

for functions $f \in C^\infty_0(\Omega_n)$, where $\Omega_n$ is the open unit ball in $\mathbb{R}^n$ and $C(\alpha,n)$ and $c(\alpha,n)$ are positive constants. We can obtain bounds for the least squares error in $f$ in terms of the least squares error in $Rf$ using interpolation inequalities [44, page 94]:

$$\|f\|_{L^2(\mathbb{R}^n)} \leq c(\beta,n) \epsilon^{1/n} \rho^{1-1/n}$$

(1.44)

where $\|Rf\|_{L^2(\mathbb{Z}^n)} \leq \epsilon$ and $\|f\|_{H^{1/2}(\mathbb{R}^n)} \leq \rho$. So we would say that the problem of reconstructing $f \in C^\infty_0(\Omega_n)$ from its Radon transform $Rf$ known for all $(s,\theta) \in \mathbb{Z}^n$ is mildly ill–posed. The same can not be said in general when we have limited data [53, 3]. That is, we only know $Rf$ for $(s,\theta) \in U$, where $U \subset \mathbb{Z}^n$ and the
compliment of $U$ in $\mathbb{Z}^n$ has non–zero measure in $\mathbb{Z}^n$ (i.e. $\int_{\mathbb{Z}^n \setminus U} dS \neq 0$, where $dS$ is the surface measure on $\mathbb{Z}^n$). For example, in limited angle tomography [18]. Limited data cases of the hyperplane Radon transform will be covered in more detail later in our introduction to microlocal analysis in section 1.7.

Under a change of coordinates, the Sobolev norms (1.40) are equivalent [1, page 63]:

**Theorem 3.** Let $\Omega' \subset \mathbb{R}^n$ and let $f \in C_0^\infty(\mathbb{R}^n)$ have support in an open set $\Omega \subset \mathbb{R}^n$. Let $u: \Omega' \to \Omega$ be a diffeomorphic map and let us extend the domain of $f \circ u$ to $\mathbb{R}^n$ such that $(f \circ u)(x) = 0$ for $x \notin \Omega'$. Then for any $\alpha \in \mathbb{R}$

$$c(n, \alpha)\|f \circ u\|_{H^\alpha(\mathbb{R}^n)} \leq \|f\|_{H^\alpha(\mathbb{R}^n)} \leq C(n, \alpha)\|f \circ u\|_{H^\alpha(\mathbb{R}^n)}$$

(1.45)

for positive constants $C(n, \alpha)$ and $c(n, \alpha)$.

We also have the theorem [44, page 204]:

**Theorem 4.** Let $v \in C_0^\infty(\mathbb{R}^n)$. Then, the map $f \to vf$ is bounded in $H^\alpha$.

We can use the above theorems to derive Sobolev space estimates and stability results for Radon transforms which are equivalent via diffeomorphism. For example, a circular arc (section 1.5) is diffeomorphic to a straight line in the plane via $u(x) = \left(\sqrt{\frac{1}{|x|^2} + 1} - \frac{1}{|x|}\right) \cdot \frac{x}{|x|}$ and $\tilde{f} = v(f \circ u)$ (equation (1.30)), where $v(x) = \frac{1}{\sqrt{|x|^2 + 1}}$, is equivalent to $f$ on Sobolev norms $H^\alpha$ for $\alpha \in \mathbb{R}$ provided $f$ is smooth and of compact support. Hence the circular arc transform and the hyperplane Radon transform in two dimensions share the same stability properties on $C_0^\infty(\mathbb{R}^n)$ and we can use (1.43) to obtain Sobolev space estimates for the circular arc transform. Such estimates have also been derived by Palamodov in [50]. This idea will be applied often throughout this thesis and will be used to obtain stability and injectivity results for a disc transform and a spindle torus transform.

### 1.7 Microlocal analysis

Radon transforms which define the integrals over a set of hypersurfaces $H$ in $X \subset \mathbb{R}^n$ detect the singularities of a function at points $x \in X$ intersected by $h \in H$ in directions normal to $h$ at $x$. Microlocal analysis studies the relationship
between the singularities of a function and those of its Radon transform, and
the ability of Radon transforms to detect singularities of a function in a given
direction. The literature covers many applications of microlocal analysis of Radon
transforms [53, 57, 23, 27, 4].

The singularities of a function and the directions in which they occur are
described by the wavefront set [20, page 16]:

**Definition 6.** Let \( A \in \mathcal{D}'(X) \) (see appendix A), where \( X \subset \mathbb{R}^n \) is open. Then
\((x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus \{0\})\) is in the complement of the wavefront set of \( A \) (denoted by \( \text{WF}(A) \)) if and only if there exists a neighbourhood \( U \) of \( x_0 \) and \( V \) of \( \xi_0 \) such that for every \( \phi \in C_0^\infty(U) \) and \( N \in \mathbb{R} \) there exists a constant \( C_N \) such that

\[
\left| \hat{(\phi A)}(\lambda \xi) \right| \leq C_N \lambda^{-N} \tag{1.46}
\]

for all \( \xi \in V \) and \( \lambda > 1 \).

The wavefront set of a distribution on \( X \) can be thought of as a subset of the cotangent bundle \( T^*X \). This follows the intuitive idea that the elements of \( \text{WF}(A) \) are the point–normal vector pairs on submanifolds of \( X \) where \( A \) has
singularities. For example if \( A \) is the characteristic function on the unit ball in \( \mathbb{R}^3 \), then

\[
\text{WF}(A) = \{(x, \xi) \in T^*\mathbb{R}^3 : x \in \mathbb{S}^2, |x \times \xi| = 0\} \tag{1.47}
\]

That is the wavefront set of the characteristic function on a ball is the set of
points on a sphere paired with the normal vectors to the sphere.

In a given domain \( X \subset \mathbb{R}^n \), Radon transforms \( R \) which define the integrals
over a set of hypersurfaces which intersect a neighbourhood of a point \( x \in X \)
orthogonal to the directions in a neighbourhood of \( \xi \in \mathbb{S}^{n-1} \), detect singularities
at \((x, \xi) \in T^*X \). So for those distributions \( f \) with \((x, \xi) \in \text{WF}(f)\) the singularity
of \( f \) at \((x, \xi) \) is more stably reconstructed from \( Rf \), and conversely, the singularities of \( f \) in \( \text{WF}(f) \) which are not cut orthogonally in a small neighbourhood
by hypersurfaces are harder to reconstruct. In [53] Quinto formalises this idea
for the hyperplane Radon transform in two dimensions. Reviewing an example
given by Quinto, let us consider the problem of reconstructing a function from
limited angle Radon transform data. That is, let us consider the problem of re-
constructing \( f \) from \( Rf \) know for all \( s \in \mathbb{R} \) and \( \theta \in U \), where \( U \subset [0, 2\pi] \) is open
and \( U = (U + \pi) \mod 2\pi \). In this case the solution is still unique [44]. However,
the only singularities of \( f \) that are reconstructed stably are those with directions
in $U$. That is, the wavefronts $(x, \xi) \in WF(f)$ which are stably reconstructed are those where $\xi = \Theta, \theta \in U$, where $\Theta = (\cos \theta, \sin \theta)$ and when $\xi = \Theta$ for $\theta \notin U$ the singularity is not stably resolved.

For example let us consider reconstructing the characteristic function

$$f(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{otherwise,} \end{cases}$$

(1.48)

where $D = \{x \in \mathbb{R}^2 : |x| < 1\}$, from $Rf$ known for $s \in \mathbb{R}$ and $\theta \in [0, \pi/2]$. Then the singularities which are resolved stably are in $S = \{(x, \xi) : x = (\cos \psi, \sin \psi), \xi = (\cos \psi, \sin \psi), 0 < \psi < \pi/2\}$ and the wavefronts $(x, \xi) \in WF(f) \setminus S$ are not detected. See figures 1.4 and 1.5.

Let us now consider reconstructing the function $f_1$ represented in figure 1.6 of three oscillating layers from $Rf_1$ known for $s \in \mathbb{R}$ and $\theta \in [\pi/4, 3\pi/4]$. The wavefront set of $f_1$ consists of the set of points on a set of lines and two directions $\xi_1 = (0, 1)$ and $\xi_2 = (1, 0)$ (except at the corners where the wavefront set is in all directions), the latter in the direction of the oscillations. Here with the limitations to the data, the cone of directions $\{(\cos \psi, \sin \psi) : \psi \in [-\pi/4, \pi/4]\}$ surrounding $\xi_2 = (1, 0)$ are not cut orthogonally by hyperplanes and the jump discontinuities between the layers fail to reconstruct. See figure 1.7. If we reorientate the layers by $90^\circ$ and reconstruct from the same data, then as the direction of the oscillations $\xi_1 \in \{(\cos \psi, \sin \psi) : \psi \in [\pi/4, 3\pi/4]\}$ the jump discontinuities are resolved. See figures 1.8 and 1.9. In [53] Quinto gives local Sobolev space estimates for his results and in [57] these are generalised to $n$ dimensions. This is to say that the points and directions in which a function is stably reconstructed in some Sobolev space from its Radon transform are those which are cut orthogonally in a small neighbourhood by hyperplanes.

### 1.7.1 Fourier integral operators

For those Radon transforms $R$ which are Fourier Integral Operators (FIO’s), we can analyse their microlocal properties by considering the canonical relations of $R$ [20]:

**Definition 7.** Let $X \subset \mathbb{R}^{n_x}, Y \subset \mathbb{R}^{n_y}$ be open sets. A Fourier integral operator
is an operator $A : C^\infty_0(X) \to \mathcal{D}'(Y)$ of the form
\begin{equation}
Af(y) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^n_x} e^{i\phi(x,y,\xi)} a(x,y,\xi) f(x)dx d\xi,
\end{equation}
where $\phi$ is a phase function (see Appendix A), and $a \in S^m_p((X \times Y) \times \mathbb{R}^N)$ is a symbol (see Appendix A).

**Definition 8.** The canonical relation of an FIO with phase function $\phi$ is defined as
\begin{equation}
C = \{(y,\eta, x,\omega) \in (Y \times \mathbb{R}^n_y \setminus 0) \times (X \times \mathbb{R}^n_x \setminus 0) : (x,y,\omega) \in \Sigma_\phi, \\
\omega = -d_x\phi(x,y,\xi), \eta = d_y\phi(x,y,\xi) \},
\end{equation}
where $\Sigma_\phi = \{(x,y,\xi) \in X \times Y \times \mathbb{R}^N \setminus 0 : d_\xi\phi = 0 \}$ is the critical set of $\phi$.

**Definition 9.** A canonical relation $C \subset T^*Y \setminus 0 \times T^*X \setminus 0$ is a called a local canonical graph if every point $(y_0,\eta_0, x_0,\xi_0)$ of $C$ has a neighborhood of the form $\Gamma_Y^0 \times \Gamma_X^0$, with $\Gamma_Y^0$ and $\Gamma_X^0$ conic open neighborhoods of $(y_0,\eta_0)$ and $(x_0,\xi_0)$ in $T^*Y \setminus 0$ and $T^*X \setminus 0$ respectively, such that $C \cap (\Gamma_Y^0 \times \Gamma_X^0)$ is the graph of a conic symplectomorphism of $\Gamma_Y^0$ onto $\Gamma_X^0$.

**Definition 10.** The order of an FIO $A : C^\infty_0(X) \to \mathcal{D}'(Y)$ with symbol order $m$ is defined as
\begin{equation}
O(A) = m + \frac{N}{2} - \frac{n_X + n_Y}{4}.
\end{equation}

An FIO with $X = Y$ and canonical relation $C \subset \Delta$, where $\Delta$ is the diagonal in $T^*X \times T^*X$ is called a pseudodifferential operator. A pseudodifferential operator $A$ which is also elliptic (see [28, page 41] for the definition of ellipticity) is invertible up to modulo smoothing by [28, Theorem 4.1] (a microlocal inversion or a parametrix of $A$ exists). In that case the inversion of $A$ is continuous of order $-O(A)$ in Sobolev spaces (a mildly ill–posed inversion).

The normal operator of an elliptic FIO whose canonical relation is a graph is an elliptic pseudodifferential differential operator by [31, Theorem 4.2.2]. The inversion of such FIO’s is therefore stable in Sobolev spaces. For example, the hyperplane Radon transform in two dimensions is an elliptic FIO order $-1/2$ [54]:
\begin{equation}
Rf(s,\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} e^{i(s-\theta x)} f(x)dx d\sigma.
\end{equation}
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The canonical relations of $R$ are therefore
\[ C = \left\{ (s, \theta; \sigma \Theta^T \cdot x), (x, \sigma \Theta) : x \in \mathbb{R}^2, (s, \theta) \in \mathbb{R} \times S^1, \sigma \in \mathbb{R} \setminus \{0\}, s - x \cdot \Theta = 0 \right\}, \]
(1.53)

where $\Theta = (\cos \theta, \sin \theta)$. The left projection $\pi_L : C \to T^* (\mathbb{R} \times S^1)$ of $R$,
\[ \pi_L(x, \sigma, \theta) = (x \cdot \Theta, \theta, \sigma, \sigma \Theta^T \cdot x) \]
(1.54)
is a local diffeomorphism as its Jacobian
\[ J_{\pi_L} = \begin{pmatrix} \Theta^T & (x \cdot \Theta^T, 0) \\ \sigma (\Theta^T)^T & (-\sigma (x \cdot \Theta), x \cdot \Theta^T) \\ 0_{2 \times 2} & I_{2 \times 2} \end{pmatrix} \]
(1.55)
has determinant $\det J_{\pi_L} = \det(\Theta, \sigma \Theta^T) = \sigma$. Hence, as $\sigma \neq 0$, the canonical relation $C$ is a local canonical graph \[63\]. It is true that the adjoint operator of an FIO with canonical relation $C$ is also an FIO with canonical relation $C^T$ (the inverse relation of $C$) \[20\]. As $C$ is a graph, the composition of relations:
\[ C^T \circ C \subset \Delta \]
(1.56)
and hence the normal operator for the hyperplane Radon transform in two dimensions $R^* R$ is an elliptic pseudodifferential operator order $-1$. From here we arrive at the same result as equation (1.43), namely that the inverse of $R$ in two dimensions is continuous in Sobolev spaces order $1/2$. This is an example of how we can obtain Sobolev space stability estimates for a Radon transform by an analysis of its microlocal properties.

In cases where the left projection $\pi_L$ of an FIO fails to be a local diffeomorphism on a set $\Sigma$ ($\pi_L$ drops rank on $\Sigma$), then the right projection $\pi_R$ of the canonical relation also drops rank on $\Sigma$ and singularities occur (e.g. of fold or blowdown type \[27, 23, 22\]). For example in \[23\], the authors consider a scattering operator $F$ in Synthetic Aperture Radar (SAR). In limited data cases, the left and right projections are shown to have blowdown type singularities. The canonical relation $C$ of $F$ is shown satisfy $C^T \circ C \in \Delta \cup \Lambda$, where $\Lambda$ is the flow out (See chapter 4, Definition 12) from the right projection of $C$ on a set of singularities $\Sigma$. In this case, the image artefacts are described by the Hamiltonian flow.
of $\pi_{R}(\Sigma)$ (the wavefronts in $\pi_{R}(\Sigma)$ are not stably resolved). In [4], the microlocal properties of a linearized forward scattering operator in SAR are considered. The left and right projections exhibit fold and blowdown singularities respectively and the artefacts are not described by a flowout. In that case the canonical relation of the normal operator is shown to lie in $\Delta \cup C_1 \cup C_2 \cup C_3$, where $C_1$, $C_2$ and $C_3$ describe reflection artefacts. Later in chapter 4, we will analyse the microlocal properties of an FIO of interest in CST and show that the image artefacts are rotations described by a flowout.
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Figure 1.4: Characteristic function $f$.

Figure 1.5: Characteristic function reconstruction from limited angle Radon transform data in the absence of noise. The edges at $x = (\cos \psi, \sin \psi)$ for $\psi \in [\pi/2, \pi] \cup [3\pi/2, 2\pi]$ are blurred out forming an eye shape.
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Figure 1.6: Horizontal layer function $f_1$ and line profile through pixel 50.

Figure 1.7: $f_1$ reconstruction and horizontal line profile through pixel 50.
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Figure 1.8: $f_1$ reorientated and vertical line profile through pixel 50.

Figure 1.9: Reorientated $f_1$ reconstruction and vertical line profile through pixel 50.
1.8 An introduction to the papers

Our exploration into CST in this thesis arose from scatter modelling work with the CASE sponsor of the program; Rapiscan Systems. The original idea was to model the scatter for the Real Time Tomography (RTT) series of X-ray scanners developed by the company (for the use in threat detection for airport baggage screening) to investigate whether the scatter was useful data as opposed to being just an unwanted noise. The modelling of scatter had been considered previously in [65, 66] as a means to remove the scattered noise. With the development of high resolution energy sensitive detectors by Prof. Robert Cernik at Manchester University [11], this led us to the consideration of energy sensitive arrays for the RTT and from there to the works of Palamodov, Cormack, Norton, Nguyen and Truong in CST [50, 13, 48, 46]. The scatter modelling work for the energy sensitive RTT geometry combined with the ideas of Norton and Cormack provided the starting point for the first paper of this thesis on X-ray CST. Also, with the widespread use of energy sensitive detectors in other imaging applications [21, 7, 42], CST techniques are becoming of more general relevance. The papers presented in this thesis introduce a new set of Radon transforms in CST and provide an analysis of their injectivity, stability and microlocal properties.

The layout of the thesis is as follows. First, in paper 1 [67] (chapter 2), we introduce a novel disc transform and explain its application in CST and in particular to the RTT design. We show the equivalence of the disc transform and the hyperplane Radon transform in two dimensions and from this, the injectivity of the disc transform is proven and Sobolev space estimates are obtained. Having covered CST problems in two dimensions in our first paper, this leads into the three dimensional case in paper 2 [69] (chapter 3), where we introduce a new three dimensional CST modality. Here we describe the surface of Compton scatterers in three dimensions and define a new spindle torus transform (spindle transform). Injectivity results are provided for a generalization of the spindle transform, where we give an explicit left inverse for a Radon transform which describes the integrals of a function over the surfaces of revolution of a class of symmetric $C^1$ curves. As the stability aspects of the spindle transform are not covered here, this gives a starting point for our final paper [68] (chapter 4), where we provide an analysis of the stability of the spindle transform from a microlocal perspective. Here the spindle transform is shown to be a paired Lagrangian type operator with blowdown–blowdown singularities and the image artefacts are identified. We
later provide verification for our results by simulation and construct a filter to reduce the image artefacts present in a reconstruction from spindle transform data.

With the distinct gap in the literature on the theoretical and practical aspects of CST, this thesis adds a significant and novel chapter in the field and hopes to prompt further research and interest. The new acquisition geometries considered in this thesis may also form the basis of a new generation of practical security and medical scanning equipment.

1.8.1 X-ray Compton scattering tomography

Here we cover the uniqueness and stability aspects of a disc transform, which describes the two dimensional CST problem for the RTT X-ray scanner geometry developed by Rapiscan Systems for use in airports in baggage screening (the RTT system is equipped with polychromatic sources and energy sensitive arrays). A circle transform for a monochromatic source and energy sensitive detector pair has been considered previously in [48], for an acquisition geometry where the source remains fixed at the origin and the detector is moved laterally along the $x$ axis. The circle transform is first introduced in a classical paper by Cormack [16]. Although at the time Cormack did not know of its application in CST, the explicit inversion formula he provides is a corollary to his results on the hyperplane Radon transform. In later works [13, 14] the circle transform is shown to belong to a more general class of integral transforms, which give the integrals of a function over a class of $\alpha$ and $\beta$ curves. While the literature has its focus purely on the monochromatic source case, an explanation of polychromatic case is necessary given the widespread use of multi-energy sources in application (e.g. medical CT or security screening, where typically an X-ray tube is used to generate photons) [60, 58, 43]. We address this here and show that the single and multi–energy cases differ by a derivative in the energy variable. This suggests that although a unique solution is still possible, the polychromatic problem is inherently more highly ill–posed.

In the primary result of our paper, the disc transform is shown to have a close relationship to the hyperplane Radon transform in two dimensions. That is, after we differentiate with respect to the energy variable and make a change of coordinates, the disc transform data is shown to be equivalent to Radon transform data via diffeomorphism, similarly to the circular arc transform reviewed earlier
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(Section 1.5). Through this equivalence and established theory on the Radon transform, we derive uniqueness results and Sobolev space estimates for the disc transform. We show that our problem is mildly ill posed, but more highly ill-posed than the hyperplane Radon transform given the extra derivative step required in the inversion. Gelfand explores a similar problem related to integrals over volumes in [25]. He considers a half plane transform which, similarly to the disc transform has a close relation to the hyperplane Radon transform through a derivative step. Although there is no known application of the half plane transform at this stage.

The RTT design [61, 66] considered in this paper has its focus primarily on the transmission CT reconstruction, where the attenuation coefficient is recovered from straight line integral data in a helical or multi-helical source acquisition geometry. Given the fixed nature of the detectors, there are a collection of detectors in the dark field which have so far not been considered for reconstructive purposes (the scatter is often regarded as an unwanted noise). Our method proposes to use the extra scattered data to obtain additional information regarding the target electron density. Later we show how to use the full RTT data set (scatter plus transmission) to reconstruct the effective atomic number \( Z \) of the target by reading from electron cross section curves and we provide simulations to test our inversion method. Other methods which fuse Compton scatter and transmission data have been considered recently in [56]. Here the photoelectric absorption coefficient and mass density of the target are reconstructed simultaneously via a gradient descent method. We propose to reconstruct the attenuation coefficient and electron density separately and then to combine our resulting images to form an effective \( Z \) image.

1.8.2 Three dimensional Compton scattering tomography

The second paper focuses on a three dimensional CST problem. Three dimensional imaging from Compton scattered data is considered in the literature [41, 47, 64], where the scattered intensity from a gamma ray source is measured using a Compton camera with the aim of reconstructing the source intensity. Here it is shown that the scattered intensity can be modelled as a weighted integral of the source intensity over cones with their apex on a plane scattering surface, and explicit reconstruction formulae are derived for a cone transform. We consider the acquisition geometry of a single cone beam source and detector which lie opposite one another on a line through the origin and are rotated on the unit
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sphere $S^2$, and we aim to reconstruct an electron density supported on a hollow unit ball. Here we explain how the Compton scattered intensity can be modelled as a weighted integral of the electron density over a spindle torus (the surface of revolution of a circular arc) and we introduce a new spindle torus transform and a generalized spindle transform which gives the weighted integrals of a function over the surfaces of revolution of a class of symmetric $C^1$ curves. Our main result proves the injectivity of the generalized spindle transform by providing an explicit expression for the harmonic coefficients of our solution function in terms of the generalized spindle data. Explicit reconstruction formulae for a general class of integral transforms in $n$ dimensional space are given in [52]. The conditions for an otherwise arbitrary generating function to yield a left inverse for a generalised Funk–Radon transform are reviewed in section 1.3. It is the case that the spindle transform is not included in this class of integral transforms as its left projection drops rank. This is proven later in chapter 4 when we consider the microlocal properties of the spindle transform.

Although the main section of the paper focuses on the monochromatic source case, where the photons are restricted to scatter on a spindle torus surface, we also consider the polychromatic source case in three dimensions and introduce a new spindle interior, apple and apple interior transform. The apple transforms describe the exterior problem and model the backscattered (scattering angles greater than $\pi/2$) intensity from a density on the exterior of the unit ball. As in the two dimensional polychromatic case for CST, we find that an extra derivative step in the energy variable is necessary to obtain a solution and we provide injectivity results for the additional three transforms considered. In the final section of the paper a simulation study of the spindle and spindle interior transforms is conducted, where we provide reconstructions of a non-trivial test phantom with varying levels of pseudo random noise.

1.8.3 Microlocal analysis of a spindle transform

As the stability of the spindle transform was not covered in our previous work, we give a novel microlocal analysis of the stability of the spindle transform in the final paper. We start by proving the equivalence via diffeomorphism of the spindle torus transform to a Radon transform which gives the integrals of a function over cylinders with their axis of revolution through the origin, and from there we consider the microlocal properties of the cylindrical Radon transform. The
cylinder transform is shown to be an FIO of order $-1$ and its canonical relation $C$ is calculated. On first analysis it is clear that the canonical relation does not form a local graph and we have difficulty resolving the radial wavefronts $(x, \xi)$ for $x \times \xi = 0$, which are cut orthogonally by the degenerate cylinders (lines through the origin). This theory is confirmed later, as when we calculate the Jacobian of the left projection of $C$, it is shown to drop rank by 1 on $\Sigma = \{x \cdot \theta = 0 : x \in B_{\epsilon_1,\epsilon_2}^3, \theta \in S^2\}$ for some $0 < \epsilon_1 < \epsilon_2 < 1$, where $B_{\epsilon_1,\epsilon_2}^3 = \{x \in \mathbb{R}^3 : \epsilon_1 < |x| < \epsilon_2\}$, $\theta$ is the direction of the axis of revolution of the cylinder and $x$ is a point in the image space. So the right projection on $\Sigma$ describes the wavefronts which are not resolved $\pi_R(\Sigma) = \{(x, \xi) : x \times \xi = 0\}$.

The left and right projections are shown to exhibit blowdown type singularities on $\Sigma$ and using the theorems presented in [23], we show that the canonical relation of the normal operator $C^T \circ C \in \Delta \cup \Lambda \cup \tilde{\Delta}$, where $\Delta$ is the diagonal, $\Lambda$ is the flow out from $\pi_R(\Sigma)$ and $\tilde{\Delta}$ is a reflection through the origin. The reflection artefact is intuitive, as the odd functions are in the null space of the cylinder transform. That is, a reconstruction from cylinder transform data is the projection of the function onto its even component. We explicitly calculate the flow out of $\pi_R(\Sigma)$, which is shown to describe a full set of rotations of the vector–cotangent vector pairs $(x, \xi) \in T^*B_{\epsilon_1,\epsilon_2}^3$ about the origin. This implies that there are artefacts in the image which are a blurring or smearing out over spheres centred at the origin. The diffeomorphism which defines the equivalence between the cylinder and spindle transforms is rotationally invariant. So the rotation artefacts are present in a reconstruction from spindle transform data. While the literature has its focus purely on a rigorous analysis of the microlocal properties of Radon transforms and the mathematical proof of the presence of image artefacts [23, 27, 4, 22], a verification of such artefacts is yet to be considered. In our work we provide a novel verification of the suspected artefacts from a spindle transform inversion by simulation. Here we apply the normal operator of the spindle transform to a delta function (approximated as a characteristic function), where it is shown to be smeared out over a sphere in the reconstruction. This is as predicted by $\Lambda$. In [23] Felea et al. construct a filter to reduce the strength of an image artefact in a SAR problem, with the idea being to find a pseudodifferential operator (the filter) which vanishes to a high enough order on $\pi_L(\Sigma)$ (the left projection on the set of singularities). Similarly, we construct a pseudodifferential operator $Q$ to reduce the strength of the rotation artefact and show how it can be applied
directly to the spherical harmonics of the data. To verify our theory, we again reconstruct a delta function, but this time we apply $Q$ before backprojecting the spindle transform data. It is clear from our results that the strength of the artefact is significantly reduced.

To finish the paper we give simulated reconstructions of a spherical layered shell centred at the origin and a plane layered shell. The idea of the spherical shell is to give a worse case scenario microlocally as the wavefronts $\{(x, \xi) : x \times \xi = 0\}$ are the hardest to identify. As expected, the spherical shell reconstruction is not clear and in particular the jump discontinuities between the layers are not well resolved. Whereas in the plane shell case the singularities are better identified. The effects of applying $Q$ as a pre–conditioner to a CGLS and Landweber iteration are also investigated.
Bibliography


Chapter 2

X–ray Compton scattering tomography
X–ray Compton scattering tomography

By James Webber

Abstract

We lay the foundations for a new fast method to reconstruct the electron density in X–ray scanning applications using measurements in the dark field. This approach is applied to a type of machine configuration with fixed energy sensitive (or resolving) detectors, and where the X–ray source is polychromatic. We consider the case where the measurements in the dark field are dominated by the Compton scattering process. This leads us to a 2D inverse problem where we aim to reconstruct an electron density slice from its integrals over discs whose boundaries intersect the given source point. We show that a unique solution exists for smooth densities compactly supported on an annulus centred at the source point.

Using Sobolev space estimates we determine a measure for the ill posedness of our problem based on the criterion given by Natterer in [13]. In addition, with a combination of our method and the more common attenuation coefficient reconstruction, we show under certain assumptions that the atomic number of the target is uniquely determined.

We test our method on simulated data sets with varying levels of added pseudo random noise.

1 Introduction

In this paper we investigate the potential for the use of incoherent scattered data for 2D reconstruction in X–ray scanning applications. The use of scattered data for image reconstruction is considered in the literature, typically for applications in gamma ray imaging, where the photon source is monochromatic [1, 2, 3]. However, in many applications (e.g. security screening of baggage) a type of X–ray tube is often used that generates a polychromatic spectrum of initial photon energies (see section 3 for an example spectrum). There has been recent interest in the use of energy sensitive detectors in tomography [4, 5], and in the present paper their application is key to the ideas presented.

Our main goal is to show that the electron density may be reconstructed analytically using the incoherent scattered data and to lay the foundations for a practical
reconstruction method based on our theory. We apply our method to a machine config-
uration commonly used in X-ray CT. In addition, by use of the reconstructed density
values in conjunction with an attenuation coefficient reconstruction, we show under
the right assumptions that the atomic number of the target is uniquely determined.

For a photon incident upon an electron Compton (incoherently) scattering at an
angle $\omega$ with initial energy $E_\lambda$, the scattered energy $E_s$ is given by the equation:

$$E_s = \frac{E_\lambda}{1 + (E_\lambda/E_0)(1 - \cos \omega)} \quad (1)$$

where $E_0 \approx 511\text{keV}$ is the electron rest energy. Equation (1) implies that $\omega$
remains fixed for any given $E_s$ and $E_\lambda$. So in the case of a monochromatic source (single
energy source), assuming only single scatter events, for every fixed measured energy
$E_s$ (possible to measure if the detectors are energy-resolved) the locus of scattering
points is a circular arc intersecting the source and detector in question. For example,
refer to [1, 3].

In an X–ray tube a cathode is negatively charged and electrons are accelerated by a
large voltage ($E_{\text{max}} \text{kV}$) towards a positively charged target material (e.g. Tungsten).
A small proportion of the initial electron energy ($\approx 1\%$) is converted to produce
photons. Due to energy conservation, the resulting photon energies are no more than
$E_{\text{max}} \text{keV}$. So in the polychromatic source case, again assuming only single scatter
events, for each given data set (photon intensity recorded with energy $E_s$), the set
of scatterers lie on a collection of circular arcs intersecting the source and detector
points. Together these form a toric section in which the photons scatter, with a
maximum scattering angle $\omega_{\text{max}}$ given by:

$$\cos (\omega_{\text{max}}) = 1 - \frac{E_0 (E_{\text{max}} - E_s)}{E_s E_{\text{max}}} \quad (2)$$

See figure 1 below:

Figure 1: A toric section $T$ in which the photons scatter with tips at source and
detector points $s$ and $d$.

In the present paper we consider a setup consisting of a ring of fixed energy sensitive
detectors and a single rotating fan beam polychromatic source (multi–energy source).
See figure 2. With this setup we can measure photon intensity in the dark field
(detectors not in direct exposure to the X–ray beam). We image an electron density
$f : \mathbb{R}^2 \rightarrow \mathbb{R}$ compactly supported within the detector ring (the blue and green circle
in figure 2), with $f \geq 0$. If we assume an equal scattering probability throughout the
region $R = D \cap \text{supp}(f)$ leaving only the electron density to vary, and if we assume
that the majority of scattering events occur within $R$, then in this case the integral of $f$
over $D$ is approximately determined by the scattered intensity recorded at the detector $d$ with some fixed energy $E_s$. See the appendix for an example application where these approximations are valid. With these assumptions and with suitable restrictions on the support of $f$, we aim to reconstruct $f$ from its integrals over discs whose boundaries intersect a fixed point, namely the source at a given position along its scanning path.

In section 2, we present a disc transform and go on to prove our main theorem (Theorem 1), which explains the relationship between our transform and the straight line Radon transform. As a corollary to this theorem, with known results on the Radon transform, we show that a unique solution exists on the domain of smooth functions compactly supported on an annulus centred at the origin. Here based on the criterion of Natterer in [13] and using the theory of Sobolev space estimates, we determine a measure for the ill posedness of our problem.

In section 3, we discuss a possible means to approximate the physical processes such as to allow for the proposed reconstruction method. Here we also present a least squares fit for the total cross section (scattering plus absorption) in terms of $Z$ (the atomic number). From this, we show that $Z$ is uniquely determined by the attenuation coefficient and electron density.

In section 4 we apply our reconstruction formulae to simulated data sets, with varying levels of added pseudo random noise. This is applied to the given machine configuration. We recover a simple water bottle cross section image (a circular region of uniform density 1) and reconstruct the atomic number in each case using the curve fit presented in section 3. To give an example reconstruction of a target not of uniform density, we also present reconstructions of a simulated hollow tube cross section.
2 A disc transform

In this section we aim to recover a smooth function compactly supported on an annulus centred at the origin \(O\) from its integrals over discs whose boundaries intersect \(O\) (the given source position).

Let \(D_{p,\phi}\) denote the set of points on the disc whose boundary intersects the origin, with centre given in polar coordinates as \((p/2, \phi)\). See figure 3. Let \(C^\infty(\Omega)\) be the set of smooth functions on \(\Omega \subseteq \mathbb{R}^n\) and let \(C^\infty_0(\Omega)\) denote the set of smooth functions compactly supported on \(\Omega\). Let \(Z^+ = \mathbb{R}^+ \times S^1\) and for a function in the plane \(f : \mathbb{R}^2 \to \mathbb{R}\), let \(F : Z^+ \to \mathbb{R}\) be defined as \(F(\rho, \theta) = f(\rho \cos \theta, \rho \sin \theta)\). Then we define the disc transform \(\mathcal{D}_1 : C^\infty_0(\mathbb{R}^2) \to C^\infty(Z^+)\) as:

\[
\mathcal{D}_1 f (p, \phi) = \int_{D_{p,\phi}} f \, dA = \int_{\pi/2}^{\pi/2} \int_0^{\cos \frac{\phi}{\rho}} \rho F(\rho, \theta + \phi) \, d\rho d\theta
\]  

(3)

Figure 3: A disc \(D_{p,\phi}\) is illustrated. Its boundary intersects the origin \(O\).

After making the change of variables:

\[
\rho = r \cos \psi, \quad \theta = \psi, \quad d\rho d\theta = \cos \psi \, dr d\psi
\]

in equation (3), we have:

\[
\mathcal{D}_1 f (p, \phi) = \int_0^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r \cos^2 \psi F(r \cos \psi, \psi + \phi) \, d\psi dr
\]

(5)

We now present further definitions which will be important in the following subsection (section 2.1), where we provide our Sobolev space estimates. Let \(Z = \mathbb{R} \times S^1\) denote the unit cylinder in \(\mathbb{R}^3\). Then we define \(\mathcal{D}_2 : C^\infty_0(\mathbb{R}^2) \to C^\infty(Z)\) as follows:

\[
\mathcal{D}_2 f (p, \phi) = \begin{cases} 
\mathcal{D}_1 f (p, \phi) & p > 0 \\
\frac{\mathcal{D}_1 f (0^+, \phi) - \mathcal{D}_1 f (0^+, \phi + \pi)}{2} & p = 0 \\
-\mathcal{D}_1 f (-p, \phi + \pi) & p < 0
\end{cases}
\]

(6)

which is not continuous as a function of \(p\). We can remove this discontinuity by adding the function:

\[
c(\phi) \, \text{sgn} (p) = \begin{cases} 
c(\phi) & p > 0 \\
n0 & p = 0 \\
-c(\phi) & p < 0
\end{cases}
\]

(7)
where \( c(\phi) = -\frac{(D_1 f(0^+,\phi_+\pi) + D_1 f(0^+,\phi))}{2} \). We define \( \mathcal{D} : C_0^\infty(\mathbb{R}^2) \rightarrow C_\infty(Z) \) as:
\[
\mathcal{D} f(p,\phi) = D_2 f(p,\phi) + c(\phi) \text{sgn}(p)
\]  
(8)

Let \( L_{p,\phi} = \{(x,y) \in \mathbb{R}^2 : x \cos \phi + y \sin \phi = p\} \) be the set of points on a line. Then we define the Radon transform \( R : C_0^\infty(\mathbb{R}^2) \rightarrow C_\infty(Z) \) as:
\[
R f(p,\phi) = \int_{L_{p,\phi}} f(x) \, ds
\]  
(9)

We are now in a position to prove our main theorem, where we give the explicit relation between \( \mathcal{D} \) and the Radon transform \( R \) for smooth functions on an annulus.

**Theorem 1.** Let \( A_{r_1,r_2} = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\} \) be the annulus centred on \( O \) with inner radius \( r_1 > 0 \) and outer radius \( r_2 \). Let \( f \in C_0^\infty(A_{r_1,r_2}) \) for some \( r > 1 \) and let \( \tilde{f} \in C_0^\infty(A_{1/r,1}) \) be defined as \( \tilde{f}(x) = \frac{1}{4\pi} f \left( \frac{x}{|x|} \right) \). Then \( \frac{\partial}{\partial p} \mathcal{D} f = -R \tilde{f} \).

**Proof.** Let \( \tilde{F} \) and \( F \) be defined as \( \tilde{F}(\rho,\theta) = \tilde{f}(\rho \cos \theta, \rho \sin \theta) \) and \( F(\rho,\theta) = f(\rho \cos \theta, \rho \sin \theta) \). Then from our definition of \( \tilde{f} \), we have \( \tilde{F}(\rho,\theta) = \frac{1}{\rho} F \left( \frac{1}{\rho}, \theta \right) \). Now we have:
\[
\frac{\partial}{\partial p} \mathcal{D}_1 f(p,\phi) = -\frac{1}{p^3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \psi F \left( \frac{\cos \psi}{p}, \psi + \phi \right) d\psi
\]
\[
= -p \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tilde{F} \left( \frac{p}{\cos \psi}, \psi + \phi \right) \frac{d\psi}{\cos^2 \psi}
\]
\[
= -R \tilde{f}(p,\phi) \text{ for } p \geq 0
\]
and hence \( \frac{\partial}{\partial p} \mathcal{D} f(0,\phi) = \frac{\partial}{\partial p} \mathcal{D}_1 f(0,\phi) = \frac{\partial}{\partial p} \mathcal{D}_1 f(0,\phi + \pi) \). So the partial derivative of \( \mathcal{D} f \) with respect to \( p \) exists and is continuous for all \( p \in \mathbb{R} \), and \( \frac{\partial}{\partial p} \mathcal{D} f = -R \tilde{f} \). \( \Box \)

We now give an explicit expression for \( f \) in terms of the disc transform \( \mathcal{D} f \). First we recall the classical inversion formula obtained by Radon in [14].

**Theorem 2.** Let \( f \in C_0^\infty(\mathbb{R}^2) \) and let \( R f \in C_\infty(Z) \) be defined as in equation (9). Then:
\[
f(x) = \frac{1}{4\pi^2} \int_\mathbb{R} \int_{S^1} \frac{\partial}{\partial p} R f(x \cdot \Phi + t, \phi) \, d\phi dt, \quad \forall x \in \mathbb{R}^2
\]  
(11)

where \( \Phi = (\cos \phi, \sin \phi) \).

**Corollary 1.** Let \( f \in C_0^\infty(A_{1,r}) \) for some \( r > 1 \) and let \( \mathcal{D} f \in C_\infty(Z) \) be defined as in equation (8). Then:
\[
f(x) = -\frac{1}{4\pi^2 |x|^4} \int_\mathbb{R} \int_{S^1} \frac{\partial^2}{\partial p^2} \mathcal{D} f \left( \frac{x}{|x|^2}, \Phi + t, \phi \right) \, d\phi dt, \quad \forall x \in \mathbb{R}^2
\]  
(12)

where \( \Phi = (\cos \phi, \sin \phi) \).

**Proof.** This follows immediately as a consequence of Theorem 1 and Theorem 2. \( \Box \)

We now aim to prove injectivity of the disc transform \( \mathcal{D} \) on the domain of smooth functions compactly supported on an annulus. First we state Helgason’s support theorem [6].
Theorem 3. Let \( X \) be a compact convex set in \( \mathbb{R}^n \) and let \( f \) be continuous on \( \mathbb{R}^n \). If \( Rf = 0 \) for all \( p \) and \( \phi \) such that \( L_{p,\phi} \cap X = \emptyset \) and \( f \) is rapidly decreasing, in the sense that:

\[
|x|^k f(x) \to 0 \quad \text{as} \quad |x| \to \infty \quad \forall k \in \mathbb{N} \quad (13)
\]

then \( f(x) = 0 \) for all \( x \notin X \).

Corollary 2. Let \( f \in C_0^\infty(A_{1,\pi}) \) for some \( r > 1 \), and let \( Z_r = \{(p, \phi) \in Z : 1/r < p < 1, \phi \in [0,2\pi]\} \). Then \( f \) is uniquely determined by \( Df \) known for all \( (p, \phi) \in Z_r \).

Proof. Let:

\[
f \in \{ f \in C_0^\infty(A_{1,\pi}) : Df = 0 \quad \text{for all} \quad (p, \phi) \in Z_r \} \quad (14)
\]

and let \( \tilde{f} \) be defined as in Theorem 1. Then by Theorem 1, we have:

\[
\tilde{f} \in \{ f \in C_0^\infty(A_{1/r,1}) : Rf = 0 \quad \text{for all} \quad (p, \phi) \in Z_r \} \quad (15)
\]

and hence \( \tilde{f} \) is rapidly decreasing. Let \( X = \{ x \in \mathbb{R}^2 : |x| \leq 1/r \} \). Then \( X \) is clearly compact and convex. By (15), \( R\tilde{f} = 0 \) for all \( p \) and \( \phi \) such that \( L_{p,\phi} \cap X = \emptyset \). So by Helgason’s support theorem, we have that \( \tilde{f}(x) = 0 \) for all \( x \notin X \). The result follows.

For the proposed machine configuration, we can define the set of points within the detector ring formally as \( D_r = \{(x,y) \in \mathbb{R}^2 : x^2 + (y - (r + 1)/2)^2 < (r - 1)^2/4 \} \), where \( r > 1 \) depends on the machine specifications (i.e. the detector ring radius and the source path radius). See figure 4. Given the positioning of the detector ring relative to a source on the source ring (\( D_r \) is a disc bounded away from the source (the origin)), the electron density has support on an annulus \( A_{1,r} \). This is important from a uniqueness perspective as it is necessary for the target function to be rapidly decreasing for Helgason’s support theorem to hold.

We now have:

Corollary 3. Let \( f \in C_0^\infty(D_r) \). Let \( \partial D_r \) denote the boundary of \( D_r \), and let \( R_{p,\phi} = D_{1/r,\phi} \cap D_r \). Then the values of \( D_1 f \) for \( p \) and \( \phi \) such that:

\[
R_{p,\phi} \neq \emptyset \quad \text{and} \quad \partial D_{1/r,\phi} \cap \partial D_r \neq \emptyset \quad (16)
\]

determine \( f \) uniquely.

Proof. We consider two cases. If \( D_r \subset D_{1/r,\phi} \) then \( D_1 f (p, \phi) = D_1 f (\tfrac{1}{r}, \pi/2) \), which is known as condition (16) is satisfied for \( p = 1/r \) and \( \phi = \pi/2 \). If \( D_r \cap D_{1/r,\phi} = \emptyset \), then \( D_1 f (p, \phi) = 0 \). In any other case, \( D_1 f \) is known by our assumption. Hence we have a full data set for \( D_1 f \) and hence for \( Df \). The result follows from Corollary 2.

So for the proposed application, we see from the above corollaries that for any given source position, the incoherent scattered data is sufficient to reconstruct the target density uniquely.
2.1 Sobolev space estimates

In this section we provide Sobolev space estimates for the disc transform $D$. From these we obtain an upper bound for the least squares error in our solution $f$ in terms of $\epsilon$, where $\epsilon$ is an upper bound for the least squares error in our measurements. First we define our Sobolev spaces and the norms which will be used in our estimates.

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain and let $L^2(\Omega)$ denote the set of square integrable functions on $\Omega$. We define the Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ as:

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$$

Then we can define Sobolev spaces $H^\alpha(\mathbb{R}^n)$ of real degree $\alpha \in \mathbb{R}$ as:

$$H^\alpha(\mathbb{R}^n) = \{\text{tempered distributions } f : (1 + |\xi|^2)^{\alpha/2} \hat{f}(\xi) \in L^2(\mathbb{R}^n)\}$$

with the norm:

$$\|f\|_{H^\alpha(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\alpha/2} |\hat{f}(\xi)|^2 d\xi$$

For functions on the cylinder $Z \subset \mathbb{R}^3$, we have the norm:

$$\|f\|^2_{H^\alpha(Z)} = \int_S \int_{\mathbb{R}} (1 + \sigma^2)^\alpha |\hat{f}(\sigma, \phi)|^2 d\sigma d\phi$$

where the Fourier transform of $Rf$ is taken with respect to the variable $p \in \mathbb{R}$. We now state some preliminary results on Sobolev spaces and the Radon transform which will be used in our theorems.

**Theorem 4** ([13, page 11]). For $f \in S(\mathbb{R}^2)$, where $S(\mathbb{R}^2)$ is the Schwartz space on $\mathbb{R}^2$, we have:

$$\widehat{Rf}(\sigma, \phi) = (2\pi)^{-1/2} \hat{f}(\sigma \Phi) \quad \sigma \in \mathbb{R}$$

where $\Phi = (\cos \phi, \sin \phi)$ and the Fourier transform of $Rf$ is taken with respect to the $p$ variable only.

**Theorem 5** ([13, page 203]). Let $k = (k_1, \ldots, k_n)$ be some multi index and let $D^k = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$, where the $\frac{\partial}{\partial x}$ are defined in the weak sense. Let $m$ be a positive integer and let $\sigma \in (0, 1)$. Then for $\alpha = m + \sigma$, the norm (19) is equivalent to the norm:

$$\|f\|^2_{H^\alpha(\Omega)} = \|f\|^2_{H^m(\Omega)} + \sum_{|k|=m} \int_{\Omega} \int_{\Omega \times \Omega} \left| \frac{D^k f(x) - D^k f(y)}{|x-y|^{n+2\sigma}} \right|^2 dx dy$$

when $\Omega = \mathbb{R}^n$.

We now prove a slice theorem for the disc transform $D$.

**Lemma 1.** Let $f \in C_0^\infty(A_{1,r})$ for some $r > 1$ and let $\hat{f}$ be defined as in Theorem 1. Then we have:

$$-i\sigma \hat{Df}(\sigma, \phi) = (2\pi)^{-1/2} \hat{f}(\sigma \Phi) \quad \sigma \in \mathbb{R}$$

where $\Phi = (\cos \phi, \sin \phi)$ and the Fourier transform of $Df$ is taken with respect to the $p$ variable.
Proof. Let \( D_2 \) and \( c(\phi) \) be as defined in section 2. Then we have:

\[
\hat{Rf}(\sigma,\phi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} R\tilde{f}(p,\phi) e^{-ip\sigma} dp \\
= (2\pi)^{-1/2} \left[ \int_{-\infty}^{0} R\tilde{f}(p,\phi) e^{-ip\sigma} dp + \int_{0}^{\infty} R\tilde{f}(p,\phi) e^{-ip\sigma} dp \right] \\
= -(2\pi)^{-1/2} \left[ \int_{-\infty}^{0} \frac{\partial}{\partial p} D_2 f(p,\phi) e^{-ip\sigma} dp + \int_{0}^{\infty} \frac{\partial}{\partial p} D_2 f(p,\phi) e^{-ip\sigma} dp \right]
\]

where the last step follows from Theorem 1. After integrating by parts we have:

\[
\hat{Rf}(\sigma,\phi) = -i\sigma \left( \frac{2c(\phi)}{i\sigma (2\pi)^{1/2}} + D_2 f(\sigma,\phi) \right) = -i\sigma D f(\sigma,\phi)
\]

The result follows from the Fourier slice theorem.

In [13, page 92] Natterer explains why it is reasonable to consider picture densities as functions \( f \) of compact support in \( H^\alpha(\mathbb{R}^n) \) with \( \alpha < 1/2 \). He then gives a bound for the least squares error in his reconstruction from plane integral data in terms of \( \rho \), where \( \|f\|_{H^\alpha} \leq \rho \). With this in mind we will show that the map \( f \to \hat{f} \) is bounded and has a bounded inverse from \( H^\alpha \to H^\alpha \) for \( 0 < \alpha < 1 \). First, from [13, page 204], we have the lemma:

**Lemma 2.** Let \( \chi \in C_0^\infty(\mathbb{R}^n) \) and let \( f \in H^\alpha(\mathbb{R}^n) \). Then the map \( f \to \chi f \) is bounded in \( H^\alpha \) for any \( \alpha \in \mathbb{R} \).

Now we have our result:

**Lemma 3.** Let \( D_r \) be as defined in section 2. Let \( f \in C_0^\infty(D_r) \) for some \( r > 1 \) and let \( \hat{f} \) be defined as in Theorem 1. Then there exist constants \( c(\alpha) \) and \( C(\alpha) \) such that:

\[
c(\alpha) \|\hat{f}\|_{H^\alpha(\mathbb{R}^2)} \leq \|f\|_{H^\alpha(\mathbb{R}^2)} \leq C(\alpha) \|\hat{f}\|_{H^\alpha(\mathbb{R}^2)}
\]

(26)

for any \( 0 < \alpha < 1 \).

**Proof.** Let \( \chi_{D_r} \in C_0^\infty(\mathbb{R}^2) \) be 1 on \( D_r \) and let \( \chi = \chi_{D_r}|x|^4 \). Then by Lemma 2, we have:

\[
c_1(\alpha) \|f\|_{H^\alpha(\mathbb{R}^2)}^2 \geq \|\chi_{D_r}|x|^4 f\|_{H^\alpha(\mathbb{R}^2)}^2 \\
= \|\chi_{D_r}|x|^4 f\|_{H^\alpha(\mathbb{R}^2)}^2 \\
= \|\hat{f}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|x|^4 f(x) - |y|^4 f(y)|^2}{|x-y|^{2+2\alpha}} dxdy \geq (1/r^4) \|\hat{f}\|_{L^2(\mathbb{R}^2)}^2 + (1/r^2)^{2\alpha-2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\hat{f}(x) - \hat{f}(y)|^2 |x-y|^{2+2\alpha} dxdy \geq c_2(\alpha) \|\hat{f}\|_{H^\alpha(\mathbb{R}^2)}^2
\]

for \( 0 < \alpha < 1 \). This proves the left hand inequality. The right hand inequality can be proven in a similar way.

Before proving the main theorem of this section we state the interpolation inequality for Sobolev spaces on \( \mathbb{R}^n \) [13, page 203].
Lemma 4. Let \( f \in H^\gamma (\mathbb{R}^n) \). Then we have:
\[
\| f \|_{H^\gamma(\mathbb{R}^n)} \leq \| f \|_{H^\beta(\mathbb{R}^n)} \beta - \gamma \| f \|_{H^\alpha(\mathbb{R}^n)}^{\alpha - \gamma \beta - \alpha} \]
(28)
for any \( \alpha < \gamma < \beta \).

Now we have our main theorem for this section:

Theorem 6. Let \( f \in C_0^\infty (D_r) \) for some \( r > 1 \). Then we have:
\[
\| f \|_{L^2(\mathbb{R}^2)} \leq c(\beta) \rho^{3/2 \beta} \| Df \|_{H^\beta([-1,1] \times S^1)}^{\beta \alpha / 3}\]
(29)
for any \( 0 < \beta < 1 \) with \( \| f \|_{H^\beta} \leq \rho \).

Proof. Let \( \tilde{f} \) be defined as in Theorem 1 and let \( \Phi = (\cos \phi, \sin \phi) \). Then we have:
\[
2 \| Df \|_{L^2([-1,1] \times S^1)}^2 = 2 \int_{S^1} \int_{-1}^1 |Df(p, \phi)|^2 d\phi dp
\]
\[
\geq \int_{S^1} \int_{-1}^1 |Df(p, \phi) + Df(-p, \phi)|^2 d\phi dp
\]
\[
= \int_{S^1} \int_{-\infty}^{\infty} |\tilde{Df}(\sigma, \phi) + \tilde{Df}(-\sigma, \phi)|^2 d\sigma d\phi
\]
\[
= 2\pi \int_{S^1} \int_{-\infty}^{\infty} \frac{1}{\sigma^2} |\tilde{f}(\sigma \Phi) - \tilde{f}(-\sigma \Phi)|^2 d\sigma d\phi \quad \text{by Lemma 1}
\]
\[
\geq 2\pi \int_{S^1} \int_{-\infty}^{\infty} (1 + \sigma^2)^{-1} |\tilde{f}(\sigma \Phi) - \tilde{f}(-\sigma \Phi)|^2 d\sigma d\phi
\]
\[
= 4\pi \int_{S^1} \int_{0}^{\infty} (1 + \sigma^2)^{-1} |\tilde{f}(\sigma \Phi) - \tilde{f}(-\sigma \Phi)|^2 d\sigma d\phi
\]
(30)

After making the substitution \( \xi = \sigma \Phi \), we have:
\[
\frac{1}{2\pi} \| Df \|_{L^2([-1,1] \times S^1)}^2 \geq \int_{\mathbb{R}^2} |\xi|^{-1} (1 + |\xi|^2)^{-1} |\tilde{f}(\xi) - \tilde{f}(-\xi)|^2 d\xi
\]
\[
\geq \| \tilde{f} - \tilde{f}^- \|_{H^{-3/2}(\mathbb{R}^2)}^2
\]
(31)

where \( \tilde{f}^- (x) = \tilde{f}(-x) \). Applying the interpolation inequality with \( \alpha = -3/2 \) and \( \gamma = 0 \), yields:
\[
2 \| f \|_{L^2(\mathbb{R}^2)} \leq 2 \| \tilde{f} \|_{L^2(\mathbb{R}^2)}
\]
\[
= \| \tilde{f} - \tilde{f}^- \|_{L^2(\mathbb{R}^2)} \quad \text{since} \quad f \in C_0^\infty (D_r)
\]
\[
\leq \| \tilde{f} - \tilde{f}^- \|_{H^{-3/2}(\mathbb{R}^2)}^{\beta / 3} \| \tilde{f} - \tilde{f}^- \|_{H^\beta(\mathbb{R}^2)}^{3/2}
\]
\[
\leq (2\pi)^{-1/2} \| f \|_{L^2([-1,1] \times S^1)}^{\beta / 3} \| f \|_{H^\beta(\mathbb{R}^2)}^{3/2}
\]
\[
\leq c(\beta) \| Df \|_{H^\beta(\mathbb{R}^2)}^{\beta / 3} \| f \|_{H^\beta(\mathbb{R}^2)}^{3/2}
\]
\[
\leq c(\beta) \rho^{3/2} \| Df \|_{L^2([-1,1] \times S^1)}^{3/2}
\]
(32)

for any \( 0 < \beta < 1 \) with \( \| f \|_{H^\beta} \leq \rho \). \qed
Corollary 4. Let \( f \in C^\infty_0(D_r) \) for some \( r > 1 \) and let \( g = Df \). Let \( g^\epsilon \in L^2([-1,1] \times S^1) \) be such that \( \|g^\epsilon - g\|_{L^2([-1,1] \times S^1)} < \epsilon \). Then for any \( f_1, f_2 \in C^\infty_0(D_r) \) which satisfy \( \|Df - g^\epsilon\|_{L^2([-1,1] \times S^1)} < \epsilon \), we have:

\[
\|f_1 - f_2\|_{L^2(\mathbb{R}^2)} \leq c(\beta) \rho^{3/2} \epsilon^{3/2} \beta^{3/2}
\]

for any \( 0 < \beta < 1 \) with \( \|f_i\|_{H^\beta} \leq \rho \) for \( i = 1, 2 \).

We can interpret this last corollary to mean that given some erroneous data \( g^\epsilon \) which differs in the least squares sense from \( Df \) absolutely by \( \epsilon \), the least squares error in our solution is bounded above by \( c(\beta) \rho^{3/2} \epsilon^{3/2} \beta^{3/2} \) for some constant \( c(\beta) \) with the a-priori knowledge that \( \|f\|_{H^\beta} \leq \rho \).

In [13] Natterer uses the value \( \beta/(\alpha + \beta) \) as a measure for the ill posedness of his problem and gives his criteria for a linear inverse problem to be modestly, mildly or severely ill posed. If we set \( \beta \) close to \( 1/2 \), then based on these criteria the above arguments would suggest that our problem is mildly ill posed, but more ill posed than the inverse Radon transform, which we would expect given that the disc transform \( D \) is a degree smoother than \( R \). That is, there is an extra derivative step going from disc transform data to Radon transform data (the value of \( \alpha \) for the disc transform is \(-3/2\) which is \( 1 \) less than the associated value for the Radon transform).

Another source of error in our solution can be due to limited sampling of the data. In practice the number of detectors will be finite. Let us parameterize the set of points on the detector ring \( \partial D_r = \{(x,y) \in \mathbb{R}^2 : x^2 + (y - (r+1)/2)^2 = (r-1)^2/4\} \) in terms of a polar angle \( \theta \), and let the finite set of polar angles \( \Theta = \{\theta_1, \ldots, \theta_n\} \) determine a finite set of detector positions \( \{d_1, \ldots, d_n\} \in \partial D_r \). See figure 4. Then for every \( \phi \in [0,2\pi] \) we can sample \( Df(p, \phi) \) for:

\[
p = p_j = \frac{r \cos \theta_j \sin \phi + (1 + r \sin \theta_j) \sin \phi}{r^2 + 1 + 2 r \sin \theta_j}, \quad 1 \leq j \leq n
\]
where \( p_j \) is such that \( \{ \frac{1}{2} ((r - 1) \cos(\theta_j), (1 + \sin \theta_j)r + 1 - \sin \theta_j) \} \subset \partial D \cap \partial D_r \) for \( 1 \leq j \leq n \).

We have:

**Lemma 5** ([13, page 204]). Let \( \Omega \subset \mathbb{R}^n \) be bounded and sufficiently regular. For \( h > 0 \) let \( \Omega_k \) be a finite subset of \( \Omega \) such that \( d(\Omega, \Omega_k) \leq h \), where \( d \) is the Hausdorff distance metric between sets. Let \( \alpha > n/2 \) where \( \alpha = m + \sigma \) for some integer \( m \) and \( 0 < \sigma < 1 \), and for \( f \in H^\alpha (\Omega) \) define the seminorm:

\[
|f|_{H^\alpha(\Omega)}^2 = \sum_{|k|=m} \iint_{\Omega \times \Omega} \frac{|D^k f(x) - D^k f(y)|^2}{|x - y|^{n+2\sigma}} \, dx \, dy
\]

Then there is a constant \( c \) such that:

\[
\|f\|_{L^2(\Omega)} \leq c h^\alpha |f|_{H^\alpha(\Omega)}
\]

for every \( f \in H^\alpha (\Omega) \) which is zero on \( \Omega_k \).

**Theorem 7** ([13, page 42]). Let \( \Omega_n \) be the unit ball in \( \mathbb{R}^n \). For every \( \alpha \) there exist positive constants \( c(\alpha, n) \) and \( C(\alpha, n) \) such that for \( f \in C_0^\infty (\Omega_n) \)

\[
c(\alpha, n) \|f\|_{H^\alpha(\Omega_n)} \leq \|Rf\|_{H^\alpha(\Omega_n)} \leq C(\alpha, n) \|f\|_{H^\alpha(\Omega_n)}
\]

From these we have the theorem:

**Theorem 8.** For each \( \phi \in [0, 2\pi] \) let \( I_\phi \subset [-1, 1] \) be a finite subset of the unit interval. Let:

\[
h = \sup_\phi d(I_\phi, [-1, 1])
\]

where \( d \) is the Hausdorff distance metric. Let \( f \in C_0^\infty (D_r) \) and let \( \|f\|_{H^\alpha} < \rho \) with \( 0 < \alpha < 1 \). For \( \phi \in [0, 2\pi] \) fixed, let \( \mathcal{D}f_\phi : \mathbb{R} \to \mathbb{R} \) be defined as \( \mathcal{D}f_\phi(p) = \mathcal{D}f(p, \phi) \). If, for every \( \phi \in [0, 2\pi] \), \( \mathcal{D}f_\phi \) is zero on \( I_\phi \), then there exists a constant \( c(\alpha) \) such that:

\[
\|f\|_{L^2(\mathbb{R}^2)} \leq c(\alpha) h^\alpha \rho
\]

**Proof.** Let \( \tilde{f} \) be defined as in Theorem 1 and let \( \| \cdot \|_{H^\alpha} \) be the seminorm defined in Lemma 5. Let \( R\tilde{f}_\phi : \mathbb{R} \to \mathbb{R} \) be defined as \( R\tilde{f}_\phi(p) = R\tilde{f}(p, \phi) \). For every \( \phi \in [0, 2\pi] \), let \( \mathcal{D}f_\phi \) be zero on \( I_\phi \). Then, we have:

\[
\|\mathcal{D}f\|_{L^2([-1,1] \times S^1)}^2 = \int_{S^1} \|\mathcal{D}f_\phi\|^2_{L^2([-1,1])} \, d\phi
\]

\[
\leq c_1 h^{2\alpha+3} \int_{S^1} \|\mathcal{D}f_\phi\|^2_{H^{\alpha+3/2}([-1,1])} \, d\phi \quad \text{by Lemma 5}
\]

\[
= c_1 h^{2\alpha+3} \int_{S^1} \|R\tilde{f}_\phi\|^2_{H^{\alpha+1/2}([-1,1])} \, d\phi \quad \text{by Theorem 1}
\]

\[
\leq c_1 h^{2\alpha+3} \|R\tilde{f}\|^2_{H^{\alpha+1/2}(\mathbb{R}^2)}
\]

\[
\leq c_2 (\alpha) h^{2\alpha+3} \|\tilde{f}\|^2_{H^\alpha(\mathbb{R}^2)} \quad \text{by Theorem 7}
\]

\[
\leq c_3 (\alpha) h^{2\alpha+3} \|f\|^2_{H^\alpha(\mathbb{R}^2)} \quad \text{by Lemma 3}
\]

\[
\leq c_3 (\alpha) h^{2\alpha+3} \rho^2
\]
for $0 < \alpha < 1$ with $\|f\|_{H^\alpha} \leq \rho$. Applying Theorem 6, gives:

$$\|f\|_{L^2(\mathbb{R}^2)} \leq c_4(\alpha) \rho^{\frac{3}{2\alpha}} \|Df\|_{L^2([-1,1] \times S^1)}$$

(41)

which completes the proof. \[\square\]

This last result tells us that given a finite set of detectors with a disc diameter sampling determined by equation (34) and with $h$ being a measure of the uniformity of the sample, the least squares error in our solution is bounded above by $c(\alpha) h^\alpha \rho$ with the a-priori knowledge that $\|f\|_{H^\alpha} \leq \rho$ for some $0 < \alpha < 1$.

### 3 The physical model

In this section we present an accurate physical model and a possible approximate model which allows for the proposed reconstruction method. We consider an intensity of photons scattering from a point $u$ as illustrated in figure 5. The points $s$, $d$ and $v$

![Figure 5: A scattering event with initial photon energy $E_\lambda$ from a source $s$ scattered to $d$ with energy $E_s$. The dashed line displays the original path of the photon to a detector $v$.](image)

are the centre points of the source and detectors respectively. The intensity of photons scattered from $u$ to $d$ with energy $E_s$ is:

$$I(u, d, E_s) = I_0(E_\lambda) \exp \left(-\int_{l_1} \mu_{E_\lambda} \right) n_e(u) dV$$

$$\times \frac{d\sigma}{d\Omega}(E_s, \omega) S(q) \exp \left(-\int_{l_2} \mu_{E_s} \right) d\Omega_{u,d}$$

(42)

where $I_0$ is the initial intensity, which depends on the energy $E_\lambda$ (see figure 6 for an example polychromatic spectrum). $\mu_E$ is the linear attenuation coefficient, which is dependant on the energy $E$ and the atomic number of the target material. Here $n_e(u) dV$ is the number of electrons in a volume $dV$ around the scattering point $u$. So $n_e$ (number of electrons per unit volume) is the quantity to be reconstructed. $l_1$ and $l_2$ are the line segments connecting $s$ to $u$ and $u$ to $d$ respectively.

The Klein-Nishina differential cross section $d\sigma/d\Omega$, is defined by:

$$\frac{d\sigma}{d\Omega}(E_s, \omega) = \frac{r_0^2}{2} \left( \frac{E_s}{E_\lambda} \right)^2 \left( \frac{E_s}{E_\lambda} + \frac{E_\lambda}{E_s} - 1 + \cos^2 \omega \right)$$

(43)
where \( r_0 \) is the classical electron radius. This predicts the scattering distribution for a photon off a free electron at rest. Given that the atomic electrons typically are neither free nor at rest, a correction factor is included, namely the incoherent scattering function \( S(q) \). Here \( q = \frac{E_s}{hc} \sin (\omega/2) \) is the momentum transferred by a photon with initial energy:

\[
E_\lambda = \frac{E_s}{1 - (E_s/E_0) (1 - \cos \omega)}
\]

scattering at an angle \( \omega \), where \( h \) is Planck’s constant and \( c \) is the speed of light. The scattering function \( S \) is dependant on the atomic number \( Z \). See figure 7. We set \( Z = Z_{avg} \) to some average atomic number as an approximation.

For \( Z_{avg} = 45 \) (Rhodium) we have the expression:

\[
S(q) = 1 - \frac{1.023}{(1 + 0.458q)^{2.509}}
\]

To acquire equation (45) and the curves given in figure 7 we have extended the least squares fits given in [7] to the values of \( S(q) \) given in [8]. For the proposed application, we are typically interested in X-ray tube voltages \( V \) in the range \( 60 \leq V \leq 200kV \), and the measured scatter will likely be backscatter (scattering angles \( \omega > \pi/2 \)). Let \( S_{Z_1}(q) \) and \( S_{Z_2}(q) \) denote the incoherent scattering functions for the atomic numbers \( Z_1 \) and \( Z_2 \). Then we measure the difference in \( S_{Z_1} \) and \( S_{Z_2} \) via the relative error:

\[
\epsilon = 100 \times \frac{\| S_{Z_1} - S_{Z_2} \|_{L^2(Q)}}{\| S_{Z_1} \|_{L^2(Q)}}
\]

where \( Q \) is the range of momentum transfer values \( q \) of interest. Let \( Z_1 = 3 \) and \( Z_2 = 45 \), and let the scattering angle \( \omega \) satisfy \( \omega > \pi/2 \). Then, for three commonly used tube voltages \( V \), we calculate the momentum transfer range \( Q \) and the error \( \epsilon \).
Figure 7: We have plotted $S(q)$ for a collection of atomic numbers. These are presented in the table below.

<table>
<thead>
<tr>
<th>$V$ (keV)</th>
<th>$Q$ (Å$^{-1}$)</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>[3.08, 4.84]</td>
<td>8.51</td>
</tr>
<tr>
<td>100</td>
<td>[4.77, 8.07]</td>
<td>4.57</td>
</tr>
<tr>
<td>150</td>
<td>[6.61, 12.10]</td>
<td>2.34</td>
</tr>
</tbody>
</table>

So we can see that for a sufficiently high tube voltage, the approximation error in equation (45) is relatively low (< 5%).

Let $A$ denote the detector area. Then the solid angle subtended by $u$ and $d$ is:

$$d\Omega_{u,d} = \frac{A \cdot r \cdot n}{4\pi |r|^3}$$ (47)

where $r = d - u$ and $n$ is the unit vector normal to the detector surface.

To model the incoming and scattered ray attenuation, an a-priori calculation of the linear attenuation coefficient $\mu$ is possible with the proposed machine configuration. This is the standard 2D X-ray CT problem. In [11, page 73] a fast means to approximate the ray attenuation is presented, using the recorded straight through data. From [11, page 73], we have the approximation:

$$\exp \left( - \int_{l_1} \mu_{E_\lambda} \right) \exp \left( - \int_{l_2} \mu_{E_\lambda} \right) \approx \exp \left( - \int_{l_v} \mu_{E_\lambda} \right)$$ (48)

where $l_v$ is the line segment from $s$ to the detector in the forward direction $v$ (see figure 5). By the Beer-Lambert law, we have:

$$\frac{I_v (E_\lambda)}{I_0 (E_\lambda)} = \exp \left( - \int_{l_v} \mu_{E_\lambda} \right)$$ (49)

where $I_v (E_\lambda)$ is the recorded straight through intensity. As the approximation error of equation (48) is dependant on unknown factors, such as the object shape and size, it
is difficult for us to suitably quantify the error. With this in mind, although we would sacrifice computational efficiency, an explicit linear attenuation coefficient calculation is preferable.

Given our machine geometry and proposed reconstruction method, it is difficult to include the more accurate model stated above as an additional weighting to our integral equations (as in done in [3] for example) while allowing for the same inversion formula. So we average equation (42) over the scattering region $R_{p,\phi} = D_{1/\phi} \cap D_r$, for each $p$ and $\phi$. Here $D_{p,\phi}$ and $D_r$ are as defined in section 2, where $r$ is fixed depending on the machine specifications.

Let $I(u,d,E_s) = I(u;p,\phi) = P(u;p,\phi) n_\epsilon(u)\,dV$. Here $P = P(u;p,\phi)$ depends on the scattering point $u$ and $p$ and $\phi$ as defined in section 2. When $R_{p,\phi} \neq \emptyset$, $p$ and $\phi$ determine the detector position $d$ and the measured energy $E_s$.

We have:

$$P_{\text{avg}}(p,\phi) = \frac{1}{A(R_{p,\phi})} \iint_{R_{p,\phi}} P(u;p,\phi)\,du$$  \hfill (50)

which gives the average of $P$ over $R_{p,\phi}$. Here $A(R_{p,\phi})$ denotes the area of $R_{p,\phi}$. Let $n_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ have support contained in $D_r$, and let $I_M(d,E_s) = I_M(p,\phi)$ denote the scattered intensity we measure at a detector $d$ with energy $E_s$. Then we approximate:

$$I_M(p,\phi) \approx I_{\text{avg}}(p,\phi) = s P_{\text{avg}}(p,\phi) \mathcal{D}_1 n_\epsilon(p,\phi)$$  \hfill (51)

where $s$ is the (constant) slice thickness. So, to account for the physical modelling, we would divide the data $I_M$ by $sP_{\text{avg}}$ to calculate approximate values for $\mathcal{D}_1 n_\epsilon$ and hence for $\mathcal{D} n_\epsilon$.

Let:

$$I_{\text{pred}}(p,\phi) = s \iint_{R_{p,\phi}} P(u;p,\phi)\,n_\epsilon(u)\,du$$  \hfill (52)

be the scattered intensity predicted for an arbitrary density $n_\epsilon$, and let:

$$I_1(C,p,\phi) = sC \iint_{R_{p,\phi}} P(u;p,\phi)\,du$$  \hfill (53)

be the predicted scattered intensity for a constant density $C$ over $R_{p,\phi}$. Then, assuming the measured intensity is as predicted by the model (52) (i.e. $I_M = I_{\text{pred}}$), the absolute error in the approximation (51) would satisfy:

$$|I_M(p,\phi) - I_{\text{avg}}(p,\phi)| \leq \left| I_1 \left( \max_{u \in R_{p,\phi}} n_\epsilon(u), p, \phi \right) - I_1 \left( \min_{u \in R_{p,\phi}} n_\epsilon(u), p, \phi \right) \right|$$  \hfill (54)

for all $(p,\phi) \in \mathbb{Z}^+$. So provided that the range of the density values is small over the majority of scattering regions considered, the averaged model (51) given above will have a similar level of accuracy to the more precise model given in equation (52).

### 3.1 Determining the atomic number

With the proposed machine configuration, we can show that the data collected in the light field determines the linear attenuation coefficient $\mu$ uniquely (this is the standard 2D reconstruction problem). With the additional information provided by
our theory, we show under the right assumptions that the atomic number of the target is determined uniquely by the full data (light plus dark field).

The electron density \( n_e \) and the linear attenuation coefficient \( \mu \) are related via the formula:

\[
\mu (E, Z) = n_e \sigma_e (E, Z)
\]  
(55)

where \( \sigma_e \) is the total cross section per electron. The cross section \( \sigma_e \) is continuous and monotone increasing as a function of \( Z \) on \([1, Z_{\text{max}}]\), where \( Z_{\text{max}} \propto \sqrt{E} \). For example, when \( E = 100\text{keV} \) we can calculate \( Z_{\text{max}} = 86 \). With this in mind, we fit a smooth curve to known values of \( \sigma_e \) given in the atomic data tables [9]. This allows us to calculate values of \( \sigma_e \) for non integer \( Z \). See figure 8.

![Figure 8](image.png)

**Figure 8:** We have presented our fit for \( \sigma_e \) for \( E = 100\text{keV} \) up to \( Z = 86 \) where a sudden dip in the \( \sigma_e \) values occurs. The tabulated values of \( \sigma_e \) given in [9] are shown alongside the fitted curve.

The formula for the fit presented for a fixed energy \( E = 100\text{keV} \) is:

\[
\sigma_e (Z) = \sigma_p (Z) + \sigma_s (Z) = \left( 1.51 \times 10^{-6} Z^{4.72-0.22 \log Z} \right) + \left( 0.49 + 7.90 \times 10^{-4} \left( 1 - Z^{-0.50} \right) Z^{1.57} \right)
\]  
(56)

This was obtained via a combination of the formula for \( \sigma_s \) (the total scattering cross section) presented by Jackson and Hawkes in [10] and the suggested fit for \( \sigma_p \) (the photoelectric cross section) given in [9]. Fits for energies other than \( E = 100\text{keV} \) are also possible via the same fitting method.

From our theory we know that the data determines \( \mu_E \) and \( n_e \) uniquely, where \( E \leq E_{\text{max}} \). If we assume that atomic numbers \( Z \geq Z_{\text{max}} \) are not present in the target material, then it is clear from the above arguments that the atomic number of the target is uniquely determined. Without this assumption the atomic number would be limited to a range of values. If we reconstruct both \( \mu_E \) and \( n_e \) for a suitably high energy \( E \), we can then calculate values for \( Z \) from our curve fit for \( \sigma_e \). We will test this additional method in our results also.
To test our reconstruction methods, let us consider the water bottle cross section \( f \) and the corresponding function \( \tilde{f} \) represented in figure 9. We calculate values of \( Df \) for \( p \) in the range \([0, 1]\) and for \( \phi \in \{ \frac{\pi}{180}, \ldots, 2\pi \} \). These were calculated using the exact formula for the area of intersection of two discs. We approximate the derivative of \( Df \) with respect to \( p \) as the finite difference:

\[
\frac{\partial}{\partial p} Df(p, \phi) = \frac{Df(p + h, \phi) - Df(p, \phi)}{h}
\]  

(57)

for a chosen step size \( h \). To reconstruct \( \tilde{f} \) we apply the Matlab function “iradon”, which filters (choosing from a selection of filters pre-coded by Matlab) and backprojects the projection data \( R\tilde{f} = -\frac{\partial}{\partial p} Df \) to recover \( \tilde{f} \). This is an implementation of the filtered backprojection algorithm [12, chapter 3]. We then make the necessary change in coordinates to produce our density image \( f \). In the absence of noise we find our results to be satisfactory. See figure 10.

Let us now perturb the calculated values of \( Df \) slightly such as to simulate random noise. We multiply each exact value of \( Df \) by a pseudo random number in the range \([1 - \%err/100, 1 + \%err/100]\) (we use the C++ function “rand” to generate random numbers), where \( \%err \) is the desired amount of percentage error to be added. In this case, even with a relatively small amount of added noise, the data must be smoothed sufficiently before applying approximation (57). To smooth the data, we apply a simple moving average filter and calculate any intermediate values via a shape preserving cubic interpolation method (“pchip” interpolation in Matlab). We expect this interpolation method to preserve the monotonicity of the data (monotone decreasing) as a function of \( p \). To illustrate this technique we refer to figure 11. We have presented our reconstructions after smoothing with 2\%, 10\% and 50\% added noise in figures 12, 13 and 14.

Here, we have reconstructed \( f \) from a single viewpoint, using data collected from a single source projection. With the proposed machine configuration however, there are a number of views from which \( f \) may be reconstructed. So we take an average over 360 views (for source positions at equal \( \pi/180 \) intervals over the range \([0, 2\pi]\)). Our results are presented in figures 15 and 16. Here we see an improvement in the signal-to-noise-ratio. The rotational symmetry of \( f \) about the centre of the circular region of \( f \)'s support is also recovered.

Let \( f_{avg} \) be the average of the non zero pixel values shown in the left hand image of figure 15 and let \( Z = 7.420 \) be the effective atomic number for water. Then we can calculate \( f_{avg} \approx 1.033 \) and using equation (55) we can calculate the total cross section to be:

\[
\sigma_e(E, 7.420) = \frac{\mu(E, 7.420)}{1.033} = 0.493
\]  

(58)

for \( E = 100\text{keV} \) assuming no additional error. Based on our curve fit for \( \sigma_e(100, Z) \), this would yield a reconstructed atomic number of \( Z = 0.886 \), which differs from the accepted value by 88\%. For the remaining averaged density reconstructions the \( f_{avg} \) and \( Z \) values are given in the figure caption.

We have presented reconstructions of a density which is homogeneous where it is not known to be zero. To give an inhomogeneous example, we have presented...
reconstructions with varying levels of added noise of a simulated hollow tube cross section in figures 17 and 18.

We can summarize our method as follows:

1. Measure the scattered intensity energy $E_s$ and divide by $P_{avg}$ and the slice thickness to calculate values for $Df$.

2. Smooth the data sufficiently and apply approximation (57) to calculate values for $R\hat{f}$.

3. Reconstruct $\hat{f}$ by filtered backprojection and recover $f$ from the definition given in Theorem 1.

4. Average over a number of source views to improve the image quality and set $f$ to 0 outside its support.
Figure 9: A water bottle cross section $f$ is simulated as a circular region of uniform density 1 on the left. The function $\tilde{f}$ as defined in Theorem 1 is shown on the right.

Figure 10: A reconstruction of $\tilde{f}$ in the absence of added noise is shown on the left. We have backprojected from 180 views with the default Ram-Lak cropped filter. The corresponding pixel values of $f$ are presented on the right. Both $f$ and $\tilde{f}$ are set to 0 outside of their support.
Figure 11: On the left we have plotted values of $Df(p,0)$ for $p \geq 0$ with 10% random noise added. On the right we have applied a simple moving average filter to the simulated data and taken a subsample of the smoothed data before interpolating as specified earlier. The exact values are presented alongside the fitted values in the right hand figure.

Figure 12: On the left we have a reconstruction of $\tilde{f}$ after smoothing with 2% added noise. We have again backprojected from 180 views, although here we have multiplied the standard ramp filter by a Hamming window to reduce the high frequency noise. The corresponding pixel values for $f$ are presented on the right.
Figure 13: On the left, a reconstruction of $\tilde{f}$ after smoothing with 10% added noise. We have multiplied the ramp filter by a Hamming window and backprojected from 180 views. The corresponding pixel values for $f$ are displayed on the right.

Figure 14: A reconstruction of $\tilde{f}$ after smoothing with 50% added noise is shown on the left. We have multiplied the ramp filter by a Hamming window and backprojected from 180 views. The corresponding pixel values for $f$ are displayed on the right.
Figure 15: On the left, an average reconstruction of $f$ is shown with no noise added to each dataset before reconstruction. For the right hand image 2% random noise was added to each dataset before reconstruction. In this case $f_{\text{avg}} = 0.853$ which gives an atomic number value of $Z = 13.3$.

Figure 16: On the left, an average reconstruction of $f$ is shown with 10% noise added to each dataset before reconstruction. Here $f_{\text{avg}} = 0.865$ which gives an atomic number value of $Z = 12.9$. For the right hand image 50% random noise was added to each dataset before reconstruction. In this case $f_{\text{avg}} = 0.863$ which gives a reconstructed atomic number value of $Z = 13.0$. 
Figure 17: On the left we have a simulated hollow tube $f_t$. On the right is an averaged reconstruction of $f_t$ with no noise added to each dataset.

Figure 18: We have presented averaged reconstructions of $f_t$ with 10% and 50% added noise in the left and right hand images respectively.
5 Conclusion

We have proposed a new fast method to determine the electron density in X-ray scanning applications, with a fixed energy sensitive detector machine configuration where it is possible to measure photon intensity in the dark field. We have shown that the density may be reconstructed analytically using the Compton scattered intensity. This method does not require the photon source to be monochromatic as is the case in recent literature, which is important from a practical standpoint as it may not be reasonable to assume a monochromatic source in some applications. Also if the source is monochromatic we cannot gain any insight into the energy dependence of the attenuation coefficient, which would rule out the recent advances in image rendering [4, 5], where a combination of multivariate and cluster analysis can be used to render a colour X-ray image.

Using Sobolev space estimates, we have determined an upper bound for the least squares error in our solution in terms of the least squares error in our data. This work is based on the approach taken by Natterer in [13].

We have shown, under the right assumptions, that the atomic number of the target is determined uniquely by the full data. With this theory in place we intend to pursue a more practical means to reconstruct the atomic number \(Z\), as the graph reading method used in the present paper was ineffective in giving an accurate reconstruction for \(Z\).

We summarize our method to recover the density image in section 4 and we reconstruct a simulated water bottle cross section via a possible practical implementation of this method. In this simple case the smoothing method (simple moving average) applied was effective and we were able to reconstruct a circular cross section of approximately uniform density. Although in the presence of noise the pixel values of our reconstructed density image on average differed from the original values by as much as 15%. We have also provided reconstructions of a simulated hollow tube cross section. In this case the inner edge of the tube cross section appeared quite blurred in the reconstruction when noise was added to the simulated data. We performed a number of trial reconstructions with different randomly generated datasets. The results presented in this paper are typical of our trial results.

We hope also to test our methods through experiment. For example, if we were to take an existing X-ray machine of a similar configuration to that discussed in the present paper, and attach energy sensitive detectors alongside the existing detectors or if we were to replace them, then we could see how closely our forward problem models the intensity of photons measured in the dark field in practice.

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Appendix – The RTT80; An example application in threat detection

The RTT80 (real time tomography) X–ray scanner is a switched source, offset detector CT machine designed with the aim to scan objects in real time. Developed by Rapiscan systems, the RTT80 is currently used in airport security screening of baggage.

The RTT80 consists of a single fixed ring of polychromatic X–ray sources and multiple offset rings of detectors, with a conveyor belt and scanning tunnel (within which the scanned object would be placed) passing through the centre of both sets of rings. See figure 19. If the detectors are energy sensitive, then in this case we have the problem of reconstructing a density slice supported within the scanning tunnel from its integrals over toric sections, with tips at the source and detector locations. We wish to check whether it is reasonable to approximate a set of toric section integrals as integrals over discs whose boundaries intersect a given source point, as then we can apply our proposed reconstruction method to reconstruct the density slice analytically.

Let us refer to figure 20 and let $D_{p,\phi}$ be defined as in section 2. We define the toric sections $T_{p,\phi}^1 = D_{p,\phi} \cap D_{p,\phi}$, $T_{p,\phi}^2 = D_{p,\phi} \cap D_{p,\phi}$, $T_{p,\phi}^3 = D_{p,\phi} \cup D_{p,\phi}$ and $T_{p,\phi}^4 = D_{p,\phi} \cup D_{p,\phi}$. Let $A(S)$ denote the area of a set $S \subseteq \mathbb{R}^2$ and let $T \subseteq \mathbb{R}^2$ denote the set of points within our ROI (region of interest, i.e. the scanning tunnel). For a large sample of discs, we will check for every disc $D_{p,\phi}$ in the sample, whether $\exists i \in \{1, 2, 3, 4\}$ such that $A(D_{p,\phi} \cap T) \approx A(T_{p,\phi}^i \cap T)$.

Let $D_r$ be defined as in section 2. Then if we consider the machine specifications for the RTT80, we can calculate $r = 6.75$ and the difference in radius between the detector ring and the scanning tunnel to be 0.375. See figure 19. For our test, we consider a sample of 36000 discs with diameters $p = 1.375 + \frac{5(i-1)}{99}$ for $1 \leq i \leq 100$ and $\phi = \frac{\pi j}{180}$ for $1 \leq j \leq 360$. We have chosen $p \in [1.375, 6.375]$ and $\phi \in [0, 2\pi]$ values in a range sufficient to determine a unique density slice image for densities supported on $T$. Refer to Corollary 2. For each of our chosen $p$ and $\phi$ value pairs, the difference:

$$
\min_{1 \leq i \leq 4} (A(D_{p,\phi} \cap T) - A(T_{p,\phi}^i \cap T)) \approx 10^{-16}
$$

was found to be negligible. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an example density slice with support contained in $T$. Then for any disc $D_{p,\phi}$ in our sample we have:

$$
\int_{D_{p,\phi}} f = \int_{D_{p,\phi} \cap T} f = \int_{T_{p,\phi}^i \cap T} f = \int_{T_{p,\phi}^i} f
$$

which holds for some $i \in \{1, 2, 3, 4\}$. So, the integral of $f$ over $D_{p,\phi}$ is equal to at least one of four toric section integrals over $f$. Assuming also that there is little error implied by our physical approximations (these are discussed in detail in section 3), the integral (60) would be determined approximately by at least one of four data sets, namely the photon intensity measured for two possible energy levels at two possible detector locations ($d_{p,\phi}^1$ or $d_{p,\phi}^2$). Thus, given that the inverse disc transform is only mildly ill posed (this was determined to be the case in section 2.1, based on the criteria given by Natterer in [13]), it seems that we should be able obtain a satisfactory density image reconstruction in this application.

In airport baggage screening, we are interested in identifying a given material as either a threat or non-threat. Let $n_e$ be the electron density and let $Z$ denote the...
effective atomic number. We define the threat space to be the set of materials with 
\((n_e, Z) \in T\), where \(T \subseteq [0, \infty) \times [1, 100]\) is the class of threat \((n_e, Z)\) pairs. For a given
suspect material, we can apply the methods presented in this paper to reconstruct 
\(n_e\) and \(Z\). Then if \((n_e, Z) \in T\), we can identify the suspect material as a potential
threat. We note that although we failed to obtain an accurate \(Z\) reconstruction in the
present paper, we aim to show that a more precise determination of \(Z\) is possible in
future work. Also, the reconstruction methods we have presented should be fast to
implement as they are largely based on the filtered back-projection algorithm. This
is important in an application such as airport baggage screening, as we require the
threat detection method we apply to not only be accurate in threat identification, but
to also be an efficient process.
Figure 19: The RTT80 machine configuration is displayed. The source-detector ring offset is relatively small and will be modelled as zero. The RTT80’s relative dimensions (the source ring, detector ring and scanning tunnel radii) are presented to the left of the diagram.

Figure 20: The RTT80 configuration is displayed. The origin is denoted by $O$ as in section 2. This is where a source is located. A disc $D_{p,\phi}$ with boundary intersecting $O$ and two detector points $d_{p,\phi}^1$ and $d_{p,\phi}^2$ is shown to have a non empty intersection with the set of points in our ROI (the scanning tunnel). The disc $D_{p,\phi}^1$ is the reflection of $D_{p,\phi}$ in the line segment connecting $O$ to $d_{p,\phi}^1$. Similarly for $D_{p,\phi}^2$. The union of a disc and its reflection about one of its chords is a toric section (e.g. $D_{p,\phi} \cup D_{p,\phi}^1$), which is where the photons scatter by the explanation given in the introduction. This is why we consider these reflections.
References


Chapter 3

Three dimensional Compton scattering tomography
Three dimensional Compton scattering tomography

By James W. Webber and William R.B. Lionheart

Abstract

We propose a new acquisition geometry for electron density reconstruction in three dimensional X-ray Compton imaging using a monochromatic source. This leads us to a new three dimensional inverse problem where we aim to reconstruct a real valued function $f$ (the electron density) from its integrals over spindle tori. We prove injectivity of a generalized spindle torus transform on the set of smooth functions compactly supported on a hollow ball. This is obtained through the explicit inversion of a class of Volterra integral operators, whose solutions give us an expression for the harmonic coefficients of $f$. The polychromatic source case is later considered, and we prove injectivity of a new spindle interior transform, apple transform and apple interior transform on the set of smooth functions compactly supported on a hollow ball.

A possible physical model is suggested for both source types. We also provide simulated density reconstructions with varying levels of added pseudo random noise and model the systematic error due to the attenuation of the incoming and scattered rays in our simulation.

1 Introduction

In this paper we lay the foundations for a new three dimensional imaging technique in X-ray Compton scattering tomography. Recent publications present various two dimensional scattering modalities, where a function in the plane is reconstructed from its integrals over circular arcs [2, 3, 4]. Three dimensional Compton tomography is also considered in the literature, where a gamma source is reconstructed from its integrals over cones with a fixed axis direction [5, 6, 7]. In [8], Truong and Nguyen give a history of Compton scattering tomography, from the point by point reconstruction case in earlier modalities to the circular arc transform modalities in later work. Here we present a new three dimensional scattering modality, where we aim to reconstruct the electron density (the number of electrons per unit volume) from its integrals over the surfaces of revolution of circular arcs. This work provides the theoretical basis
for a new form of non invasive density determination which would be applied in fields
such as fossil imaging, airport baggage screening and more generally in X-ray spec-
troscopic imaging. Our main goal is to show that a unique three dimensional density
reconstruction is possible with knowledge of the Compton scattered intensity with our
proposed acquisition geometry, and to provide an analytic expression for the density
in terms of the Compton scattered data.

A spindle torus is the surface of revolution of a circular arc. Specifically we define:

\[ T_r = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \left( r - \sqrt{x_1^2 + x_2^2} \right)^2 + x_3^2 = 1 + r^2 \right\} \]  

(1)

to be the spindle torus, radially symmetric about the \( x_3 \) axis, with tube centre offset
\( r \geq 0 \) and tube radius \( \sqrt{1 + r^2} \). Let \( B_{0,1} \) denote the unit ball in \( \mathbb{R}^3 \). Then we define
the spindle \( S_r = T_r \cap B_{0,1} \) as the interior of the torus \( T_r \) and we define the apple
\( A_r = T_r \setminus S_r \) as the remaining exterior. See figure 1.

In three dimensions, the surface of scatterers in Compton scattering tomography is
the surface of revolution of a circular arc about its circle chord (see section 3). Equiv-
alently, the surface of scattering points is a spindle torus with an axis of revolution
\( sd \), whose tube radius and tube centre offset are determined by the distance \( |sd| \) and
the scattering angle \( \omega \). In [3], Nguyen and Truong present an acquisition geometry
in two dimensions for a monochromatic source (e.g. a gamma ray source) and energy
resolving detector pair, where the source and detector remain at a fixed distance op-
posite one another and are rotated about the origin on the curve \( S^1 \) (the unit circle).
Here, the dimensionality of the data is two (an energy variable and a one dimensional
rotation). Taking our inspiration from Nguyen and Truong’s idea, we propose a novel
acquisition geometry in three dimensions for a single source and energy resolving de-
tector pair, which are rotated opposite each other at a fixed distance about the origin
on the surface \( S^2 \) (the unit sphere). Our data set is three dimensional (an energy
variable and a two dimensional rotation).

As illustrated in figure 1, the forward scattered (for scattering angles \( \omega \leq \pi/2 \))
intensity measured at the detector \( d \) for a given energy \( E_s \), can be written as a weighted
integral over the spindle \( S_r \) (the measured energy \( E_s \) determines \( r \)). With this in mind
we aim to reconstruct a function supported within the unit ball from its weighted
integrals over spindles. Similarly, the backscattered (\( \omega > \pi/2 \)) intensity can be given
as the weighted integral over the apple \( A_r \). We also consider the exterior problem,
where we aim to reconstruct a function supported on the exterior of the unit ball from
its weighted integrals over apples.

In section 2 we introduce a new spindle transform for the monochromatic forward
scatter problem and introduce a generalization of the spindle transform which gives the
integrals of a function over the surfaces of revolution of a particular class of symmetric
curves. We prove the injectivity of the generalized spindle transform on the domain
of smooth functions compactly supported on the intersection of a hollow ball with the
upper half space \( x_3 > 0 \). We show that our problem can be decomposed as a set of one
dimensional inverse problems, which we then solve to provide an explicit expression
for the harmonic coefficients of \( f \). In section 2.1 we introduce a new toric interior
transform for the polychromatic forward scatter problem and prove its injectivity on
the domain of smooth functions compactly supported on the intersection of a hollow
ball and \( x_3 > 0 \). A new apple and apple interior transform are also introduced in
Figure 1: A spindle torus slice with height $H$, tube radius $R$ and tube centre offset $r$ has tips at source and detector points $s$ and $d$. The spindle $S$ and the apple $A$ are highlighted by a solid green line and a dashed blue line respectively. The spherical scanning region highlighted has unit radius.

section 2.2 for the monochromatic and polychromatic backscatter problem. Their injectivity is proven on the domain of smooth functions compactly supported on the intersection of the exterior of the unit ball and $x_3 > 0$.

In section 3 we discuss possible approaches to the physical modelling of our problem for the case of a monochromatic and polychromatic photon source, and explain how this relates to the theory presented in section 2.

In section 4 we provide simulated density reconstructions of a test phantom via a discrete approach. We simulate data sets using the equations given in section 2 and apply our reconstruction method with varying levels of added pseudo random noise. We also simulate the added effects due to the attenuation of the incoming and scattered rays in our data and see how this systematic error effects the quality of our reconstruction.
2 A spindle transform

We will now parameterize the set of points on a spindle $S_r$ in terms of spherical coordinates $(\rho, \theta, \varphi)$ and give some preliminary definitions before going on to define our spindle transform later in this section.

Consider the circular arc $C$ as illustrated in figure 2. By the cosine rule, we have:

$$R^2 = \rho^2 + r^2 + 2r\rho \sin \varphi$$

and hence:

$$\rho = \sqrt{r^2 \sin^2 \varphi + 1 - r \sin \varphi}.$$  \hspace{1cm} (3)

Let $B_{\epsilon_1, \epsilon_2} = \{x \in \mathbb{R}^3 : \epsilon_1 < |x| < \epsilon_2\}$ denote the set of points on a hollow ball with inner radius $\epsilon_1$ and outer radius $\epsilon_2$, and let $S^2$ denote the unit sphere. Let $Z = \mathbb{R}^+ \times S^2$. For a function $f : \mathbb{R}^3 \to \mathbb{R}$, let $F : Z \to \mathbb{R}$ be its polar form $F(\rho, \theta, \varphi) = f(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi)$. We parameterize $h \in \text{SO}(3)$ (the group of rotations about the origin of $\mathbb{R}^3$) in terms of Euler angles $\alpha$ and $\beta$, $h = U(\alpha)V(\beta)$, where $U$
and $V$ are rotations about the $x_3$ and $x_2$ axis respectively as in [11, page 206]. We define an action of the rotation group $\text{SO}(3)$ on the set of real valued functions $f$ on $\mathbb{R}^3$ in the natural way as $(h \cdot f)(x) = f(hx)$, and define an action on the polar form as $(h \cdot F) = (h \cdot f)$.

The arc element for the circular arc $C$ is given in [3] as:

$$\text{d}v = \rho \sqrt{\frac{1 + r^2}{1 + r^2 \sin^2 \varphi}} \text{d}\varphi$$

(4)

and the area element for the spindle $S_r$ is:

$$\text{d}A = \rho \sin \varphi \text{d}v \text{d}\theta.$$  

(5)

We define the spindle transform $S : C_0^\infty(\mathbb{R}^3) \to C_0^\infty(Z)$ as:

$$Sf(r, \alpha, \beta) = \int_0^{2\pi} \int_0^\pi \rho^2 \sin \varphi \sqrt{\frac{1 + r^2}{1 + r^2 \sin^2 \varphi}} (h \cdot F)(\rho, \theta, \varphi) |_{\rho = \sqrt{\frac{\rho^2 + (r \sin \varphi)^2}{\rho^2 \sin \varphi + 1 - r \sin \varphi}}} \text{d}\varphi \text{d}\theta,$$

(6)

where $h = U(\alpha)V(\beta)$.

This transform belongs to a larger class of integral transforms as we will now show. Let $p \in C^1([0, \frac{\pi}{2}])$ be a curve parameterized by a colatitude $\varphi \in [0, \frac{\pi}{2}]$. Then we define the class of symmetric curves $\rho \in C^1([0, 1] \times [0, \pi])$ by:

$$\rho(r, \varphi) = p(\sin^{-1}(r \sin \varphi)).$$

(7)

From this we define the generalized spindle transform $S_{w,p} : C_0^\infty(\mathbb{R}^3) \to C_0^\infty([0, 1] \times S^2)$ as:

$$S_{w,p}f(r, \alpha, \beta) = \int_0^{2\pi} \int_0^\pi w(r, \varphi) \rho^2 \sin \varphi \sqrt{\rho^2 + \left(\frac{\text{d}\rho}{\text{d}\varphi}\right)^2} (h \cdot F)(\rho, \theta, \varphi) |_{\rho = \rho(r, \varphi)} \text{d}\varphi \text{d}\theta,$$

(8)

where the weighting $w : [0, 1] \times [0, \pi] \to \mathbb{R}$ is dependant only on $r$ and $\varphi$. The additional weighting $w$ will represent terms required for physical modelling which we will explain later in section 3.

We now aim to prove injectivity of the generalized spindle transform $S_{w,p}$ on the set of smooth functions on a hollow ball. For integers $l \geq 0$, $|m| \leq l$, we define the spherical harmonics $Y_l^m$ as in [11, page 207]. With this normalisation the $Y_l^m$ form an orthonormal basis for $L^2(S^2)$. From [9], we have the theorem:

**Theorem 1.** Let $f \in C^\infty(\mathbb{R}^3)$ and let:

$$F_l^m = \int_{S^2} F \overline{Y_l^m} \text{d}\tau,$$

(9)

where $\text{d}\tau$ is the surface measure on the sphere. Then the series:

$$F_N = \sum_{0 \leq l \leq N} \sum_{|m| \leq l} F_l^m Y_l^m$$

(10)

converges uniformly absolutely on compact subsets of $Z$ to $F$. 


We now show how the problem of reconstructing a density \( f \) from its integrals \( S_{w,p}f \) for some curve \( p \in C^1([0, \frac{\pi}{2}]) \) can be broken down into a set of one dimensional inverse problems to solve for the harmonic coefficients \( f_{lm} \):

**Lemma 1.** Let \( f \in C_0^\infty(\mathbb{R}^3) \), let \( w : [0, 1] \times [0, \pi] \to \mathbb{R} \) be a weighting and let \( p \in C^1([0, \frac{\pi}{2}]) \) be a curve, then:

\[
S_{w,p}f_{lm}(r) = 2\pi \int_0^\pi w(r, \varphi) \rho \sin \varphi \sqrt{\rho^2 + \left( \frac{d\rho}{d\varphi} \right)^2} F_{lm}(\rho) P_l(\cos \varphi) \mid_{\rho = \rho(r, \varphi)} \, d\varphi, \tag{11}
\]

where

\[
S_{w,p}f_{lm} = \int_{S^2} (S_{w,p}f) \bar{Y}_l^m \, d\tau,
\]

the \( F_{lm} \) are defined as in Theorem 1 and \( P_l \) is a Legendre polynomial degree \( l \).

**Proof.** Let \( \tau \in S^2 \) and let \( h \in SO(3) \) act on a spherical harmonic \( Y_l^m \) by \( (h \cdot Y_l^m)(\tau) = Y_l^m(h\tau) \). Then we have:

\[
S_{w,p}f(r, \alpha, \beta) = \int_0^{2\pi} \int_0^\pi w(r, \varphi) \rho \sin \varphi \sqrt{\rho^2 + \left( \frac{d\rho}{d\varphi} \right)^2} (h \cdot F) (\rho, \theta, \varphi) \mid_{\rho = \rho(r, \varphi)} \, d\varphi d\theta
= \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} \int_0^{2\pi} \int_0^\pi w(r, \varphi) \rho \sin \varphi \sqrt{\rho^2 + \left( \frac{d\rho}{d\varphi} \right)^2} F_{lm}(\rho) (h \cdot Y_l^m)(\theta, \varphi) \, d\varphi d\theta,
\]

where \( h = U(\alpha)V(\beta) \).

We may write \( h \cdot Y_l^m \) as a linear combination of spherical harmonics of the same degree [11, page 209]:

\[
(h \cdot Y_l^m)(\tau) = \sum_{|n| \leq l} Y_l^n(\tau) D_{n,m}^{(l)}(h^{-1}), \tag{14}
\]

where the block diagonal entries \( D_{n,m}^{(l)} \) are defined:

\[
D_{n,m}^{(l)}(h) = D_{n,m}^{(l)}(U(\alpha)V(\beta)U(\gamma)) = e^{-i\nu} d_{n,m}^{(l)}(\cos \beta) e^{-i\nu} \tag{15}
\]

and \( d_{n,m}^{(l)} \) is given, for \( m = 0 \), by:

\[
d_{n,0}^{(l)}(\cos \beta) = (-1)^n \sqrt{\frac{(l-n)!}{4\pi(l+n)!}} P_n^m(\cos \beta). \tag{16}
\]

Here the \( D^l \) are the irreducible blocks which form the regular representation of \( SO(3) \).

After the expansion (14) is substituted into equation (13), we see that the inserted sum is zero unless \( n = 0 \) and we have:

\[
S_{w,p}f(r, \alpha, \beta) = 2\pi \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} \sqrt{\frac{(2l+1)}{4\pi}} D_{0,m}^{(l)}(h^{-1}) \int_0^\pi w(r, \varphi) \rho \sin \varphi \sqrt{\rho^2 + \left( \frac{d\rho}{d\varphi} \right)^2} F_{lm}(\rho) P_l(\cos \varphi) \, d\varphi.
\]

(17)
Since the regular representation of $\text{SO}(3)$ is unitary, we have $D_{l,m}(h^{-1}) = \overline{D_{m,l}(h)}$ and from the above we have:

$$Y_l^m(\alpha, \beta) = \sqrt{\frac{(2l+1)}{4\pi}} \overline{D_{m,l}(h)}(h).$$

(18)

It follows that:

$$S_{w,p}f(r,\alpha,\beta) = 2\pi \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} Y_l^m(\alpha, \beta) \int_0^\pi w(r,\varphi)\rho \sin \varphi \sqrt{\rho^2 + \left(\frac{d\rho}{d\varphi}\right)^2} F_{lm}(\rho)P_l(\cos \varphi) d\varphi.$$  

(19)

From which we have:

$$S_{w,p}f_{lm}(r) = \int_{S^2} S_{w,p}f Y_l^m d\tau$$

$$= 2\pi \int_0^\pi w(r,\varphi)\rho \sin \varphi \sqrt{\rho^2 + \left(\frac{d\rho}{d\varphi}\right)^2} F_{lm}(\rho)P_l(\cos \varphi) |_{\rho=r(r,\varphi)} d\varphi$$

(20)

and the result is proven. 

In his classical paper [20], Funk uses a similar expansion in spherical harmonics to invert the Funk transform, and in [21] Cormack and Quinto give an inversion formula for a spherical Radon transform using an expansion in higher dimensional Gegenbauer polynomials. Our inversion formula is similar to that of [21], in the sense that we will require a further integral equation type inversion of (20) to obtain our solution.

We now go on to show that, for functions $f$ supported on the intersection of a hollow ball and the upper half space, the set of one dimensional integral equations derived above are uniquely solvable for $f_{lm}$ for every $l \in \mathbb{N}$, $|m| \leq l$. For the moment we focus on the reconstruction of the coefficients $f_{lm}$ for even $l$.

We now have our main theorem:

**Theorem 2.** Let $p \in C^4([0,\frac{\pi}{2}])$ be such that $p(x) \neq 0$ for all $x \in [0,\frac{\pi}{2}]$, and let $g \in C^1([0,1])$ be defined as $g(x) = p(\sin^{-1} x)$. Let $A = \{\rho \in [0,1] : g(\rho) \in [0,\epsilon_1] \cup [\epsilon_2,\epsilon_3]\}$ and let $w : [0,1] \times [0,\pi] \to \mathbb{R}$ be a weighting such that $W(r,\rho) = w(r,\sin^{-1}(\rho/r)) + w(r,\pi - \sin^{-1}(\rho/r))$ and its first order partial derivative with respect to $r$ is continuous on $\{(r,\rho) : r \in A, \min(A) \leq \rho \leq r\}$ and $W(r,\rho) \neq 0$ on $A$, where $\min(A)$ is the minimum value of $A$. Let $f \in C^0_{\text{loc}}(B_{0,\epsilon_1} \cup B_{\epsilon_2,\epsilon_3})$, where $\epsilon_1 < p(0) < \epsilon_2$. Then $S_{w,p}f$ determines the harmonic coefficients $F_{lm}$ uniquely for $\rho \in g([0,1])$ for all $l \in \{2k : k \in \mathbb{N}\}$, $|m| \leq l$.

**Proof.** Given that the Legendre polynomials $P_l$ are symmetric ($P_l(x) = P_l(-x)$, $x \in \mathbb{R}$) for even $l$, we have:

$$\frac{1}{2\pi} S_{w,p}f_{lm}(r) = \int_0^{\frac{\pi}{2}} (w(r,\varphi) + w(r,\pi - \varphi)) \rho \sin \varphi \sqrt{\rho^2 + \left(\frac{d\rho}{d\varphi}\right)^2} F_{lm}(\rho)P_l(\cos \varphi) |_{\rho=r(r,\varphi)} d\varphi$$

$$= \int_0^{\frac{\pi}{2}} (w(r,\varphi) + w(r,\pi - \varphi)) g(\rho) \sin \varphi \sqrt{g(\rho)^2 + r^2 \cos^2 \varphi} g'(\rho)^2 F_{lm}(g(\rho))P_l(\cos \varphi) |_{\rho=r\sin \varphi} d\varphi.$$  

(21)
After making the substitution $\rho = r \sin \varphi$, equation (21) becomes:

$$\frac{1}{2\pi} S_{w,p} f_{lm}(r) = \int_0^r \frac{F_{lm}(g(\rho)) K_l(r, \rho)}{\sqrt{r - \rho}} d\rho$$

(22)

a Volterra integral equation of the first kind with weakly singular kernel, where:

$$K_l(r, \rho) = W(r, \rho) \rho g(\rho) \sqrt{g(\rho)^2 + (r^2 - \rho^2) g'(\rho)^2} \frac{P_l \left( \sqrt{1 - \frac{\rho^2}{r^2}} \right)}{r \sqrt{r - \rho}}.$$  

(23)

As $F_{lm} \circ g$ is zero for $\rho$ close to 0, given our prior assumptions regarding $W$ and given that $g \in C^1([0, 1])$, we can see that $K_l$ and its first order derivative with respect to $r$ is continuous on the support of $F_{lm} \circ g$.

Multiplying both sides of equation (22) by $1/\sqrt{z - r}$ and integrating with respect to $r$ over the interval $[0, z]$, yields:

$$\frac{1}{2\pi} \int_0^z \frac{S_{w,p} f_{lm}(r)}{\sqrt{z - r}} dr = \int_0^z F_{lm}(g(\rho)) \left[ \int_0^r \frac{K_l(r, \rho)}{\sqrt{z - r} \sqrt{r - \rho}} dr \right] d\rho$$

$$= \int_0^z F_{lm}(g(\rho)) Q_l(z, \rho) d\rho$$

(24)

after changing the integration order. Making the substitution $r = \rho + (z - \rho)t$, gives:

$$Q_l(z, \rho) = \int_\rho^z \frac{K_l(r, \rho)}{\sqrt{z - r} \sqrt{r - \rho}} dr$$

$$= \int_0^1 \frac{K_l(\rho + (z - \rho)t, \rho)}{\sqrt{t} \sqrt{1 - t}} dt,$$

(25)

from which we have:

$$Q_l(z, z) = \pi K_l(z, z) = W(z, z) \frac{\pi c_l g(z)^2}{\sqrt{2z}},$$

(26)

where

$$c_l = P_l(0) = \frac{(-1)^{(l/2)}}{2^l} \left( \begin{array}{c} l/2 \\ l/2 \end{array} \right).$$

(27)

So by our assumptions that $p$ is non zero and $W$ is non zero on the diagonal, $Q_l(z, z) \neq 0$ on the support of $F_{lm} \circ g$.

Differentiating both sides of equation (24) with respect to $z$ and rearranging gives:

$$v_{lm}(z) = \int_0^z F_{lm}(g(\rho)) H_l(z, \rho) d\rho + F_{lm}(g(z)),$$

(28)

which is a Volterra type integral equation of the second kind, where

$$v_{lm}(z) = \frac{1}{2\pi^2 K_l(z, z)} \frac{d}{dz} \int_0^z \frac{S_{w,p} f_{lm}(r)}{\sqrt{z - r}} dr$$

(29)

and

$$H_l(z, \rho) = \frac{1}{\pi K_l(z, z)} \frac{d}{dz} Q_l(z, \rho).$$

(30)
Given the continuity of \( H_l \) on the support of \( F_{lm} \circ g \) and the continuity of \( F_{lm} \circ g \) on \([0, 1]\), the Neumann series associated with equation (28) converges and we may write our solution:

\[
F_{lm}(g(z)) = \int_0^z R_l(z, \rho) v_{lm}(\rho) d\rho + v_{lm}(z),
\]

where the resolvent kernel:

\[
R_l(z, \rho) = \sum_{i=1}^{\infty} H_{l,i}(z, \rho)
\]

is defined by:

\[
H_{l,1}(z, \rho) = H_l(z, \rho), \quad H_{l,i}(z, \rho) = \int_0^z H_l(z, x) H_{l,i-1}(x, \rho) dx \quad \forall i \geq 2.
\]

Since the series converges, the solution is unique and we may reconstruct \( F_{lm} \) explicitly for \( \rho \in g([0, 1]) \).

The above Theorem uses well established theory in the literature [14] regarding the solution to weakly singular Volterra equations of the first kind. This theory is also applied in [15] to obtain an explicit inversion formula for a circular Radon transform.

Now we have a theorem regarding those functions \( f \in C_0^\infty(\mathbb{R}^3) \) whose support lies in the upper half space \( x_3 > 0 \):

**Theorem 3.** Let \( U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\} \) and let \( f \in C_0^\infty(U) \). Then the even coefficients \( F_{lm} \) for \( l \in \{2k : k \in \mathbb{N}\} = \mathbb{N}_e, |m| \leq l \) determine \( F \) uniquely and:

\[
F(\rho, \theta, \varphi) = 2 \sum_{l \in \mathbb{N}_e} \sum_{|m| \leq l} F_{lm}(\rho) Y_{ml}^m(\theta, \varphi)
\]

for \( \rho \geq 0, 0 \leq \theta \leq 2\pi \) and \( 0 \leq \varphi \leq \frac{\pi}{2} \).

**Proof.** We can write \( F \) as the sum of its even and odd coefficients:

\[
F(\rho, \theta, \varphi) = F_o(\rho, \theta, \varphi) + F_e(\rho, \theta, \varphi) = \sum_{\text{odd } l \in \mathbb{N}} \sum_{|m| \leq l} F_{lm}(\rho) Y_{ml}^m(\theta, \varphi) + \sum_{\text{even } l \in \mathbb{N}} \sum_{|m| \leq l} F_{lm}(\rho) Y_{ml}^m(\theta, \varphi).
\]

Then from our assumption, we have:

\[
-F_e(\rho, \theta, \varphi) = F_o(\rho, \theta, \varphi) = \sum_{\text{odd } l \in \mathbb{N}} \sum_{|m| \leq l} F_{lm}(\rho) Y_{ml}^m(\theta, \varphi)
\]

\[
= \sum_{m \in \mathbb{Z}} \left[ \sum_{\text{odd } l \geq |m|} c(l, m) F_{lm}(\rho) P_l^m(\cos \varphi) \right] e^{i m \theta}
\]

\[
= \sum_{m \in \mathbb{Z}} F_{m}^m(\rho, \varphi) e^{i m \theta}
\]

for all \( \rho \geq 0, 0 \leq \theta \leq 2\pi \) and \( \frac{\pi}{2} \leq \varphi \leq \pi \), where:

\[
c(l, m) = (-1)^m \sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}}.
\]
We have:

\[-F_e^m(\rho, \varphi) = -\frac{1}{2\pi} \int_0^{2\pi} F_e(\rho, \theta, \varphi) e^{-im\theta} d\theta = F_o^m(\rho, \varphi) \text{ for } \rho \geq 0, \frac{\pi}{2} \leq \varphi \leq \pi. \] (38)

The associated Legendre polynomials \(P_l^m\) are symmetric when \(l + m\) is even and antisymmetric otherwise. It follows that:

\[F_o(\rho, \theta, \varphi) = \sum_{m \in \mathbb{Z}} (-1)^{m+1} F_o^m(\rho, \pi - \varphi) e^{im\theta}\]

\[= \sum_{m \in \mathbb{Z}} (-1)^m F_e^m(\rho, \varphi) e^{im\theta}\]

for \(\rho \geq 0, 0 \leq \theta \leq 2\pi\) and \(0 \leq \varphi \leq \frac{\pi}{2}\). The result follows.

**Corollary 1.** Let \(f \in C_0^{-\infty}(B_{\epsilon_1, \epsilon_2} \cap U)\) for some \(0 < \epsilon_1 < \epsilon_2 < 1\). Let \(\epsilon_1 \leq \epsilon \leq \epsilon_2\) and let \(\delta = \frac{1-\epsilon}{2}\). Then \(Sf\) known for \(0 \leq r \leq \delta\) and for all \((\alpha, \beta) \in S^2\) determines \(f\) uniquely for \(\epsilon \leq |x| \leq 1\).

**Proof.** Let \(p(\varphi) = \sqrt{1 + \delta^2 \sin^2 \varphi - \delta \sin \varphi}\) and let \(w \equiv 1\). Then \(Sf(\delta r, \alpha, \beta) = S_{w,p} r(\alpha, \beta)\) for \(r \in [0, 1], (\alpha, \beta) \in S^2\). The result follows from Theorems 2 and 3.

So provided our density is supported in the upper half space and the conditions of Theorem 2 are met we find that a unique density reconstruction is possible with knowledge of weighted spindle data. Later in section 3 we will show how the scattered intensity we measure can be written as a weighted spindle transform with a weighting which satisfies the conditions of Theorem 2.

### 2.1 A toric interior transform; the polychromatic case

In the previous section we considered the three dimensional Compton scatter tomography problem for a monochromatic source and energy sensitive detector pair. The polychromatic source case is covered here and we consider the modifications needed in our model to describe a full spectrum of initial photon energies.

In an X-ray tube electrons are accelerated by a large voltage \(E_m\text{keV}\) towards a target material and photons are emitted. Due to conservation of energy, the emitted photons have energy no greater than \(E_m\text{keV}\). So for a given measured energy \(E_s\), where \(\frac{E_m}{1 - 2E_m/E_s} < E_s < E_m\), the set of scatterers is the union of spindle tori corresponding to scattering angles \(\omega\) in the range \(0 < \omega < \cos^{-1}\left(1 - \frac{E_s(E_m - E_s)}{E_s E_m}\right)\) (corresponding to energies \(E\) in the range \(E_s < E < E_m\)). That is, the set of scatterers is a torus interior:

\[I_r = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \left(r - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 < 1 + r^2\right\}, \] (40)

where \(r\) is determined by the scattered energy measured \(E_s\). See [1] for an explanation of the two dimensional case.
With this in mind we define the spindle interior transform $I : C^\infty_0(\mathbb{R}^3) \to C^\infty_0(\mathbb{Z})$ as:

$$I f(r, \alpha, \beta) = \int_0^{2\pi} \int_0^\pi \int_0^r \rho^2 \sin \varphi (h \cdot F)(\rho, \theta, \varphi) d\rho d\varphi d\theta,$$

and we have the uniqueness theorem for a weighted spindle interior transform:

**Theorem 4.** Let $f \in C^\infty_0(B_{\epsilon_1, \epsilon_2} \cap U)$ for some $0 < \epsilon_1 < \epsilon_2 < 1$ and and let $\delta_1 = \frac{1-\epsilon_1^2}{2\epsilon_1}$.

Let $\tilde{f}$ be defined as:

$$\frac{1}{\sqrt{\frac{(1-|x|^2)^2}{4|x|^4} + 1}} \tilde{f}(x) = \left(1 - \frac{|x|^2}{2|x|^2} - \frac{(1-|x|^2)^2}{4|x|^4} + 1 \right) f(x).$$

Define the weighted interior transform:

$$I_w f(r', \alpha, \beta) = \int_0^{r'} \int_0^{2\pi} \int_0^\pi w(r', t, \varphi) \rho^2 \sin \varphi \left( h \cdot \tilde{F} \right)(\rho, \theta, \varphi) \left|_{\rho=\frac{\sin^2 \frac{\theta}{2} + 1}{\sin^2 \frac{\theta}{2}}} \right. d\varphi d\theta dt,$$

where $h = U(\alpha)V(\beta)$ and $r' = 1/r$. Let us suppose that the weighting $w$ satisfies the following:

1. $w$ can be decomposed as $w(r', t, \varphi) = w_1(r', t)w_2(t, \varphi)$.
2. $W_2(t, \rho) = w_2 \left( 1/t, \sin^{-1}(\rho/t) \right) + w_2 \left( 1/t, \pi - \sin^{-1}(\rho/t) \right)$ and its first order partial derivative with respect to $t$ is continuous on the triangle $T = \{(t, \rho) : \epsilon_1 \leq t \leq \epsilon_2, \epsilon_1 \leq \rho \leq t\}$ and $W_2(\rho, \rho) \neq 0$ on $[\epsilon_1, \epsilon_2]$.
3. $w_1(r', t)$ and its first order partial derivatives are bounded on $0 \leq r' < \infty$ and $w_1(r', r') \neq 0$ for $0 \leq r' < \infty$.

Then $I_w f$ known for all $r' \in [0, \infty)$ and for all $(\alpha, \beta) \in S^2$ determines $f$ uniquely for $0 < |x| < 1$.

**Proof.** We have:

$$I_w f(r', \alpha, \beta) = \int_0^{r'} w_1(r', t)G(t, \alpha, \beta) dt,$$

where

$$G(t, \alpha, \beta) = \int_0^{2\pi} \int_0^\pi w_2(t, \varphi) \rho^2 \sin \varphi \left( h \cdot \tilde{F} \right)(\rho, \theta, \varphi) \left|_{\rho=\frac{\sin^2 \frac{\theta}{2} + 1}{\sin^2 \frac{\theta}{2}}} \right. d\varphi d\theta.$$

Differentiating both sides of equation (44) with respect to $r'$ and rearranging yields:

$$g(r', \alpha, \beta) = \int_0^{r'} L(r', t)G(t, \alpha, \beta) dt + G(r', \alpha, \beta),$$

where

$$g(r', \alpha, \beta) = -\frac{d}{dr'} I_w f(r', \alpha, \beta) \frac{w_1(r', r')}{w_1(r', r')}.$$
and
\[ L(r', t) = \frac{\partial}{\partial r'} w_1(r', t) \]
\[ w_1(r', r') \cdot (48) \]
Given our prior assumptions regarding \( w_1 \), we can solve the Volterra equation of the second kind (46) uniquely for \( G_{in}(t) \) for \( 0 < t < \infty \). From which we have:
\[ G \left( \frac{1}{\delta_1t}, \alpha, \beta \right) = \int_0^{2\pi} \int_0^\pi w_2 \left( \frac{1}{\delta_1t}, \varphi \right) \frac{\delta_1 t \rho^2 \sin \varphi}{\sqrt{\delta_1^2 t^2 \sin^2 \varphi + 1}} (h \cdot \tilde{F}) (\rho, \theta, \varphi) \mid_{\rho = \sqrt{\delta_1^2 t^2 \sin^2 \varphi + 1 - \delta_1 t \sin \varphi}} d\varphi d\theta \]
\[ = \frac{\delta_1 t}{\sqrt{1 + \delta_1^2 t^2}} S_{u_3,p} \tilde{f} (t, \alpha, \beta) \]
(49)
for \( t \in (0, 1] \), where \( w_3(t, \varphi) = w_2 \left( \frac{1}{\delta_1 t}, \varphi \right) \) and \( p(\varphi) = \sqrt{1 + \delta_1^2 \sin^2 \varphi} - \delta_1 \sin \varphi \). By our assumptions regarding \( w_2 \), the weighting \( w_3 \) satisfies the conditions of Theorem 2, as does the curve \( p \). The result follows from Theorems 2 and 3.

**Corollary 2.** Let \( f \in C^\infty_0 (B_{\epsilon_1, \epsilon_2} \cap U) \) for some \( 0 < \epsilon_1 < \epsilon_2 < 1 \). Then \( I f \) as defined in equation (41) known for all \( r \in [0, \infty) \) and for all \( (\alpha, \beta) \in S^2 \) determines \( f \) uniquely for \( 0 < |x| < 1 \).

**Proof.** Let \( \tilde{f} \) be defined as in Theorem 4. Then, after making the substitution \( \rho = \sqrt{\sin^2 \varphi + 1 - \sin \varphi} \) in equation (41), we have:
\[ I f (r, \alpha, \beta) = \int_0^1 \int_0^{2\pi} \int_0^\pi \frac{\rho^2 \sin \varphi}{t\sqrt{\sin^2 \varphi + 1}} (h \cdot \tilde{F}) (\rho, \theta, \varphi) \mid_{\rho = \sqrt{\sin^2 \varphi + 1 - \frac{\sin \varphi}{t}}} d\varphi d\theta dt \]
\[ = I_w f (1/r, \alpha, \beta) \]
(50)
for all \( r \in (0, \infty) \) and for all \( (\alpha, \beta) \in S^2 \), where the weighting \( w \equiv 1 \). The result follows from Theorem 4.

We can see from the above that the difference in the inversion process for the spindle transform and spindle interior transform is essentially a derivative step in the \( r' \) variable (\( r' \) is determined by the scattered energy). That is, after we differentiate with respect to \( r' \) and make a change of coordinates the spindle interior data is reduced to spindle data. We see this idea also in the literature in Compton scattering tomography [1], where the authors consider the polychromatic source case in two dimensions and provide an explicit left inverse for a disc transform.

The advantage of using a polychromatic source (e.g. an X-ray tube) over a monochromatic source, which would most commonly be some type of gamma ray source, is that they have a significantly higher output intensity and so the data acquisition is faster. They are also safer to handle and use and are already in use in many fields of imaging. The downside, when compared to using a monochromatic source, would be a decrease in efficiency of our reconstruction algorithm and the added differentiation step required in the inversion process. This makes the polychromatic problem more ill posed, and so small errors in our measurements would be more greatly amplified in the reconstruction. We should consider both source types for further testing to determine an optimal imaging technique.
2.2 The exterior problem

Here we consider the exterior problem, and show that a full set of apple integrals (which represent the backscattered intensity) are sufficient to reconstruct a compactly supported density on the exterior of the unit ball.

2.2.1 An apple transform

For backscattered photons (for scattering angles $\omega > \pi/2$) the surface of scatterers is an apple $A_r$. Refer back to figure 1. The spherical coordinates $(\rho, \theta, \varphi)$ of the points on $A_r$ can be parameterized as follows:

$$\rho = \sqrt{r^2 \sin^2 \varphi + 1 + r \sin \varphi}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi.$$ (51)

We define the apple transform $\mathcal{A} : C_0^\infty(\mathbb{R}^3) \to C_0^\infty(Z)$ as:

$$\mathcal{A}f(r, \alpha, \beta) = \int_0^{2\pi} \int_0^\pi \rho^2 \sin \varphi \sqrt{1 + r^2 \sin^2 \varphi} (h \cdot F)(\rho, \theta, \varphi) \, d\rho \, d\theta,$$ (52)

where $h = U(\alpha)V(\beta)$. We have the uniqueness theorem for the apple transform for functions supported on the exterior of the unit ball:

**Theorem 5.** Let $f \in C_0^\infty(B_{\epsilon_1, \epsilon_2} \cap U)$ for some $1 < \epsilon_1 < \epsilon_2 < \infty$. Let $\epsilon_1 \leq \epsilon \leq \epsilon_2$ and let $\delta = \frac{2-1}{2\epsilon}$. Then $\mathcal{A}f$ known for $0 \leq r \leq \delta$ and for all $(\alpha, \beta) \in S^2$ determines $f$ uniquely for $1 \leq |x| \leq \epsilon$.

**Proof.** Let $p(\varphi) = \sqrt{1 + \delta^2 \sin^2 \varphi} + \delta \sin \varphi$ and let $w \equiv 1$. Then $\mathcal{A}f(\delta r, \alpha, \beta) = S_{w,p}f(r, \alpha, \beta)$ for $r \in [0,1]$, $(\alpha, \beta) \in S^2$. The result follows from Theorems 2 and 3. \hfill \Box

2.2.2 An apple interior transform

Similar to our discussion at the start of section 2.1, if the source is polychromatic the set of backscatterers is an apple interior. Hence we define the apple interior transform $\mathcal{AI} : C_0^\infty(\mathbb{R}^3) \to C_0^\infty(Z)$:

$$\mathcal{AI}f(r, \alpha, \beta) = \int_0^{2\pi} \int_0^\pi \rho^2 \sin \varphi (h \cdot F)(\rho, \theta, \varphi) d\rho d\varphi,$$ (53)

and we have a theorem for its injectivity:

**Theorem 6.** Let $f \in C_0^\infty(B_{\epsilon_1, \epsilon_2} \cap U)$ for some $1 < \epsilon_1 < \epsilon_2 < \infty$. Let $\epsilon_1 \leq \epsilon \leq \epsilon_2$ and let $\delta = \frac{2-1}{2\epsilon}$. Then $\mathcal{AI}f$ known for $0 \leq r \leq \delta$ and for all $(\alpha, \beta) \in S^2$ determines $f$ uniquely for $1 \leq |x| \leq \epsilon$.

**Proof.** Let $\tilde{f}$ be defined as:

$$\tilde{f}(x) = \frac{1}{\sqrt{\frac{|x|^2-1}{4|x|^2} + 1}} \left( \frac{\frac{|x|^2-1}{4|x|^2}}{\sqrt{\frac{|x|^2-1}{4|x|^2} + 1}} + \frac{|x|^2-1}{2|x|^2} \right) f(x).$$ (54)
Then by Leibniz rule we have:
\[ r \frac{d}{dr} A\hat{f}(r, \alpha, \beta) = \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin \varphi \sqrt{1 + r^2 \sin^2 \varphi} (h \cdot \tilde{F})(\rho, \theta, \varphi) \big|_{\varphi = \sqrt{r^2 \sin^2 \varphi + 1 + r \sin \varphi}} d\varphi d\theta \]
\[ = \frac{1}{\sqrt{1 + r^2}} \hat{A} \tilde{f}(r, \alpha, \beta). \]  
(55)

So \( A\hat{f} \) known for \( 0 \leq r \leq \delta \) and for \((\alpha, \beta) \in S^2\) determines \( A\tilde{f} \) on the same set after differentiating. The result follows from Theorem 5.

\[ \square \]

3 A physical model

Here we explain how the theory presented in the previous section relates to what we measure in a practical setting. Compton scattering is the process which describes the inelastic scattering of photons with charged particles (usually electrons). A loss in photon energy occurs upon the collision. This is known as the Compton effect. The energy loss is dependant on the initial photon energy and the angle of scattering and is described by the following equation:
\[ E_s = \frac{E_\lambda}{1 + (E_\lambda/E_0) (1 - \cos \omega)}. \]  
(56)

Here \( E_s \) is the energy of the scattered photon which had an initial energy \( E_\lambda \), \( \omega \) is the scattering angle and \( E_0 \approx 511 \text{keV} \) is the electron rest energy. Typically Compton scattering refers to the scattering of photons in the mid energy range. That is, the scattering of X-rays and gamma rays with photon energies ranging from 1keV up to 1MeV. Forward Compton scattering is the scattering of photons with scattering angles \( \omega \leq \pi/2 \). Conversely, backscatter refers to the scattering of photons at angles \( \omega > \pi/2 \).

If the photon source is monochromatic (\( E_\lambda \) is fixed) and the detector is energy resolving (we can measure photon intensity at a given scattered energy \( E_s \)), then, in a given plane, the locus of scattering points is a circular arc connecting the source and detector points. See figure 3. Hence, in three dimensions, the surface of scatterers is the surface of revolution of a circular arc, namely a spindle torus.

We consider an intensity of photons scattering from a point \( x \) as illustrated in figure 3. The points \( s \) and \( d \) are the centre points of the source and detector respectively. The intensity of photons scattered from \( x \) to \( d \) with energy \( E_s \) is given by [16, page 73]:
\[ I(E_s) = I_0(E_\lambda) e^{-\int_{L_{sx}} \mu(E_\lambda, Z) f(x) \, dV} \times \frac{d\sigma}{d\Omega}(E_s, \omega) e^{-\int_{L_{xd}} \mu(E_\lambda, Z) S(q, Z) \, d\Omega}, \]  
(57)

where \( Z \) denotes the atomic number, \( \omega \) the scattering angle and \( L_{sx} \) and \( L_{xd} \) are the line segments connecting \( s \) to \( x \) and \( x \) to \( d \) respectively. \( f(x) \) denotes the electron density (number of electrons per unit volume) at the scattering point \( x \) and \( dV \) is the volume measure. \( f \) is the quantity to be reconstructed.

Let \( W_k(E_\lambda) \) denote the incident photon flux (number of photons per unit area per unit time), energy \( E_\lambda \), measured at a fixed distance \( D \) from the source. Then the incident photon intensity \( I_0 \) can be written [16, page 49]:
\[ I_0(E_\lambda) = \frac{t D^2 W_k(E_\lambda)}{(\rho \cos \varphi + 1)^2 + \rho^2 \sin^2 \varphi}, \]  
(58)
Figure 3: A scattering event occurs on the circular arc $C$ with initial photon energy $E_{\lambda}$ from the source $s$ to the detector $d$ with energy $E_s$.

where $t$ is the emission time.

The Klein-Nishina differential cross section $d\sigma/d\Omega$ [17], is defined by:

$$
\frac{d\sigma}{d\Omega} (E_s, \omega) = \frac{r_0^2}{2} \left( \frac{E_s}{E_{\lambda}} \right)^2 \left( \frac{E_s}{E_{\lambda}} + \frac{E_{\lambda}}{E_s} - 1 + \cos^2 \omega \right),
$$

(59)

where $r_0$ is the classical electron radius and $\cos \omega = r/\sqrt{1 + r^2}$. This predicts the scattering distribution for a photon off a free electron at rest. Given that the atomic electrons typically are neither free nor at rest, a correction factor is included, namely the incoherent scattering function $S(q, Z)$ [10]. Here $q = \frac{E_{\lambda}}{hc} \sin (\omega/2)$ is the momentum transferred by a photon with initial energy:

$$
E_{\lambda} = \frac{E_s}{1 - (E_s/E_0)(1 - \cos \omega)}
$$

(60)

scattering at an angle $\omega$, where $h$ is Planck’s constant and $c$ is the speed of light. As the scattering function $S$ is dependant on the atomic number $Z$, we set $Z = Z_{av}$ to some average atomic number as an approximation and interpolate values of $S(q, Z_{av})$ from the tables given in [10].

The solid angle subtended by $x$ and $d$ may be approximated as [4]:

$$
d\Omega = \frac{A}{4\pi} \times \frac{1 - \rho \cos \varphi}{(1 - \rho \cos \varphi)^2 + \rho^2 \sin^2 \varphi},
$$

(61)

where $A$ is the detector area.

The exponential terms in equation (57) account for the attenuation of the incoming and scattered rays by the Beer–Lambert law [18, 19]. We approximate:

$$
e^{-\int_{L_{sx}} \mu(E_{\lambda}, Z) d\ell} e^{-\int_{L_{xd}} \mu(E_s, Z) d\ell} \approx 1.
$$

(62)

In general this approximation is unrealistic. For example in medical CT, if we were scanning a relatively large (e.g. the size of someone’s head) mass of organic material at a low energy (e.g. 50keV), the absorption would play a significant role and the above model would over approximate the data. However in an application where the objects
are smaller (centimetres in diameter) and where we can scan at a higher energy (≈ 1MeV), the effects due to absorption would be less prevalent and Compton scattering would be the dominant interaction. For example, in airport baggage screening typical hand luggage would be a small bag containing a few low effective Z densities (e.g. acetone (a nail varnish solvent), water, some plastic (polyethylene) etc.) and the rest may be clothes or air. Here also, as we are not worried about dosage, we can scan at high energies (e.g. using a high voltage X-ray tube or high emission energy gamma ray source). We will simulate the error due to attenuation later, in section 4, and show the effects of neglecting the attenuation in our reconstruction.

3.1 The monochromatic case

Let our density \( f \) be supported on \( B_{\epsilon_1, \epsilon_2} \cap U \) for some \( 0 < \epsilon_1 < \epsilon_2 < 1 \) and let \( \delta_2 = \frac{1 - r^2}{2r^2} \). If the source \( s \) is monochromatic (\( E_\lambda \) remains fixed), then the forward Compton scattered intensity measured is:

\[
I(r, \alpha, \beta) = c t D^2 W_k(E_\lambda) \frac{d\sigma_c}{d\Omega}(r) S_{w,p} f(r, \alpha, \beta),
\]

where \( p \in C^1([0, \pi]) \) is defined by \( p(\varphi) = \sqrt{1 + \delta_2^2 \sin^2 \varphi - \delta_2 \sin \varphi} \), \( c \) is a constant thickness and:

\[
\frac{d\sigma_c}{d\Omega}(r) = \frac{d\sigma}{d\Omega}(E_s, \omega) S(q, Z_{avg})
\]

depends only on \( r \). Here the variable \( r \in [0, 1] \) determines the scattered energy \( E_s \) and \( (\alpha, \beta) \in S^2 \) determines the source and detector position. The weighting \( w \) is given by:

\[
w(r, \varphi) = \frac{A(1 - \rho \cos \varphi)}{4\pi \left[ (1 - \rho \cos \varphi)^2 + \rho^2 \sin^2 \varphi \right] \left[ (\rho \cos \varphi + 1)^2 + \rho^2 \sin^2 \varphi \right]},
\]

where \( \rho = \sqrt{1 + r^2 \sin^2 \varphi - r \sin \varphi} \). It is left to the reader to show that the weighting \( w \) satisfies the conditions given in Theorem 2. After dividing through by the physical modelling terms in equation (63), we can invert the weighted spindle transform \( S_{w,p} f \) as in Theorem 2 to obtain an analytic expression for \( f \) in terms of the Compton scattered intensity \( I \).

3.2 The polychromatic case

Here we have a spectrum of incoming photon energies. See figure 4. The Compton scattered intensity measured for a tube centre offset \( r = 1/r' \) for a source and detector position \( (\alpha, \beta) \in S^2 \) may be written:

\[
I(r', \alpha, \beta) = st D^2 I_{w} f(r', \alpha, \beta),
\]

with the weighting:

\[
w(r', t, \varphi) = \left[ W_k(r', t) \frac{d\sigma_c}{d\Omega}(r', t) \right] w_2(t, \varphi) = w_1(r', t) w_2(t, \varphi),
\]

where \( w_2 = w \) as in equation (65). Here the scattering probability \( \frac{d\sigma_c}{d\Omega} \) and the photon flux \( W_k \) are dependant on \( r' \) and the integration variable \( t \) as in subsection 2.1. While
the weighting \( w \) is separable as in Theorem 4, the incident photon flux \( W_k(r', r') = W_k(E_m) = 0 \) for all \( r' \), where \( E_m \) is the maximum spectrum energy. Hence \( w_1(r', r') \equiv 0 \) and \( w \) fails to meet the conditions of Theorem 4. To deal with this, let:

\[
\begin{align*}
\text{w}_{\text{avg}}(r') &= \frac{1}{r'} \int_0^{r'} w_1(r', t) \, dt \\
\text{w}_{\text{app}}(r', t, \varphi) &= w_{\text{avg}}(r') w_2(t, \varphi).
\end{align*}
\]

and let \( \text{w}_{\text{app}}(r', t, \varphi) = \text{w}_{\text{avg}}(r') w_2(t, \varphi) \). Then, although we sacrifice some accuracy in our forward model, \( \text{w}_{\text{app}} \) would satisfy the conditions given in Theorem 4 and we can obtain an analytic expression for the density. The error in our approximation is bounded by:

\[
\left| w_{\text{avg}}(r') - w_1(r', t) \right| \leq \left| \max_{t \in [0, r']} w_1(r', t) - \min_{t \in [0, r']} w_1(r', t) \right| \quad (69)
\]

for all \( r' \geq 0, 0 \leq t \leq r' \). So provided the changes in the incident photon energy \( E_s \) (\( E_s \) is determined by \( t \)) are negligible over the range \( t \in [0, r'] \), the error in \( \text{w}_{\text{app}} \) would be small.

4 Simulations

Here we provide reconstructions of a test phantom density at a low resolution using simulated datasets of the unweighted spindle and spindle interior transforms, and simulate noise as additive pseudo Gaussian noise. To simulate data for each transform we discretize the integrals in equations (6) and (41), and solve the Tikhonov least squares problem:

\[
\arg \min_x \| A x - b \|^2 + \lambda^2 \| x \|^2 \quad (70)
\]
for some regularisation parameter $\lambda > 0$, where $A$ is the discrete forward operator of
the transform considered, $x$ is the vector of density pixel values and $b = Ax$ is the
simulated transform data. We simulate perturbed data $b'$ via:

$$b' = b + \epsilon \frac{g\|b\|}{\sqrt{n}},$$

(71)

where $g$ is a pseudo random vector of samples from the standard Gaussian distribution
and $n$ is the number of entries in $b$. The proposed noise model has the property that $\|b - b'\|/\|b\| \approx \epsilon$, so a noise level of $\epsilon$ is $\epsilon \times 100\%$ relative error.

We consider the test phantom displayed in figure 5. The unit cube is discretised
into $50 \times 50 \times 50$ pixels and a hollow ball, some stairs and a low density block with a
metal sheet passing through it are placed in the upper unit hemisphere. Slice profiles
are given in the right hand figure to display the metal sheet and the hollow shell. We
sample 25 values of $r$ (corresponding to spindles with heights $H \in \{0.02 + 0.04i : 0 \leq i < 25\}$), 45 values of $\alpha \in \{\frac{2\pi i}{45} : 0 \leq i < 45\}$ and 45 values of $\beta \in \{\frac{\pi}{90}(1 + i) : 0 \leq i < 45\}$. So we have an underdetermined sparse system matrix $A$ with 50625 rows and 125000 columns. To reconstruct we solve the regularised problem (70) using
the conjugate gradient least squares algorithm (CGLS) and pick our regularisation
parameter $\lambda$ via a manual approach (for $0 < \lambda < 200$ depending on the level of noise).
In each reconstruction presented, any negative values are set to zero and the iteration
number and noise level are given in the figure caption.

In figure 6 we have presented a reconstruction of the test phantom in the absence
of added noise. Given the symmetries involved in our geometry, the density has been
rotated about the $x_3$ axis and reflected in the $x_1x_2$ plane in the reconstruction. To
better visualise the reconstruction we set our reconstructed images to 0 in the lower
half space. In figure 7 we display the same zero noise reconstruction but with the voxel
values set to zero in the lower hemisphere.

Figures 7–12 show test phantom reconstructions from spindle and spindle interior
transform data with varying levels of added noise. In the absence of noise (figures 7
and 10) the reconstructions are ideal. However in the presence of noise we notice a
harsher degradation in image quality when reconstructing with spindle interior data.
This is because the spindle interior problem is inherently more ill posed due to the
extra differentiation step required in the inversion process. At a low noise level (figure
11), while the shape of the objects can be deciphered and the ball appears to be hollow,
the reconstructions are not clear, in particular the lower density block and the metal
sheet start to become lost in the reconstruction. At a higher noise level (figure 12), the
artefacts in the reconstruction are severe, the ball no longer appears to be hollow and
the metal sheet fails to reconstruct. So, although the practical advantages of using a
polychromatic source such as an X-ray tube or linear accelerator are clear (i.e. reduced
data acquisition time, easy and safe to use etc.), a low noise level would need to be
maintained to achieve a satisfactory image quality. The spindle transform inversion
performs better when there is added noise. At a low noise level (figure 8), the size and
shape of the objects is clear and the image contrast is good. At a higher noise level,
the background noise in the image is amplified and the ends of metal sheet are hard
to identify. We notice, in the presence of noise, that the thinner object (namely the
metal sheet) is hardest to reconstruct. This is as we’d expect since smaller densities
are determined by the higher frequency harmonic components which degrade faster
with noise.
To simulate data for the reconstructions presented in figures 5–12 we applied the discrete operator $A$ to the vector of test phantom pixels $x$ and added Gaussian noise. We now include the effects due to the attenuation of the incoming and scattered rays in our simulated data and see how this affects the quality of our reconstruction. For our first example, we set the size of the scanning cube to be $20 \times 20 \times 20\text{cm}^3$ (each pixel is $4 \times 4 \times 4\text{mm}^3$) and the phantom materials are polyethylene (low density block), water (the stairs), rubber (the ball) and Aluminium (the metal sheet). We model the source as $^{60}\text{Co}$, a monochromatic gamma ray source widely used in security screening applications (e.g. screening of freight shipping containers), which has an emission energy of 2824keV. We also add 1% Gaussian noise as in equation (71) to simulate random error as well as the systematic error due to attenuation effects. Our results are presented in figure 13. Here the phantom reconstruction is satisfactory with only minor artefacts appearing in the image. So if we scan a low effective $Z$ target of a small enough size with a high energy source, the effects due to attenuation can be neglected while maintaining a satisfactory image quality. For our next example, we set the scanning cube size to be $50 \times 50 \times 50\text{cm}^3$ (each pixel is $1 \times 1 \times 1\text{cm}^3$) and the phantom materials are Teflon (low density block), PVC (stairs), polyoxymethylene (ball) and steel (the metal sheet). The source is as in our last example and again we add a further 1% Gaussian noise to the simulated data after we have accounted for the attenuative effects. Our results are presented in figure 14. Here, with a larger, higher density target, the effects due to attenuation are significant and the artefacts in the reconstruction are more severe.
Figure 5: Test phantom.

Figure 6: Test phantom reconstruction from spindle transform data no noise, 2000 iterations.
Figure 7: Test phantom reconstruction from spindle transform data no noise, set to zero in lower hemisphere, 2000 iterations.

Figure 8: Test phantom reconstruction from spindle transform data, noise level $\epsilon = 0.01$, 2000 iterations.
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Figure 9: Test phantom reconstruction from spindle transform data, noise level $\epsilon = 0.05$, 2000 iterations.

Figure 10: Test phantom reconstruction from spindle interior data no noise, 2000 iterations.
Figure 11: Test phantom reconstruction from spindle interior data, noise level $\epsilon = 0.01$, 2000 iterations.

Figure 12: Test phantom reconstruction from spindle interior data, noise level $\epsilon = 0.05$, 2000 iterations.
Figure 13: Small, low effective $Z$ test phantom reconstruction from spindle transform data with attenuation effects added and noise level $\epsilon = 0.01$, 2000 iterations.

Figure 14: Large, high effective $Z$ test phantom reconstruction from spindle transform data with attenuation effects added and noise level $\epsilon = 0.01$, 2000 iterations.
5 Conclusions and further work

We have presented a new acquisition geometry for three dimensional density reconstruction in Compton imaging with a monochromatic source and introduced a new spindle transform and a generalization of the spindle transform for the surfaces of revolution of a class of symmetric $C^1$ curves. The generalized spindle transform was shown to be injective on the domain of smooth functions $f$ supported on the intersection of a hollow ball with the upper half space $x_3 > 0$. In section 2 it was shown that our problem could be decomposed into a set of one dimensional inverse problems to solve for the harmonic coefficients of a given density, which we then went on to solve via the explicit inversion of a class of Volterra integral operators. Later in section 2.1 we considered the problem for a polychromatic source and introduced a new spindle interior transform and proved its injectivity on the set of smooth functions compactly supported on the intersection of a hollow ball and $x_3 > 0$. We also considered the exterior problem for backscattered photons with a monochromatic and polychromatic photon source, where in section 2.2 we introduced a new apple and apple interior transform. Their injectivity was proven on the set of smooth functions compactly supported on the intersection of the exterior of the unit ball and $x_3 > 0$. We note that in Palamodov’s paper on generalized Funk transforms [12], although he provides an explicit inverse for a fairly general family of integral transforms over surfaces in three dimensional space, the spindle transform is excluded from this family of transforms.

In section 3 we discussed a possible approach to the physical modelling of our forward problem for both a monochromatic and polychromatic source. In the monochromatic source case, we found that the Compton scattered intensity resembled a weighted spindle transform (as in Theorem 2) which could be solved explicitly via repeated approximations. In the polychromatic source case, with a more accurate forward model, we found that an analytic reconstruction was not possible. So we suggested a simplified model in order to obtain an analytic expression for the density, and gave some simple error estimates for our approximation.

Test phantom reconstructions were presented in section 4 using simulated datasets from spindle and spindle interior transform data with varying levels of added pseudo random Gaussian noise. When reconstructing with spindle transform data the reconstructions were satisfactory for low noise levels (1%) but we saw a significant degradation in the image quality with a higher noise level (5%). In particular the low density (blue) block was distorted in the image and the ends of the metal sheet were hard to identify. We saw a harsher reduction in image quality in the presence of added noise when reconstructing from spindle interior data. This was as expected given the increased instability of the spindle interior problem.

In the latter part of section 4, we provided further reconstructions of our test phantom image in the presence of a systematic error due to the attenuative effects of the incoming and scattered rays. We found, when scanning a small, low effective $Z$ target with a high energy source, that the attenuative effects could be neglected while maintaining a good image quality. We also gave an example where the target materials were larger and higher effective $Z$. Here the effects due to attenuation were significant and we saw a more severe reduction in the image quality. The simulation work presented here gives a verification of our inversion formulae and investigates the effects due to attenuation and random noise. The noise and attenuation effects are shown in some cases to severely hamper the quality of the reconstruction and so at its
current stage the work presented here does not yet constitute a robust reconstruction algorithm. In further work we aim to refine our reconstruction method to better deal with noisy data. This would include an improvement to our noise model (Poisson noise would be more appropriate for low counts) and regularisation method (combined generalized Tikhonov and total variation) and an inverse solver that is suited to such a penalty term such as Chambolle–Pock [22].

An inverse problem is said to be mildly ill posed if it is stable in some Sobolev norms and severely ill posed if it is not stable in any Sobolev space. Numerical evidence suggests that the inverse problem for the spindle transform is severely ill posed, but that does not rule out Compton scattering tomography as a practical technique. Indeed one of the most widely used X-ray imaging techniques in non-destructive testing and dentistry uses a circle scan cone beam. Finch [23] shows that, while the forward problem for this is injective for a large enough circle and detector, the inverse problem is severely ill posed. Jump discontinuities on smooth surfaces where no rays are measured tangential to the surface cannot be stably recovered with this geometry. Users of such systems are typically aware of this limitation and the artefacts it produces in their volume images. When matrix based methods are used for solution, regularization must be included that constrains the solution to be smooth in the problematic directions. Microlocal analysis proves an effective tool for understanding the directions in which reconstruction problems for integral transforms are unstable and preliminary results for the spindle transform appear in the preprint [24].

Although the injectivity of the exterior problem is covered here, simulations of an exterior density reconstruction are left for future work.

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References


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Chapter 4

Microlocal analysis of a spindle transform
Abstract

An analysis of the stability of the spindle transform, introduced in [16], is presented. We do this via a microlocal approach and show that the normal operator for the spindle transform is a type of paired Lagrangian operator with “blowdown–blowdown” singularities analogous to that of a limited data synthetic aperture radar (SAR) problem studied by Felea et. al. [4]. We find that the normal operator for the spindle transform belongs to a class of distributions $I^{p,l}_\Delta(\Delta,\Lambda) + I^{p,l}_{\tilde{\Delta}}(\tilde{\Delta},\Lambda)$ studied by Felea and Marhuenda in [4, 10], where $\tilde{\Delta}$ is reflection through the origin, and $\Lambda$ is associated to a rotation artefact. Later, we derive a filter to reduce the strength of the image artefact and show that it is of convolution type. We also provide simulated reconstructions to show the artefacts produced by $\Lambda$ and show how the filter we derived can be applied to reduce the strength of the artefact.

1 Introduction

Here we present a microlocal analysis of the spindle transform, first introduced by the authors in [16], which describes the Compton scattering tomography problem in three dimensions for a monochromatic source and energy sensitive detector pair. Compton scattering is the process in which a photon interacts in an inelastic collision with a charged particle. As the collision is inelastic, the photon undergoes a loss in energy, described by the equation

$$E_s = \frac{E_\lambda}{1 + (E_\lambda/E_0)(1 - \cos \omega)} ,$$

(1)

where $E_s$ is the energy of the scattered photon which had an initial energy $E_\lambda$, $\omega$ is the scattering angle and $E_0 \approx 511\text{keV}$ is the electron rest energy. For $E_s$ and $E_\lambda$ fixed (i.e if the source is monochromatic and we can measure $E_s$), the scattering angle $\omega$ remains fixed and, in three dimensions, the surface of scatterers is the surface of revolution of a circular arc [16]. The surface of revolution of a circular arc is a spindle torus. We define

$$T_r = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \left(r - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 = 1 + r^2\}$$

(2)
to be the spindle torus, radially symmetric about the $x_3$ axis, with tube centre offset $r \geq 0$ and tube radius $\sqrt{1 + r^2}$. See figure 1 which displays a rotation of $T_r$.

In [13] Norton considered the problem of reconstructing a density supported in a quadrant of the plane from the Compton scattered intensity measured at a single point detector moved laterally along the axis away from a point source at the origin. Here the curve of scatterers is a circle. He considers the circle transform

$$\tilde{S}f(r, \theta) = \int_0^{2\pi} \int_0^\pi \rho^2 \sin \varphi \sqrt{\frac{1 + r^2}{1 + r^2 \sin^2 \varphi}} (h \cdot F)(\rho, \psi, \varphi) \big|_{\rho = \sqrt{r^2 \sin^2 \varphi + 1 - r \cos \varphi}} \, d\varphi \, d\psi,$$

where $F(\rho, \psi, \varphi) = f(\rho \cos \psi \sin \varphi, \rho \sin \psi \sin \varphi, \rho \cos \varphi)$ is the spherical polar form of $f : \mathbb{R}^3 \to \mathbb{R}$ and $h \in \text{SO}(3)$ describes the rotation of the north pole to $\theta$, where $h$ defines a group action on real-valued functions in the natural way $(h \cdot f)(x) = f(hx)$. They show that a left inverse to $\tilde{S}$ exists through the explicit inversion of a set of one-dimensional Volterra integral operators, and show that the null space of $\tilde{S}$ consists of those functions whose even harmonic components are zero (odd functions). However the stability of the spindle transform was not considered. We aim to address this here from a microlocal perspective. In [4], various acquisition geometries are considered for synthetic aperture radar imaging of moving objects. In each case the microlocal properties of the forward operator and its normal operator are analysed.
three of the four cases considered the Schwartz kernel of the normal operator was shown to belong to a class of distributions associated to two cleanly intersecting Lagrangians $P_{p,l}(\Delta, \Lambda)$ (that is, the wavefront set of the kernel of the normal operator is contained in $\Delta \cup \Lambda$). We show a similar result for the spindle transform $S^*S$, although the diagonal $\Delta$ is replaced by the disjoint union $\Delta \cup \tilde{\Delta}$ where $\tilde{\Delta}$ is reflection through the origin. We also determine the associated Lagrangian $\Lambda$. The authors of [4] suggest a way to reduce the size of the image artefact microlocally by applying an appropriate pseudodifferential operator as a filter before applying the backprojection operator. Similarly we derive a suitable filter for the spindle transform and show how it can be applied using the spherical harmonics of the data.

In section 2.1 we show that $S$ is equivalent to a weighted cylinder transform $C$, which gives the weighted integrals over cylinders with an axis of revolution through the origin. After this we prove that $C$ is a Fourier integral operator and we compute its canonical relation. Later in section 2.2 we present our main theorem (Theorem 3), where we show that $C^*C$ belongs to a class of distributions $P_{p,l}(\Delta, \Lambda) + P_{p,l}(\tilde{\Delta}, \Lambda)$, where $\tilde{\Delta}$ is a reflection and the Lagrangian $\Lambda$ is associated to a rotation artefact.
In section 3, we adopt the ideas of Felea et al. in [4] and derive a suitable pseudodifferential operator $Q$ which, when applied as a filter before applying the backprojection operator of the cylinder transform, reduces the artefact intensity in the image. We show that $Q$ can be applied by multiplying the harmonic components of the data by a factor $c_l$, which depends on the degree $l$ of the component, and show how this translates to a spherical convolution of the data with a distribution on the sphere $h$.

Simulated reconstructions from spindle transform data are presented in section 4. We reconstruct a small bead of constant density by unfiltered backprojection and show that the artefacts produced by $\Lambda$ in our reconstruction. We then reconstruct the same density by filtered backprojection, applying the filter $Q$ as an intermediate step, and show how the size of the artefacts are reduced in the image. Later we provide reconstructions of densities of oscillating layers using the conjugate gradient least squares (CGLS) method and Landweber iteration. We arrange the layers as spherical shells centred at the origin and as planes and compare our results. We also investigate the effects of applying the filter $Q$ as a pre-conditioner, prior to implementing CGLS and the Landweber method.

2 The microlocal properties of $S$ and $S^*S$

Here we investigate the microlocal properties of the spindle transform and its normal operator. We start by showing the equivalence of $S$ to a cylinder transform $C$, and how we can write $S$ and $C$ as Fourier integral operators. Then we calculate the canonical relation $C$ of $C$ and show that the left and right projections of $C$ drop rank by 1 and have blowdown singularities. From here and using the Theorems of Felea et al. [4] we show that $C^*C$ belongs to a class of distributions associated to two cleanly intersecting Lagrangians $\Lambda_1$ and $\Lambda_2$ ($C^*C \in I^{p,l}(\Lambda_1,\Lambda_2)$ is a paired Lagrangian operator). First we give some preliminaries.

Let $B_{\epsilon_1,\epsilon_2}^n = \{ x \in \mathbb{R}^n : 0 < \epsilon_1 < |x| < \epsilon_2 < 1 \}$ denote the set of points on a hollow ball with inner radius $\epsilon_1$ and outer radius $\epsilon_2$. Let $Z^n = \mathbb{R} \times S^{n-1}$ denote the $n$-cylinder and, for $X \subset \mathbb{R}^n$ an open set, let $\mathcal{D}'(X)$ denote the vector space of distributions on $X$, and let $\mathcal{E}'(X)$ denote the vector space of distributions with compact support contained in $X$. We next recall some definitions.

**Definition 1** ([8, Definition 7.1.1]). For a function $f$ in the Schwartz space $S(\mathbb{R}^n)$ we define the Fourier transform and its inverse in terms of angular frequency as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx,$$

$$\mathcal{F}^{-1}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) d\xi. \quad (6)$$

**Definition 2** ([8, Definition 7.8.1]). Let $m, \rho, \delta \in \mathbb{R}$ with $0 \leq \rho \leq 1$ and $\delta = 1 - \rho$. Then we define $S^m_\rho(X \times \mathbb{R}^n)$ to be the set of $a \in C^\infty(X \times \mathbb{R}^n)$ such that for every compact set $K \subset X$ and all multi–indices $\alpha, \beta$ the bound

$$|\partial_\xi^\beta \partial_x^\alpha a(x,\xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \quad x \in K, \ \xi \in \mathbb{R}^n, \quad (7)$$

holds for some constant $C_{\alpha,\beta,K}$. The elements of $S^m_\rho$ are called *symbols* of order $m$, type $\rho$. 


Definition 3 ([9, Definition 21.2.15]). A function \( \phi = \phi(x, \xi) \in C^\infty(X \times \mathbb{R}^N \setminus 0) \) is a phase function if \( \phi(x, \lambda \xi) = \lambda \phi(x, \xi), \forall \lambda > 0 \) and \( d \phi \neq 0 \). A phase function is clean if the critical set \( \Sigma_\phi = \{(x, \xi) : d_x \phi(x, \xi) = 0\} \) is a smooth manifold with tangent space defined by \( dd_\xi \phi = 0 \).

By the implicit function theorem the requirement for a phase function to be clean is satisfied if \( dd_\xi \phi \) has constant rank.

Definition 4 ([9, Definition 21.2.15] and [7, Section 25.2]). Let \( X \subset \mathbb{R}^{n_x}, Y \subset \mathbb{R}^{n_y} \) be open sets. The canonical relation parametrised by a clean phase function \( \phi \in C^\infty((X \times Y) \times \mathbb{R}^N \setminus 0) \) is defined as

\[
C = \{(y, \eta, (x, \omega)) \in (Y \times \mathbb{R}^{n_y} \setminus 0) \times (X \times \mathbb{R}^{n_x} \setminus 0) : (x, y, \omega) \in \Sigma_\phi, \omega = -d_x \phi(x, y, \xi), \eta = d_y \phi(x, y, \xi) \},
\]

where \( \Sigma_\phi = \{(x, y, \xi) : d_x \phi(x, y, \xi) = 0 \} \) is the critical set of \( \phi \).

For the definition of canonical relation see [9, Definition 21.2.12], although this will not be important for us.

Definition 5 ([7, Section 25.2] or [2, Section 2.4]). Let \( X \subset \mathbb{R}^{n_x}, Y \subset \mathbb{R}^{n_y} \) be open sets. A Fourier integral operator (FIO) of order \( m + N/2 - (n_x + n_y)/4 \) is an operator \( A : C_0^\infty(X) \to \mathcal{D}'(Y) \) with Schwartz kernel given locally by oscillatory integrals of the form

\[
K_A(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x, y, \xi)} a(x, y, \xi) d\xi,
\]

where each \( \phi \) is a clean phase function and \( a \in S^m_\rho((X \times Y) \times \mathbb{R}^N) \) a symbol, and all of the phase functions in the local representations parametrise pieces of a single canonical relation \( C \subset (Y \times \mathbb{R}^{n_y} \setminus 0) \times (X \times \mathbb{R}^{n_x} \setminus 0) \).

If \( Y \) and \( X \) are manifolds without boundary, then an operator \( A : C_0^\infty(X) \to \mathcal{D}'(Y) \) is a Fourier integral operator if its Schwartz kernel can be represented locally in coordinates by oscillatory integrals of the form (9), and the canonical relations of the phase functions for the local representations all lie within a single immersed Lagrangian submanifold of \( T^*Y \times T^*X \). For much more detail on Fourier integral operators and their definition see [7].

2.1 The spindle transform as an FIO

Recall the author’s acquisition geometry in [16] (displayed in figure 1). We have the implicit equation

\[
(r + |x \times \theta|)^2 + (x \cdot \theta)^2 = 1 + r^2
\]

for the set of points on a spindle with tube centre offset \( r \) and axis of revolution given by \( \theta \in S^2 \). With this in mind we define

\[
h(s, x, \theta) = \frac{4|x \times \theta|^2}{(1 - |x|^2)^2} - s,
\]

and then we can write the spindle transform \( \mathcal{S} : C_0^\infty(B_{t_1, t_2}^3) \to C^\infty((0, 1) \times S^2) \) as

\[
\mathcal{S}f(s, \theta) = \int_{B_{t_1, t_2}^3} \delta \left( \frac{4|x \times \theta|^2}{(1 - |x|^2)^2} - s \right) \frac{f(x)}{\|\nabla x h(s, x, \theta)\|} dx,
\]

for the set of points on a spindle with tube centre offset \( r \) and axis of revolution given by \( \theta \in S^2 \). With this in mind we define

\[
h(s, x, \theta) = \frac{4|x \times \theta|^2}{(1 - |x|^2)^2} - s,
\]

and then we can write the spindle transform \( \mathcal{S} : C_0^\infty(B_{t_1, t_2}^3) \to C^\infty((0, 1) \times S^2) \) as

\[
\mathcal{S}f(s, \theta) = \int_{B_{t_1, t_2}^3} \delta \left( \frac{4|x \times \theta|^2}{(1 - |x|^2)^2} - s \right) \frac{f(x)}{\|\nabla x h(s, x, \theta)\|} dx,
\]
where \( s = 1/r^2 \) and \( \delta \) is the Dirac–delta function. Note that
\[
\nabla_x h(s, x, \theta) = 8(1 - |x|^2) + 16 |x \times \theta|^2 x
\]
(13)
is smooth, bounded above, and does not vanish on \((0,1) \times B^3_{\epsilon_1, \epsilon_2} \times S^2\). We define the backprojection operator \( S^* : C^\infty((0,1) \times S^2) \to C^\infty(B^3_{\epsilon_1, \epsilon_2}) \) as
\[
S^*g(x) = \int_{S^2} \frac{g\left(\frac{4|x \times \theta|^2}{(1-|x|^2)^2}, \theta\right)}{|\nabla_x h(s, x, \theta)|} d\Omega,
\]
(14)
where \( d\Omega \) is the surface measure on \( S^2 \).

**Proposition 1.** The backprojection operator \( S^* \) is the adjoint operator to \( S \).

**Proof.** Let \( g \in C^\infty((0,1) \times S^2) \) and \( f \in C^\infty_0(B^3_{\epsilon_1, \epsilon_2}) \) and let \( \langle \cdot, \cdot \rangle_{L^2((0,1) \times S^2)} \) denote the \( L^2 \) inner product on \( L^2((0,1) \times S^2) \). Then (in the third step note that \( \nabla_x h \) does not actually depend on \( s \))
\[
\langle g, Sf \rangle_{L^2((0,1) \times S^2)} = \int_{S^2} \int_0^1 g(s, \theta)Sf(s, \theta)dsd\Omega
\]
\[
= \int_{S^2} \int_0^1 g(s, \theta) \int_{B^3_{\epsilon_1, \epsilon_2}} \frac{\delta\left(\frac{4|x \times \theta|^2}{(1-|x|^2)^2} - s\right)}{|\nabla_x h(s, x, \theta)|} f(x)dxdsd\Omega
\]
\[
= \int_{B^3_{\epsilon_1, \epsilon_2}} \int_{S^2} \frac{g\left(\frac{4|x \times \theta|^2}{(1-|x|^2)^2}, \theta\right)}{|\nabla_x h(s, x, \theta)|} d\Omega f(x)dx
\]
\[
= \int_{B^3_{\epsilon_1, \epsilon_2}} S^*g(x)f(x)dx = \langle S^*g, f \rangle_{L^2((0,1) \times S^2)},
\]
which completes the proof. \( \square \)

Let \( v(x) = \left(\frac{1}{1 + \frac{|x|^2}{|x|^2}} - \frac{1}{|x|^2}\right) \cdot \frac{x}{|x|}\) and set \( \alpha_i = 2\epsilon_i/(1 - \epsilon_i^2) \) for \( i = 1 \) or \( 2 \) so that when \( |x| = \alpha_i, \ |v(x)| = \epsilon_i \). Then, after making the substitution \( x \to v(x) \) in equation (12), we have
\[
Sf(s, \theta) = \int_{B^3_{\alpha_1, \alpha_2}} \frac{|\det(J_v)|}{|\nabla_x h(s, v(x), \theta)|} \frac{\delta\left(|x \times \theta|^2 - s\right)}{|\nabla_x h(s, v(x), \theta)|} f \left( \sqrt{1 + \frac{1}{|x|^2} - \frac{1}{|x|^2}} \cdot \frac{x}{|x|} \right) dx
\]
\[
= \int_{B^3_{\alpha_1, \alpha_2}} \frac{\delta\left(|x|^2 - (x \cdot \theta)^2 - s\right)}{|\nabla_x h(s, v(x), \theta)|} \tilde{f}(x)dx,
\]
(16)
where
\[
\tilde{f}(x) = |\det(J_v)| f \left( \sqrt{1 + \frac{1}{|x|^2} - \frac{1}{|x|^2}} \cdot \frac{x}{|x|} \right),
\]
(17)
where \( J_v \) denotes the Jacobian of \( v \). We define the weighted cylinder transform \( C : C^\infty_0(B^3_{\alpha_1, \alpha_2}) \to C^\infty((0,1) \times S^2) \) as
\[
Cf(s, \theta) = \int_{B^3_{\alpha_1, \alpha_2}} \frac{\delta\left(|x|^2 - (x \cdot \theta)^2 - s\right)}{\sqrt{|\nabla_x h(s, v(x), \theta)|}} f(x)dx
\]
(18)
and its backprojection operator $C^* : C^\infty((0,1) \times S^2) \rightarrow C^\infty(B^3_{a_1,a_2})$:
\[ C^* g(x) = \int_{S^2} \frac{g(|x \times \theta|^2, \theta)}{|\nabla v h(s,v(x),\theta)|} d\Omega. \] (19)

As in Proposition 1, we can show that $C^*$ is the formal adjoint to $C$.

The above is to say that the spindle transform is equivalent, via the diffeomorphism $x \rightarrow (\sqrt{1 + \frac{1}{|x|^2} - \frac{1}{|\theta|^2}} \cdot \frac{x}{|x|}, \theta)$, to the transform $C$ which defines the weighted integrals over cylinders with radius $\sqrt{s}$ and axis of rotation through the origin with direction $\theta$. With this in mind we consider the microlocal properties of the cylinder transform $C$ and its normal operator $C^*C$ for the remainder of this section.

First, we characterise $C$ as a Fourier integral operator in the next theorem.

**Theorem 1.** The cylinder transform $C$ is a Fourier integral operator order $-1$ with canonical relation
\[ C = \{ ((s, \alpha, \beta), (\sigma, 2\sigma(x \cdot \theta_\alpha)(x \cdot \theta), 2\sigma(x \cdot \theta_\beta)(x \cdot \theta)); (x, 2\sigma(x - (x \cdot \theta)\theta)) : x \in B^3_{a_1,a_2}, \]
\[ s \in (0,1), \sigma \in \mathbb{R} \setminus 0, \theta \in S^2, |x|^2 - (x \cdot \theta)^2 - s = 0 \}, \] (20)

where $(\alpha, \beta) \in \mathbb{R}^2$ provide a local parameterization of $\theta$, $\partial_\alpha \theta = \partial_\beta \theta$.

**Proof.** The delta function may be written as the oscillatory integral
\[ \delta(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\sigma s} d\sigma. \] (21)

Thus, by equation (18) we have
\[ Cf(s, \theta) = \int_{B^3_{a_1,a_2}} \frac{\delta(|x|^2 - (x \cdot \theta)^2 - s)}{|\nabla v h(s, v(x), \theta)|} f(x) \, dx \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{B^3_{a_1,a_2}} \frac{e^{i(s-|x|^2+(x \cdot \theta)^2)\sigma}}{|\nabla v h(s, v(x), \theta)|} f(x) \, dx d\sigma, \] (22)

and from this we see that $C$ has the form of an FIO with phase function
\[ \phi(x, s, \theta, \sigma) = (s - |x|^2 + (x \cdot \theta)^2)\sigma, \] (23)

and amplitude
\[ a(x, s, \theta, \sigma) = \frac{1}{2\pi} |\nabla v h(s, v(x), \theta)|^{-1}, \] (24)

where the single phase variable is $\sigma$. Since $\phi$ is homogeneous order 1 in $\sigma$ and $d_\sigma \phi = \sigma$ is nonvanishing for $\sigma \neq 0$, $\phi$ is indeed a phase function. According to Definition 5 we must also show that $\phi$ is clean, and we can do this by noting that $\partial_\theta \partial_\sigma \phi = 1$, and so $d_\sigma d_\sigma \phi$ has constant rank 1. By the comment after Definition 3 this is sufficient to prove that $\phi$ is a clean phase function.

Also as we noted above and is evident from the formula (13), $|\nabla v h(s, v, \theta)|$ is smooth, bounded from above, bounded from below and larger than zero when $v \in B^3_{\epsilon_1,\epsilon_2}$. Also, $x \mapsto v(x)$ is a diffeomorphism from $B^3_{a_1,a_2}$ to $B^3_{\epsilon_1,\epsilon_2}$. Therefore, since $a$
also does not depend on the phase variable $\sigma$, $a \in S^0_0((B_{\alpha_1,\alpha_2} \times (0,1) \times S^2) \times \mathbb{R})$. Hence we can conclude that $C$ is an FIO of order $0 + \frac{1}{2} - \frac{1}{4}(3 + 3) = -1$.

Now suppose that $\theta \in S^2$ is parametrized by $\alpha$ and $\beta \in \mathbb{R}$ (for example using standard spherical coordinates). Then

$$\phi(x, s, \alpha, \beta, \sigma) = (s - |x|^2 + (x \cdot \theta)^2)\sigma,$$

and the derivatives of $\phi$ are

$$d_x\phi = -2\sigma(x - (x \cdot \theta)\theta), \quad d_\alpha\phi = 2\sigma(x \cdot \theta_\alpha)(x \cdot \theta), \quad d_\beta\phi = 2\sigma(x \cdot \theta_\beta)(x \cdot \theta),$$

$$d_s\phi = \sigma, \quad d_\sigma\phi = s - |x|^2 + (x \cdot \theta)^2.$$

From Definition 4, it follows that the canonical relation of $C$ is:

$$C = \left\{ ((s, \alpha, \beta), (\hat{s}, \hat{\alpha}, \hat{\beta}); (x, \xi)) : x \in B^3_{\alpha_1,\alpha_2}, \right.$$

$$s \in (0, 1), \sigma \in \mathbb{R}\backslash{0}, \theta \in S^2, \xi = -d_x\phi, \hat{s} = d_s\phi, \hat{\alpha} = d_\alpha\phi, \hat{\beta} = d_\beta\phi, d_\sigma\phi = 0 \left\}$$

$$= \left\{ ((s, \alpha, \beta), (\sigma, 2\sigma(x \cdot \theta_\alpha)(x \cdot \theta), 2\sigma(x \cdot \theta_\beta)(x \cdot \theta)); x, 2\sigma(x - (x \cdot \theta)\theta)) : x \in B^4_{\alpha_1,\alpha_2}, \right.$$

$$s \in (0, 1), \sigma \in \mathbb{R}\backslash{0}, \theta \in S^2, |x|^2 - (x \cdot \theta)^2 - s = 0 \left\},$$

which completes the proof. \qed

### 2.2 $C^*C$ as a paired Lagrangian operator

If we analyse the canonical relation $C$ given in Theorem 1, we can see that it is non-injective as the points on the ring $\{ x \in \mathbb{R}^3 : |x|^2 - s = 0, x \cdot \theta = 0 \}$ map to $((s, \theta), (\sigma, 0))$ if we fix $s$ and $\sigma$. Let $C^*$ be the canonical relation of $C^*$ and let $\Delta$ denote the diagonal. Then, given the non-injectivity of $C$, $C^* \circ C \notin \Delta$ and $C^*C$ is not a pseudodifferential operator, or even an FIO. In this section we show that the Schwarz kernel of $C^*C$ instead belongs to a class of distributions $I^{p,l}(\Delta, \Lambda) + I^{p,l}(\widetilde{\Delta}, \Lambda)$ studied in [4, 10]. First we recall some definitions and theorems from [4].

**Definition 6.** Two submanifolds $M, N \subset X$ intersect cleanly if $M \cap N$ is a smooth submanifold and $T(M \cap T) = TM \cap TN$

**Definition 7.** We define $I^m(C)$ to be the set of Fourier integral operators, $A : \mathcal{E}'(X) \to \mathcal{D}'(Y)$, of order $m$ with canonical relation $C \subset (T^*Y \backslash 0) \times (T^*X \backslash 0)$

Recall the definitions of the left and right projections of a canonical relation.

**Definition 8.** Let $C$ be the canonical relation associated to the FIO $A : \mathcal{E}'(X) \to \mathcal{D}'(Y)$. Then we denote $\pi_L$ and $\pi_R$ to be the left and right projections of $C$, $\pi_L : C \to T^*Y\backslash{0}$ and $\pi_R : C \to T^*X\backslash{0}$.

We have the following result from [7].

**Proposition 2.** Let $\dim(X) = \dim(Y)$. Then at any point in $C$:

(i) if one of $\pi_L$ or $\pi_R$ is a local diffeomorphism, then $C$ is a local canonical graph;
(ii) If one of the projections \( \pi_R \) or \( \pi_L \) is singular (drops rank), then so is the other. The type of the singularity may be different (e.g. fold or blowdown \([3]\)) but both projections drop rank on the same set
\[
\Sigma = \{(y, \eta; x, \xi) \in C : \det(d\pi_L) = 0\} = \{(y, \eta; x, \xi) \in C : \det(d\pi_R) = 0\}. \tag{28}
\]

Now we have the definition of a blowdown singularity and the definitions of a nonradial and involutive submanifold:

**Definition 9.** Let \( M \) and \( N \) be manifolds of dimension \( n \) and let \( f : N \to M \) be a smooth function. \( f \) is said to have a blowdown singularity of order \( k \in \mathbb{N} \) along a smooth hypersurface \( \Sigma \subset M \) if \( f \) is a local diffeomorphism away from \( \Sigma \), \( df \) drops rank by \( k \) at \( \Sigma \), \( \ker(df) \subset T(\Sigma) \), and the determinant of the Jacobian matrix vanishes to order \( k \) at \( \Sigma \).

**Definition 10.** A submanifold \( M \subset T^*X \) is nonradial if \( \rho \notin (TM)^\perp \), where \( \rho = \sum \xi_i \partial_{\xi_i} \).

**Definition 11.** A submanifold \( M \subset T^*X \), \( M = \{(x, \xi) : p_i(x, \xi) = 0, 1 \leq i \leq k\} \) is involutive if the differentials \( dp_i, i = 1, \ldots, k \), are linearly independent and the Poisson brackets satisfy \( \{p_i, p_j\} = 0 \), \( i \neq j \).

From \([5]\), we have the definition of the flowout.

**Definition 12.** Let \( \Gamma = \{(x, \xi) : p_i(x, \xi) = 0, 1 \leq i \leq k\} \) be a submanifold of \( T^*X \). Then the flowout of \( \Gamma \) is given by \( \{(x, \xi; y, \eta) \in T^*X \times T^*X : (x, \xi) \in \Gamma, (y, \eta) = \exp(\sum_{i=1}^k t_i H_{p_i})(x, \xi), t \in \mathbb{R}^k\} \), where \( H_{p_i} \) is the Hamiltonian vector field of \( p_i \).

Finally, for two cleanly intersecting Lagrangians \( \Lambda_0 \) and \( \Lambda_1 \), we define the \( I^\mu(\Lambda_0, \Lambda_1) \) classes as in \([4, 6]\). We now state a result of \([10, \text{Theorem 1.2}]\) concerning the composition of FIO’s with blowdown–blowdown singularities.

**Theorem 2.** Let \( C \subset (T^*X \setminus 0) \times (T^*X \setminus 0) \) be a canonical relation which satisfies the following:

(i) away from a hypersurface \( \Sigma \subset C \), the left and right projections \( \pi_L \) and \( \pi_R \) are diffeomorphisms;

(ii) at \( \Sigma \), both \( \pi_L \) and \( \pi_R \) have blowdown singularities of order \( k \);

(iii) \( \pi_L(\Sigma) \) and \( \pi_R(\Sigma) \) are nonradial and involutive.

If \( A \in I^m(C) \) and \( B \in I^{m'}(C^*) \), then \( BA \in I^{m+m'+k\frac{1}{2} - k\frac{1}{2}}(\Delta, \Lambda_{\pi_R(\Sigma)}) \), where \( \Delta \) is the diagonal and \( \Lambda_{\pi_R(\Sigma)} \) is the flowout of \( \pi_R(\Sigma) \).

While it is not explicitly proven in \([10]\), we note that the diagonal \( \Delta \) and the flowout \( \Lambda_{\pi_R(\Sigma)} \) are always cleanly intersecting, and so the class \( I^{m+m'+k\frac{1}{2} - k\frac{1}{2}}(\Delta, \Lambda_{\pi_R(\Sigma)}) \) mentioned in Theorem 2 is well-defined. In the application of Theorem 2 to specific cases, such as we consider here, it is not necessary to prove that this intersection is clean.

We now have our main Theorem.
**Theorem 3.** Let $C$ be the canonical relation of the cylinder transform $\mathcal{C}$. Then the left and right projections of $C$ have blowdown singularities of order 1 along a codimension 1 submanifold $\Sigma$, $\pi_L(\Sigma)$ and $\pi_R(\Sigma)$ are involutive and nonradial, and $C\circ C \in I^{-2,0}(\Delta, \Lambda) + I^{-2,0}(\tilde{\Delta}, \Lambda)$, where $\Delta$ is the diagonal in $T^*B^3_{a_1,a_2} \times T^*B^3_{a_1,a_2}$, $\tilde{\Delta} = \{(x, -\xi;x, \xi) : (x, \xi) \in T^*B^3_{a_1,a_2}\}$, and $\Lambda$ is the flowout of $\pi_R(\Sigma)$.

**Proof.** From Theorem 1 we have the canonical relation of the cylinder transform

$$C = \{(s, \alpha, \beta), (\sigma, 2\sigma(x \cdot \theta_\alpha)(x \cdot \theta), 2\sigma(x \cdot \theta_\beta)(x \cdot \theta)); x, 2\sigma(x - (x \cdot \theta)\theta) : x \in B^3_{a_1,a_2}, \quad s \in (0, 1), \sigma \in \mathbb{R}\setminus0, \theta \in S^2, |x|^2 - (x \cdot \theta)^2 = s = 0\},$$

(29)

where $\alpha$ and $\beta$ parameterize $\theta \in S^2$. Suppose we use standard spherical coordinates centred at any given point on $S^2$ (e.g. when centred at $(1, 0, 0)$ these would be defined by $\theta = (\cos \alpha \cos \beta, \sin \alpha \cos \beta, \sin \beta)$) with the notation

$$\partial_\alpha \theta = \theta_\alpha, \quad \partial_\beta \theta = \theta_\beta.$$  

(30)

With such a parameterization, $\{\theta, \theta_\alpha, \theta_\beta\}$ is an orthogonal basis for $\mathbb{R}^3$ and $|\theta_\beta| = 1$, $|\theta_\alpha| = \cos \beta$. Furthermore

$$\det \begin{pmatrix} \theta^T \\ \theta_\alpha^T \\ \theta_\beta^T \end{pmatrix} = \cos \beta.$$  

(31)

Using these coordinates on $S^2$, we can also parametrise $C$ by $(x, \alpha, \beta, \sigma)$ where $x \in B^3_{a_1,a_2}$ is such that $|x|^2 - (x \cdot \theta)^2 = (\cos \beta)^2(x \cdot \theta_\alpha)^2 + (x \cdot \theta_\beta)^2 \in (0, 1)$, and $(\alpha, \beta)$ are in the domain of the coordinates for $S^2$. With this parametrization of $C$ the left projection is given by

$$\pi_L(x, \alpha, \beta, \sigma) = \left(|x|^2 - (x \cdot \theta)^2, \alpha, \beta, \sigma, 2\sigma(x \cdot \theta_\alpha)(x \cdot \theta), 2\sigma(x \cdot \theta_\beta)(x \cdot \theta)\right).$$  

(32)

First note that $\pi_L(x, \alpha, \beta, \sigma) = \pi_L(-x, \alpha, \beta, \sigma)$, and so $\pi_L$ is not injective. However, as we shall see, except for on the set $\Sigma = \{x \cdot \theta = 0\}$, $\pi_L$ is exactly two-to-one. Indeed, suppose that $x \cdot \theta \neq 0$ and $\pi_L(x, \alpha, \beta, \sigma) = \pi_L(-x', \alpha', \beta', \sigma')$. Then $\alpha = \alpha'$, $\beta = \beta'$, $\sigma = \sigma'$,

$$|x|^2 - (x \cdot \theta)^2 = |x'|^2 - (x' \cdot \theta)^2 \Leftrightarrow (\cos \beta)^2(x \cdot \theta_\alpha)^2 + (x \cdot \theta_\beta)^2 = (\cos \beta')^2(x' \cdot \theta_\alpha)^2 + (x' \cdot \theta_\beta)^2,$$

and (using the fact that $\sigma = \sigma' \neq 0$)

$$(x \cdot \theta)(x \cdot \theta_\alpha, x \cdot \theta_\beta) = (x' \cdot \theta)(x' \cdot \theta_\alpha, x' \cdot \theta_\beta).$$

Since $x \cdot \theta \neq 0$, and $(\cos \beta')^2(x' \cdot \theta_\alpha)^2 + (x' \cdot \theta_\beta)^2 \neq 0$, we can combine these to see that $x = \pm x'$.

Now let us analyze $D\pi_L$ to show that $\pi_L$ is a local diffeomorphism away from $\Sigma$. Letting $I_{n \times n}$ and $0_{n \times n}$ denote the $n \times n$ identity and zero matrices respectively, after a permutation of rows, the differential of $\pi_L$ is

$$D\pi_L = \begin{pmatrix} 2(x \cdot \theta - (x \cdot \theta) \theta^T) \\ 2\sigma((x \cdot \theta_\alpha)\theta^T + (x \cdot \theta)\theta_\alpha^T) \\ 2\sigma((x \cdot \theta_\beta)\theta^T + (x \cdot \theta)\theta_\beta^T) \\ 0_{3 \times 3} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ I_{3 \times 3} \end{pmatrix},$$  

(33)

where $r_1, r_2, r_3$ are rows of matrices.
where

\[ r_1 = -(2(x \cdot \theta_\alpha)(x \cdot \theta), 2(x \cdot \theta_\beta)(x \cdot \theta), 0), \]
\[ r_2 = (2\sigma((2(x \cdot \theta \alpha)(x \cdot \theta_\alpha)), 2\sigma((x \cdot \theta_\alpha)(x \cdot \theta_\beta) + (x \cdot \theta_\alpha_\beta)(x \cdot \theta)), (x \cdot \theta_\alpha)(x \cdot \theta)\), \]
\[ r_3 = (2\sigma((x \cdot \theta_\beta)(x \cdot \theta_\alpha) + (x \cdot \theta_\alpha)(x \cdot \theta)), 2\sigma((x \cdot \theta_\beta)(x \cdot \theta_\beta) + (x \cdot \theta_\beta)(x \cdot \theta)), (x \cdot \theta_\beta)(x \cdot \theta)), \]

(34)

and \( \theta_{\alpha\alpha} = \partial_{\alpha\alpha} \theta, \theta_{\beta\beta} = \partial_{\beta\beta} \theta \) and \( \theta_{\alpha\beta} = \partial_{\alpha\beta} \theta. \)

We can now calculate the determinant of \( D\pi_L \) as follows:

\[
\det D\pi_L = \left| \begin{array}{ccc}
2(x^T - (x \cdot \theta)\theta^T) & 2\sigma((x \cdot \theta_\alpha)\theta^T + (x \cdot \theta)\theta_\alpha^T) & 0 \\
2\sigma((x \cdot \theta_\beta)\theta^T + (x \cdot \theta)\theta_\beta^T) & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right| \]

\[
= \frac{1}{\cos \beta} \det \left| \begin{array}{ccc}
2(x^T - (x \cdot \theta)\theta^T) & 2\sigma((x \cdot \theta_\alpha)\theta^T + (x \cdot \theta)\theta_\alpha^T) & 0 \\
2\sigma((x \cdot \theta_\beta)\theta^T + (x \cdot \theta)\theta_\beta^T) & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right| \]

(35)

This is zero when \( x \cdot \theta = 0 \) or \( (x \cdot \theta_\alpha)^2 + (x \cdot \theta_\beta)^2 \cos^2 \beta = 0 \). The latter case corresponds to \( x \) and \( \theta \) are parallel, which we do not consider (\( x \) and \( \theta \) are parallel only when the cylinder is degenerate, i.e. when \( s = 0 \)), and so \( \pi_L \) is a local diffeomorphism away from the manifold \( \Sigma = \{x \cdot \theta = 0\} \).

Finally we show that the singularities of \( \pi_L \) on \( \Sigma \) are blowdown of order 1. Indeed, on \( \Sigma \) we have

\[
d\det D\pi_L = \frac{8\sigma^2}{\cos \beta} \left( (x \cdot \theta_\alpha)^2 + (x \cdot \theta_\beta)^2 \cos^2 \beta \right) (\theta \cdot dx + (x \cdot \theta_\alpha)d\alpha + (x \cdot \theta_\beta)d\beta), \]

(36)

and the kernel of \( D\pi_L \) on \( \Sigma \) is

\[
\text{span} \{((x \cdot \theta_\beta)\theta_\alpha - (x \cdot \theta_\alpha)\theta_\beta) \cdot \nabla_x \} \subset \ker \{d\det D\pi_L\}. \]

(37)

So the left projection \( \pi_L \) drops rank by 1 on \( \Sigma \) and its critical points on \( \Sigma \) are blowdown type singularities. Furthermore,

\[
\pi_L(\Sigma) = \{\hat{\alpha} = \hat{\beta} = 0\}, \]

(38)

which is involutive and nonradial (here \( \hat{\alpha} \) and \( \hat{\beta} \) are the dual variables of \( \alpha \) and \( \beta \)).

Using the same parameterization of \( C \) as above, the right projection is given by

\[
\pi_R(x, \alpha, \beta, \sigma) = (x, 2\sigma(x \cdot (x \cdot \theta))\sigma), \]

(39)

and its differential is

\[
D\pi_R = \left( \begin{array}{ccc}
I_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\
2\sigma(I_{3 \times 3} - \theta_\beta^T) & -2\sigma((x \cdot \theta_\alpha)\theta + (x \cdot \theta)\theta_\alpha) & -2\sigma((x \cdot \theta_\beta)\theta + (x \cdot \theta)\theta_\beta) \\
0_{3 \times 1} & 2(x - (x \cdot \theta)\theta) & 0_{3 \times 1} \\
\end{array} \right). \]

(40)
The determinant can now be calculated as

\[ \det D \pi_R = \det \left( -2\sigma((x \cdot \theta_\alpha)\theta + (x \cdot \theta)\theta_\alpha), -2\sigma((x \cdot \theta_\beta)\theta + (x \cdot \theta)\theta_\beta), 2(x - (x \cdot \theta)\theta) \right) \]

\[ = \frac{1}{\cos \beta} \det \begin{pmatrix} \theta_\alpha^T & \theta_\beta^T \end{pmatrix} \left( -2\sigma((x \cdot \theta_\alpha)\theta + (x \cdot \theta)\theta_\alpha), -2\sigma((x \cdot \theta_\beta)\theta + (x \cdot \theta)\theta_\beta), 2(x - (x \cdot \theta)\theta) \right) \]

\[ = \frac{1}{\cos \beta} \det \begin{pmatrix} -2\sigma(x \cdot \theta_\alpha) & -2\sigma(x \cdot \theta_\beta) & 0 \\ -2\sigma(x \cdot \theta) \cos^2 \beta & 0 & 2(x \cdot \theta_\alpha) \\ 0 & -2\sigma(x \cdot \theta) & 2(x \cdot \theta_\beta) \end{pmatrix} \]

\[ = -\frac{8\sigma^2}{\cos \beta} (x \cdot \theta) ((x \cdot \theta_\alpha)^2 + (x \cdot \theta_\beta)^2 \cos^2 \beta) . \]  

(41)

Hence, on \( \Sigma \)

\[ \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \theta} \]

\[ \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \theta} \]

So \( \pi_R \) drops rank by 1 on \( \Sigma = \{ x \cdot \theta = 0 \} \) and its singularities are blowdown type as the kernel of \( D \pi_R \) on \( \Sigma \) is

\[ \text{span} \left\{ (x \cdot \theta_\beta) \frac{\partial}{\partial \alpha} - (x \cdot \theta_\alpha) \frac{\partial}{\partial \beta} \right\} \subset \text{ker} \{ \det D \pi_R \} . \]  

(43)

Moreover, we have

\[ \pi_R(\Sigma) = \{ x \times \xi = 0 \} = \{ (x, \xi) : p_i(x, \xi) = 0, 1 \leq i \leq 3 \}, \]

(44)

where \( p_1(x, \xi) = x_1 \xi_2 - x_2 \xi_1, p_2(x, \xi) = x_1 \xi_3 - x_3 \xi_1 \) and \( p_3(x, \xi) = x_2 \xi_3 - x_3 \xi_2 \).

The Hamiltonian vector fields of the \( p_i \) are given by

\[ H_{p_1} = -x_2 \partial_x + x_1 \partial_{x_2} - \xi_2 \partial_{\xi_1} + \xi_1 \partial_{\xi_2}, \]

\[ H_{p_2} = -x_3 \partial_x + x_1 \partial_{x_3} - \xi_3 \partial_{\xi_1} + \xi_1 \partial_{\xi_3}, \]

\[ H_{p_3} = -x_2 \partial_x + x_2 \partial_{x_3} - \xi_3 \partial_{\xi_2} + \xi_2 \partial_{\xi_3}. \]  

(45)

Let \( \rho = \sum_{i=1}^{3} \xi_i \partial_{\xi_i} \). Then, as \( x = t \xi \) for some \( t \in \mathbb{R} \), we can see that \( \rho \notin \text{span}\{H_{p_1}, H_{p_1}, H_{p_3}\} \), so \( \pi_R(\Sigma) \) is nonradial.

To check that \( \pi_R(\Sigma) \) is involutive, we first check that the Poisson brackets satisfy \( \{ p_i, p_j \} = 0, i \neq j \):

\[ \{ p_1, p_2 \} = H_{p_1} p_2 = \xi_2 x_3 - x_2 \xi_3 = 0 \]

\[ \{ p_1, p_3 \} = H_{p_1} p_3 = -\xi_1 x_3 + x_1 \xi_3 = 0 \]

\[ \{ p_2, p_3 \} = H_{p_2} p_3 = \xi_1 x_2 - x_1 \xi_2 = 0. \]  

(46)

Furthermore, if we work locally in a neighbourhood away from \( x_1 = 0 \), then \( p_1, p_2 = 0 \implies p_3 = 0 \), so we need only consider the dependance of the differentials of \( p_1 \) and \( p_2 \). \( dp_1 \) and \( dp_2 \) are linearly independant if and only if the Hamiltonian vector fields of \( p_1 \) and \( p_2 \) are linearly independant. But \( \text{span}\{H_{p_1}, H_{p_2}\} \) has dimension 2, so \( \pi_R(\Sigma) \) is involutive. So the conditions of Theorem 2 are satisfied except for the
Corollary 1. Let \( \pi_R \) be the right projection of \( \mathcal{C} \) and let \( \Sigma = \{x : \theta = 0\} \). Then the flowout of \( \pi_R(\Sigma) \) is

\[
\Lambda = \{(x, \xi; \mathcal{O}(x, \xi)) : x \in B_{2\alpha_1,2\alpha_2}^3, \quad \xi \in \mathbb{R}^3 \setminus 0, \quad x \times \xi = 0, \quad \mathcal{O} \in \Delta(SO_3 \times SO_3)\},
\]

(47)

Here \( \mathcal{O}(x, \xi) = (Ox, O\xi) \) where \( O \in SO_3 \) is any rotation.

Proof. Working locally away from \( x_1 = 0 \), we have

\[
\pi_R(\Sigma) = \{(x, \xi) : p_i(x, \xi) = 0, 1 \leq i \leq 2\},
\]

(48)

where \( p_1(x, \xi) = x_1 \xi_2 - x_2 \xi_1 \) and \( p_2(x, \xi) = x_1 \xi_3 - x_3 \xi_1 \). Letting \( H_z = z_1 H_{p_1} + z_2 H_{p_2} \), by definition, the flowout of \( \pi_R(\Sigma) \) is \( \Lambda = \{(x, \xi; y, \eta) \in T^*X \times T^*X : (x, \xi) \in \pi_R(\Sigma), (y, \eta) = \exp(H_z)(x, \xi), z \in \mathbb{R}^2\} \).

We can write \( H_z \) as

\[
H_z = (x, \xi) \begin{pmatrix} H^T & 0_{3 \times 3} \\ 0_{3 \times 3} & H^T \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_\xi \end{pmatrix},
\]

(49)

where \( \partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^T \), \( \partial_\xi = (\partial_{\xi_1}, \partial_{\xi_2}, \partial_{\xi_3})^T \) and

\[
H = \begin{pmatrix} 0 & -z_1 & -z_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix}.
\]

(50)

The flow of \( H_z \) is thus given by the system of linear ODE’s

\[
\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} H & 0_{3 \times 3} \\ 0_{3 \times 3} & H \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix},
\]

(51)

with initial conditions \( (x(0), \xi(0)) = (x_0, \xi_0) \). Then the solution to (51) at time \( t' = 1 \) is

\[
\begin{pmatrix} x(1) \\ \xi(1) \end{pmatrix} = \begin{pmatrix} e^H & 0_{3 \times 3} \\ 0_{3 \times 3} & e^H \end{pmatrix} \begin{pmatrix} x_0 \\ \xi_0 \end{pmatrix},
\]

(52)

and hence the flow of \( H_z \) can be computed as the exponential of the matrix \( H \).

Now, if we parameterize \( z_1 = t \cos \omega \) and \( z_2 = t \sin \omega \) in terms of standard polar coordinates, then \( H = tG = t(ba^T - ab^T) \), where \( a = (1, 0, 0)^T \) and \( b = (0, \cos \omega, \sin \omega)^T \). Let \( P = -G^2 = aa^T + bb^T \). Then \( P^2 = P \) (\( P \) is idempotent) and \( PG = GP = G \). From this it follows that

\[
e^H = e^{iG} = I_{3 \times 3} + G \sin t + G^2(1 - \cos t).
\]

(53)
We can write
\[
e^H = I_{3 \times 3} + G \sin t + G^2 (1 - \cos t) = \begin{pmatrix}
\cos t & -\sin t \cos \omega & -\sin t \sin \omega \\
\sin t \cos \omega & \sin^2 \omega + \cos t \cos^2 \omega & -\cos t \cos \omega \sin \omega \\
\sin t \sin \omega & -\cos t \cos \omega \sin \omega & \cos^2 \omega + \cos t \sin^2 \omega
\end{pmatrix},
\]
where \( \omega \in [0, 2\pi] \) and \( t \in [0, \infty) \). It follows that
\[
e^H e^T_1 = e^H (1, 0, 0)^T = (\cos t, \sin t \cos \omega, \sin t \sin \omega)^T.
\]
The right hand side of equation (55) is the standard parameterization of \( S^2 \) in terms of spherical coordinates, where \( t \) is the polar angle from the \( x \) axis pole and \( \omega \) is the angle of rotation in the \( yz \) plane (azimuth angle). So \( e^H \) defines a full set of rotations on \( S^2 \) and hence, given a vector–conormal vector pair \((x, \xi)\), the Lagrangian \( \Lambda \) includes the rotation of \( x \) and \( \xi \) over the whole sphere. This completes the proof. \( \Box \)

The above results tell us that the wavefront set of the kernel of the normal operator \( \mathcal{C}^* \mathcal{C} \) is contained in \( \Delta \cup \bar{\Delta} \cup \Lambda \), where \( \Delta \) is the diagonal, \( \bar{\Delta} \) the diagonal composed with reflection through the origin, and \( \Lambda \) is the flowout from \( \pi_R(\Sigma) \), which is a rotation by Corollary 1. Also microlocally \( \mathcal{C}^* \mathcal{C} \in I^{-2}(\Delta \setminus \Lambda) + I^{-2}(\bar{\Delta} \setminus \Lambda) \) and \( \mathcal{C}^* \mathcal{C} \in I^{-2}(\Lambda \setminus (\Delta \cup \bar{\Delta})) \), which implies that the strength of the artefacts represented by \( \Lambda \) are the same as the image intensity on \( \Sigma = \{x \cdot \theta = 0\} \). We will give examples of the artefacts implied by \( \Lambda \) later in our simulations in section 4, but in the next section we shall show how to reduce the strength of this artefact microlocally.

3 Reducing the strength of the image artifact

Here we derive a filter \( Q \), which we show can be applied to reduce the strength of the image artefact \( \Lambda \) for the cylinder tranform \( \mathcal{C} \). We further show how \( Q \) can be applied as a spherical convolution with a distribution \( h \) on the sphere, which we will determine.

Using the ideas of [4], our aim is to apply a filtering operator \( Q : \mathcal{E}'((0, 1) \times S^2) \to \mathcal{E}'((0, 1) \times S^2) \), whose principal symbol vanishes to some order \( s \) on \( \pi_L(\Sigma) \), to \( \mathcal{C} \) before applying the backprojection operator \( \mathcal{C}^* \). From [4], we have the following theorem.

**Theorem 4.** Let \( A \in I^m(\mathcal{C}) \) be such that both the left projections of \( A \), \( \pi_L \) and \( \pi_R \) are diffeomorphisms except on a set \( \Sigma \) where they drop rank by \( k \) and have blowdown singularities at \( \Sigma \). Further let \( \pi_L(\Sigma) \) and \( \pi_R(\Sigma) \) be involutive and nonradial. Let \( Q \) be a pseudodifferential operator of order 0 whose principal symbol vanishes to order \( s \) on \( \pi_L(\Sigma) \). Then \( A^* QA \in I^{2m+\frac{k+1}{2} - s, s - \frac{k-1}{2}}(\Delta, \Lambda) \), where \( \Delta \) is the diagonal and \( \Lambda \) is the flowout from \( \pi_R(\Sigma) \).

Let \( \Delta_{S^2} \) denote the Laplacian on \( S^2 \), and \( I \) the identity operator. Then we will take
\[
Q = -\Delta_{S^2} (I - \Delta_{S^2})^{-1},
\]
whose symbol vanishes to order 2 on
\[
\pi_L(\Sigma) = \{\hat{\alpha} = \hat{\beta} = 0\}.
\]
CHAPTER 4. MICROLOCAL ANALYSIS

There are two technical issues with the application of Theorem 4 in our case. One is the fact, which we already mentioned in the proof of Theorem 3 that \( \pi_L \) is two-to-one away from \( \Sigma \). We can deal with this in the same way we dealt with it in the proof of Theorem 4 by restricting \( C \) to small neighbourhoods of each point, and including a reflection through the origin to handle neighbourhoods of points \( x_0 \) and \(-x_0\).

The other issue is that this operator \( Q \), defined by (56), is not a pseudodifferential operator on \( Y = (0, 1) \times S^2 \) since differentiation of its symbol in the dual angular variables does not increase the decay in the \( \hat{s} \) direction. However, this objection can be overcome by noting that

\[
Q \left( 1 - \Delta_{S^2} \right) \left( 1 - \Delta_{S^2} - \partial^2_{\hat{s}} \right)^{-1} = -\Delta_{S^2} \left( 1 - \Delta_{S^2} - \partial^2_{\hat{s}} \right)^{-1}. 
\]

Both \( \Psi_1 \) and \( \Psi_2 \) are then pseudodifferential operators, and \( \Psi_1 \) is elliptic and of order zero. Thus, if \( \Psi_1^{-1} \) is a pseudodifferential parametrix for \( \Psi_1 \), we have

\[
Q = \Psi_2 \Psi_1^{-1} + R
\]

where \( R \) is an operator with smooth kernel and \( \Psi_2 \Psi_1^{-1} \) is a pseudodifferential operator satisfying the hypotheses of Theorem 4. From this the results of Theorem 4 hold when \( Q \) is given by (56).

We thus have, upon applying the filter \( Q \) to \( C \) before applying \( C^* \) that \( C^*QC \in I^{-4,2}(\Delta, \Lambda) + I^{-4,2}(\hat{\Delta}, \Lambda) \). So \( C^*QC \in I^{-2}(\Delta, \Lambda) + I^{-2}(\hat{\Delta}, \Lambda) \) and \( C^*QC \in I^{-4}(\Lambda \setminus (\Delta \cup \hat{\Delta})) \) and the strength of the artefact is reduced and is now less than the strength of the image.

We now show how the filter \( Q \) can be applied as a convolution with a distribution \( h \) on the sphere. First we give some definitions and theorems on spherical harmonic expansions. For integers \( l \geq 0, |m| \leq l \), we define the spherical harmonics \( Y_l^m \) as

\[
Y_l^m(\alpha, \beta) = (-1)^m \sqrt{(2l+1)(l-m)! \over 4\pi(l+m)!} P_l^m(\cos \beta) e^{im\alpha},
\]

where

\[
P_l^m(x) = (-1)^m (1 - x^2)^{m/2} {d^m \over dx^m} P_l(x)
\]

and

\[
P_l(x) = {1 \over 2} \sum_{k=0}^l \binom{l}{k}^2 (x-1)^{l-k}(x+1)^k
\]

are Legendre polynomials of degree \( l \). The spherical harmonics \( Y_l^m \) are the eigenfunctions of the Laplacian on \( S^2 \), with corresponding eigenvalues \( c_l = -l(l+1) \). So \( \Delta_{S^2} Y_l^m = c_l Y_l^m \). From [15] we have the following theorem.

**Theorem 5.** Let \( F \in C^\infty(Z^3) \) and let

\[
F_{lm} = \int_{S^2} F \bar{Y}_l^m \, d\Omega,
\]

where \( d\Omega \) is the surface measure on the sphere. Then the series

\[
F_N = \sum_{0 \leq l \leq N} \sum_{|m| \leq l} F_{lm} Y_l^m
\]

converges uniformly absolutely on compact subsets of \( Z^3 \) to \( F \).
So after writing the cylinder transform $C$ in terms of its spherical harmonic expansion, we can apply the filter $Q$ as follows:

$$QCf(s, \theta) = -\Delta_{S^2} (I - \Delta_{S^2})^{-1} \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} C_{lm}(s) Y_l^m(\theta)$$

$$= \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} \frac{-c_l}{1 - c_l} C_{lm}(s) Y_l^m(\theta),$$

where $C_{lm} = \int_{S^2} \bar{C}_l Y_l^m d\Omega$.

For a function $f$ on the sphere and $h$, a distribution on the sphere, we define the spherical convolution [1]

$$(f *_{S^2} h)(\theta) = \int_{g \in SO(3)} f(g \omega) h(g^{-1} \theta) dg,$$

where $\omega$ is the north pole, and from [1] we have the next theorem.

**Theorem 6.** For functions $f, h \in L^2(S^2)$, the harmonic components of the convolution is a pointwise product of the harmonic components of the transforms:

$$(f *_{S^2} h)_{lm} = 2\pi \sqrt{\frac{4\pi}{2l + 1}} h_l F_{lm}(s, \theta),$$

For our case, this gives the following.

**Theorem 7.** Let $Q = -\Delta_{S^2} (I - \Delta_{S^2})^{-1}$ and let $F \in C_0^\infty(Z^3)$. Then

$$QF = (F *_{S^2} h),$$

where $h$ is defined by

$$h(\theta) = \sum_{l \in \mathbb{N}} h_l \sum_{|m| \leq l} Y_l^m(\theta),$$

where

$$h_l = \frac{l(l+1) a_l}{l(l+1)+1}$$

and $a_l = \frac{1}{2\pi} \cdot \sqrt{\frac{2l+1}{4\pi}}$.

**Proof.** From equation (63) we have

$$QF(s, \theta) = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} \frac{l(l+1)}{l(l+1)+1} F_{lm}(s) Y_l^m(\theta)$$

$$= \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} 2\pi \sqrt{\frac{4\pi}{2l + 1}} h_l F_{lm}(s) Y_l^m(\theta).$$

Defining $h_{lm} = h_l$ for all $l \in \mathbb{N}$, $|m| \leq l$, we have by Theorem 6

$$QF(s, \theta) = \int_{g \in SO(3)} F(s, g\omega) h(g^{-1} \theta) dg$$

$$= (F *_{S^2} h)(s, \theta),$$

where $h(\theta) = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} h_{lm} Y_l^m(\theta) = \sum_{l \in \mathbb{N}} h_l \sum_{|m| \leq l} Y_l^m(\theta)$, which completes the proof. \[\square\]
4 Simulations

Given the equivalence of the spindle transform $S$ and the cylinder transform $C$, and given also that the diffeomorphism defining their equivalence $v(x) = \left(\sqrt{1 + \frac{1}{|x|^2}} - \frac{1}{|x|}\right) \cdot |x|$, depends only on $|x|$, the artefacts described by the Lagrangian $\Lambda$ apply also to the normal operator of the spindle transform $S^* S$. Here we simulate the image artefacts produced by $\Lambda$ in image reconstructions from spindle transform data and show how the filter we derived in section 3 can be used to reduce these artefacts. We also provide simulated reconstructions of densities which we should find difficult to reconstruct from a microlocal perspective (i.e. densities whose wavefront set is in directions normal to the surface of a sphere centred at the origin), and investigate the effects of applying the filter $Q$ as a pre-conditioner, prior to implementing some discrete solver, in our reconstruction.

To conduct our simulations we consider the discrete form of the spindle transform as in [16], and solve the linear system of equations

$$Ax = b,$$

where $A$ is the discrete operator of the spindle transform, $x$ is the vector of pixel values and $b = Ax$ is the vector of spindle transform values (simulated as an inverse crime). To apply the filter $Q$ derived in section 3, we decompose $b$ into its first $L$ spherical harmonic components and then multiply each component by the filter components $h_l/a_l = \frac{l(l+1)}{2(l+1)-1}$, $0 \leq l \leq L$ before recomposing the series.

Consider the small bead of constant density pictured in figure 2. In figure 3 we present a reconstruction of the small bead by unfiltered backprojection (represented as an MIP image to highlight the bead). Here we see artefacts described by the Lagrangian $\Lambda$ as, in the reconstruction, the bead is smeared out over the sphere. If we apply the filter $Q$ to the spherical components of the data and sum over the first $L = 25$ components before backprojecting, then we see a significant reduction in the strength of the artefact, the density is more concentrated around the small bead and the image is sharper. See figure 4. If we simply truncate the harmonic series of our data before backprojecting without a filter, this has a regularising effect and the level of blurring around the sphere is reduced. However we still see the artefacts due to $\Lambda$. See figure 5. The line profiles in figures 2–5 have been normalised. We note that in the reconstructions presented, the object is reflected through the origin in the reconstruction. This is as predicted by the Lagrangian $\tilde{\Delta}$. The reflection artefact seems intuitive given the symmetries involved in our geometry. It was shown in [16] that the null space of $S$ consists of odd functions (i.e. functions whose even harmonic components are zero), and so what we see in the reconstruction is the projection of the density onto its even components. We also see this effect in the reconstructions presented in [16].

Now let us consider the layered spherical shell segment phantom (the layers have values oscillating between 1 and 2) centred at the origin, shown in figure 6. We reconstruct the phantom by applying CGLS implicitly to the normal equations (i.e. we avoid a direct application of $A^T A$) with 1% added Gaussian noise and regularise our solution using Tikhonov regularisation. See figure 8. Here the image quality is not clear and the layers seem to blur into one, and the jump discontinuities in the image are not reconstructed adequately. However if we arrange the layers as sections of planes
and perform the same reconstruction (see figures 7 and 10), then the image quality is significantly improved, and the jump discontinuities between the oscillating layers are clear. This is as expected, as the wavefront set of the spherical density is contained in \( \pi_R(\Sigma) \), so we see artefacts in the reconstruction. When the layers are arranged as planes this is not the case and we see an improvement in the reconstruction. In figure 9 we have investigated the effects of applying the filter \( Q \) as a pre-conditioner prior to a CGLS implementation. To obtain the reconstruction, we solved the system of equations \( Q^{\frac{1}{2}}Ax = Q^{\frac{1}{2}}b \) using CGLS with 1% added Gaussian noise. The filter has the effect of smoothing the radial singularities in the reconstruction. Here we see that the outer shell is better distinguished than before but the inner shells fail to reconstruct and overall the image quality is not good.

In figures 11, 12 and 13 we have presented reconstructions of the layered spherical shell and layered plane phantoms by Landweber iteration, with 1% Gaussian noise. Here the jump discontinuities in the spherical shell reconstruction are clearer. However as the Landweber method applies the normal operator \( (A^TA) \) at each iteration, we see the artefacts predicted by \( \Lambda \) in the reconstruction and the spherical segment is blurred out over spheres centred at the origin. The artefacts are less prevalent in the plane phantom reconstruction. Although we do see some blurring at the plane edges. In figure 13 we have applied \( Q^{\frac{1}{2}} \) as a pre-conditioner to a Landweber iteration. Here it is not clear that we see a reduction in the spherical artefact and there is a loss in clarity due to the level of smoothing.
CHAPTER 4. MICROLOCAL ANALYSIS

Figure 2: Small bead.

Figure 3: Bead reconstruction by backprojection.
Figure 4: Bead reconstruction by filtered backprojection, with $L = 25$ components.

Figure 5: Bead reconstruction by backprojection, truncating the data to $L = 25$ components.
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Figure 6: Layered spherical shell segment phantom, centred at the origin.

Figure 7: Layered plane phantom.

Figure 6: Layered spherical shell segment phantom, centred at the origin.
Figure 7: Layered plane phantom.
Figure 8: Layered spherical shell segment CGLS reconstruction.

Figure 9: Layered spherical shell segment CGLS reconstruction, with $Q^{1/2}$ used as a pre-conditioner and no added Tikhonov regularisation.
Figure 10: Layered plane reconstruction by CGLS.

Figure 11: Layered plane reconstruction by Landweber iteration.
Figure 12: Spherical shell reconstruction by Landweber iteration.

Figure 13: Spherical shell reconstruction by Landweber iteration, with $Q^{1/2}$ used as a pre-conditioner.

Figure 12: Spherical shell reconstruction by Landweber iteration.

Figure 13: Spherical shell reconstruction by Landweber iteration, with $Q^{1/2}$ used as a pre-conditioner.
5 Conclusions and further work

We have presented a microlocal analysis of the spindle transform introduced in [16]. An equivalence to a cylinder transform $C$ was proven and the microlocal properties of $C$ were studied. We showed that $C$ was an FIO whose normal operator belonged to a class of distributions $I^{-2,0}(\Delta, \Lambda)+I^{-2,0}(\tilde{\Delta}, \Lambda)$, where $\tilde{\Delta}$ is a reflection through the origin and $\Lambda$ is a flowout from the right projection of the canonical relation of $C$, which we calculated explicitly. In section 3, we showed how to reduce the size of the rotation artefact associated to $\Lambda$ microlocally, through the application of a pseudodifferential operator $Q$, and showed that $Q$ could be applied as a spherical convolution with a distribution $h$ on the sphere, or using spherical harmonics. We provided simulated reconstructions to show the artefacts produced by $\Lambda$, and showed how applying $Q$ reduced the artefacts in the reconstruction. Reconstructions of densities of oscillating layers were provided using CGLS and a Landweber iteration. We also gave reconstructions of a spherical layered shell centred at the origin, using $Q^{1/2}$ as a pre–conditioner, prior to a CGLS and Landweber implementation and compared our results.

In future work we aim to derive an inversion formula of either a filtered backprojection or backprojection filter type. That is, we aim to determine whether there exists an operator $A$ such that either $S^*A$ or $AS^*$ is a left inverse for $S$. After which we could see how the filter $Q$ derived here may be involved in the inversion process. We also aim to assess if Sobolev space estimates can be derived for the spindle transform to gain a further understanding of its stability.

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References


Chapter 5

Conclusions and further work

This thesis introduces a new set of Radon transforms of interest in Compton scattering tomography and presents an analysis of their injectivity, stability and microlocal properties. The results presented in this thesis lay the mathematical foundations for new types of scanning equipment in CST and we hope the work presented here sparks further research and interest in the field. The main results prove the injectivity of a disc transform and a generalized spindle transform, which describe source–detector acquisition geometries in CST. The spindle transform is three dimensional and describes an acquisition geometry of a single source and detector which remain opposite one another and are rotated about the origin on the surface of a sphere. In general the manifold of spindle tori in $\mathbb{R}^3$ is seven dimensional. That is, a spindle torus in $\mathbb{R}^3$ is determined by a rotation (two dimensions), a translation (three dimensions) and by the radius and offset of the torus tube. In further work we aim to consider further the properties of the seven dimensional spindle transform. Namely, the Radon transform which gives the integrals of a function in $\mathbb{R}^3$ over any spindle torus in $\mathbb{R}^3$ (we would have to think about which functions this could be defined for). In this thesis we have considered the three dimensional subset $T$ of all spindle tori $S$ in $\mathbb{R}^3$, for which both points of self intersection of $S$ lie on $S^2$. Our injectivity results prove that $T$ is a set of injectivity for the seven dimensional spindle transform on the set of smooth functions compactly supported on a ball. The RTT system (considered in chapter 2) has the ability to measure scatter given the nature of its fixed detectors. The acquisition geometry for the RTT describes a set of source points on a helical path and an array of fixed detectors on $Z^2$ which image a density supported on the interior of $Z^2$. The scatter for the RTT is therefore given
by the set of integrals of the electron density function over spindle tori which have points of self intersection on a helix and a cylinder. This describes a four dimensional subset of the full spindle transform data, one which is of particular interest in security screening and threat detection. We would like to know if this is a set of injectivity for the spindle transform and more generally what other sets constitute injectivity sets for the spindle transform. One open question is, can we fully characterize the injectivity sets for the spindle transform, as is done in the two dimensional case for the circle transform by Agranovsky and Quinto [2]?

In chapter 2, the disc transform inversion is shown to be continuous in Sobolev spaces of order \( 3/2 \) (a mildly ill posed inversion) through an equivalence with the hyperplane Radon transform in two dimensions. The spindle transform has more complicated stability properties which are investigated in the final paper from a microlocal perspective. The spindle transform is shown to detect all wavefronts of the image space excluding those which lie in a radial direction (the vector–cotangent vector pairs which are parallel). This would suggest that the spindle transform inversion is continuous in some Sobolev space in directions that are non–radial, but fails to resolve the radial singularities stably. This hypothesis is investigated in the final paper, where we give reconstructions of a layered spherical shell centred at the origin from spindle transform data. Although the results agree with our hypothesis, we leave a proof of this for further work. We aim also to analyse the stability properties of the seven dimensional spindle transform in further work. For the acquisition geometry considered in chapter 3, we had trouble resolving the singularities which were cut orthogonally by the degenerate spindle tori (straight lines through the origin) and the operator was a special type of paired Lagrangian with blowdown–blowdown singularities with artefacts described by a flow out. We wonder if this microlocal property were to translate to other acquisition geometries. For example, with the RTT’s geometry, would we be able to recover singularities orthogonal to lines connected by points on the source helix and detector cylinder? Are spindle transforms necessarily paired Lagrangian type operators? Clearly an inversion from seven dimensional spindle transform data is more stable than the inversion considered paper 3, and we can imagine a dataset for which all the wavefronts of the image space are cut orthogonally by non–degenerate spindle tori (e.g. reconstructing a function on a ball from all spindle tori which intersect the ball). So it would seem the answer to the second question is no in general as there are spindle transforms which are
pseudodifferential operators. However this remains to be proven, and for more specific acquisition geometries, such as the RTT geometry, a further analysis is required of the microlocal properties.

The practical application of our results to fields in X-ray and gamma ray imaging (e.g. airport security screening of baggage) are investigated through simulation. In the presence of an additive pseudo random Gaussian noise, the reconstruction results from disc transform data (paper 2) were as expected and a satisfactory reconstruction was obtainable with the correct amount of regularisation. However, as the spindle transform is more highly ill-posed, the inversion results from spindle transform data (chapter 3) were less satisfactory at higher noise levels, where we noticed a loss in clarity in some of the image features (the lower densities were harder to identify, there was blurring and the background noise was amplified). The reconstructions were obtained using the Conjugate Gradient Least Squares (CGLS) algorithm with Tikhonov regularisation. For future work, we suggest to precondition (perhaps using some variant of the filter $Q$ derived in chapter 4) our problem before solving using a least squares solver (e.g. CGLS, gradient descent) to make the inversion algorithm more robust. A powerful regulariser such as total variation minimization could also be applied to handle the high levels of noise.

To obtain the explicit inversion formulae presented in chapters 2 and 3, the forward problems were assumed to be linear and the effects of the attenuation of the incoming and scattered radiation were neglected. This approximation can be a large contributor to the systematic noise, which was simulated in paper 3. We found that when the attenuation effects were added to our simulated noise, the reconstruction produced severe artefacts in some cases (e.g. when the scanned targets were too large or highly attenuating). A modification to our inversion method is required to address this problem, which we aim to complete in further work. Approximate treatments of the attenuation are suggested in [46, 48] but the effects due to attenuation are not investigated nor are the suggested methods tested. As the results obtained in chapter 3 of this thesis yield a good initial approximation to the density map when attenuative effects are ignored, perhaps an implementation of an iterative approach such as suggested in [48] would be a good starting point to treat the attenuative effects for our method.

The application of CST to areas in security screening and threat detection was investigated in chapter 2. We showed with a combination of a more traditional
CHAPTER 5. CONCLUSIONS AND FURTHER WORK

X-ray CT reconstruction and a CST reconstruction, that the atomic number $Z$ of the target was uniquely determined. However our inversion method was unstable given the small variation in the electron cross section for low $Z$ ($1 < Z < 20$) at high energies $E$ (we considered a single energy of $E = 100\text{keV}$). With energy sensitive capabilities, the transmission CT data determines the attenuation coefficient $\mu(E, Z)$ for a range of energies. Given the increased variation in the electron cross section as a function of $Z$ for low $Z$ at lower energies ($E < 20\text{keV}$), the lower energy data is more useful for determining the lower $Z$ material (typically materials of interest in airport security screening are $Z < 20$). We propose to use the cross section data across a range of known energies (low and high) to find an optimal solution for $Z$. Also, we wonder what other material properties may be derived using CST data. Can the weight contributions of the constituent elements be calculated, or can we determine the empirical or chemical formula of the target material? Naturally this information would be of great use in threat detection (explosives have a specific proportion of Hydrogen, Carbon, Oxygen and Nitrogen content). The attenuation coefficient and electron density are linear combinations of the coefficients and densities of the constituent elements by the mixture rule [33]. So the problem of reconstructing the constituent element proportions is a further linear inversion. Further analysis of this problem is required however, to put it into practice.

Accurate radiation dosage determinations are necessary in various forms of cancer therapy treatment (e.g. beta therapy or proton therapy). The electron density and the Linear Energy Transfer (LET) of a material (related closely to the dose level) are proportional by the Bethe formula [9]. Given this close relationship between the tissue electron density and dosage from proton radiation, the author in particular aims to investigate further the application of CST in proton therapy dosage determination. Current techniques in dosage determination use a traditional X–ray CT reconstruction to locate the tumour, then the attenuation map is used to approximate the LET and dose. We aim to use the electron density information recovered from CST combined with Monte Carlo simulation methods to give a more accurate determination of the LET, dosage level and spread of the radiation dosage about the patient.
Appendix A

Definitions

For $X \subset \mathbb{R}^n$ we denote the following:

(i) $C^k(X)$ the set of functions on $X$ with $k$ continuous derivatives.

(ii) $C^k_0(X)$ the set of functions compactly supported on $X$ with $k$ continuous derivatives.

(iii) $C^k_B(X) = \{ f \in C^k(X) : D^\alpha f \ \text{bounded on} \ X \ \text{for} \ |\alpha| \leq k \}$, where $\alpha$ is a mult–index.

(iv) $L^p(X) = \{ f : X \to \mathbb{R} : \int_X |f|^p < \infty \}$

(v) $L^\infty(X) = \{ f : X \to \mathbb{R} : |f(x)| \leq C \ a.e. \ on \ X \}$

Definition 11. The Schwartz space on $\mathbb{R}^n$ is defined as

$$S(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \ \forall \alpha, \beta \in \mathbb{Z}^n_+ \},$$

where $\alpha$ and $\beta$ are multi–indices and

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

We denote $\mathcal{D}'(X)$ to be the dual of the Schwartz space on $X$, namely the set of tempered distributions on $X$.

Definition 12. The Schwartz space on $\mathbb{Z}^n$ is defined:

$$S(\mathbb{Z}^n) = \left\{ g \in C^\infty(\mathbb{Z}^n) : s^l \frac{\partial^k}{\partial s^k} g(s, \theta) \ \text{bounded}, \ l, k = 0, 1, \ldots \right\}.$$
Definition 13. Let $X$ and $Y$ be two non-empty subsets of a metric space $(M,d)$. We define their Hausdorff distance by

$$d_H(X,Y) = \max\{\sup_{x \in X} \inf_{y \in Y} d(x,y), \sup_{y \in Y} \inf_{x \in X} d(x,y)\}. \quad (A.4)$$

Definition 14. For $f \in C([a,b])$, a Volterra integral operator of the first kind has the form

$$Vf(y) = \int_a^y K(x,y)f(x)dx, \quad a \leq y \leq b. \quad (A.5)$$

If the kernel $K$ has the form

$$K(x,y) = K_1(x,y)(x-y)^{-\alpha}, \quad (A.6)$$

for some $0 < \alpha < 1$ then $K$ is said to be weakly singular.

Definition 15. Let $X \subset \mathbb{R}^n$ and let $u \in L^1(X)$. Then $v^k \in L^1(X)$ is called the $k^{th}$ weak derivative of $u$, written $v^k = D^k u$, if

$$\int_X u(x)D^k \phi(x)dx = (-1)^{|k|} \int_X v^k(x)\phi(x)dx, \quad \forall \phi \in C^\infty_0(\Omega) \quad (A.7)$$

where $k \in \mathbb{Z}^n_+$ is a multi index and $D^k = \frac{\partial^k_1}{\partial x_1^1} \cdots \frac{\partial^k_n}{\partial x_n^n}$.

Definition 16. For integers $l \geq 0$, $|m| \leq l$, we define the spherical harmonics $Y_{lm}^m$ as:

$$Y_{lm}^m(\theta, \varphi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \varphi) e^{im\theta}, \quad (A.8)$$

where

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \quad (A.9)$$

and

$$P_l(x) = \frac{1}{2^l} \sum_{k=0}^l \binom{l}{k}^2 (x-1)^{l-k}(x+1)^k \quad (A.10)$$

are the Legendre polynomials of degree $l$.

The spherical harmonics $Y_{lm}^m$ form an orthonormal basis for $L^2(S^2)$, with the inner product:

$$\langle f, g \rangle = \int_{S^2} f \bar{g} d\Omega = \int_0^\pi \int_0^{2\pi} f(\theta, \varphi) \bar{g}(\theta, \varphi) \sin \varphi \, d\theta d\varphi. \quad (A.11)$$
where $d\Omega$ is the surface measure on $S^2$.

**Definition 17.** The Chebyshev polynomials of the first kind order $l$ are defined as

$$T_l(x) = \begin{cases} \cos(l \cos^{-1} x) & \text{if } |x| \leq 1 \\ \cosh(l \cosh^{-1} x) & \text{if } x \geq 1 \\ (-1)^l \cosh(l \cosh^{-1} -x) & \text{if } x \leq 1. \end{cases}$$

(A.12)

**Definition 18.** The cotangent bundle of a differentiable manifold $M$ is defined as

$$T^\ast M = \{(x, \xi) : x \in M, \xi \in T^\ast_x M\}$$

(A.13)

where $T^\ast_x M$ is the cotangent space to $M$ at $x$. The tangent bundle $TM$ of $M$ is the dual bundle to $T^\ast M$.

**Definition 19.** Let $(M, g)$ be a Riemannian manifold and let $i : X \to M$ be an immersion. Then the normal bundle of $X$ is defined as

$$NX = \{(x, \xi) : x \in X, \xi \in T_{i(x)}(M, g)(\xi, \eta) = 0 \quad \forall \eta \in T_x X\}.$$  

(A.14)

The conormal bundle $N^\ast X$ is the dual bundle to $NX$.

**Definition 20.** For a function $f$ in the Schwartz space $S(\mathbb{R}^n)$ we define its Fourier transform:

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx,$$

(A.15)

**Definition 21.** The Hilbert transform in $\mathbb{R}$ is defined by

$$Hf(x) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy, \quad f \in L^2(\mathbb{R})$$

(A.16)

where P.V. denotes the Cauchy principle value.

**Definition 22.** Let $m, \rho, \delta \in \mathbb{R}$ with $0 \leq \rho \leq 1$ and $\delta = 1 - \rho$. Then we define $S^m_\rho(X \times \mathbb{R}^n)$ to be the set of $a \in C^\infty(X \times \mathbb{R}^n)$ such that for every compact set $K \subset X$ and all multi–indices $\alpha, \beta$ the bound

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha, \beta, K} (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \quad x \in K, \ \xi \in \mathbb{R}^n,$$

(A.17)

holds for some constant $C_{\alpha, \beta, K}$. The elements of $S^m_\rho$ are called symbols of order $m$, type $\rho$. 

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*APPENDIX A. DEFINITIONS*
Definition 23. A function $\phi = \phi(x, \xi) \in C^\infty(X \times \mathbb{R}^N\setminus 0)$ is a phase function if $\phi(x, \lambda \xi) = \lambda \phi(x, \xi)$, $\forall \lambda > 0$ and $d\phi \neq 0$. 