Symmetric and exterior powers of representations of cyclic groups.

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In this thesis we will consider symmetric and exterior powers of modular representations of cyclic groups of prime power orders. It is shown that some pth exterior powers can be expressed in terms of symmetric powers of subgroup modules quotiented out by pth powers. An expression for this quotient in terms of exterior powers is proved. Numerical data is given for small exterior powers at the primes 3 and 5.

Declaration and Copyright

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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About the author

I graduated from the University of Manchester in 2014 with a first class honours M(Math&Phys) in mathematics and physics. In my undergraduate project I studied Hopf algebras and Drinfeld twists.

Chapter 1

Introduction

In this project we will consider modular representations of cyclic groups of prime power orders where the characteristic of the field divides the order of the group. We will mostly be concerned with the exterior powers of indecomposable modules but will also consider symmetric powers and tensor products with the aim of finding formulae for their direct sum decompositions. Tensor products are generally well understood and there are several algorithms for computing their decomposition, for example [1]. G. Almkvist and R.Fossum decomposed symmetric and exterior powers for groups of prime order [2]. A formula for small (less than the characteristic of the field) exterior powers of tensor products was found by I. Hughes and G. Kemper [3]. They used this to calulate small exterior powers. P. Symonds proved a formula for symmetric powers of indecomposable modules in terms of exterior powers, up to induced summands [4] and with F. Himstedt proved an iterative formula for exterior powers in characteristic two [5]. Also Adams operations are of interest as they encode the same information as the symmetric and exterior powers. R.M. Bryant showed Adams operations in powers not divisible by the characteristic of the field are equal to psi operations, so are homomorphisms on the representation ring [6]. R.M. Bryant and M. Johnson used this and a similar approach to Hughes and Kemper to calulate these Adams operations [7].

In chapter 2 the key definitions and background lemmas are reviewed including the relationship between symmetric and exterior powers. Some side lemmas are proved in chapter 3. In chapter 4 some interesting new theorems are proved. The first theorem expresses the pth exterior power of indecomposable modules of dimension greater than the order of the group minus the order of the maximal proper subgroup, in terms of symmetric powers of the subgroup's modules quotiented out by pth powers where p is the characteristic of the field. The rest of the chapter generalises the technique used by Himstedt and Symonds [5] to describe this quotient in the characteristic 2 case, to the odd characteristic case. This allows this quotient to be expressed in terms of exterior powers of indecomposable modules. In chapter 5 conjectures which describe a pattern appearing in the numerical data, are given. Appendix A contains tables of numerical data (for symmetric and exterior powers and Adams operations) calculated during the project. In appendix B, the program code is given which was used to calculate the tensor product using the algorithm from [1].

Chapter 2

Background

This chapter contains basic definitions and lemmas used throughout this document and goes on to give the known relationship between symmetric and exterior powers of indecomposable modules of cyclic groups of prime power order.

In this thesis we will only consider modules of finite dimension.

Definition 2.1. Let G be a group and k be a field. For kG modules V and W the direct sum $V \oplus W$ and tensor product $V \otimes W = V \otimes_k W$ are given kG module structure by $g(v \oplus w) = (gv) \oplus (gw),\ g(v \otimes w) = (gv) \otimes (gw) \qquad \forall g \in G, v \in V, w \in W$.

Definition 2.2. The nth symmetric power of a module V, $S^n(V)$ is given kG module structure by $g(w_1 w_2 ... w_n) = (gw_1)(gw_2)...(gw_n), w_i \in V$.

The nth exterior power of a module V, $\Lambda^n(V)$ is given kG module structure by $g(w_1 \wedge w_2 \wedge \cdots \wedge w_n) = (gw_1) \wedge (gw_2) \wedge \cdots \wedge (gw_n), w_i \in V$.

Lemma 2.3. *Let C_{p^n} be a cyclic group of order p^n and k a field of characteristic p. Then the indecomposable kC_{p^n} modules are isomorphic to Jordan blocks with eigenvalue 1, dimension i for $0 < i \leqslant p^n$ (see [9]).*

Let V_i be the Jordan block of dimension i and eigenvalue 1. Then V_i has basis $\{v_j \mid 1 \leqslant j \leqslant i\}$ where for a fixed generator g of C_{p^n}, $gv_j = v_j + v_{j-1}, gv_1 = v_1$ and let $v_j = 0$ for $j \leqslant 0$.

Definition 2.4. A kG module V is induced if and only if there exists a proper subgroup H of G and a kH module W such that V is induced from W i.e. $V \cong kG \otimes_{kH} W$.

Let $W \uparrow_H^G = kG \otimes_{kH} W$.

Lemma 2.5. *V_i is induced if and only if $p|i$.*

Proof. If V_i is induced then $V_i \cong W \uparrow_{C_{p^m}}^{C_{p^n}}$, $m < n$ so $i = |V_i| = |W \uparrow_{C_{p^m}}^{C_{p^n}}| = p^{n-m}|W|$ therefore $p|i$. If $p|i$ then $i = pj$ and $V_i \cong V_j \uparrow_{C_{p^{n-1}}}^{C_{p^n}}$ so V_i is induced. □

We now let $G \cong C_{p^n}$.

Definition 2.6. Let $\cong_{ind}$ mean isomorphic modulo induced modules and $\cong_{proj}$ mean isomorphic modulo projective modules. I.e. $V \cong_{ind} W$ if and only if there exists I_1, I_2 induced modules such that $I_1 \oplus V \cong I_2 \oplus W$ and $V \cong_{proj} W$ if and only if there exists P_1, P_2 projective modules such that $P_1 \oplus V \cong P_2 \oplus W$.

Lemma 2.7. *(see [4] lemma 3.7) For $p \nmid r$, $S^r(V_{pu})$ is induced; also for $p^n \nmid s$, $S^s(V_{p^n})$ is induced and $S^{ip^n}(V_{p^n}) \cong_{ind} V_1$.*

The following lemma is proved in Symonds [4] (corollary 3.11).

Lemma 2.8. *For $G \cong C_{p^n}, r < p^n$ and $p^{n-1} \leqslant t \leqslant p^n$.*

$$S^r(V_t) \cong_{ind} \Omega_{p^n}^{-r} \Lambda^r(V_{p^n - t})$$

where Ω_{p^n} is the syzygy or Heller operator, $\Omega_{p^n} V_i \cong V_{p^n - i}$.

Lemma 2.9. *(see [4] lemma 1.1) For a C_{p^n} module V and a submodule V' of codimension 1, $x \in V \setminus V'$, $S^r(V) \cong \sum_i a^i T^{r-p^n i}(V)$, where $a = \prod_{g \in C_{p^n}} gx$ and $T^i(V)$ the submodule of $S^i(V)$ spanned by the monomials in the elements of V' and x that are not divisible by x^{p^n}.*

Lemma 2.10. *(see [4] theorem 1.2) For $r \geqslant p^n$, $i \leqslant p^n$; $T^r(V_i)$ is induced.*

Definition 2.11. The representation ring R_{kG} is the group of integer combinations of isomorphism classes of indecomposable modules with multiplication given by $V \times W = \sum_{r \in R_{kG}} a_r r$ where a_r is the coefficient of r in the decomposition of $V \otimes W$.

Remark 2.12. (see [10]) As C_{p^n} has a unique maximal proper subgroup, the induced modules form an ideal in the representation ring.

Definition 2.13. For a kG module V, define formal power series

$$\lambda_t(V) = 1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots$$

$$\sigma_t(V) = 1 + S^1(V)t + S^2(V)t^2 + \cdots .$$

Definition 2.14. The Adams operations ψ_Λ and ψ_S are linear maps from R_{kG} to R_{kG} defined by

$$\psi^1_\Lambda(V)t - \frac{1}{2}\psi^2_\Lambda(V)t^2 + \frac{1}{3}\psi^3_\Lambda(V)t^3 - \cdots = \log \lambda_t(V)$$

$$\psi^1_S(V)t + \frac{1}{2}\psi^2_S(V)t^2 + \frac{1}{3}\psi^3_S(V)t^3 + \cdots = \log \sigma_t(V)$$

where log is defined by the formal expansion of $\log(1+x)$.

Remark 2.15. Rearranging the above definition gives

$$\lambda_t(V) = \exp\left(\psi^1_\Lambda(V)t - \frac{1}{2}\psi^2_\Lambda(V)t^2 + \frac{1}{3}\psi^3_\Lambda(V)t^3 - \dots\right)$$

$$\sigma_t(V) = \exp\left(\psi^1_S(V)t + \frac{1}{2}\psi^2_S(V)t^2 + \frac{1}{3}\psi^3_S(V)t^3 + \dots\right).$$

Lemma 2.16. *(see [6] theorem 5.4) For $p \nmid r$*

$$\psi^r_\Lambda = \psi^r_S.$$

Lemma 2.17. *(see [8] theorem 6.2) For $G \cong C_{p^n},\ n > 0$ and $p^{n-1} \leqslant t \leqslant p^n$,*

$$\psi^r_S(V_t) \cong_{proj} (-1)^{r-1}\Omega^r_{p^n}\psi^r_\Lambda(V_{p^n-t}) + (r,p^n)V_{p^n/(r,p^n)}.$$

Lemma 2.18. *(see [1] lemmas 2.4, 2.3 and G-3,G-5) For $q = p^m$:*
If $0 \leqslant r \leqslant q, 1 \leqslant a \leqslant p$,

$$V_r \otimes V_{aq-1} \cong V_{aq-r} + (r-1)V_{aq}$$

If $0 \leqslant r \leqslant q, 0 \leqslant a < p$,

$$V_r \otimes V_{aq+1} \cong V_{aq+r} + (r-1)V_{aq}.$$

If $q \leqslant r \leqslant pq, r = r_0q + r_1, 0 \leqslant r_1 < q$,

$$V_r \otimes V_{q-1} \cong (r_1-1)V_{(r_0+1)q} + V_{(r_0+1)q-r_1} + (q-r_1-1)V_{r_0q}.$$

If $q \leqslant r \leqslant (p-1)q, r = r_0q + r_1, 0 \leqslant r_1 < q$,

$$V_r \otimes V_{q+1} \cong V_{r-q} + (r_1-1)V_{(r_0+1)q} + V_{(r_0+1)q-r_1} + (q-r_1-1)V_{r_0q} + V_{r+q}.$$

Definition 2.19. A chain complex of modules K_i,

$$0 \overset{d_{r+1}}{\to} K_r \overset{d_r}{\to} K_{r-1} \overset{d_{r-1}}{\to} K_{r-2} \overset{d_{r-2}}{\to} K_{r-3} \overset{d_{r-3}}{\to} \dots \overset{d_3}{\to} K_2 \overset{d_2}{\to} K_1 \overset{d_1}{\to} K_0$$

is defined to be

1. exact at i, if $\text{im}(d_{i+1})$=$\ker(d_i)$
2. separated at i, if $\text{im}(d_{i+1})\to K_i$ factors through a projective
3. separated, if it is separated at i for $0 \leqslant i \leqslant r$.

Lemma 2.20. *(see [4] proposition 3.3) Let*

$$0 \overset{d_{r+1}}{\to} K_r \overset{d_r}{\to} K_{r-1} \overset{d_{r-1}}{\to} K_{r-2} \overset{d_{r-2}}{\to} K_{r-3} \overset{d_{r-3}}{\to} \dots \overset{d_3}{\to} K_2 \overset{d_2}{\to} K_1 \overset{d_1}{\to} K_0$$

be a chain complex which is separated and exact at j for $0 < j \leqslant r$ then there exists a projective module P such that

$$H_0 \oplus P \cong K_0 \oplus \Omega^{-1}K_1 \oplus \Omega^{-2}K_2 \oplus \Omega^{-3}K_3 \oplus \Omega^{-4}K_4 \oplus \cdots \oplus \Omega^{-r}K_r$$

where

$$H_0 = \frac{K_0}{\text{im}(d_1)}$$

is the degree 0 homology.

Definition 2.21. Let W be a submodule of V; then the Koszul complex $K^r(V, W)$ is

$$\dots \overset{d_3}{\to} \Lambda^2(W) \otimes S^{r-2}(V) \overset{d_2}{\to} \Lambda^1(W) \otimes S^{r-1}(V) \overset{d_1}{\to} S^r(V),$$

where $d_i(w_1 \wedge \cdots \wedge w_i \otimes s) = \sum_{j=1}^{i}(-1)^{i-j} w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_i \otimes w_j s$ and $\hat{w}_j$ denotes that w_j is removed.
The complex $K^r(V, W^{(p)})$ is

$$\dots \overset{d_3}{\to} \Lambda^2(W^{(p)}) \otimes S^{r-2p}(V) \overset{d_2}{\to} \Lambda^1(W^{(p)}) \otimes S^{r-p}(V) \overset{d_1}{\to} S^r(V)$$

where $W^{(p)} = \text{span}_k\{w^p \in S^p(W) \mid w \in W\}$
and $d_i(w_1 \wedge \cdots \wedge w_i \otimes s) = \sum_{j=1}^{i}(-1)^{i-j} w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_i \otimes w_j s$.

Lemma 2.22. *(see [4] lemma 3.5 and [5] lemma 2.7) The complexes $K^r(V, W)$ and $K^r(V, W^{(p)})$ are exact at j for $j > 0$.*

Lemma 2.23. *(see [4] theorem 3.10) For $0 \leqslant t \leqslant p^n - p^{n-1}$, the complex $K^r(V_{p^n}, V_t)$ is separated.*

Definition 2.24. For a kG module homomorphism $f : A \to B$, f is split injective modulo induced summands if there exists an induced module X and a homomorphism $f' : A \to X$ such that $(f, f') : A \to B \oplus X$ is split injective.

Lemma 2.25. *(see [5] lemma 3.9) For homomorphisms $f : A \to B$ and $g : B \to C$ of kG modules*

1. *if f and g are split injective modulo induced summands then so is $g \circ f$*

2. *if $g \circ f$ is split injective modulo induced summands then so is f.*

Lemma 2.26. *(see [5] lemma 3.10) Let $f : A \to B$ be a map of kG modules and $A = A' \oplus A''$, where A' has only non-induced summands and A'' is induced. Let i denote the inclusion of A' in A. Then f is split injective modulo induced summands if and only if $f \circ i$ is split injective.*

Definition 2.27. For a kG module M define

$$M/G = k \otimes_{kG} M.$$

Lemma 2.28. *If*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow a & & \downarrow b \\ C & \xrightarrow{g} & D \end{array}$$

commutes then there exists a map $c : \operatorname{coker}(f) \to \operatorname{coker}(g)$ such that

$$\begin{array}{ccc} B & \to & \operatorname{coker}(f) \\ \downarrow b & & \downarrow c \\ D & \to & \operatorname{coker}(g) \end{array}$$

commutes.

Proof. Consider $c : \operatorname{coker}(f) \to \operatorname{coker}(g)$, $c(x + \operatorname{Im}(f)) = b(x) + \operatorname{Im}(g)$ which is well defined as

$$b(\operatorname{Im}(f)) = \operatorname{Im}(b \circ f) = \operatorname{Im}(g \circ a) \subseteq \operatorname{Im}(g).$$

□

Lemma 2.29. *(see [11] Exercises 2.6.8) Let $n \in \mathbb{N}$ and p a prime. Suppose that $n = n_0 + n_1 p + \cdots + n_k p^k$ where $0 \leqslant n_i < p$. The largest power of p which divides $n!$ is $p^{\nu(n)}$ where*

$$\nu(n) = \sum_{i=0}^{k} \left\lfloor \frac{n}{p^i} \right\rfloor = n_1 + n_2 \frac{(p^2 - 1)}{(p-1)} + \cdots + n_k \frac{(p^k - 1)}{(p-1)}.$$

Chapter 3

Two Lemmas

The first lemma in this chapter gives the formula for a particular tensor product and the other lemma provides a neat little formula for the exterior square of a tensor product.

Let V_i be defined as in chapter 2.

Lemma 3.1. *Let $i, r \in \mathbb{N}$, then*

$$V_r \otimes V_{p^i-1} \cong_{ind} V_{(r_0+1)p^i-r_1} \qquad r = r_0 p^i + r_1, 0 \leqslant r_1 < p^i.$$

Proof. From lemma 2.18, we have

$$V_r \otimes V_{p^i-1} \cong V_{p^i-r} + (r-1)V_{p^i} \qquad 1 \leqslant r \leqslant p^i$$

and

$$V_b \otimes V_{aq+1} \cong V_{aq+b} + (r-1)V_{aq} \qquad 0 \leqslant a < p, 0 \leqslant b \leqslant q, q = p^j.$$

In particular the lemma is true for $r \leqslant p^i$. For the case $p|r$ then $p|r_1$ and

$$V_r \otimes V_{p^i-1} \cong_{ind} 0 \cong_{ind} V_{(r_0+1)p^i-r_1}.$$

For $r > p^i$ and $p \nmid r$, assume the lemma is true for all $r' < r$.
Write $r = aq + b$ where $0 \leqslant b < q, q = p^j < r < p^{j+1}$ so $j \geqslant i$, $a < p$ and $b < r$.
As $p^i|(r - r_1)$ we have $p^i|(aq + b - r_1)$. But $p^i|q$ so $p^i|(b - r_1)$, therefore

$$b = b_0 p^i + r_1,\ aq + b_0 p^i = r_0 p^i.$$

Also

$$p \nmid r \Rightarrow r_1 \neq 0$$

and

$$b_0p^i < q \Rightarrow b_0 < p^{j-i} \Rightarrow b_0 + 1 \leqslant p^{j-i} \Rightarrow (b_0+1)p^i \leqslant q \Rightarrow (b_0+1)p^i - b_1 \leqslant q.$$

Then, as V_{aq} is induced, we have from lemma 2.18,

$$V_r \cong_{ind} V_{aq+1} \otimes V_b$$

and

$$V_r \otimes V_{p^i-1} \cong_{ind} V_{aq+1} \otimes V_b \otimes V_{p^i-1} \cong_{ind} V_{aq+1} \otimes V_{(b_0+1)p^i-r_1}$$

by the inductive assumption, so using lemma 2.18,

$$V_r \otimes V_{p^i-1} \cong_{ind} V_{aq+(b_0+1)p^i-r_1} \cong_{ind} V_{(r_0+1)p^i-r_1}.$$

Therefore the lemma is true for all r by induction. □

Lemma 3.2. *For $p > 2$*

$$\Lambda^2(V \otimes W) \cong S^2(V)\Lambda^2(W) \oplus S^2(W)\Lambda^2(V).$$

Proof. Let

$$m : S^2(V)\Lambda^2(W) \oplus S^2(W)\Lambda^2(V) \to \Lambda^2(V \otimes W)$$

be the linear map such that

$$m(v_1v_2 \otimes w_1 \wedge w_2) = (v_1 \otimes w_1) \wedge (v_2 \otimes w_2) - (v_1 \otimes w_2) \wedge (v_2 \otimes w_1)$$

$$m(w_1w_2 \otimes v_1 \wedge v_2) = (v_1 \otimes w_1) \wedge (v_2 \otimes w_2) + (v_1 \otimes w_2) \wedge (v_2 \otimes w_1)$$

where $v_1, v_2 \in V$ and $w_1, w_2 \in W$ which is well defined as

$$\begin{aligned} m(v_2v_1 \otimes w_1 \wedge w_2) &= (v_2 \otimes w_1) \wedge (v_1 \otimes w_2) - (v_2 \otimes w_2) \wedge (v_1 \otimes w_1) \\ &= -(v_1 \otimes w_2) \wedge (v_2 \otimes w_1) + (v_1 \otimes w_1) \wedge (v_2 \otimes w_2) \\ &= m(v_1v_2 \otimes w_1 \wedge w_2) \end{aligned}$$

similarly

$$\begin{aligned} m(v_1v_2 \otimes w_2 \wedge w_1) &= -m(v_1v_2 \otimes w_1 \wedge w_2) \\ m(w_2w_1 \otimes v_1 \wedge v_2) &= m(w_1w_2 \otimes v_1 \wedge v_2) \\ m(w_1w_2 \otimes v_2 \wedge v_1) &= -m(w_1w_2 \otimes v_1 \wedge v_2). \end{aligned}$$

Also

$$(v_1 \otimes w_1) \wedge (v_2 \otimes w_2) = m(2^{-1}(v_1v_2 \otimes w_1 \wedge w_2 + w_1w_2 \otimes v_1 \wedge v_2))$$

therefore m is onto. Let $a = dim(V), b = dim(W)$ then

$$\begin{aligned} dim(S^2(V)\Lambda^2(W) \oplus S^2(W)\Lambda^2(V)) &= \frac{a(a+1)}{2}\frac{b(b-1)}{2} + \frac{a(a-1)}{2}\frac{b(b+1)}{2} \\ &= \frac{ab}{4}((a+1)(b-1) + (a-1)(b+1)) \\ &= \frac{ab}{4}(2ab-2) = \frac{ab(ab-1)}{2} \\ &= dim(\Lambda^2(V \otimes W)) \end{aligned}$$

so m is 1-to-1. As G acts diagonally on all terms, m is a homomorphism hence it is an isomorphism. □

Chapter 4

Theorems

In this chapter some of the work in Himstedt and Symonds [5] will be generalised to the odd characteristic case, but first a new theorem is given which motivated some of this work. Let $G = C_{p^n}$ generator g. Also we will use V_i from chapter 2.

Theorem 4.1. *For $0 \leqslant a < p^{n-1}$,*

$$\Lambda^p(V_{p^n-a}) \cong_{proj} \Omega_{p^n}^{p-1}\left(V_{p^{n-1}-a} \oplus \Omega_{p^n}\frac{S^p(V_a)}{V_a^{(p)}}\right),$$

where $V_a^{(p)} = \mathrm{span}_k\{v^p \in S^p(V_a) \mid v \in V_a\}$.

Proof. Let $f : V_{p^{n-1}} \to V_{p^n} \oplus S^p(V_{p^n})$ be the kG homomorphism such that

$$f(v_{p^{n-1}}) = (v_{p^{n-1}}, v_{p^n}(hv_{p^n})...(h^{p-1}v_{p^n}))$$

where $h = g^{p^{n-1}}$. This is well defined as

$$h(v_{p^{n-1}}, v_{p^n}(hv_{p^n})...(h^{p-1}v_{p^n})) = (v_{p^{n-1}}, v_{p^n}(hv_{p^n})...(h^{p-1}v_{p^n}))$$

and $\mathrm{Hom}_G(V_{p^{n-1}}, X) \cong X^H$.

Let $e : V_{p^n} \oplus S^p(V_{p^n}) \to V_{p^{n-1}}$ be the linear map such that for any $0 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_p < p^n$

$$e(w, (g^{b_1}v_{p^n})(g^{b_2}v_{p^n})\dots(g^{b_p}v_{p^n})) = \begin{cases} g^{b_1}v_{p^{n-1}}, & \text{if } b_{i+1} = b_1 + p^{n-1}i \text{ for all } 0 \leqslant i < p \\ 0, & \text{otherwise.} \end{cases}$$

where $w \in V_{p^n}$. We know that $\{g^i v_{p^n} \mid 0 \leqslant i < p^n\}$ is a basis for V_{p^n}.

For $b_p < p^n - 1$ we have

$$\begin{aligned} e(gw, g((g^{b_1}v_{p^n})(g^{b_2}v_{p^n})\dots(g^{b_p}v_{p^n}))) &= e(w, (g^{b_1+1}v_{p^n})(g^{b_2+1}v_{p^n})\dots(g^{b_p+1}v_{p^n})) \\ &= \begin{cases} g^{b_1+1}v_{p^{n-1}}, & \text{if } b_{i+1}+1 = b_1+1+p^{n-1}i \text{ for all } 0 \leqslant i < p \\ 0, & \text{otherwise} \end{cases} \\ &= g\begin{cases} g^{b_1}v_{p^{n-1}}, & \text{if } b_{i+1} = b_1+p^{n-1}i \text{ for all } 0 \leqslant i < p \\ 0, & \text{otherwise} \end{cases} \\ &= ge(w, (g^{b_1}v_{p^n})(g^{b_2}v_{p^n})\dots(g^{b_p}v_{p^n})). \end{aligned}$$

For $b_p = p^n - 1$, if $b_{p-1} = b_p$ then

$$e(gw, g(g^{b_1}v_{p^n})(g^{b_2}v_{p^n})\dots(g^{b_p}v_{p^n})) = 0 = ge(w, (g^{b_1}v_{p^n})(g^{b_2}v_{p^n})\dots(g^{b_p}v_{p^n})).$$

Otherwise $b_{p-1} < p^n - 1$ and using $g^{p^{n-1}}v_{p^{n-1}} = v_{p^{n-1}}$ we obtain

$$\begin{aligned} &e(gw, g((g^{b_1}v_{p^n})(g^{b_2}v_{p^n})\dots(g^{b_{p-1}}v_{p^n})(g^{p^n-1}v_{p^n}))) \\ &\quad= e(w, v_{p^n}(g^{b_1+1}v_{p^n})(g^{b_2+1}v_{p^n})\dots(g^{b_{p-1}+1}v_{p^n})) \\ &\quad= \begin{cases} g^0 v_{p^{n-1}}, & \text{if } b_i+1 = p^{n-1}i \text{ for all } 0 < i < p \\ 0, & \text{otherwise} \end{cases} \\ &\quad= g\begin{cases} g^{p^{n-1}-1}v_{p^{n-1}}, & \text{if } b_i = p^{n-1}i-1 \text{ for all } 0 < i < p \\ 0, & \text{otherwise} \end{cases} \\ &\quad= g\begin{cases} g^{b_1}v_{p^{n-1}}, & \text{if } b_{i+1} = b_1+p^{n-1}i \text{ for all } 0 \leqslant i < p \\ 0, & \text{otherwise} \end{cases} \\ &\quad= ge(w, (g^{b_1}v_{p^n})(g^{b_2}v_{p^n})\dots(g^{b_p}v_{p^n})) \end{aligned}$$

thus e is a kG homomorphism.

Let $P^p(V_{p^n}) = \operatorname{coker} f$. Note that f is split as $e \circ f = \mathrm{id}$, therefore $V_{p^{n-1}} \oplus P^p(V_{p^n}) \cong V_{p^n} \oplus S^p(V_{p^n})$. But $S^p(V_{p^n}) \cong_{proj} V_{p^{n-1}}$ by lemma 2.7 and restriction, so $V_{p^{n-1}} \oplus P^p(V_{p^n}) \cong_{proj} V_{p^{n-1}}$. Hence $P^p(V_{p^n}) \cong_{proj} 0$, i.e. $P^p(V_{p^n})$ is projective.

Let f_1 be the quotient map from $V_{p^n} \oplus S^p(V_{p^n})$ to $P^p(V_{p^n})$. Consider the complex

$$0 \overset{d_{p+1}}{\to} \Lambda^p(V_{p^n-a}) \overset{d_p}{\to} \Lambda^{p-1}(V_{p^n-a}) \otimes V_{p^n} \overset{d_{p-1}}{\to} \dots \overset{d_2}{\to} V_{p^n-a} \otimes S^{p-1}(V_{p^n}) \overset{f_1\circ(0,d_1)}{\to} P^p(V_{p^n}),$$

where the d_i are the maps from $K^p(V_{p^n}, V_{p^n-a})$. This is exact at j for $1 < j \leqslant p$ as

$K^p(V_{p^n}, V_{p^n-a})$ is exact at j. Also $\ker(f_1 \circ d_1) = d_1^{-1}(\ker(f_1)) = d_1^{-1}(\ker(f_1) \cap \operatorname{Im}(d_1))$; but $\ker(f_1) \cap \operatorname{Im}(d_1) = 0$, so $\ker(f_1 \circ d_1) = d_1^{-1}(0) = \ker(d_1)$, therefore the complex is exact at 1. Also all but one of the terms are projective by lemma 2.7; this forces separability.

Therefore by lemma 2.20

$$\Omega_{p^n}^{-p}\Lambda^p(V_{p^n-a}) \cong_{proj} \frac{P^p(V_{p^n})}{\operatorname{Im}(f_1 \circ (0, d_1))}. \tag{4.1}$$

Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & V_{p^n-a} \otimes S^{p-1}(V_{p^n}) & & & & \\
 & & & & \downarrow (0, d_1) & & & & \\
0 & \to & V_{p^{n-1}} & \xrightarrow{f} & V_{p^n} \oplus S^p(V_{p^n}) & \xrightarrow{f_1} & P^p(V_{p^n}) & \to & 0 \\
 & & \downarrow \mathrm{id} & & \downarrow q_2 & & \downarrow q & & \\
0 & \to & V_{p^{n-1}} & \xrightarrow{f'} & V_{p^n} \oplus S^p(V_a) & \xrightarrow{f_1'} & H_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $q_2 = (\mathrm{id}, S^p(e))$ for e a natural homomorphism from V_{p^n} to V_a where $e(v_{p^n}) = v_a$, $f'(v_{p^{n-1}}) = (v_{p^{n-1}}, v_a(hv_a)...(h^{p-1}v_a))$ and $H_0 = \operatorname{coker} f'$. As
$q_2(f(v_{p^{n-1}})) = q_2(v_{p^{n-1}}, v_{p^n}(hv_{p^n})...(h^{p-1}v_{p^n})) = (v_{p^{n-1}}, v_a(hv_a)...(h^{p-1}v_a)) = f'(v_{p^{n-1}})$, applying lemma 2.28 shows that there exists q such that the diagram commutes.

Let $v \in \ker\, q$. Then there exists $v' \in V_{p^n} \oplus S^p(V_{p^n})$ such that $f_1(v') = v$. As $f_1'(q_2(v')) = q(f_1(v')) = q(v) = 0$, there exists $v'' \in V_{p^{n-1}}$ such that $f'(v'') = q_2(v')$. But $f_1(v' - f(v'')) = v$ and $q_2(v' - f(v'')) = 0$, so $v' - f(v'') \in \ker\, q_2 = \operatorname{Im}\, d_1$. Therefore $v \in \operatorname{Im}(f_1 \circ d_1)$ and $\ker\, q \subseteq \operatorname{Im}(f_1 \circ d_1)$. Also $q \circ f_1 \circ d_1 = f_1' \circ q_2 \circ d_1 = 0$, so $\operatorname{Im}(f_1 \circ d_1) \subseteq \ker\, q$. Therefore $\operatorname{Im}(f_1 \circ d_1) = \ker\, q$ so $H_0 \cong \frac{P^p(V_{p^n})}{\operatorname{Im}(f_1 \circ d_1)}$.

As V_a is fixed by h, $f'(v_{p^{n-1}}) = (v_{p^{n-1}}, v_a^p)$.

Consider

$$f'(v_{p^{n-1}-i}) = (g-1)^i f'(v_{p^{n-1}}) = ((g-1)^i v_{p^{n-1}}, (g-1)^i v_a^p).$$

If $i \geqslant a$ then $(g-1)^i v_a^p = 0$, hence $f'(v_{p^{n-1}-i}) = (v_{p^{n-1}-i}, 0)$ so $V_{p^{n-1}-a} \oplus 0 \subset \operatorname{Im}(f')$.

Also

$$(g-1)^{p^{n-1}} S^p(V_a) = 0,\ (g-1)^{p^n - p^{n-1} + a} V_{p^n} = V_{p^{n-1}-a}.$$

So as $p^n - p^{n-1} \geqslant p^{n-1}$,

$$(g-1)^{p^n-p^{n-1}+a}(V_{p^n} \oplus S^p(V_a)) = V_{p^{n-1}-a} \oplus 0 \subset \mathrm{Im}(f').$$

Hence

$$(g-1)^{p^n-p^{n-1}+a} H_0 = 0.$$

Therefore H_0 can be considered as a $R = k[X]/(X^{p^n-p^{n-1}+a})$ module, where $Xv = (g-1)v$ for $v \in H_0$.

Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V_{p^{n-1}-a} & \overset{k_1}{\to} & V_{p^n} & \to & V_{p^n-p^{n-1}+a} & \to & 0 \\
 & & \downarrow j_1 & & \downarrow i_1 & & \downarrow & & \\
0 & \to & V_{p^{n-1}} & \overset{f'}{\to} & V_{p^n} \oplus S^p(V_a) & \to & H_0 & \to & 0 \\
 & & \downarrow j_2 & & \downarrow i_2 & & \downarrow & & \\
0 & \to & V_a^{(p)} & \overset{\mathrm{id}}{\to} & S^p(V_a) & \to & \frac{S^p(V_a)}{V_a^{(p)}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where $j_1(v_i) = v_i$, $j_2(v_i) = v^p_{i-(p^{n-1}-a)}$, $k_1(v_i) = v_i$, $i_1(v_i) = (v_i, 0)$ and $i_2(v,w) = w$. The left hand side of this diagram commutes as
$f'(j_1(v_i)) = (v_i, 0) = i_1(k_1(v_i))$, $i_2(f'(v_{p^{n-1}})) = v_a^p = j_2(v_{p^{n-1}})$ and so lemma 2.28 allows us to fill the right hand side to commute and be exact. From this diagram we get the short exact sequence

$$0 \to V_{p^n-p^{n-1}+a} \to H_0 \to \frac{S^p(V_a)}{V_a^{(p)}} \to 0$$

which also is a R module complex. But $V_{p^n-p^{n-1}+a}$ is projective and injective as a R module, so

$$H_0 \cong V_{p^n-p^{n-1}+a} \oplus \frac{S^p(V_a)}{V_a^{(p)}}.$$

Substituting in equation 4.1 for H_0 gives

$$\Lambda^p(V_{p^n-a}) \cong_{proj} \Omega^p_{p^n}(V_{p^n-p^{n-1}+a} \oplus \frac{S^p(V_a)}{V_a^{(p)}}).$$

□

Example 4.2. For $p = 5, a = 2, n = 2$

$$\frac{S^5(V_2)}{V_2^{(5)}} = \frac{T^5(V_2)}{V_1^{(5)}} \cong \Omega_5 V_1 \oplus bV_5 \cong V_4 \oplus bV_5.$$

Checking dimensions gives $b = 0$ so

$$\frac{S^5(V_2)}{V_2^{(5)}} \cong V_4$$

therefore

$$\Lambda^5(V_{23}) = \Lambda^5(V_{5^2-2}) \cong_{proj} V_{5^1-2} + \Omega_{25} V_4 \cong_{proj} V_3 + V_{21}$$

so $\Lambda^5(V_{23}) \cong V_3 + V_{21} + 1345V_{25}$.

Lemma 4.3. *Let P be a projective kC_{p^n} module, $v \in P$ and $0 \leqslant b \leqslant p^n$. If $(g-1)^b v = 0$ then there exists $m \in P$ such that $(g-1)^{p^n-b} m = v$.*

Proof. If $P = V_{p^n}$ then $v = \sum_{i=1}^{p^n} a_i v_i$. So $(g-1)^b v = \sum_{i=1+b}^{p^n} a_i v_{i-b} = 0$ therefore $a_i = 0$ for $i > b$ i.e. $v = \sum_{i=1}^{b} a_i v_i$. Let $m = \sum_{i=1}^{b} a_i v_{i+p^n-b}$ then $(g-1)^{p^n-b} m = v$.
For general P:
We have $P \cong cV_{p^n}$ so $v = (w_1, w_2, \ldots, w_c)$ where $w_j \in V_{p^n}$. As the copies of V_{p^n} in P are disjoint, $(g-1)^b w_j = 0$ so there exists $m_j \in V_{p^n}$ such that $(g-1)^{p^n-b} m_j = w_j$. So let $m = (m_1, m_2, \ldots, m_c)$ then $(g-1)^{p^n-b} m = v$. □

Lemma 4.4. *For $p^{n-1} < a \leqslant p^n, n \geqslant 2$
the inclusion $V_{a-p^{n-1}}^{(p)} \to S^p(V_a)$ is separated over $G = C_{p^n}$.*

Proof. For $a = p^n$, $V_{a-p^{n-1}}^{(p)} \subset V_{p^n}^{(p)} \subset S^p(V_a)$ so $V_{a-p^{n-1}}^{(p)} \to S^p(V_a)$ is separated.
For $a < p^n$:
Let $q = p^{n-1}$ and $h = g^q$. Write $a = q(p-b) + c,\ 0 \leqslant c < q$, so $0 < b < p$.
Let $V'_p = \operatorname{span}_k\{v'_i \mid 1 \leqslant i \leqslant p\}$ be the indecomposable projective $k\langle h\rangle$-module where $hv'_i = v'_i + v'_{i-1}$. Let $V'_b = \operatorname{span}_k\{v'_i \mid 1 \leqslant i \leqslant b\}$.

Consider $K^p(V'_p, V'_b)$:

$$\Lambda^b(V'_b) \otimes S^{p-b}(V'_p) \xrightarrow{d_b} \ldots \xrightarrow{d_3} \Lambda^2(V'_b) \otimes S^{p-2}(V'_p) \xrightarrow{d_2} \Lambda^1(V'_b) \otimes S^{p-1}(V'_p) \xrightarrow{d_1} S^p(V'_p).$$

As all the terms left of d_1 are projective by lemma 2.7 and also are exact,
$d_1(S^{p-1}(V'_p) \otimes V'_b)$ is projective over $\langle h\rangle$.
As $(h-1)^b v_b'^p = 0$, by lemma 4.3 there exists $m \in d_1(S^{p-1}(V'_p) \otimes V'_b)$ such that $(h-1)^{p-b} m = v_b'^p$.

Define $f : V'_p \to V_{pq} \downarrow_{C_p}$ by $f(v'_i) = v_{qi-c}$.
Note that $(h-1)^{p-b} S^p(f)(m) = S^p(f)(v'^p_b) = v^p_{qb-c}$. Also using d_1 from $K(*,*)$,

$$S^p(f)(d_1(S^{p-1}(V'_p) \otimes V'_b)) \subset d_1(S^{p-1}(V_{pq}) \otimes V_{qb-c}).$$

Let $l = qb - c$, $v = (g-1)^{q-c} S^p(f)(m)$. Then

$$\begin{aligned}(g-1)^{pq-l-q} v &= (g-1)^{pq-(qb-c)-q}(g-1)^{q-c} S^p(f)(m) = (g-1)^{pq-qb} S^p(f)(m) \\ &= (h-1)^{p-b} S^p(f)(m) = v^p_{qb-c}.\end{aligned}$$

So there exists $v \in d_1(S^{p-1}(V_{pq}) \otimes V_l)$ such that $(g-1)^{pq-l-q} v = v^p_l$.
Consider

$$v^p_{(p-1)q} - v \in d_1(S^{p-1}(V_{pq}) \otimes V_{(p-1)q}) \subset S^p(V_{pq}).$$

Then

$$(g-1)^{pq-l-q}(v^p_{(p-1)q} - v) = v^p_{(p-1)q-(pq-l-q)} - v^p_l = 0.$$

By lemma 2.23, $d_1(S^{p-1}(V_{pq}) \otimes V_{(p-1)q}) \to S^p(V_{pq})$ factors through a projective so by lemma 4.3 there exists $w \in S^p(V_{pq})$ such that $(g-1)^{l+q} w = v^p_{(p-1)q} - v$.
Let $e : V_{pq} \to V_{pq-l}$, $e(v_i) = v_{i-l}$, so $S^p(e) S^p(V_{pq}) = S^p(V_{pq-l})$
and $S^p(e) d(S^{p-1}(V_{pq}) \otimes V_l) = 0$. Consider

$$\bar{w} = S^p(e)(w) \in S^p(V_{pq-l}).$$

We have

$$(g-1)^{l+q} \bar{w} = v^p_{(p-1)q-l} - 0.$$

Also $(g-1)^{pq-1}\bar{w} = (g-1)^{(p-1)q-l-1}(g-1)^{l+q}\bar{w} = (g-1)^{(p-1)q-l-1} v^p_{(p-1)q-l} = v^p_1 \neq 0$, so $\bar{w}$ generates a projective module W. Also $pq - l = pq - (qb-c) = q(p-b) + c = a$ and $(p-1)q - l = a - q$ so $V^{(p)}_{a-p^{n-1}} \subset W \subset S^p(V_a)$. □

Lemma 4.5. *For $G \cong C_p$, $1 \leqslant b \leqslant p$, $r \in \mathbb{N}$; $K^r(V_b, V^{(p)}_{b-1})$ is separated.*

Proof. Let x, a and T be defined as in lemma 2.9. Consider the complex $K'^r(V_b, V^{(p)}_{b-1})$:

$$\cdots \to \Lambda^2(V^{(p)}_{b-1}) \otimes T^{r-2p}(V_b) \to \Lambda^1(V^{(p)}_{b-1}) \otimes T^{r-p}(V_b) \to T^r(V_b)$$

which is well defined as $x \notin V_{b-1}$ and exact at j for $j > 0$. For $r \geqslant p$, $T^r(V_b)$ is projective therefore all but one of the terms in $K'^r(V_b, V^{(p)}_{b-1})$ are projective so $K'^r(V_b, V^{(p)}_{b-1})$ is separated. But

$$K^r(V_b, V^{(p)}_{b-1}) \cong \sum_i a^i K'^{(r-pi)}(V_b, V^{(p)}_{b-1})$$

so $K^r(V_b, V^{(p)}_{b-1})$ is separated. □

Definition 4.6. For a module V, define

$$\breve{S}(V) = \frac{S(V)}{V^{(p)}}$$

and for $p^{n-1} < i \leqslant p^n$,

$$\widetilde{S}(V_i) = \frac{S(V_i)}{V_{i-p^{n-1}}^{(p)}}.$$

For $r \in \mathbb{N}$, if $p^{m-1} < r < p^m$, consider the partition of the first r natural numbers into two blocks, the first consisting of the first p^{m-1} numbers. If $r = p^m$ consider the partition of the first r natural numbers into p blocks each consisting of p^{m-1} consecutive numbers.

Definition 4.7. The subgroup Q_r of the symmetric group Σ_r is defined by

$$Q_r := \begin{cases} \Sigma_{\{1,\dots,p^{m-1}\}} \times \Sigma_{\{p^{m-1}+1,\dots,r\}}, & \text{if } p^{m-1} < r < p^m \\ \Sigma_{\{1,\dots,p^{m-1}\}} \times \cdots \times \Sigma_{\{(p-1)p^{m-1}+1,\dots,p^m\}} \rtimes B, & \text{if } r = p^m \end{cases}$$

where $\Sigma_I = \{\pi \in \Sigma_r \mid \pi(i) = i \text{ for all } i \notin I\}$, $B \cong \Sigma_p$ permutes the blocks preserving order within them.

Lemma 4.8. *For $r > p$ the p-cycles in Q_r are in*

$$\begin{cases} \Sigma_{\{1,\dots,p^{m-1}\}} \cup \Sigma_{\{p^{m-1}+1,\dots,r\}}, & \text{if } p^{m-1} < r < p^m \\ \Sigma_{\{1,\dots,p^{m-1}\}} \cup \cdots \cup \Sigma_{\{(p-1)p^{m-1}+1,\dots,p^m\}}, & \text{if } r = p^m \end{cases}.$$

Proof. For $r = p^m$, let $a \in Q_r$ be a p-cycle. Then $a = b\rho_1\rho_2\dots\rho_p$ where $\rho_i \in \Sigma_{\{(i-1)p^{m-1}+1,\dots,ip^{m-1}\}}$ and $b \in B$. Consider $cp+d \in \{1, 2, \dots, r\}$ where $1 \leqslant d \leqslant p$. Then $a(cp+d) = b(\rho_c(cp+d)) = b(cp)+(\rho_c(cp+d)-cp)$ and $1 \leqslant (\rho_c(cp+d)-cp) \leqslant p$. If $a(cp+d) = cp+d$ then $b(cp) = cp$ so $b(cp+d) = b(cp)+d = cp+d$ therefore fixed points of a are fixed points of b. Since a has $p^m - p$ fixed points, b has at least $p^m - p$ fixed points. As $r > p$, $p^m > p$ we have $p^m-p > (p-2)p^{m-1}$. The only such element in B is 1 so $b = 1$ therefore $a \in \Sigma_{\{1,\dots,p^{m-1}\}} \times \cdots \times \Sigma_{\{(p-1)p^{m-1}+1,\dots,p^m\}}$. But $\Sigma_{\{(i-1)p^{m-1}+1,\dots,ip^{m-1}\}}$ are disjoint, so as a is a cycle, $a \in \Sigma_{\{1,\dots,p^{m-1}\}} \cup \cdots \cup \Sigma_{\{(p-1)p^{m-1}+1,\dots,p^m\}}$.

For $p^{m-1} < r < p^m$, $\Sigma_{\{1,\dots,p^{m-1}\}}$ and $\Sigma_{\{p^{m-1}+1,\dots,r\}}$ are disjoint so the cycles in Q_r are in $\Sigma_{\{1,\dots,p^{m-1}\}} \cup \Sigma_{\{p^{m-1}+1,\dots,r\}}$. □

Lemma 4.9.

$$p \nmid |\Sigma_r : Q_r|.$$

Proof. From lemma 2.29 the largest power of p which divides $|\Sigma_r|$ is $p^{\nu(r)}$.
For $p^{m-1} < r < p^m$, the largest power of p which divides $|Q_r|$ is $p^{\nu(r-p^{m-1})+\nu(p^{m-1})}$ but $\nu(r) - \nu(r - p^{m-1}) = \frac{(p^{m-1}-1)}{(p-1)} = \nu(p^{m-1})$ so the largest power of p which divides $|\Sigma_r : Q_r|$ is $p^{\nu(r)-(\nu(r-p^{m-1})+\nu(p^{m-1}))} = p^0$ therefore $p \nmid |\Sigma_r : Q_r|$.
For $r = p^m$, the largest power of p which divides $|Q_r|$ is $p^{p\nu(p^{m-1})+\nu(p)}$ but $p\nu(p^{m-1}) + \nu(p) = p\frac{(p^{m-1}-1)}{(p-1)} + 1 = \frac{(p^m-p)+(p-1)}{(p-1)} = \frac{(p^m-1)}{(p-1)} = \nu(p^m)$ so the largest power of p which divides $|\Sigma_r : Q_r|$ is $p^{\nu(p^m)-(p\nu(p^{m-1})+\nu(p))} = p^0$ therefore $p \nmid |\Sigma_r : Q_r|$. □

Definition 4.10.

$$\text{tr} : V^{\otimes r}/\Sigma_r \to V^{\otimes r}/Q_r$$

$$\text{tr } x = \frac{1}{|\Sigma_r : Q_r|} \sum_{\pi \in Q_r \backslash \Sigma_r} \pi x$$

where V/H is the coinvariants of V over H.

We can view $S^r(V) = V^{\otimes r}/\Sigma_r$. Let $L^r(V) = V^{\otimes r}/Q_r$. Then

$$L^r(V) \cong \begin{cases} S^{p^{m-1}}(V) \otimes S^{r-p^{m-1}}(V), & \text{if } p^{m-1} < r < p^m \\ S^p(S^{p^{m-1}}(V)) & \text{if } r = p^m. \end{cases}$$

Let d be the natural map from $S^*(V) \otimes S^*(V)$ to $S^*(V)$, let W be a submodule of V. Let

$$X_W^r(V) = \begin{cases} d(W^{(p)} \otimes S^{p^{m-1}-p}(V)) \otimes S^{r-p^{m-1}}(V) & \\ \qquad + S^{p^{m-1}}(V) \otimes d(W^{(p)} \otimes S^{r-p^{m-1}-p}(V)), & \text{if } p^{m-1} < r < p^m \\ d(d(W^{(p)} \otimes S^{p^{m-1}-p}(V)) \otimes S^{p-1}(S^{p^{m-1}}(V))) & \text{if } r = p^m. \end{cases}$$

a submodule of $L^r(V)$. Also

$$L^r(V_a)/X^r_{V_{a-p^{n-1}}}(V_a) \cong \begin{cases} \widetilde{S}^{p^{m-1}}(V_a) \otimes \widetilde{S}^{r-p^{m-1}}(V_a), & \text{if } p^{m-1} < r < p^m \\ S^p(\widetilde{S}^{p^{m-1}}(V_a)) & \text{if } r = p^m. \end{cases}$$

Lemma 4.11. *For $r > p$,*

$$\text{tr } d(W^{(p)} \otimes S^{r-p}(V)) \subset X_W^r(V).$$

Proof. We lift the domain of tr to $V^{\otimes r}$. Let $w \in W, s_1 s_2 ... s_{r-p} \in S^{r-p}(V)$, let $x = w^{\otimes p} \otimes s_1 \otimes \cdots \otimes s_{r-p}$ and let $\alpha = (12...p)$; then $\alpha x = x$.

Let $K = \langle \alpha \rangle$; then

$$\text{tr } x = \frac{1}{|\Sigma_r : Q_r|} \sum_{\pi \in Q_r \backslash \Sigma_r} \pi x = \frac{1}{|\Sigma_r : Q_r|} \sum_{\tau \in Q_r \backslash \Sigma_r / K} \quad \sum_{\sigma \in (\tau^{-1} Q_r \tau \cap K) \backslash K} \tau \sigma x$$

$$= \frac{1}{|\Sigma_r : Q_r|} \sum_{\tau \in Q_r \backslash \Sigma_r / K} |(\tau^{-1} Q_r \tau \cap K) \backslash K| \tau x.$$

Now $|(\tau^{-1}Q_r\tau \cap K)\backslash K| \in \{1, p\}$, but $p = 0$ in k so the only terms that need to be considered are the $|(\tau^{-1}Q_r\tau \cap K)\backslash K| = 1$ terms. For $|(\tau^{-1}Q_r\tau \cap K)\backslash K| = 1$

$$K \subset \tau^{-1} Q_r \tau$$

$$\tau \alpha \tau^{-1} \in Q_r.$$

But $\tau\alpha\tau^{-1}$ is a p cycle and from lemma 4.8 all the p cycles in Q_r fix the elements of all but one block. Therefore, as $\tau\alpha\tau^{-1}$ permutes the copies of w in τx, there are p copies of w in one block so $\tau x \in X^r_W(V)$ hence

$$\text{tr } x \in X^r_W(V).$$

□

Theorem 4.12. *For $p^{n-1} \leqslant a \leqslant p^n$, $r, n \in \mathbb{N}$:*
$K^r(V_a, V^{(p)}_{a-p^{n-1}})$ is separated over $G = C_{p^n}$.

Proof. For $n = 1$ the theorem is true by lemma 4.5.
For $n > 1$ assume the theorem for $H = C_{p^{n-1}}$. As

$$K^r(V_a, V^{(p)}_{a-p^{n-1}}) \downarrow_H = \bigoplus_{\sum j_i = r} \bigotimes_{i=1}^{p} K^{j_i}(V_{a_i}, V^{(p)}_{a_i - p^{n-2}}),$$

where $V_a \downarrow_H = \bigoplus_{i=1}^p V_{a_i}$, $K^r(V_a, V^{(p)}_{a-p^{n-1}})$ is separated over H.

For $a = p^{n-1}$, $K^r(V_a, V^{(p)}_{a-p^{n-1}})$ has only one non-zero term so it is separated. Thus we assume $a > p^{n-1}$.
For $i < p$, $K^i(V_a, V^{(p)}_{a-p^{n-1}})$ has only one nonzero term therefore is separated.
$K^p(V_a, V^{(p)}_{a-p^{n-1}})$ is separated over G by lemma 4.4.
Let $V = V_a,\ W = V^{(p)}_{a-p^{n-1}}$.
Let $r > p$ and assume that for all $i < r$, $K^i(V, W)$ is separated over G. So the inclusion $\text{Im}(d_{1,i}) \to S^i(V)$ factors through a projective P^i.

Case 1, separated at j for $j > 0$:
Consider

$$d_{j+1,r} : S^{r-pj-p}(V) \otimes \Lambda^{j+1}(W) \to S^{r-pj}(V) \otimes \Lambda^j(W).$$

The inclusion $\mathrm{Im}(d_{j+1,r}) \to S^{r-pj}(V) \otimes \Lambda^j(W)$ is the composition of inclusions

$$\mathrm{Im}(d_{j+1,r}) \to \mathrm{Im}(d_{1,r-pj}) \otimes \Lambda^j(W) \to S^{r-pj}(V) \otimes \Lambda^j(W),$$

where the last map factors through the projective $P^{r-pj} \otimes \Lambda^j(W)$.

Therefore $K^r(V_a, V^{(p)}_{a-p^{n-1}})$ is separated at j.

Case 2, separated at 0:
$\widetilde{S}(V_a)$ is the degree 0 homology of $K(V_a, V^{(p)}_{a-p^{n-1}})$.
Let $f_{i,1}$ be the quotient map from $S^i(V_a)$ to $\widetilde{S}^i(V_a)$ and $f_{i,2}$ a projection map from $S^i(V_a)$ to P^i. Let $f_i : S^i(V_a) \to \widetilde{S}^i(V_a) \oplus P^i,\ f_i = f_{i,1} \oplus f_{i,2}$.
Let $M_i = \ker(f_{i,2})$; then $S^i(V_a) = M_i \oplus P^i$. Also $\ker(f_{i,1}) = \mathrm{Im}(d_{1,i}) \subset P^i$ so $\widetilde{S}(V_a) = f_{i,1}(M_i) \oplus f_{i,1}(P^i)$.
Let $u_i : f_{i,1}(M_i) \oplus f_{i,1}(P^i) \to S^i(V_a)$ be given by $u_i(f_{i,1}(m) \oplus f_{i,1}(x)) = m$ and let u'_i be the inclusion of P^i in $S^i(V_a)$ so $(u_i \oplus u'_i) \circ f_i = \mathrm{id}$. Therefore $S^i(V_a) \xrightarrow{f_i} \widetilde{S}^i(V_a) \oplus P^i$ is split injective.

Let $\widetilde{L}^i(V_a) = L^i(V_a)/X^i_{V_{a-p^{n-1}}}(V_a)$ so

$$\widetilde{L}^r(V) \cong \begin{cases} \widetilde{S}^{p^{m-1}}(V) \otimes \widetilde{S}^{r-p^{m-1}}(V) & \text{if } p^{m-1} < r < p^m \\ S^p(\widetilde{S}^{p^{m-1}}(V)) & \text{if } r = p^m. \end{cases}$$

For $p^{m-1} < r < p^m$, the map

$$\begin{aligned} f_{p^{m-1}} \otimes f_{r-p^{m-1}} : L^r(V) &\to (\widetilde{S}^{p^{m-1}}(V_a) \oplus P^{p^{m-1}}) \otimes (\widetilde{S}^{r-p^{m-1}}(V_a) \oplus P^{r-p^{m-1}}) \\ &\cong \widetilde{L}^r(V) \oplus (\widetilde{S}^{p^{m-1}}(V) \otimes P^{r-p^{m-1}}) \oplus (P^{p^{m-1}} \otimes \widetilde{S}^{r-p^{m-1}}(V)) \\ &\quad \oplus (P^{p^{m-1}} \otimes P^{r-p^{m-1}}) \end{aligned}$$

is split injective, therefore

$$f' : L^r(V) \to \widetilde{L}^r(V), \qquad f' = f_{p^{m-1},1} \otimes f_{r-p^{m-1},1}$$

is split injective modulo induced summands.
For $r = p^m$

$$S^p(f_{p^{m-1}}) : L^r(V) \to S^p(\widetilde{S}^{p^{m-1}}(V_a) \oplus P^{p^{m-1}})$$

is split injective but

$$S^p(\widetilde{S}^{p^{m-1}}(V_a) \oplus P^{p^{m-1}}) = \widetilde{L}^r(V) \oplus \bigoplus_{j=1}^{p} S^j(P^{p^{m-1}}) \otimes S^{p-j}(\widetilde{S}^{p^{m-1}}(V_a))$$

and $S^j(P^{p^{m-1}})$ is induced as $n > 1$. Therefore

$$f' : L^r(V) \to \widetilde{L}^r(V), \qquad f' = S^p(f_{p^{m-1},1})$$

is split injective modulo induced summands.

So for all r

$$f' : L^r(V) \to \widetilde{L}^r(V)$$

is split injective modulo induced summands. By lemma 4.11, tr : $\widetilde{S}^r(V_a) \to \widetilde{L}^r(V_a)$ is well defined.

So

$$\mathrm{tr} \circ f_{r,1} = f' \circ \mathrm{tr}$$

is split injective modulo induced summands therefore $f_{r,1}$ is split injective modulo induced summands .

Let $A' \oplus A'' = S^r(V_a)$, where A' has only non-induced summands and A'' is induced. Let $f_r : S^r(V_a) \to \widetilde{S}^r(V_a) \oplus A''$, $f_r = f_{r,1} + i''$ where $i''(a' + a'') = a''$ for $a' \in A'$ and $a'' \in A''$. By lemma 2.26, $A' \overset{f_{r,1}}{\to} \widetilde{S}^r(V_a)$ is split injective so there exists a map l' such that $l' \circ f_{r,1} = \mathrm{id}_{A'}$.

Let $d_1' : \mathrm{Im}(d_1) \to S^r(V_a)$. Let $l = (l', \mathrm{id}_{A''})$ so $l \circ f_r = \mathrm{id}_{S^r(V_a)}$ then $d_1' = l \circ f_r \circ d_1' = l \circ (f_r - f_{r,1}) \circ d_1' = l \circ i'' \circ d_1'$ as $f_{r,1} \circ d_1' = 0$. Also d_1' is separated on restriction to H so $i'' \circ d_1'$ is separated on restriction to H but A'' is induced so $i'' \circ d_1'$ is separated over G therefore d_1' is separated over G.

So $K^r(V_a, V^{(p)}_{a-p^{n-1}})$ is separated.

Therefore by induction on r then n, $K^r(V_a, V^{(p)}_{a-p^{n-1}})$ is separated. □

Theorem 4.13. *For $p^{n-1} < a \leqslant p^n, r \geqslant 0$,*

$$\widetilde{S}^r(V_a) \cong \sum_{i \geqslant 0, pi \leqslant r} S^i(V^{(p)}_{p^{n-1}}) \otimes \check{S}^{r-pi}(V_a).$$

Proof. Consider $\{y_0, ..., y_{p^{n-1}-1}, x_{p^{n-1}}, ..., x_{a-1}\}$ in V_a where $y_i = g^i v_a$, $x_i = (g-1)^i v_a$. Then $y_i = v_a + i v_{a-1} + \cdots + v_{a-i}$ and $x_i = v_{a-i}$, therefore $\{y_0, ..., y_{p^{n-1}-1}, x_{p^{n-1}}, ..., x_{a-1}\}$ is linearly independent and so form a basis for V_a. Also $V^{(p)}_{p^{n-1}}$ has a basis $\{\bar{y}_0^p, ..., \bar{y}^p_{p^{n-1}-1}\}$ where $\bar{y}_i = g^i \bar{y}_0 = g^i v_{p^{n-1}}$.

Let

$$f : \widetilde{S}^r(V_a) \to \sum_{i=0}^{r'} S^i(V_{p^{n-1}}^{(p)}) \otimes \breve{S}^{r-pi}(V_a)$$

be the linear map such that

$$f\left(\left(\prod_{i=0}^{p^{n-1}-1} y_i^{pr_i+s_i}\right) w\right) = \left(\prod_{i=0}^{p^{n-1}-1} \bar{y}_i^{pr_i}\right) \otimes \left(\prod_{i=0}^{p^{n-1}-1} y_i^{s_i}\right) w$$

where w is a monomial in $\{x_{p^{n-1}}, ..., x_{a-1}\}$, $0 \leqslant s_i < p$, $\sum(pr_i + s_i) + \deg(w) = r$. As any monomial in $\widetilde{S}^r(V_a)$ can be written in the form $\left(\prod_{i=0}^{p^{n-1}-1} y_i^{pr_i+s_i}\right) w$ and any monomial in $S^i(V_{p^{n-1}}^{(p)}) \otimes \breve{S}^{r-pi}(V_a)$ can be written in the form $\left(\prod_{i=0}^{p^{n-1}-1} \bar{y}_i^{pr_i}\right) \otimes \left(\prod_{i=0}^{p^{n-1}-1} y_i^{s_i}\right) w$, we see that f is a bijection.

As $y_i = g^i v_a$, $\quad gy_{p^{n-1}-1} = g^{p^{n-1}} v_a = v_a + x_{p^{n-1}} = y_0 + x_{p^{n-1}}$ and $g\bar{y}_{p^{n-1}-1} = \bar{y}_0$ we have

$$g\left(\left(\prod_{i=0}^{p^{n-1}-1} \bar{y}_i^{pr_i}\right) \otimes \left(\prod_{i=0}^{p^{n-1}-1} y_i^{s}\right) w\right) = \left(\prod_{i=0}^{p^{n-1}-2} \bar{y}_{i+1}^{pr_i}\right) \bar{y}_0^{pr_{p^{n-1}-1}} \otimes g\left(\prod_{i=0}^{p^{n-1}-1} y_i^{s_i}\right) w.$$

Let $r_{top} = r_{p^{n-1}-1}, s_{top} = s_{p^{n-1}-1}$. In $\widetilde{S}^r(V_a)$, $x_{p^{n-1}}^p = 0$, therefore

$$gy_{p^{n-1}-1}^{pr_{top}+s_{top}} = (gy_{p^{n-1}-1}^{pr_{top}})(gy_{p^{n-1}-1}^{s_{top}}) = ((y_0 + x_{p^{n-1}})^p)^{r_{top}}(gy_{p^{n-1}-1}^{s_{top}}) = y_0^{pr_{top}}(gy_{p^{n-1}-1}^{s_{top}})$$

and for $i < p^{n-1} - 1$

$$gy_i^{pr_i+s_i} = y_{i+1}^{pr_i+s_i} = y_{i+1}^{pr_i}(gy_i^{s_i}).$$

Hence

$$g\left(\left(\prod_{i=0}^{p^{n-1}-1} y_i^{pr_i+s_i}\right) w\right) = \left(\prod_{i=0}^{p^{n-1}-2} y_{i+1}^{pr_i}\right) y_0^{pr_{p^{n-1}-1}} g\left(\prod_{i=0}^{p^{n-1}-1} y_i^{s_i}\right) w.$$

As $gy_i^j = y_{i+1}^j$ for $i < p^{n-1} - 1$, $gy_{p^{n-1}-1}^j = (y_0 + x_{p^{n-1}})^j = \sum_c \binom{j}{c} y_0^c x_{p^{n-1}}^{j-c}$, and $gw = \sum_d w_d$, where the w_d are monomials in $\{x_{p^{n-1}}, ..., x_a\}$, we obtain

$$g\left(\left(\prod_{i=0}^{p^{n-1}-1} y_i^{pr_i+s_i}\right) w\right) = \sum_{c,d} \binom{j}{c} y_0^{pr_{p^{n-1}-1}+c} \left(\prod_{i=0}^{p^{n-1}-2} y_{i+1}^{pr_i+s_i}\right) x_{p^{n-1}}^{j-c} w_d,$$

where $j = s_{p^{n-1}-1}$.

Therefore

$$f\left(g\left(\left(\prod_{i=0}^{p^{n-1}-1} y_i^{pr_i+s_i}\right)w\right)\right) = \sum_{c,d}\binom{j}{c}\bar{y}_0^{pr_{p^{n-1}-1}}\left(\prod_{i=0}^{p^{n-1}-2}\bar{y}_{i+1}^{pr_i}\right)\otimes y_0^c\left(\prod_{i=0}^{p^{n-1}-2} y_{i+1}^{s_i}\right)x_{p^{n-1}}^{j-c}w_d$$

$$= \left(\prod_{i=0}^{p^{n-1}-2}\bar{y}_{i+1}^{pr_i}\right)\bar{y}_0^{pr_{p^{n-1}-1}}\otimes g\left(\prod_{i=0}^{p^{n-1}-1} y_i^{s_i}\right)w$$

$$= g\left(\left(\prod_{i=0}^{p^{n-1}-1}\bar{y}_i^{pr_i}\right)\otimes\left(\prod_{i=0}^{p^{n-1}-1} y_i^{s}\right)w\right) = gf\left(\left(\prod_{i=0}^{p^{n-1}-1} y_i^{pr_i+s_i}\right)w\right).$$

Hence $fg = gf$, so f is a kG-homomorphism therefore, as f is a bijection, f is an isomorphism.

□

Corollary 4.14. *For $p^{n-1} < a \leqslant p^n$*

$$\widetilde{\sigma}_t(V_a) \cong \sigma_{t^p}(V_{p^{n-1}})\breve{\sigma}_t(V_a)$$

$$\breve{\sigma}_t(V_a) \cong_{ind} (1 - t^{p^n})\widetilde{\sigma}_t(V_a)$$

where

$$\widetilde{\sigma}_t(V) = 1 + \widetilde{S}^1(V)t + \widetilde{S}^2(V)t^2 + \cdots$$

$$\breve{\sigma}_t(V) = 1 + \breve{S}^1(V)t + \breve{S}^2(V)t^2 + \cdots$$

Proof. The t^r coefficients in

$$\widetilde{\sigma}_t(V_a) \cong \sigma_{t^p}(V_{p^{n-1}})\breve{\sigma}_t(V_a)$$

correspond with theorem 4.13 so the formula is equivalent to the theorem.

By lemma 2.7 $\sigma_t(V_{p^{n-1}}) \cong_{ind} 1 + t^{p^{n-1}} + t^{2p^{n-1}} + \cdots$ so $\sigma_{t^p}(V_{p^{n-1}}) \cong_{ind} 1 + t^{p^n} + t^{2p^n} + \cdots$ then $(1 - t^{p^n})\sigma_{t^p}(V_{p^{n-1}}) \cong_{ind} 1$ therefore

$$\breve{\sigma}_t(V_a) \cong_{ind} (1 - t^{p^n})\sigma_{t^p}(V_{p^{n-1}})\breve{\sigma}_t(V_a) \cong_{ind} (1 - t^{p^n})\widetilde{\sigma}_t(V_a).$$

□

Corollary 4.15. *For $p^{n-1} < a \leqslant p^n$,*

$$\breve{\sigma}_t(V_a)\cong_{ind}\lambda^{\Omega}_{t^p}(V_{a-p^{n-1}})\lambda^{\Omega}_t(V_{p^n-a})$$

where

$$\lambda^{\Omega}_t(V) = 1 + \Omega\Lambda^1(V)t + \Lambda^2(V)t^2 + \Omega\Lambda^3(V)t^3 + \cdots .$$

Proof. Applying lemma 2.20 to theorem 4.12 gives

$$\widetilde{S}^r(V_a)\cong_{ind} \sum_{i\geqslant 0,\ pi\leqslant r} \Omega^{-i}\Lambda^i(V^{(p)}_{a-p^{n-1}}) \otimes S^{r-pi}(V_a)$$

which can be rewritten as

$$\widetilde{\sigma}_t(V_a)\cong_{ind}\lambda^{\Omega}_{t^p}(V_{a-p^{n-1}})\sigma_t(V_a)$$

then by corollary 4.14

$$\breve{\sigma}_t(V_a)\cong_{ind}(1-t^{p^n})\widetilde{\sigma}_t(V_a)\cong_{ind}(1-t^{p^n})\lambda^{\Omega}_{t^p}(V_{a-p^{n-1}})\sigma_t(V_a)$$

and by lemma 2.8

$$\breve{\sigma}_t(V_a)\cong_{ind}\lambda^{\Omega}_{t^p}(V_{a-p^{n-1}})\lambda^{\Omega}_t(V_{p^n-a}).$$

□

Example 4.16. Let $p = 3, a = 13, n = 3$ then from table A.1,

$$\lambda_t(V_{13-9})\cong_{ind}1 + V_4t + (V_1 + V_5)t^2 + V_4t^3 + t^4$$

$$\lambda_t(V_{27-13})\cong_{ind}1+V_{14}t+(V_1+V_5+V_{13}+V_{17}+V_{25})t^2+(V_4+V_{10}+2V_{14}+V_{16}+V_{20}+V_{22})t^3+\cdots$$

so

$$\lambda^{\Omega}_t(V_{13-9})\cong_{ind}1 + V_{23}t + (V_1 + V_5)t^2 + V_{23}t^3 + t^4$$

$$\lambda^{\Omega}_t(V_{27-13})\cong_{ind}1+V_{13}t+(V_1+V_5+V_{13}+V_{17}+V_{25})t^2+(V_{23}+V_{17}+2V_{13}+V_{11}+V_7+V_5)t^3+\cdots$$

therefore

$$\lambda^{\Omega}_{t^p}(V_{a-p^{n-1}})\lambda^{\Omega}_t(V_{p^n-a})\cong_{ind}1+V_{13}t + (V_1 + V_5 + V_{13} + V_{17} + V_{25})t^2 \\ +(V_{23} + V_{17} + 2V_{13} + V_{11} + V_7 + V_5 + V_{23})t^3 + \cdots$$

which corresponds with the numerical calulations for $\breve{S}^3(V_{13})$ in appendix A.

Chapter 5

Conjectures

In this chapter some conjectures are given. Let $G = C_{p^n}$ and V_i be again defined as in chapter 2.

Conjecture 5.1. *For $p > 2$*

$$\psi^p_\Lambda(V_a) - \psi^p_\Lambda(V_{2q-a}) + \psi^p_\Lambda(V_{2q+a}) - \cdots - \psi^p_\Lambda(V_{(p-1)q-a}) + \psi^p_\Lambda(V_{(p-1)q+a}) \cong_{ind} pV_a$$

where $q = p^i$, $0 \leqslant a \leqslant q$.

Remark 5.2. This fits numerical data for C_{25} and C_{27}. See table A.7 and table A.9.

Let h be a generator of C_3.
Note that $(h-1)^2 S^3(V_1) = (h-1)^2 V_1^3 = (h-1)^2 V_1^2 \otimes V_2 = 0$,
$(h-1)^2 V_2^2 \otimes V_1 = \operatorname{span}_k\{v_1 \otimes v_1 \otimes v_1\}$, $(h-1)^2 S^3(V_2) = \operatorname{span}_k\{v_1 v_1 v_1\}$ and
$(h-1)^2 V_2^3 = \operatorname{span}_k\{v_1 \otimes v_1 \otimes v_1, v_1 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_1\}$. This motivated the following conjecture.

Conjecture 5.3. *For $p = 3$, $0 \leqslant a \leqslant q$, $q = p^{n-1}$, $G = C_{pq}$ there exists exact kG module complexes*

$$0 \to \Lambda^3(V_a) \xrightarrow{\alpha} \left(\frac{S^2(V_a) \otimes V_{q+a}}{X}\right)^H \to (h-1)^2 S^3(V_{q+a}) \to 0$$

$$0 \to \breve{S}^3(V_a) \xrightarrow{\beta} \left(\frac{\Lambda^2(V_a) \otimes V_{q+a}}{Y}\right)^H \to (h-1)^2 \Lambda^3(V_{q+a}) \to 0$$

$$0 \to V_a \otimes \Lambda^2(V_a) \to V_a \otimes \Lambda^2(V_q) \to \left(\frac{S(V_a, V_q) \otimes V_{q+a}}{Z}\right)^H \to ((h-1)S^3(V_{q+a}))^H \to 0$$

where $h = g^q$, $H = \langle h \rangle$,

$$X = \operatorname{span}_k\{v_i v_j \otimes v_k + v_k v_i \otimes v_j + v_j v_k \otimes v_i\} \subset S^2(V_a) \otimes V_a,$$

$$v_i \wedge v_j \wedge v_k \xrightarrow{\alpha} v_iv_j \otimes v_k - v_kv_i \otimes v_j + X,$$

$$Y = \mathrm{span}_k\{v_i \wedge v_j \otimes v_k + v_k \wedge v_i \otimes v_j + v_j \wedge v_k \otimes v_i \mid i \neq j \neq k \neq i\} \subset \Lambda^2(V_a) \otimes V_a,$$

$$v_iv_jv_k \xrightarrow{\beta} v_i \wedge v_j \otimes v_k - v_k \wedge v_i \otimes v_j + Y,$$

$$S(V_a, V_q) = \mathrm{span}_k\{vw \mid v \in V_a, w \in V_q\},$$

$$Z = \mathrm{span}_k\{v_iv_j \otimes v_k + v_kv_i \otimes v_j + v_jv_k \otimes v_i\}$$

$$\cup \mathrm{span}_k\{v_kv_i \otimes v_j - v_kv_j \otimes v_i \mid i, j, k \leqslant a, i \neq k \neq j\} \subset S^2(V_a) \otimes V_a$$

and the middle terms are projective over G/H. It would follow from these that

$$\Lambda^3(V_a) \cong_{proj} \Omega_q(h-1)^2 S^3(V_{q+a})$$

$$\breve{S}^3(V_a) \cong_{proj} \Omega_q(h-1)^2 \Lambda^3(V_{q+a})$$

$$V_a \otimes \Lambda^2(V_a) \cong_{proj} \Omega_q^2((h-1)S^3(V_{q+a}))^H.$$

We also conjecture that these formulae can be used to express $S^3(V_{q+a})$ in terms of $\Lambda^3(V_a)$, $\breve{S}^3(V_{q-a})$ and $V_a \otimes \Lambda^2(V_a)$ modulo induced summands.

Example 5.4. Let $p = 3$, $n = 3$. Write $S^3(V_{14}) \cong \bigoplus_{i=1}^{27} a_iV_i$ then $(h-1)^2S^3(V_{14}) \cong \bigoplus_{i=19}^{27} a_iV_{i-18}$ and $((h-1)S^3(V_{14}))^H \cong \bigoplus_{i=10}^{18} a_iV_{i-9} \oplus \bigoplus_{i=19}^{27} a_iV_9$. Also $\Lambda^3(V_{13}) \cong_{ind} \Omega_{27}S^3(V_{14}) \cong_{ind} \bigoplus_{i=1}^{26} a_iV_{27-i}$ so $(h-1)^2\Lambda^3(V_{13}) \cong_{ind} \bigoplus_{i=1}^{8} a_iV_{9-i}$. If we assume conjecture 5.3 then using $\Lambda^*(V_5)$ from table A.1 and $\breve{S}^3(V_4)$ from table A.4,

$$\bigoplus_{i=19}^{27} a_iV_{i-18} \cong_{proj} \Omega_9\Lambda^3(V_5) \cong_{ind} \Omega_9(V_7) \cong_{ind} V_2,$$

$$\bigoplus_{i=10}^{18} a_iV_{i-9} \cong_{ind} V_5 \otimes \Lambda^2(V_5) \cong_{ind} V_5 \otimes (V_7) \cong_{ind} V_5,$$

$$\bigoplus_{i=1}^{8} a_iV_{9-i} \cong_{ind} \Omega_9\breve{S}^3(V_4) \cong_{ind} \Omega_9(V_2 \oplus V_8) \cong_{ind} V_7 \oplus V_1$$

so

i	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26
a_i	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0

.

Therefore $S^3(V_{14}) \cong_{ind} V_2 \oplus V_8 \oplus V_{14} \oplus V_{20}$ and $\Lambda^3(V_{13}) \cong_{ind} V_7 \oplus V_{13} \oplus V_{19} \oplus V_{25}$ which corresponds with the numerical calulations.

The same method can be used to predict the structure of Λ^3 and S^3 in cases that are too big to do numerically.

Example 5.5. From table A.3

$$\Lambda^3(V_{35})\cong_{ind} V_1 \oplus V_{23} \oplus V_{77},$$

$$V_{35} \otimes \Lambda^2(V_{35})\cong_{ind} V_{35} \otimes V_{55}\cong_{ind} V_{35}$$

and by corollary 4.15,

$$\breve{S}^3(V_{46})\cong_{ind}\Lambda^1(V_{46-27}) \otimes \Lambda^0(V_{81-46}) \oplus \Lambda^0(V_{46-27}) \otimes \Lambda^3(V_{81-46})$$

$$\cong_{ind} V_{19} \oplus \Lambda^3(V_{35})\cong_{ind} V_1 \oplus V_{19} \oplus V_{23} \oplus V_{77}.$$

Using this, conjecture 5.3 predicts

$$S^3(V_{116})\cong_{ind}(V_1 \oplus V_{19} \oplus V_{23} \oplus V_{77}) \oplus V_{116} \oplus (V_{166} \oplus V_{220} \oplus V_{242})$$

and

$$\Lambda^3(V_{127})\cong_{ind}(V_1 \oplus V_{23} \oplus V_{77}) \oplus V_{127} \oplus (V_{166} \oplus V_{220} \oplus V_{224} \oplus V_{242}).$$

Remark 5.6. Using conjecture 5.3 and theorem 4.1 at the prime 3, the third exterior power can be expressed in terms of $\Lambda^{\leqslant 3}$, $S^{\leqslant 3}$ and $\breve{S}^{\leqslant 3}$ of modules for a proper subgroup. Also using lemma 2.8 and corollary 4.15, $S^{\leqslant 3}$ and $\breve{S}^{\leqslant 3}$ can be expressed in terms of $\Lambda^{\leqslant 3}$. For indecomposable modules with the exception of V_{3^m} and $V_{2\times 3^m}$, if the first digit of the dimension in ternary is 1 then conjecture 5.3 can be used, if the first digit of the dimension in ternary is 2 then theorem 4.1 can be used.

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Appendix A

Tables

In the following tables, decompositions of modules are represented by listing the dimensions of the indecomposable summands with their multiplicities given as superscripts. e.g. $1^{45}, 4, 5^2$ represents $45V_1 \oplus V_4 \oplus 2V_5$.

Blank cells represent induced modules. Values in brackets and "??????" represent modules which were too big for magma.

Tables A.1 through A.3 and A.5 were calculated using the magma function `ExteriorPower` to construct the whole module which was then decomposed using `PrimaryInvariantFactors`.

Table A.4 was calculated by using the magma function `SymmetricPower` to construct the symmetric power then quotienting down to $\breve{S}$ and finally decomposing with `PrimaryInvariantFactors`.

Tables A.6 through A.10 were calculated using tables A.1 through A.3 and A.5 and calculations of the tensor product made using the code in appendix B.

$p=3$	Λ^1	Λ^2	Λ^3	Λ^4	Λ^5
V_1	1				
V_2	2	1			
V_3			1		
V_4	4	$1, 5$	4	1	
V_5	5	7	7	5	1
V_6			8		
V_7	7		$1, 7$	$1, 7$	
V_8	8	1	2	7	2
V_9					
V_{10}	10	$1, 17$	$2, 10$	$1, 11$	$2, 16$
V_{11}	11	19	$1, 7, 25$	$11, 17^2$	$7, 19, 25$
V_{12}			$8, 26$		
V_{13}	13	$7, 11, 19, 23$	$7, 13, 19, 25$	$1, 13^3$	$1^2, 5^2, 7, 11, 17^2,$ $19, 23, 25^2$
V_{14}	14	$1, 5, 13, 17, 25$	$4, 10, 14^2,$ $16, 20, 22$	$1^2, 5^2, 11,$ $13^2, 17^2, 23, 25$	$4, 8, 10, 14^3,$ $16^2, 20, 22^2$
V_{15}			$1, 19$		
V_{16}	16	$1, 17$	$16, 22$	$1, 13$	$14, 22$
V_{17}	17	19	23	13	5
V_{18}					
V_{19}	19		$1, 23$	$5, 19$	
V_{20}	20	1	$2, 22$	$5, 19$	$2, 22$
V_{21}			19		
V_{22}	22	$1, 5$	$4, 20, 22$	$1, 5, 19, 23, 25$	$2, 4^2, 8, 20, 22$
V_{23}	23	7	$5, 19, 25$	$5^2, 19, 23, 25$	??????
V_{24}			26		??????
V_{25}	25		$7, 25$	$1, 19$	??????
V_{26}	26	1	8	19	??????

Table A.1: Exterior power of C_{27} modules modulo induced modules.

$p=3$	Λ^1	Λ^2	Λ^3
V_{26}	26	1	8
V_{27}			
V_{28}	28	$1, 53$	$8, 28$
V_{29}	29	55	$7, 25, 79$
V_{30}			$26, 80$
V_{31}	31	$7, 47, 55, 59$	$5, 19, 25, 31, 73, 79$
V_{32}	32	$1, 5, 49, 53, 61$	$4, 20, 22, 28, 32^2, 34, 74, 76$
V_{33}			$19, 73$
V_{34}	34	$1, 53$	$2, 22, 34, 76$
V_{35}	35	55	$1, 23, 77$
V_{36}			
V_{37}	37	$19, 35, 55, 71$	$23, 37, 55, 77$
V_{38}	38	$1, 17, 37, 53, 73$	$16, 22, 34, 38^2, 52, 56, 70, 76$
V_{39}			$1, 19, 55, 73$
V_{40}	40	$1, 5, 13, 17, 25, 29, 37,$ $41, 49, 53, 61, 65, 73, 77$	$4, 10, 14^2, 16, 20, 22, 28, 32^2,$ $34, 38^2, 40^5, 44^2, 46, 50^2,$ $52, 58^2, 64, 68^2, 70, 74, 76$
V_{41}	41	$7, 11, 19, 23, 31, 35,$ $43, 47, 55, 59, 67, 71, 79$	$7, 13, 19, 25, 31, 37, 41^4,$ $43, 49, 59, 61, 67, 73, 79$
V_{42}			$8, 26, 62, 80$
V_{43}	43	$19, 35, 55, 71$	$1, 7, 25, 43, 55, 61^2, 79$
V_{44}	44	$1, 17, 37, 53, 73$	$2, 10, 28, 44^2, 46, 56, 62, 64$
V_{45}			
V_{46}	46	$1, 53$	$2, 46, 56, 64$
V_{47}	47	55	$1, 7, 55, 61, 65$
V_{48}			$8, 62$
V_{49}	49	$7, 47, 55, 59$	$7, 49, 61, 67$
V_{50}	50	$1, 5, 49, 53, 61$	$4, 46, 50^2, 52, 58, 68$
V_{51}			$1, 55$
V_{52}	52	$1, 53$	$52, 70$

Table A.2: Exterior power of C_{81} modules modulo induced modules.

$p=3$	Λ^1	Λ^2	Λ^3
V_{51}			$1,55$
V_{52}	52	$1,53$	$52,70$
V_{53}	53	55	71
V_{54}			
V_{55}	55		$1,71$
V_{56}	56	1	$2,70$
V_{57}			55
V_{58}	58	$1,5$	$4,58,68$
V_{59}	59	7	$5,61,67$
V_{60}			??????
V_{61}	61		??????
V_{62}	62	1	??????
V_{63}			??????
V_{64}	64	$1,17$	??????
V_{65}	65	19	??????
V_{66}			??????
V_{67}	67	$7,11,19,23$	??????
V_{68}	68	$1,5,13,17,25$	??????
V_{69}			??????
V_{70}	70	$1,17$	??????
V_{71}	71	19	??????
V_{72}			??????
V_{73}	73		??????
V_{74}	74	1	??????
V_{75}			??????
V_{76}	76	$1,5$	??????
V_{77}	77	7	??????
V_{78}			??????
V_{79}	79		??????
V_{80}	80	1	??????

Table A.3: Exterior power of C_{81} modules modulo induced modules.

$p=3$	$\breve{S}^3$
V_1	
V_2	2
V_3	1
V_4	$2,8$
V_5	$5,7$
V_6	8
V_7	5
V_8	4
V_9	
V_{10}	$4,26$
V_{11}	$5,11,25$
V_{12}	$8,26$
V_{13}	$5,7,11,13^2,17,23^2$
V_{14}	$2,8,14,20,22$
V_{15}	$1,19$
V_{16}	$2,20^2,26$
V_{17}	$17,19,25$
V_{18}	
V_{19}	$17,25$
V_{20}	$16,20,26$
V_{21}	19
V_{22}	$14,20$
V_{23}	$13,23$
V_{24}	26
V_{25}	11
V_{26}	10

Table A.4: $\breve{S}$ of C_{27} modules modulo induced modules.

$p=5$	Λ^1	Λ^2	Λ^3	Λ^4	Λ^5
V_1	1				
V_2	2	1			
V_3	3	3	1		
V_4	4	1	4	1	
V_5					1
V_6	6	$1, 9$	$4, 6$	$1, 9$	6
V_7	7	$3, 7, 11$	$1, 7, 9, 13$	$1, 7, 9, 13$	$3, 7, 11$
V_8	8	$1, 9, 13$	$8, 12, 16$	$1, 9, 13, 17$	$8, 12, 16$
V_9	9	11	19	21	21
V_{10}					22
V_{11}	11	$11, 19$	$1, 9, 19, 21$	$9, 11$	$11, 21$
V_{12}	12	$1, 9, 13, 17, 21$	$4, 6, 8, 12^2,$ $16, 18, 24$	$1^2, 7, 9^2, 13^3,$ $17^2, 19, 21$	$4, 6^2, 8^2, 12^4,$ $16^2, 18^2, 24$
V_{13}	13	$3, 7, 11, 19, 23$	$7, 11,$ $13^2, 17$	$1^2, 7, 9^2, 13,$ $17, 19, 21, 23$	$1, 7, 11, 13^2, 17$
V_{14}	14	$1, 9, 21$	14	1	2
V_{15}					3
V_{16}	16	$1, 9$	$16, 24$	$1, 9$	$2, 16$
V_{17}	17	$3, 7, 11$	$13, 17, 19, 21$	$1, 7, 9, 13$	$1, 11, 17, 21, 23$
V_{18}	18	$1, 9, 13$	$6, 12, 18$	$1, 9, 13, 17$	$6, 12, 18, 22$
V_{19}	19	11	9	21	23
V_{20}					24
V_{21}	21				$1, 23$
V_{22}	22	1			$2, 22$
V_{23}	23	3	21		$(3, 21)$
V_{24}	24	1	24	1	(4)

Table A.5: Exterior power of C_{25} modules, modulo induced modules. Values in brackets are calculated using theorem 4.1.

$p=5$	ψ^1_Λ	ψ^2_Λ	ψ^3_Λ
V_1	1^1	1^1	1^1
V_2	2^1	$1^{-1}, 3^1$	$2^{-1}, 4^1$
V_3	3^1	$1^1, 3^{-1}$	$1^1, 3^{-1}$
V_4	4^1	1^{-1}	4^1
V_5			
V_6	6^1	$1^{-1}, 9^{-1}, 11^1$	$4^1, 14^{-1}, 16^1$
V_7	7^1	$1^1, 3^{-1}, 7^{-1}, 9^1, 11^{-1}, 13^1$	$1^1, 3^{-1}, 11^{-1}, 13^1, 17^{-1}, 19^1$
V_8	8^1	$1^{-1}, 3^1, 7^1, 9^{-1}, 11^1, 13^{-1}$	$2^{-1}, 4^1, 12^1, 14^{-1}, 16^1, 18^{-1}$
V_9	9^1	$1^1, 9^1, 11^{-1}$	$1^1, 11^{-1}, 19^1$
V_{10}			
V_{11}	11^1	$1^1, 9^1, 11^{-1}, 19^{-1}, 21^1$	$1^1, 11^{-1}, 21^1$
V_{12}	12^1	$1^{-1}, 3^1, 7^1, 9^{-1}, 11^1,$ $13^{-1}, 17^{-1}, 19^1, 21^{-1}, 23^1$	$2^{-1}, 4^1, 12^1, 14^{-1}, 22^{-1}, 24^1$
V_{13}	13^1	$1^1, 3^{-1}, 7^{-1}, 9^1, 11^{-1},$ $13^1, 17^1, 19^{-1}, 21^1, 23^{-1}$	$1^1, 3^{-1}, 11^{-1}, 13^1, 21^1, 23^{-1}$
V_{14}	14^1	$1^{-1}, 9^{-1}, 11^1, 19^1, 21^{-1}$	$4^1, 14^{-1}, 24^1$
V_{15}			
V_{16}	16^1	$1^{-1}, 9^{-1}, 11^1$	$6^1, 14^{-1}, 24^1$
V_{17}	17^1	$1^1, 3^{-1}, 7^{-1}, 9^1, 11^{-1}, 13^1$	$7^{-1}, 9^1, 11^{-1}, 13^1, 21^1, 23^{-1}$
V_{18}	18^1	$1^{-1}, 3^1, 7^1, 9^{-1}, 11^1, 13^{-1}$	$6^1, 8^{-1}, 12^1, 14^{-1}, 22^{-1}, 24^1$
V_{19}	19^1	$1^1, 9^1, 11^{-1}$	$9^1, 11^{-1}, 21^1$
V_{20}			
V_{21}	21^1	1^1	21^1
V_{22}	22^1	$1^{-1}, 3^1$	$22^{-1}, 24^1$
V_{23}	23^1	$1^1, 3^{-1}$	$21^1, 23^{-1}$
V_{24}	24^1	1^{-1}	24^1
V_{25}			

Table A.6: Adams operations of C_{25} modules, modulo induced modules.

$p=5$	ψ^4_Λ	ψ^5_Λ
V_1	1^1	1^1
V_2	3^{-1}	4^{-2}
V_3	3^1	1^3
V_4	1^{-1}	4^{-4}
V_5		1^5
V_6	$1^{-1}, 19^{-1}, 21^1$	$4^{-4}, 24^{-2}$
V_7	$3^1, 17^1, 23^{-1}$	$1^3, 21^4$
V_8	$3^{-1}, 17^{-1}, 23^1$	$4^{-2}, 24^{-6}$
V_9	$1^1, 19^1, 21^{-1}$	$1^1, 21^8$
V_{10}		22^5
V_{11}	$1^1, 9^{-1}, 11^1, 19^1, 21^{-1}$	$1^3, 21^8$
V_{12}	$3^{-1}, 7^1, 13^{-1}, 17^{-1}, 23^1$	$4^{-6}, 24^{-6}$
V_{13}	$3^1, 7^{-1}, 13^1, 17^1, 23^{-1}$	$1^9, 21^4$
V_{14}	$1^{-1}, 9^1, 11^{-1}, 19^{-1}, 21^1$	$2^5, 4^{-2}, 24^{-2}$
V_{15}		3^5
V_{16}	$1^{-1}, 19^{-1}, 21^1$	$2^5, 4^{-2}, 24^{-4}$
V_{17}	$3^1, 17^1, 23^{-1}$	$1^9, 21^8$
V_{18}	$3^{-1}, 17^{-1}, 23^1$	$4^{-6}, 22^5, 24^{-2}$
V_{19}	$1^1, 19^1, 21^{-1}$	$1^3, 21^1, 23^5$
V_{20}		24^5
V_{21}	1^1	$1^5, 21^1, 23^5$
V_{22}	3^{-1}	$2^5, 22^5, 24^{-2}$
V_{23}	3^1	$3^5, 21^8$
V_{24}	1^{-1}	$4^5, 24^{-4}$
V_{25}		

Table A.7: Adams operations of C_{25} modules, modulo induced modules.

$p=3$	ψ^4_Λ	ψ^5_Λ
V_1	1^1	1^1
V_2	1^{-1}	2^1
V_3		
V_4	$1^{-1}, 5^{-1}, 7^1$	4^1
V_5	$1^1, 5^1, 7^{-1}$	5^1
V_6		
V_7	1^1	7^1
V_8	1^{-1}	8^1
V_9		
V_{10}	$1^{-1}, 17^{-1}, 19^1$	10^1
V_{11}	$1^1, 17^1, 19^{-1}$	11^1
V_{12}		
V_{13}	$1^1, 5^1, 7^{-1}, 11^{-1}, 13^1, 17^1, 19^{-1}, 23^{-1}, 25^1$	13^1
V_{14}	$1^{-1}, 5^{-1}, 7^1, 11^1, 13^{-1}, 17^{-1}, 19^1, 23^1, 25^{-1}$	14^1
V_{15}		
V_{16}	$1^{-1}, 17^{-1}, 19^1$	16^1
V_{17}	$1^1, 17^1, 19^{-1}$	17^1
V_{18}		
V_{19}	1^1	19^1
V_{20}	1^{-1}	20^1
V_{21}		
V_{22}	$1^{-1}, 5^{-1}, 7^1$	22^1
V_{23}	$1^1, 5^1, 7^{-1}$	??????
V_{24}		??????
V_{25}	1^1	??????
V_{26}	1^{-1}	??????
V_{27}		??????

Table A.8: Adams operations of C_{81} modules, modulo induced modules.

$p=3$	ψ^1_Λ	ψ^2_Λ	ψ^3_Λ
V_1	1^1	1^1	1^1
V_2	2^1	1^{-1}	2^{-2}
V_3			1^3
V_4	4^1	$1^{-1}, 5^{-1}, 7^1$	$2^{-2}, 8^{-2}$
V_5	5^1	$1^1, 5^1, 7^{-1}$	$1^1, 7^4$
V_6			8^3
V_7	7^1	1^1	$1^3, 7^4$
V_8	8^1	1^{-1}	$2^3, 8^{-2}$
V_9			
V_{10}	10^1	$1^{-1}, 17^{-1}, 19^1$	$2^3, 8^{-2}, 26^{-2}$
V_{11}	11^1	$1^1, 17^1, 19^{-1}$	$1^3, 7^4, 25^4$
V_{12}			$8^3, 26^3$
V_{13}	13^1	$1^1, 5^1, 7^{-1}, 11^{-1}, 13^1, 17^1, 19^{-1}, 23^{-1}, 25^1$	$1^1, 7^4, 19^4, 25^4$
V_{14}	14^1	$1^{-1}, 5^{-1}, 7^1, 11^1, 13^{-1}, 17^{-1}, 19^1, 23^1, 25^{-1}$	$2^{-2}, 8^{-2}, 20^1, 26^{-2}$
V_{15}			$1^3, 19^3$
V_{16}	16^1	$1^{-1}, 17^{-1}, 19^1$	$2^{-2}, 20^{-2}, 22^3$
V_{17}	17^1	$1^1, 17^1, 19^{-1}$	$1^1, 19^1, 23^3$
V_{18}			
V_{19}	19^1	1^1	$1^3, 19^1, 23^3$
V_{20}	20^1	1^{-1}	$2^3, 20^{-2}, 22^3$
V_{21}			19^3
V_{22}	22^1	$1^{-1}, 5^{-1}, 7^1$	$4^3, 20^1, 26^{-2}$
V_{23}	23^1	$1^1, 5^1, 7^{-1}$	$5^3, 19^4, 25^4$
V_{24}			26^3
V_{25}	25^1	1^1	$7^3, 25^4$
V_{26}	26^1	1^{-1}	$8^3, 26^{-2}$
V_{27}			
V_{28}	28^1	$1^{-1}, 53^{-1}, 55^1$	$8^3, 26^{-2}, 80^{-2}$
V_{29}	29^1	$1^1, 53^1, 55^{-1}$	$7^3, 25^4, 79^4$
V_{30}			$26^3, 80^3$
V_{31}	31^1	$1^1, 5^1, 7^{-1}, 47^{-1}, 49^1, 53^1, 55^{-1}, 59^{-1}, 61^1$	$5^3, 19^4, 25^4, 73^4, 79^4$
V_{32}	32^1	$1^{-1}, 5^{-1}, 7^1, 47^1, 49^{-1}, 53^{-1}, 55^1, 59^1, 61^{-1}$	$4^3, 20^1, 26^{-2}, 74^1, 80^{-2}$

Table A.9: Adams operations of C_{81} modules, modulo induced modules.

$p=3$	ψ^1_Λ	ψ^2_Λ	ψ^3_Λ
V_{33}			$19^3, 73^3$
V_{34}	34^1	$1^{-1}, 53^{-1}, 55^1$	$2^3, 20^{-2}, 22^3, 74^{-2}, 76^3$
V_{35}	35^1	$1^1, 53^1, 55^{-1}$	$1^3, 19^1, 23^3, 73^1, 77^3$
V_{36}			
V_{37}	37^1	$1^1, 17^1, 19^{-1}, 35^{-1}, 37^1,$ $53^1, 55^{-1}, 71^{-1}, 73^1$	$1^1, 19^1, 23^3, 55^4, 73^1, 77^3$
V_{38}	38^1	$1^{-1}, 17^{-1}, 19^1, 35^1, 37^{-1},$ $53^{-1}, 55^1, 71^1, 73^{-1}$	$2^{-2}, 20^{-2}, 22^3, 56^1, 74^{-2}, 76^3$
V_{39}			$1^3, 19^3, 55^3, 73^3$
V_{40}	40^1	$1^{-1}, 5^{-1}, 7^1, 11^1, 13^{-1}, 17^{-1}, 19^1,$ $23^1, 25^{-1}, 29^{-1}, 31^1, 35^1, 37^{-1}, 41^{-1},$ $43^1, 47^1, 49^{-1}, 53^{-1}, 55^1, 59^1,$ $61^{-1}, 65^{-1}, 67^1, 71^1, 73^{-1}, 77^{-1}, 79^1$	$2^{-2}, 8^{-2}, 20^1, 26^{-2}, 56^{-2},$ $58^3, 62^{-2}, 74^1, 80^{-2}$
V_{41}	41^1	$1^1, 5^1, 7^{-1}, 11^{-1}, 13^1, 17^1, 19^{-1},$ $23^{-1}, 25^1, 29^1, 31^{-1}, 35^{-1}, 37^1, 41^1,$ $43^{-1}, 47^{-1}, 49^1, 53^1, 55^{-1}, 59^{-1},$ $61^1, 65^1, 67^{-1}, 71^{-1}, 73^1, 77^1, 79^{-1}$	$1^1, 7^4, 19^4, 25^4, 55^1,$ $59^3, 61^4, 73^4, 79^4$
V_{42}			$8^3, 26^3, 62^3, 80^3$
V_{43}	43^1	$1^1, 17^1, 19^{-1}, 35^{-1}, 37^1,$ $53^1, 55^{-1}, 71^{-1}, 73^1$	$1^3, 7^4, 25^4, 55^3, 61^7, 79^4$
V_{44}	44^1	$1^{-1}, 17^{-1}, 19^1, 35^1, 37^{-1},$ $53^{-1}, 55^1, 71^1, 73^{-1}$	$2^3, 8^{-2}, 26^{-2}, 56^3, 62^1, 80^{-2}$
V_{45}			
V_{46}	46^1	$1^{-1}, 53^{-1}, 55^1$	$2^3, 8^{-2}, 56^3, 62^{-2}, 64^3$
V_{47}	47^1	$1^1, 53^1, 55^{-1}$	$1^3, 7^4, 55^3, 61^4, 65^3$
V_{48}			$8^3, 62^3$
V_{49}	49^1	$1^1, 5^1, 7^{-1}, 47^{-1}, 49^1,$ $53^1, 55^{-1}, 59^{-1}, 61^1$	$1^1, 7^4, 55^1, 61^4, 67^3$
V_{50}	50^1	$1^{-1}, 5^{-1}, 7^1, 47^1, 49^{-1},$ $53^{-1}, 55^1, 59^1, 61^{-1}$	$2^{-2}, 8^{-2}, 56^{-2}, 62^{-2}, 68^3$
V_{51}			$1^3, 55^3$
V_{52}	52^1	$1^{-1}, 53^{-1}, 55^1$	$2^{-2}, 56^{-2}, 70^3$
V_{53}	53^1	$1^1, 53^1, 55^{-1}$	$1^1, 55^1, 71^3$
V_{54}			
V_{55}	55^1	1^1	$1^3, 55^1, 71^3$
V_{56}	56^1	1^{-1}	$2^3, 56^{-2}, 70^3$
V_{57}			55^3
V_{58}	58^1	$1^{-1}, 5^{-1}, 7^1$	$4^3, 56^{-2}, 62^{-2}, 68^3$
V_{59}	59^1	$1^1, 5^1, 7^{-1}$	$5^3, 55^1, 61^4, 67^3$

Table A.10: Adams operations of C_{81} modules, modulo induced modules.

Appendix B

Code

The following code, written in pascal for compiling with free pascal (`http://www.freepascal.org`), implements the algorithm for the tensor product from [1].

```
PROGRAM ddd;
Uses math;
{$OVERFLOWCHECKS ON}

var c,c1,t,x,tx:word;
    r,s,mrs,rsd:word;
    ga,pga:word;
    be:word;
    r0,r1,s0,s1:Longint;
    k:word;
    nu:qword;
    f:text;
const p=5;
     n=2;
     pn=25;//p**n
     ppn=p*pn;
var vs,pvs:array[1..ppn] of qword;
begin
 Assign(f,'p=5-n=2--7.csv');
 rewrite(f);
 for r:=1 to pn do begin
  for s:=1 to pn do begin
   mrs:=min(r,s);
   rsd:=max(r,s)-mrs;
   fillqword(vs,pn,0);
   for c:=1 to mrs do begin
      vs[rsd+2*c-1]:=1;
   end;
   ga:=1;pga:=p;
   while pga<=max(r,s) do begin inc(ga);pga:=pga*p end;
   for be:=ga downto 1 do begin
      pvs:=vs;
      divmod(r,pga,r0,r1);
      divmod(s,pga,s0,s1);
      k:=1;
```

```
      repeat
         if (pvs[k*pga] <> 0) or ((pvs[k*pga+1] <> 0) and (pvs[k*pga-1] <> 0)) then begin
            if odd(k) xor odd(r0+s0) then
               nu:=abs(pga-(r1+s1))
            else
               nu:=abs(r1-s1);
            for t:= k*pga-nu to k*pga+nu do begin
               vs[t]:=0;
            end;
              vs[k*pga]:=nu;
         end;
         inc(k);
         if k mod p=0 then inc(k);
      until k*pga>pn;
      pga:=pga div p;
   end;
   write(f,',"');
   for c:=1 to pn do
      if (vs[c]<>0) and ((c mod p)<>0) then
         for t:=1 to vs[c] do
            write (f,c,',');
   write(f,'"');
  end;
   writeln(f);
 end;
 Close(f);
end.
```