A NEW PERSPECTIVE ON THE CLASSICAL COURNOT DUOPOLY

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Abstract. The paper provides new conditions for the existence, uniqueness, and symmetry of pure-strategy Nash equilibrium in the classical Cournot duopoly.

1. Introduction. The Cournot duopoly model created more than 180 years ago remains one of the most widely used frameworks for the study of imperfect competition and partial equilibrium in such diverse fields as industrial organization, public economics, environmental economics and international trade. This model is surprisingly versatile and lends itself to multiple approaches yielding a variety of important insights (see e.g. Friedman [5] and Vives [15] for monograph treatments of the extensive literature on this topic). In the present paper, we take a fresh look at the classical Cournot duopoly and extend some basic results on the existence, uniqueness and symmetry of Cournot-Nash equilibrium. Previous results of a similar nature were obtained in the work by McManus [8], Roberts and Sonnenschein [11], Novshek [10], Amir [1], Amir and Lambson [2], and Ewerhart [4] among many others.\(^1\)

We focus on a version of the Cournot game with nonlinear inverse demand and symmetric linear costs. There are two firms, players 1 and 2 in the game, maximizing profits

\[
\pi_i(q_1, q_2) = q_i P(Q) - c q_i \quad [i = 1, 2, \ Q = q_1 + q_2]
\]

where \(q_i \geq 0, i = 1, 2,\) are the quantities of a homogeneous good produced by the firms and \(c q_i (c > 0)\) are production costs. The function \(P(Q)\) specifies the market clearing price (inverse demand), depending on the total quantity produced \(Q = q_1 + q_2.\) This defines a symmetric game of two players with payoff functions \(\pi_i(q_1, q_2)\) and the common strategy set consisting of all non-negative numbers \(q_i.\) We

\(^1\) These studies have all relied on the lattice-theoretic approach, tacitly in the case of the first two papers, and explicitly in the case of the last two ones. The main ideas, notions and results concerning this approach are covered in some detail in Topkis [14] and Vives [15]. Instead, Szidarovszky and Yakowitz [12] rely on the standard approach via continuous reaction curves.
are interested in questions of existence, uniqueness and symmetry of pure-strategy Nash equilibria in the game described.

We shall make use of the following assumptions:

(C) There exists \( \bar{Q} > 0 \) such that the function \( P(Q) \) is upper semicontinuous on \([0, \bar{Q}]\) and satisfies the following conditions: (i) \( P(\bar{Q}) = c \); (ii) if \( Q > \bar{Q} \), then \( P(Q) < c \); (iii) if \( Q < \bar{Q} \), then \( P(Q) > c \).

(C1) The function \( P(Q) \) is differentiable on the interval \((0, \bar{Q})\).

(C2) The function \( P(Q) \) is twice differentiable on \((0, \bar{Q})\), and for every \( 0 \leq q^* < \bar{Q}/2 \) the function \( \pi(q, q^*) = qP(q+q^*) - cq \) is concave with respect to \( q \) on \([0, \bar{Q}-q^*] \).

Under assumption (C), we prove the existence of a Nash equilibrium in the Cournot game. Under assumptions (C) and (C1), we show that every Nash equilibrium is symmetric. If conditions (C), (C1) and (C2) hold, then the equilibrium is proved to be unique. All the mathematical techniques used in the proofs are absolutely elementary.

The above set of assumptions differs in a number of respects from the standard ones. The only requirements on the function \( P(Q) \) needed for the existence of equilibrium are that it be upper semicontinuous on \([0, \bar{Q}]\), equal to \( c \) at \( \bar{Q} \), and strictly greater than \( c \) on \([0, \bar{Q})\). Outside the "effective domain" \([0, \bar{Q}] \), no conditions except \( P(Q) < c \) are imposed. None of the familiar requirements of (quasi-) concavity of the players’ payoff functions with respect to their own output are needed for existence. We do not employ in our analysis the methods of supermodular games (see, e.g., Amir [1] and Topkis [14]) and we do not impose all the assumptions that would be needed for the use of those methods. However, our condition (C) is akin to the "single crossing property" playing an important role in the works cited.

The existence proof is based on the method of ordinal potentials proposed by Monderer and Shapley [9]. We show that under assumption (C) the ordinal potential \( F(q_1, q_2) = q_1q_2[P(q_1 + q_2) - c] \) attains its maximum on the (non-compact) set of all strategy profiles \((q_1, q_2) > 0\), and a strategy profile where this maximum is attained forms a pure-strategy Nash equilibrium.

The usual assumption of the concavity of the payoff function with respect to the player’s action appears for the first time in (C2), the assumption guaranteeing uniqueness. This assumption is quite standard. But what is not standard is that (C2) is used nowhere except the proof of uniqueness.

Furthermore, we do not rely upon the customary assumption that inverse demand be downward-sloping. The explanation is as follows. Indeed, it is needed that \( P'(Q) < 0 \), but only at those particular points \( Q = Q^* \) that correspond to the equilibrium total output \( Q^* = q_1^* + q_2^* \). It turns out that the inequality \( P'(Q^*) < 0 \) holds automatically in equilibrium as long as our assumptions (C) and (C1) hold. This property turns out to be sufficient to prove the symmetry of Nash equilibrium and its uniqueness by using the following elementary fact. If the derivative of a differentiable function is strictly negative at each of its roots, then it cannot have more than one root.

Finally, it should be stressed that the assumption of differentiability of \( P(Q) \) stated in (C1) plays quite a substantial role in the present context. This might seem at first glance strange because conditions of this kind are typically regarded as "regularity assumptions", irrelevant to the essence of the problem. However, it turns out that the role of the assumption of differentiability in the present model is critical. We construct an example (see Section 5) showing that, if there is at least...
one point at which the function $P(Q)$ is not differentiable, while satisfying all the other assumptions, then symmetry and uniqueness might break down, yielding a continuum of symmetric and asymmetric equilibria.

The remainder of the paper is organized as follows. Section 2 contains a proof of the existence theorem. Sections 3 and 4 discuss symmetry and uniqueness, respectively. Section 5 concludes with a counterexample.

2. Existence. We begin with the observation that the existence of Cournot-Nash equilibrium follows from assumption (C), in light of the fact that symmetric Cournot oligopoly admits an ordinal potential (Monderer and Shapley, 1996).

**Theorem 1.** If the inverse demand function $P(Q)$ satisfies condition (C), then a pure-strategy Nash equilibrium in the Cournot game exists.

**Proof.** Consider the following function

$$F(q_1, q_2) = q_1 q_2 [P(q_1 + q_2) - c].$$

By virtue of the upper semicontinuity of $P(Q)$, this function is upper semicontinuous on the closed bounded set $M = \{(q_1, q_2) : 0 \leq q_1 \leq \bar{Q}, 0 \leq q_2 \leq \bar{Q}\}$. Consequently, $F(q_1, q_2)$ attains its maximum on $M$ at some point $(q_1^*, q_2^*)$. We will prove that $(q_1^*, q_2^*)$ is a Nash equilibrium.

Let us first show that $q_1^* > 0$, $q_2^* > 0$ and $F(q_1^*, q_2^*) > 0$. Let $Q$ be any quantity satisfying $0 < Q < \bar{Q}/2$. Then $(Q, Q) \in M$, and we have

$$F(q_1^*, q_2^*) \geq F(Q, Q) = Q^2 [P(2Q) - c] > 0.$$  

The last inequality holds because $2Q < \bar{Q}$ (see condition (C)). Thus $q_1^* > 0$ and $q_2^* > 0$: otherwise, $q_1^* = 0$ or $q_2^* = 0$, and then $F(q_1^*, q_2^*) = 0$.

Observe that the inequality

$$F(q_1, q_2) \leq F(q_1^*, q_2^*) \tag{1}$$

holds for all $(q_1, q_2) \geq 0$, and not only for $(q_1, q_2) \in M$. Indeed, if $(q_1, q_2)$ does not belong to $M$, then $q_1 > \bar{Q}$ or $q_2 > \bar{Q}$, and so $q_1 + q_2 > \bar{Q}$, which implies $P(q_1 + q_2) < c$ by virtue of condition (C). Therefore, $F(q_1, q_2) = q_1 q_2 [P(q_1 + q_2) - c] \leq 0$. But we have shown that $F(q_1^*, q_2^*) > 0$, which implies (1).

By setting $q_2 = q_2^*$ in (1), we get

$$q_1 q_2^* [P(q_1 + q_2^*) - c] = F(q_1, q_2^*) \leq F(q_1^*, q_2^*) = q_1^* q_2^* [P(q_1^* + q_2^*) - c],$$

which yields

$$q_1 [P(q_1 + q_2^*) - c] \leq q_1^* [P(q_1^* + q_2^*) - c],$$

since $q_2^* > 0$. Analogously, by setting $q_1 = q_1^*$ in (1), we find

$$q_2 [P(q_1 + q_2^*) - c] \leq q_2^* [P(q_1^* + q_2^*) - c].$$

Consequently, $(q_1^*, q_2^*)$ is a Nash equilibrium. \( \square \)

**Remark 1.** The present approach to the issue of existence of Cournot equilibrium translates what is inherently a fixed point problem into one involving the maximum of an upper semi-continuous function (the ordinal potential) being achieved on a compact set. The classical approach to existence relies on Kakutani’s fixed point theorem and imposes assumptions that guarantee that each firm’s profit function is quasi-concave in its own action (see e.g., Friedman [5]). An alternative and newer approach, lattice-theoretic in nature, is based instead on Tarski’s fixed point theorem (Tarski [13]) and requires some form of supermodularity of each firm’s profit function; see Novshek [10], Amir [1], and Ewerhart [4].

Theorem 2. Under assumptions (C) and (C1), every Nash equilibrium \((q_1^*, q_2^*)\) is symmetric: \(q_1^* = q_2^*\).

Proof. Let \((q_1^*, q_2^*)\) be a Nash equilibrium. Let us first show that \(q_1^* > 0\) and \(q_2^* > 0\). By virtue of the symmetry of the model, it is sufficient to establish the former inequality. Suppose \(q_1^* = 0\) and observe that \(q_2^* < Q\). Suppose the contrary: \(q_2^* \geq Q\). By the definition of a Nash equilibrium,

\[
q_2[P(q_2 + q_1^*) - c] \leq q_2^*[P(q_2^* + q_1^*) - c], \quad q_2 \geq 0.
\]

If \(q_1^* = 0\) and \(q_2^* \geq Q\) (as we have assumed), then it should be

\[
q_2[P(q_2) - c] \leq q_2^*[P(q_2^* + q_1^*) - c] \leq 0
\]

for all \(q_2 \geq 0\), which is not true, for example, for \(q_2 = Q/2\). This contradiction proves that \(q_2^* < Q\).

By setting \(q_1 = (Q - q_2^*)/2\), we get \(q_1 > 0\) and

\[
q_1 + q_2^* = (Q - q_2^*)/2 + q_2^* = (Q + q_2^*)/2 < Q.
\]

Consequently,

\[
0 < q_1[P(q_1 + q_2^*) - c] \leq q_1^*[P(q_1^* + q_2^*) - c] = 0,
\]

which is a contradiction \((0 < 0)\). Thus \(q_1^* > 0\) and \(q_2^* > 0\).

We next show that if \((q_1^*, q_2^*)\) is a Nash equilibrium, then \(q_1^* + q_2^* < Q\). Suppose the contrary: \(q_1^* + q_2^* \geq Q\). If \(q_1^* + q_2^* > Q\), then, by setting \(q_1 = 0\) in the equilibrium condition for player 1 and using the inequalities \(q_1^* > 0\) and \(q_1^* + q_2^* > Q\), we obtain

\[
0 = q_1[P(q_1 + q_2) - c] \leq q_1^*[P(q_1^* + q_2^*)] < 0,
\]

which is a contradiction. If \(q_1^* + q_2^* = Q\), then we can set \(q_1 = q_1^*/2 (> 0)\), which yields

\[
0 < q_1[P(q_1 + q_2^*) - c] \leq q_1^*[P(q_1^* + q_2^*) - c] = 0.
\]

because \(q_1 + q_2^* = q_1^*/2 + q_2^* < Q\). The contradiction obtained proves that \(q_1^* + q_2^* < Q\).

Since \((q_1^*, q_2^*)\) is a Nash equilibrium, the function \(\pi(q_1, q_2) = q_1[P(q_1 + q_2) - c]\) attains its maximum with respect to \(q_1 \geq 0\) at the point \(q_1 = q_1^*\), which belongs to \((0, Q - q_2^*)\) (indeed, we have shown above that \(0 < q_1^* < q_1^* + q_2^* < Q\)). The function \(\pi(\cdot, q_2^*)\) is differentiable at \(q_1 = q_1^*\) because \(P(Q)\) is differentiable on \((0, Q)\) and \(0 < q_1^* + q_2^* < Q\). Consequently, we can use the first order optimality condition \(\partial \pi(q_1^*, q_2^*)/\partial q_1 = 0\), which gives

\[
P(q_1^* + q_2^*) - c + q_1^*P'(q_1^* + q_2^*) = 0. \tag{2}
\]

By virtue of the analogous considerations applied to the function \(\pi(q_1^*, q_2) = q_2[P(q_2 + q_1^*) - c]\), we find

\[
P(q_1^* + q_2^*) - c + q_2^*P'(q_1^* + q_2^*) = 0. \tag{3}
\]

Since \(q_1^* + q_2^* < Q\), we have \(P(q_1^* + q_2^*) - c > 0\). Furthermore, we have shown that \(q_2^* > 0\). Therefore, relation (3) (or (2)) implies that

\[
P'(q_1^* + q_2^*) < 0. \tag{4}
\]

Dividing (2) and (3) by \(P'(q_1^* + q_2^*)\), we get

\[
q_1^* = \frac{-P(q_1^* + q_2^*) - c}{P'(q_1^* + q_2^*)} = q_2^*, \tag{5}
\]
which terminates the proof. \qed

4. Uniqueness.

**Theorem 3.** Under assumptions (C), (C1) and (C2), there is only one Nash equilibrium (which is symmetric).

**Proof.** By virtue of Theorem 2, every equilibrium is symmetric. When proving this theorem under assumptions (C) and (C1), we have shown that if \((q^*, q^*)\) is a Nash equilibrium in the game at hand, then \(q^*\) satisfies \(0 < q^* < Q/2\),

\[
L(q^*) := P(2q^*) - c + q^* P'(2q^*) = 0 \tag{6}
\]

(see (3)) and

\[
P'(2q^*) < 0 \tag{7}
\]

(see (4)). Equation (6) was obtained as a necessary condition for a maximum of the differentiable function

\[
\pi(q) := \pi(q, q^*) = q[P(q + q^*) - c], \quad 0 < q < \bar{Q} - q^*,
\]

which has to be satisfied as long as \(q^*\) forms a symmetric Nash equilibrium.

By virtue of assumption (C2), the function \(\pi(q)\) is concave on \((0, Q - q^*)\). Consequently, equation (6) is not only a necessary, but also a sufficient condition for its maximum. Thus if (6) holds for some \(0 < q^* < Q/2\), then

\[
q[P(q + q^*) - c] = \pi(q) \leq \pi(q^*) = q^*[P(2q^*) - c] \tag{8}
\]

for all \(0 < q < \bar{Q} - q^*\). This implies that \(\pi(q) \leq \pi(q^*)\) for all \(q \geq 0\). Indeed, inequality (8) can be extended to \(q = 0\) and \(q = \bar{Q} - q^*\) by using the concavity of \(\pi(q)\) on \([0, \bar{Q} - q^*]\). If \(q > \bar{Q} - q^*\), then

\[
\pi(q) = q[P(q + q^*) - c] < 0 < q^*[P(2q^*) - c] = \pi(q^*).
\]

Thus any \(0 < q^* < \bar{Q}/2\) satisfying (6) forms a symmetric Nash equilibrium, and conversely, any \(q^* \geq 0\) forming a symmetric Nash equilibrium satisfies \(0 < q^* < Q/2\), (6) and (7).

The derivative of the function \(L(q) = P(2q) - c + qP'(2q)\) equals

\[
L'(q) = 3P'(2q) + 2qP''(2q).
\]

Further, for the function \(\mu(q) := q[P(q) - c] = \pi(q, 0)\), we have

\[
\mu''(q) = [qP'(q) + P(q) - c]' = qP''(q) + 2P'(q),
\]

and so \(\mu''(2q) = 2qP''(2q) + 2P'(2q)\). This yields

\[
L'(q) = P''(2q) + [2P'(2q) + 2qP''(2q)] = P''(2q) + \mu''(2q) \leq P'(2q) \tag{9}
\]

for each \(0 < q < \bar{Q}/2\). The inequality in (9) holds by virtue of the concavity of \(\mu(q)\) on the interval \(0 < q < \bar{Q}\), following from (C2).

As we have shown, \(q^* \geq 0\) forms a symmetric Nash equilibrium if and only if \(q^*\) is a solution to the equation \(L(q^*) = 0\) in the interval \(0 < q^* < \bar{Q}/2\). Moreover, any such solution satisfies inequality (7) and hence, by virtue of (9), the inequality \(L'(q^*) < 0\). Thus the derivative at any root of the differentiable function \(L(q)\) in some interval is strictly negative. This implies that this function cannot have more than one root, because if \(L(q_1) = L(q_2) = 0\) at some points \(q_1 < q_2\) where \(L(q_1) < 0\) and \(L(q_2) < 0\), then for some \(q\) in the interval \((q_1, q_2)\) we have \(L(q) = 0\) and \(L'(q) \geq 0\). Consequently, there is only one solution to the equation \(L(q^*) = 0\), and thus only one symmetric Nash equilibrium. \qed
Remark 2. A general approach to the uniqueness of Cournot equilibrium in the asymmetric case, based on degree theory, is developed in Kolstad and Mathiesen [7]; see also Gaudet and Salant [6].

5. Multiple equilibria: An example. We provide an example showing that the assumption of differentiability of \( P(Q) \) is essential for the validity of Theorems 2 and 3. Put \( P_1(Q) = 2/(1 + Q) \), \( P_2(Q) = 3/(1 + 2Q) \) and consider the following inverse demand function:

\[
P(Q) = \min_{i=1,2} P_i(Q) = \begin{cases} 
P_1(Q), & \text{if } 0 \leq Q \leq 1, \\
P_2(Q), & \text{if } Q > 1.
\end{cases}
\]

Define the marginal cost as an inverse demand function:

\[
P(Q) = 1.\]  
\[
\Phi(Q) = \begin{cases} 
\Phi_1(Q), & \text{if } 0 \leq Q \leq 1, \\
\Phi_2(Q), & \text{if } Q > 1.
\end{cases}
\]

This function (denote it by \( \Phi(Q) \)) is concave. Consequently, in order to show that it attains its maximum on \( [0, 1] \), it suffices to verify that its derivative at \( q = 1 \) is non-negative. We have

\[
q_1^* + q_2^* = 1, \quad \frac{9}{20} \leq q_1^* \leq \frac{11}{20}, \quad \frac{9}{20} \leq q_2^* \leq \frac{11}{20}.
\]

forms a Cournot-Nash equilibrium.

Proof. We have to verify that \( q_1 \{ P(q_1 + q_2^*) - c \} \leq q_1^* \{ P(q_1^* + q_2^*) - c \} \) for all \( q_1 \geq 0 \).

1st case. Let \( q_1 \leq q_1^* \). Then \( q_1 + q_2^* \leq 1 \) and \( q_1 \{ P(q_1 + q_2^*) - c \} = q_1 \{ P_1(q_1 + q_2^*) - c \} \) as we have shown, this function (denote it by \( \phi(q_1) \)) is concave. Consequently, in order to prove that it attains its maximum on \( [0, q_1^*] \) at \( q_1 = q_1^* \), it suffices to verify that its derivative at \( q_1 = q_1^* \) is non-negative. We have

\[
\phi'(q_1^*) = q_1^* P_1'(q_1^* + q_2^*) + P_1(q_1^* + q_2^*) - c = -q_1^*/2 + 3/10 = 1/40 \geq 0.
\]

2nd case. Let \( q_1 \geq q_1^* \). Then \( q_1 + q_2^* \geq 1 \) and \( q_1 \{ P(q_1 + q_2^*) - c \} = q_1 \{ P_2(q_1 + q_2^*) - c \} \). This function (denote it by \( \psi(q_1) \)) is concave, and in order to show that it attains
its maximum on \([q_1^*, \infty]\) at \(q_1 = q_1^*\), it is sufficient to show that its derivative at \(q_1 = q_1^*\) is non-positive. We have

\[
\psi'(q_1^*) = q_1^* P_2'(q_1^* + q_2^*) + P_2(q_1^* + q_2^*) - c = -2q_1^*/3 + 3/10 = 0.
\]

The proof is complete. □

REFERENCES


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