STOCHASTIC BEAM EQUATION OF JUMP TYPE:
EXISTENCE AND UNIQUENESS

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This thesis explores one kind of equation used to model the physics behind one beam with two ends fixed. Initially, Woinowsky Krieger sets a nonlinear partial differential equation (PDE) model by attaching one nonlinear term to the classic linear beam equation. From Zdzislaw Brezezniak, Bohdan Maslowski, Jan Seidler, they demonstrate this model mixed with one Brownian motion term describing random fluctuation. After stochastic modifications, this model becomes more accurate to the behaviors of beam vibrations in reality, and theoretically, the solution has better properties. In this thesis, the model includes more complex noises which cover the condition of random uncontinuous disturbance in the language of Poisson random measure. The major breakthrough of this work is the proof of existence and uniqueness of solutions to this stochastic beam equation and solves the flaws of previous work on proof.
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Chapter 1

Introduction

Beam equation (also known as engineer’s beam equation or classical beam equation or Euler-Bernoulli beam theory) is quite important in industry for engineers, for example the development of the Eiffel Tower and Ferris wheel in the late 19th century. There is even another more generalized theory called Timoshenko beam theory [36][37].

For mathematicians, there are several different models. For example, the simplest beam equation is as below

$$\partial^2_t u(t,x) + \gamma \partial^4_x u = 0.$$  

With one more non-linear term added, it is the Woinowsky Krieger’s model. To be precise, it is given as following

$$\partial^2_t u(t,x) + \gamma \partial^4_x u - [\alpha + \beta \int_0^l | \partial_y u(t,y) |^2 \, dy] \partial^2_x u = 0.$$  

The physics meaning of all the coefficient involved will be explained later in our literature survey.

As a stochastic beam equation, Zdzislaw Brzezniak, Bohdan Maslowski, Jan Seidler and P. L. Chow, etc worked on a model like following

$$\partial^2_t u(t,x) + \gamma \partial^4_x u - [\alpha + \beta \int_0^l | \partial_y u(t,y) |^2 \, dy] \partial^2_x u = \sigma(t,x,\dot{u},\partial_x u) \, d\omega_t,$$

where it is added one stochastic term, Brownian motion $\omega_t(\cdot)$. With Brownian motion involved, it becomes non-trivial of finding a proper ‘solution’ in a proper space.

In this dissertation, it consists of two parts. Our first part is a literature review on the topic of beam equation. The second part is about the existence and uniqueness of stochastic beam equation driven by Brownian motion and Poisson random measures.
CHAPTER 1. INTRODUCTION

In our first part, we will review the works of beam equation on Partial Differential Equation (PDE) side and Stochastic Partial Differential Equation (SPDE) side.

We will start from explaining the physics background of one kind of non-linear beam equation. It is based on Woinowsky’s work [7]. In following subsection, we review the work of R. W. Dickey [8] where they extended the topic by giving another kind of solution. Then, we also mention the work of J. M. Ball [16]. They extend the work of Dickey [8] in several directions. They show that the solutions satisfy an energy equation and it depends continuously on the initial data.

For the problem of stability on beam equation, we review the works of J. M. Ball [17] and R. W. Dickey [13]. Dickey discussed the asymptotic stability of equilibrium solutions of beam equation. In J. M. Ball’s paper [17], they used topological methods to show that the solution converges in a suitable topology to an equilibrium position of the beam under certain conditions. In the end, we briefly review the work of W. Fitzgibbon [14].

After those results on the PDE side, we summarize some results on the SPDE side. We will firstly introduce the works of Z. Brzeniak, B. Maslowski, and J. Seidler [9]. They obtained some results on existence and uniqueness of solutions for the stochastic beam equation. Finally, our survey is ended up by presenting the work of Z. Brzezniak, M. Ondrejat, and J. Seidler on invariant measure [10], where they proved the existence of an invariant measure for the stochastic extensible beam equation.

In our second part, we consider one class of non-linear stochastic beam equation driven by Brownian motion and Poisson random measure. To be precise, our equation is given as below

\[
\partial_t^2 u(t, x) + \gamma \partial_x^4 u - [\alpha + \beta \int_0^t |\partial_y u(t, y)|^2 \, dy] \partial_x^2 u
\]

\[= \sigma(t, x, \dot{u}, \partial_x u) \, d\omega(t) + \int_X g(t, x, \dot{u}, \partial_x u) \tilde{N}(dt \, dq).\]

This part consists of five sections. In Section 1, some definitions and theorems are presented that are required for later sections. We then state some fundamental settings of spaces and equations in Section 2, before using these to build an energy equation based on Ito’s formula in Section 3. Finally, with the help of this energy functional, we establish the existence and uniqueness of solutions of stochastic beam equation driven by Poisson random measures in Section 4 for additive noise case and Section 5
for general case. The method of this part is strongly inspired by the one in [5].
Chapter 2

Literature Survey

First of all, I will show the physics background of beam equation that I worked on, it bases on the work of Woinwsky Krieger [7]. Except Woinowsky, Eisley [11] also work on beam equation while the model he worked on is one term different compared to the Woinowsky’s model. Dickey [12] extend Eisley’s work to infinite dimension space case.

Then our survey goes to the work of J.M. Ball on the topic of initial-boundary value problem [16] and stability [17]. Following Ball’s work, W.E. Fitzgibbon [14] gave some results on boundedness and global existence of solutions for beam equation.

On SPDE side, We reviewed the work of Zdzislaw Brzezniak, Bohdan Maslowski, and Jan Seidler who solve the problem of existence and uniqueness by Lyapunov’s method [9]. In addition, some results gotten by Z. Brzezniak, M. Ondrejat, and J. Seidler on invariance measure also attracted our attention [10].

2.1 Beam Equation as a PDE

2.1.1 Physics Background

The research of beam equation that we are studied, starts from Woinwsky Krieger [7], who build a mathematical model from physics and he solved that equation by offering one kind of periodic solution.

The equation that they solved is as below

\[ B \frac{\partial^4 y}{\partial x^4} = - \mu \frac{\partial y^2}{\partial t^2} + (S_0 + S_1) \frac{\partial^2 y}{\partial x^2}. \]  

(2.1)
It defines the deflection of the vibrating bar, in the absence of transverse load. In the equation above, \( E \) is Young’s modulus of material, \( I \) is the cross-sectional moment of inertia, \( B = EI \) is the flexural rigidity of bar, \( \mu \) stands for the vibration mass per unit length of bar, \( S_0 \) is the initial axial tensile force of bar, \( S_1 \) is the axial tensile force due to deflection, \( y \) is the instantaneous deflection of any point \( x \) of bar, \( t \) stands for time.

The usual theory of vibration of the bars is based on the assumption that one end of the bar, being free to move in an axial direction, an extensionless deflection of the bar is obtained. In technical practice engineers often have to deal with immovable end hinges, or with hinges connected with supports in such a manner that, as the ends approach each other, a tensile force is produced in the bar which is proportional to the amount of that motion. Further, they assume an initial tensile force and an extensibility of the bar. The deflection of the bar does not need to be small comparison with its transverse dimensions; however, it must be small enough to represent the curvature of the deflected bar by the approximate expression \( \partial^2 y / \partial x^2 \).

The amount of approach of both hinged ends of the bar due to the deflection is

\[
\Delta l = \frac{1}{2} \int_0^l \left( \frac{\partial y}{\partial x} \right)^2 \, dx. \tag{2.2}
\]

Now, the axial force \( S_1 \) produces an elongation of the bar

\[
\Delta l = S_1 \left( \frac{l}{EA} - \frac{1}{\beta} \right) = \frac{S_1 l}{EA} \left( 1 - \frac{EA}{l \beta} \right), \tag{2.3}
\]

\( l \) is the length of the bar, \( A \) is the cross-sectional area of bar, \( \beta \) is the spring constant of supports of bar relative to axial displacement, then they omitted the constant \( 1/\beta \). If the actual constant \( 1/\beta \neq 0 \), they replaced the actual area \( A \) in their final results by the reduced value

\[
A' = \frac{A}{1 - \frac{EA}{l \beta}}.
\]

Equating the Expressions (2.2), (2.3), they have

\[
S_1 = EA \frac{\Delta l}{l} = \frac{B}{2lr^2} \int_0^l \left( \frac{\partial y}{\partial x} \right)^2 \, dx, \tag{2.4}
\]

where \( r = \sqrt{I/A} \) is the radius of gyration. Substituting the last expression in Equation (2.1), they obtain

\[
\frac{\partial^2 y}{\partial t^2} = \frac{B}{\mu} \frac{\partial y^4}{\partial x^4} + \frac{1}{\mu} \left[ S_0 + \frac{B}{2lr^2} \int_0^l \left( \frac{\partial y}{\partial x} \right)^2 \, dx \right] \frac{\partial^2 y}{\partial x^2}, \tag{2.5}
\]
then the equation has been built up.

**Periodic Solution**

After model built, they give a periodical solution, and it is a Fourier series solution. Putting

\[ y = a \psi \sin \frac{n_0 \pi x}{l}, \]

where \( \psi \) is a function of \( t \), and \( a \) is half amplitude of vibration.

The solution of this equation is

\[ p^2 = \frac{n_0^4 \pi^4 B}{\mu l^4} \left( 1 + \frac{\alpha^2}{4} \right) + \frac{n_0^2 \pi^2 S_0}{\mu l^2}, \]

\[ \psi = cn[p(t + t_0), k], \quad n_0 = 1, 2, 3, \ldots \]

where \( k \) is the modulus of the elliptic function and \( t_0 \) is a constant of integration, \( n_0 \) is a positive integers, \( \alpha = a/r \). In addition, \( cn(\cdot, \cdot) \) is the elliptic cosine function.

The period of the function \( cn(pt, k) \) is

\[ K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \]

and the corresponding frequency is

\[ \omega = \frac{\pi p}{2K}. \]

In the absence of axial forces (they can get this case assuming \( S_0 = 0, r \to \infty \)) and thus obtaining \( \alpha = 0, k = 0, K = \pi/2 \). The frequency becomes

\[ \omega_0 = p_0, \]

where \( p_0^2 = n^4 \pi^4 B/\mu l^4 \).

By Equation (2.4) the tensile force due to the deflection alone is

\[ S_1 = \frac{B \psi^2}{2 tr^2} \int_0^l \left( \frac{an \pi}{l} \right)^2 \cos^2 \frac{n \pi x}{l} \, dx = \frac{P_n \alpha^2}{4} cn^2(pt, k). \]

Hence the frequency of this force is equal to \( 2\omega \) and its maximum value is

\[ \max S_1 = \frac{P_n \alpha^2}{4}, \]

\( P_n \) being that Euler load which corresponds to the orthogonal function \( y = \sin(n \pi x/l) \).
2.1.2 Different Solutions

This subsection is based on the work of R.W. Dickey [13].

The deflection $W(x,t)$ of an extensible beam under an axial force $H$ satisfies the non-linear partial integro-differential equation like the one of Woinowsky [7],

$$\frac{\partial^2 W}{\partial t^2} + \frac{EI}{\rho} \frac{\partial^4 W}{\partial x^4} - \left( \frac{H}{\rho} + \frac{EA_c}{2\rho L} \int_0^L W_\xi(\xi, t)^2 \, d\xi \right) \frac{\partial^2 W}{\partial x^2} = 0,$$

(2.6)

where $E$ is the Young's modulus, $I$ is the cross-sectional moment of inertia, $\rho$ is the density, $L$ is the length, and $A_c$ is the cross-sectional area. The assumption of hinged ends implies that $W(x,t)$ satisfies the boundary conditions

$$W(0, t) = W_{xx}(0, t) = W(L,t) = W_{xx}(L,t) = 0.$$

The initial conditions will be

$$W(x, 0) = f(x),$$
$$W_t(x, 0) = g(x).$$

The vibration of non-linear beams has been treated by W. Krieger [7] and Eisley [11]. Their papers shows the existence of special solutions of the form

$$W(x,t) = T_j(t) \sin \frac{j\pi}{L} x,$$

when $H \geq 0$. In this paper, they use the results in [12] to discuss solutions of the form

$$W(x,t) = \sum_{j=1}^\infty T_j(t) \sin \frac{j\pi}{L} x.
$$

(2.7)

In addition the assumption that $H \geq 0$ will be dropped, i.e, the case of the compressed beam will be included.

If (2.7) is to be a formal solution of (2.6), it follows that the functions $T_j$ must be solutions of the infinite system of non-linear ordinary differential equation (cf.[12])

$$T_j'' + j^2(a_0j^2 + a_1 + a_2 \sum_{l=1}^\infty l^2 T_l^2)T_j = 0, \quad j = 1, 2, \cdots, \infty,
$$

(2.8)

where

$$a_0 = \frac{EI\pi^4}{\rho L^4}, \quad a_1 = \frac{H\pi^2}{\rho L^2}, \quad a_2 = \frac{EA_c\pi^4}{4\rho L^4}.
$$

(2.9)

In addition, the fact that (2.7) must satisfy the initial conditions (2.9) implies that $T_j$ must satisfy

$$T_j(0) = \beta_j = \frac{2}{L} \int_0^L f(x) \sin(j\pi x/L) \, dx,$$

(2.10)

$$T_j'(0) = \gamma_j = \frac{2}{L} \int_0^L g(x) \sin(j\pi x/L) \, dx.
$$

(2.11)
Elementary Solutions

Before discussing solutions of (2.6) of the form (2.7), it is convenient to begin by searching for more elementary solutions. In particular, (2.6) will have a solution of the form (cf. [7], [11])

$$W(x,t) = T_N(t) \sin(N \pi x/L),$$

if $T_N(t)$ satisfies the ordinary differential equation

$$T_N'' + (a_0 N^4 + a_1 N^2)T_N + a_2 N^4 T_N^3 = 0,$$

(2.12)

The constants $a_0$ and $a_2$ are positive, therefore, if $a_0 N^4 + a_1 N^2 \geq 0$, Equation (2.12) is Duffing’s equation with a hard spring characteristic (cf. [38]). It is well known that the (unique) solution of Duffing’s equation exists and is periodic with period depending on the initial data $a_0, a_1, a_2$, and $N$. If $a_0 N^4 + a_1 N^4 < 0$, or equivalently

$$H/EI < -N^2 \pi^2 / L^2.$$  

(2.13)

Equation (2.12) still has periodic solutions. However, the qualitative behaviour of these solutions is different from the solutions of the duffing equation. Note that (2.13) is the requirement that the magnitude of the compressive force be greater than the critical buckling load of the linear beam theory. In order to describe the solutions of (2.12) when the inequality (2.13) is satisfied, multiply (2.12) by $T_N'$ and integrate the resulting equation to show

$$T_N'(t)^2 + (a_0 N^4 + a_1 N^2)T_N^2 + \frac{a_2 N^4}{2} T_N^4 = h,$$

where

$$h = T_N'(0)^2 + (a_0 N^4 + a_1 N^2)T_N(0)^2 + \frac{a_2 N^4}{2} T_N(0)^4.$$

The equation (2.6) will also have solutions of the form

$$W(x,t) = \sum_{j=1}^{N} T_j(t) \sin j \pi x/L.$$

If the functions $T_j$ satisfy the system of equations (cf. [12])

$$T_j'' + j^2(a_0 j^2 + a_1 + a_2 \sum_{i=1}^{N} l^2 T_i^2)T_j = 0, \quad j = 1, 2, \ldots, N.$$  

(2.14)

There is no difficulty in using the method of successive approximation (cf. [39]) to show that the system (2.14) has a solution in some interval $0 \leq t < t_1$. 

Theorem 2.1.1. The solution of (2.14) exists for all \( t \geq 0 \).

General solutions

Define the functions \( T_{j,N} \) to be solutions of (2.14) satisfying the initial conditions (2.9) for \( j \leq N \) and \( T_{j,N} \) for \( j > N \). The functions \( T_{j,N} \) are also solutions of (2.6) and satisfy the initial conditions (2.9) for \( J \leq N \) and \( T_{j,N}(0) = T_{j,N}'(0) = 0 \) for \( j > N \), i.e.

\[
T''_{j,N} + j^2 A_{j,N} T_{j,N} = 0,
\]

where

\[
A_{j,N} = a_0 j^2 + a_1 + a_2 \sum_{i=1}^{\infty} l^2 l_i^2 T_{l_i,N}^2.
\]

The first step will be to show that \(| A_{j,N} | \) and \(| A_{j,N}' | \) is bounded independent of \( N \).

Lemma 2.1.1. \(| A_{j,N} | \) is uniformly bounded (independent of \( N \)) if

\[
\sum_{j=1}^{\infty} \gamma_j^2 < \infty,
\]

\[
\sum_{j=1}^{\infty} j^4 \beta_j^2 < \infty.
\]

Lemma 2.1.2. \(| A_{j,N}' | \) is uniformly bounded (independent of \( N \)) if

\[
\sum_{j=1}^{\infty} \gamma_j^2 < \infty.
\]

Lemma 2.1.3. \( T_{j,N} \) converges to \( T_j \) on the interval \( 0 \leq t \leq t^* < \infty \).

Then the question of existence of solutions to (2.8) is settled by

Theorem 2.1.2. There exists a solution of (2.8) satisfying the initial conditions (2.10) if the initial conditions satisfy

\[
\sum_{j=1}^{\infty} \gamma_j^2 < \infty.
\]
2.1.3 Initial-Boundary Value Problem

This subsection is based on the work of J.M. Ball [16]. In this paper, their equation settings are as below

\[
\partial_t^2 u(t, x) + \alpha \partial_x^4 u - (\beta + k \int_0^l |\partial_y u(t, y)|^2 \, dy)\partial_x^2 u = 0 \quad (2.15)
\]

This present paper extends the work of Dickey’s in several directions. They deal with both the cases of hinged ends and that of clamped (or built-in) ends for which

\[
u(0, t) = u(l, t) = u_x(0, t) = u_x(l, t) = 0.
\]

In both cases they use techniques of Lions [28] to prove the existence of weak solutions to the initial-boundary value problem for (2.15). They then show that these solutions satisfy an energy equation and depend continuously on the initial data in a way which implies that the solution for given initial data is unique.

The Galerkin method used converges to the solution for an arbitrary basis of the appropriate function spaces. They next prove that when the initial data is sufficiently smooth and satisfies appropriate compatibility conditions, the resulting solution is a classical solution of (2.15).

In the hinged end case, the compatibility conditions are linear, but in clamped-end case, they are non-linear and this makes the regularity proof less straightforward. In a series of papers, Antman[33-35] has used the direct method of the calculus of variations to prove the existence of stable equilibrium configurations for rods and shells with a Cosserat structure.

The models used by Antman incorporate both geometric non-linearities, due to large deflections, and the effects of non-linear stress-strain laws. Antman obtains qualitative results on the nature of buckled states. Convexity assumptions analogous to those of Coleman and Noll (see [26]) are essential for the existence proofs. In a similar way they used a monotonicity property (Lemma 6 in [16]) to establish the convergence of the non-linear term in (2.15).

**Hinged Ends**

The model for the deflection \( u(x, t) \) which they discuss is

\[
\ddot{u} + \alpha u^{(4)} - (\beta + k \, |u^{(1)}|^2)u^{(2)} = 0,
\]  

(2.16)
where $\alpha = EI/\rho, \beta = H/\rho, k = EA_c/2\rho l$ and $H = EA \Delta /l$, where $E$ is the Young’s modulus, $I$ is the cross-sectional second moment of area, $\rho$ the density and $A_c$ the cross-sectional area.

In this case, they establish the existence of weak solutions of the equation (2.15), where the initial conditions are as below

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = u_1(x),$$  \hspace{1cm} (2.17)

With the boundary condition

$$u(0, t) = u(l, t) = u^{(2)}(0, t) = u^{(2)}(l, t) = 0.$$  \hspace{1cm} (2.18)

Let $\Omega$ be the open interval $(0, l)$ of $\mathbb{R}$, where $l > 0$ is the length of the beam in its unstressed case. Let $L^2(\Omega)$ be the Hilbert space of real valued Lebesgue measurable functions.

**Theorem 2.1.3.** If $u_0 \in S_2, u_1 \in L^2(\Omega)$, then there exists $u \equiv u(x, t)$ with

$$u \in L^\infty(0, T; S_2),$$

$$\dot{u} \in L^\infty(0, T; L^2(\Omega)),$$

such that $u$ satisfies the initial condition (2.17) and the equation (2.16) in the sense that

$$(\ddot{u}, \phi) + \alpha(u^{(2)}, \phi^{(2)}) - (\beta + k | u^{(1)} |^2) (u^{(2)}, \phi) = 0, \quad \forall \phi \in S_2,$$  \hspace{1cm} (2.19)

where $S_2 = H^1_0(\Omega) \cap H^2(\Omega)$.

**Theorem 2.1.4.** Suppose $u, v$ are two solutions of (2.19) with

$$u, v \in L^\infty(0, T; S_2),$$

$$\dot{u}, \dot{v} \in L^\infty(0, T; L^2(\Omega)),$$

and suppose that $u, v$ satisfy the initial conditions

$$u(0) = u_0, \dot{u}(0) = u_1, v(0) = v_0, \dot{v}(0) = v_1,$$

with $u_0, v_0 \in S_2$ and $u_1, v_1 \in L^2(\Omega)$. Let $\omega = u - v$. Then

$$| \ddot{\omega}(t) |^2 + \alpha | \omega^{(2)}(t) |^2 \leq \| u_1 - v_1 \|^2 + \alpha | u_0^{(2)} - v_0^{(2)} |^2 \exp(Kt),$$

where $K$ is a continuous function of $| u_0^{(2)} |, | u_1 |, | v_0^{(2)} |$ and $| v_1 |$. 


CHAPTER 2. LITERATURE SURVEY

Theorem 2.1.5. The unique solution in Theorem 2.1.4 satisfies the energy equation

\[ | \dot{u} |^2 + \alpha | u^{(2)} |^2 + \beta | u^{(1)} |^2 + (k/2) | u^{(1)} |^4 = h. \]

Theorem 2.1.6. Suppose \( u_0 \in S_1 \) and \( u_1 \in S_2 \). Then there exists a unique function \( u \) such that

\[
\begin{align*}
\dot{u} + \alpha u^{(4)} - (\beta + k | u^{(1)} |)u^{(2)} &= 0 \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)), \\
\ddot{u} &\in L^\infty(0, T; L^2(\Omega)), \quad u \in C_1(\bar{Q}) \cap [C^5(\bar{\Omega}) \times C^2([0, T])] ,
\end{align*}
\]

where \( Q = \Omega \times (0, T) \).

**Clamped Ends**

Here the initial conditions are the same as it is in the case of hinged ends, but boundary conditions are different, it changed to

\[ u(0, t) = u(l, t) = u^{(1)}(0, t) = u^{(1)}(l, t) = 0. \]

Theorem 2.1.8. If \( u_0 \in H^2_0(\Omega), u_1 \in L^2(\Omega), \) then there exists \( u = u(x, t) \) with

\[
\begin{align*}
\dot{u} &\in L^\infty(0, T; H^2_0(\Omega)), \\
\ddot{u} &\in L^\infty(0, T; L^2(\Omega)),
\end{align*}
\]

such that \( u \) satisfies the initial conditions

\[ u(x, 0) = u_0(x), \]
\[ \dot{u}(x, 0) = u_1(x), \]

and the equation

\[ \ddot{u} + \alpha u^{(4)} - (\beta + k | u^{(1)} |^2)u^{(2)} = 0, \]

in the sense that

\[ (\ddot{u}, \varphi) + \alpha (u^{(2)}, \varphi^{(2)}) - (\beta + k | u^{(1)} |^2)(u^{(2)}, \varphi) = 0, \quad \forall \varphi \in H^2_0(\Omega). \]
Theorem 2.1.9. Suppose \( u, v \) satisfy the initial conditions
\[
  u(0) = u_0, \dot{u}(0) = u_1, v(0) = v_0, \dot{v}(0) = v_1,
\]
with
\[
  u_0, v_0 \in H^2_0(\Omega) \quad \text{and} \quad u_1, v_1 \in L^2(\Omega).
\]
Set \( \omega = u - v. \) Then
\[
  |\dot{\omega}(t)|^2 + \alpha |\omega^{(2)}|^2 \leq \| u_1 - v_1 \|^2 + \alpha \left| u_0^{(2)} - v_0^{(2)} \right| \exp(K_1 t)
\]
where \( K_1 \) is a continuous function of \( |u_0^{(2)}|, |u_1|, |v_0^{(2)}| \) and \( |v_1| \).

Theorem 2.1.10. The unique solution in Theorem 2.1.8 satisfies the energy equation.

Lemma 2.1.4. There are constants \( C_i \) such that for all \( f \in X = H^2_0(\Omega) \cap H^4(\Omega) \),
\[
  |f^{(i)}| \leq C_i |f^{(i+1)}|, \quad i = 0, 1, 2, 3.
\]

Theorem 2.1.11. If \( u_0 \in X, u_1 \in H^2_0(\Omega) \) then there exists a unique function \( u \equiv u(x,t) \) with
\[
  u \in L^\infty(0, T; X), \\
  \dot{u} \in L^\infty(0, T; H^2_0(\Omega)), \\
  \ddot{u} \in L^\infty(0, T; L^2(\Omega)),
\]
such that \( u \) satisfies the initial conditions
\[
  u(x, 0) = u_0(x), \\
  \dot{u}(x, 0) = u_1(x),
\]
and the equation
\[
  \ddot{u} + \alpha u^{(4)} - (\beta + k |u^{(1)}|^2)u^{(2)} = 0 \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)).
\]

Theorem 2.1.12. Let \( f \) be a continuous real valued function on \([0, T]\) such that
\[
  f, \dot{f}, \ddot{f} \in L^\infty(0, T).
\]

Let \( u_0 \in H^b(\Omega) \) with
\[
  u_0 = u_0^{(1)} = \alpha u_0^{(4)} - f(0)u_0^{(2)} = \alpha u_0^{(5)} - f(0)u_0^{(3)} = 0 \quad \text{at} \quad x = 0, 1,
where $\alpha > 0$. Let

$$u_1 \in X = H^2_0(\Omega) \cap H^4(\Omega).$$

Then there exists a unique function $y \equiv y(x, t)$ with

\begin{align*}
y &\in L^\infty(0, T; H^2_0(\Omega) \cap H^6(\Omega)), \\
\dot{y} &\in L^\infty(0, T; X), \\
\ddot{y} &\in L^\infty(0, T; H^2_0(\Omega)), \\
\dddot{y} &\in L^\infty(0, T; L^2(\Omega)), \\
y &\in C^1(\bar{Q}) \cap \left[ C^5(\bar{\Omega}) \times C^2([0, T]) \right],
\end{align*}

such that $y$ satisfies the linear equation

$$\dddot{y} + \alpha y^{(4)} - f(t)y^{(2)} = 0,$$

and the initial conditions

$$y(0) = u_0, \quad \dot{y}(0) = u_1.$$

Lemma 2.1.5. Let

$$\gamma = f(0)/\alpha.$$

Consider the ordinary differential equation $L\omega = \lambda \omega$ subject to the boundary conditions $\omega = \omega^{(4)} = 0$ at $x = 0$ and $x = l$, where

$$L\omega \equiv \omega^{(4)} - \gamma \omega^{(2)}.$$

Then

1. There exist an infinity of eigenvalues $\lambda_i$ whose absolute values are unbounded and for which zero is not an accumulation point.

2. To each eigenvalue $\lambda_i$ corresponds a unique normalized eigenfunction $\omega_i$. For convenience, enumerate the $\lambda_i$ so that $0 < |\lambda_1| \leq |\lambda_2| \leq \cdots$. Zero ($=\lambda_0$) may be an eigenvalues, in which case let $\omega_0$ be the corresponding non-trivial eigenfunction.

3. The normalized eigenfunctions $\omega_i$ form a basis of $L^2(\Omega)$. Any $g \in L^2(\Omega)$ can be expanded in a series $g(x) = \sum_i (\omega_i, g)\omega_i(x)$, convergence holding in $L^2(\Omega)$.

Lemma 2.1.6. Let $M$ be the subspace of $L^2(\Omega)$ generated by $\omega_0$ if $\lambda = 0$ is an eigenvalue of $L$, and be empty otherwise. Let $M^\perp$ be the orthogonal complement of $M$ in $L^2(\Omega)$. Then for all $y \in M^\perp \cap X$,

$$|y| \leq |\lambda_1|^{-1} |Ly|.$$
Lemma 2.1.7. For all $y \in M^\perp \cap X$, $|y^{(4)}| \leq C \, |Ly|$.

Theorem 2.1.13. $\{\omega_j\}$ is a basis of $X$ and of $Y$.

Theorem 2.1.14. Let $u_0 \in H^6(\Omega)$ and satisfy

$$\alpha u_0^{(4)} - (\beta + k \, |u_0^{(1)}|^2)u_0^{(2)} = \alpha u_0^{(5)} - (\beta + k \, |u_0^{(1)}|^2)u_0^{(3)} = 0 \quad \text{at} \quad x = 0, l.$$ 

Let $u_1 \in X$. Then the unique solution $u$ in Theorem 2.1.12 is such that

- $u \in L^\infty(0, T; H^2_0(\Omega) \cap H^6(\Omega))$,
- $\dot{u} \in L^\infty(0, T; X)$,
- $\ddot{u} \in L^\infty(0, T; H^2_0(\Omega))$,
- $\dddot{u} \in L^\infty(0, T; L^2(\Omega))$,

and $u \in C^1(\bar{Q}) \cap [C^5(\bar{\Omega}) \times C^2([0, T])]$. 
2.1.4 Stability

This section is a review about J. M. Ball’s work [17].

In J. M. Ball’s paper, they use topological methods to study the asymptotic properties of one kind of beam equation. Compared to the one treated by Woinowsky, the one they worked on is like below

\[
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - (\beta + k \int_0^l \left( \frac{\partial u(\xi, t)}{\partial \xi} \right)^2 \frac{\partial^2 u}{\partial x^2} \\
+ \gamma \frac{\partial^5 u}{\partial x^4 \partial t} - \sigma \int_0^l \frac{\partial u}{\partial \xi} \frac{\partial^2 u}{\partial \xi \partial t} \, d\xi \frac{\partial^2 u}{\partial x^2} + \delta \frac{\partial u}{\partial t} = 0,
\]

\(2.20\)

When \(\gamma = \sigma = \delta = 0\), their equation reduces to the equation studied in [16]. They modify the model by introducing terms to account for the effects of internal (structural) and external damping. Specifically, they assume that the beam is linearly viscoelastic and that the (possibly negative) external damping is proportional to the velocity.

Their method can cater for a variety of damping terms. In particular they may be simply adapted to the case when \(\gamma = \delta = 0\) and \(\delta > 0\) in (2.20).

Their aim is to show that as \(t \to \infty\), provided \(\delta\) is not large and negative, any solution of (2.20) converges in a suitable topology to an equilibrium position of the beam. When the beam is constrained to lie along the \(x\)-axis, an axial force \(H\) is set up. (\(H\) is taken as positive when tensile) If \(H \geq H_E\), the Euler load of the beam, then only equilibrium position is the trivial one. In this case they give an elementary proof that if \(\delta > \delta_0\), any solution converges to the trivial position in the energy norm. The constant \(\delta_0\) is negative and depends on the boundary conditions. When \(H < H_E\), there are \(2n + 1\) equilibrium positions for some \(n \geq 1\) and there is no obvious criterion for determining to which state a particular solution will converge. If an orbit in a dynamical system on a reflexive Banach space \(B\) belongs to a compact set then this ensures that the limit set is non-empty and invariant.

In Ball’s case, they prove using the methods of Lions [28] that (2.20) defines a weakly continuous dynamical system on a suitable Banach space. However, the natural Lyapunov function, the total energy of the beam, unfortunately fails to be weakly continuous, and so the Slemrod’s result is invalid.
Ball’s method involves studying functions representing energy lost through dissipation. It turns out to be simple to prove that if $\delta > \delta_0$, any orbit converges weakly to an equilibrium position. This is a similar result to Slemrod’s for the modified Van der Pol equation. They then show that convergence to an equilibrium position also takes place in the energy norm. For the hinged beam they show that if $\delta < \delta_0$ and $H \geq H_E$ then there is at least one periodic solution to (2.20). One result they get is that for the case $H < H_E, \delta > \delta_0$ the two equilibrium states which minimize the potential energy of beam are (dynamically) stable. There are some relevant papers by Hsu [29-32].

Their model settings are following.

Consider transverse motion, at small strains, in the $X - Y$ plane, of a linear viscoelastic beam in a viscous medium whose resistance is proportional to the velocity. They neglect rotational inertia and shear deformation. In the reference, stress-free state the beam occupies the interval $[0, l]$ of the X-axis. The ends are then fixed at $(0, 0)$ and $(l + \Delta, 0)$. Let an arbitrary point $P$ of the neutral axis, whose position is $(x, 0)$ in the reference state, be displaced to $(x + \omega, u)$. For their model the axial force $N$ and the bending moment $M$ are given by

$$N = \frac{E A \Delta}{l} + \frac{E A}{2l} |u'|^2 + \frac{A \eta}{l} (u', \dot{u'})$$

$$M = -E I u'' - \eta I \dot{u}'',$$

where $E$ is the Young’s modulus, $A$ the cross-sectional area, $\eta$ the effective viscosity, and $I$ the cross-sectional second moment of area. The equation of motion in the $Y$-direction is

$$\rho \ddot{u} + E I \dot{u}''' + \eta \ddot{u}''' - \left[ \frac{E A \Delta}{l} + \frac{E A}{2l} |u'|^2 + \frac{A \eta}{l} (u', \dot{u}'') \right] + \rho \delta \dot{u} = 0,$$

which is (2.20) with $\alpha = E I / \rho, \beta = E A \Delta / l \rho, \gamma = \eta I / \rho$ and $k = E A / 2l \rho, \rho = A \eta / l \rho$. In (2.21) $\rho$ is the mass per unit length in the reference configuration and $\delta$ the coefficient of external damping.

Their aim is to study the initial-boundary value problem consisting of (2.20), the initial conditions

$$u(0) = u_0, \quad \dot{u}(0) = u_1,$$

and the boundary conditions corresponding either to hinged ends, when

$$u = u'' = 0, \quad x = 0, l,$$
or to clamped ends, when
\[ u = u' = 0, \quad x = 0, l \] (2.24)
(2.23) is a sufficient, though not necessary, condition for a smooth enough \( u \) to have zero bending moment at \( x = 0, l \).

The equilibrium states of the beam have been studied by, for example, Reiss [27] and satisfy \( \alpha u'''' = (\beta + k | \ u' |^2)u'' \) subject to either (2.23) or (2.24).

Any non-zero equilibrium position \( v_j \) is an eigenfunction satisfying \( \alpha v_j''' + \lambda_j v_j'' = 0 \) subject to the relevant boundary conditions, where \( | v_j' |^2 = -(\beta + \lambda_j)/k \).

The positive sequence \( \{ \lambda_j \} \) is strictly increasing and has no finite accumulation point. So if \( \beta \geq -\lambda_1 \), there are no non-zero equilibrium positions, while if \( \beta < \lambda_1 \), there are \( 2n \) such corresponding to those \( \lambda_j < -\beta, \lambda_1 = \alpha \pi^2/l^2 \) or \( 4\alpha \pi^2/l^2 \) for hinged or clamped ends, respectively.

Define the load \( H \) by \( H = EA\Delta/l \), and the Euler buckling load \( H_E \) by \( H_E = -\lambda_1 \rho \). Then the conditions \( \beta \geq -\lambda_1 \) and \( \beta < -\lambda_1 \) are equivalent to \( H \geq H_E \) and \( H < H_E \), respectively.

The potential energy \( V(u) \) of the beam when deflection equals \( u \) is
\[ V(u) = (\alpha/2) | u'' |^2 + (\beta/2) | u' |^2 + (k/4) | u' |^4 . \]

Therefore, for \( H < H_E \),
\[ V(v_j) = -(\beta + \lambda_j)^2/4k, \quad j = 1, 2, \ldots, n. \]

It is easy to prove that the zero position \( (H \geq H_E) \) and \( v_1, -v_1 (H < H_E) \) minimize \( V \). If \( H < H_E \) and \( j > 1 \) then neither \( v_j \) nor \( -v_j \) minimize \( V \) locally, since a little calculation shows that
\[ V(v_j + \epsilon v_1) - V(v_j) = \frac{\epsilon^2}{2} | v_j' |^2 + \frac{\epsilon^4 k}{4} | v_j' |^4 , \]
which is negative if \( \epsilon \) is small enough. Similarly, if \( H < H_E \), the zero position does not minimize \( V \) locally.

Then they get some results about existence, uniqueness and regularity in the case of clamped ends.

They assume that \( \alpha, k, \sigma, k > 0 \) while they do not restrict \( \delta \), in other word, admitting the possibility of large negative external damping.
Lemma 2.1.8. Let $X$ be a Banach space. If $f \in L^2(0,T;X)$ and $f \in L^2(0,T;X)$, then $f$, possibly after redefinition on a set of measure zero, is continuous from $[0,T] \rightarrow X$. Indeed, for almost all $s,t \in [0,T]$,
\[
    f(t) - f(s) = \int_s^t \dot{f}(\sigma) \, d\sigma.
\]

Lemma 2.1.9. Let $X,Y$ be Banach spaces with $X$ continuous embedded in $Y$. Suppose $X$ is reflexive. If $f \in L^\infty(0,T;X)$ and $\dot{f} \in L^2(0,T;X)$ and then $f$ is continuous map of $[0,T]$ into $\tilde{X}$.

In their first two theorems they prove the existence of weak solutions to the initial-boundary value problem and show that these solutions have continuity properties with respect to time and the initial data.

Theorem 2.1.15. Suppose $u_0 \in H^2_0(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exists a function $u \equiv u(x,t)$ with $u \in L^\infty(0,T;H^2_0(\Omega))$ and $u \in L^\infty(0,T;L^2(\Omega) \cap L^2(0,T;H^2_0(\Omega)))$, such that $u$ satisfies the initial conditions (2.22) and (2.20) in the sense that
\[
    (\ddot{u}, \phi) + \alpha(u'', \phi'') - (\beta + k | u'|^2)(u'', \phi) + \gamma(\ddot{u}'', \phi'')
\]
\[
    -\sigma(u', \dot{u})(u'', \phi) + \delta(\dot{u}, \phi) = 0, \quad \forall \phi \in H^2_0(\Omega).
\]

Theorem 2.1.16. Let $u$ be as in the previous Theorem, and let $v$ be another such weak solution with initial condition $v(0) = v_0$, $\dot{v}(0) = v_1$, where $v_0 \in H^2_0(\Omega)$ and $v_1 \in L^2(\Omega)$. Set $y = u - v$. Then
\[
    |\dot{y}(t)|^2 + \alpha |y''(t)|^2 + 2\gamma \int_0^t |\dot{y}''| \, ds
\]
\[
    \leq \| u_1 - v_1 \|^2 + \alpha |u''_0 - v''_0|^2 \exp(Kt)
\]
where $K$ is a continuous function of $T$, $|u''_0|, |u_1|, |v''_0|, |v_1|$. In particular, the solution $u$ in previous is unique. Furthermore, the function $u : [0,T] \rightarrow H^2_0(\Omega)$ and $\dot{u} : [0,T] \rightarrow L^2(\Omega)$ are continuous and satisfy the energy equation
\[
    E(t) + \gamma \int_0^t |\dot{u}''|^2 \, ds + \sigma \int_0^t (u', \dot{u})^2 \, ds + \delta \int_0^t |\dot{u}|^2 \, ds = E(0),
\]
where $E(t) \equiv E(u(t)) = \frac{1}{2} |\dot{u}(t)|^2 + (\alpha/2) |u''(t)|^2 + (\beta/2) |u'(t)|^2 + (k/4) |u'(t)|^4$.
Theorem 2.1.17. Suppose \( u_0 \in X \) and \( u_1 \in H^2_0(\Omega) \). Then there exists a unique function \( u \equiv u(x,t) \) with

\[
\begin{align*}
\dot{u} &\in L^\infty(0,T;H^2_0(\Omega) \cap L^2(0,T;X)), \\
\ddot{u} &\in L^2(0,T;L^2(\Omega)), \\
\end{align*}
\]

such that \( u \) satisfies the initial conditions (2.22) and the equation

\[
\ddot{u} + \alpha u''' - (\beta + k |u'|^2)u'' + \gamma u''' - \sigma(u',u'')u'' + \delta u = 0
\]
in \( L^2(0,T;L^2(\Omega)) \).

Weak Dynamical Systems

By a dynamical system on a Banach space \( B \) we mean a function \( \omega : \mathbb{R}^+ \times B \rightarrow B \) which satisfies

\[
\begin{align*}
(i)\omega^t : \phi &\rightarrow \omega(t,\phi) \text{ is continuous } \forall t \in \mathbb{R}^+, \\
(ii)\omega^\phi : t &\rightarrow \omega(t,\phi) \text{ is continuous } \forall \phi \in B, \\
(iii)\omega(0,\phi) &= \phi, \forall \phi \in B, \\
(iv)\text{ (semigroup property) } \omega(t + \tau,\phi) &= \omega(t,\omega(\tau,\phi)) \forall t, \tau \in \mathbb{R}^+, \phi \in B. \\
\end{align*}
\]

Let \( \Sigma = H^2_0(\Omega) \times L^2(\Omega) \). \( L \) is a Hilbert space under the ‘energy’ norm

\[
\| \{ \psi, \chi \} \|_\Omega = [\| \chi \|^2 + \alpha |\phi'|^2]^{1/2}.
\]

Theorem (2.1.15)(2.1.16)(2.1.17) show that \( \{u, \dot{u}\} \) generates a dynamical system on \( \Sigma \) and on the space \( X \times H^2_0(\Omega) \).

A weak dynamical system on a reflexive space \( B \) is a function \( \omega : \mathbb{R}^+ \times B \rightarrow B \) which satisfies

\[
\begin{align*}
(i)\omega^t : \phi &\rightarrow \omega(t,\phi) \text{ is (sequentially) weakly continuous } \forall t \in \mathbb{R}^+, \\
(ii)\omega^\phi : t &\rightarrow \omega(t,\phi) \text{ is continuous map of } [0,T] \text{ into } \tilde{B} \forall \phi \in B, \\
(iii)\omega(0,\phi) &= \phi, \\
(iv)\omega(t + \tau,\phi) &= \omega(t,\omega(\tau,\phi)) \forall t, \tau \in \mathbb{R}^+, \phi \in B. \\
\end{align*}
\]

The positive orbit \( 0^+(\phi) \) through \( \phi \in B \) is defined by \( 0^+ = \bigcup_{t \geq 0} \omega(t,\phi) \). A set \( M \) in \( B \) is an invariant set of the weak dynamical system \( \omega \) if for each \( \phi \in M \), there
exists function $W \equiv W(s, \phi)$ such that

(a) $W(s, \phi) \in M \forall s \in R,$

(b) $W(0, \phi) = \phi,$

$\forall \sigma \in R, \omega(t, W(\sigma, \phi)) = W(t + \sigma, \phi) \forall t \in R^+.$

$\forall \phi \in B,$ define the weak limit set $\tilde{\Omega}(\phi)$ of an orbit through $\phi$ by

$$\tilde{\Omega}(\phi) = \{\psi \in B \mid \exists \{t_n\} \nearrow, t_n \to \infty, n \to \infty, \text{such that } \omega(t_n, \phi) \to \psi \in B, n \to \infty\}.$$ 

**Theorem 2.1.18.** Let $B$ be a separable, reflexible Banach space, and let $\omega$ be a weak dynamical system on $B$. Also let $\phi \in B$ be such that $0^+(\phi)$ is bounded in $B$. Then $\tilde{\Omega}(\phi)$ is a non-empty, weakly compact, invariant, weakly connected set in $B$.

**Asymptotic Behaviour of the Clamped Beam**

Here they strict the condition on $\delta$ as

$$\delta > \delta_0 = -\nu_0 \gamma / l^4,$$

where $\nu = \nu_0$ is the lowest eigenvalue of $y'''' = (\mu / l^4)y$ subject to $y = y' = 0, at x = 0, l$.

$\mu_0$ has the approximate value 500.56. It is well known that $| y'' | \geq (\mu_0 / l^4) | y |^2, \forall y \in H_0^2(\Omega)$.

They have shown that the unique weak solution $u$ of

They first show that in the case $H \geq H_E$, the beam approaches the undeflected equilibrium position.

**Theorem 2.1.19.** If $H \geq H_E$, then $\omega(t, \phi_0) \to \{0, 0\}$ strongly in $\Sigma$ as $t \to \infty$.

**Theorem 2.1.20.** If a non-negative measurable function $f$ satisfies

(i) $f, \dot{f} \in L^\infty(R^+),$

(ii) $\int_0^t f(s) \, ds \leq C (\text{independent of } t \in R^+),$

then $f(t) \to 0, t \to \infty.$

**Theorem 2.1.21.** $\forall H, \omega(t, \phi_0) \to \{\nu, 0\}$ strongly in $\Omega$, where $\nu$ is an equilibrium position.

**Lemma 2.1.10.** Let $H < H_E$, and suppose that $\Gamma$ is a continuous arc in $H_0^2(\Omega)$ joining $v_1$ and $-v_1$. Then there is a point $\psi$ on $\Gamma$ with $V(\psi) \geq V(v^*)$ where $v^* = v_2$ if $n \geq 2$ and $v^* = 0$ if $n = 1$. 
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Theorem 2.1.22. If \( H < H_E \) then \( v_1 \) and \( -v_1 \) are stable.

**Hinged-End Boundary Conditions**

Under these conditions, they consider the initial-boundary value problem for (2.20), when the initial conditions \( u(0) = u_0, \dot{u}(0) = u_1 \) with the boundary conditions for hinged ends \( u = u'' = 0 \) at \( x = 0, l \).

Let \( m \) be a positive integer.

Define \( G_m = \{ u \in H^{2m}(\Omega) : u^{2r} \in H^1_0(\Omega), 0 \leq r \leq m - 1 \} \), so that

\[
G_1 = H^1_0(\Omega) \cap H^2(\Omega), \
G_2 = \{ u \in H^4(\Omega) : u, u'' \in H^1_0(\Omega) \}.
\]

\( G_1, G_2, G_3 \) equal to the Hilbert spaces \( S_2, S_1, S_0 \) in [14], respectively.

In these theorems following, no restrictions on \( \delta \) but \( \alpha, k, \gamma, \sigma > 0 \).

**Theorem 2.1.23.** Suppose \( u_0 \in G_1, u_1 \in L^2(\Omega) \). Then there exists \( u \) with

\[
u \in L^\infty(0, T; G_1),
\]

\[
\dot{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; G_1),
\]

satisfying the initial conditions \( u(0) = u_0, \dot{u}(0) = u_1 \), and

\[
(\ddot{u}, \phi) + \alpha(u'', \phi'') - (\beta + k |u'|^2)(u'', \phi) + \gamma(u'', \phi'') - \sigma(u', \dot{u})(u'', \phi) + \delta(\dot{u}, \phi) = 0
\]

\[\forall \phi \in G_1.\]

**Theorem 2.1.24.** Let \( u \) be as it in the previous theorem, and let \( v \) be another such weak solution with initial conditions \( v(0)v_0, \dot{v}(0) = v_1 \), where \( v_0 \in G_1 \) and \( v_1 \in L^2(\Omega) \). Set \( y = u - v \). Then

\[
|\dot{y}(t)|^2 + \alpha |y''(t)|^2 + 2\gamma \int_0^t |\dot{y}''|^2 \, ds \leq [(|u_1 - v_1|^2 + \alpha |u_0'' - v_0''|^2)] \exp(K_1t),
\]

where \( K_1 \) is a continuous function of \( T \), \( |v''|, |v_1|, |v_0''|, \) and \( |v_1| \).

**Theorem 2.1.25.** Suppose \( r \geq 2 \) and that \( u_0 \in G_r, u_1 \in G_{r-1} \). Then

\[
u \in L^\infty(0, T; G_r)
\]

\[
\dot{u} \in L^\infty(0, T; G_{r-1}) \cap L^2(0, T; G_r)
\]

if \( r \geq 4 \)

\[
\partial^j u/\partial t^j \in L^\infty(0, T; G_{r-j} - 1) \cap L^2(0, T; G_{r-j}), \quad 2 \leq j \leq r - 2,
\]

if \( r \geq 3 \)

\[
\partial^{r-1} u/\partial t^{r-1} \in L^\infty(0, T; L^2) \cap L^2(0, T; G_1),
\]

and

\[
\partial^r u/\partial t^r \in L^2(Q).
\]

Furthermore,

\[
u \in C^{r-2}(\bar{Q}) \cap [C^{2r-1}(\bar{\Omega} \times C^{r-1}([0, T])].
\]
Theorem 2.1.26. \( \{u, \dot{u}\} \) forms a weak dynamical system on \( \Sigma_1 \).

Theorem 2.1.27. \( \{u(t), \dot{u}(t)\} \rightarrow \{v, 0\} \) strongly in \( \Sigma_1 \), where \( v \) is an equilibrium position.
2.1.5 Existence and Boundedness

In this section, W.E. Fitzgibbon’s work [14] will be reviewed. They use the theory of semi groups of linear operators and the theory of cosine operators to provide a variation of parameters representation of solutions to the equation governing the transverse motion of an extensible beam and to attain the $L^2$ boundedness of these solutions. They consider the beam equation like before, with the boundary conditions of the form

$$u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0.$$  \tag{2.25}

This boundary conditions correspond to the ends of the beam being hinged. Boundary conditions appropriate for a beam with clamped ends would have the form

$$u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0.$$  \tag{2.26}

Their methods apply equally well to clamped ends but they just discuss the condition of form (2.26). They treat the equation as an abstract second order differential equation. One way to view abstract second order equations is to rewrite them as first order system. For rewriting the equation as an abstract equation, they require that $\alpha, \kappa > 0$, then sign of $\beta$ is unrestricted. Without generality, they assume $\alpha = 1$. Define $X = L^2[0, 1]$. They define $A : X \to X$ by the equation

$$Au = u_{xxxx},$$

$$D(A) = \left\{ u \in X \mid u_x, u_{xx}, u_{xxx} \text{ are absolutely continuous}, \begin{array}{l} u_{xxxx} \in X \text{ and } u(0) = u(1) = u_x(0) = u_x(1) = 0 \end{array} \right\}.$$  

It is known from [15] that $A$ so defined is a positive self-adjoint operator on $X$. The eigenvalues are of $\{ \lambda_n = (n\pi)^4 \mid n \in \mathbb{Z}^+ \}$ and the corresponding eigenvectors are $\{ z_n(s) = \sqrt{2} \sin n\pi s \mid n \in \mathbb{Z}^+, s \in [0, 1] \}$ Moreover, they get the following explicit spectral representation for $A$:

$$Au = \sum_{n=1}^{\infty} (n\pi)^4 < u, z_n > z_n.$$  \tag{2.27}

Fractional powers of $A$ are also positive self-adjoint operators and by be computed

$$A^\gamma u = \sum_{n=1} (n\pi)^4 \gamma < u, z_n > z_n.$$  \tag{2.28}
While

\[ A^{1/2} u = \sum_{n=1}^{\infty} (n\pi)^2 < u, z_n > z_n = -u_{xx}, \]  
(2.29)

and

\[ A^{1/4} u = \sum_{n=1}^{\infty} (n\pi) < u, z_n > z_n = -u_{xx}, \]  
(2.30)

They observe that

\[
\int_0^1 |u_x|^2 = \sum_{n=1}^{\infty} (n\pi)^2 < u, z_n >^2 = <A^{1/4} u, A^{1/4} u> = \|A^{1/4} u\|^2. 
\]  
(2.31)

Then they define a non-linear function \( F : X \to X \) by

\[ Fu = (\beta + \kappa < A^{1/4} u, A^{1/4} u >) A^{1/2} u. \]  
(2.32)

With this settings the beam equation becomes

\[ \ddot{x}(t) + Ax(t) + Fx(t) = 0, x(0) = \varphi, \dot{x}(0) = \psi. \]  
(2.33)

Converting equation(2.33) to a first order system. Toward this end they make \( D(A) \) into Banach space \( X_A \) by imposing the Euclidean graph norm

\[ \|x\|_A = (\|Ax\|^2 + \|x\|^2)^{1/2}, \]  
(2.34)

Then they introduce the Banach space \( \hat{X} \) by defining

\[ \hat{X} = X_A \times X \]

with

\[ \|[\varphi, \psi]\|_\hat{X} = (\|\varphi\|_A^2 + \|\psi\|^2)^{1/2}. \]

They define an operator \( \hat{A} : \hat{X} \to \hat{X} \) using the operator matrix

\[
\begin{pmatrix}
0 & I \\
-A & 0
\end{pmatrix}
\]  
(2.35)

with \( D(A) = D(A) \times D(A^{1/2}) \). It is known from [23] that \( \hat{A} \) defined like this is the infinitesimal generator of a strongly continuous group \( \{T(t) \mid \infty < t < \infty\} \) of linear transformations on \( \hat{X} \). The non-linear operator \( F \) is used to define an operator \( F : X \to X \) in the following way:
\[ \hat{F}\hat{u} = \begin{pmatrix} 0 \\ Fu \end{pmatrix}, \quad \forall \hat{u} = [u,v] \in \hat{X}. \quad (2.36) \]

It should be clear that there exists a positive non-decreasing function \( L(\cdot) \), so that
\[ \| \hat{F}u_1 - \hat{F}u_2 \|_{\hat{X}} \leq L(R) \| \hat{u}_1 - \hat{u}_2 \|_{\hat{X}} \] whenever \( \hat{u}_1, \hat{u}_2 \in \hat{X} \) and \( \sup \| \hat{u}_i \|_{\hat{X}} \leq R \).

They seek solutions to the first order differential equation:
\[ \hat{u}'(t) = \hat{A}\hat{u}(t) - \hat{F}\hat{u}(t), \quad \hat{u}(0) = \hat{x}_0 = [\varphi, \psi] \in D(\hat{A}). \quad (2.37) \]

If \( \pi_1 \) and \( \pi_2 \) project \( \hat{X} \) onto its first and second coordinates, respectively, they attain that (17) has matrix representation
\[ \frac{d}{dt} \begin{pmatrix} \pi_1 \hat{u}(t) \\ \pi_2 \hat{u}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\hat{A} & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \hat{u}(t) \\ \pi_2 \hat{u}(t) \end{pmatrix} - \begin{pmatrix} 0 \\ F(\pi_1 \hat{u}(t)) \end{pmatrix}. \quad (2.38) \]

**Theorem 2.1.28.** Let \( \hat{A} \) and \( \hat{F} \) be defined via (14) and (15) respectively. If \( \hat{x}_0 = [\varphi, \psi] \in D(\hat{A}) \), then there exists a unique continuous function \( \hat{u} : [0, \infty) \to \hat{X} \) which satisfies
\[ \hat{u}(t) = \hat{T}(t)\hat{x}_0 - \int_0^t \hat{T}(t-s) \hat{F}u(s) \, ds, \quad \hat{u}(0) = \hat{x}_0. \]

For a.e. \( t \in [0, \infty) \), \( \hat{u}'(t) \) exists and satisfies (17); consequently, there exists \( x(\cdot) : [0, \infty) \to X \) which satisfies (11) for a.e. \( t \in [0, \infty) \). Moreover, there exists a constant \( K \) which depends on \( \beta, \kappa, \| \psi \| \) and \( \| A^{1/2} \varphi \| \) so that
\[ \sup_{t \in [0, \infty)} \| \hat{x}(t) \| \leq K, \]
and
\[ \sup_{t \in [0, \infty)} \| A^{1/2}x(t) \| \leq K. \]

**Cosine Operators.**

**Definition 2.1.1.** A one-parameter family \( \{ \hat{C}(t) \mid -\infty < t < \infty \} \) of bounded linear operators mapping a Banach space \( X \) into itself is called a strongly continuous cosine family if and only if
1. \( \hat{C}(s+t) + \hat{C}(s-t) = 2\hat{C}(s)\hat{C}(t), \forall s, t \in (-\infty, \infty); \)
2. \( \hat{C}(0) = I; \)
3. \( \hat{C}(t)x \) is continuous in \( t \) for fixed \( x \in X \).
The infinitesimal generator of a strongly continuous cosine family \( \{C(t) \mid -\infty < T < \infty \} \) is the operator \( A : X \to X \) defined by the equation

\[
Ax = \frac{d^2}{dt^2}C(0)x, D(A) = \{ x \in X : C(t)x \in C^2 \}.
\]

In [27], it is shown that the infinitesimal generators may be equivalently characterized as

\[
Ax = \lim_{h \to 0} \frac{2(C(h)x) - x}{h^2}, \quad D(A) = \left\{ x \in X : \lim_{h \to 0} \frac{2(C(h)x) - x}{h^2} \text{ exists} \right\}.
\]

Associated with every strongly continuous cosine family \( \{C(t) \mid -\infty < t < \infty \} \),
we have a strongly continuous sine family \( \{S(t) \mid -\infty < t < \infty \} \), where \( S(t) \) is defined by the equation:

\[
S(t)x = \int_0^t C(s) \, ds.
\]

Webb and Travis [24] show that if \( F(\cdot) : [0, \infty) \to X \) is a continuously differentiable function, then there exists a unique function \( \omega : [0, \infty) \to X \) satisfying

\[
\omega(t) = C(t)x + S(t)y + \int_0^t S(t-s)F(s) \, ds
\]

Moreover, \( \omega(\cdot) \in C^2 \) and satisfies the abstract inhomogeneous equation

\[
\frac{d^2\omega(t)}{dt^2} = Aw(t) + F(t), \omega(0) = x, \omega'(0) = y.
\]

The operator \( A \) defined by (2.35) is self-adjoint. Therefore from [23], it is known that \( -A \) is the infinitesimal generator of a family of cosine operators. Then we can get the result below.

**Theorem 2.1.29.** Let \( X = L^2[0, 1] \) and suppose that \( A : X \to X \) is defined via (2.35).

The operator \( -A \) is the infinitesimal generator of a strongly continuous cosine family \( \{C(t) \mid -\infty < x < \infty \} \). Let \( \{S(t) \mid -\infty < t < \infty \} \) be the associated sine family. If \( ([\varphi, \psi] \in D(A) \times D(A^{1/2}) \) and \( F(\cdot) : [0, \infty) \to X \) is defined by (1.24), then the solution to (1.10) has variation of parameters representation

\[
x(t) = C(t)\varphi + S(t)\psi - \int_0^t S(t-s)f \, ds, \quad t \geq 0,
\]

\[
x(\cdot) : [0, \infty) \to X \in C^2.
\]
2.2 Beam Equation as a SPDE

2.2.1 Existence, Uniqueness and Stability

This subsection is a review of Brzezniak, B. Maslowski and J. Seidler’s work [9].

This paper treated the non-linear beam equation with noise fluctuations driven by Brownian motion in infinite dimension space (Hilbert space). They treated a wider class of abstract stochastic beam equation with a non-local term and non-linear damping

\[ u_{tt} + A^2 u + g(u, u_t) + m(\| B^{1/2} u \|^2)Bu = \sigma(u, u_t)\dot{W} \]

in Hilbert space \( H \), where the operators \( A \) and \( B \) are positive self-adjoint and \( W \) is an (infinite-dimensional) Wiener process (details will be discussed later).

The core part of their method is Lyapunov function (can be understood as energy). By their selected Lyapunov function, they can prove that there exist a non-explosion mild solution to that non-linear beam equation. After that, they prove the stability of that solution.

Their settings are like below.

Let \( H \) be a separable Hilbert space, the norm and inner product of which are denoted by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \), respectively. Suppose that

(A1) \( A : \text{Domain}(A) \to H \) and \( B : \text{Domain}(B) \to H \) are self-adjoint operators in \( H \), \( B > 0 \), \( A \geq \mu I \) for some \( \mu > 0 \), \( \text{Domain}(B) \supseteq \text{Domain}(A) \) and \( B \in \mathcal{L}(\text{Domain}(A), H) \), Domain \( (A) \) being endowed with the graph norm \( \| x \|_{\text{Domain}(A)} = \| Ax \| \).

(A2) \( W \) is a (possibly cylindrical) Wiener process in another real separable Hilbert space \( U \) with a covariance operator \( Q \), defined on a stochastic basis \( (\Omega, \mathcal{F}, (\mathcal{F}_t), P) \) such that \( \mathcal{F}_0 \) contains all \( P \)-null sets.

(A3) \( m \in C([0, \infty)) \) is a non-negative function, \( m \geq 0 \).

Let the space \( \text{Rng}Q^{1/2}[45] \) be endowed with its natural Hilbert space structure,
and let $\mathcal{L}_2$ denote the space of Hilbert-Schmidt operators. Set

$$\mathcal{H} = \text{Domain}(A) \times H, \quad \left\| \begin{array}{c} x \\ y \end{array} \right\|^2_H = \| Ax \|^2 + \| y \|^2,$$

$$\text{Domain}(U) = \text{Domain}(A^2) \times \text{Domain}(A), \quad \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix},$$

$$F: \mathcal{H} \rightarrow \mathcal{H}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -m(\| B^{1/2}x \|^2)Bx - g(x,y) \end{pmatrix}.$$ 

Assume that

(A4) $\sigma: \mathcal{H} \rightarrow \mathcal{L}_2(\text{Rng}Q^{1/2}, H)$ is Lipschitz continuous on bounded set in $\mathcal{H}$ and of a linear growth, that is

$$\exists L_\sigma < \infty, \forall \tau \in \mathcal{H}, \quad \| \sigma(\tau)Q^{1/2} \|_{HS} \leq L_\sigma(1 + \| \tau \|_H),$$

$$\| (\sigma(\tau) - \sigma(\eta))Q^{1/2} \|_{HS} \leq L_\sigma(N) \| \tau - \eta \|_H,$$

by $\| \|_{HS}$ we denote the norm of both $\mathcal{L}_2(U, H)$ and $\mathcal{L}_2(U, \mathcal{H})$. Obviously, this implies that the mapping

$$\Sigma: \mathcal{H} \rightarrow \mathcal{L}_2(\text{Rng}Q^{1/2}, \mathcal{H}), \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \sigma(x,y) \end{pmatrix}$$

is Lipschitz on bounded subsets of $\mathcal{H}$ and of a linear growth(with the same constants).

Further, suppose that $g$ satisfies the following growth estimate:

(A5) $g: \mathcal{H} \rightarrow H$ is Lipschitz on bounded subsets of $\mathcal{H}$ and there exists a $L_g \in (0, \infty)$ such that

$$\langle y, g(x,y) \rangle \geq -L_g(1 + \| \tau \|^2_H), \text{ for any } \tau = (x,y)^T \in \mathcal{H}.$$

Then set $u_0 = (u_0, u_1)^T$. After this settings, their equations are changed to

$$du = (\mathcal{U}u + F(u))dt + \Sigma(u)dW; \quad (2.39)$$

$$u(0) = u_0. \quad (2.40)$$

The components of the solution $u$ will be denoted by $u = (u, u_t)^T$.

**Main Result**
Theorem 2.2.1. Suppose that the hypotheses $(A1)-(A5)$ are satisfied and $u_0 : \Omega \to \mathcal{H}$ is $\mathcal{F}_0 -$ measurable. Let $u$ be the unique maximal local mild solution to (2.38)-(2.39) with lifespan $\tau$, then $\tau = +\infty$, $\mathbb{P} -$ almost surely. i.e. there exists a unique mild solution $u$ to (2.38)-(2.39) on $[0,\infty)$ and $u \in C([0,\infty);\mathcal{H})$, $\mathbb{P} -$ almost surely.

Set 
$$M(s) = \int_0^s m(r) \, dr, s \geq 0,$$
and define a mapping $\epsilon$ from the set of all random variable $\vartheta = (v, z)^T : \Omega \to \mathcal{H}$ into $[0,\infty)$ by
$$\epsilon(\vartheta) = E\|\vartheta\|^2_\vartheta + M(\|B^{1/2} v\|^2).$$

Corollary 1. Let the hypotheses $(A1)-(A5)$ be satisfied. If we set
$$C = 2(L_g + L^2_\sigma),$$
then
$$\epsilon(u(t)) \leq e^{Ct}(2 + \epsilon(u(0)))$$
holds for all $t \geq 0$ whenever $u$ is a solution (2.38)-(2.39) the initial datum $u_0$ of which satisfies $\epsilon(u_0) < \infty$.

Remark

Theorem 2.2.1 implies, in particular, that there is a unique global mild solution to (2.38) (with continuous trajectories) for every deterministic initial condition $u(0) = \tau \in \mathcal{H}$ and it follows from [26], Theorem 27, that (2.38) defines a Markov process $(u, \mathbb{P}_\tau)$ on $\mathcal{H}$.

Corollary 2. Assume $(A1)-(A5)$. Then the Markov process $(u, \mathbb{P}_\tau)$ associated with (2.38) is Feller, the function
$$\tau \mapsto \mathbb{E}_\tau \varphi(u(t))$$
being continuous on $\mathcal{H}$ for every bounded continuous function $\varphi : \mathcal{H} \to \mathbb{R}$ and for all $t \geq 0$.

After these results, they also attain some results on stability of solutions.

The problem (1.1) with the damping term $g$ of the form
$$g(\tau) = \beta y, \text{ for some } \beta \geq 0 \text{ and all } \tau = (x, y)^T \in \mathcal{H} \quad (2.41)$$
will be considered. Moreover, we enhance the linear growth hypothesis on \( g \to \exists \ R_\delta < \infty, \ \forall \tau \in \mathcal{H} \parallel \delta(\tau)Q^{1/2} \parallel_{HS} \leq R_\delta \parallel \tau \parallel_{\mathcal{H}}. \) (2.42)

With this, we can imply that (2.38) admits a trivial solution \( u \equiv 0. \)

**Theorem 2.2.2.** Suppose that (A1)-(A4),(2.41),(2.42) are satisfied, and \( R_\delta^2 < \beta, \) \( \alpha > 0, \) such that

\[ ym(y) \geq \alpha, M(y), \forall y \geq 0, \]

then the zero solution to (2.38) is exponentially mean-square stable and exponentially stable with probability one: there exist constants \( C < 0, \lambda > 0 \) such that if \( u \) is a solution to (2.38)-(2.39)satisfying \( \mathcal{E}(u_0) < \infty, \) then we have:

\[ \mathbb{E} \parallel u(t) \parallel_{\mathcal{H}}^2 \leq Ce^{-\lambda t}\varepsilon(u_0), \] (2.43)

for every \( \lambda^* \in (0, \lambda), \) a \( P \)-almost surely finite function \( t_0 : \Omega \to [0, \infty) \) may be found, such that

\[ \parallel u(t) \parallel_{\mathcal{H}}^2 \leq Ce^{-\lambda^* t}\varepsilon(u_0), \quad \forall t \leq t_0, P - a.e. \] (2.44)

**Theorem 2.2.3.** Under the hypotheses of theorem 2.2.2, the zero solution of (2.38) is stable in probability: for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for any solution \( u \) to (2.38)-(2.39) with \( \varepsilon(u_0) < \infty \)

\[ \mathbb{P} \{ \parallel u(0) \parallel_{\mathcal{H}} > \delta \} < \delta \rightarrow \mathbb{P} \left\{ \sup_{t \geq 0} \parallel u(t) \parallel_{\mathcal{H}} > \epsilon \right\} < \epsilon \]

from that abstract form to stochastic non-linear beam equation.
2.2.2 Invariant Measures for Stochastic Beam Equation

In this subsection, we introduce the work of Z. Brzeźniak, M. Ondrejat, and J. Seidler [10].

Their approach is based on the classical Krylov-Bogolyubov procedure. Let $X$ be a separable Hilbert space and $U = (U_t)$ a transition semigroup on $X$. There exists an invariant measure for $U$, provided the semigroup is Feller, that is, $U_t$ maps the space $C_b(X)$ of all bounded continuous functions on $X$ into itself for all $t \geq 0$, and the set of measures

$$\left\{ \frac{1}{T_n} \int_0^{T_n} U_s^* \nu \, ds, \; n \geq 1 \right\}$$

(2.45)

is tight on $X$ for some $T_n \nearrow \infty$ and a probability measure $\nu$ on $X$. Transition semigroups associated with stochastic partial differential equations maybe quite often easily shown to be Feller but tightness of the set (2.45) is a difficult problem for equations with solutions of low spatial regularity like beam and wave equations.

The situation completely changes if the space $X$ is endowed with its weak topology. The tightness of the set (2.45) follows from existence of a solution that is bounded in probability (in the mean) on $[\tau, \infty)$ for some $\tau > 0$, a property which can be verified by direct calculations with Lyapunov functions in many cases.

On the other hand, it is not obvious why $U_t f$ should be weakly continuous for a bounded weakly continuous function $f$ on $X$.

In fact, except for linear equations, only sequential weak continuity can be usually established.

They denote by $\mathcal{b} \omega$ the bounded weak topology on $X$, i.e. the finest topology that agrees with the weak topology on every closed ball.

Note that a real function on $X$ is $\mathcal{b} \omega - continuous$ if and only if it is sequentially weakly continuous and if and only if its restriction to any ball is weakly continuous.

Hence to carry out the Krylov-Bogolyubov procedure in $X$ with its weak topology, it is necessary to check that $U_t(C_b(X, \mathcal{b} \omega)) \subset C_b(X, \mathcal{b} \omega), \forall t \geq 0$. They call the transition semigroups with this property $\mathcal{b} \omega - Feller$.

It is straightforward to prove that a $\mathcal{b} \omega - Feller$ semigroup such that the set (2.45) is tight on $(X, \mathcal{b} \omega)$ has an invariant probability measure (see [40]), however, it is not that obvious to identify stochastic PDEs for which the associated transition semigroups are $\mathcal{b} \omega - Feller$. As far as they knew, the first to address this problem
was A.Ichikawa\[41\] who considered equations with coefficients depending only on finite
dimensional projections of solutions.

G.Leha and G.Ritter \[43,44\] studied thoroughly (yet in different terms) general
results concerning \(b\omega - \text{Feller}\) and related semigroups.

In the field of stochastic PDEs, however, they considered only a special stochastic
reaction-diffusion equation.

In \[40\], the \(b\omega - \text{Feller}\) property was shown for semigroups corresponding to
parabolic problems in bounded domains and to equations reducible to \(b\omega - \text{Feller}\)
one via the Girsanov transform, neither of these results applies to hyperbolic or beam
equation.

They establish \(b\omega - \text{Feller}\) property of transition semigroups corresponding to
stochastic non-linear beam and wave equation by a different way, whose main ingredi-
ents are \(b\omega - \text{continuity}\) of non-linear terms on \(X\) (if the target space is endowed with
a suitable weak-type topology, this follows from the fact the equations of the second
order in time are dealt with), uniform boundedness in probability on compact inter-
vals for solutions starting in a given ball, and results on convergence of sequences of
local martingales (invoked in a form that was used in \[42\] to construct weak solutions
of stochastic differential equations). Combining results on the \(b\omega - \text{Feller}\) property
with fairly standard estimates obtained in terms of Lyapunov functions they arrive at
theorems on existence of invariant measures.

**Main results**

They consider the stochastic extensible beam equation, which is

\[
\begin{align*}
    u_{tt} + A^2 u + g(u, u_t) + m(\|B^{1/2}u\|^2)Bu = G(u) \, dW. \\
\end{align*}
\]  

(2.46)

Assuming

(b1)

A and B are self-adjoint operators on a separable Hilbert space \(H\), \(W\) is a standard
cylindrical Wiener process on a real separable Hilbert space \(\mathcal{X}\), defined on a stochastic
basis \((\mathcal{O}, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})\);

(b2)

\(B > 0, A \geq \mu I\) for some \(\mu > 0\), \(\text{Domain}(A) \subset \text{Domain}(B)\) and
\(B \in \mathcal{L}(\text{Domain}(A), H)\);
(b3) \[ m : \mathbb{R}_+ \to R_+ \text{ is } C^1 \text{ and } \beta \geq 0; \]

(b4) 
\[ G : \text{Domain}(A) \to \mathcal{L}_2(\mathcal{X}, H) \] such that there exist constants \( L \) and \((L_n)\) such that, \( \forall x, y, z \in \text{Domain}(A), \)

\[ \| G(x) \|_{\mathcal{L}_2(\mathcal{X}, H)} \leq L(1 + \| x \|_{\text{Domain}(A)}), \]

\[ (2.47) \]

\[ \| G(y) - G(z) \|_{\mathcal{L}_2(\mathcal{X}, H)} \leq L_n \| y - z \|_{\text{Domain}(A)} \]

\[ (2.48) \]

holds for every \( \| y \|_{\text{Domain}(A)} \leq n, \| z \|_{\text{Domain}(A)} \leq n, \forall n \in \mathbb{N}. \)

Here \( \mathcal{L}_2 \) denotes the ideal of Hilbert-Schmidt operators, \( \text{Domain}(A) \) is equipped with the graph norm and (2.46) is interpreted in a standard way as a system of two first-order equations in the state space \( X = \text{Domain}(A) \times H. \)

It was proved in [9] that, under the hypotheses (b1)-(b4), there exists a pathwise unique mild solution to (2.46) for any deterministic initial condition in \( X \) and (2.46) defines a Feller Markov process on \( X \) with a transition semigroup \( U. \)

To show the \( b\omega - \) Feller property, they use two additional assumptions:

(b5) \( \text{Domain}(B) \) is compactly embedded into \( H; \)

(b6) \( G : (\text{Domain}(A), \| \cdot \|_{\text{Domain}(B_1/2)}) \to \mathcal{L}_2(\mathcal{X}, H) \) is continuous.

Then, another hypothesis

(b7) \( \beta > 0 \) and \( \| G \|_{\mathcal{L}_2(\mathcal{X}, H)} \) is bounded on \( \text{Domain}(A) \) or \( L^2 < \beta, \) where \( L \) is the constant introduced in (b4).

**Theorem 2.2.4.** Let the hypotheses (b1)-(b6) be satisfied. Then the Markov transition semigroup \( U \) defined by (2.46) is \( b\omega - \) Feller.

If in addition the hypothesis (b7) holds, then there exists an invariant probability measure for \( U. \)

They have an assumption of \( b\omega - \) continuity:

\( f_\gamma, g_\gamma, \| g_\gamma \|_{l_2} \) are sequentially weakly continuous on \( X \) for every \( \gamma, i \in \mathbb{N} \) (8.1).

**b\omega - FellerSemigroup**

(1) Let (5.1) have global solution \( (O^x, G^x, (G^x_i), \mathbb{P}^x, W^x, u^x) \) on \( \mathbb{R}_+ \) with \( u^x(0) = x \) for every \( x \in X. \)

(2) Let weak uniqueness hold for (5.1) in the class of solutions with initial law \( \delta_x, \) when \( x \in X. \)
(3) For $\forall \epsilon > 0, \forall \tau > 0, \forall r > 0, \exists R > 0$ such that $P_x[\sup_{t \in [0,\tau]} \| u^x(t) \|_X \geq R] \leq \epsilon$ holds for $\forall \| x \|_X \leq r$. Define the Markov operators for bounded Borel functions $\psi : X \to \mathbb{R}$

$$ (U_t \psi)(x) = \int_{O^x} \psi(u^x(t)) \, dP^x, \quad t \in \mathbb{R}_+, $$

(2.49)

and denote by $\nu^x$ the law of $(u^x|_{[0,T]}, W^x|_{[0,T]})$ on $\mathcal{F}^T$ and extend it to $\mathcal{B}_*(\Omega)$.

There are some settings before next theorem.

Define a separable Hilbert space $\chi$, an infinitesimal generator $A$ of a $C_0$-semi-group $(e^{At})_{t \geq 0}$ on $X$, Borel mappings $F : X \to X, G : X \to L_2(\chi, X)$, a stochastic basis $(O, \mathcal{G}, (\mathcal{G}_t), \nu)$, a standard cylindrical $(\mathcal{G}_t)$–Wiener process $W$ on $\chi$ and an equation

$$ du = (Au + F(u)) \, dt + G(u) \, dW, $$

(2.50)

and an assumption on boundedness condition, which is

$$ F : X \to X \quad \text{and} \quad G : X \to L_2(\chi, X) $$

(2.51)

are bounded on bounded sets of $X$.

**Theorem 2.2.5.** Let (1)-(3) above hold, let (2.51) and (8.1) be satisfied and let $t_n \to t$ in $\mathbb{R}_+, x_n \to x$ weakly in $X$ and $\psi : X \to \mathbb{R}$ is a bounded sequentially weakly continuous function. Then $U_{t_n} \psi(x_n) \to U_t \psi(x)$.

**Invariant Measure**

**Theorem 2.2.6.** Under the assumptions of Theorem 2.2.5, let there exists a global solution $(O, \mathcal{G}, (\mathcal{G}_t), \nu, u, W)$ of (2.50) such that, for every $\epsilon > 0$, there exists $R > 0$ and

$$ \limsup_{t \to \infty} \frac{1}{t} \int_0^t \nu[\| u(s) \|_X \geq R] \, ds \leq \epsilon, $$

then there exists an invariant measure for $(U_t)_{t \geq 0}$ defined in (2.49).

**Stochastic Beam Equation**

Consider the equation

$$ u_{tt} + A^2 u + \beta u_t + m(\| B^{1/2} u \|_H^2) Bu = G(u) \, dW, $$

(2.52)

with the hypotheses (b1)-(b4) set up in Section 1.1, and define $X = \text{Dom} A \times H$.

**Remark**
If \( C \) is a closed operator on \( H \), then we consider \( \text{Dom}C \) as a Hilbert space with 
\[
\| x \|_{\text{Dom}C}^2 = \| x \|_X^2 + \| Cx \|_X^2, \forall x \in \text{Dom}C.
\]

**Remark**

They define \( \text{Dom}\, A^2 = \text{Dom}A^2 \times \text{Dom}A \),

\[
A = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}, \quad F(u, v) = \begin{pmatrix} 0 \\ -m(\| B^{1/2}u \|_2^2)Bu - \beta v \end{pmatrix},
\]

\[
G(u, v)\xi = \begin{pmatrix} 0 \\ G(u)\xi \end{pmatrix},
\]

and rewrite (2.52) as a stochastic evolution equation

\[
d\phi = (A\phi + F(\phi))\, dt + G(\phi)\, dW
\]

in the Hilbert space \( X \). On the other hand, let \( (h_\gamma)_{\gamma \in N} \) be some dense subset in \( \text{Dom}A^2 \), \( (\xi_i) \) an orthonormal basis in \( \chi \) and define, for \( \gamma, i \in N \) and \( (u, v) \in X \),

\[
f_\gamma(u, v) = -\langle u, A^2h_\gamma \rangle_H - m(\| B^{1/2}u \|_H^2)Bu + \beta v, h_\gamma \rangle_H,
\]

\[
f_{-\gamma}(u, v) = \langle v, h_\gamma \rangle_H,
\]

\[
g_{\gamma,i}(u, v) = \langle G(u)\xi_i, h_\gamma \rangle_H, \quad g_{-\gamma}(u, v) = 0,
\]

\[
\varphi_{-\gamma}(u, v) = \langle u, h_\gamma \rangle_H, \quad \varphi_{\gamma}(u, v) = \langle v, h_\gamma \rangle_H.
\]

Now, according to Remark, the equation (11.2) is equivalent to

\[
d\varphi_{\gamma}(u(t)) = f_\gamma(u(t))\, dt + \sum_{i=1}^{\infty} g_{\gamma,i}(u(t))\, dW_i, \quad t \in I, \gamma \in N
\]

where \( W_i = W(\xi_i) \).

**Remark**

By [9], the equation (2.52) has a global solution for every \( (G_0) - \text{measurable} \) \( X - \text{valued} \) initial condition, path-wise uniqueness holds and every solution has \( X - \text{continuous} \) paths almost surely.

**Weak sequential continuity**

Consider the additional hypotheses (b5) and (b6) introduced in Section 1.1.

**Theorem 2.2.7.** Let (b1),(b2),(b5)and (b6)hold. Then \( f_\gamma, g_{\gamma,i} \) and \( \| g_\gamma \|_2^2 \) are sequentially weakly continuous for every \( \gamma \in \mathbb{Z} \setminus \{0\} \) and \( i \in N \).
Weak Tightness

Under (b1)-(b4), define \( M(r) = \int_0^r m(s) \, ds \) for \( r \geq 0 \),

\[
V(\omega) = \frac{1}{2} \| \omega \|_X^2 + M(\| B^{1/2} \omega_1 \|), \quad \omega = (\omega_1, \omega_2) \in X,
\]

\( q_k = \inf \{ V(\omega) : \| \omega \|_X \geq k \} \), let \( \phi^x \) be the unique global mild solution of (2.53) with \( \mathbb{P}[\phi^x(0) = x] = 1 \) and \( \tau^x_k = \inf \{ t \geq 0 : \| \phi^x(t) \|_X \} \) for \( x \in X \).

**Lemma 2.2.1.** Let \( R > 0 \) and \( \epsilon > 0 \). Then there exists a maximal compact \( K \) in \( C([0, T]; X_\omega) \) such that \( \mathbb{P}[\phi^x \in K] > 1 - \epsilon \) holds whenever \( \| x \|_X \leq R \).

Weak Sequential Feller Property

Under (b1)-(b6), \( \phi^x \) denotes the unique mild solution of (2.53) starting from \( x \in X \) and we define \( U_t F(x) = \mathbb{E} F(\phi^x(t)) \) for \( (t, x) \in \mathbb{R} \times X \).

**Theorem 2.2.8.** Let (b1)-(b6) hold. Then \((U_t)_{t \geq 0}\) is a semigroup on bounded Borel measurable functions on \( X \) and if \( t_n \to t \) in \( \mathbb{R}_+ \) and \( x_n \to x \) weakly in \( X \) and \( F \) is a bounded sequentially weakly continuous function on \( X \) then \( U_{t_n} F(x_n) \to U_t F(x) \).

**Theorem 2.2.9.** Let (b1)-(b7) hold. Then there exists an invariant measure for (2.52).
Chapter 3

Stochastic Beam Equation with Jumps

In this part, our concerns go to the problems of existence and uniqueness of solutions for one particular kind of stochastic beam equation. This part will consist of five sections. In Section 1, some definitions and theorems are presented that are required for later sections. We then state some fundamental settings of spaces and equations in Section 2, before using these to build an energy function based on Ito’s formula in Section 3. Finally, with the help of this energy functional, we establish the existence and uniqueness of solutions of stochastic beam equation driven by Poisson random measures in Section 4 and Section 5 respectively.

Our approach is similar to that in [5]. However, we have to use a different truncation of the non-linear operator involved and modify some of the arguments.
3.1 Preliminaries

We are firstly going to state some definitions and theorems which will be required for later sections. This part is divided into three subsections, with the first subsection introducing definitions from stochastic calculus as well as Itô's formula, the second focusing predominantly on the jump term and essential definitions of point processes, and the final subsection describing the equations and the famous fixed point theorem of Banach.

3.1.1 Stopping times, martingales, and related

Definition 1 A stopping time relative to the filtration \( (\mathcal{F}_t) \) is a map on \( \Omega \) with values in \([0, \infty] \), such that, for every \( t \),

\[
\{ T \leq t \} \in \mathcal{F}_t.
\]

Definition 2 A real-valued process \( X_t, t \in T \), adapted to \( (\mathcal{F}_t) \) is a submartingale (with respect to \( \mathcal{F}_t \)) if

i) \( E[X_t^+] < \infty, \forall t \in T \);

ii) \( E[X_t | \mathcal{F}_s] \geq X_s \), a.s. for every pair \( s, t \) such that \( s < t \).

A process \( X \) such that \( -X \) is a submartingale is called a supermartingale and a process which is both a sub- and a supermartingale is a martingale.

In other words, a martingale is an adapted family of integrable random variables \( \{X_t, t \geq 0\} \) such that:

\[
\int_A X_s \, dP = \int_A X_t \, dP
\]
for every pair \( s, t \) with \( s < t \) and \( A \in \mathcal{F}_s \).

Definition 3 A process \( X \) is progressively measurable (with the respect to the filtration \( (\mathcal{F}_t) \)) if for every \( t \) the mapping \((s, \omega) \to X_s(\omega)\) from \([0, t] \times \Omega \to (E, \mathcal{E})\) is \( \mathcal{B}([0, t]) \times \mathcal{F}_t - \) measurable.

Definition 4 A real function \( f(t, x, \omega) \) defined on \([0, \infty) \times X \times \Omega\) is called \( (\mathcal{F}_t) - \) predictable if the mapping \((t, x, \omega) \to f(t, x, \omega)\) is \( \mathcal{G}/\mathcal{B}(R) - \) measurable where \( \mathcal{G} \) is the smallest \( \sigma \) - field on \([0, \infty) \times X \times \Omega\) with respect to which all \( g \) having the following properties are measurable:

(i) for each \( t > 0, (x, \omega) \to g(t, x, \omega) \) is \( \mathcal{B} \times \mathcal{F}_t - \) measurable;

(ii) for each \( (x, \omega), t \to g(t, x, \omega) \) is left continuous.
3.1.2 Poisson point processes, semi-martingales and Ito’s formula

Let \((X, \mathcal{B}_X)\) be a measurable space. Let \(M\) be the totality of non-negative (possibly infinite) integral-valued measures on \((X, \mathcal{B}_X)\) and \(\mathcal{B}_M\) be the smallest \(\sigma\)-field on \(M\) with respect to which

\[
\mu \in M \mapsto \mu(B) \in \mathbb{Z}^+ \cup \{\infty\}, B \in \mathcal{B}_X
\]

are measurable.

**Definition 5** An \((M, \mathcal{B}_B)\)-valued random variable \(\mu\) (a mapping \(\mu : \Omega \to M\) defined on a probability space \((X, \mathcal{F}, \mathcal{B})\) which is \(\mathcal{F}/\mathcal{B}_M\)-measurable) is called a Poisson random measure if:

(i) For each \(B \in \mathcal{B}_X\), \(\mu(B)\) is Poisson distributed, i.e.

\[
P(\mu(B) = n) = \lambda(B)^n e^{-\lambda(B)} / n!, n = 0, 1, 2, \cdots,
\]

where \(\lambda(B) = E(\mu(B)), B \in \mathcal{B}_X\);

(ii) If \(B_1, B_2, \cdots, B_n \in \mathcal{B}_X\) are disjoint, then \(\mu(B_1), \mu(B_2), \cdots, \mu(B_n)\) are mutually independent.

**Definition 6** Let \((X, \mathcal{B}_X)\) be a measurable space. We call a mapping \(p\),

\[
p : D_p \subset (0, \infty) \to X,
\]

a point function. Where \(D_p\) is a countable subset of \((0, \infty)\).

\(p\) defines a counting measure \(N_p(\, dt \, dx)\) on \((0, \infty) \times X\) by

\[
N_p((0, t] \times U) = \# \{s \in D_p : s \leq t, p(s) \in U\}, t > 0, U \in \mathcal{B}_X.
\]

A point process is built by randomizing the definition of a point function. Let \(\Pi_X\) be the set of all point functions on \(X\) and \(\mathcal{B}(\Pi_X)\) be the smallest \(\sigma\)-field on \(\Pi_X\) with respect to which all

\[
p \to N_p((0, t] \times U), t > 0, U \in \mathcal{B}_X
\]

are measurable.
Definition 7 A point process $p$ on $X$ is a $(\Pi_X, \mathcal{B}(\Pi)_X)$-valued random variable, that is a mapping
\[ p : \Omega \rightarrow \Pi_X \]
defined on a probability space $(\Omega, \mathcal{F}, P)$ which is $\mathcal{F}/\mathcal{B}(\Pi)_X$-measurable.

A point process $p$ is called stationary if for every $t > 0$, $p$ and $\theta_tp$ have the same probability law, where $\theta_tp$ is defined by
\[ D_{\theta_t p} = \{ s \in (0, \infty); s + t \in D_p \}, \]
\[ (\theta_t p)(s) = p(t + s) \]
A point process $p$ is called Poisson if $N_p(dt \, dx)$ is a Poisson random measure on $(0, \infty) \times X$. A Poisson point process is stationary if and only if its intensity measure
\[ n_p(dt \, dx) = E[N_p dt \, dx) \]
is of the form
\[ n_p(dt \, dx) = dtn(dx) \]
for some $\sigma$-finite measure $n(dx)$ on $(X, \mathcal{B}_X)$. $n(dx)$ is called the characteristic measure of $p$.

From now on, we assume that $p$ is a stationary Poisson random measure with characteristic measure $n(dx)$. Let $\Gamma_p = \{ U \in \mathcal{B}(X); E[N_p(t, U)] < \infty, \forall t > 0 \}$. Then it is known (See [4]) that $t \rightarrow \hat{N}_p(t, U) = N_p(t, U) - \hat{N}(t, U)$ is a martingale for $U \in \Gamma_p$. The random measure $\{ \hat{N}_p(t, U) \}$ is the compensator of the point process $p$. With our assumptions, $\hat{N}_p(t, U) = E[N_p(t, U)] = dtn(dx)(cf.[4]).$

Introduce the following spaces.
\[ F_p = \{ f(t, x, \omega) ; f \text{ is } (\mathcal{F}_t) - \text{predictable and for each } t > 0, \int_0^t \int_X |f(s, x, \omega)| N_p(ds \, dx) < \infty \text{ a.s.} \} \]
\[ F^2_p = \{ f(t, x, \omega) ; f \text{ is } (\mathcal{F}_t) - \text{predictable and for each } t > 0, \]
\[ E[\int_0^t \int_X |f(s, x, \omega)|^2 \, dsn(dx)] < \infty \text{ a.s.} \} \]
\[ F^{2, loc}_p = \{ f(t, x, \omega) ; f \text{ is } (\mathcal{F}_t) - \text{predictable and there exists a sequence of } (\mathcal{F}_t) \text{- stopping times } \sigma_n \text{ such that } \sigma_n \uparrow \infty \text{ a.s. and } I_{[0, \sigma_n]}(t)f(t, x, \omega) \in F^2_p, \ n = 1, 2, 3, \ldots \} \]
Recall from [4] that for $f \in \mathbf{F}^2_{p,loc}$, the stochastic integral
\[ \int_0^t \int_X f(s, x, \omega) \tilde{N}_p(\,ds, \,dx) \]
can be defined as a local martingale such that
\[ \int_0^t \int_X I_{[0,\sigma_n]}(s)f(s, x, \omega) \tilde{N}_p(\,ds, \,dx), t \geq 0 \]
is a square integrable martingale for every $n \geq 1$.

Suppose the following are given:
(i) $M^i(t) \in \mathcal{M}^{c,loc}_2, (i = 1, 2, \cdots, d)$, where $\mathcal{M}^{c,loc}_2$ stands for the space of continuous, locally square integrable martingales;
(ii) $A^i(t) (i = 1, 2, \cdots, d)$: a continuous ($\mathcal{F}_t$)-adapted process whose almost sample functions are of bounded variation on each finite interval and $A^i(0) = 0$;
(iii) $f^i \in \mathbf{F}_p, g^j \in \mathbf{F}^2_{p,loc}$ and $f^i(t, x, \omega)g^j(t, x, \omega) = 0, i, j = 1, 2, \cdots, d$; furthermore, we assume that $g(t, x, \omega)$ is bounded, i.e. a constant $M > 0$ exists such that
\[ |g^i(t, x, \omega)| \leq M \quad \forall i, t, x, \omega \]
(iv) $X^i(0) (i = 1, 2, \cdots, d)$: an $\mathcal{F}_0$-measurable random variable.

Define a $d$-dimensional semi-martingale $X(t) = (X^1(t), X^2(t), \cdots, X^d(t))$ by:
\[ X^i(t) = X^i + M^i(t) + A^i(t) \]
\[ + \int_0^{t^+} \int_X f^i(s, x, \cdot) \tilde{N}_p(\,ds, \,dx) + \int_0^{t^+} \int_X g^i(s, x, \cdot) \tilde{N}_p(\,ds, \,dx), \]
\[ i = 1, 2, \cdots, d. \]

Denote also $f = (f^1, f^2, \cdots, f^d)$ and $g = (g^1, g^2, \cdots, g^d)$.

The following is the Ito formula proved in [4].

**Theorem 8** (Ito formula) Let $F$ be a function of class $C^2$ on $\mathbb{R}^d$ and $X(t)$ a $d$-dimensional semi-martingale as defined above. Then the stochastic process $F(X(t))$
is also a semi-martingale (with respect to \((\mathcal{F}_t)_{t \geq 0}\)) and the following formula holds:

\[
F(X(t)) - F(X(0)) = \sum_{i=1}^d \int_0^t F'_i(X(s)) \, dM^i(s) + \sum_{i=1}^d \int_0^t F'_i(X(s)) \, dA^i(s) \\
+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t F''_{ij}(X(s)) \, d\langle M^i, M^j \rangle(s) \\
+ \int_0^{t^+} \int_X \{F(X(s^-) + f(s, x, \cdot)) - F(X(s^-))\} \, N_p(ds \, dx) \\
+ \int_0^{t^+} \int_X \{F(X(s^-) + g(s, x, \cdot)) - F(X(s^-))\} \, \tilde{N}_p(ds \, dx) \\
+ \int_0^t \int_X \{F(X(s) + g(s, x, \cdot)) - F(X(s)) - \sum_{i=1}^d g^i(s, x, \cdot)F'_i(X(s))\} \, \hat{N}_p(ds \, dx).
\]

### 3.1.3 Sobolev spaces, Banach fixed point theorem

Let \( U \subset \mathbb{R}^d \) be an open subset.

**Definition 9** Suppose \( u, v \in L^1_{loc}(U) \) and \( \alpha \) is a multi-index. We say that \( v \) is the \( \alpha^{th} \) weak partial derivative of \( u \), written

\[
D^\alpha u = v,
\]

provided

\[
\int_U uD^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v\phi \, dx,
\]

for all test functions \( \phi \in C_\infty(U) \).

**Definition 10** \( W^{k,p}(U) \) consists of all locally integrable functions \( u : U \to \mathbb{R} \) such that for each multi-index \( \alpha \) with \(| \alpha | \leq k\), \( D^\alpha u \) exists in the weak sense and belongs to \( L^p(U) \).

**Definition 11** Let \( f : U \subseteq X \to Y \) be a given operator defined on an open neighbourhood of the point \( u \), where \( X \) and \( Y \) are Banach spaces over \( \mathbb{R} \).

(i)

The differential \( df(u) \) of \( f \) at the point \( u \) exists iff there is a linear bounded operator denoted by

\[
df(u) : X \to Y,
\]

such that

\[
f(u + h) - f(u) = df(u)h + o(\| h \|), \, h \to 0
\]
holds for all $h \in X$ in some open neighbourhood of $h = 0$ in $X$.

Synonymously, we also use $f'(u)$ instead of $df(u)$ and we call $f'(u)$ the $F$-derivative of $f$ at the point $u$.

(ii)
The second differential $d^2f(u)$ of $f$ at the point $u$ exists if and only if there is a bilinear bounded operator denoted by

$$d^2f(u) : X \times X \to Y,$$

such that

$$df(u + h)k - df(u)k = d^2f(u)hk + r$$

with

$$\sup_{\|k\| \leq 1} \| r(u; h, k) \| = o(\|h\|), h \to 0$$

hold for all $k \in X$ and all $h$ in some open neighbourhood of $h = 0$ in $X$.

We will use the fixed point theorem in our proof, which is stated below.

**Theorem 3.1.1.** Assume $A : X \to X$ is a non-linear mapping, and suppose that

$$\| A[u] - A[\tilde{u}] \| \leq \gamma \| u - \tilde{u} \|, (u, \tilde{u} \in X),$$

for some constant $\gamma < 1$, then $A$ has a unique fixed point.

Finally we recall the Holder inequality:

**Theorem 3.1.2.** Assume $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then if $u \in L^p(U)$, $v \in L^q$, we have

$$\int_U |uv| \, dx \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}.$$
3.2 Non-linear stochastic beam equation

The equations of interest take the form:

\[
\begin{align*}
\partial_t^2 u(t, x) \, dt + \gamma \partial_x^4 u \, dt & - [\alpha + \beta \int_0^l | \partial_y u(t, y) |^2 \, dy] \partial_x^2 u \\
= & F(t, x, \partial_t u, \partial_x u)
\end{align*}
\]

with associated boundary conditions

\[
\begin{align*}
u(t, 0) = u(t, l) = 0, u(0, x) = \phi_0(x), \\
\partial_x u(t, 0) = \partial_x u(t, l) = 0, \partial_t u(0, x) = \phi_1(x),
\end{align*}
\]

for \(0 \leq t \leq T, 0 \leq x \leq l\), where \(\partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial x}\), \(\alpha, \beta, \gamma\) are positive constants and \(\phi_0\) and \(\phi_1\) are given functions.

For convenience, we write \(\dot{u} = \partial_t u\).

The right-hand side takes the form

\[
\begin{align*}
F(t, x, \partial_t u, \partial_x u) = f(t, x, \dot{u}, \partial_x u) \\
+ & \sigma(t, x, \dot{u}, \partial_x u) \omega(t) + \int_X g(t, x, \dot{u}, \partial_x u, q) \tilde{N}_p(\, dt \, dq),
\end{align*}
\]

where \(f\) represents the mean force, \(\{\sigma(t, x, \cdot, \cdot)\}, \{g(t, x, \cdot, \cdot, \cdot)\}\) are given mappings. \(\{\omega(t)\}\) is a one dimensional Brownian motion.

Then the equation can be written as

\[
\begin{align*}
\partial_t^2 u(t, x) \, dt + \gamma \partial_x^4 u \, dt & - [\alpha + \beta \int_0^l | \partial_y u(t, y) |^2 \, dy] \partial_x^2 u \\
= & \sigma(t, x, \dot{u}, \partial_x u) \, d\omega_t + \int_X g(t, x, \dot{u}, \partial_x u) \, \tilde{N}_p(\, dt \, dq)
\end{align*}
\]

\[
\begin{align*}
u(t, 0) = u(t, l) = 0, u(0, x) = \phi_0(x), \\
\partial_x u(t, 0) = \partial_x u(t, l) = 0, \partial_t u(0, x) = \phi_1(x).
\end{align*}
\]

Let us now introduce the Hilbert spaces \(L^2 = L^2(0, l), H_0^1 = H_0^1(0, l)\) and \(H_0^2 = H_0^2(0, l)\), where \(H_0^1\) and \(H_0^2\) stand for Sobolev spaces of order one and two, respectively, satisfying the homogeneous boundary condition.

Denote by \(H^{-k}\) the dual space of \(H_0^k\) with \(k = 1, 2\). Let \(\langle \cdot, \cdot \rangle\) denote the \(L^2\) – inner product. The norm \(\| \cdot \|\) is on \(L^2\). Then it can be seen that the following inclusion relation holds: \(H_0^2 \subset H_0^1 \subset L^2 \subset H^{-1} \subset H^{-2}\).
For convenience, we define the two operators:

\[ Au = \alpha \partial^2_x u - \gamma \partial^4_x u, \]
\[ B(u) = \beta \left( \int_0^1 |\partial_x u|^2 \, dx \right) \partial^2_x u, \]

where \( A : H_0^2 \to H^{-2} \) is linear and continuous, and \( B : H_0^1 \to H^{-1} \) is non-linear and locally Lipschitz continuous.
3.3 Existence and uniqueness: the case of additive noise

In this case, the basic idea is to use the energy function. After building this function, we use the fixed point theorem to show existence of solutions, including the techniques of truncation and stopping time. Firstly, we will show how we build the energy function from Ito’s formula, then use it to show the existence of a solution for the beam equation. The process of truncation is difficult and needs to be treated with care.

3.3.1 Energy equation

Consider the following stochastic beam equation:

\[ \begin{align*}
    d\dot{u}(t, x) &= -\gamma \partial_x^4 u(t, x) dt + \left[ \alpha + \beta \int_0^t |\partial_y u(t, y)|^2 \, dy \right] \partial_x^2 u(t, x) dt + f(t, x) dt \\
    &\quad + \sigma(t, x)d\omega_t + \int_X g(t, x, q) \tilde{N}_p(\, dt \, dq). 
\end{align*} \]  

(3.5)

Define our energy process as

\[ e(t, u) = \frac{1}{2} \left[ \|\dot{u}(t)\|^2 + (\alpha + \beta/2\|\partial_x u\|^2)\|\partial_x u\|^2 + \gamma \|\partial_x^2 u\|^2 \right], \]  

(3.6)

then we have the following result.

**Theorem 11** It holds that

\[ e(t, u) \]

\[ = e(0, u) + \int_0^t \left[ (f_s, \dot{u}_s) + \frac{1}{2} \|\sigma_s\|^2 \right] \, ds + \int_0^t (\dot{u}_s, \sigma_s) \, dw_s \\
+ \frac{1}{2} \int_0^{t+} \int_X \left[ 2(g(s, q, \omega), \dot{u}(s^-)) + \|g(s, q, \omega)\|^2 \right] \tilde{N}_p(\, ds \, dq) \\
+ \frac{1}{2} \int_0^t \int_X \|g(s, q, \omega)\|^2 \, ds \, dn(q). \]  

(3.7)
Proof. Applying the Ito’s formula, we get

\[
\|\hat{u}(t, \cdot)\|^2 - \|\hat{u}(0)\|^2 \\
= 2 \int_0^t (\hat{u}, \sigma_s) \, d\omega_s + 2 \int_0^t (\hat{u}, Au + Bu + f_s) \, ds + \int_0^t \|\sigma_s\|^2 \, ds \\
+ \int_0^{t^+} \int_X \|\hat{u}(s-) + g(s, q, \omega)\|^2 - \|\hat{u}(s-)\|^2 \tilde{N}_p(\, ds \, dq) \\
+ \int_0^t \int_X \{\|\hat{u}(s) + g(s, q, \omega)\|^2 - \|\hat{u}(s)\|^2\} \, ds \, dn(q) \\
- 2(\hat{u}(s), g(s, q, \omega)) \, ds \, dn(q).
\]

By integration by parts, we have

\[
\int_0^t (\hat{u}(s), Au + Bu) \, ds \\
= \int_0^t \partial_s(u(s), Au(s) + Bu(s)) \, ds \\
- \int_0^t (u(s), \partial_s[Au(s) + B(u(s))]) \, ds \\
= (u(t), Au(t) + B(u(t))) - (u(0), Au(0) + Bu(0)) \\
- \int_0^t (u(s), \partial_s[Au(s) + Bu(s)]) \, ds \\
= (u(t), Au(t)) + (u(t), Bu(t)) - (u(0), Au(0) + Bu(0)) \\
- \int_0^t (u(s), \partial_s[Au(s) + B(s)u(s)]) \, ds \\
= -\alpha\|\partial_x u(t)\|^2 - \gamma\|\partial_x^2 u(t)\|^2 - \beta\|\partial_x u(t)\|^4 \\
+ \alpha\|\partial_x u(0)\|^2 + \gamma\|\partial_x^2 u(0)\|^2 + \beta\|\partial_x u(0)\|^4 \\
+ \frac{1}{2}\alpha\int_0^t \frac{\partial}{\partial s}(\|\partial_x u(s)\|^2) \, ds \\
+ \frac{1}{2}\gamma\int_0^t \frac{\partial}{\partial s}(\|\partial_x^2 u(s)\|^2) \, ds \\
+ \frac{3}{4}\beta\int_0^t \frac{\partial}{\partial s}(\|\partial_x u(s)\|^4) \, ds \\
= -\frac{1}{2}\alpha\|\partial_x u(t)\|^2 - \frac{1}{2}\gamma\|\partial_x^2 u(t)\|^2 - \frac{1}{4}\beta\|\partial_x u(t)\|^4 \\
+ \frac{1}{2}\alpha\|\partial_x u(0)\|^2 + \frac{1}{2}\gamma\|\partial_x^2 u(0)\|^2 + \frac{1}{4}\beta\|\partial_x u(0)\|^4.
\]

As

\[
e(t, u) = 1/2[\|\hat{u}(t)\|^2 + (\alpha + \beta/2\|\partial_x u\|^2)\|\partial_x u\|^2 + \gamma\|\partial_x^2 u\|^2],
\]

(3.9)
substitute the above expression back into (3.8), we can get

\[ 2 \int_0^t (\dot{u}, \sigma_s) \, d\omega_s - \alpha \| \partial_x u(t) \|^2 - \gamma \| \partial_x^2 u(t) \|^2 - \frac{1}{2} \beta \| \partial_x u(t) \|^4 \\
+ \alpha \| \partial_x u(0) \|^2 + \gamma \| \partial_x^2 u(0) \|^2 + \frac{1}{2} \beta \| \partial_x u(0) \|^4 \\
+ \int_0^t \| \sigma_s \|^2 \, ds \]

(3.10)

as

\[ \| \dot{u}(s-) + g(s, q, \omega) \|^2 - \| \dot{u}(s-) \|^2 = g(s, q, \omega)^2 + 2(g(s, q, \omega), \dot{u}(s-)) \]

\[ \| \dot{u}(s) + g(s, q, \omega) \|^2 - \| \dot{u}(s) \|^2 = g(s, q, \omega)^2 + 2(g(s, q, \omega), \dot{u}(s)), \]

then

\[ e(t, u) = e(0, u) + \int_0^t [(f_s, \dot{u}_s) \, ds + \frac{1}{2} \| \sigma_s \|^2] \, ds + \int_0^t (\dot{u}_s, \sigma_s) \, d\omega_s \]

(3.12)

\[ + \frac{1}{2} \int_0^{t+} \int_{\mathcal{X}} [2(g(s, q, \omega), \dot{u}(s-)) + \| g(s, q, \omega) \|^2] \tilde{N_p} \, (ds \, dq) \]

(3.13)

\[ + \frac{1}{2} \int_0^t \int_{\mathcal{X}} \| g(s, q, \omega) \|^2 \, ds \, dn(q). \]

(3.14)

Set

\[ e_L(t, u) = \frac{1}{2} \| \dot{u}(t) \|^2 + \alpha \| \partial_x u(t) \|^2 + \gamma \| \partial_x^2 u(t) \|^2 . \]

To establish the existence and uniqueness, we need to introduce the following space.

\( v \) is progressively measurable such that and \( v \in Y \)

\[ Y = \{ v \mid v \in L^2(\Omega; L^\infty((0, T); H^2_0)), \dot{v} \in L^2(\Omega; L^\infty((0, T); L^2)) \} , \]

equipped with the norm

\[ \| v \|_Y = \left\{ E \sup_{0 \leq t \leq T} [e_L(t, v)e^{-\eta^2 t}] \right\}^{1/2} . \]
3.3.2 Main result

For $r > 0$, choose $Φ_r(·) ∈ C_0^∞(R)$ such that

\[
\begin{cases}
   Φ_r(z) = 1, & \text{if } \|z\| ≤ r^2, \\
   Φ_r(z) = 0, & \text{if } \|z\| ≥ r^2 + 1.
\end{cases}
\]

Define $B_r(φ) = β∂_2 x φ(∥∂_x φ∥∧r)^2 Φ_r(∥∂_2 x φ∥^2)$. We have the following result.

**Lemma 12.** There exists a constant $C_r$ such that

\[
\|B_r(φ) - B_r(ψ)\| \leq C_r[\|∂_x φ - ∂_x ψ\| + \|∂_2 x φ - ∂_2 x ψ\|].
\]  

(3.15)

**Proof.** We have

\[
\|B_r(φ) - B_r(ψ)\| \\
\leq β[(∥∂_x φ∥ ∧ r)^2 - (∥∂_x ψ∥ ∧ r)^2] ∥∂_x φ∥ Φ_r(∥∂_2 x φ∥^2) \\
+ β[∥∂_x ψ∥ ∧ r)^2] ∥∂_x φ∥ Φ_r(∥∂_2 x φ∥^2) - ∂_2 x ψ φ_r(∥∂_2 x ψ∥^2)] \\
\leq C_{1,r}β∥∂_x φ∥ - ∥∂_x ψ∥ \\
+ C_{2,r}∥∂_2 x φ - ∂_2 x ψ\| \\
\leq C_r \left[∥∂_x φ - ∂_x ψ\| + ∥∂_2 x φ - ∂_2 x ψ\|\right],
\]

(3.16)

where we have used the fact that

$F_r(·) : h → hΦ_r(∥h∥^2)$

is a Lipschitz mapping from $H$ into $H$ because

\[
\|DF_r(h)\|_{L_∞(H)} \leq |Φ_r(∥h∥^2)| + 2∥h∥^2 |Φ'_r(∥h∥^2)| ≤ L_r,
\]

for some constant $L_r$, independent of $h$. Here $L_∞(H)$ stands for the space of bounded linear operators from $H$ into $H$.

**Theorem 13.** Assume for any $T > 0$,

\[
E[∫_0^T \|σ\|^2(s)ds] < ∞, \quad E \left\{ ∫_0^T ∫_X \|g(s, ·, q, \cdot, \cdot)\|^2 dμ(q) ds \right\} < ∞,
\]

(3.17)
then there exists a solution $u$ of the non-linear stochastic equations

$$\partial_t^2 u(t, x) \, dt + \gamma \partial_x^4 u \, dt - [\alpha + \beta \int_0^t | \partial_y u(t, y) |^2 \, dy] \partial_x^2 u \, dt - f(t, x) \, dt$$

$$= \sigma_t(x) \, d\omega_t + \int_X g(t, x, q, \omega) \tilde{N}_p(\, dt \, dq),$$

$$u(t, 0) = u(t, l) = 0, u(0, x) = \phi_0(x),$$

$$\partial_x u(t, 0) = \partial_x u(t, l) = 0, \partial_t u(0, x) = \phi_1(x).$$

**Proof**

For a fixed $r > 0$ and for a progressively measurable process $u_t$ taking values in $H_0^2$, we define a new process $v_t = R_r(u_t)$ as one solution of the linear equation:

$$\dot{v}_t = [Av_t + B_r(u_t) + f_t] \, dt + \sigma_s \, d\omega_s + \int_X g(t, x, q, \omega) \tilde{N}_p(\, dt \, dq),$$

with the data $\dot{v}_0, v_0$ and $f$ satisfying

$$\dot{v}_0 = \phi, v_0 = \phi_0,$$

$$\phi \in H_0^2, \phi_0 = L^2, f \in L^2((0, T) \times (0, l)).$$

Set $\tilde{v}(t) = \dot{v}_t$, then the above equation can be written as a system of stochastic evolution equation:

$$\begin{align*}
\dot{v}(t) &= \tilde{v}(t) \, dt, \\
\dot{\tilde{v}}(t) &= [Av_t + B_r(u_t) + f_t] \, dt + \sigma_s \, d\omega_s + \int_X g(t, x, q, \omega) \tilde{N}_p(\, dt \, dq). \quad (3.18)
\end{align*}$$

The existence and uniqueness of the solution of such system follows from [6].

Let $v^i = R_r(v^i), i = 1, 2$, and $\tilde{v} = v^1 - v^2$. We have the following formula:

$$[e_L(t, \tilde{v})] = \int_0^t (\tilde{v}(s), B_r(u_1(s)) - B_r(u_2(s))) \, ds. \quad (3.19)$$

Indeed, subtracting the two equations satisfied by $v^i$, we have

$$\dot{\tilde{v}}(t) = A\tilde{v}(t) \, dt + [B_r(u_1(t)) - B_r(u_2(t))] \, dt,$$
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This implies that

\[ \dot{\varphi}(s, \dot{v}) \]

\[ = \partial_s [\text{1/2} \| \dot{\varphi}(s) \|^2 + \alpha/2 \| \partial_x \dot{v}(s) \|^2 + \gamma/2 \| \partial_x^2 \dot{v}(s) \|^2] \]

\[ = \left\{ \begin{array}{l}
\dot{\varphi}, \dot{\varphi} + \alpha(\partial_x \dot{v}, \partial_x \dot{v}) + \gamma(\partial_x^2 \dot{v}, \partial_x^2 \dot{v}) \end{array} \right\} ds \]

\[ = \left\{ \begin{array}{l}
(\dot{\varphi} + \alpha(\partial_x \dot{v}, \partial_x \dot{v}) + \gamma(\partial_x^2 \dot{v}, \partial_x^2 \dot{v})) \end{array} \right\} ds \]

\[ = \left\{ \begin{array}{l}
(\dot{\varphi} + \alpha(\partial_x \dot{v}, \partial_x \dot{v}) + \gamma(\partial_x^2 \dot{v}, \partial_x^2 \dot{v})) \end{array} \right\} ds \]

where we used the fact that \( (\dot{\varphi}, \dot{\varphi}) = -\alpha/2 \| \partial_x \dot{v}(s) \|^2 - \gamma/2 \| \partial_x^2 \dot{v}(s) \|^2 \)

Applying the chain rule, we obtain

\[ e_L(t, \dot{v})e^{-\eta^2 t} = \int_0^t e^{-\eta^2 s} d e_L(s, \dot{v}) - \eta^2 \int_0^t e_L(s, \dot{v})e^{-\eta^2 s} ds \]

\[ = \int_0^t e^{-\eta^2 s}(\dot{\varphi}, B_r(u_1) - B_r(u_2)) ds - \eta^2 \int_0^t e_L(s, \dot{v})e^{-\eta^2 s} ds \]

\[ \leq \frac{1}{2\eta^2} \int_0^t e^{-\eta^2 s}\|B_r(u_1(s)) - B_r(u_2(s))\|^2 ds + \frac{1}{2\eta^2} \int_0^t e^{-\eta^2 s}\|\dot{\varphi}\|^2 ds \]

\[ - \eta^2 \int_0^t e_L(s, \dot{v})e^{-\eta^2 s} ds \]

\[ \leq \frac{1}{2\eta^2} \int_0^t e^{-\eta^2 s}\|B_r(u_1(s)) - B_r(u_2(s))\|^2 ds. \] (3.20)

This implies that

\[ \sup_{0 \leq t \leq T} [e_L(t, \dot{v})e^{-\eta^2 t}] \]

\[ \leq \frac{1}{2\eta^2} \int_0^T e^{-\eta^2 s}\|B_r(u_1(s)) - B_r(u_2(s))\|^2 ds. \] (3.21)

In view of (3.15), we have

\[ \|B_r(u_1(s)) - B_r(u_2(s))\|^2 \]

\[ \leq \|\partial_x u_1(s) - \partial_x u_2(s)\|^2 + \|\partial_x^2 u_1(s) - \partial_x^2 u_2(s)\|^2 \]

\[ \leq \tilde{C}_r e_L(s, u_1 - u_2). \] (3.22)
It follows from (3.21), (3.22) that
\[
\| R_r(u^1) - R_r(u^2) \|^2_V = E \sup_{0 \leq t \leq T} [e_L(t, \tilde{v}) e^{-\eta t}] \\
\leq \tilde{C}_r T E \sup_{0 \leq s \leq T} [e_L(s, u_1 - u_2) e^{-\eta s}] \\
= \frac{\tilde{C}_r T}{2\eta^2} \| u^1 - u^2 \|^2_V . \tag{3.23}
\]
Namely,
\[
\| R_r(u_1) - R_r(u_2) \|_V \leq \sqrt{\frac{\tilde{C}_r T}{2\eta^2}} \| u_1 - u_2 \|_V, u_1, u_2 \in V. \tag{3.24}
\]
Choose \( \eta \) sufficiently large so that \( \sqrt{\frac{\tilde{C}_r T}{2\eta^2}} < 1 \) and hence the mapping \( R_r : V \rightarrow V \) is a contraction from the Banach space \( V \) into itself. By the fixed point theorem, there is a point \( u^* \in V \) such that
\[ u^* = R_r(u^*), \]
or \( u^* \) is the solution of the following equation:
\[
d\dot{u}^*_t = [Au_t^* + B_r(u_t^*) + f_t] dt + \sigma_t \, d\omega_t + \int_X g(t, \cdot, q) \, d\tilde{N}_p(\, dt \, dq),
\]
and moreover
\[ u^* \in D([0, T]; H^2_0), \quad \dot{u}^* \in D([0, T], L^2). \]
For \( r > 0 \), introduce the stopping time
\[ \tau_r = \inf \{ t > 0 : \| \dot{u}^* \| \geq r, \text{ or } \| \partial_x u^* (t) \| \geq r, \text{ or } \| \partial_x^2 u^* (t) \| \geq r \}, \]
by the uniqueness, it follows that if \( r_2 \geq r_2 \), then \( \tau_{r_1} \leq \tau_{r_2} \) and \( u^{r_1}(t) = u^{r_2}(t) \) for \( t \in [0, \tau_{r_1}) \). Define
\[ u(t, x) = u^*(t, x), \quad \text{for } t \in [0, \tau_r), \]
then \( u(t), t \geq 0 \) is a solution to the stochastic beam equation on the interval \( [0, \tau) \), where \( \tau = \lim_{r \to \infty} \tau_r \).

Now it remains to show \( \tau = \infty \) a.s.. We will show that for any \( T > 0 \), \( P(\tau \geq T) = 1 \).

To prove this, we use the energy equation to deduce that
\[ e(t \wedge \tau_r, u) \]
\[ \leq e(0, u) + \int_{t \wedge \tau_r}^{t} \left[ 1/2 \| f_s \|^2 + 1/2 \| \dot{u}_{s \wedge \tau_r} \|^2 + 1/2 \| \sigma_s \|^2 \right] ds \]
\[ + \int_{0}^{t \wedge \tau_r} (\dot{u}_s, \sigma_s) d\omega_s \]
\[ + 1/2 \int_{0}^{t \wedge \tau_r} \int_{X} \left[ 2(g(s, q, \omega), \dot{u}(s-)) + \| g(s, q, \omega) \|^2 \right] \tilde{N}_p(\omega) ds d\omega \]
\[ + 1/2 \int_{0}^{t \wedge \tau_r} \int_{X} \| g(s, q, \omega) \|^2 ds dn(q). \]

Taking the expectation on both sides we obtain for \( t \leq T \),
\[
E[e(t \wedge \tau_r, u)] \leq e(0, u) + E\left[ \int_{0}^{t \wedge \tau_r} 1/2 \| f_s \|^2 ds \right] + \left[ e(s \wedge \tau_r, u) \right] ds + E[1/2 \int_{0}^{t \wedge \tau_r} \| \sigma_s \|^2 ds]
\]
\[ + 1/2 E\left[ \int_{0}^{t \wedge \tau_r} \int_{X} \| g(s, \cdot, q, \omega) \|^2 ds dn(q) \right] \]
\[ \leq e(0, u) + E\left[ \int_{0}^{T} 1/2 \| f_s \|^2 ds \right] + E\left[ \int_{0}^{t \wedge \tau_r} e(s \wedge \tau_r, u) \right] ds + E[1/2 \int_{0}^{T} \| \sigma_s \|^2 ds]
\]
\[ + 1/2 E\left[ \int_{0}^{T} \int_{X} \| g(s, \cdot, q, \omega) \|^2 ds dn(q) \right], \]

applying the Gronwall inequality, we obtain that
\[
E[e(T \wedge \tau_r, u)] \leq \left\{ e(0, u) + E\int_{0}^{T} 1/2 \| f_s \|^2 ds \right\} + E[1/2 \int_{0}^{T} \| \sigma_s \|^2 ds]
\]
\[ + 1/2 E\left[ \int_{0}^{T} \int_{X} \| g(s, \cdot, q, \omega) \|^2 ds dn(q) \right] \}
e^{CT}
\[ \leq C_T e^{CT}. \] (3.25)

From the definitions of the energy process \( e(t, u) \) and the stopping time \( \tau_r \), we have
\[
e(\tau_r, u) \geq (1 \wedge (\alpha + \beta) \wedge \gamma)^{r_2/2}. \] (3.26)

thus it follows from (3.25), (3.26) that
\[
(1 \wedge (\alpha + \beta) \wedge \gamma)^{r_2/2} P(\tau_r \leq T) \leq E[e(T \wedge \tau_r, u), \tau_r \leq T]
\]
\[ \leq E[e(T \wedge \tau_r, u)] \leq C_T e^{CT}. \] (3.27)
Letting $r \to \infty$ we obtain that

$$P(\tau \leq T) = \lim_{r \to \infty} P(\tau_r \leq T)$$

$$\leq \lim_{r \to \infty} \frac{1}{(1 \wedge (\alpha + \beta) \wedge \gamma)^{\frac{r^2}{2}}} C_T e^{CT} = 0. \quad (3.28)$$

Since $T > 0$ is arbitrary, we conclude that $\tau = \infty$ a.s. This proves the existence.

**Uniqueness**

*Proof.* Let $u_1$ and $u_2$ be two solutions. For $r > 0$, define the stopping times.

$$\tau^1_r = \inf \{ t > 0 : \| \dot{u}_1 \| \geq r, \text{or} \, \| \partial_x u_1(t) \| \geq r, \text{or} \, \| \partial^2_x u_1(t) \| \geq r \},$$

$$\tau^2_r = \inf \{ t > 0 : \| \dot{u}_2 \| \geq r, \text{or} \, \| \partial_x u_2(t) \| \geq r, \text{or} \, \| \partial^2_x u_2(t) \| \geq r \}.$$  

Set $\tau_r = \tau^1_r \wedge \tau^2_r$. Then it holds that

$$u_1(t) = R_r(u_1)(t), \quad u_2(t) = R_r(u_2)(t), \quad \text{for} \quad t \in [0, \tau_r).$$

Since $R_r(\cdot)$ is a contraction mapping, it follows that $u_1(t) = u_2(t)$ for $t \in [0, \tau_r)$. Let $r \to \infty$ to conclude the uniqueness. \qed
3.4 Existence and uniqueness: general case

In this section, we consider the following general stochastic beam equation:

\begin{align*}
\frac{d}{dt}u(t,x) &= -\gamma \partial_x^4 u + \alpha \int_0^t |\partial_y u(t,y)|^2 \, dy \partial_x^2 u \, dt \\
&+ f(u(t), \partial_x u(t)) \, dt + \sigma(u(t), \partial_x u(t)) \, d\omega_t \\
&+ \int_X g(u(t), \partial_x u(t), q) \tilde{N}_p(\, dt \, dq).
\end{align*}

Introduce the following conditions:

**H.1** \(f(\cdot, \cdot), \sigma(\cdot, \cdot) : H \times H \rightarrow H\) are measurable and satisfy

\begin{align*}
\{\|f(\phi_1, \psi_1) - f(\phi_2, \psi_2)\| + \|\sigma(\phi_1, \psi_1) - \sigma(\phi_2, \psi_2)\|\} \\
\leq C_1(\|\phi_1 - \phi_2\| + \|\psi_1 - \psi_2\|), \quad \phi_1, \psi_1, \phi_2, \psi_2 \in H = L^2(0, l),
\end{align*}

where \(C_1\) is some constant.

**H.2** \(g(\cdot, \cdot, \cdot) : H \times H \times X \rightarrow H\) is a measurable mapping that satisfies

\begin{align*}
\int_X \|g(\phi_1, \psi_1, q) - g(\phi_2, \psi_2, q)\|^2 \, n(\, dq) \\
\leq C_2(\|\phi_1 - \phi_2\|^2 + \|\psi_1 - \psi_2\|^2), \quad \phi_1, \psi_1, \phi_2, \psi_2 \in H,
\end{align*}

for some constant \(C_2\).

**Theorem 14.** Suppose the assumptions (H.1), (H.2) hold, then there exists a unique solution \(u\) to the stochastic beam equation (3.4) that satisfies

\begin{align*}
u &\in L^2(\Omega; D([0, T); H^2_0)), \\
\dot{u} &\in L^2(\Omega; D([0, T); H)).
\end{align*}

**Proof.** Introduce the following space:

\begin{align*}
Z &= \{v; \; v \quad \text{is progressively measurable such that} \quad v \in L^\infty([0, T]; L^2(\Omega; H^2_0)), \\
\dot{v} &\in L^\infty([0, T]; L^2(\Omega; H^2_0))\},
\end{align*}

(3.29)

equipped with the norm

\[\|v\|_Z = \left\{\sup_{0 \leq t \leq T} e^{-\eta^2 t} E[e_L(t, v)]\right\}^{1/2},\]
for every $r > 0$. Let $B_r(\cdot)$ be the truncation mapping of $B(\cdot)$ defined in the proof of Theorem 13. We first show that there exists a unique solution to the stochastic evolution equation:

\begin{align}
\dot{u}_r(t,x) &= Au_r(t) dt + B_r(u_r(t)) dt \\
&+ f(\dot{u}_r(t), \partial_x u_r(t)) dt + \sigma(\dot{u}_r(t), \partial_x u_r(t)) d\omega_t \\
&+ \int_X g(\dot{u}_r(t), \partial_x u_r(t), q) \tilde{N}_p(\dt dq).
\end{align}

(3.30)

We will use a fixed point argument. Let $u \in Z$, and consider the linear stochastic equation:

\begin{align}
\dot{v}(t,x) &= Av(t) dt + B_r(u(t)) dt \\
&+ f(\dot{v}(t), \partial_x u(t)) dt + \sigma(\dot{v}(t), \partial_x u(t)) d\omega_t \\
&+ \int_X g(\dot{v}(t), \partial_x u(t), q) \tilde{N}_p(\dt dq).
\end{align}

(3.31)

The unique solution $v$ gives a mapping from space $Z$ into $Z$, which is denoted by $v = S_r(u)$. The fixed point of the mapping $S_r(\cdot)$ will be a solution to equation (3.30). Let $u^1, u^2 \in Z$ and $v^1 = S_r(u^1), v^2 = S_r(u^2)$. Putting $\tilde{v} = v^1 - v^2$ we have

\begin{align}
\dot{\tilde{v}}(t,x) &= Av(t) dt + (B_r(u^1(t)) - B_r(u^2(t))) dt \\
&+ [f(\dot{v}^1(t), \partial_x u^1(t)) - f(\dot{v}^2(t), \partial_x u^2(t))] dt \\
&+ [\sigma(\dot{v}^1(t), \partial_x u^1(t)) - \sigma(\dot{v}^2(t), \partial_x u^2(t))] d\omega_t \\
&+ \int_X [g(\dot{v}^1(t), \partial_x u^1(t), q) - g(\dot{v}^2(t), \partial_x u^2(t), q)] \tilde{N}_p(\dt dq),
\end{align}
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applying the Ito’s formula, we have

\[
\|\hat{v}(t, \cdot)\|^2 = 2 \int_0^t (\hat{v}, \left[\sigma(u^1(s), \partial_x u^1(s)) - \sigma(u^2(s), \partial_x u^2(s))\right] d\omega_s) + 2 \int_0^t (\hat{v}, A\hat{v}(s) + B_r(u^1(s)) - B_r(u^2(s))) ds + 2 \int_0^t (\hat{v}, [f(u^1(s), \partial_x u^1(s)) - f(u^2(s), \partial_x u^2(s))]) ds + \frac{1}{2} \int_0^t \|\sigma(u^1(s), \partial_x u^1(s)) - \sigma(u^2(s), \partial_x u^2(s))\|^2 ds + \frac{1}{2} \int_0^{t+} \int_X [\|\hat{v}(s-)+g(u^1(s), \partial_x u^1(s), q) - g(u^2(s), \partial_x u^2(s), q)\|^2] \tilde{N}_p(\, ds \, dq) - \frac{1}{2} \int_0^t \int_X \|\hat{v}(s-)(\cdot)\|^2] \tilde{N}_p(\, ds \, dq) + \frac{1}{2} \int_0^t \int_X \|g(u^1(s), \partial_x u^1(s), q) - g(u^2(s), \partial_x u^2(s), q)\|^2 ds \, dn(q).
\]

(3.32)

Noting

\[
(\hat{v}, A\hat{v}(s)) = -\frac{\alpha}{2} \partial_s \|\partial_x \hat{v}(s)\|^2 - \frac{\gamma}{2} \partial_x \|\partial_x^2 \hat{v}(s)\|^2,
\]

(3.35) becomes

\[
e_L(t, \hat{v}) = \int_0^t (\hat{v}, \left[\sigma(u^1(s), \partial_x u^1(s)) - \sigma(u^2(s), \partial_x u^2(s))\right] d\omega_s) + \int_0^t (\hat{v}, B_r(u^1(s)) - B_r(u^2(s))) ds + \int_0^t (\hat{v}, [f(u^1(s), \partial_x u^1(s)) - f(u^2(s), \partial_x u^2(s))]) ds + \frac{1}{2} \int_0^t \|\sigma(u^1(s), \partial_x u^1(s)) - \sigma(u^2(s), \partial_x u^2(s))\|^2 ds + \frac{1}{2} \int_0^{t+} \int_X [\|\hat{v}(s-)+g(u^1(s), \partial_x u^1(s), q) - g(u^2(s), \partial_x u^2(s), q)\|^2] \tilde{N}_p(\, ds \, dq) - \frac{1}{2} \int_0^t \int_X \|\hat{v}(s-)(\cdot)\|^2] \tilde{N}_p(\, ds \, dq) + \frac{1}{2} \int_0^t \int_X \|g(u^1(s), \partial_x u^1(s), q) - g(u^2(s), \partial_x u^2(s), q)\|^2 ds \, dn(q).
\]

(3.33)

Take the expectation in the above equation and use the chain rule to get
Similarly, using the assumptions (H.1) and (H.2) we deduce that

\[ E[e_L(t, \bar{v})]e^{-\eta^2 t} \]

\[ = - \int_0^t E[e_L(s, \bar{v})]e^{-\eta^2 s} \eta^2 s ds \]

\[ + E \left[ \int_0^t e^{-\eta^2 s}(\dot{v}, B_r(u^1(s)) - B_r(u^2(s))) ds \right] \]

\[ + E \left[ \int_0^t e^{-\eta^2 s}[f(u^1(s), \partial_x u^1(s)) - f(u^2(s), \partial_x u^2(s))] ds \right] \]

\[ + E \left[ \frac{1}{2} \int_0^t e^{-\eta^2 s}\|\sigma(u^1(s), \partial_x u^1(s)) - \sigma(u^2(s), \partial_x u^2(s))\|^2 ds \right] \]

\[ + E \left[ \frac{1}{2} \int_X e^{-\eta^2 s}\|g(u^1(s), \partial_x u^1(s), q) - g(u^2(s), \partial_x u^2(s), q)\|^2 ds d\eta(q) \right], \tag{3.34} \]

by Young’s inequality, we have

\[ (\dot{v}, B_r(u^1(s)) - B_r(u^2(s))) \]

\[ \leq \frac{1}{2}\eta^2\|\dot{v}(s)\|^2 + \frac{1}{2\eta^2}\|B_r(u^1(s)) - B_r(u^2(s))\|^2 \]

\[ \leq \frac{1}{2}\eta^2 e_L(s, \bar{v}) + C_{1, \eta} e_L(s, u^1 - u^2), \tag{3.35} \]

where we have used the fact that

\[ \|B_r(u^1(s)) - B_r(u^2(s))\|^2 \leq C(\|\partial_x u^1(s) - \partial_x u^2(s)\|^2 + \|\partial_x^2 u^1(s) - \partial_x^2 u^2(s)\|^2). \]

Using Young’s inequality and the assumption (H.1) we have

\[ (\dot{v}, f \left( \dot{u}^1(s), \partial_x u^1(s) \right) - f \left( \dot{u}^2(s), \partial_x u^2(s) \right)) \]

\[ \leq \frac{1}{2}\eta^2\|\dot{v}(s)\|^2 + \frac{1}{2\eta^2}\|f \left( \dot{u}^1(s), \partial_x u^1(s) \right) - f \left( \dot{u}^2(s), \partial_x u^2(s) \right)\|^2 \]

\[ \leq \frac{1}{2}\eta^2 e_L(s, \bar{v}) + C_{1, \eta} e_L(s, u^1 - u^2). \tag{3.36} \]

Similarly, using the assumptions (H.1) and (H.2) we deduce that

\[ \|\sigma \left( \dot{u}^1(s), \partial_x u^1(s) \right) - \sigma \left( \dot{u}^2(s), \partial_x u^2(s) \right)\|^2 \]

\[ \leq C_3 e_L(s, u^1 - u^2), \tag{3.37} \]
CHAPTER 3. BEAM EQUATION WITH JUMPS

\[
\int_X g \left( \dot{u}^1(s), \partial_x u^1(s), q \right) - g \left( \dot{u}^2(s), \partial_x u^2(s), q \right) \|^2 \, dn(q) \leq C_4 e_L(s, u^1 - u^2). \tag{3.38}
\]

Substitute (3.34)–(3.37) back into (3.33) to get that

\[
E[e_L(t, \tilde{v})] e^{-\eta^2 t} \leq C_\eta \int_0^t e^{-\eta^2 s} E[e_L(s, u^1 - u^2)] \, ds, \tag{3.39}
\]

In particular, (3.38) implies that \( \|v^1 - v^2\|^2_Z \leq C_\eta T \|u^1 - u^2\|^2_Z \). This shows that the mapping \( S_r(\cdot) \) is continuous from the space \( Z \) into the space \( Z \). Let \( u^0 \in Z \) and define recursively \( u^n = S_r(u^{n-1}) \), for \( n = 1, ..., \). It follows from (3.38) that

\[
E[e_L(t, u^{n+1} - u^n)] e^{-\eta^2 t} \\
\leq C_\eta \int_0^t e^{-\eta^2 s} E[e_L(s, u^n - u^{n-1})] \, ds \\
\leq C^2_\eta \int_0^t ds_1 \int_0^{s_1} e^{-\eta^2 s_2} E[e_L(s_2, u^{n-1} - u^{n-2})] \, ds_2 \\
\leq \cdots C^n_\eta \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} e^{-\eta^2 s_n} E[e_L(s_n, u^1 - u^0)] \, ds_n \\
\leq \frac{C^n_\eta t^n}{n!} \|u^1 - u^0\|^2_Z,
\]

hence we have

\[
\|u^{n+1} - u^n\|^2_Z \leq \frac{C^n_\eta t^n}{n!} \|u^1 - u^0\|^2_Z,
\]

for all \( n \geq 0 \). This particularly implies that \( u^n, n \geq 0 \) is a Cauchy sequence in the space \( Z \). We denote the limit of this sequence by \( u_r \). Letting \( n \to \infty \), we see that \( u_r \) is a fixed point of the mapping \( S_r(\cdot) \) and hence a solution of the truncated equation (3.4). Next we will patch \( u_r \) together to get a solution of the stochastic beam equation. Introduce the stopping time

\[
\tau_r = \inf \{ t > 0 : \|\dot{u}_r\| \geq r, \|\partial_x u_r(t)\| \geq r, \|\partial^2_x u_r(t)\| \geq r \},
\]

by the uniqueness, it follows that if \( r_2 \geq r_2 \), then \( \tau_{r_1} \leq \tau_{r_2} \) and \( u_{r_1}(t) = u_{r_2}(t) \) for \( t \in [0, \tau_{r_1}) \). Define

\[
u(t, x) = u^*(t, x), \quad \text{for} \quad t \in [0, \tau_r),
\]

then \( u(t), t \geq 0 \) is a solution to the stochastic beam equation (3.4) on the interval \([0, \tau)\), where \( \tau = \lim_{r \to \infty} \tau_r \).
Now it remains to show $\tau = \infty$ a.s.. This will be done if show that for any $T > 0$, $P(\tau \geq T) = 1$. By the energy equation (3.7) we have

$$e(t \wedge \tau_r, u)$$

$$= e(0, u) + \int_0^{t \wedge \tau_r} (f(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r)), \dot{u}(s \wedge \tau_r)) \, ds$$

$$+ \int_0^{t \wedge \tau_r} \frac{1}{2} \| \sigma(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r)) \|^2 \, ds$$

$$+ \int_0^{t \wedge \tau_r} (\dot{u}(s \wedge \tau_r), \sigma(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r)) \, dw_s$$

$$+ \frac{1}{2} \int_0^{t \wedge \tau_r} \int_X | \sigma(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r), q) \|^2 \tilde{N}_p(\, ds \, dq)$$

$$+ \frac{1}{2} \int_0^{t \wedge \tau_r} \int_X \| g(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r), q) \|^2 \, ds \, dn(q),$$

using the linear growth condition of the coefficients and taking expectation in (3.39) we obtain

$$E[e(t \wedge \tau_r, u)]$$

$$= e(0, u) + E\left[ \int_0^{t \wedge \tau_r} (f(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r)), \dot{u}(s \wedge \tau_r)) \, ds \right]$$

$$+ E\left[ \int_0^{t \wedge \tau_r} \frac{1}{2} \| \sigma(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r)) \|^2 \, ds \right]$$

$$+ \frac{1}{2} E\left[ \int_0^{t \wedge \tau_r} \int_X \| g(\dot{u}(s \wedge \tau_r), \partial_x u(s \wedge \tau_r), q) \|^2 \, ds \, dn(q) \right]$$

$$\leq e(0, u) + C E\left[ \int_0^{t \wedge \tau_r} [1 + \| \dot{u}(s \wedge \tau_r) \|^2 + \| \partial_x u(s \wedge \tau_r) \|^2] \, ds \right]$$

$$\leq C \int_0^{t} [1 + E[e(s \wedge \tau_r, u)]] \, ds,$$

by Grownwall’s inequality, it follows that

$$E[e(T \wedge \tau_r, u)] \leq C_T,$$

where $C_T$ is a constant, independent of $r$. Now from here we can follow the same argument as in the proof of Theorem to show that $\tau = \infty$ a.s. The uniqueness can be proved similarly as in Theorem. The proof is complete.


