MODEL THEORY OF HOLOMORPHIC FUNCTIONS IN AN O-MINIMAL SETTING

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Given an o-minimal structure on the real field, we consider an elementary extension to a non-archimedean field $R$, and interpret the algebraically closed field $K = R[\sqrt{-1}]$ on this extension. We construct two pregeometries on $K$: one by considering images under $\mathbb{C}$-definable holomorphic functions, and the other by considering images under proper restrictions of $\mathbb{C}$-definable holomorphic functions together with algebraic functions (i.e. zeros of polynomials).

We show that these two pregeometries are the same, generalising a result of A. Wilkie for complex holomorphic functions. We also do some work towards generalising another result of his on local definability of complex holomorphic functions to our non-archimedean setting.
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Introduction

Given a totally ordered set \((M, <)\) and a first-order structure on \(M\) in which the total order is definable, the notion of “interior of a definable set under the order topology” will be definable as well. This allows several topological concepts to have definable analogues in such a first-order structure.

In the case of o-minimal structures extending real-closed fields, not only topological concepts have definable analogues, but also some notions from real analysis, like differentiability, are definable. This allows for analogues of results like the Inverse Function Theorem to hold on these real-closed fields (cf. [6] for an excellent collection of results for this “tame” definable topology).

For any such o-minimal structure \(\mathcal{R} = (R; 0, 1, +, \cdot, <, \ldots)\) extending a real-closed field, the algebraic closure \(K = R[\sqrt{-1}]\) can be interpreted in \(\mathcal{R}\) via \((x, y) \mapsto x + \sqrt{-1}y\); the multiplication map in \(K\) is definable in \(\mathcal{R}\), and the norm topology on \(K\) is equivalent to the product topology on \(R^2\). Working in this setting, Peterzil and Starchenko defined the concept of definable holomorphic function, and proved that many of the results from standard complex analysis have a counterpart in what they called “complex-like” o-minimal analysis (cf. [13], [9] and [10]).

From a model theoretical point of view, Wilkie in ([13]) showed that, given a collection \(\mathcal{F}\) of holomorphic functions in \(\mathbb{C}\), any holomorphic function locally definable from \(\mathcal{F}\) in a neighbourhood of a generic point can be obtained from \(\mathcal{F}\) via the “basic” operations of composition, partial derivation, Schwarz reflection and implicit functions in one dependent variable. He does so by proving that the natural pregeometry associated with all locally definable holomorphic functions is determined by these four operations.

The aim of this work is to obtain an analogue of Wilkie’s result to non-archimedean extensions of \(\mathbb{C}\). Namely, we aim to find a collection of “basic” \(\mathbb{C}\)-definable holomorphic
functions such that the natural pregeometry associated with \( \mathbb{C} \)-definable holomorphic functions in \( K \) is generated by these basic functions.

Chapter 1 is a brief list of selected results from o-minimality and “complex-like” o-minimal analysis that will be used later in the text.

Chapter 2 is about preparation theorems for complex analytic functions. These results transfer to our non-archimedean extensions of \( \mathbb{C} \), provided we restrict ourselves to \( \mathbb{C} \)-definable holomorphic functions. It includes a proof of the Weierstrass Preparation Theorem for a parametrised family of complex analytic functions, which gives a uniform lower bound for the radius of convergence of the analytic functions given by Weierstrass Preparation. This stronger version is not used in the remainder of the thesis, but it is one of the steps that would be needed for the main results to work without parameters. Unfortunately, for the other Preparation Theorem from this Chapter, Denef-van den Dries Preparation, we were not able to find a similar uniform version, and thus the main results from this work need complex parameters.

In Chapter 3 we define our collection of “basic” functions and operations: \( \mathbb{C} \)-definable holomorphic functions and zeros of polynomials. In order to apply the preparation theorems of the previous Chapter to these functions, we only consider the image of infinitesimal elements under them.

In Chapter 4 we construct a pregeometry from the functions and operations of the previous Chapter. Unlike the standard complex case, this pregeometry is not just given by the images under our holomorphic functions and zeros of polynomials, we need to add some extra elements to deal with the fact that a “complex” element may have its real and imaginary parts infinitesimally close to each other (infinitesimal even when compared with the real and imaginary parts themselves). Moreover, it may be the case that iterating our basic operations from the previous Chapter will interfere with the proof of a Exchange Property for our pregeometry: as a simple example, consider two infinitesimal elements \( a \) and \( b \) in our non-archimedean extension of \( \mathbb{C} \). Then \( \exp(a + b) \) and \( \exp(a) \) belong to the closure under holomorphic functions, and thus \( \exp(b) = \frac{\exp(a + b)}{\exp(a)} \) is in the closure of \( \{a, b\} \). The problem here arises as \( \exp(b) \) does not depend on \( a \), but \( a \) appears in the particular iteration of functions we used to get to \( \exp b \). To work around this problem we construct a new closure based on the operator from the previous Chapter, where we account for holomorphic functions
which are, in a sense, constant on some variables.

Finally, in Chapter 5 we construct the pregeometry associated with \( \mathbb{C} \)-definable holomorphic functions in general, and show that it is the same as the pregeometry defined in the previous Chapter. However, for similar reasons as for the other pregeometry, this “natural” pregeometry needs some extra elements added to it (basically, a definable holomorphic function might be constant when we fix some of its variables and in some cases we will need to add these constant value to the closure). We obtain a result on the natural pregeometries analogous to Wilkie’s, but in this non-archimedean setting it does not translate to a result on the local behaviour of \( \mathbb{C} \)-definable holomorphic functions in terms of our basic functions.

However, we conjecture that these “extra elements” we add to both closures to turn them into pregeometries are actually included in the respective original closures, and in particular there is a local behaviour result for \( \mathbb{C} \)-definable holomorphic functions similar to Wilkie’s.

Throughout this work, the word ‘definable’ means ‘definable with parameters’, unless a specific subset of parameters is mentioned.
Chapter 1

Some background results on o-minimality

1.1 O-minimality

Let $\mathcal{M} = (M; <, \ldots)$ be a first-order structure where the relation symbol $<$ is interpreted as a total order relation on $M$, and let $M_0 \subseteq M$ be the collection of definable elements in $M$. As the order relation is definable, the intervals $(-\infty, a) = \{x \in M : x < a\}$, $(a, b) = \{x \in M : a < x < b\}$ and $(b, \infty) = \{x \in M : b < x\}$, are definable for any $a, b \in M_0$. An o-minimal structure is a first-order structure for which these are the only basic 1-dimensional definable sets. More formally,

a structure $\mathcal{M} = (M; <, \ldots)$ is said to be o-minimal if the only definable subsets of $M$ are finite unions of points and intervals.

The original example of a non-trivial o-minimal structure is the real field:

**Theorem 1.1.1 (Tarski).** The real field $(\mathbb{R}; 0, 1, +, \cdot, <)$ is o-minimal.

The set $(M, <)$ can be naturally equipped with the order topology. As open intervals with definable endpoints are definable, we can equip $\mathcal{M}$ with a weak analogue of the order topology over definable sets – arbitrary unions of definable open intervals are not definable in general. We can extend this notion of definable topology to subsets of $M^n$ via the product topology, and we can define
a definable set $A \subseteq M^n$ is said to be *definably open* if for every $\bar{x} = (x_1, \ldots, x_n) \in A$ there exist elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$ such that $\prod_{i=1}^n (a_i, b_i) \subseteq A$ and $a_i < x_i < b_i$ for all $i$.

This is a definable notion, and thus the topological concepts of *closed set*, *interior*, *closure* and *connectedness* have a definable analogue. Moreover, given a definable function $f : (a, b) \to M$, the appropriate one-sided limits of $f(x)$ as $x \to a$ and $x \to b$ are definable, and as such we have the notion of *continuous function*. Definable continuous functions relate to our definable topological concepts much like the original topological continuous functions do - for example, the preimage of a definably open set under a definable continuous function is definably open, and the image of a definably connected set under a definable continuous function is definably connected. A natural problem, then, is to obtain analogues of results from real analysis for arbitrary real-closed fields. A fundamental tool for doing so is the cell decomposition theorem, which is the topic of the next section.

### 1.2 Other properties of o-minimal structures

Given a definable set $A \subseteq M^n$, the cell decomposition theorem will allow us to partition $A$ into finitely many cells – definable sets of a standard form. We begin by establishing precisely what these building blocks are:

A cell $C \subseteq M$ is either a definable singleton or a definable open interval $(a, b)$, where the endpoints might be $\pm \infty$. A cell $C \subseteq M^n$ is either the graph of a cell $D \subseteq M^{n-1}$ under a definable continuous function or the set of points in $D \times M$ whose last coordinate is bounded above and/or below by graphs of definable continuous functions on $D$.

Cells are definable and definably homeomorphic to $(0, 1)^d$ for some $d$.

**Theorem 1.2.1** (Cell decomposition ([6], Chapter 3, §2)).

- Given finitely many definable sets $A_1, \ldots, A_r \subseteq M^n$, there exists a partition of $M^n$ into finitely many cells $C_1, \ldots, C_m$ such that, for each $i$ and each $j$, either $C_i \subseteq A_j$ or $C_i \cap A_j = \emptyset$. 
• Given a definable function $f : A \subseteq M^n \to M$, there exists a partition of $A$ into finitely many cells $C_1, \ldots, C_m$ such that $f|_{C_i}$ is continuous for every $i$.

As an application of cell-decomposition, we have the following result, which we will need later on:

**Lemma 1.2.2.** Let $A \subseteq M^n$. If the topological closure $\bar{A}$ has interior, then $A$ has interior as well.

**Proof.** By the cell-decomposition theorem, we have finitely many cells $C_1, \ldots, C_r$ and $D_1, \ldots, D_m$ such that $C_i \subseteq A$ for all $i$ and $D_j \subseteq \bar{A} \setminus A$ for all $j$. As $\bar{A}$ has interior, at least one of the $C_i$ or $D_j$ must have interior; but if $x \in D_j$ is an interior point of $D_j$ then there exists a definable open set $U$ such that $x \in U \subseteq D_j \subseteq \bar{A} \setminus A$, and thus $x$ would not lie in $\bar{A}$. Thus at least one of the $C_i$ must have interior. \qed

A final property of o-minimal structures that will be needed later is definable choice:

**Theorem 1.2.3** (Definable choice ([6], Chapter 6 §1)). Let $\mathcal{M} = (M; <, 0, +, \ldots)$ be an o-minimal ordered group. Then for every $m, n \geq 0$ and every definable subset $A \subseteq M^{m+n}$ there exists a definable function $f : \pi(A) \to M^n$ such that $\text{graph}(f) \subseteq A$, where $\pi : M^{m+n} \to M^m$ is the projection onto the first $m$ coordinates.

This last result, among other things, implies that, for any $A \subseteq M$, every $A$-definable set contains at least one $A$-definable element.

### 1.3 O-minimal “complex-like” analysis

Let $\mathcal{R} = (R; 0, 1, <, +, \ldots)$ be an o-minimal structure on a real closed field, and let $K = R[\sqrt{-1}]$ be the algebraic closure of $R$. We can identify $R^2$ with $K$ via $(x, y) \mapsto x + \sqrt{-1}y$, and under this identification the “complex” multiplication map $K \times K \to K$ is definable in $\mathcal{R}$. Similarly, maps from $K^n$ to $K$ can be interpreted as maps from $R^{2n}$ to $R^2$, and we can define the concept of complex differentiability at a point either as a component-wise limit (which is definable in our base ordered field) or via the Cauchy-Riemann equations for each pair of “real” and “imaginary” components of a map, and these two definitions coincide. A definable function $f : A \subseteq K^n \to K$
with domain $A$ definably open is said to be holomorphic if it is differentiable on every point of $A$.

In order to obtain analogues of results from complex analysis for these holomorphic functions we will need the concept of winding number, as the standard tool used in complex analysis - integration - is not available to us in this definable setting.

- Let $\sigma : [0, 1] \subseteq \mathbb{R} \rightarrow K$ be definable, continuous, and such that $\sigma(0) = \sigma(1)$. We will refer to the image of any such $\sigma$ as a definable closed oriented curve in $K$. If $\sigma$ is injective on $[0, 1)$, we will additionally say that the curve is simple closed.

- Given a definable simple closed oriented curve $C$ in $K$, the set $K \setminus C$ consists of two definably connected components, with exactly one of them bounded as a subset of $K$. We will refer to this bounded definably connected component as the interior of $C$.

- A definable simple closed oriented curve $C$ in $K$ is said to be star-shaped if there exists a point $p$ in the interior of $C$ such that, for every $x \in C$, the line segment between $p$ and $x$ (without the endpoints) is contained in the interior of $C$.

- Let $C$ be a definable closed oriented curve in $K$, and let $f : C \rightarrow S^1 = \{z \in K : |z| = 1\}$ be definable and continuous. The winding number of $f$, denoted by $W_C(f)$, is the number of counter-clockwise turns of $f(z)$ around $S^1$, as $z$ varies continuously in $C$. This definition makes sense for all $f$ and $C$, and is always an integer.

- Let $C$ be a definable closed oriented curve in $K$, and let $f : C \rightarrow K$ be definable and continuous. For any $w \in K \setminus f(C)$ we define the winding number of $f$ around $w$ by

$$W_C(f, w) = W_C\left(\frac{f(z) - w}{|f(z) - w|}\right).$$

The formal definition of winding number can be found in [9], where the authors first define it from a notion of universal covering for $S^1$, and then show it behaves analogously to the corresponding concept of winding number from complex analysis.

The following technical result is key to linking the notions of winding numbers and $K$-differentiability:
Lemma 1.3.1 ([9], Lemma 2.30). Let $C$ be a definable, star-shaped, simple closed curve whose interior is $D$. Let $f : D \cup C \to K$ be a definable continuous function, which is holomorphic in $D$ except possibly for a definable subset $L \subseteq D$ of $R$-dimension less than 2. Let $W$ be a definably connected component of $K \setminus f(C)$. The following are equivalent:

(1) $W \cap f(D) \neq \emptyset$,

(2) $W \subseteq f(D)$.

(3) There exists $w \in W$ such that $W_C(f, w) \neq 0$.

(4) For all $w \in W$, $W_C(f, w) \neq 0$.

From this result we can obtain most of the properties of $K$-differentiable definable functions. For example, if $C$, $D$ and $f$ are as in the Lemma, and $z_0$ is a point in $D$, then either $f(z_0) \in f(C)$ or there exists a connected component $W$ of $K \setminus f(C)$ such that $f(z_0) \in W \subseteq f(D)$. In particular, $f(z_0)$ lies in the topological interior of $f(D)$. This implies the Maximum Principle for holomorphic functions. We will highlight two of these properties, as we will need them later:

Theorem 1.3.2 ([9], Theorem 2.40). Let $U \subseteq K$ be a definably open set, and let $f : U \to K$ be definable and holomorphic. Then $f$ is infinitely differentiable on $U$.

Theorem 1.3.3 ([9], Theorem 2.33). Let $U \subseteq K$ be a definably connected open set, and let $f : U \to K$ be definable and holomorphic. If there exists $w_0 \in K$ such that $f(z) = w_0$ for infinitely many $z \in U$, then $f(z) = w_0$ for all $z \in U$.

A final result we will require on holomorphic functions is a $K$-differentiable version of the Implicit Function Theorem:

Theorem 1.3.4 ([6], Chapter 7, §2). Let $U \subset K^{m+n}$ be a definable open set and let $f_1, \ldots, f_n : U \to K$ be definable holomorphic functions. Let $(x_0, y_0) \in U$ be such that $f_i(x_0, y_0) = 0$ for all $i$ and such that the matrix $[\frac{\partial f_i}{\partial y_k}(x_0, y_0)]_{1 \leq i, k \leq n}$ is invertible. Then there exist open definable neighbourhoods $V$ of $x_0$ and $W$ of $y_0$ and a definable, holomorphic map $\phi : V \to W$ such that $V \times W \subseteq U$, $\phi(x_0) = y_0$ and such that for all $(x, y) \in V \times W$ we have

$$f_1(x, y) = \ldots = f_n(x, y) = 0 \Leftrightarrow y = \phi(x).$$
Chapter 2

Preparation Theorems

Let $\mathcal{R} = (\mathbb{R}; 0, 1, +, \cdot, <, \ldots)$ be an o-minimal structure over the real field. As described in Chapter 1, we will work on the structure on the field $\mathbb{C}$ generated via the identification $\mathbb{C} = \mathbb{R}^2$. From now on we will work over this structure $\mathcal{R}$ on the real numbers (and elementary extensions of it), as we will need the concept of infinitesimal element in order to guarantee that the Theorems from this chapter, all of which deal with ‘standard’ elements, hold over elementary extensions.

2.1 Weierstrass Preparation

Given a $\mathbb{C}$-valued function $f(x)$, we will say that $f$ is of order $s$ at the origin if $s$ is the least integer for which $\frac{d^s f}{dx^s}(0)$ is nonzero. The following theorem, the Weierstrass Preparation Theorem, is already known to hold for definable analytic functions in $\mathbb{C}$ (cf. [10]):

**Theorem 2.1.1** (Weierstrass Preparation). Let $f(t, \bar{z})$ be a $\mathbb{C}$-valued definable, analytic function on a neighbourhood of the origin such that $f(t, \bar{0})$ is of order $s$ at the origin. There exist definable, analytic functions $h_1(\bar{z}), \ldots, h_s(\bar{z}), u(t, \bar{z})$, defined in a neighbourhood of the origin, such that

$$f(t, \bar{z}) = \left( t^s + \sum_{i=1}^s h_i(\bar{z}) t^{s-i} \right) u(t, \bar{z}).$$

Moreover, $u(0, 0) \neq 0$.

We will prove a slightly stronger version of this theorem. We are able to find a lower bound for the size of the neighbourhood of the origin in where these functions
are defined. Moreover, this bound is definable on the same parameters the starting function \( f \) is:

**Theorem 2.1.2.** Let \( D \subseteq \mathbb{C}^{1+n} \) be a \( \emptyset \)-definable, open neighbourhood of the origin; let \( f(x, \bar{y}, \bar{p}) : D \times P \to \mathbb{C} \) be a \( \emptyset \)-definable function such that, for each \( \bar{p} \in P \), the function \( f_{\bar{p}}(x, \bar{y}) = f(x, \bar{y}, \bar{p}) \) is analytic in \( D \), and let \( s \geq 0 \) be an integer such that each \( f_{\bar{p}}(x, \bar{y}) \) is regular in \( x \) of order \( s \) at the origin. Then there exists a \( \emptyset \)-definable \( \mathbb{R}^+ \)-valued function \( \varphi \) such that, for every \( \bar{p} \in P \), the representation of \( f_{\bar{p}} \) given by the Weierstrass Preparation Theorem 2.1.1 holds in the disc \( \{ (x, \bar{y}) : \|(x, \bar{y})\| \leq \varphi(\bar{p}) \} \).

In order to prove Theorem 2.1.2, we’ll follow the proof of the Weierstrass Preparation Theorem as presented in [12]. We’ll start by proving the Division Theorem, obtaining a lower bound for the radius of the neighbourhood of the origin where it holds:

Write

\[ P_s(t, \lambda) = t^s + \sum_{i=1}^{s} \lambda_i t^{s-i}. \]

We will need the following bound on the roots of a monic polynomial:

**Theorem 2.1.3 (Fujiwara’s Bound (cf. [2], [11])).** Let \( P = z^n + a_1 z^{n-1} + \cdots + a_n \) be a monic polynomials with complex coefficients. For every root \( z_0 \) of \( P \) we have

\[ |z_0| \leq 2 \max \left\{ |a_1|, \sqrt{a_2}, \ldots, \sqrt[n-1]{a_{n-1}}, \sqrt[n]{a_n} \right\}. \]

**Lemma 2.1.4 (Division).** Fix \( s \geq 0 \). For any open neighbourhood \( D \) of the origin in \( \mathbb{C}^{1+n+s} \) there exists a positive rational number \( q \) such that, given any \( \mathbb{C} \)-valued function \( F(t, z, \lambda) \) analytic in \( D \), there exist analytic functions

\[ A_i : \{ (z, \lambda) : \|(z, \lambda)\| < q \} \subseteq \mathbb{C}^{n+s} \to \mathbb{C}, \quad 1 \leq i \leq s \]

and

\[ Q : \{ (t, z, \lambda) : \|(t, z, \lambda)\| < q \} \subseteq \mathbb{C}^{1+n+s} \to \mathbb{C} \]

such that

\[ F(t, z, \lambda) = P_s(t, \lambda)Q(t, z, \lambda) + \sum_{i=1}^{s} A_i(z, \lambda)t^{s-i}. \]
Proof. Let \( r \in \mathbb{Q}^+ \) be such that \( B[0,r] \subseteq D \), and choose \( q \) positive, rational such that

\[
r > \begin{cases} 
2q & \text{for } s = 0, 1 \\
2 \max \left\{ q, q^{\frac{1}{s-1}}, \left( \frac{q}{2} \right)^{\frac{1}{s}} \right\} & \text{for } s \geq 2
\end{cases}
\]

Then, via Fujiwara’s Bound 2.1.3, for any tuple \( \lambda \) with \( \| \lambda \| < q \) every zero of the polynomial \( P_s(u, \lambda) \) will lie in \( B(0,r) \). Let \( \gamma \) be the circle \( |u| = r \) winding once around the origin in \( \mathbb{C} \), and we define

\[
Q(t, z, \lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(u, z, \lambda)}{P_s(u, \lambda)(u-t)} du
\]

\[
A_i(z, \lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{P_i(u, \lambda)F(u, z, \lambda)}{P_s(u, \lambda)} du.
\]

\( Q \) is well-defined and analytic in \( \{ (t, z, \lambda) : \| (t, z, \lambda) \| < q \} \subseteq \mathbb{C}^{1+n+s} \) due to the choice of \( \gamma \), and similarly the \( A_i \) are well-defined and analytic in \( \{ (z, \lambda) : \| (z, \lambda) \| < q \} \subseteq \mathbb{C}^{n+s} \). As

\[
F(t, z, \lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(u, z, \lambda)}{u-t} du
\]

(by Cauchy) the conclusion follows from the identity

\[
\frac{1}{u-t} = \frac{P_s(t, \lambda)}{P_s(u, \lambda)(u-t)} + \sum_{i=1}^{s} \frac{P_{i-1}(u, \lambda)}{P_s(u, \lambda)} t^{s-i}.
\]

\( \Box \)

In order to obtain the desired lower bounds for the radius of the domain in the Weierstrass Preparation Theorem, we need lower bounds for the Implicit Function Theorem first:

**Theorem 2.1.5 ([3]).**

- Let \( \psi(\vec{x}, z) \) be a \( \mathbb{C} \)-valued function of \( n+1 \) complex variables, analytic in a neighbourhood of \((\vec{0},0)\). Let \( |\frac{\partial \psi}{\partial \vec{x}}(\vec{0},0)| = a \neq 0 \), and let \( |\psi(\vec{0},z)| \leq M \) on \( B \), where \( B = \{ (\vec{x}, z) : |(\vec{x}, z)| \leq R \} \). Then \( z = g(\vec{x}) \) is an analytic function of \( \vec{x} \) in the ball

\[
|\vec{x}| \leq \Phi_1(M, a, R; \psi) = \frac{1}{M} \left( ar - \frac{Mr^2}{R^2 - rR} \right), \text{ where } r = \min \left( \frac{R}{2}, \frac{aR^2}{2M} \right).
\]

- Let \( \psi_1, \ldots, \psi_m(\vec{x}, \vec{z}) \) be \( \mathbb{C} \)-valued functions of \( n+m \) complex variables, analytic in a neighbourhood of \((\vec{0},\vec{0})\). Let \( |\frac{\partial \psi_i}{\partial \vec{x}_j}|_{1 \leq i, j \leq m} = a_m \neq 0 \), \( |\frac{\partial \psi_i}{\partial \vec{z}_j}|_{2 \leq i, j \leq m} = a_{m-1} \neq 0 \)
and let $|\psi(0,z)| \leq M$ on $B$, where $B = \{(\bar{x}, z) : |(\bar{x}, z)| \leq R\}$. Then $\bar{z} = g(\bar{x})$ is an analytic function of $\bar{x}$ in the ball

$$|ar{x}| \leq \Phi_m(\psi_1, \ldots, \psi_m) = \Phi_1(M, a_m, \min(R, \Phi_{m-1}(\psi_2, \ldots, \psi_m)); \psi_1).$$

Proof of Theorem 2.1.2. Define $q$ and $\gamma$ for $D$ and $s$ as in the previous Lemma. For each $p \in P$ we write

$$f_p(x, \bar{y}) = P_s(x, \lambda)Q_p(x, \bar{y}, \lambda) + \sum_{i=1}^{s} A_{i,p}(\bar{y}, \lambda)x^{s-i} \quad (2.1)$$

where $Q_p, A_{i,p}$ are given by Lemma 2.1.4 in the case that $F(t, z, \lambda) = f_p(x, \bar{y})$. Moreover, by Theorem 2.55 in [9], for each $i = 1, \ldots, s$ there exists a $\emptyset$-definable function $A_i(\bar{y}, \lambda, p)$ such that

$$A_i(-, -, p) = A_{i,p}(-, -)$$

for every $p \in P$.

Let $c : P \to \mathbb{C}$ be defined by

$$c(p) = \frac{1}{s!} \frac{\partial^s f_p}{\partial x^s}(0, \bar{0}).$$

Note that $c$ is $\emptyset$-definable, and $c(p) \neq 0$ for all $p$. From Equation 2.1 we obtain

$$\frac{\partial A_{i,p}}{\partial \lambda_j}(0, 0) = \begin{cases} -c(p) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and thus we can apply the Implicit Function Theorem to get, for each $p \in P$, $p$-definable, analytic functions $H_{j,p}(\bar{y})$ such that $A_{i,p}(\bar{y}, H_{j,p}(\bar{y})) = 0$ and

$$f_p(x, \bar{y}) = P_s(x, H_{j,p}(\bar{y}))Q(x, \bar{y})$$

on a neighbourhood of the origin. We’re interested in getting a lower bound for the radius of this neighbourhood. We will use the bounds for the domain of the Implicit Function Theorem shown in Theorem 2.1.5. To do so, let $M : P \to \mathbb{R}^+$ be $\emptyset$-definable satisfying both $M(p) > \frac{q|c(p)|}{2}$ and $|A_i(0, \lambda, p)| \leq M(p)$ for all $i = 1, \ldots, s$ and all $|\lambda| \leq \frac{q}{2}$. Using notation from Theorem 2.1.5, $\varphi(p) = \Phi_s(A_{1,p}, \ldots, A_{s,p})$ is definable, and is the required bound. 

$\square$
Lemma 2.1.6. Let $D \subseteq \mathbb{C}^{1+n}$ be a $\emptyset$-definable, open neighbourhood of the origin, and let $f(x, \bar{y}, \bar{p}) : D \times P \to \mathbb{C}$ be a $\emptyset$-definable function such that, for each $\bar{p} \in P$, the function $f_{\bar{p}}(x, \bar{y}) = f(x, \bar{y}, \bar{p})$ is analytic in $D$. For any $s \geq 0$, the set

$$P_s = \{ p \in P : f_p(x, \bar{y}) \text{ is regular in } x \text{ of order } s \text{ at the origin} \}$$

is definable.

Proof. Fix $s \geq 0$. The set $P_s$ consists exactly of all $p \in P$ such that $\frac{\partial^s f_p}{\partial x^s}(0, \bar{0}) = 0$ for all $i < s$ and $\frac{\partial^s f_p}{\partial x^s}(0, \bar{0}) \neq 0$. Each of these conditions is clearly first-order definable, and thus their intersection, $P_s$, is so as well. \qed

Combining this result with Theorem 2.1.2 we obtain

Corollary 2.1.7. Let $D \subseteq \mathbb{C}^{1+n}$ be a $\emptyset$-definable, open neighbourhood of the origin; let $f(x, \bar{y}, \bar{p}) : D \times P \to \mathbb{C}$ be a $\emptyset$-definable function such that, for each $\bar{p} \in P$, the function $f_{\bar{p}}(x, \bar{y}) = f(x, \bar{y}, \bar{p})$ is analytic in $D$, and let $s \geq 0$ be an integer such that each $f_{\bar{p}}(x, \bar{y})$ is regular in $x$ of order at most $s$ at the origin. Then there exists a $\emptyset$-definable $\mathbb{R}^+$-valued function $\varphi$ such that, for every $\bar{p} \in P$, the representation of $f_{\bar{p}}$ given by the Weierstrass Preparation Theorem holds in the disc $\{ (x, \bar{y}) : \|(x, \bar{y})\| \leq \varphi(\bar{p}) \}$.

Note that this is an improvement of Theorem 2.1.2 as it works for families of regular functions at the origin, as long as the order is uniformly bounded. This uniform bound on the radius of convergence of the Weierstrass Preparation Theorem is original, but as mentioned in the Introduction is not used in its full generality in the later results of this work.

2.2 The rings $\mathcal{O}_n$

Let $\mathcal{O}_n$ be the ring of all $\emptyset$-definable functions on $n$ variables $x_1, \ldots, x_n$ which are analytic in a neighbourhood of the origin in $\mathbb{C}^n$.

We will need some definitions from commutative algebra first:

- Let $A$ be a ring and $M$ an $A$-module. We say that $M$ is $A$-flat if for every integer $r \geq 1$ and every pair of $r$-tuples $(a_i)_{1 \leq i \leq r}$ in $A$ and $(m_i)_{1 \leq i \leq r}$ in $M$ satisfying
Theorem 2.2.1. For every $n \in \mathbb{N}$, the ring $\mathcal{O}_n[[x_{n+1}]]$ is faithfully flat as a $\mathcal{O}_{n+1}$-module.

Lemma 2.2.2. $\mathcal{O}_n$ is Noetherian.

Proof. By induction on $n$. For $n = 0$, $\mathcal{O}_0$ is the field of $\emptyset$-definable constants, and thus Noetherian. For the inductive step, assume $\mathcal{O}_n$ is Noetherian, and let $I \subseteq \mathcal{O}_{n+1}$ be an ideal. We will show that $I$ is finitely generated.

Given $f \in I$, by the Weierstrass Preparation Theorem 2.1.1 there exist a polynomial $g \in \mathcal{O}_n[x_{n+1}]$ and a unit $u \in \mathcal{O}_{n+1}$ such that $f = gu$. By the induction hypothesis and the Hilbert Basis Theorem, $\mathcal{O}_n[x_{n+1}]$ is Noetherian, and thus $I \cap \mathcal{O}_n[x_{n+1}]$ is finitely generated. As $g = u^{-1}gu = u^{-1}f \in I$, $g$ belongs to $I \cap \mathcal{O}_n[x_{n+1}]$, and is then generated by those finitely many generators. And so is $f$. \hfill \Box

Proposition 2.2.3 ([1], Proposition 10.14). Let $R$ be a Noetherian ring. Then $R[[X]]$ is flat as a $R[X]$-module.

Theorem 2.2.4. For every $n \in \mathbb{N}$, the ring $\mathcal{O}_n[[x_{n+1}]]$ is flat as a $\mathcal{O}_{n+1}$-module.

Proof. Fix $r \geq 1$ and elements $f_i \in \mathcal{O}_{n+1}$ and $P_i \in \mathcal{O}_n[x_{n+1}]$ such that $\sum_i f_i P_i = 0$. 

$\sum_i a_i m_i = 0$ there exist an integer $s \geq 1$ and elements $(b_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ in $A$ and $(n_j)_{1 \leq j \leq s}$ in $M$ such that

$$\sum_{i=1}^r a_i b_{ij} = 0 \quad \text{for } 1 \leq j \leq s, \quad \text{and}$$

$$\sum_{j=1}^s b_{ij} n_j = m_i \quad \text{for } 1 \leq i \leq r.$$

Let $A$ be a ring and $M$ an $A$-module. We say that $M$ is $A$-faithfully flat if $M$ is $A$-flat, and for every maximal ideal $m$ of $A$ we have $mM \neq M$.

These definitions, in particular the second one, are not the usual definitions from algebra for these concepts. They are, however, equivalent to the standard definitions (cf. [7], Chapter 2 §2 and §3).

We aim to prove the following:

Theorem 2.2.1. For every $n \in \mathbb{N}$, the ring $\mathcal{O}_n[[x_{n+1}]]$ is faithfully flat as a $\mathcal{O}_{n+1}$-module.

Lemma 2.2.2. $\mathcal{O}_n$ is Noetherian.

Proof. By induction on $n$. For $n = 0$, $\mathcal{O}_0$ is the field of $\emptyset$-definable constants, and thus Noetherian. For the inductive step, assume $\mathcal{O}_n$ is Noetherian, and let $I \subseteq \mathcal{O}_{n+1}$ be an ideal. We will show that $I$ is finitely generated.

Given $f \in I$, by the Weierstrass Preparation Theorem 2.1.1 there exist a polynomial $g \in \mathcal{O}_n[x_{n+1}]$ and a unit $u \in \mathcal{O}_{n+1}$ such that $f = gu$. By the induction hypothesis and the Hilbert Basis Theorem, $\mathcal{O}_n[x_{n+1}]$ is Noetherian, and thus $I \cap \mathcal{O}_n[x_{n+1}]$ is finitely generated. As $g = u^{-1}gu = u^{-1}f \in I$, $g$ belongs to $I \cap \mathcal{O}_n[x_{n+1}]$, and is then generated by those finitely many generators. And so is $f$. \hfill \Box

Proposition 2.2.3 ([1], Proposition 10.14). Let $R$ be a Noetherian ring. Then $R[[X]]$ is flat as a $R[X]$-module.

Theorem 2.2.4. For every $n \in \mathbb{N}$, the ring $\mathcal{O}_n[[x_{n+1}]]$ is flat as a $\mathcal{O}_{n+1}$-module.

Proof. Fix $r \geq 1$ and elements $f_i \in \mathcal{O}_{n+1}$ and $P_i \in \mathcal{O}_n[x_{n+1}]$ such that $\sum_i f_i P_i = 0$. 

$\sum_i a_i m_i = 0$ there exist an integer $s \geq 1$ and elements $(b_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ in $A$ and $(n_j)_{1 \leq j \leq s}$ in $M$ such that

$$\sum_{i=1}^r a_i b_{ij} = 0 \quad \text{for } 1 \leq j \leq s, \quad \text{and}$$

$$\sum_{j=1}^s b_{ij} n_j = m_i \quad \text{for } 1 \leq i \leq r.$$
By the Weierstrass Preparation Theorem 2.1.1, there exist units \( u_i \in \mathcal{O}_{n+1} \) and polynomials \( g_i \in \mathcal{O}_n[x_{n+1}] \) such that, for all \( i \), \( f_i = u_ig_i \). Now, as \( \mathcal{O}_n[[x_{n+1}]] \) is \( \mathcal{O}_n[x_{n+1}] \)-flat (by Proposition 2.2.3) there exist an integer \( s \geq 1 \) and elements \( (b_{ij})_{1 \leq i \leq r, 1 \leq j \leq s} \) in \( \mathcal{O}_n[x_{n+1}] \) and \( (Q_j)_{1 \leq j \leq s} \) in \( \mathcal{O}_n[[x_{n+1}]] \) such that

\[
\sum_{i=1}^{r} g_i b_{ij} = 0 \quad \text{for } 1 \leq j \leq s, \text{ and }
\]

\[
\sum_{j=1}^{s} b_{ij} Q_j = u_i P_i \quad \text{for } 1 \leq i \leq r.
\]

We can then define, for \( 1 \leq i \leq r, 1 \leq j \leq s \) and \( 1 \leq k \leq r \), \( b_{ijk}' = \frac{b_{ij}}{u_i u_k} \in \mathcal{O}_{n+1} \) and \( Q_{jk}' = \frac{u_i Q_j}{r} \). We then have

\[
\sum_{i=1}^{r} f_i b_{ijk}' = \sum_{i=1}^{r} u_i g_i \frac{b_{ij}}{u_i u_k} = \frac{1}{u_k} \sum_{i=1}^{r} g_i b_{ij} = 0 \quad \text{for } 1 \leq j \leq s, 1 \leq k \leq r, \text{ and }
\]

\[
\sum_{j=1}^{s} \sum_{k=1}^{r} b_{ijk}' Q_{jk}' = \sum_{j=1}^{s} \sum_{k=1}^{r} \frac{b_{ij}}{u_i u_k} \frac{u_k Q_j}{r} = \frac{1}{u_i} \sum_{j=1}^{s} \sum_{k=1}^{r} \frac{b_{ij} Q_j}{r} = P_i \quad \text{for } 1 \leq i \leq r
\]

which concludes the proof. \( \square \)

**Proof of Theorem 2.2.1.** As we have already proved flatness, we need to show that for every maximal ideal \( m \leq \mathcal{O}_{n+1} \) we have \( m\mathcal{O}_n[[x_{n+1}]] \neq \mathcal{O}_n[[x_{n+1}]] \). This follows from the fact that, for every such ideal \( m \) and every \( f \in m \), when writing \( f \) as a power series in \( x_{n+1} \) the coefficient of \( x_{n+1}^0 \), say \( c_0(x_1, \ldots, x_n) \), must satisfy \( c_0(0, \ldots, 0) = 0 \). In particular, for no such \( m \) can we have \( 1 \in m\mathcal{O}_n[[x_{n+1}]] \). \( \square \)

For faithfully flat morphisms of rings we have the following result:

**Fact 2.2.5** ([7], 4.C (ii)). Let \( f : A \to B \) be a faithfully flat morphism of rings. Then for any ideal \( I \leq A \) we have \( IB \cap A = I \).

From this Fact and Theorem 2.2.1 we obtain the following, which we will need in the next section:

**Corollary 2.2.6.** Fix \( n \geq 0, r \geq 1 \). Let \( \sum_{i=1}^{r} c_i y_i = c_0 \) be a linear equation in the \( y_i \), with coefficients in \( \mathcal{O}_{n+1} \). If this equation has a solution in \( \mathcal{O}_n[[x_{n+1}]] \), then it has a solution in \( \mathcal{O}_{n+1} \).

**Proof.** Let \( I \) be the ideal \( (c_1, \ldots, c_r) \leq \mathcal{O}_{n+1} \). We want to show that \( c_0 \) belongs in \( I \). By the hypothesis of the Corollary, \( c_0 \) is in \( \mathcal{O}_{n+1} \); as the equation given has a solution in \( \mathcal{O}_n[[x_{n+1}]] \), then \( c_0 \) belongs in \( I\mathcal{O}_n[[x_{n+1}]] \) as well. \( \square \)
2.3 Denef - van den Dries Preparation

In this section we aim to prove the following preparation theorem, shown by Denef and van den Dries for the real case in [5]:

**Theorem 2.3.1** (Denef - van den Dries Preparation). Let \( f(x, y) \) be a \( \emptyset \)-definable function, analytic in a neighbourhood \( U \subseteq \mathbb{C}^{n+m} \) of the origin, where \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_m) \). There exists an integer \( d > 0 \) such that

\[
    f(x, y) = \sum_{|i| < d} a_i(x) y^i u_i(x, y)
\]

where the \( a_i \) are \( \emptyset \)-definable functions analytic in \( \pi_n(U) \), the projection of \( U \) to its first \( n \) coordinates, and the \( u_i \) are invertible elements of \( \mathcal{O}_{n+m} \).

**Proof.** The proof is a straightforward adaptation of the original proof in [5] for \( \mathbb{R} \). We proceed by induction on \( m \). For \( m = 0 \) it is trivial, for \( m = 1 \) write

\[
    f(x, y) = \sum_{i \in \mathbb{N}} a_i(x) y^i,
\]

where the \( a_i \) are \( \emptyset \)-definable and analytic in \( \pi_n(U) \).

As \( \mathcal{O}_n \) is Noetherian, the ideal generated by the \( a_i(x) \) is generated by a finite number of them, and thus there exist an integer \( d \) and functions \( b_{ij}(x) \in \mathcal{O}_n \) for \( i < d \) and \( j \geq d \) such that, for every \( j \geq d \),

\[
    a_j(x) = \sum_{i < d} b_{ij}(x) a_i(x). \tag{2.2}
\]

Consider the following linear equation on the unknowns \( z_i \):

\[
    f(x, y) = \sum_{i < d} a_i(x) y^i (1 + y z_i).
\]

This equation has coefficients in \( \mathcal{O}_{n+1} \) and, by Equation 2.2, is solved in \( \mathcal{O}_n[[y]] \) by \( z_i = \sum_{j \geq d} b_{ij}(x) y^{j-i-1} \). By Corollary 2.2.6, there exist functions \( g_i(x, y) \in \mathcal{O}_{n+1} \) such that

\[
    f(x, y) = \sum_{i < d} a_i(x) y^i (1 + y g_i(x, y));
\]

taking \( u_i(x, y) = 1 + y g(x, y) \) we conclude this case.
For \( m > 1 \), writing \( \bar{y} = (y_2, \ldots, y_m) \) and applying the inductive hypothesis, we obtain
\[
f(x, y) = \sum_{|i| < d} a_i(x, y_1) \bar{y}^i u_i(x, y)
\]
for some \( d > 0 \), where the \( a_i \) are \( \emptyset \)-definable functions analytic in \( \pi_{n+1}(U) \), the projection of \( U \) to its first \( n + 1 \) coordinates, and the \( u_i \) are invertible elements of \( \mathcal{O}_{n+m} \).

Applying the case \( m = 1 \) again, there exists \( c > 0 \) such that, for each \( |i| < d \),
\[
a_i(x, y_1) = \sum_{j < c} b_j(x) y_1^j v_{ij}(x, y_1),
\]
where the \( b_i \) are \( \emptyset \)-definable functions analytic in \( \pi_n(U) \) and the \( v_{ij} \) are invertible elements of \( \mathcal{O}_{n+1} \).

Thus
\[
f(x, y) = \sum_{|i| < d} \sum_{j < c} b_j(x) \bar{y}_i^j y_1^j (u_i(x, y)v_{ij}(x, y_1)).
\]
Chapter 3

The algebraic-analytic closure

Let $\mathcal{R} = (\mathbb{R}; 0, 1, +, \cdot, <, \ldots)$ be an o-minimal structure over the real field, and let $\mathcal{R}'$ be an elementary extension of $\mathcal{R}$ with base set $R$, a non-archimedean real closed field. Set $K = R[\sqrt{-1}]$, the algebraic closure of $R$, and consider the structure generated on $K$ via the identification $K = R^2$, as described in Chapter 1. For $n \in \mathbb{N}$, let $\mathcal{O}_n(\mathbb{C})$ be the ring of $\mathbb{C}$-definable functions holomorphic in a $\mathbb{C}$-definable neighbourhood of the origin in $K^n$.

3.1 The closure function

For a set $A \subseteq K$, we define the algebraic-analytic closure of $A$, $\text{Alg/An}(A)$ as the smallest subset of $K^n$ containing $A$ such that

- if $a_1, \ldots, a_n$ are infinitesimal elements of $\text{Alg/An}(A)$ and $F \in \mathcal{O}_n(\mathbb{C})$, then $F(a_1, \ldots, a_n)$ is in $\text{Alg/An}(A)$; and

- $\text{Alg/An}(A)$ is algebraically closed.

We can consider the algebraic-analytic closure as an iteration of alternating algebraic closures and closures under certain definable analytic functions. Formally, given a set $X \subseteq K$, we define

- $\text{alg}(X)$ as the algebraic closure of $X$; and

- the infinitesimal image of $X$, $\text{in}(X)$, as the set of all $a \in K$ such that there exist
infinitesimal elements $c_1, \ldots, c_n \in X$ and a function $F \in \mathcal{O}_n(\mathbb{C})$ satisfying
\[ a = F(c_1, \ldots, c_n). \]

We iterate these two operators by inductively defining the sets $N_i(X)$ and $G_i(X)$, for $i \in \mathbb{N}$:
\[
G_0(X) = X; \\
N_i(X) = \text{in}(G_i(X)); \\
G_{i+1}(X) = \text{alg}(N_i(X)).
\]

This generates the following chain of subsets, which we will use frequently in this chapter:
\[
X = G_0(X) \subseteq N_0(X) \subseteq G_1(X) \subseteq \ldots \subseteq G_i(X) \subseteq N_i(X) \subseteq G_{i+1}(X) \subseteq \ldots \tag{3.1}
\]

Then, from the very definition of the algebraic-analytic closure, we have

**Proposition 3.1.1.** For any $A \subseteq K$,
\[
\text{Alg/An}(A) = \bigcup_{i \in \mathbb{N}} N_i(A) = \bigcup_{i \in \mathbb{N}} G_i(A).
\]

**Lemma 3.1.2.** Let $A, B$ be subsets of $K$ satisfying $A \subseteq B$. Then, for all $i \in \mathbb{N}$, $N_i(A) \subseteq N_i(B)$ and $G_i(A) \subseteq G_i(B)$.

**Proof.** By induction on the subsets $N_i$ and $G_i$, ordered as in (3.1). The base case, $G_0(A) \subseteq G_0(B)$, is trivial; for the inductive steps, fix $i \geq 0$. For any $a \in N_i(A)$ there exist $c_1, \ldots, c_n \in G_i(A)$ and $F \in \mathcal{O}_n(\mathbb{C})$ such that $F(c_1, \ldots, c_n) = a$. As, by induction hypothesis, $c_1, \ldots, c_n \in G_i(B)$ as well, then $a \in N_i(B)$ and $N_i(A) \subseteq N_i(B)$.

Finally, $G_{i+1}(A) \subseteq G_{i+1}(B)$ follows from $N_i(A) \subseteq N_i(B)$ and the fact that the algebraic closure is a known closure operator.

The previous Lemma implies the following:

**Corollary 3.1.3.** Let $A, B$ be subsets of $K$ satisfying $A \subseteq B$. Then $\text{Alg/An}(A) \subseteq \text{Alg/An}(B)$.

**Lemma 3.1.4.** Let $A$ be a subset of $K$. Then $\text{Alg/An}(\text{Alg/An}(A)) \subseteq \text{Alg/An}(A)$. 

Proof. By definition of Alg/An, Alg/An(A) = an(Alg/An(A)) = alg(Alg/An(A)). Thus the entire chain in Equation 3.1 for Alg/An(A) is constant equal to Alg/An(A).

A closure operator on a non-empty set $X$ is a map $c : \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies

1. for all $A, B \subseteq X$, if $A \subseteq B$ then $c(A) \subseteq c(B)$;
2. for all $A \subseteq X$, $A \subseteq c(A)$; and
3. for all $A \subseteq X$, $c(A) = c(c(A))$.

Proposition 3.1.5. Alg/An is a closure operator.

Proof. The three properties of a closure operator are satisfied by Alg/An; they follow, respectively, from Corollary 3.1.3, Proposition 3.1.1, and Lemma 3.1.4.

Proposition 3.1.6. For every $a \in \text{Alg/An}(A)$ there exists a finite subset $A_0$ of $A$ such that $a \in \text{Alg/An}(A_0)$.

Proof. By induction on the subsets $N_i(A)$ and $G_i(A)$, ordered as in (3.1). Fix $a \in A$, and let $X$ be the smallest subset among the $N_i(A)$ and $G_i(A)$ containing $a$.

- For $X = G_0(A)$, then $a \in A$, and we have $a \in \text{Alg/An}({a})$.
- For $X = N_i(A)$ for some $i \geq 0$, there exist elements $c_1, \ldots, c_n \in G_i(A)$ and a function $F \in \mathcal{O}_n(\mathbb{C})$ such that $a = F(c_1, \ldots, c_n)$. By induction hypothesis, there exist finite sets $C_1, \ldots C_n \subseteq A$ such that

\[ c_j \in \text{Alg/An}(C_j), \quad \text{for} \quad 1 \leq j \leq n. \]

Then $a \in \text{Alg/An}(C_1 \cup \cdots \cup C_n)$.

- For $X = G_{i+1}(A)$ for some $i \geq 0$, let $c_1, \ldots, c_n \in N_i(A)$ be such that $a$ is a zero of $\sum c_j x^j$. By induction hypothesis, there exist finite sets $C_1, \ldots, C_n \subseteq A$ such that

\[ c_j \in \text{Alg/An}(C_j), \quad \text{for} \quad 1 \leq j \leq n. \]

Then $a \in \text{Alg/An}(C_1 \cup \cdots \cup C_n)$. 

\qed
3.2 Elimination of variables via resultants

Before proceeding to show more properties of the algebraic-analytic closure, we will need to prove some results related to the resultant of two polynomials. Given a unique factorisation domain $A$ and non-constant polynomials $f, g \in A[x]$, the resultant of $f$ and $g$ with respect to $x$, $\text{Res}(f, g, x)$ is an integer polynomial on the coefficients of $f$ and $g$ (depending, as an integer polynomial, only on the degrees of $f$ and $g$) that satisfies, among others, the following properties:

- $\text{Res}(f, g, x) = 0$ if and only if $f$ and $g$ have a common factor of positive degree;
- there exist $p, q \in A[x]$, not both zero, such that $\text{Res}(f, g, x) = pf + qg$; and
- if $B$ is a ring of functions over $A$ and $f(x, \bar{y}), g(x, \bar{y}) \in B[x]$ are monic and non-constant as polynomials on $x$, then $\text{Res}(f(x, \bar{y}), g(x, \bar{y}), x)$ is a function on the variables $\bar{y}$, and, for every $\bar{y}_0$ in its domain,
  \[ \text{Res}(f(x, \bar{y}), g(x, \bar{y}), x)(\bar{y}_0) = \text{Res}(f(x, \bar{y}_0), g(x, \bar{y}_0), x). \]

See ([4], Chapter 3, §5 and §6) for details.

We will need these properties in the following result:

**Proposition 3.2.1.** Let $f(\bar{x}, y), g(\bar{x}, y) \in \mathcal{O}_n(\mathbb{C})[y]$ be monic, and let $(\bar{\alpha}, \beta) \in K^{n+1}$ be infinitesimals such that $f(\bar{\alpha}, \beta) = g(\bar{\alpha}, \beta) = 0$, $\frac{\partial f}{\partial y}(\bar{\alpha}, \beta) \neq 0$ and such that the rank of the matrix

\[
\begin{bmatrix}
\frac{\partial f}{\partial x_j}(\bar{\alpha}, \beta) \\
\frac{\partial g}{\partial x_j}(\bar{\alpha}, \beta)
\end{bmatrix}_{1 \leq j \leq n}
\]

is two. Then there exist a function $h(\bar{x}) \in \mathcal{O}_n(\mathbb{C})$ and constants $C, D \in K$ (which do not depend on $j$), with $C$ non-zero, such that $h(\bar{\alpha}) = 0$ and, for any $1 \leq j \leq n$,

\[
\frac{\partial h}{\partial x_j}(\bar{\alpha}) = C \left( \frac{\partial g}{\partial x_j}(\bar{\alpha}, \beta) + D \frac{\partial g}{\partial y}(\bar{\alpha}, \beta) \right).
\]

**Proof.** If $g(\bar{x}, y)$ is constant as a polynomial on $y$, then we may take $C = 1$, $D = 0$ and $h(\bar{x}) = g(\bar{x}, 0)$. If it is not, we construct the resultant $\text{Res}(f, g, y) \in \mathcal{O}_n(\mathbb{C})$.

Suppose this resultant is identically zero as a function on $\bar{x}$. Then there exist functions $p, \tilde{f}, \tilde{g} \in \mathcal{O}_n(\mathbb{C})[y]$ with $\deg_y p > 0$ such that $f = p\tilde{f}$, $g = p\tilde{g}$. We claim that
p(\bar{\alpha}, \bar{\beta}) \neq 0. Indeed, if it were zero, the matrix of partial derivatives of \( f \) and \( g \) at \((\bar{\alpha}, \bar{\beta})\) would be

\[
\begin{bmatrix}
\tilde{f}(\bar{\alpha}, \bar{\beta}) \left[ \frac{\partial p}{\partial x_j}(\bar{\alpha}, \bar{\beta}) \right]_{1 \leq j \leq n} & \tilde{f}(\bar{\alpha}, \bar{\beta}) \left[ \frac{\partial p}{\partial y}(\bar{\alpha}, \bar{\beta}) \right] \\
\tilde{g}(\bar{\alpha}, \bar{\beta}) \left[ \frac{\partial p}{\partial x_j}(\bar{\alpha}, \bar{\beta}) \right]_{1 \leq j \leq n} & \tilde{g}(\bar{\alpha}, \bar{\beta}) \left[ \frac{\partial p}{\partial y}(\bar{\alpha}, \bar{\beta}) \right]
\end{bmatrix}
\]

contradicting the fact that its rank must be two. Then \( \tilde{f}(\bar{\alpha}, \bar{\beta}) = \tilde{g}(\bar{\alpha}, \bar{\beta}) = 0 \), and

\[
\begin{bmatrix}
\left[ \frac{\partial \tilde{f}}{\partial x_j}(\bar{\alpha}, \bar{\beta}) \right]_{1 \leq j \leq n} & \left[ \frac{\partial \tilde{f}}{\partial y}(\bar{\alpha}, \bar{\beta}) \right] \\
\left[ \frac{\partial \tilde{g}}{\partial x_j}(\bar{\alpha}, \bar{\beta}) \right]_{1 \leq j \leq n} & \left[ \frac{\partial \tilde{g}}{\partial y}(\bar{\alpha}, \bar{\beta}) \right]
\end{bmatrix} = \frac{1}{p(\bar{\alpha}, \bar{\beta})} \begin{bmatrix}
\left[ \frac{\partial f}{\partial x_j}(\bar{\alpha}, \bar{\beta}) \right]_{1 \leq j \leq n} & \left[ \frac{\partial f}{\partial y}(\bar{\alpha}, \bar{\beta}) \right] \\
\left[ \frac{\partial g}{\partial x_j}(\bar{\alpha}, \bar{\beta}) \right]_{1 \leq j \leq n} & \left[ \frac{\partial g}{\partial y}(\bar{\alpha}, \bar{\beta}) \right]
\end{bmatrix}.
\]

Thus \( \tilde{f} \) and \( \tilde{g} \) also satisfy the hypotheses of this Proposition; as \( 0 < \deg_y \tilde{f} < \deg_y f \), we cannot repeat this process indefinitely, and so there exist \( P, Q, R \in \mathcal{O}_n(\mathbb{C})[y] \) such that \( f = PQ, g = PR, \frac{\partial Q}{\partial y}(\bar{\alpha}, \bar{\beta}) \neq 0, P(\bar{\alpha}, \bar{\beta}) \neq 0 \) and \( \text{Res}(Q, R, y) \neq 0 \).

Let \( h(\bar{x}) = \text{Res}(Q, R, y) \). There exist \( h_1, h_2 \in \mathcal{O}_n(\mathbb{C})[y] \), not both zero at \((\bar{\alpha}, \bar{\beta})\), such that \( h = Qh_1 + Rh_2 \). We claim that \( h_2(\bar{\alpha}, \bar{\beta}) \neq 0 \). This follows from the equality

\[
0 = \frac{\partial h}{\partial y}(\bar{\alpha}, \bar{\beta}) = h_1(\bar{\alpha}, \bar{\beta}) \frac{\partial Q}{\partial y}(\bar{\alpha}, \bar{\beta}) + h_2(\bar{\alpha}, \bar{\beta}) \frac{\partial R}{\partial y}(\bar{\alpha}, \bar{\beta})
\]

and the fact that \( \frac{\partial Q}{\partial y}(\bar{\alpha}, \bar{\beta}) \neq 0 \).

Clearly \( h(\bar{\alpha}) = 0 \); by the Implicit Function Theorem, there exists a function \( \psi \), holomorphic in a neighbourhood \( U \) of \( \alpha \), such that \( Q(\bar{x}, \psi(\bar{x})) \) for all \( \bar{x} \in U \). Thus, over \( U \),

\[
h(\bar{x}) = h_2(\bar{x}, \psi(\bar{x}))R(\bar{x}, \psi(\bar{x}))
\]

and, for \( 1 \leq j \leq n \),

\[
\frac{\partial h}{\partial x_j}(\bar{\alpha}) = h_2(\bar{\alpha}, \bar{\beta}) \left( \frac{\partial R}{\partial x_j}(\bar{\alpha}, \bar{\beta}) + \frac{\partial R}{\partial y}(\bar{\alpha}, \bar{\beta}) \frac{\partial \psi}{\partial x_j}(\bar{\alpha}) \right).
\]

The conclusion follows from the fact that

\[
\frac{\partial g}{\partial x_j}(\bar{\alpha}, \bar{\beta}) = P(\bar{\alpha}, \bar{\beta}) \frac{\partial R}{\partial x_j}(\bar{\alpha}, \bar{\beta})
\]

for all \( 1 \leq j \leq n \); that

\[
\frac{\partial g}{\partial y}(\bar{\alpha}, \bar{\beta}) = P(\bar{\alpha}, \bar{\beta}) \frac{\partial R}{\partial y}(\bar{\alpha}, \bar{\beta})
\]

and that \( P(\bar{\alpha}, \bar{\beta}) \neq 0 \).

We will refer to this \( h \) as the reduction of \( g \) from \( f \) with respect to the variable \( y \). It will be the key to the inductive step of the main Theorem of the next section.


3.3 Closure under implicit functions

In this section we will show that, given \( m \) functions \( f_1, \ldots, f_m \in \mathcal{O}_{n+m} (\mathbb{C}) \) and a tuple of infinitesimals \( \tilde{\alpha} \in K^n \), then any infinitesimal \( \tilde{\beta} \in K^m \) which is a non-trivial solution to the system \( f_1(\tilde{\alpha}, \tilde{y}) = \ldots = f_m(\tilde{\alpha}, \tilde{y}) = 0 \) belongs to the algebraic-analytic closure of \( \tilde{\alpha} \). This is analogous to the Implicit Function Theorem, yet it cannot be directly obtained from said Theorem, as the implicitly defined functions are not necessarily holomorphic in a neighbourhood of the origin.

We will first define a change of variables we will need later, in order to be able to apply the Weierstrass Preparation Theorem 2.1.1 to the functions we are interested in:

**Lemma 3.3.1.** Let \( U \) be a definable neighbourhood of the origin in \( K^m \), and let \( f(x_1, \ldots, x_m) : U \to K \) be a definable, holomorphic function of the form

\[
    f(\bar{x}) = \bar{x}^k u_k(\bar{x}) + \sum_{|i| < d, i \neq k} c_i \bar{x}^i u_i(\bar{x})
\]

for some definable, holomorphic functions \( u_i \), some constants \( c_i \in K \), some integer \( d > 0 \) and some multi-index \( k \) with \( |k| < d \). Let \( \tilde{a} = (a_1, \ldots, a_m) \in U \) be such that \( f(\tilde{a}) = 0 \) and \( u_i(\tilde{a}) \neq 0 \) for every multi-index \( i \). Then there exists a \( \emptyset \)-definable polynomial bijection \( \Phi \) over \( K^m \), depending only on \( m \) and \( d \), such that

- \( \Phi \) maps tuples of infinitesimals to tuples of infinitesimals;
- the inverse \( \Phi^{-1} : K^m \to K^m \) is also polynomial;
- the \( m \)-th component of \( \Phi(x_1, \ldots, x_m) \) is \( x_m \);
- \( \Phi(\bar{0}) = \bar{0} \); and
- the function \( g(\bar{x}) = f(\Phi(\bar{x})) \) is regular with respect to \( x_m \) at the origin.

**Proof.** Consider the function

\[
    \Phi(x_1, \ldots, x_m) = (x_1 + (x_m)^{d-1}, x_2 + (x_m)^{d-2}, \ldots, x_m - x_m^d, x_m).
\]

It is straightforward to verify that

\[
    \Psi(x_1, \ldots, x_m) = (x_1 - (x_m)^{d-1}, x_2 - (x_m)^{d-2}, \ldots, x_m - x_m^d, x_m)
\]
satisfies $\Psi = \Phi^{-1}$, and thus we only need to show that $f(\Phi(\bar{x}))$ is regular with respect to $x_m$ at the origin. Indeed, let $l = (l_1, \ldots, l_m)$ be the lexicographically smallest multi-index such that $\bar{x}^l$ has non-zero coefficient in

$$f(\bar{x}) = \bar{x}^k u_k(\bar{x}) + \sum_{|i| < d, i \neq k} c_i \bar{x}^i u_i(\bar{x})$$

(we know this $l$ exists, for $k$ is such that $\bar{x}^k$ has non-zero coefficient). As $|l| < d$, for any multi-index $t = (t_1, \ldots, t_m)$ lexicographically larger than $l$ and satisfying $|t| < d$, we have

$$l_1 d^{m-1} + l_2 d^{m-2} + \cdots + l_{m-1} d + l_m < t_1 d^{m-1} + t_2 d^{m-2} + \cdots + t_{m-1} d + t_m$$

and thus the power series $f(\Phi(0, \ldots, 0, x_m))$ has order $l_1 d^{m-1} + l_2 d^{m-2} + \cdots + l_{m-1} d + l_m$ at the origin.

This bijection was shown by J. Denef and L. van den Dries in the proof of Lemma 4.13 in [5].

We can now show the closure under implicit functions:

**Theorem 3.3.2.** Let $f_1(\bar{x}, \bar{y}), \ldots, f_m(\bar{x}, \bar{y}) \in \mathcal{O}_{n+m}(\mathbb{C})$ and let $(\bar{\alpha}, \bar{\beta}) \in K^{n+m}$ be infinitesimals such that

- $f_1(\bar{\alpha}, \bar{\beta}) = \ldots = f_m(\bar{\alpha}, \bar{\beta}) = 0$; and,
- the Jacobian determinant $\left| \frac{\partial f_j}{\partial y_j}(\bar{\alpha}, \bar{\beta}) \right|$ is non-zero.

Then $\bar{\beta} \in \text{Alg}/\text{An}(\{\bar{\alpha}\})$.

**Proof.** By induction on $m$. By the Denef - van den Dries Preparation Theorem 2.3.1, there exists an integer $d > 0$ such that, for every $1 \leq j \leq m$, there exist $a_{ij} \in \mathcal{O}_n(\mathbb{C})$ and $u_{ij} \in \mathcal{O}_{n+1}(\mathbb{C})^*$, with $i$ ranging over all multi-indices $(i_1, \ldots, i_m) \in \mathbb{N}^m$ with $|i| < d$, such that

$$f_j(\bar{x}, \bar{y}) = \sum_{|i| < d} a_{ij}(\bar{x}) \bar{y}^i u_{ij}(\bar{x}, \bar{y}).$$

Fix $j \in \{1, \ldots, m\}$. At least one of the partial derivatives $\frac{\partial f_j}{\partial y_j}(\bar{\alpha}, \bar{\beta})$ (for some $l$) must be non-zero, and thus at least one of the $a_{ij}(\bar{\alpha})$ must be non-zero as well; let $k$ be such that $|a_{kj}(\bar{\alpha})|$ is maximal among the $a_{ij}(\bar{\alpha})$. Theorem 3.3.2
For $|i| < d$, $i \neq k$, we write $\frac{a_{ij}(\alpha)}{a_{kj}(\alpha)} = S_i + c_i$, where $S_i \in \mathbb{C}$ and $c_i$ is an infinitesimal element of $K$. Consider the $\mathbb{C}$-definable function

$$
\tilde{f}_j(\bar{x}, \bar{y}, \bar{z}) = \bar{y}^k u_{kj}(\bar{x}, \bar{y}) + \sum_{\bar{i} \neq \bar{k}} (S_i + z_i) \bar{i}^t u_{ij}(\bar{x}, \bar{y}).
$$

Clearly, this function is definable and holomorphic in a neighbourhood of the origin. Moreover,

$$
\tilde{f}_j(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \frac{1}{a_{kj}(\alpha)} \left( \sum_{|i| < d} a_{ij}(\alpha) \bar{\beta}^i u_{ij}(\bar{\alpha}, \bar{\beta}) \right) = 0
$$

and, for all $1 \leq l \leq m$,

$$
\frac{\partial \tilde{f}_j}{\partial y_l}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \frac{1}{a_{kj}(\alpha)} \frac{\partial f_j}{\partial y_l}(\bar{\alpha}, \bar{\beta}).
$$

Note that $c_i \in \text{Alg/An}(\{\bar{\alpha}\})$ for all $i$.

We have obtained a system of $m$ such functions $\tilde{f}_1, \ldots, \tilde{f}_m$, and an infinitesimal tuple $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ such that $f_j(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = 0$ for all $i$, and $\bar{\gamma} \in \text{Alg/An}(\{\bar{\alpha}\})$. Additionally, the Jacobian matrix $\left[ \frac{\partial \tilde{f}_j}{\partial y_j}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \right]$ is obtained from the matrix $\left[ \frac{\partial f_j}{\partial y_j}(\bar{\alpha}, \bar{\beta}) \right]$ by multiplying each row by a non-zero constant, and thus the Jacobian determinant $\left| \frac{\partial \tilde{f}_j}{\partial y_j}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \right|$ is non-zero.

Let $\Phi$ be the $\emptyset$-definable, entire bijection defined in Lemma 3.3.1 for our current values of $m$ and $d$. For every $j$, the function $g_j(\bar{x}, \bar{y}, \bar{z}) = f_j(\bar{x}, \Phi(\bar{y}), \bar{z})$ is regular with respect to the variable $y_m$ at the origin; writing $\bar{\delta} = \Phi^{-1}(\bar{\beta})$, we have $g_j(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) = 0$, for all $j$, and, as $\Phi$ is invertible, the Jacobian determinant $\left| \frac{\partial g_j}{\partial y_j}(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) \right|$ is non-zero.

By the Weierstrass Preparation Theorem 2.1.1, for each $g_i$, there exists a function $p_i(\bar{x}, \bar{y}, \bar{z})$ which is regular on the variable $y_m$, monic, and its coefficients are $\mathbb{C}$-definable functions of the variables $\bar{x}, \bar{z}, y_1, \ldots, y_{m-1}$, holomorphic in a neighbourhood of the origin. Moreover, for every $j$, $p_j(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) = 0$, and the Jacobian matrix $\left[ \frac{\partial p_j}{\partial y_j}(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) \right]$ is obtained from the matrix $\left[ \frac{\partial g_j}{\partial y_j}(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) \right]$ by multiplying each row by a non-zero constant, and thus the Jacobian determinant $\left| \frac{\partial p_j}{\partial y_j}(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) \right|$ is non-zero.

By reindexing the $p_i$ if necessary, we may assume that $\frac{\partial p_m}{\partial y_m}(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) \neq 0$. Thus $p_m$ is a polynomial of positive degree on $y_m$, and thus

$$
\beta_m = \delta_m \in \text{Alg/An}(\{\bar{\alpha}, \bar{\gamma}, \delta_1, \ldots, \delta_{m-1}\}) = \text{Alg/An}(\{\bar{\alpha}, \delta_1, \ldots, \delta_{m-1}\}).
$$

In particular, this proves the case $m = 1$. For $m > 1$, we define, for every $1 \leq j \leq m-1$, $h_l$ to be the reduction of $p_j$ from $p_m$ with respect to the variable $y_m$ (defined in the
previous section). The $h_j$ are functions on the variables $(\bar{x}, y_1, \ldots, y_{m-1}, \bar{z})$, for all $j$, $h_j(\bar{\alpha}, \delta_1, \ldots, \delta_{m-1}, \bar{\gamma}) = 0$ and, by Proposition 3.2.1, there exist constants $C_j, D_j \in K$, with the $C_j$ non-zero, such that

$$\begin{bmatrix}
\frac{\partial h_i}{\partial y_j}(\bar{\alpha}, \delta_1, \ldots, \delta_{m-1}, \bar{\gamma}) \\
0
\end{bmatrix} M = \begin{bmatrix}
\frac{\partial p_i}{\partial y_j}(\bar{\alpha}, \bar{\delta}, \bar{\gamma}) \\
\frac{\partial p_m}{\partial y_j}(\bar{\alpha}, \bar{\delta}, \bar{\gamma})
\end{bmatrix}
\begin{bmatrix}
\text{diag}(C_i) \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
D_i
\end{bmatrix}$$

for some column matrix $M$. Thus the Jacobian determinant $\left| \begin{bmatrix}
\frac{\partial h_i}{\partial y_j}(\bar{\alpha}, \delta_1, \ldots, \delta_{m-1}, \bar{\gamma}) \\
0
\end{bmatrix}
\right|$ is non-zero and, by inductive hypothesis,

$$\delta_1, \ldots, \delta_{m-1} \in \text{Alg/An}(\{\bar{\alpha}, \bar{\gamma}\}) = \text{Alg/An}(\{\bar{\alpha}\}).$$

The conclusion follows from the fact that $\bar{\beta} \in \text{Alg/An}(\{\bar{\delta}\})$. \qed
Chapter 4

The pregeometry of algebraic-analytic functions

In the previous chapter, we defined a (finitary) closure in $K$ via algebraic closure and images of infinitesimal elements under $\mathbb{C}$-definable functions holomorphic around the origin. In this chapter we aim to turn this closure into a pregeometry. In order to do this, we will need to extend the definition of the closure. We begin by studying a known pregeometry on $K$.

4.1  The pregeometry of derivations

For our purposes, a *derivation* on a subfield $T$ of $K$ is a $\mathbb{Q}[\sqrt{-1}]$-linear map $\delta : T \to T$. Given a differentiable $K$-valued function $F(x_1, \ldots x_n)$, we say that a complex derivation $\delta$ respects $F$ if, for all $\vec{a} = (a_1, \ldots, a_n)$ in the domain of $F$,

$$\delta(F(\vec{a})) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\vec{a})\delta(a_i).$$

Given a collection of $K$-valued holomorphic functions $\mathcal{F}$ and a subfield $T$ of $K$, the set of all derivations that respect every element of $\mathcal{F}$ will be denoted by $\text{Der}_T\mathcal{F}$. Note that this set is a $T$-vector space. For any set $X \subseteq K$, we define a closure operator from these derivations: the set $\text{DD}_F(X)$ will consist of all elements $x \in K$ for which there exists a finite subset $X_0 \subseteq X$ such that every $\delta \in \text{Der}_K(\mathcal{F})$ which vanishes in $X_0$ will vanish at $x$ as well.
Proposition 4.1.1. For any collection $F$ of holomorphic $K$-valued functions, the operator $DD_F$ is a pregeometry.

Proof. We adapt Wilkie’s proof from [13], Lemma 3.2, which is similar to our statement but for $K = \mathbb{C}$. We need to show that $DD_F$ satisfies the Exchange Property. Let $A \subseteq K$ be finite, and let $a, b \in K$ be such that $a \notin DD_F(A)$ and $b \notin DD_F(A \cup \{b\})$. As $a \notin DD_F(A)$, there exists a derivation $\delta_1 \in \text{Der}_K(F)$ such that $\delta_1(x) = 0$ for all $x \in A$ and $\delta_1(a) \neq 0$. Similarly, there exists a derivation $\delta_2 \in \text{Der}_K(F)$ such that $\delta_2(a) = 0$, $\delta_2(x) = 0$ for all $x \in A$ and $\delta_2(b) \neq 0$.

Then the derivation $\delta = \delta_2(b) \cdot \delta_1 = \delta_1(b) \cdot \delta_2$ belongs to the $K$-vector space $\text{Der}_K(F)$, vanishes in $A \cup \{b\}$ and satisfies $\delta(a) \neq 0$. Thus $a \notin DD_F(A \cup \{b\})$.

Let $\Delta$ be the collection of all $K$-valued $\mathbb{C}$-definable holomorphic functions, defined on neighbourhoods of the origin on any number of variables. Note that any complex derivation that respects the functions in $\Delta$ must respect the global function $f(z_1, z_2) = z_1 z_2$. In particular, any such derivation satisfies the multiplication rule, and thus we have the following:

Lemma 4.1.2. Let $\delta \in \text{Der}_K(\Delta)$ and $a_1, \ldots, a_n \in K$ be such that $\delta(a_i) = 0$ for all $i$. For any element $b$ in the algebraic closure of $\{a_1, \ldots, a_n\}$, $\delta(b) = 0$.

By definition of $\text{Der}_K(\Delta)$, given any derivation $\delta$ in it and infinitesimals $\alpha_1, \ldots, \alpha_n \in K$ such that $\delta(\alpha_i) = 0$ for all $i$, then $\delta(F(\bar{\alpha})) = 0$ for any $F \in \mathcal{O}_n(\mathbb{C})$. We have shown the following:

Proposition 4.1.3. For any set $X \subseteq K$, $\text{Alg/An}(X) \subseteq DD_\Delta(X)$. \hfill \Box

Unfortunately, in order for the reverse inclusion to hold we will have to enlarge the definition of the algebraic-analytic closure. We will see why on the next sections.

4.2 $\bar{\alpha}$-chains and generating functions

Given $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \in K^n$ infinitesimal and an element $w \in \text{Alg/An}(\{\bar{\alpha}\})$, there exists a finite sequence of elements $(e_1, \ldots, e_r)$ in $K$ such that $e_r = w$ and, for every $1 \leq i \leq r$, the element $e_i$ belongs to either the analytic image or the algebraic closure of $\{\bar{\alpha}, e_1, \ldots, e_{i-1}\}$.
Formally, given integers \( n \geq 0 \) and \( r > 0 \), and a subset \( A \subseteq \{1, \ldots, r\} \), we will say that a sequence of \( \mathbb{C} \)-definable holomorphic functions \((F_1, \ldots, F_r)\) is *admissible* if for every \( 1 \leq i \leq r \), the function \( F_i \) acts on \( n + i \) variables \((x_1, \ldots, x_n, y_1, \ldots, y_i)\), and satisfies one of the following:

- either \( F_i(\bar{x}, y_1, \ldots, y_i) = y_i - G_i(\bar{x}, y_1, \ldots, y_{i-1}) \), where \( G_i \) is a \( \mathbb{C} \)-definable function, holomorphic in a neighbourhood of the origin, and the only variables among the \( y_j \) to actually appear in it are such that \( j \in A \);
- or \( F_i(\bar{x}, y_1, \ldots, y_i) = \sum_{l=0}^{d} c_l y_l^i \) for some \( d > 0 \), where the \( c_l \) belong to the set \( \{x_1, \ldots, x_n, y_1, \ldots, y_{n-1}\} \).

Given an admissible sequence of functions \( \bar{F} \) (for fixed \( n \), \( r \) and \( A \)) and a tuple of infinitesimals \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \), we say that a sequence of elements of \( K \), \((e_1, \ldots, e_r)\), is an \( \alpha \)-chain (defined by \( \bar{F} \)) if

- for every \( j \in A \), \( e_j \) is an infinitesimal element of \( K \);
- for every \( 1 \leq i \leq r \), \( F_i(\bar{\alpha}, e_1, \ldots, e_i) = 0 \); and
- for every \( 1 \leq i \leq r \), \( \frac{\partial F_i}{\partial y_i}(\bar{\alpha}, e_1, \ldots, e_i) \neq 0 \).

The following result is clear from the definition of \( \text{Alg/An} \):

**Lemma 4.2.1.** Let \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \in K^n \) be infinitesimal. Then, for any \( w \in K \), \( w \) belongs to \( \text{Alg/An}(\{\bar{\alpha}\}) \) if and only if there exists an \( \bar{\alpha} \)-chain that terminates in \( w \).

Given an infinitesimal tuple \( \bar{\alpha} \in K^n \) and an \( \bar{\alpha} \)-chain \((e_1, \ldots, e_r)\) defined by an admissible sequence \( \bar{F} \), by the Implicit Function Theorem there exist \( D_0 \), a definable open neighbourhood of \( \bar{\alpha} \); for \( 1 \leq i \leq r - 1 \), open sets \( D_i \), definable open neighbourhoods of \( e_i \), and definable holomorphic functions \( G_1, \ldots, G_r \) such that, for every \( 1 \leq i \leq r \), \( G_i \) is holomorphic on \( D_0 \times D_1 \times \ldots \times D_{i-1} \) and \( G_i(\bar{\alpha}, e_1, \ldots, e_{i-1}) = e_i \). Moreover, for every \( i \) such that \( e_i \) is infinitesimal, we can use the fact that all the functions involved in this composition are continuous to possibly shrink the sets \( D_j \) so that the image of \( D_0 \times D_1 \ldots \times D_{i-1} \) under \( G_i \) consists entirely of infinitesimal values.

In particular, given an infinitesimal tuple \((\bar{\alpha}, \bar{\beta}) \in K^{n+m} \) and an \((\bar{\alpha}, \bar{\beta})\)-chain \((e_1, \ldots, e_r)\), there exist a definable open neighbourhood of \( \bar{\beta} \), \( B \), and definable holomorphic functions \( g_1, \ldots, g_r : B \to K \) such that, for all \( i \), \( g_i(\bar{\beta}) = e_i \) and, for every
\( y \in B \), the sequence \((g_1(\bar{y}), \ldots, g_r(\bar{y}))\) is an \((\bar{\alpha}, \bar{y})\)-chain, and all these chains are defined by the same admissible sequence of functions. We will refer to these functions \(g_i\) as the generating functions of the chain \((e_1, \ldots, e_r)\) from \(\bar{\alpha}\).

### 4.3 A weak Exchange Property for Alg/An

In the spirit of Proposition 4.1.3, given an infinitesimal tuple \(\bar{\alpha} \in K^n\) and an infinitesimal \(\beta \not\in \text{Alg/An}\{\bar{\alpha}\}\), we attempt to construct a derivation \(\delta \in \text{Der}_K(\Delta)\) satisfying \(\delta(\alpha_i) = 0\) and \(\delta(\beta) = 1\). We begin by defining this derivation over chains: given an \((\bar{\alpha}, \beta)\) chain \((e_1, \ldots, e_r)\) defined by a sequence \(F\), we can define \(\delta(e_i)\) recursively from the relations

\[
F_i(\bar{\alpha}, \beta, e_1, \ldots, e_i) = 0
\]

and the fact that \(\delta\) respects functions in \(\Delta\) and polynomials. So far, this definition of \(\delta\) is dependent on the chain used, and it satisfies the following:

**Lemma 4.3.1.** Let \(\bar{\alpha}, \beta\) and \(\delta\) be as above. Let \((e_1, \ldots, e_r)\) an \((\bar{\alpha}, \beta)\)-chain and \(g_1(y), \ldots, g_r(y)\) the generating functions of this chain from \(\alpha\). Then \(\delta(e_i) = g'_i(\beta)\).

**Proof.** Let \((F_i(\bar{x}, y, z_1, \ldots, z_i))\) be the sequence of functions defining the chain. Then, for all \(i\),

\[
\delta(e_i) = - \frac{\partial F_i}{\partial y}(\bar{\alpha}, \beta, e_1, \ldots, e_i) + \sum_{j<i} \frac{\partial F_i}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i) \delta(e_j)
\]

And, from the construction of the \(g_i\) from the Implicit Function Theorem,

\[
g'_i(\beta) = - \frac{\partial F_i}{\partial y}(\bar{\alpha}, \beta, e_1, \ldots, e_i) + \sum_{j<i} \frac{\partial F_i}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i) g'(\beta)
\]

The proof follows by induction on \(i\). \(\Box\)

Before showing that \(\delta\) is well-defined on \(\text{Alg/An}\{\bar{\alpha}, \beta\}\), we will establish a property of the Jacobian matrix of an admissible sequence of functions that will be needed regularly:

**Proposition 4.3.2.** Let \(\bar{\alpha}, \beta\) and \(\delta\) be as above, and let \((e_1, \ldots, e_r)\) be an \((\bar{\alpha}, \beta)\)-chain defined by a sequence of functions \((F_i(\bar{x}, y, z_1, \ldots, z_i))\). Then the \(r \times r\) matrix of partial derivatives

\[
M = \left[ \begin{array}{c|c} \frac{\partial F_i}{\partial y}(\bar{\alpha}, \beta, e_1, \ldots, e_i) & \frac{\partial F_i}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i) \\ \hline & \end{array} \right]_{1 \leq i, j \leq r}
\]

\[
M = \left[ \begin{array}{c|c} \frac{\partial F_i}{\partial y}(\bar{\alpha}, \beta, e_1, \ldots, e_i) & \frac{\partial F_i}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i) \\ \hline & \end{array} \right]_{1 \leq i, j \leq r}
\]
is equivalent under elementary column operations to a matrix of the form
\[
\begin{bmatrix}
0 & L \\
c & A
\end{bmatrix}
\]
where \(L\) is a \((r-1) \times (r-1)\) lower triangular matrix with non-zero determinant, \(A\) is a \(1 \times (r-1)\) row matrix and \(c \in K\) satisfies
\[
c = -\delta(e_r) \frac{\partial F_r}{\partial z_r}(\bar{\alpha}, \beta, e_1, \ldots, e_r).
\]

Moreover, all elementary column operations involved in this equivalence are adding, to the first column, a multiple of one of the other columns. In particular, all columns other than the first remain unchanged.

**Proof.** By the definition of a chain, the matrix \(M\) is of the form
\[
\begin{bmatrix}
B & L' \\
a & C
\end{bmatrix}
\]
for some lower triangular \((r-1) \times (r-1)\) matrix \(L'\) with non-zero determinant, some matrices \(B, C\) and some constant \(a\). Let \(N(1) = \left[ \frac{\partial F_i}{\partial y}(\bar{\alpha}, \beta, e_1, \ldots, e_i) \right]_{1 \leq i \leq r}\) be the leftmost column of \(M\). We recursively define \(N(j+1)\), for \(1 \leq j \leq r\) as the column vector given by
\[
N(j + 1) = N(j) - \frac{N(j)_j}{\frac{\partial F_j}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_j)} \left[ \frac{\partial F_i}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i) \right]_{1 \leq i \leq r}
\]
where \(N(j)_j\) is the \(j\)-th coordinate of the vector \(N(j)\). Note that \(N(j + 1)\) is obtained from \(N(j)\) by adding a multiple of one of the columns of \(M\) (but not the leftmost one) to it, and thus if we replace the leftmost column of \(M\) with any of the \(N(j)\), the resulting matrix is equivalent to \(M\). These column vectors \(N(j)\) satisfy the following:

**Lemma 4.3.3.** For every pair \((i, j)\) with \(1 \leq i < j \leq r\), \(N(j)_i = 0\).

**Proof.** By induction on \(j - i\). It is clear, from the recursive definition of \(N(j+1)\), that \(N(j+1)_j = N(j)_j - N(j)_j = 0\). For \(i < j\), we have \(N(j+1)_i = N(j)_i\), as the partial derivative \(\frac{\partial F_i}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i)\) vanishes. \(\square\)

**Lemma 4.3.4.** For every pair \((i, j)\) with \(1 \leq j \leq i \leq r\),
\[
N(j)_i = -\sum_{l=j}^{i} \delta(e_l) \frac{\partial F_l}{\partial z_l}(\bar{\alpha}, \beta, e_1, \ldots, e_i).
\]
Proof. By induction on \( j \). For \( j = 1 \), as \( N(1) = \frac{\partial F}{\partial y}(\bar{\alpha}, \beta, e_1, \ldots, e_i) \), this follows from \( F_i(\bar{\alpha}, \beta, e_1, \ldots, e_i) = 0 \) and the fact that \( \delta \) respects both multiplication and the functions \( F_i \). For the inductive step, if \( j + 1 \leq i \) then

\[
N(j+1)_i = N(j)_i - N(j)_j \frac{\partial F_j}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i)
\]

\[
= N(j)_i + \delta(e_j) \frac{\partial F_j}{\partial z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i) \frac{\partial F_i}{\delta z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i)
\]

\[
= - \sum_{l=j}^i \delta(e_l) \frac{\partial F_j}{\partial z_l}(\bar{\alpha}, \beta, e_1, \ldots, e_i) + \delta(e_j) \frac{\partial F_i}{\delta z_j}(\bar{\alpha}, \beta, e_1, \ldots, e_i)
\]

\[
= - \sum_{l=j+1}^i \delta(e_l) \frac{\partial F_i}{\partial z_l}(\bar{\alpha}, e_1, \ldots, e_i).
\]

From these two results, we conclude that replacing the leftmost column of \( M \) with \( N(r) \) verifies the required conditions.

Theorem 4.3.5. Given infinitesimals \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \in K^n \) and \( \beta \in K \) such that \( \beta \not\in \text{Alg/An}(\{\bar{\alpha}\}) \), there exists a unique derivation \( \delta \in \text{Der}_{\text{Alg/An}(\{\bar{\alpha}, \beta\})}(\Delta) \) satisfying \( \delta(\beta) = 1 \) and \( \delta(\alpha_i) = 0 \) for all \( i \).

Proof. Given \( w \in \text{Alg/An}(\{\bar{\alpha}, \beta\}) \), we have already mentioned how to define \( \delta(w) \) from an \( \{\bar{\alpha}, \beta\}\)-chain that terminates in \( w \). We will show now that this definition does not depend on the chain.

Indeed, let \( (e_1, \ldots, e_r) \) and \( (f_1, \ldots, f_s) \) be two \( \{\bar{\alpha}, \beta\}\)-chains, defined from functions \( (F_1, \ldots, F_r) \) and \( (G_1, \ldots, G_s) \), respectively, and suppose that \( e_r = f_s \). Write \( (\bar{x}, y, z_1, \ldots, z_i) \) for the variables of \( F_i \) and \( (\bar{x}, y, w_1, \ldots, w_i) \) for the variables of \( G_i \); by the previous Proposition the matrix

\[
\begin{bmatrix}
\frac{\partial F_i}{\partial y} & \frac{\partial F_i}{\partial z_j} & 0 & \frac{\partial F_i}{\partial z_r} & 0 \\
\frac{\partial G_i}{\partial y} & 0 & \frac{\partial G_i}{\partial w_j} & 0 & \frac{\partial G_i}{\partial w_r}
\end{bmatrix}_{1 \leq i \leq r}
\]

at the point \((\bar{x}, y, \bar{z}, \bar{w}) = (\bar{\alpha}, \beta, \bar{e}, \bar{f})\) is equivalent to the matrix

\[
M = \begin{bmatrix}
0 & \frac{\partial F_i}{\partial z_j} & 0 & \frac{\partial F_i}{\partial z_r} & 0 \\
\delta(e_i) \frac{\partial F_i}{\partial z_j} & 0 & \frac{\partial G_i}{\partial w_j} & 0 & \frac{\partial G_i}{\partial w_r} \\
0 & 0 & \frac{\partial G_i}{\delta w_j} & 0 & \frac{\partial G_i}{\delta w_r}
\end{bmatrix}_{1 \leq j \leq r-1}
\]

\[
-\frac{\partial G_i}{\delta w_i} & 0 & \frac{\partial G_i}{\delta w_j} & 0 & \frac{\partial G_i}{\delta w_r}
\end{bmatrix}_{1 \leq j \leq s-1}
\]

\[
-\frac{\partial \delta}{\delta w_i} & 0 & \frac{\partial \delta}{\delta w_j} & 0 & \frac{\partial \delta}{\delta w_r}
\end{bmatrix}_{1 \leq j \leq s-1}
\]

\[
\frac{\partial \delta}{\delta w_i} & 0 & \frac{\partial \delta}{\delta w_j} & 0 & \frac{\partial \delta}{\delta w_r}
\end{bmatrix}_{1 \leq j \leq s-1}
\]
at the same point, where $\delta_1$ is $\delta(e_r)$ as defined via the chain $(e_1, \ldots, e_r)$ and $\delta_2 = \delta(f_s)$, defined via $(f_1, \ldots, f_s)$. To see this, note that

$$
\begin{bmatrix}
0 & \left[ \frac{\partial F_i}{\partial z_j} \right]_{1 \leq i \leq r-1, 1 \leq j \leq r-1} \\
-\delta_1 \frac{\partial F_r}{\partial z_r} & \left[ \frac{\partial F_r}{\partial z_j} \right]_{1 \leq j \leq r-1}
\end{bmatrix}
$$

can be obtained from

$$
\begin{bmatrix}
\left[ \frac{\partial F_i}{\partial y} \right]_{1 \leq i \leq r} & \left[ \frac{\partial F_i}{\partial z_j} \right]_{1 \leq i \leq r, 1 \leq j \leq r-1} \\
-\delta_2 \frac{\partial G_s}{\partial y} & \left[ \frac{\partial G_s}{\partial z_j} \right]_{1 \leq j \leq r-1}
\end{bmatrix}
$$

via adding, to the first column, multiples of the other columns. We also obtain

$$
\begin{bmatrix}
\left[ \frac{\partial G_i}{\partial y} \right]_{1 \leq i \leq s} & \left[ \frac{\partial G_i}{\partial z_j} \right]_{1 \leq i \leq s, 1 \leq j \leq s-1} \\
-\delta_2 \frac{\partial G_s}{\partial y} & \left[ \frac{\partial G_s}{\partial z_j} \right]_{1 \leq j \leq s-1}
\end{bmatrix}
$$

from

$$
\begin{bmatrix}
\left[ \frac{\partial G_i}{\partial y} \right]_{1 \leq i \leq s} & \left[ \frac{\partial G_i}{\partial z_j} \right]_{1 \leq i \leq s, 1 \leq j \leq s-1}
\end{bmatrix}
$$

in the same way.

But if the determinant of the matrix $M$ is non-zero, then, by Theorem 3.3.2, $\beta \in \text{Alg/An}(\bar{\alpha})$, contradicting the hypothesis of this Theorem. Thus $\delta_1 = \delta_2$.

Using the same argument, we have a limited version of the Exchange Property for Alg/An:

**Corollary 4.3.6.** Let $\bar{\alpha} \in K^n$ and $\beta \in K$ infinitesimals be such that $\beta \notin \text{Alg/An}(\bar{\alpha})$, and let $\delta$ be the derivation on $\text{Alg/An}(\bar{\alpha} \cup \{\beta\})$ satisfying $\delta(\{\bar{\alpha}\}) = \{0\}$ and $\delta(\beta) = 1$. If $w \in \text{Alg/An}(\{\bar{\alpha}, \beta\})$ is such that $\delta(w) \neq 0$, then $\beta \in \text{Alg/An}(\{\bar{\alpha}, w\})$.

**Proof.** Let $(e_1, \ldots, e_r)$ be an $(\{\bar{\alpha}, \beta\})$ chain such that $e_r = w$. Consider the matrix $M$, defined as in the statement of Proposition 4.3.2. As $\delta(w) \neq 0$, this matrix has non-zero determinant, and we conclude by Theorem 3.3.2. 

The problem, however, arises when studying elements in $\text{Alg/An}(\{\bar{\alpha}, \beta\})$ with zero derivation. To prove the full Exchange Property for Alg/An, we would like to show that for $\bar{\alpha}$, $\beta$ and $\delta$ be as before, if $w \in \text{Alg/An}(\{\bar{\alpha}, \beta\})$ is such that $\delta(a) = 0$, then $a \in \text{Alg/An}(\{\bar{\alpha}\})$. This was not possible; instead we proved the following weaker result, that still contains the fact that in this case $a$ does not depend on the parameter $\beta$: 
Proposition 4.3.7. Let $\bar{\alpha}$, $\beta$ and $\delta$ be as above, and let $a \in \text{Alg/An}\{\bar{\alpha}, \beta\}$ satisfy $\delta(a) = 0$. Then there exists a definable open set $D \subseteq K$ such that 

$$a \in \bigcap_{x \in D} \text{Alg/An}\{\bar{\alpha}, x\}.$$ 

Proof. Let $(e_1, \ldots, e_r)$ be an $(\bar{\alpha}, \beta)$-chain such that $e_r = a$, and let $(F_1, \ldots, F_s)$ be the functions that define it. Consider the functions

$$G_i(x, y, z_1, \ldots, z_i, w_1, \ldots, w_i) = \frac{\partial F_i}{\partial y}(x, y, z_1, \ldots, z_i) + \sum_{j=1}^{i} w_j \frac{\partial F_i}{\partial z_j}(x, y, z_1, \ldots, z_i).$$

The functions $(\bar{F}, \bar{G})$ define the chain $(e_1, \ldots, e_r, g'_1(\beta), \ldots, g'_r(\beta))$, where the $g_i$ are the generating functions of $(\bar{e})$ from $\bar{\alpha}$. As the generating functions of a chain are unique in a neighbourhood of $\beta$ (as they are given by compositions of implicit functions), the generating functions for the new chain $(\bar{e}, \bar{g}'(\beta))$ are $(g_1, \ldots, g_r, g'_1, \ldots, g'_r)$.

But then, by Lemma 4.3.1, $g''_r(\beta) = \delta(g'_r(\beta)) = \delta(\delta(a)) = 0$. Iterating this process, we conclude that every derivative of $g_r$ at $\beta$ vanishes, and as it is holomorphic, is is the constant function equal to $a$ on its domain. As it is a generating function, there exists an open neighbourhood $D$ of $\beta$ such that $a = g_r(x) \in \text{Alg/An}\{\bar{\alpha}, x\}$ for all $x \in D$. \hfill $\Box$

This result will motivate a modified version of the definition of the closure in the next section.

### 4.4 The completed algebraic-analytic closure

Based on the last result of the previous section, we will define a new closure operator: given a finite set $A \subseteq K$, we define its completed algebraic-analytic closure, denoted by $\overline{\text{Alg/An}}(A)$, to be the set of all $w \in K$ such that there exist an integer $n \geq 1$ and a definable open set $D \subseteq K^n$ satisfying

$$w \in \bigcap_{x \in D} \overline{\text{Alg/An}}(A \cup \{x_1, \ldots, x_n\}).$$

For a general $A \subseteq K$, an element $w \in K$ is in $\overline{\text{Alg/An}}(A)$ if and only if $w \in \overline{\text{Alg/An}}(A_0)$ for some finite $A_0 \subseteq A$.

For each element $x \in D$ there exists a generating function $g_x$ such that $g_x(x) = w$. In a sufficiently saturated model, $D$ will contain a generic point, and so we may shrink
D to an open subset \( D' \) such that \( w \in \bigcap_{x \in D'} \text{Alg/An}(A \cup \{ \bar{x} \}) \) is given by the same generating function from \( A \) as \( x \) varies in \( D' \).

**Lemma 4.4.1.** \( \overline{\text{Alg/An}} \) is a finitary closure operator on \( K \).

**Proof.** Most of the properties we need to verify follow easily from the fact that \( \text{Alg/An} \) is a closure; it only remains to prove that \( \overline{\text{Alg/An}}(\overline{\text{Alg/An}}(A)) \subseteq \overline{\text{Alg/An}}(A) \) for any \( A \subseteq K \).

Let \( w \in \overline{\text{Alg/An}}(\overline{\text{Alg/An}}(A)) \). There exist a finite \( B = \{ b_1, \ldots, b_m \} \subseteq \overline{\text{Alg/An}}(A) \) and a definable open \( D \subseteq K^n \) satisfying \( w \in \bigcap_{x \in D} \text{Alg/An}(B \cup \{ \bar{x} \}) \). As each \( b_i \) is in \( \overline{\text{Alg/An}}(A) \), there exist a finite \( A_0 \subseteq A \) and definable open sets \( D_i \subseteq K^{n_i} \) such that \( b_i \in \bigcap_{x \in D_i} \text{Alg/An}(A_0 \cup \{ \bar{x} \}) \). Then

\[
B \subseteq \bigcap_{x \in \prod D_i} \text{Alg/An}(A_0 \cup \{ \bar{x} \})
\]

and

\[
w \in \bigcap_{x \in D \times \prod D_i} \text{Alg/An}(A_0 \cup \{ \bar{x} \}).
\]

We can now show that \( \overline{\text{Alg/An}} \) is a pregeometry on \( K \):

**Theorem 4.4.2.** Let \( a_1, \ldots, a_m, b, w \in K \) be such that \( w \in \overline{\text{Alg/An}}(\{ \bar{a}, b \}) \) and \( w \not\in \overline{\text{Alg/An}}(\{ \bar{a} \}) \). Then \( b \in \overline{\text{Alg/An}}(\{ \bar{a}, w \}) \).

**Proof.** Let \( D \subseteq K^n \) be definable open and such that \( w \in \bigcap_{x \in D} \text{Alg/An}(\{ \bar{a}, b, \bar{x} \}) \). As \( b \not\in \overline{\text{Alg/An}}(\{ \bar{a} \}) \), the set \( \{ \bar{x} \in D : b \in \text{Alg/An}(\{ \bar{a}, \bar{x} \}) \} \) must be nowhere dense in \( D \) (by Lemma 1.2.2). By shrinking \( D \) we may assume that \( b \not\in \text{Alg/An}(\{ \bar{a}, \bar{x} \}) \) for all \( \bar{x} \in D \).

As mentioned at the beginning of this section, we may additionally assume that there exists a generating function \( f \) (from \( \bar{a}, b \)) on \( D \) such that \( f(\bar{x}) = w \) for all \( x \). Fix \( \bar{x}_0 \in D \). There exist a definable neighbourhood of \( \bar{x}_0 \), \( D' \), a definable neighbourhood of \( b, E \) and a definable holomorphic generating function \( g \) on \( E \times D' \) such that \( g(\bar{y}, \bar{x}) \in \text{Alg/An}(\{ \bar{a}, y, \bar{x} \}) \) for all \( (\bar{y}, \bar{x}) \in E \times D' \). Additionally, \( g(b, \bar{x}) = w \) for all \( \bar{x} \in D' \).

Fix \( \bar{x} \in D \). As \( b \not\in \text{Alg/An}(\{ \bar{a}, \bar{x} \}) \), we can define a derivation on it as in Theorem 4.3.5, taking the value 1 at \( b \) and 0 at \( \{ \bar{a}, \bar{x} \} \). Then by Corollary 4.3.6 and Proposition
the set $D'$ partitions into the following two definable subsets, depending on whether $\delta(w)$ is zero or not in the corresponding derivation:

$$N = \{ \bar{x} \in D' : b \in \text{Alg/An}(\{ \bar{a}, w, \bar{x} \}) \}$$

and

$$Z = \{ \bar{x} \in D' : \forall y \in E \ (g(y, \bar{x}) = w) \}.$$

To conclude the Theorem we need to show that $N$ has interior. Indeed, if $Z$ has interior, then the set $\{(y, \bar{x}) \in E \times D' : g(y, \bar{x}) = w \}$ has interior, $g$ is constantly $w$ on its domain and $w \in \bigcap_{(y, \bar{x}) \in E \times D'} \text{Alg/An}(\{ \bar{a}, y, \bar{x} \})$, contradicting the fact that $w \notin \text{Alg/An}(\{ \bar{a} \})$. And by Lemma 1.2.2, as $D'$ is open at least one of $N$ or $Z$ must have interior.

We can now conclude:

**Theorem 4.4.3.** $\text{Alg/An}$ is a pregeometry on $K$. More accurately, $\text{Alg/An}$ and $\text{DD}_\Delta$ are the same pregeometry on $K$.

**Proof.** Fix $X \subseteq K$. We need to show that $\text{DD}_\Delta(X) \subseteq \text{Alg/An}(X)$.

Given $a \notin \text{Alg/An}(X)$, let $B \cup \{a\}$ be a $\text{Alg/An}$-basis of $K$. We have $X \subseteq \text{Alg/An}(B)$, and we aim to construct a derivation $\delta \in \text{Der}_K(\Delta)$ which vanishes in $B$ and satisfies $\delta(a) = 1$. This would imply that $a \notin \text{DD}_\Delta(X)$ as desired.

We obtain said derivation $\delta$ as follows: for each finite subset $B_0$ of $B$, let $\delta_{B_0} \in \text{Der}_{\text{Alg/An}(B_0 \cup \{a\})}(\Delta)$ be the unique derivation on $\text{Alg/An}(B_0 \cup \{a\})$ vanishing in $B_0$ and satisfying $\delta_{B_0}(a) = 1$ (cf. Theorem 4.3.5). Given $c \in K$, let $B_1$ be a finite subset of $B$ such that $c \in \text{Alg/An}(B_1 \cup \{a\})$. We let $\delta(c) = \delta_{B_1}(c)$. To see that $\delta$ is well-defined, let $B_2$ be another finite subset of $B$ such that $c \in \text{Alg/An}(B_2 \cup \{a\})$. Then, by the uniqueness clause in Theorem 4.3.5, $\delta_{B_1 \cup B_2} |_{\text{Alg/An}(B_1 \cup \{a\})} = \delta_{B_1}$ and similarly $\delta_{B_1 \cup B_2} |_{\text{Alg/An}(B_2 \cup \{a\})} = \delta_{B_2}$ which implies that $\delta_{B_1}(c) = \delta_{B_2}(c)$. 

Chapter 5

The definable holomorphic closure over $K$

5.1 The closure operator

Given a subset $A$ of $K$, we define the definable holomorphic closure of $A$, denoted by $\text{hcl}(A)$, as the set of all $x \in K$ such that there exist an integer $n \geq 0$, a $\mathbb{C}$-definable function $f : K^n \to K$ and elements $a_1, \ldots, a_n \in A$ satisfying

(i) $f(a_1, \ldots, a_n) = x$; and

(ii) $f$ is holomorphic in a neighbourhood of $(a_1, \ldots, a_n)$.

In this section, we will show that $\text{hcl}$ is a finitary closure operator. Let $A$ and $B$ be subsets of $K$ such that $A \subseteq B$. Then $A \subseteq \text{hcl}(A) \subseteq \text{hcl}(B)$ follows easily from the definition of $\text{hcl}$.

Proposition 5.1.1. Let $A$ be a subset of $K$. Then $\text{hcl}(\text{hcl}(A)) \subseteq \text{hcl}(A)$.

Proof. The composition of holomorphic functions is holomorphic. \hfill \Box

Corollary 5.1.2. $\text{hcl}$ is a finitary closure operator. \hfill \Box

5.2 Parameters for the Implicit Function Theorem

In order to prove that this closure is compatible with implicit functions, we need to make sure that the functions given by the Implicit Function Theorem are $\mathbb{C}$-definable.
We will establish some notation first: for every \( n \geq 1 \), \( \pi_{n-1} : K^n \to K^{n-1} \) will be the projection onto the first \( n - 1 \) coordinates; and given a set \( A \subseteq K^{n+1} \) and a point \( x \in \pi_n(A) \), \( A_x \) will denote the fibre \( \{ y \in K : (x, y) \in A \} \). We have the following results:

**Lemma 5.2.1.** Let \( A \subseteq K^{n+1} \) be \( \emptyset \)-definable and such that \( A_x \) is finite for every \( \bar{x} \in \pi_n(A) \). Given \((\bar{x}_0, y_0) \in A\), there exists a \( \emptyset \)-definable function \( \psi : \pi_n(A) \to K \) satisfying \( \text{graph}(\psi) \subseteq A \) and \( \psi(\bar{x}_0) = y_0 \).

**Proof.** By induction on \(|A_{\bar{x}_0}|\). By definable choice, there exists \( f : \pi_n(A) \to K \) \( \emptyset \)-definable such that \( \text{graph}(f) \subseteq A \). Assume \( f(x_0) \neq y_0 \). The set \( R_A = \pi_n(A) \setminus \text{graph}(f) \) is \( \emptyset \)-definable and non-empty, and the set

\[
B = (A \setminus \text{graph}(f)) \cup \{(\bar{x}, f(\bar{x})) : \bar{x} \in (\pi_n(A) \setminus R_A)\}
\]

is \( \emptyset \)-definable, contains \((\bar{x}_0, y_0)\) and satisfies \( \pi_n(A) = \pi_n(B) \). Moreover, for every \( \bar{x} \in \pi_n(B) \) the fibre \( B_\bar{x} \) is finite, and \(|B_{\bar{x}_0}| = |A_{\bar{x}_0}| - 1\). The conclusion follows. \( \square \)

**Proposition 5.2.2.** Let \( A \subseteq K^{n+1} \) be \( \emptyset \)-definable and such that \( A_\bar{x} \) is finite for every \( \bar{x} \in \pi_n(A) \). Let \((\bar{x}_0, y_0) \in A\), \( D \subseteq \pi_n(A) \) a neighbourhood of \( \bar{x}_0 \) and \( F : D \to K \) be such that

- \( F(\bar{x}_0) = y_0 \);
- \( \text{graph}(F) \subseteq A \);
- \( F \) is continuous on \( D \); and
- \( \text{graph}(F) \) is a connected component of \( A \cap (D \times K) \).

Then there exist a \( \emptyset \)-definable function \( \psi : \pi_n(A) \to K \) and a neighbourhood \( D' \subseteq D \) of \( \bar{a} \) satisfying \( \text{graph}(\psi) \subseteq A \) and \( \psi|_{D'} = F \).

**Proof.** By the previous Lemma, there exists \( f : \pi_n(A) \to K \) satisfying \( \text{graph}(f) \subseteq A \) and \( f(\bar{x}_0) = y_0 \). By cell decomposition on the “real” and “imaginary” parts of \( f \), there exist finitely many \( \emptyset \)-definable cells \( C_1, \ldots, C_m \) that partition \( \pi_n(A) \) and such that each restriction \( f|_{C_i} \) is continuous. Let \( C'_1, \ldots, C'_r \) the cells which intersect \( D \), and let \( C = C'_1 \cup \cdots \cup C'_r \) be their union. We proceed by induction on \( M_C = \max_{\bar{x} \in C} |A_{\bar{x}}| \).
For $i = 1, \ldots, r$, let $D_i = D \cap C'_i$. As $\text{graph}(F(D_i))$ is a connected component of $A \cap (D_i \times K)$ and both $F$ and $f$ are continuous over $D_i$, then either $F$ and $f$ coincide on $D_i$ or their graphs are disjoint. If $f$ and $F$ coincide on every $D_i$ we are done. If not, then we must have $M_C > 1$. For each $i$ we define $B_i = \text{graph}(f|_{C'_i})$ if $F$ and $f$ coincide on $D_i$; else, we define $B_i = (A \cap (C'_i \times K)) \setminus \text{graph}(f|_{C'_i})$.

Each $B_i$ is $\emptyset$-definable, and thus $B = B_1 \cup \ldots \cup B_r$ is $\emptyset$-definable as well. Moreover, $B \subseteq A$, $D \subseteq \pi_n(B)$, $\text{graph}(F)$ is a connected component of $B \cap (D \times K)$ and the cardinality of the fibres $B_x$ is bounded above by $M_C - 1$. By induction, there exists a $\emptyset$-definable function $\psi : \pi_n(B) \to K$ such that $\text{graph}(\psi) \subseteq B$ and $\psi$ and $F$ coincide on $D$. We extend this $\psi$ to $\pi_n(A)$ by setting $\psi = f$ on $\pi_n(A) \setminus \pi_n(B)$.  

We can now prove the desired property of the implicit function theorem:

**Corollary 5.2.3.** Let $F(\bar{x}, y) : K^{n+1} \to K$ be a $\emptyset$-definable function, and let $(\bar{a}, b) \in K^{n+1}$ be such that $F$ is holomorphic in a neighbourhood of $(\bar{a}, b)$, $F(\bar{a}, b) = 0$ and $\frac{\partial F}{\partial y}(\bar{a}, b) \neq 0$. Then there exist a $\emptyset$-definable function $G : K^n \to K$ and an $\bar{a}$-definable neighbourhood $D$ of $\bar{a}$ such that

- $G(\bar{a}) = b$;

- $F(\bar{x}, G(\bar{x})) = 0$ for all $x \in D$; and

- $G$ is holomorphic in $D$.

**Proof.** Let $H \subseteq K^{n+1}$ be the open, $\emptyset$-definable set of points around which $F$ is holomorphic. Let $Z \subseteq H$ be the zero set of $F$, and let $A \subseteq Z$ be the collection of points $(\bar{x}, y) \in Z$ such that $y$ is isolated in $Z_{\bar{x}}$. Then $A$ is $\emptyset$-definable, and for each $\bar{x} \in \pi_n(A)$ the fibre $A_{\bar{x}}$ is finite.

By the Implicit Function Theorem, there exist an $\{\bar{a}\}$ neighbourhood of $\bar{a}$, $D$, and a holomorphic function $\phi : D \to K$ such that $\text{graph}(\phi)$ is a connected component of $A \cap (D \times K)$. The conclusion follows from Proposition 5.2.2.  

5.3 The complete pregeometry

Similarly to the previous chapter, we define the **completed definable holomorphic closure** of a finite set $A \subseteq K$, denoted by $\text{hcl}(A)$, as the collection of all $w \in K$ for which
there exists a definable open set $D \subseteq K^n$ (for some $n \geq 0$) such that

$$w \in \bigcap_{\bar{x} \in D} \text{hcl}(A \cup \{x_1, \ldots, x_n\}).$$

As in the previous Chapter (cf. paragraph before Lemma 4.4.1), $\overline{\text{hcl}}(\{a_1, \ldots, a_m\})$ is the set of all $w \in K$ for which there exist a definable open set $D \subseteq K^n$ (for some $n \geq 0$) and a $\emptyset$-definable function $F : K^{m+n} \to K$ such that $F$ is holomorphic in a neighbourhood of $\{\bar{a}\} \times D$ and $F(\bar{a}, \bar{x}) = w$ for all $\bar{x} \in D$.

For general sets $A \subseteq K$, we say that $w \in \overline{\text{hcl}}(A)$ if and only if there exists a finite subset $A_0 \subseteq A$ such that $w \in \overline{\text{hcl}}(A_0)$.

By exactly the same reasoning as in Lemma 4.4.1, we have

**Lemma 5.3.1.** $\overline{\text{hcl}}$ is a finitary closure operator on $K$.

To show that $\overline{\text{hcl}}$ is actually a pregeometry, we will first need a weaker version of the Exchange Property for $\text{hcl}$:

**Proposition 5.3.2.** Let $A \subseteq K$, $a, b \in K$ be such that $a \in \text{hcl}(A \cup \{b\})$. Then at least one of the following holds:

(i) $b \in \text{hcl}(A \cup \{a\})$;

(ii) $a \in \overline{\text{hcl}}(A)$.

**Proof.** Let $\bar{c} = (c_1, \ldots, c_n) \in K^n$ and $F(\bar{x}, y) : K^{n+1} \to K$ $\emptyset$-definable be such that $F(c_1, \ldots, c_n, b) = a$ and $F$ is holomorphic in a neighbourhood of $(\bar{c}, b)$. If all the partial derivatives $\frac{\partial^k F}{\partial y^k}$ vanish at $(\bar{c}, b)$, there exists a neighbourhood $D$ of $b$ where $F(\bar{c}, -)$ is constant equal to $a$, and thus $a \in \overline{\text{hcl}}(A)$.

Otherwise, let $k \geq 1$ be the least positive integer such that $\frac{\partial^k F}{\partial y^k}(\bar{c}, b) \neq 0$. We define a new $\emptyset$-definable function $G(\bar{x}, y)$ as follows: if $k = 1$, we set $G = F$; else, we let $G = F + \frac{\partial^{k-1} F}{\partial y^{k-1}}$. Then $G(\bar{c}, b) = a$ and $\frac{\partial G}{\partial y}(\bar{c}, b) \neq 0$. By Corollary 5.2.3, we conclude that $b \in \text{hcl}(\{\bar{c}, a\})$. \hfill $\square$

From this ‘weak Exchange’ we obtain the Exchange Property for $\overline{\text{hcl}}$:

**Theorem 5.3.3.** Let $A \subseteq K$, $a, b \in K$ be such that $a \in \overline{\text{hcl}}(A \cup \{b\})$ but $a \not\in \overline{\text{hcl}}(A)$. Then $b \in \overline{\text{hcl}}(A \cup \{a\})$. 
Proof. Let $c_1, \ldots, c_n \in A$, $D \subseteq K^m$ definable, open and $F(\bar{x}, y, \bar{z}) : K^{n+1+m} \to K$ $\emptyset$-definable be such that $F$ is holomorphic in a neighbourhood of $\bar{c} \times \{b\} \times D$ and $F(\bar{c}, b, \bar{z}) = a$ for all $\bar{z} \in D$.

By the previous Proposition, $D$ is covered by the following two definable sets:

$$N = \{ \bar{z} \in D : b \in \text{hcl}(\{\bar{c}, a, \bar{z}\}) \}$$

and

$$Z = \{ \bar{z} \in D : a \in \text{hcl}(\{\bar{c}, \bar{z}\}) \}.$$ 

If $Z$ were to have interior, then $a \in \text{hcl}(\{\bar{c}\})$, contradicting the hypotheses of this Theorem; thus, by Lemma 1.2.2, $N$ has interior, and $b \in \text{hcl}(\{\bar{c}, a\})$. \qed

We have shown the following:

**Corollary 5.3.4.** $\text{hcl}$ is a pregeometry on $K$.

### 5.4 Relation to the other pregeometries

As in the previous Chapter, let $\Delta$ be the collection of $\mathbb{C}$-definable $K$-valued functions holomorphic in neighbourhoods of the origin (on any number of variables). We will show that $\text{DD}_\Delta$, the pregeometry of derivations on $K$, coincides with the completed definable holomorphic closure. This will imply that $\text{hcl}$ is also identical to the pregeometry of algebraic-analytic functions defined previously.

**Lemma 5.4.1.** Let $\delta \in \text{Der}_K(\Delta)$, $F : K^n \to K$ a $\mathbb{C}$-definable function and $\bar{a} \in K^n$ such that $F$ is holomorphic in a neighbourhood of $\bar{a}$. Then

$$\delta(F(\bar{a})) = \sum_{i=1}^{n} \delta(a_i) \frac{\partial F}{\partial x_i}(\bar{a}).$$

Proof. Let $H \subseteq K^n$ be the $\mathbb{C}$-definable set of points around which $F$ is holomorphic, and let $H'$ be the connected component of $H$ containing $\bar{a}$. By definable choice, there exists a $\mathbb{C}$-definable element $\bar{c} \in H'$. Then the function $G(\bar{x}) = F(\bar{x} + \bar{c})$ is holomorphic in a neighbourhood of the origin, and, for every $i$, $\frac{\partial G}{\partial x_i}(\bar{x}) = \frac{\partial F}{\partial x_i}(\bar{x} + \bar{c})$ as well. Then

$$\delta(F(\bar{a})) = \delta(G(\bar{a} - \bar{c})) = \sum_{i=1}^{n} (\delta(a_i) - \delta(c_i)) \frac{\partial G}{\partial x_i}(\bar{a} - \bar{c}) = \sum_{i=1}^{n} \delta(a_i) \frac{\partial F}{\partial x_i}(\bar{a})$$

since the derivation is zero at $\mathbb{C}$-definable points. \qed
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Theorem 5.4.2. For any set $X \subseteq K$, $\overline{\text{hcl}}(X) = \text{DD}_{\Delta}(X)$.

Proof. Let $\delta \in \text{Der}_K(\Delta)$, $a_1, \ldots, a_n \in X$ and $F : K^n \to K$ a $\mathbb{C}$-definable function holomorphic in a neighbourhood of $\bar{a}$. Suppose that $\delta(a_i) = 0$ for all $i$. By the previous Lemma, $\delta(F(\bar{a})) = 0$, thus $\overline{\text{hcl}}(X) \subseteq \text{DD}_{\Delta}(X)$.

For the converse inclusion, let $w \notin \overline{\text{hcl}}(X)$. We will construct a derivation $\delta \in \text{Der}_K(\Delta)$ that vanishes on $X$ and such that $\delta(w) = 1$. Let $B \cup \{w\}$ be a $\overline{\text{hcl}}$-basis for $K$. Then $X \subseteq \overline{\text{hcl}}(B)$ and, for every $c \in K$, there exist elements $b_1, \ldots, b_n \in B$, a definable open set $D \subseteq K^m$ and a $\mathbb{C}$-definable function $F(\bar{x}, y, \bar{z}) : K^{n+1+m} \to K$, holomorphic in a neighbourhood of $\bar{b} \times \{w\} \times D$, such that $F(\bar{b}, w, \bar{z}) = c$ for all $\bar{z} \in D$.

We claim that the value of $\frac{\partial F}{\partial y}(\bar{b}, w, \bar{z})$ depends only on $c$. Indeed, let $D' \subseteq K^{m'}$ be definable and open, and let $G(\bar{x}, y\bar{z}) : K^{n+1+m'} \to K$ be a $\mathbb{C}$-definable function, holomorphic in a neighbourhood of $\bar{b} \times \{w\} \times D'$, such that $G(\bar{b}, w, \bar{z}) = c$ for all $\bar{z} \in D'$, and suppose there exists an element $(\bar{z}, \bar{z}') \in D \times D'$ such that $\frac{\partial F}{\partial y}(\bar{b}, w, \bar{z}) \neq \frac{\partial G}{\partial y}(\bar{b}, w, \bar{z}')$.

Then, by continuity of $F$ and $G$, there exists a definable, open $E \subseteq D \times D'$ such that $\frac{\partial F}{\partial y}(\bar{b}, w, \bar{z}) \neq \frac{\partial G}{\partial y}(\bar{b}, w, \bar{z}')$ for all $(\bar{z}, \bar{z}') \in E$, and thus, by 5.2.3, $w \in \overline{\text{hcl}}(\{\bar{b}\})$, contradicting the fact that $B \cup \{w\}$ is a $\overline{\text{hcl}}$-independent set.

The fact that $\delta$ respects all functions in $\Delta$ (and, indeed, all $\mathbb{C}$-definable holomorphic functions) follows from the chain rule for differentiation. \qed
Bibliography


