THE INFLUENCE OF BIAXIAL LOADING ON THE ASSESSMENT OF STRUCTURES WITH DEFECTS

A thesis submitted to the University of Manchester for the degree of Doctor of Engineering in the Faculty of Science and Engineering

2017

CAROLINE MEEK

SCHOOL OF MECHANICAL, AEROSPACE AND CIVIL ENGINEERING
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Abstract

Caroline Meek, The University of Manchester, Doctor of Engineering

The influence of biaxial loading on the assessment of structures with defects

Assessments of structures with postulated or existing defects are generally carried out using standards and engineering assessment procedures. Assessments of this type involve comparing an applied force, in this case the crack driving force, with a material property, which in this case is the material’s resistance to fracture, its fracture toughness.

The crack driving force \( J \) can be calculated directly or implicitly by using a failure assessment diagram. Assessments can be based on either the initiation of the growth of a crack or, when dealing with ductile fracture, on an amount, e.g. 2 mm, of ductile tearing.

Material fracture toughness values are obtained by testing high constraint specimens such as deeply cracked compact tension specimens and single edge notched bend specimens under uniaxial loading conditions. The high constraint of the test specimens provides conservative measurements of the fracture toughness for most applications.

However, this assumption of conservatism is not necessarily applicable where there are biaxial loading conditions. The literature concerning assessments of such components mainly discusses whether uniaxial loading conditions provide conservative estimates of fracture toughness. Crack driving forces under biaxial loading can be overestimated, leading to a loss of conservatism.

Conversely, biaxial loading could be beneficial and thus an approach that is consistently conservative has implications for the cost and time involved in the consequences of prematurely assessing or predicting the failure of a structure or component.

This research considers the effects of biaxial loading on all the parameters involved in the integrity assessment of structures, components and specimens with defects. These parameters include the crack driving force, material fracture toughness, internal stresses and limit loads. It will address their relative effects on the determination of failure when compared with the assumption of uniaxial loading.

The methods used will be analytical, using the equations and theories of standard solid mechanics, fracture mechanics and existing advice in R6 and the literature, and numerical using finite element analyses. Experimental, analytical and numerical work in the literature will be assessed and discussed and their outcomes compared with the findings of this research.

The overall aim is to provide more explicit advice on the assessment of defects in components under biaxial loading in the R6 procedure.
Declaration

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Acknowledgements

Firstly, I would like to thank my academic supervisors, Professor Bob Ainsworth and Professor Andrew Sherry for all their time, support, guidance and encouragement, for generously sharing their expertise and experience and for the opportunities to present our work at international conferences.

Thank you to the EPSRC and EDF Energy for funding this project through the EngD Centre at the University of Manchester and to my industrial supervisor Dr Peter Budden and his team at EDF Energy for their input.

Sincere thanks to all in the Structural Integrity group at AMEC Foster Wheeler, especially John Sharples and Dr Peter James, who have provided me with office accommodation, technical equipment, academic advice, friendship and moral support throughout.

Dr David Stanley and Caroline Lalley of the EngD Centre, and Beverley Knight of the School of MACE at the University of Manchester have guided me through the technical and administrative processes involved in undertaking this doctorate, and many thanks are extended to them.

I would like to thank all my family and friends for their love and support. In addition, I am grateful to my father, Ron, for his general acquaintance with matters mathematical, my many sisters and their families for encouragement, advice and chocolate and all my lovely friends for the same; in particular Dr Victoria Moody and Dr Joanna Denbigh for inspiring me and convincing me to do this in the first place.

Finally, I would like to thank my wonderful husband, Stephen, for his vast resources of love, support, kindness, encouragement, patience, inspiration and technical expertise in all matters IT related. I’m not sure I could have done this without him.

Dedication

I dedicate this thesis to my beautiful little son, Elliot, who, despite his inclusion in this page, has been of no practical help whatsoever and without whose arrival in January 2017 you would have been reading this a lot sooner.
Papers

Journal


Conference

- C. Meek and R. A. Ainsworth, “Ductile fracture assessment of plates under biaxial loading”, ASME Pressure Vessels and Piping Division Conference (PVP2014), Anaheim, California, USA, 2014
- C. Meek and R. A. Ainsworth, “The effect of biaxial loading on the limit load of cracked plates”, ASME Pressure Vessels and Piping Division Conference (PVP2015), Boston, Massachusetts, USA, 2015
- C. Meek and R. A. Ainsworth, “Biaxial loading effects on large-scale cruciform bending specimens”, Structural Mechanics in Reactor Technology (SMiRT-23), Manchester, UK, 2015
Nomenclature

a Half crack width
A₂ Constraint parameter
b Specimen thickness
B Biaxial load ratio = \( \sigma_1 / \sigma_2 \)
B' Biaxial displacement ratio = \( \delta_1 / \delta_2 \)
Cₗ Out-of-plane constraint
E Young's modulus
E' Elastic modulus
Plane stress \( E' = E \),
Plane strain \( E' = E/(1 - v^2) \)
E₂n Euler numbers
\( f_1(L_r) \) Function defining R6 Option 1 failure assessment curve
\( f_2(L_r) \) Function defining R6 Option 2 failure assessment curve
\( f_3(L_r) \) Function defining R6 Option 3 failure assessment curve
\( f_{ij} \) Angular functions in elastic stress field equation
\( f(\beta) \) Limit load function
F Loading applied during cruciform testing
F₀ Reference load
G Energy release rate
G_c Critical energy release rate
H Half plate length
Iₙ Dimensionless integral parameter
J J-integral, elastic-plastic crack tip characterising parameter
Jₑ Elastic component of J-integral
J_{JC} Critical (fracture initiation) value of J in-plane strain
J_{mat} Material fracture toughness in terms of J
J_{pl} Plastic component of J-integral
\( k_{(i)} \) Stress intensity factor per unit value of load (i)
K  Stress intensity factor
K  Material constant in Ramberg-Osgood equation
K_i  Mode I stress intensity factor
K_{II}  Mode II stress intensity factor
K_{III}  Mode III stress intensity factor
K_{IC}  Critical stress intensity factor for mode I loading
K_{mat}  Material fracture toughness
K_{mat}^c  Material fracture toughness modified by constraint
K_r  Proximity to LEFM failure
L_r  Proximity to plastic collapse
L_r^{max}  Maximum value of L_r defining cut-off for the failure assessment curve
m  Parameter defining influence of constraint on fracture toughness
M  Applied bending moment
n  Work hardening exponent (Ramberg-Osgood)
n_j  Unit normal vector
n_L  Normalised limit load
n_N  Normalised limit end force
n_p  Normalised limit pressure
N  Applied force
N  Applied end force on a cylinder
N_L  Limit force
p  Applied internal pressure
P_{app}  Applied loading
P(i)  Loading related to displacement δ(i)
P_L  Limit loading
Q  Elastic-plastic constraint parameter
Q(T)  Q estimated using functions of T and limit load
r, θ  Crack tip polar coordinates
$RF_1$ FEA output reaction force in x direction

$RF_2$ FEA output reaction force in y direction

$r_p$ Radius of crack tip plastic zone

$R_i$ Inner radius of cylinder

$R_m$ Mean cylinder radius

$R_o$ Outer radius of cylinder

$S_{11}$ FEA output stress field components in x direction

$S_{22}$ FEA output stress field components in y direction

$S_{33}$ FEA output stress field components in z direction

$t$ Plate thickness, cylinder wall thickness

$T$ Elastic constraint parameter, stress parallel to the crack

$T_i$ Traction force vector components

$U_1$ FEA output displacement field components in x direction

$U_2$ FEA output displacement field components in y direction

$u_i$ Displacement in direction i

$v$ Velocity along slip-line

$w$ Strain energy density

$W$ Half plate width (centre cracked plate)

$W$ Plate width (single edge cracked plate)

$x, y, z$ Cartesian coordinates

$z$ $1 - a/W$

$\alpha$ Work hardening coefficient (Ramberg-Osgood)

$\alpha$ Parameter defining influence constraint on fracture toughness

$\alpha$ Radial cylinder crack width, $a/t$

$\beta$ Normalised crack size $a/W$ for limit load calculations

$\beta$ Normalised constraint parameter

$\beta_Q$ Normalised constraint parameter based on $Q$

$\beta_T$ Normalised constraint parameter based on $T$
\( \gamma \) Surface energy

\( \gamma \) Von Mises yield stress multiplier = \( 2/\sqrt{3} \)

\( \Gamma \) Path around the crack tip

\( \delta_1 \) Remote displacement applied parallel to crack

\( \delta_2 \) Remote displacement applied normal to crack

\( \delta_{ij} \) Applied displacement

\( \delta_{ij}^{uc}, \delta_{ij}^{ac} \) Uncracked body displacement,

\( \delta_{e(i)}, \delta_{e}^{c} \) Elastic displacement due to the crack

\( \delta_{pl(i)}, \delta_{pl}^{c} \) Plastic displacement due to the crack

\( \delta_{ij} \) Kronecker delta

\( \delta_{y} \) Value of displacement \( \delta_2 \) when stress \( \sigma_2 = \sigma_y \)

\( \varepsilon \) Strain

\( \varepsilon_1 \) Strain in x direction

\( \varepsilon_2 \) Strain in y direction

\( \varepsilon_3 \) Strain in z direction

\( \varepsilon_0 \) Normalising strain \( \sigma_y /E \)

\( \varepsilon_{ref} \) Reference strain

\( \varepsilon_t \) True strain

\( \varepsilon_y \) Strain at yield

\( \eta \) \( t/R_m \) cylinder wall thickness ratio

\( \theta \) Angle made by slip-line to horizontal

\( \theta \) Half angle of circumferential defect in cylinder

\( \kappa \) Crack tip displacement field parameter

\( \mu \) Shear modulus

\( \nu \) Poisson's ratio

\( \sigma \) Stress

\( \sigma_0 \) Reference stress value, usually equal to the yield stress

\( \sigma_1 \) Remote applied uniform stress parallel to crack

\( \sigma_2 \) Remote applied uniform stress normal to crack
\( \sigma_3 \) Stress in z direction

\( \sigma_{11}, \sigma_{xx} \) Stress field components in x direction

\( \sigma_{22}, \sigma_{yy} \) Stress field components in y direction

\( \sigma_{33}, \sigma_{zz} \) Stress field components in z direction

\( \sigma_{ssy} \) Small-scale yielding stress field for crack opening stress component \( \sigma_{22} \)

\( (\sigma_{2})_L \) Limit load solution

\( (\sigma_{2})_{L}^{lb} \) Lower bound solution for limit load

\( (\sigma_{2})_{L}^{ub} \) Upper bound solution for limit load

\( \sigma_a \) Axial stress

\( \sigma_f \) Stress leading to brittle fracture

\( \sigma_h \) Hoop stress

\( \sigma_{ij} \) Components of stress tensor

\( \sigma_{ij}^{ssy} \) Small-scale yielding stress field

\( \sigma_m \) Membrane stress

\( (\sigma_m)_L \) Limit membrane stress

\( \sigma_{ref} \) Reference stress

\( \sigma_t \) True stress

\( \sigma_u \) Ultimate tensile stress

\( \sigma_{vm} \) Equivalent von Mises stress

\( \sigma_{tr} \) Tresca stress

\( \sigma_y \) Yield stress

\( \bar{\sigma} \) Flow stress

\( \tau_{ij} \) Shear stress

\( \tau_{max} \) Maximum shear stress

\( \tau_y \) Yield shear stress

\( \varphi \) Angle subtended at the centre of a circle by a circular slip-line

\( \dot{\omega} \) Angular velocity along the slip-line
### Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full Form</th>
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<tbody>
<tr>
<td>AFOSR</td>
<td>Air Force Office of Scientific Research</td>
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<tr>
<td>AGR</td>
<td>Advanced Gas-cooled Reactor</td>
</tr>
<tr>
<td>ASME</td>
<td>The American Society of Mechanical Engineers</td>
</tr>
<tr>
<td>ASTM</td>
<td>The American Society for Testing and Materials</td>
</tr>
<tr>
<td>BS</td>
<td>British Standard</td>
</tr>
<tr>
<td>BARC</td>
<td>Bhabha Atomic Research Centre</td>
</tr>
<tr>
<td>BNFL</td>
<td>British Nuclear Fuels Ltd</td>
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<tr>
<td>CAPM</td>
<td>Certified Associate in Project Management</td>
</tr>
<tr>
<td>CCP</td>
<td>Centre cracked plate</td>
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<tr>
<td>CDF</td>
<td>Crack driving force</td>
</tr>
<tr>
<td>CEGB</td>
<td>Central Electricity Generating Board</td>
</tr>
<tr>
<td>CMOD</td>
<td>Crack mouth opening displacement</td>
</tr>
<tr>
<td>CTOD</td>
<td>Crack tip opening displacement</td>
</tr>
<tr>
<td>DCL</td>
<td>Displacement controlled loading</td>
</tr>
<tr>
<td>EDF</td>
<td>Électricité de France</td>
</tr>
<tr>
<td>EngD</td>
<td>Engineering Doctorate/ Dr of Engineering</td>
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<tr>
<td>ENYGF</td>
<td>European Nuclear Young Generation Forum</td>
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<tr>
<td>EPFM</td>
<td>Elastic-plastic fracture mechanics</td>
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<tr>
<td>EPP</td>
<td>Elastic-perfectly plastic</td>
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<tr>
<td>EPRI</td>
<td>Electrical Power Research Institute</td>
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<td>EPSRC</td>
<td>Engineering and Physical Sciences Research Council</td>
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<tr>
<td>ESIA</td>
<td>Engineering Structural Integrity Assessment</td>
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<tr>
<td>FAC</td>
<td>Failure assessment curve</td>
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<td>FAD</td>
<td>Failure assessment diagram</td>
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<tr>
<td>FEA</td>
<td>Finite element analysis</td>
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<td>FESI</td>
<td>The UK Forum for Engineering Structural Integrity</td>
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<tr>
<td>FFS</td>
<td>Fitness for service</td>
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<tr>
<td>FSE</td>
<td>Faculty of Science and Engineering</td>
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<tr>
<td>HRR</td>
<td>Hutchinson, Rice and Rosengren</td>
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<td>Abbreviation</td>
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<tr>
<td>HSST</td>
<td>Heavy Section Steel Technology</td>
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<tr>
<td>ICAM</td>
<td>International Centre for Advanced Materials</td>
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<tr>
<td>LEFM</td>
<td>Linear elastic fracture mechanics</td>
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<tr>
<td>MACE</td>
<td>Mechanical, Aerospace and Civil Engineering</td>
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<tr>
<td>MBL</td>
<td>Modified boundary layer</td>
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<tr>
<td>NESC</td>
<td>The Network for Evaluation of Structural Components</td>
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<tr>
<td>NNL</td>
<td>National Nuclear Laboratory</td>
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<tr>
<td>NRC</td>
<td>United States Nuclear Regulatory Commission</td>
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<tr>
<td>NUREG</td>
<td>United States Nuclear Regulatory Commission Regulation</td>
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<tr>
<td>PhD</td>
<td>Doctor of Philosophy</td>
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<tr>
<td>PMMA</td>
<td>Poly methyl methacrylate (Plexiglass)</td>
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<tr>
<td>PTS</td>
<td>Pressurised thermal shock</td>
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<tr>
<td>PWR</td>
<td>Pressurised Water Reactor</td>
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<tr>
<td>PVC</td>
<td>Polyvinyl chloride</td>
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<tr>
<td>PVP</td>
<td>Pressure Vessels and Piping</td>
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<tr>
<td>R&amp;D</td>
<td>Research and development</td>
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<tr>
<td>RPV</td>
<td>Reactor pressure vessel</td>
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<td>SCL</td>
<td>Stress controlled loading</td>
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<td>SIAP</td>
<td>Structural Integrity Assessment Procedures</td>
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<td>SIF</td>
<td>Stress intensity factor</td>
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<td>SMiRT</td>
<td>Structural Mechanics in Reactor Technology</td>
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<td>SSY</td>
<td>Small-scale yielding</td>
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<td>TWh</td>
<td>Terawatt hour</td>
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<td>The Welding Institute</td>
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Introduction

The Nuclear Engineering Doctorate (EngD) programme provides training in a range of business, enterprise, management and leadership skills, technical courses to master’s degree level and a PhD equivalent research project, undertaken with support from the Engineering and Physical Sciences Research Council (EPSRC) in collaboration with the nuclear industry.

This EngD has been carried in collaboration with EDF Energy and this thesis forms the doctoral research part of the EngD programme.

1.1 Background

EDF Energy own and manage fifteen nuclear reactors across eight sites in operation in the UK. These reactors generate approximately 19% of the country’s electricity. Another 19% of the UK’s electricity comes from renewable sources and the majority, 62%, comes from carbon-based fossil fuels such as coal, gas and oil [1].

The Climate Change Act of 2008 [2] sets out a target to reduce the net UK emissions of carbon dioxide and other greenhouse gases to no more than 80% of the levels measured in 1990. There are several ways to achieve this target, including reducing demand, reducing emissions from fossil fuel powered stations or closing them down. The closure of fossil fuel powered stations would lead to a shortfall in energy supply. One way of making up this shortfall would be to increase the supply of renewables and nuclear power.

Currently all existing reactors in the UK are due to be shut down by 2035, some as soon as 2019.

EDF Energy have proposed new reactors at Hinkley Point, Sizewell and Bradwell. Horizon has proposals for new reactors at Wylfa and Oldbury and NuGen has plans for a reactor at Moorside. However, the costs and lead times of their construction mean that it could be many years before they start to operate.

Thus, a major contribution to maintaining a secure and reliable supply of electricity is to extend the lives of the existing reactors. For continued operation of a reactor, its components and infrastructure must be shown to
be able to continue to operate safely without breaking catastrophically or leading to a leak of radioactive materials, i.e. that they maintain sufficient structural integrity.

### 1.2 Structural Integrity Assessments

During construction and operation, materials start to age. Components can develop defects by mechanisms such as creep, fatigue and stress corrosion. Such defects can grow under the action of design stresses including pressure differential, thermal stresses, residual welding stresses, gravitational loading and forces associated with lateral movement.

Consequently, components need to be inspected periodically to determine the presence and size of defects.

To determine at what level these defects are likely to lead to failure, it is necessary to establish criteria that define failure, quantify acceptable safety margins, assess material properties and either measure or predict with acceptable levels of certainty the size and nature of defects.

The documents prescribing these approaches are fitness for service (FFS) procedures. An internationally recognised guidance document for the assessment of the structural integrity of components in the UK power generation industry is the R6 defect assessment procedure [3]. This document is managed by EDF Energy and its continuing development is carried out in partnership with other leading organisations in the nuclear research industry.

### 1.3 The Failure Assessment Diagram

Within R6, structural integrity assessment is based on the two-parameter failure assessment diagram (FAD). An assessment point representing a particular defect is plotted on the diagram and its position relative to a failure assessment curve determines whether the defect is considered sufficient to fail the component and therefore deem it to be unsafe.

The FAD curve itself and assessment points are based on two parameters; a fracture parameter $K_t$, quantifying the proximity to failure by elastic fracture and a plastic limit load parameter $L_t$, quantifying proximity to failure by plastic collapse.
The fracture parameter $K_r$ compares estimates of the elastic crack driving force with the component material’s resistance to fracture. The plastic collapse load parameter $L_r$ compares the applied loading to the limit load of the structure or component. Figure 1.1 shows an example of the FADs used in R6.

R6 contains procedures for calculating these parameters and includes compendia of input values for the elastic stress intensity factor (SIF) and the limit load for a variety of geometries and loading conditions.

![Figure 1.1: Example of a failure assessment diagram (FAD)](image)

### 1.4 Biaxial Loading

Pressure vessels and piping are subject to thermal stresses, pressure differential stresses and residual stresses, for example weld residual stresses, from the manufacturing processes. There are further gravitational loading and bending forces that can lead to tensile, compressive and shearing stresses.

Many of the components are subject to stresses that act in more than one direction. Pressure vessels and pipes under differential pressure, for example, will experience both hoop (circumferential) and axial stresses. This simultaneous exposure to applied stresses in two different directions is referred to as biaxial loading.
The advice for biaxially loaded components in the compendia of stress intensity factor and limit load solutions parameters in R6 is limited to cylinders with circumferential defects under combined tension and pressure.

One approach for assessments for all other components subject to biaxial loading would be to carry these out using the parameters derived for uniaxial loading. For many engineering applications, this will be conservative. This approach however comes with potential problems.

- The level of conservatism could be high such that failure is determined pessimistically, leading to unnecessary and expensive repair or replacement of components.
- There are cases where the assumption of uniaxial loading is non-conservative, leading to overestimates of the failure loads or limiting crack size and potentially a diagnosis of safety and acceptability where failure will occur under the applied loadings.
- Overcoming these problems by using bespoke FEA assessments for each case of biaxial loading can be time and resource consuming and would often repeat work already undertaken.

### 1.5 Research Objectives

The research summarised in this thesis aims to assess the influence of biaxial loading on the crack driving force and limit load parameters that are used in fracture assessments undertaken using R6. The intention is to provide advice for fitness for service (FFS) assessments of components that contain real or postulated defects under biaxial loading conditions.

The areas investigated include the fracture parameters — stress intensity factor (SIF), \( J \)-integral and material fracture toughness; and the plastic collapse parameters — limit loads and applied stress, strain or displacement. The thesis will also examine the effects of geometry, crack size, material hardening properties, constraint and displacement controlled loading.

The analyses are a mix of theoretical and numerical research. The theoretical analyses take current validated concepts, equations and formulae that are used for uniaxially loaded geometries and extend them to
biaxial loading. The resulting equations and formulae are tested using numerical analyses using Abaqus FEA software [4]. Previous experimental work is also examined, and the results compared with those derived in these analyses.

1.6 Thesis Structure

- Chapter 2 is a description of the industrial context of the project with reference to the sponsoring company, EDF Energy. It covers its history, structure, markets, and goals and how these relate to this research. This outlines the key objectives of the work and how the results will be used by industry in general and EDF Energy in particular.

- Chapter 3 provides an overview of the theories and concepts used in the analyses and provides a critical review of the published research in the field. The review focuses on research into the effects of biaxial loading as well as the seminal papers, books and reports from which the current thinking on the subject has been determined.

- Chapter 4 summarises the aims of the research with reference to the gaps in the knowledge identified by the review in chapter 3 and introduces the parameters and models used to address these.

- Chapter 5 reports the examination of the effects of biaxial loading on limit load solutions, starting with the simplest geometry of a centre cracked plate and extends the analyses to include short centre cracked plates, double edge cracked plates, single edge cracked plates and circumferentially cracked cylinders. This chapter also illustrates how these effects on limit load influence the position of the loci on the FAD and thus the failure load and crack size for given loading conditions.

- Chapter 6 covers the effects of biaxial loading on the material fracture parameters, briefly covering the SIF and a more detailed look at the effects of biaxial loading on the J-integral for the cases covered in chapter 5. It then examines how biaxial loading affects constraint and to what degree commonly known constraint effects apply under biaxial loading and how constraint effects on the fracture toughness affect FAD assessments.

- Chapter 7 looks in more detail at one of the experimental works reviewed in chapter 3. It investigates how displacement controlled loading can yield very different results from stress controlled loading when biaxial loading is
applied and how this may have affected some of the results from the experimental work considered.

- Chapters 8 and 9 form the analysis of results, discussion and conclusion section, bringing together the findings from the previous chapters. Chapter 8 compares the results with those found in the literature during chapter 3 and describes how they have achieved the aims of the sponsorship company as discussed in chapter 2. Chapter 9 also provides suggestions and recommendations for further work.

- Appendix A contains the validation of the FEA models.

- Appendix B contains the more detailed derivations of some of the equations and formulae in chapter 5.
2 Industrial Sponsor

The research presented in this engineering doctorate has been carried out with the collaboration and support of EDF Energy as part of its R6 programme with the intention that the results of the research will be included in the R6 procedure [3]. It is the aim of EDF Energy that the addition of these results will improve the advice for assessing components with real or postulated defects under biaxial loading conditions.

2.1 About the Company

EDF Energy is an integrated energy company based in the UK. It is a subsidiary of the French state owned EDF Group (Électricité de France), the world’s biggest electricity generating company.

EDF Energy employs more than 13 000 people in the UK over its four main businesses, EDF Energy Customers, EDF Energy Generation, Nuclear New Build and the Corporate and Steering functions.

2.1.1 EDF Energy Customers

The EDF Energy Customers business buys gas and electricity wholesale to sell on to residential and business customers in the UK.

2.1.2 EDF Energy Generation

The generation business operates eight nuclear power stations, two coal-fired and one gas-fired power stations. EDF Energy Generation also owns and operates 25 wind farms and a small number of combined heat and power plants. In total, these generate around one fifth of the UK’s 303 TWh annual electricity consumption.

2.1.3 Nuclear New Build

The Nuclear New Build business was set up with plans to build new reactors at Hinkley Point C, Sizewell C and Bradwell B.

2.2 Acquisition of Nuclear Power Stations

Until 1990, the UK’s electricity generation and supply were state owned and run by the Central Electricity Generating Board (CEGB). Ahead of privatization in 1990, Scottish Nuclear was formed and the CEGB was split
into four separate companies, PowerGen, National Power, Nuclear Electric and National Grid Company.

Nuclear Electric remained in public ownership and operated five Advance Gas-cooled Reactor (AGR) stations (Dungeness B, Hartlepool, Heysham 1, Heysham 2 and Hinkley Point B), one Pressurised Water Reactor (PWR) station (Sizewell B) and eight Magnox power stations from its headquarters in Barnwood, the former site of the generation and construction division of the CEGB. Scottish Nuclear ran one Magnox station (Hunterston A) and two AGR power stations (Hunterston B and Torness).

In 1995, Nuclear Electric merged with Scottish Nuclear and then split to create two new companies. British Energy took over operation of the more modern seven AGR and one PWR stations while the older Magnox stations were taken over by Magnox Electric, which later combined with British Nuclear Fuels Ltd (BNFL) and is now run by the US based company Energy Solutions.

EDF Energy acquired British Energy in 2009 adding the eight nuclear power stations to its portfolio.

2.3 Research and Development

As referred to in chapter 1, extending the life of the operating UK reactors requires structural integrity assessments. The procedures for carrying out these assessments need to be kept up to date in line with the latest research, much of which is carried out by EDF Energy in collaboration with other industry partners and academia.

2.3.1 Collaboration with higher education

EDF Energy has research collaborations with a number of universities, which include an Alliance Group consisting of Imperial College London, Bristol, Strathclyde and Manchester as well as projects at other universities. EDF Energy supports and sponsors Engineering Doctorates, PhDs and post-doctoral projects.

EDF Energy UK R&D Centre’s Modelling and Simulation Centre is based at the University of Manchester and provides a link between research in EDF Group R&D in France and that of EDF Energy in the UK.
Collaborations with universities, as well as generating knowledge from the research, bring skills development and recruitment to a nuclear sector that needs highly trained people and will continue to do so long term.

2.3.2 R6 development within EDF Energy
The R6 document is the pre-eminent defect assessment procedure for the UK nuclear industry. The procedure was initially developed in 1976 by CEGB and was the first defect assessment procedure of its kind.

Within EDF Energy, the Infrastructure Team is part of the Assessment Technology Group (see Figure 2.1) and is responsible for the maintenance and development of R6 and R5, the assessment procedure for high temperature response [5]. The group collaborates with other departments within the Structural Integrity Branch. The main users of R6 within EDF Energy are the Structural Analysis Group who assess plant components for safety cases and return-to-service in support of continuing operation.

2.3.3 Administration of R5 and R6
R6 and R5 are currently owned and managed by EDF Energy under a Structural Integrity Assessment Procedures (SIAP) collaboration. For R6, the members of that collaboration are EDF Energy, AMEC Foster Wheeler, TWI, NRG, Rolls Royce, National Nuclear Laboratory (NNL) and Frazer-Nash. EDF Energy chairs the SIAP, R6 and R5 panels, which consist of representatives from the industry collaborators as above as well as academia. The panels are observed by the Office of Nuclear Regulation (ONR) to ensure its confidence in the R5 and R6 programmes as part of EDF Energy’s Nuclear Research Schedule.
Figure 2.1: Organisation chart for the infrastructure team at EDF Energy

The numbers in brackets indicate the numbers of people working in that section. The structure shown here is a simplified version of the overall structure with some branches that are not relevant to this project omitted for clarity.
3 Literature Review

This literature review chapter will serve two main functions. The first will be to introduce the main concepts of the science and engineering theories that have been used in this thesis and in the reviewed literature. In addition, it will present an overview of how the concepts have been shown to be affected by biaxial loading.

Detailed explanations of the theories have been kept to a minimum. This has been done to avoid repetition or paraphrasing of the contents of the textbooks and standards that have been used here on the subjects covered [3, 6-8]. The concepts specific to this research are covered in more detail as they arise in the subsequent chapters of this thesis.

For many years of fracture mechanics research and applications, there was a paucity of studies taking into account any biaxial loading. This suggests that it was assumed that the biaxial loading either made no difference to the fracture parameters or was generally beneficial. As such, where biaxial loading was not taken into account, it was assumed that solutions were conservative.

Thus the second function of this chapter is to show how gradually the research has revealed, through analytical, numerical and experimental methods, that the effects of biaxial loading on the parameters are variable and are not always beneficial.

This leads to the need for these effects to be formalised and quantified in terms of direct relationships between the fracture parameters and the absolute and relative magnitudes of applied biaxial loads.

3.1 Solid Mechanics

There are three fundamental relationships that govern the stresses and strains of a structure and the relationships between them. These are, respectively, equilibrium, compatibility and a constitutive model, the last of which, for a linear elastic material, is described by Hooke’s law.

3.1.1 Equilibrium

The normal and shear stresses on a three-dimensional element using Cartesian coordinates are shown in Figure 3.1.
In order for these stresses to be in internal equilibrium there must be no net forces or torques on the element. Thus the shear stresses must be balanced as in equation (3.1), and the net stress on any element in any direction must equal zero as in equation (3.2).

\[ \tau_{ij} = \tau_{ji} \]  

(3.1)

\[ \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \]  

(3.2)

where \( x_j \) represents the three-dimensional directions.

Any traction forces (or surface forces) must be balanced by these internal stresses, leading to the Cauchy relation as shown in equation (3.3)

\[ \sigma_{ij} n_j = T_i \]  

(3.3)

where \( n_j \) is the unit normal vector and \( T_i \) is the traction force.
3.1.2 Compatibility
Compatibility relates strains and displacements such that the displacement field $u_{ij}$ results in a unique strain tensor field $\varepsilon_{ij}$. The displacement boundary conditions on the structure must be satisfied at the boundaries. Compatibility is satisfied if the strains and displacements are related by equation (3.4).

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

(3.4)

3.1.3 Hooke’s law
For an elastic material, and within the elastic limits of an elastic-plastic material, Hooke’s Law states that for small deformations of a body, the applied force (or stress) needed to extend the body by some distance is proportional to that distance. In other words, under elastic conditions, strain, $\varepsilon$, is directly proportional to stress, $\sigma$. This is stated more generally in equation (3.5).

$$\varepsilon_{ij} = C_{ijkl} \sigma_{kl}$$

(3.5)

where $C_{ijkl}$ are the elastic constants.

For an isotropic material, the elastic constants are related to Young’s modulus, $E$, and Poisson’s ratio, $\nu$, in accordance with equations (3.6) and (3.7) for principal stresses and equation (3.8) for shear stresses. These equations are taken from Smith [9]

$$
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33}
\end{bmatrix} = \frac{1}{E} \begin{bmatrix}
1 & -\nu & -\nu \\
-\nu & 1 & -\nu \\
-\nu & -\nu & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{bmatrix}
$$

(3.6)
\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{E} \left[ \sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz}) \right] \\
\varepsilon_{yy} &= \frac{1}{E} \left[ \sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz}) \right] \\
\varepsilon_{zz} &= \frac{1}{E} \left[ \sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy}) \right] \\
\varepsilon_{yz} &= 1 + \nu \frac{\tau_{yz}}{E} \\
\varepsilon_{xz} &= 1 + \nu \frac{\tau_{xz}}{E} \\
\varepsilon_{xy} &= 1 + \nu \frac{\tau_{xy}}{E}
\end{align*}
\]

(3.7)

\[
\begin{align*}
\varepsilon_{yz} &= \frac{1 + \nu}{E} \tau_{yz} \\
\varepsilon_{xz} &= \frac{1 + \nu}{E} \tau_{xz} \\
\varepsilon_{xy} &= \frac{1 + \nu}{E} \tau_{xy}
\end{align*}
\]

(3.8)

3.1.4 Out-of-plane constraint

One measure of out-of-plane constraint, \( C_z \), is defined in terms of the ratio of the out-of-plane stress \( \sigma_{zz} \) to the in-plane stresses \( \sigma_{xx} \) and \( \sigma_{yy} \) by equation (3.9).

\[
C_z = \frac{\sigma_{zz}}{\nu (\sigma_{xx} + \sigma_{yy})}
\]

(3.9)

The two limiting values of \( C_z \) are 0 and 1 and occur when \( C_z = \sigma_{zz} = 0 \), known as plane stress or \( C_z = 1 \), known as plane strain.

3.1.5 Plane stress

Plane stress is a two-dimensional approximation whereby it is assumed that the principal stress and shear stresses in the z direction are negligible when compared with the other stresses, giving \( \sigma_{zz} = \tau_{xx} = \tau_{yz} = 0 \).

Plane stress is a suitable approximation for thin-walled elements such as plates subjected to forces parallel to them. In plane stress conditions, Hooke’s law — equations (3.7) and (3.8) — for the in-plane strains are described by equation (3.10).
\[ \varepsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \]
\[ \varepsilon_{yy} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) \]
\[ \varepsilon_{xy} = \frac{1 + \nu}{E} \tau_{xy} \]

(3.10)

The shear strains \( \varepsilon_{xz} = \varepsilon_{yz} = 0 \).

The out-of-plane strains, given by equation (3.11), would not normally need to be considered for a two-dimensional approximation but can be calculated if required at the completion of analyses.

\[ \varepsilon_{zz} = -\frac{1}{E} \nu (\sigma_{xx} + \sigma_{yy}) \]

(3.11)

3.1.6 Plane strain

Plane strain is an approximation whereby it is assumed that the strains the \( z \) direction are negligible when compared with the other strains giving \( \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \).

Plane strain is most commonly used where the \( z \) dimension is very much larger than the others, for example a thick plate, cylinder or channel. It is also appropriate in cracked components where the in-plane strains close to the crack tip are much greater than the out-of-plane strains constrained by deformation away from the crack tip.

In plane strain, Hooke's law for the in-plane strains remains as per equations (3.7) and (3.8). For the out-of-plane strains, Hooke's law can be described by equation (3.12).
\[ \varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy}) \right] = 0 \]

\[ \therefore \sigma_{zz} = \nu (\sigma_{xx} + \sigma_{yy}) \]

\[ \varepsilon_{xz} = \varepsilon_{yz} = 0 \]

(3.12)

Combining equations (3.9) and (3.12) gives an out-of-plane constraint value of \( C_z = 1. \)

Equation (3.12) shows that \( \sigma_{zz} \) can be expressed in terms of \( \sigma_{xx} \) and \( \sigma_{yy} \) thus plane strain analysis can be reduced to a two-dimensional analysis by substituting the formula for \( \sigma_{zz} \) in equation (3.12) into equation (3.7) giving equation (3.13) relating the in-plane strains to the in-plane stresses only.

\[ \varepsilon_{xx} = \frac{1 + \nu}{E} \left[ (1 - \nu)\sigma_{xx} - \nu \sigma_{yy} \right] \]

\[ \varepsilon_{yy} = \frac{1 + \nu}{E} \left[ (1 - \nu)\sigma_{yy} - \nu \sigma_{xx} \right] \]

(3.13)

The effects of biaxial loading on Hooke’s law are examined in chapter 7.

3.1.7 Elastic and plastic deformation

Elastic behaviour occurs when an applied force causes small changes in the spacing and bonds of the interatomic structure. The material returns to its original shape once the load is removed.

Polycrystalline materials such as metals generally contain irregularities, known as dislocations, in their structure. Where sufficient force is applied, the material slips along the planes of these dislocations and this relative movement of the interatomic bonds causes the bonds to break. New bonds are formed and the material maintains its new shape when the force is removed. This is plastic behaviour.
3.1.8 Stress-strain curves

The majority of the analyses in this thesis have been carried out on a theoretical linear elastic material (obeying Hooke’s law) and an elastic-perfectly plastic material.

In an elastic-perfectly plastic material stress-strain curve (Figure 3.2), there are two regions, the elastic and the plastic. In the elastic region, the stress increases linearly with the strain, with slope $E$, Young’s modulus, until the yield stress is reached. Thereafter, in the plastic region, there is no further resistance to deformation and irreversible deformation occurs with no further increase in stress.

![Stress-strain curve](image)

*Figure 3.2: Stress-strain curve for an elastic-perfectly plastic material*

Some of the analyses in this thesis have been carried out on work hardening materials, with a stress-strain curve illustrated in Figure 3.3, whereby the process of plastic deformation strengthens the material by raising the level of stress that the material can withstand.

Work hardening occurs due to the generation and movement of dislocations which can interact with each other to impede the slip. For elongation to occur, higher stresses are needed, until they reach the ultimate tensile stress $\sigma_u$.

Fracture occurs at the end of the stress-strain curve. This often follows the onset of necking, during which the cross-sectional area decreases and the
local stress increases until the material fractures, although fracture can occur without significant necking.

![Stress-strain curve for a non-linear elastic power law hardening material](image)

The theoretical elastic-plastic materials analysed in chapter 6 have stress-strain relationships that follow the Ramberg-Osgood equation [10] reproduced here in equation (3.14).

$$\varepsilon = \frac{\sigma}{E} + K \left( \frac{\sigma}{E} \right)^n$$

(3.14)

where $\varepsilon$ is the strain, $\sigma$ is the stress, $E$ is Young’s modulus and $K$ and $n$ are material dependent constants, $n$ being known as the work hardening exponent.

The two parts of the right-hand side of equation (3.14) are the elastic and plastic components of the strain respectively.

A work hardening coefficient $\alpha$ is defined in equation (3.15).
\[ \alpha = K \left( \frac{\sigma_y}{E} \right)^{n-1} \]  

(3.15)

where \( \sigma_y \) is the yield stress. Equation (3.14) can be rewritten in terms of \( \alpha \) and \( n \) as equation (3.16).

\[ \varepsilon = \frac{\sigma}{E} + \alpha \left( \frac{\sigma}{\sigma_y} \right)^n \]  

(3.16)

At yield, when \( \sigma = \sigma_y \), the elastic and plastic components of the strain are, respectively, \( \sigma_y/E \) and \( \alpha(\sigma_y/E) \), the latter known as the yield offset. This yield offset can be fixed at 0.2% by fixing \( \alpha \) at \( 0.002E/\sigma_y \).

A theoretical value of \( n = 1 \) represents an elastic material. As \( n \) tends towards infinity, the material tends towards one with elastic-perfectly plastic properties. Typical values of \( n \) for metals are between 3 and 10.

The idealised stress-strain curves of Figure 3.2 and Figure 3.3 do not show whether the removal of the applied loading results in the deformation remaining (plastic) or the material returning to its original shape (elastic) i.e. whether the material is actually non-linear elastic rather than elastic-plastic. As long as no significant unloading occurs, the non-linear elastic assumption is valid for the analysis of elastic-plastic materials.

The stress-strain data used in this thesis are for engineering stress and engineering strain such that the calculations of stress and strain are based on the original surface area and length respectively. Alternatively, the stress-strain curves can be based on true stress \( \sigma_t \), whereby the surface area at each stage is the current surface area rather than the original, and true strain \( \varepsilon_t \), whereby the specimen length at each stage is the current value rather than the original.

True stress and true strain can be calculated from the engineering stress \( \sigma \) and engineering strain \( \varepsilon \) using equation (3.17).

Assuming the volume of material remains constant,
\[ \sigma_t = (1 + \varepsilon)\sigma \]
\[ \varepsilon_t = \ln(1 + \varepsilon) \]

(3.17)

3.2 Fracture
Fracture is defined as the separation of a specimen, subjected to an imposed stress, into two or more pieces. There are two modes of fracture that engineering materials can undergo, brittle and ductile.

3.2.1 Brittle fracture
Brittle fracture generally occurs when a material breaks or cracks under stress without any significant prior plastic deformation or energy absorption.

For polycrystalline materials such as metals, the type of brittle fracture that occurs is known as cleavage fracture, in which the breaks occur along cleavage planes. These planes are the surfaces along which the interatomic bonds are weaker and break more easily.

Under brittle fracture, cracks can spread (propagate) suddenly and rapidly.

3.2.2 Ductile fracture
Ductile fracture occurs when significant plastic deformation and absorption of energy takes place before fracture.

For an uncracked ductile specimen, an initial stage of necking can occur. The ductile fracture process then follows various stages. Small cavities, known as voids (or microvoids) form (nucleate). These voids then grow and come together. This coming together is known as coalescence.

In the initially uncracked specimen, this forms a crack which continues to propagate by the void nucleation, growth and coalescence process. Failure occurs as the crack reaches the perimeter of the neck.

In a cracked specimen, the void nucleation, growth and coalescence process causes failure between the void and the original crack tip leading to the extension of the crack. This process is known as ductile tearing.
More complex modes of failure occur in the form of combinations of fracture mechanisms, such as ductile tearing (void coalescence) followed by brittle fracture or ductile tearing followed by plastic collapse.

### 3.3 Linear Elastic Fracture Mechanics

Fracture mechanics is the field of study within solid mechanics concerned with the propagation of cracks. Where there is an existing crack, the objective of the application of fracture mechanics is to determine relationships between crack geometry, specimen geometry, material properties and the nature and magnitude of the applied stresses that will cause the crack to propagate and lead to failure.

#### 3.3.1 The energy balance approach

Linear Elastic Fracture Mechanics (LEFM) theory was originally developed by Griffith whose proposed approach was that crack growth is based on a balance of strain energy and surface energy [11]. This approach assumes that a crack causes strain energy to be released at the free surface. The crack depth, $a$, increases under applied normal stress increasing the surface energy, and this depends on strain energy until the stresses applied are high enough to cause the crack to propagate. The stress at which fracture occurs, $\sigma_f$, for a sharp crack with length $a$, is given in equation (3.18).

$$
\sigma_f = \sqrt{\frac{2E\gamma}{\pi a}}
$$

(3.18)

where $E$ is the Young's modulus, $\gamma$ is the surface energy and $a$ is the crack depth.

This theory was developed by Orowan [12] and Irwin [13], who noted that while the theory held for brittle materials, it overpredicted the surface energy $\gamma$ for ductile materials, as for these materials the released strain energy was absorbed by energy dissipation due to the development of a plastic zone at the crack tip. They defined an energy release rate (the rate of change of in potential energy with crack area) $G$, also known as the crack driving force (CDF) shown in equation (3.19) for a crack length $2a$. 

$$
G = \frac{\sigma_f^2}{E}
$$

(3.19)
\[
\sigma = \sqrt{\frac{EG}{\pi a}}
\]

\[
G = \frac{\sigma^2 \pi a}{E}
\] (3.19)

Crack extension occurs when \( G \) reaches a critical value \( G_c \) such that if \( G \geq G_c \), the crack will propagate. Thus \( G_c \) is a measure of resistance to fracture, i.e. the fracture toughness of the material.

The formula for \( G_c \) is shown in equation (3.20).

\[
G_c = \frac{\sigma_f^2 \pi a_c}{E}
\] (3.20)

where \( a_c \) is the critical semi crack length.

LEFM is applicable where the size of the plastic zone at the crack tip is small relative to the size of the body.

3.3.2 The stress field approach

Williams [14] determined equations relating the opening stress fields \( \sigma_{ij} \) near to the crack tip to the polar coordinates \((r, \theta)\) relative to the crack tip, a simplified version of which is given here in equation (3.21)

\[
\sigma_{ij}(r, \theta) = \sum_{n=-\infty}^{\infty} A_n f_{ij}^{(n)}(\theta) r^n
\] (3.21)

where \( A_n \) are constants and \( f_{ij}^{(n)}(\theta) \) are angular functions of \( \theta \).
The leading term of equation (3.21) is directly proportional to r\(^{-1/2}\), thus there is a singularity as r approaches 0 and the stress theoretically approaches infinity.

3.3.3 Stress intensity factor

Irwin [13] developed the stress intensity factor (SIF), K describing the intensity of the stress state near the crack tip.

The value of K depends in part on the mode of loading. There are 3 main types of loading known as modes I, II and III. Mode I loading opens the crack as a result of applied loading perpendicular to the crack plane. Modes II and III are in-plane and out-of-plane shear opening modes respectively and will slide the crack faces relative to each other. These are illustrated in Figure 3.4.

The values of K and \(f_{ij}\) in equation (3.21) depend on these crack loading and opening modes. As the research presented in this thesis concentrates on the mode I opening, Figure 3.4 (a), it is only the values of SIF K and angular functions \(f_{ij}(\theta)\) for mode I opening that will be covered here.

![Figure 3.4: Loading modes for cracked components](image)

(a) Mode I, principal loading applied normal to the crack plane, opens crack (b) Mode II, in-plane shear, (c) Mode III, out-of-plane shear

The first term of the expansion of equation (3.21) is shown in equation (3.22).

\[
\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) + \ldots
\]  

(3.22)
where $K_I$ is the mode I stress intensity factor

Westergaard [15] developed the angular functions $f_0(\theta)$ which give the formulae for the stress fields ahead of a crack tip for mode I opening, in equation (3.23).

$$
\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[ 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right]
$$

$$
\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[ 1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right]
$$

$$
\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right)
$$

(3.23)

Values of $K_I$ vary with component geometry and crack geometry and loading. A compendium of values of $K$ can be found in R6 Section IV.3 [3]. For a crack in an infinite plate subject to a uniform mode I remote tensile stress (Figure 3.5), the mode I stress intensity factor is given by equation (3.24).

Similarly to the energy release rate, $G$, when the stress intensity factor $K_I$ reaches a critical value $K_{IC}$, the material can no longer withstand the crack tip stresses and the crack propagates. Thus $K_{IC}$ is a measure of the material fracture toughness.

$$
K_I = \sigma \sqrt{\pi a}
$$

(3.24)

The SIF $K_I$ is directly related to the energy release rate $G$. Combining equations (3.19) and (3.24) gives the relationship shown in equation (3.25).
where $E'$ is the elastic modulus such that $E' = E$ (Young’s modulus) for plane stress and $E' = E(1-\nu^2)$, where $\nu$ is Poisson’s ratio, for plane strain.

In accordance with Irwin [13] this relationship holds for all component and crack configurations.

$K$ is normally given in units of MPa$\sqrt{m}$.

Biaxial loading has been shown to have no effect on the mode I stress intensity factor $K_I$ [16, 17]. Consider a cracked plate subject to biaxial loading such that the crack is at an angle $\beta$ to the horizontal (Figure 3.6).

For a ratio of the stress parallel to the horizontal to the stress perpendicular to the horizontal is given by $B = \sigma_1/\sigma_2$ then the value of $K_I$ is given by equation (3.26), based on [18].

\[
G = \frac{K_I^2}{E'}
\]

(3.25)
Thus for a stress $\sigma_1 = B\sigma_2$ applied parallel to the crack, $\beta = \sin \beta = 0$, so $B \sin^2 \beta = 0$ and $\cos^2 \beta = 1$ for all values of B. Therefore, for a cracked plate with stresses applied remotely in the two orthogonal directions, the biaxial SIF $K_I$ will be identical to that for uniaxial loading.

Biaxial loading can, however, affect the material fracture toughness $K_{IC}$. The literature generally indicates that an effect of biaxial loading on fracture toughness has been demonstrated. Some have found an increase [19], some a decrease in fracture toughness under the biaxial loading conditions tested [20-26] while others have found a non-monotonic dependency for different biaxial ratios [27].

### 3.3.4 T-stress

The second term of the Williams [14] expansion (equation (3.21)) is a function of $r^0$, i.e. it is a constant term independent of $r$. It is shown here as the second term in equation (3.27).
\[
\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) + T\delta_{ii}\delta_{ij}
\]

(3.27)

where \( \delta_{ij} \) is the Kronecker delta.

Thus this second term describes a stress in the direction of \( \sigma_{ii} \) only, which is the stress component parallel to the plane of the crack. Like \( K \), the value of \( T \)-stress depends on the geometry, crack size and loading. A compendium of values of \( \beta_T \), which is \( T \)-stress normalised by reference stress \( \sigma_{ref} \), can be found in R6 [3] Section IV.5.

(Reference stress describes a relationship between the applied load, limit load and yield stress. It is the yield stress multiplied by the ratio of the applied load to the limit load.)

A more detailed description of the calculation of \( T \)-stress and how it is affected by biaxial loading is given in section 6.2 of this thesis. In the literature, Betegón and Hancock [28] describe the development of assessment methods using the \( T \)-stress. The effect of biaxial loading on \( T \)-stress for plates is covered in O’Dowd et al. [29].

## 3.4 Limit Analysis

The limit load of a structure is the maximum load that it can carry before plastic collapse occurs, that is that the plastic zone has become extended far enough across the structure for collapse to occur.

### 3.4.1 Von Mises stress criterion

The von Mises stress, \( \sigma_{vm} \), is an effective stress based on the theory that while stresses in all three principal directions may be below (or above) the yield stress, the material will yield or fail if the following combination of these stresses as shown in equations (3.29) to (3.31) reaches or exceeds the yield stress \( \sigma_y \).
\[ \sigma_{vm} = \frac{1}{\sqrt{2}} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 \right]^{\frac{1}{2}} + 6(\sigma_{12} + \sigma_{23} + \sigma_{31})^2 ]^{\frac{1}{2}} \]

(3.28)

In terms of principal stress only,
\[ \sigma_{vm} = \frac{1}{\sqrt{2}} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 \right]^{\frac{1}{2}} \]

(3.29)

For plane strain equation (3.29) becomes
\[ \sigma_{vm} = \frac{\sqrt{3}}{2} |(\sigma_{11} - \sigma_{22})| \]

(3.30)

and for plane stress,
\[ \sigma_{vm} = (\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2)^{\frac{1}{2}} \]

(3.31)

### 3.4.2 Tresca maximum shear stress criterion

The material starts to yield when the maximum shear stress \( \tau_{\text{max}} \) reaches or exceeds the shear yield stress \( \tau_y \).

The maximum shear stress can be calculated using equation (3.32)
\[ \tau_{\text{max}} = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} = \max \{ |\sigma_{11} - \sigma_{22}|, |\sigma_{22} - \sigma_{33}|, |\sigma_{11} - \sigma_{33}| \} / 2 \]

(3.32)

where \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) are the maximum and minimum values of principal stress.

As \( \sigma_y = 2\tau_y \), the Tresca criterion is as shown in equation (3.33).
\[ \sigma_{tr} = \max\{\left|\sigma_{11} - \sigma_{22}\right|, \left|\sigma_{22} - \sigma_{33}\right|, \left|\sigma_{11} - \sigma_{33}\right|\} \]  

(3.33)

For plane stress, \(\sigma_{33} = 0\) and the yield condition becomes

\[ \sigma_{tr} = \max\{\left|\sigma_{11} - \sigma_{22}\right|, \left|\sigma_{22}\right|, \left|\sigma_{11}\right|\} \]  

(3.34)

The Tresca yield criterion is more conservative than the von Mises criterion. Figure 3.7 displays the von Mises and Tresca yield criteria in the \(\sigma_{11}, \sigma_{22}\) plane under both plane stress and plane strain conditions. Each line represents the points where the value of the yield criterion — \(\sigma_{vm}\) or \(\sigma_{tr}\) — is equal to the material yield stress \(\sigma_y\).

In-plane stress conditions, for the Tresca criterion, neither of the applied stresses can exceed \(\sigma_y\). For the von Mises criterion, the value of \(\sigma_{22}\) (or \(\sigma_{11}\)) can exceed the yield stress \(\sigma_y\) up to a maximum value of \(\frac{2}{\sqrt{3}}\sigma_y \approx 1.155\sigma_y\). The corresponding maximum value of \(\sigma_{11}\) (or \(\sigma_{22}\)) at yield is \(\frac{1}{\sqrt{3}}\sigma_y \approx 0.575\sigma_y\).

In the case of plane strain, it can be seen from Figure 3.7 that the yield lines have a slope of 1. The loading line where \(\sigma_{11} = \sigma_{22}\) has a slope of 1 and runs parallel to the yield lines so never crosses them. From equations (3.30) and (3.33) respectively it is clear that for the equibiaxial case, for all values of \(\sigma_{22}\), the von Mises stress \(\sigma_{vm}\) and Tresca stress \(\sigma_{tr}\) are equal to zero and never reach the yield stress \(\sigma_y\).

### 3.4.3 Lower bound theorem of limit analysis

The lower bound theorem states that where a stress distribution in a body:

- Is in internal equilibrium throughout
- Balances the external loads
- Has an equivalent stress that does not exceed the yield stress throughout, then the corresponding applied load will be less than or equal to the exact external collapse load and is therefore a lower bound estimate of that load.
The upper bound theorem states that for any mechanism of deformation, equating internal dissipation of energy with the rate of work done by external forces gives an estimate of collapse load that is greater than or equal to the exact external collapse load.

Slip-line fields can be used to model plastic deformation. They are the lines of maximum shear stress and the lines along which it is assumed plastic flow takes place. They are limited by their use exclusively for plane strain conditions and their inability to incorporate strain hardening, temperature effects and elasticity. However, they are useful in the determination of limit loads.

The theory was originally developed by Prandtl [30] and further equations were derived by Hencky in 1923 and later by Geiringer [31]. Hill [32] described the use of slip-line field methods in determining the plastic deformation of a thick plate as well as introducing the upper bound theorem.
In the early years, the use of slip-lines was limited by the difficulty involved in determining their location, magnitude and direction but more recently FEA has made their determination reasonably straightforward. An explanation of how to determine the slip-line fields analytically can be found in [33]. Some deformation patterns are shown in Figure 3.8. These are based on those in [7] taken from McClintock’s work on slip-lines for cracked structures [34].

![Figure 3.8: Schematics of yielding of cracked structures](image)

(a) Small-scale yielding, (b) Slip-lines under large-scale yielding for double edge cracked plate in tension, (c) Slip-lines under large-scale yielding for single edge cracked plate in bending, (d) Slip-lines under large-scale yielding for centre cracked plate in tension.

In the analyses in chapters 5 and 6 of this thesis, upper bound estimates to the limit load have been found using the energy along slip-lines.

### 3.4.6 The influence of a defect on limit analysis

The presence of a defect in a component means that limit analysis is concerned with the loadings that the ligaments in the region of a crack can withstand before they yield plastically. This yielding of the component generally occurs when the plastic zone extends from the crack tip to a free surface.
Instinctively, the existence of a defect in a component should reduce the limit load of that component in proportion to the relative size of the defect, given the very high stresses at the crack tip and the load-carrying area of the component being reduced to that of the ligaments either side of the crack. The lower bound theorem of limit analysis can be used to show that the limit load of an uncracked body cannot be lower than that of the corresponding cracked body.

There is a compendium of examples of the limit loads of cracked bodies, using both von Mises and Tresca criteria, presented in R6 [3] Section IV.1. This compendium contains solutions for test specimens, plates, cylinders and spheres, of which many are from Miller [36], and from Lei's work on cylinders [37] and surface cracks in plates [38].

The limit loads calculated using these solutions confirm that the presence of a defect reduces the limit load of a component subjected to uniaxially applied remote tensile loading giving a limit load whose magnitude is less than the material's yield stress.

Biaxial loading is referred to very little in the R6 compendium of limit loads. If it were assumed that it is conservative to neglect biaxial loading, this would suggest that the addition of a biaxial load would increase the limit load achieved under the uniaxial loading.

There is, however, little to be found in the literature quantifying the effect of biaxial loading on limit loads before O'Dowd et al in 1999 [29] developed
equations for the limit loads of centre cracked plates (CCP) under biaxial loading.

Their equations yield values of the limit load that are greater than the uniaxial loading limit load for all $0 < B < 1$, where $B$ is the ratio of the load parallel to the crack to the loading perpendicular to the crack. This concurs with the assumption of conservatism.

In their comparison with the FEA output for these values of $B$, they identify major differences to the equations developed for the limit loads when $B = 1$, particularly for lower values of $a/W$. They conclude that their equations are not reliable predictors of limit loads when $B = 1$ for the values of $a/W$ examined.

Since O’Dowd et al., few researchers have continued to look at limit loads for biaxial loading. In 2004 Kim et al. [39] confirmed the findings with their own FEA. Since then work has mainly been carried out by Lei et al. [38, 40, 41] and Wei and Hadley [42] on plates with surface cracks. They identified a non-monotonic relationship between biaxial force ratio and limit load such that use of the uniaxial loading limit load could either underestimate or overestimate the biaxial limit load.

More research has been carried out on limit loads of biaxially loaded cylinders. This type of loading consists of axial tension and internal pressure, although internal pressure alone induces biaxial loading in the form of a hoop (circumferential) stress component and an axial stress component.

Miller’s 1988 review [36] looks at cylinders under pressure and bending and it is the basis for a lot of further research and many of the R6 solutions [3]. Among those building on this work and adding in the axial tension, Michel and Plancq [43] and Lei and Budden [44] derived solutions for thin walled cylinders. Their more recent research using FEA assessed and confirmed the validity of the solutions in R6 for thick walled cylinders [37].

These limit load analyses of cylinders show relationships between the pressure, tension and bending limit loads rather than advice on direct relationships between biaxial load ratio (hoop stress to axial) stress and limit load.
3.5 Elastic Plastic Fracture Mechanics

As described in section 3.3.1, LEFM can be applied where the loading is in the elastic range of a material such that any plastic deformation is negligible and confined to a small zone around the crack tip (small-scale yielding or SSY).

Once the area of plastic deformation that occurs before crack propagation exceeds the size determined for SSY, elastic-plastic fracture mechanics (EPFM) will more accurately describe the material’s behaviour in the larger plastic zones.

Research into the effects of biaxial loading on plasticity has generally found that biaxial loading reduces the size of the plastic zone [23, 27, 45-47].

3.5.1 Small-scale yielding

For small-scale yielding to be defined, the size of the plastic zone, \( r_p \), must be small compared to the crack size, \( a \). This relative size, for plane strain, is referred to by Larsson and Carlsson [48] as shown here in equation (3.35).

\[
2r_p = \frac{1}{3\pi} \left( \frac{K_I}{\sigma_y} \right)^2
\]

(3.35)

where \( K_I \) is the mode I stress intensity factor and \( \sigma_y \) is the yield stress, with a maximum \( K_I \) as shown in the inequality (3.36).

\[
K_I < \sigma_y \sqrt{\frac{a}{2.5}}
\]

(3.36)

This gives a maximum plastic zone radius \( r_p \) of 1/47 of the crack width \( a \). This is very small. Ren et al. [49] give a ratio of 0.2 which is considerably larger.

There is little other guidance in the literature. The general consensus is that as long there is enough material between the plastic zone and the outer
edge of the model, a plastic zone radius with a value of a/10 is considered sufficient.

However, the determination of the maximum size of the plastic zone as a proportion of the crack size is only really used as a visual guide during modelling. In practice, the departure from small-scale yielding is determined by the value of the J-integral exceeding G as defined in equation (3.25).

3.5.2 J-integral

In section 3.3, the two approaches to Linear Elastic Fracture Mechanics (LEFM), the energy based approach (3.3.1) and the stress based approach (3.3.2) were described. Similarly to LEFM, Elastic Plastic Fracture Mechanics (EPFM) can be approached in these two ways, both of which use the parameter J — or J-integral.

In LEFM, J is equivalent to the Griffith energy release rate G and is related to the stress intensity factor K as in equation (3.25). Its units are kJ/m², the rate in this case meaning per unit area rather than time. This elastic component of J is usually denoted by \( J_e \).

For the stress fields at a crack tip of non-linear, power law hardening, elastic-plastic materials, Hutchinson [50], Rice and Rosengren [51] developed equations for which the stress is scaled by J. This is known as the HRR singularity and is shown in equation (3.37).

\[
\sigma_{ij} = \sigma_y \left( \frac{J}{\alpha I_n \sigma_y \varepsilon_y r} \right)^{\frac{1}{n+1}} \overline{\sigma}_{ij}(\theta, n)
\]

(3.37)

where \( \alpha \) and \( n \) are the Ramberg-Osgood [10] material’s work hardening coefficient and exponent respectively as defined in equations (3.14) to (3.16), \( I_n \) is a dimensionless integral parameter, \( \sigma_y \) the stress at yield, \( \varepsilon_y \) the strain at yield, \( r \) and \( \theta \) the polar coordinates around the crack tip and \( \overline{\sigma}_{ij} \) is a dimensionless function of \( \theta \) and \( n \).

For an elastic material, where \( n = 1 \), the singularity determined using equation (3.37) is in \( r^{1/2} \) which relates to the stress fields defined for LEFM
in equation (3.21). In addition, like the LEFM field, the expression in $r^{-1/2}$
does not fully describe the stress field and additional terms are required.
This will be explained more in section 3.5.3.

The energy interpretation of the J-integral was developed by Rice in 1968
[35], and describes a path-independent line integral around the crack tip.
The J contour integral is given by equation (3.38).

$$ J = \int_{\Gamma} \left( wdy - T_i \frac{\partial u_i}{\partial x} \right) ds $$

(3.38)

$$ w = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} $$

(3.39)

where $w$ is the strain energy density given in equation (3.39), $T_i$ are the
traction vector components, $\Gamma$ is the contour integral path and $s$ its length.

The value of $J$ is normally found using finite element analysis (FEA) but can be estimated using the failure assessment diagram (FAD) as described in
section 3.6.3.

In 1977, Lee and Liebowitz et al. [16, 52] found, for a CCP, using FEA
methods, that for LEFM analysis the J-integral is independent of the biaxial
loading – this is to be expected as $J$ is directly proportional to $K^2$, which is
itself independent of biaxial loading. However, for nonlinear analysis, the
relationship between biaxial load ratio, $k$, applied stress and $J$ is not so
straightforward. For stresses with a magnitude of less than 0.4 of the yield
stress, $J$ decreases with higher biaxial load ratio. Above this value, $J$
decreases up to ratio $k = 1$ and then starts to increase. This study is one of
the first — and one of few — to look at a wide range of biaxial load ratios
and include those greater than unity. This allowed the research to establish
the non-monotonic effects of biaxial loading on the fracture parameters.

Wright et al.’s experiments on centre through cracked plates [25] and Lei et
al.’s analytical analysis and FEA on extended surface cracked plates [41]
showed a decrease in $J$ with increasing biaxial loading ratio.
Kim et al. [39] observed that the effect of biaxial loading on $J$ is largely driven by its effect on limit load.

### 3.5.3 Constraint parameter $Q$

Fracture toughness values are generally obtained from the testing of specimens that are high constraint, such that the plastic zone is contained in a small area around the crack tip and does not spread easily making fracture more likely under increased loading. This makes for conservative estimates of fracture toughness when applied to lower constraint specimens.

The increased value of the fracture toughness for lower constraint specimens can be represented by equation (3.40), developed by Ainsworth and O’Dowd [53].

$$
\begin{cases}
K_{mat}^c = K_{mat} & \beta L_r > 0 \\
K_{mat}^c = K_{mat}[1+\alpha(-\beta L_r)^m] & \beta L_r < 0
\end{cases}
$$

where $K_{mat}$ is the standard fracture toughness, $K_{mat}^c$ the constraint modified fracture toughness, $\alpha$ and $m$ are material parameters, $\beta$ is the normalised constraint parameter and $L_r$ the proximity of the specimen to plastic collapse (described later in section 3.6).

The normalised constraint parameter $\beta$ can be determined using T-stress (section 3.3.4) or $Q$, a parameter derived by O’Dowd and Shih [54] to account for the difference identified between the HRR stress field (equation (3.37)) and the more accurate crack tip stress fields determined using FEA. Chao et al. [55] developed a similar constraint parameter $A_2$ for use with power law hardening materials. The research presented in this thesis is based around $Q$, which will be covered in more detail here and in chapter 6.

As well as the difference between the HRR field and the measured crack tip stress field, another way of determining a value for $Q$ is shown in R6 Section III.7.5.2 [3], relating it to the stress field under small-scale yielding (SSY), reproduced here in equation (3.41).
\[
\sigma_{ij} = \sigma_{ij}^{ssy} + Q \sigma_y \delta_{ij}
\]  

(3.41)

where \(\sigma_{ij}^{ssy}\) is the SSY stress field, \(\sigma_y\) the yield stress and \(\delta_{ij}\) the Kronecker delta.

In order to determine a value for \(Q\), it is necessary first to determine the small-scale yielding stresses \(\sigma_{ij}^{ssy}\). These stresses are generally calculated using a modified boundary layer model (MBL).

The MBL model consists of a two-dimensional plane strain circular FEA model of theoretical infinite radius, which is achieved by using displacement boundary conditions. The MBL can be used to develop stress fields in the vicinity of the crack tip under high or low constraint conditions depending on the application of the relevant displacement boundary conditions.

The MBL theory and method are explained thoroughly in [49] and [56]. The particular model used in this research is described in chapter 4.

As the stress varies with distance from the crack tip, a specific distance must be selected at which to extract the value of stress \(\sigma_{22}\) from the FEA model results. The most commonly used is the value of stress at a distance \(r/(J/\sigma_y) = 2\) from the crack tip, from [54]. This gives a value for \(\sigma_{22}^{ssy}\).

For each geometry and loading condition of interest, the value of \(\sigma_{22}\), also at a distance \(r/(J/\sigma_y) = 2\) from the crack tip, is determined and the SSY stress \(\sigma_{22}^{ssy}\) is subtracted. The value thus derived is then normalised by the yield stress \(\sigma_y\) to determine \(Q\) as in equation (3.42).

\[
Q = \frac{\sigma_{22} - \sigma_{22}^{ssy}}{\sigma_y}
\]  

(3.42)

where \(\sigma_{22}\) is the stress in the crack opening direction and \(\sigma_{22}^{ssy}\) is the stress field for the small-scale yielding in this direction.
There is currently little specific advice in R6 for directly determining a value for \( Q \) for biaxially loaded specimens. There is some research however that demonstrates how \( Q \) can be estimated using T-stress for power law hardening materials [29], [57].

### 3.6 The Failure Assessment Diagram

The failure assessment diagram (FAD) is the fundamental method by which structural integrity assessments are carried out in R6.

#### 3.6.1 A brief history of the FAD

The previous sections have described some of the mechanisms that can cause a material to fail; plastic collapse, brittle fracture, ductile fracture and combinations thereof. At the two extremes are elastic fracture and plastic collapse between which lie combinations of the various failure mechanisms.

The complexity of these failure mechanisms leads to the requirement for a formal assessment procedure to address the nuclear industry’s need for the integrity of structures to be standardized.

In 1976, the Central Electricity Generating Board (CEGB) developed the R6 procedure [58] based around a two-parameter failure assessment diagram (FAD) from the concept originally introduced by Dowling and Townley [59]. The FAD is a graph with horizontal and vertical axes representing the two parameters \( L_r \), proximity to plastic collapse, and \( K_r \), proximity to elastic fracture. A failure assessment curve (FAC) is defined by \( K_r \) as a function of \( L_r \) up to a maximum \( L_r \), above which the value of the curve drops to zero.

For each selected case, the parameters \( L_r \) and \( K_r \) are calculated and a point with coordinate \(( L_r, K_r )\) is plotted on the graph. Failure is then determined by whether the point \(( L_r, K_r )\) lies within or without the area bounded by the two axes and the FAC.

The original R6 FAC used the Dugdale strip yield model [60]. The curve derived from this model has undergone various refinements and amendments to take into account residual stress, secondary stresses [61] and work hardening [62].

There are three options for the FAC as described below. The Option 1 curve is based on empirical data from fitting Option 2 curves for various materials.
and is a particularly good fit to austenitic steels with low strain-hardening rates [63].

An example Option 1 FAD is shown in Figure 1.1.

### 3.6.2 Current R6 FAD determination

The parameters for the plotted point \((L_r, K_r)\) are determined using equations (3.43) and (3.44) respectively.

\[
L_r = \frac{P_{\text{app}}}{P_L}
\]  
(3.43)

\[
K_r = \frac{K_I}{K_{\text{mat}}}
\]  
(3.44)

where \(K_r\) is the proximity to elastic fracture, \(K_I\) is the mode I stress intensity factor, \(K_{\text{mat}}\) the material fracture toughness, \(L_r\) the proximity to plastic collapse, \(P_{\text{app}}\) the applied loading and \(P_L\) the limit loading.

In the current revision of R6 [3] there are three main options for the FAC with increasing specificity and complexity and decreasing conservatism. The Option 1 curve, \(f_1(L_r)\), equation (3.45), is the most straightforward and requires no material specific information.

\[
f_1(L_r) = \left[1 + 0.5L_r^2\right]^{-1/2}\left[0.3 + 0.7e^{-0.6L_r}\right]
\]  
(3.45)

The Option 2 curve \(f_2(L_r)\), equation (3.46), requires stress-strain data for the material under consideration and was developed by Ainsworth [64] using a reference stress method for strain hardening materials.
\[ f_2(L_{\tau}) = \left( \frac{\epsilon_{\text{ref}}}{L_{\tau} \sigma_y} + \frac{L_{\tau}^3 \sigma_y}{2E \epsilon_{\text{ref}}} \right)^{-1/2} \]

(3.46)

where \( \epsilon_{\text{ref}} \) is the reference strain calculated at the reference stress \( \sigma_{\text{ref}} \) for true stress-strain data.

An Option 2 curve for a theoretical elastic-perfectly plastic material is shown together with the Option 1 curve in Figure 3.10. The material tensile properties influence only the value of the maximum \( L_{\tau} \) cut-off when using the Option 1 curve, but determine also the shape of the curve when using the Option 2 curve (see section 5.5).

The Option 3 curve, given in equation (3.47), is determined using J-integral analysis.

\[ f_3(L_{\tau}) = \left( \frac{J_e}{J} \right)^{-1/2} \]

(3.47)

where \( J_e \) is the elastic J-integral and \( J \) the elastic-plastic J-integral.

Clearly the Option 3 curve requires analysis specific to both the material and the cracked component for its input parameters.

3.6.3 FAD as J estimation scheme

An alternative way of using the FAD is to estimate \( J \), the crack driving force (CDF) from the position on the FAC given by the known variables, and assess this against the known fracture toughness \( J_{\text{mat}} \). Failure occurs when the CDF thus determined is equal to \( J_{\text{mat}} \).

\( J \) is calculated using equations (3.48) and (3.49).
\[ K_r = f(L_{r}) = \left[ \frac{J_e}{J} \right]^{1/2} \]

(3.48)

\[ J = \frac{J_e}{[f(L_{r})]^2} = \frac{K^2}{E} \left[ f(L_{r}) \right]^2 \]

(3.49)

where \( f(L_{r}) \) is the Option 1 or Option 2 FAC (equations (3.45) and (3.46)).

### 3.7 Experimental Investigations into Biaxial Loading

The main focus of this research is on the effects of biaxial loading on the CDF rather than the fracture toughness, and as such this will be the main focus of this section of the literature review.

Historically, it had been assumed for most structural applications, that the addition of biaxial loading would most likely be beneficial to the material’s resistance to fracture, such that where a component was subject to biaxial loading, the application of parameters derived from uniaxial loading would lead to conservative estimates of failure conditions.
Undertaking both large and small-scale experiments on biaxially loaded components and specimens can be hindered by the complexity and high costs of the required test pieces and rigs. Together with the assumption of conservatism, these restrictions have led to a paucity in practical investigations into the effects of biaxial loading on the failure parameters — limit load, fracture toughness, crack driving force, stress intensity factor, crack initiation and crack propagation — until relatively recently.

Specimens suitable for experiments on the effects of biaxial loading are generally in the form of plates, cylinders or cruciform specimens.

In early experiments in 1970, Kibler and Roberts [19] carried out tests for the effects of biaxial loading on fracture toughness of aluminium and PMMA (Plexiglass) centre cracked plates. These indicated that the fracture toughness increased with the addition of a biaxial load. In the 1980s Eftis, Jones et al [27, 65, 66] continued this work and showed that the increase in biaxial load leads to an increase in fracture load for the aluminium plates but a reduction in fracture load for the Plexiglass plates.

Garwood et al. [17] experimented on large-scale A533B (a nuclear pressure vessel steel) plates and found that resistance to crack growth in surface cracked plates was not affected by biaxial loading but that for through thickness cracks the resistance to crack growth reduces under equibiaxial loading. Wright et al. [25] found a 20% increase in the fracture initiation stress for the biaxially loaded specimen in their experiments on 0.36% carbon steel through cracked plates.

Practical experiments — especially large-scale experiments — to investigate the effects of biaxial loading on cylinders are few. Østby and Hellesvik [67] demonstrated that the strain capacity and the CDF both increased when a biaxial pressure load was added to an axially loaded pipe with a defect, and that the failure mode changed from buckling to fracture.

During the 1990s, a series of experiments were carried out on cruciform specimens by Bass, Theiss, McAfee et al. as part of the HSST (Heavy-Section Steel Technology) programme for the US Nuclear Regulatory Commission (NRC) [20–24]. In 2005, Bass and McAfee, led by Taylor, were also involved in more recent experiments on cruciform specimens as part of the NESC-IV project (Network for Evaluating Structural Components) [68].
They showed a significant (42%) reduction in toughness under biaxial loading.

Other experiments using cruciform specimens have been carried out by Hohe et al., who carried out smaller scale experiments that showed consistency with the results of large stage tests [69]. They also developed a formula for stress intensity factor K for biaxially loaded cruciform specimens. The Bhabha Atomic Research Centre (BARC) in Mumbai carried out large-scale cruciform tests [70]. Those subject to equibiaxial loading resisted crack extension and were able to withstand a higher load than the uniaxially loaded specimens.

Biaxial loading on the transverse arm of a cruciform specimen will be out-of-plane as far as the crack is concerned and its effect can be different from the orthogonal biaxial loading on cracked plates and cylinders. An additional observation by [70] is that the biaxial load ratio of 2:1 was not constant throughout the loading, so it is not straightforward to deduce the exact effects of a 2:1 loading from this paper.

Advances in computing power and the availability of the technology over the last 40 years have led to considerable growth in the use of finite element analyses (FEA) for modelling the effects of all kinds of complex loading, materials and geometries. Early FEA research on the effects of biaxial loading include that of Miller and Kfouri, who carried out FEA on centre cracked plates in the mid-1970s [46, 47].

Most of the research into the effects of biaxial loading since the 1990s now uses finite element models either exclusively or as validation of new theoretical and earlier experimental solutions.

### 3.8 Conclusions

The findings of this literature review can be summarised by the following conclusions.

1. There is currently little advice in the fitness for service standards regarding the assessment of structures with defects subjected to biaxial loading. There is also relatively little research into biaxial effects in general. These shortages may be due to an assumption that for failure assessments of components subject to biaxial loading, it is not necessary to consider
loadings parallel to the crack, and that the application of parameters relating to uniaxial loading will generally be conservative.

2. Assessment based on the FAD requires the determination of four main parameters, stress intensity factor, fracture toughness, applied stress and limit load. These parameters can be affected by biaxial loading to various degrees and the effect can be either beneficial or detrimental.

3. Biaxial loading has been shown to have the following effects.
   - A stress parallel to the crack has no effect on the stress intensity factor.
   - It can increase or decrease fracture toughness.
   - Tensile biaxial loading can delay the onset of plasticity
   - Tensile biaxial loading can reduce the size of the plastic zone
   - The effect of the addition of a biaxial load on the applied stress affects the T-stress in the direction of the biaxial load parallel to the crack direction.
   - The effect of biaxial loading on limit load can be complex. It varies according to the size and relative size of the biaxial loading and whether the biaxial loading is in-plane or out-of-plane.

4. The FAD can be used to estimate crack driving force $J$ and an assessment of failure made by testing the CDF against the fracture toughness. Biaxial loading has been shown to affect $J$, however it is suggested that this effect is driven by the effect of biaxial loading on the limit load.

5. Experimental work has been carried out on plates, cylinders and cruciform specimens. The complexity and expense of carrying out such experiments has meant that there are few examples available in the literature.

6. Developments in computing technology has meant that finite element analyses have been used to develop solutions to biaxial loading and also to verify previous experimental, empirical and analytical solutions.
4 Research Framework

The literature review in the previous chapter highlighted the need for the development of solutions for the dependence on biaxial loading of the stress and fracture parameters of structural integrity assessments.

This chapter will provide an overview of the research and the main models and parameters used throughout the project to accomplish the research aims.

Detailed methodology will be described as it occurs in the subsequent chapters, as each chapter follows a slightly different methodology, the developments of which are essential parts of the analyses and results.

4.1 Research Aims

The main aim of this research is to develop solutions that quantify the effects of biaxial loading on the parameters involved in the assessment of a structure, component or specimen with defects. These solutions will be mainly analytical solutions based on the solid and fracture mechanics theories as described in the previous chapter and in more detail as they arise.

FEA have been carried out to verify the analytical solutions. In some cases, for example the deformation patterns for the upper bound limit load solutions, the FEA solutions are the basis on which the theoretical solutions have been developed.

The analyses presented here first aim to describe the effect of biaxial loading on the limit load solutions. Starting with the simplest form, the centre cracked plate (CCP) in an elastic-perfectly plastic material, upper and lower limit load solutions are developed. Building on these, short plate solutions and solutions for single and double edge cracked plates are explored. These lead to an assessment of how these effects influence the FAD and hence the failure loads for components under biaxial loading.

The aim of the next chapter is to deal with the effect of biaxial loading on the other three parameters; applied stress in the form of the T-stress, stress intensity factor and the influence of biaxial loading on fracture toughness based on the Q stress constraint parameter. These will again look mainly at
a CCP. The effect on the J-integral and its influence on the FAD are also explored here.

Following on from the findings of the two previous chapters, chapter 7 examines what happens when the loading is controlled by applied displacement rather than stress. In other words, when the biaxial ratio being tested is the ratio of the displacement in the direction parallel to the crack to the displacement in the direction perpendicular to the crack as opposed to the ratio of applied remote stresses.

The aim of chapter 7 is to highlight the different outcomes in resultant stress and failure loads from the load controlled analyses when using displacement control in either FEA or experimental procedures, which could explain unexpected or misleading results.

4.2 Biaxial Loading Ratio, B

Throughout the thesis, the biaxial load ratio, B, is defined as the ratio of the remotely applied stress parallel to the crack direction, $\sigma_1$, to the remotely applied tensile loading, $\sigma_2$ (see equation (4.1) and Figure 4.1).

\[
B = \frac{\sigma_1}{\sigma_2}
\]  

(4.1)

The parallel stress, $\sigma_1$, can be either tensile or compressive. A compressive parallel stress will be indicated by a negative value of B. Similarly, $B = 0$ corresponds to uniaxial tension and $B = 1$ to equibiaxial loading.

4.3 Finite Element Analysis

Finite element analysis (FEA) is a widely used and industry-wide accepted approach for carrying out structural integrity calculations. FEA is used in this research for testing the validity of equations when its output is compared with analytical solutions and as a guide for analytical solutions using the plastic deformation patterns on which the slip-line analyses in chapter 5 are based.

Provided the right mesh and boundary conditions are used the FEA will produce accurate solutions.
The finite element analyses throughout this research were carried out using Abaqus/CAE (Complete Abaqus Environment), [4] a software application used for modelling and analysis of components and assemblies.

To determine the predicted behaviour of cracked plates under biaxial loading, FEA of model plates were performed for various plate geometries, crack sizes, materials and biaxial loading ratios.

![Figure 4.1: Biaxially loaded centre cracked plate, width 2W, height 2H and crack width 2a](image)

**4.3.1 Model parameters**

Abaqus has no units, however the scaling of the variables was such that the stresses and moduli were in MPa and the plate measurements in mm. This gave output stress values in MPa, reaction forces in N, J-integral in kJ/m² and stress intensity factor K in MPa√mm.

The models used were composed of theoretical elastic material, elastic-perfectly plastic material or power law hardening material. Both the elastic and elastic-perfectly plastic material were given a Young’s modulus $E = 200\,000$ and Poisson’s ratio $\nu = 0.3$. The elastic-perfectly plastic material was given a yield stress $\sigma_y = 100$, chosen for mathematical ease of calculations.
For modelling the centre cracked and double edge cracked plates, to take advantage of the geometric and loading symmetry, a quarter model of size \( H = 1000 \) and \( W = 500 \) was used.

For the short CCP the plate dimensions modelled were \( H = 250 \) and \( W = 500 \). The single edge cracked plates have only one plane of symmetry and were modelled using a half plate with dimensions \( H = 500 \) and \( W = 500 \).

The consistent value of \( W \) meant that the crack tip geometry and mesh properties could remain consistent throughout the different plate geometries for each value of \( a/W \).

Crack sizes, \( a \), varied from 20 (\( a/W = 0.04 \)) to 300 (\( a/W = 0.6 \)). Figure 4.2 shows the CCP with the positioning of the quarter model.

![Figure 4.2: Biaxially loaded CCP, width 2W, height 2H and crack width 2a](image)

(a) The whole CCP from which the quarter is derived. (b) The quarter plate model used in the FEA. The dotted lines represent the symmetry boundary conditions. The same quarter plate model is used for double edge cracked plates with the side symmetry boundary condition moved from the left to the right of the quarter plate.

The symmetry planes for each of the plate geometries were characterised by boundary conditions. The crack is defined by a zero stress boundary condition along the crack front as well as the crack tip properties. The crack
tip singularity was set to collapse the element side to a single node with a midside node parameter of 0.5.

### 4.3.2 Mesh

The meshed quarter plate is shown in Figure 4.3.

The mesh around the crack tip was a focus mesh bounded by a semicircle of radius 0.01. The smallest crack width tested was $a = 20$ ($a/W = 0.04$) giving a ratio of smallest mesh element size to crack width of 0.05 %, which is less than the maximum 0.1 % suggested by O’Dowd [71].

The number of mesh elements varies with plate size. Most of the analyses in the following chapters were carried out for plates with crack sizes $a/W = 0.2$ and $a/W = 0.6$. For the plates with $a/W = 0.2$, the 0.01 radius semi-circle bounds 60 CPE3 plane strain triangular elements and 80 of these elements for $a/W = 0.6$.

An outer semicircle of radius $0.6a$ surrounds the smaller semicircle and bounds 6000 CPE4R plane strain quadrilateral reduced integration elements. The meshes were biased towards the crack tip to give a higher concentration of elements there.
A further 4050 (a/W = 0.2) and 9100 (a/W = 0.6) CPE4R plane strain quadrilateral reduced integration elements with a bias towards the crack tip make up the remainder of the elements.

4.3.3 Loading

For the stress-controlled loading, a remote uniform tensile stress, $\sigma_2$, with a value of the yield stress $\sigma_y$, was assigned along the top face of the quarter plate for loading to increase to this value during the analysis. In cases where the limit load was greater than the yield stress, multiples of the yield stress were applied at the top of the plate.

For plates with biaxial loading, i.e. where $B \neq 0$, a stress parallel to the crack direction, $\sigma_1$, with value of biaxial load ratio $B$ multiplied by $\sigma_2$, was applied along the free side of the plate. The analyses were run separately for values of $B$ ranging from $-2$ to $+4$. For each analysis during stress controlled loading, the ratio $B$ remained constant as the loading increased incrementally during the run.

The analyses were run until the iterative solutions no longer converged. At this point, the model can no longer withstand the applied loading. The value of the applied remote tensile loading $\sigma_2$ was recorded as the collapse load.

Validations of these models are presented in appendix A.

4.3.4 FEA Output

The output of the FEA, in particular the plastic deformation patterns for various plate geometries, are presented in chapter 5. These are shown in order to illustrate the directions of the slip-lines. The values of the plastic strains in the different outputs vary considerably and are all visible at different scales. Thus as several of the outputs are shown together in each figure, for clarity the scale legends have been omitted as it is the shapes of the deformation patterns that are important.

4.3.5 Modified Boundary Layer Model (MBL)

To model the small-scale yielding (SSY) used in chapter 6, a circular MBL of infinite width and semi-infinite crack width with displacement controlled loading has been used.
The FEA model consisted of a 1000 mm radius semi-circular model with a radial crack. The mesh was made up of 6432 plane strain elements. The mesh immediately surrounding the crack tip consisted of 32 CPE3 3-node linear triangular elements with a side length 0.01 mm. The surrounding mesh consisted of 6400 CPE4R 4-node bilinear quadrilateral elements. The mesh is shown in Figure 4.4.

![Figure 4.4: Modified boundary layer model mesh](image)
(a) Whole model mesh, (b) Triangular mesh elements around crack tip

Displacement conditions were applied at each circumferential outer boundary node. These were calculated using the Westergaard displacement equations, taken from [7] and shown here in equations (4.2) and (4.3). The values of mode I stress intensity factor used were $K_I = 221.5 \text{ MPa}\sqrt{\text{m}}$, based on $J = 220 \text{ kJ/m}^2$, for the pipe material (SA333 Grade 6, data supplied by BARC) and both $K_I = 50 \text{ MPa}\sqrt{\text{m}}$ and $K_I = 70.71 \text{ MPa}\sqrt{\text{m}}$ (to give a value of $J$ double that for $K_I = 50 \text{ MPa}\sqrt{\text{m}}$) for the elastic-perfectly plastic material.

$$u_x = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2\sin^2\left(\frac{\theta}{2}\right)\right]$$

(4.2)

$$u_y = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right)\right]$$

(4.3)

where $u_x$ and $u_y$ are the displacements in the directions parallel to and perpendicular to the crack respectively, $\mu$ is the shear modulus determined using equation (4.4) and $\kappa$ is a parameter defined in equation (4.5).
\[ \mu = \frac{E}{2(1 + \nu)} \]  
\[ \kappa = 3 - 4\nu \]

(4.4)

(4.5)

where \( E \) is Young’s modulus and \( \nu \) is Poisson’s ratio.

The model with the boundary conditions is shown in Figure 4.5.

Figure 4.5: Modified boundary layer model boundary conditions

4.4 Mathematical Formulae

There are various mathematical formulae derived for the solutions in this research, most of which can be followed in the text. Where there are several stages to the derivation whose inclusion in the text would be cumbersome and hinder the flow of reading, more detailed and comprehensive derivations are given in appendix B.

The more complex formula for the short centre cracked plate solutions in chapter 5 were solved using Mathematica software [72]. The input and output of these are also given in appendix B.
5 Limit Loads

The limit load of a specimen, component or structure is the maximum load that it can carry before it yields to plastic collapse.

Applied to an elastic-perfectly plastic material with a yield stress $\sigma_y$, the limit load forms the denominator of the $L_r$ parameter of the failure assessment diagram (FAD).

R6 [3] Section IV.1 contains a compendium of limit load solutions for standard test specimens, plates, cylinders and spheres but with limited advice for biaxially loaded specimens.

This chapter describes how limit load solutions have been derived for biaxially loaded centre cracked plates, single and double edge cracked plates and short sided centre cracked plates in-plane strain. Limit load solutions for cylinders under biaxial loading (internal pressure and end force) are available in R6.

Loading can be stress controlled or displacement controlled and, as will be shown in chapter 7, these can give very different results for the same ratios of applied remote loading parallel to the crack to remote loading perpendicular to the crack.

For this chapter, all the loading is stress controlled and the biaxial stress ratio $B$ is as described in section 5.2.1. for cylinders, the biaxial stress ratio $B$ and the dependence of the cylinder limit loads on $B$ are evaluated in section 5.4 of this chapter.

The value of the limit load will affect a structural assessment using the FAD. If a biaxially loaded specimen has a higher limit load than the equivalent uniaxially loaded specimen, the increased limit load will lower $L_r$ and the plotted point will be moved to the left on the FAD. This can benefit the assessment by increasing the failure load or crack size, particularly in cases where the failure would have been due to plastic collapse. Conversely, a lower limit load will increase $L_r$ and decrease the load or crack size required to fail the specimen as defined by the FAD. These effects will be demonstrated in section 5.5 of this chapter.

The limit loads in this chapter are mainly expressed as the limit tensile stress $(\sigma_2)_L$ applied in the crack opening direction.
5.1 Upper and Lower Bound Limit Load Theorems

This research involves three methods of finding limit load solutions for the specimens described above; the lower bound, the upper bound and the collapse load found using finite element analysis (FEA).

The upper and lower bound theorems are described below. A detailed description of the FEA can be found in chapter 4 of this thesis.

5.1.1 Lower bound theorem

The lower bound theorem states that where a stress distribution in a body:

- Is in internal equilibrium throughout
- Balances the external loads
- Has an equivalent stress that does not exceed the yield stress throughout,

then the corresponding load is less than or equal to the exact external collapse load and is therefore a lower bound estimate of that load.

The research presented here has used a von Mises equilibrium stress field to derive the lower bound solutions.

5.1.2 Upper bound theorem

The upper bound theorem states that for any mechanism of deformation, equating internal dissipation of energy with the rate of work done by external forces gives an estimate of collapse load that is greater than or equal to the exact external collapse load.

The research presented here has used deformation along slip-lines to derive the upper bound solutions.

5.2 Centre Cracked Plates

The first set of limit load calculations is concerned with centre cracked plates with length to width ratio $H/W = 2$. The length of this plate is assumed sufficient for the vertical stress field on the crack plane to redistribute such that boundary conditions are satisfied at the top and bottom surfaces of the plate. This is not the case for short plates, which are covered in section 5.2.4.
5.2.1 Lower bound limit loads for centre cracked plates

Consider a centre cracked plate of width 2W, length 2H and a centre crack with a width 2a (Figure 5.1). The plate is subject to a remote tensile stress, $\sigma_2$, normal to the crack and a remote stress, $B\sigma_2$, parallel to the crack where $B$ is the ratio of biaxial stress. If $B$ is positive, the parallel stress is tensile and if $B$ is negative, the stress is compressive.

An equilibrium stress field is assumed such that:

- The tensile stress field perpendicular to and above the crack is equal to zero
- The tensile stress field perpendicular to the crack and above the ligament ahead of the crack is equal to $\sigma_2/(2W - 2a)/2W = \sigma_2/(1 - a/W)$
- The stress parallel to the crack is equal to $B\sigma_2$ throughout

These stress fields can be substituted into the plane strain von Mises yield criterion, equation (5.1).
\[ \sigma_{vm} = \frac{\sqrt{3}}{2} |\sigma_1 - \sigma_2| \leq \sigma_y \]

(5.1)

where \( \sigma_1 \) and \( \sigma_2 \) are the principal stresses and \( \sigma_y \) is the material yield stress.

Above the crack,

\[ \sigma_{vm} = \frac{\sqrt{3}}{2} |B\sigma_2 - 0| \leq \sigma_y \]

\[ \sigma_2 \leq \frac{2}{\sqrt{3}} |B| \sigma_y \]

(5.2)

and above the ligament,

\[ \sigma_{vm} = \frac{\sqrt{3}}{2} |B\sigma_2 - \frac{\sigma_2}{1-a/W}| \leq \sigma_y \]

\[ \sigma_2 \leq \frac{2}{\sqrt{3}} |\frac{1-a/W}{1-B(1-a/W)}| \sigma_y \]

(5.3)

This gives a lower bound solution to the limit load, equation (5.4).

\[ (\sigma_2)^{lb} = \frac{2\sigma_y}{\sqrt{3}} \min \left[ \frac{1-a/W}{|1-B(1-a/W)|}, \frac{1}{|B|} \right] \]

(5.4)

where the 'lb' superscript represents the lower bound.

For values of \( B < 1/2(1-a/W) \), the left-hand function of equation (5.4) representing the stress field above the ligament applies. The \( 1/|B| \) solution
from the stress field above the crack applies thereafter. For these higher values of B, the value of this lower bound limit load is thus dependent only on B and independent of plate or crack size. Figure 5.2 shows the estimated lower bound limit loads for $a/W = 0.2$ and $a/W = 0.6$ for a range of values of biaxial stress ratio B.

![Graphs showing estimated lower bound limit loads](image)

Figure 5.2: Lower bound limit loads using equilibrium stress fields

a) Crack size $a/W = 0.2$ and (b) $a/W = 0.6$

The limit loads for both the crack sizes follow a similar pattern. When $B < 0$, the limit load is less than that for the uniaxial case where $B = 0$. The limit load increases to a peak at the point where the $1/|B|$ solution of equation (5.4) becomes the minimum, thereafter decreasing as B increases. Thus for a range of values of B between 0 and $1/2(1 - a/W)$, the addition of the biaxial stress is beneficial to the limit load, increasing the applied load required to cause plastic collapse.

The limit loads for the plate with $a/W = 0.2$ are all higher than those for $a/W = 0.6$. Thus, there is a higher resistance to plasticity in the plates with shorter cracks as a consequence of the lower bound theorem as the stress field at collapse for the plate with the longer crack satisfies equilibrium for the plate with the shorter crack.

However for both sizes of plate the limit load decreases for values of B higher than 0.625 ($a/W = 0.2$) and 1.25 ($a/W = 0.6$).

In summary, for a biaxially loaded centre cracked plate with values of B either less than 0 or greater than a value typically around $B \approx 1$, the uniaxial limit load would overestimate the biaxial limit load. For values of B where $0 \leq B \leq 1$, using the uniaxial limit load to estimate the limit load would underestimate the biaxial limit load leading to conservatism.
This same pattern was observed by Lei et al. [41] in their work on extended surface cracked plates. The work of Wei and Hadley [42], also on surface cracks, showed again the increase in limit load for biaxial stress ratios of 0, 0.5 and 1, indicating the conservatism of the uniaxial load approach for this range of B.

5.2.2 Upper bound limit loads for centre cracked plates

Finite element analyses (FEA) have been carried out in accordance with the models described in chapter 4 for a number of values of B for plates with crack size $a/W = 0.2$ and $a/W = 0.6$. The output shown in Figure 5.3 is the equivalent plastic strain for plates of elastic-perfectly plastic material.

Two distinct patterns of deformation can be observed. For lower values of B, the slip-lines follow a classical forward slip-line pattern from the crack tip to the side of the plate. For higher values of B, the lines follow a reverse pattern from the crack tip to the centre line of the plate and back across the ligament to the side of the plate.

Between these two patterns, for $B = 1$ (for $a/W = 0.2$) and $B = 1.5$ (for $a/W = 0.6$), the deformation initially follows a reverse slip-line pattern but remains around the crack tip and does not extend to the edge of the plate.

O’Dowd et al. [29] also observed this interim pattern indicating that the assumption of a uniform stress field across the ligament is invalid.

However, as the limit load solution in equation (5.4) includes the stress field across the crack for values of $B > 1/2(1 - a/W)$, the lower bound solution presented above is valid for all B.

Consider the classical slip-line pattern for the lower values of B. Figure 5.4 shows the dimensions and velocities for $a/W = 0.2$ and $B = 0.5$. An upper bound solution to the limit load is found by equating external work done, equation (5.5) with energy dissipated along the slip-line, equation (5.6).

$$\text{Work done} = B\sigma_2 L \dot{v} \cos \theta [H - (W - a) \tan \theta] + (\sigma_2) L 2W \dot{v} \sin \theta - B\sigma_2 L \dot{v} \cos \theta [H + (W - a) \tan \theta]$$

(5.5)

where $\dot{v}$ is the velocity along the slip-line and $\theta$ is the angle of the slip-line to the horizontal.
where \( \sigma_y \) is the yield stress.

\[
\text{Energy dissipated along slip-line} \quad = \quad \frac{2\dot{v}(W - a) \sigma_y}{\cos \theta} \frac{1}{\sqrt{3}}
\]

(5.6)

where \( \sigma_y \) is the yield stress.
Equating these and simplifying (see appendix B) leads to equation (5.7)

\[
(\sigma_2)_L = \frac{2\sigma_y}{\sqrt{3}} \frac{1 - a/W}{\sin 2\theta [1 - B(1 - a/W)]}
\]  

(5.7)

The minimum value of this limit load is when \( \theta = 45^\circ \) and \( \sin 2\theta = 1 \), giving equation (5.8).

\[
\sigma_{Lub} = \frac{2\sigma_y}{\sqrt{3}} \frac{1 - a/W}{|1 - B(1 - a/W)|}
\]  

(5.8)

where the ‘ub’ superscript represents the upper bound.

This is the same as the formula for the lower bound solution for lower values of \( B \) given in equation (5.4). Thus an exact solution to the limit load has been determined for the lower values of \( B \).

For the reverse slip-line pattern, the dimensions in Figure 5.5 apply.

The methodology is the same as for forward slip-lines. Equating external work done with energy dissipated along the slip-line, equation (5.9), (see appendix B) and solving for \( \theta = 45^\circ \) gives equation (5.10).

\[
B(\sigma_2)_L \dot{\gamma} \cos \theta [H + (W + a) \tan \theta] - (\sigma_2)_L 2W \dot{\gamma} \sin \theta
- B(\sigma_2)_L \dot{\gamma} \cos \theta [H - (W + a) \tan \theta]
= \frac{2\dot{\gamma}(W + a) \sigma_y}{\cos \theta \sqrt{3}}
\]  

(5.9)

\[
(\sigma_2)_{Lub} = \frac{2\sigma_y}{\sqrt{3}} \left[ \frac{1 + a/W}{|B(1 + a/W) - 1|} \right]
\]  

(5.10)
The overall upper bound solution for centre cracked plates is given in equation (5.11).

\[
(s_2)^{lb} = \frac{2\sigma_y}{\sqrt{3}} \min \left\{ \frac{1 - a/W}{|1 - B(1 - a/W)|}, \frac{1 + a/W}{|B(1 + a/W) - 1|} \right\}
\]  

(5.11)

The threshold value of B for which the forward slip-line solution is no longer the minimum and the reverse solution takes over is at B = 1 / [1 - (a/W)^2]. This occurs at B = 1.042 for a/W = 0.2 and B = 1.563 for a/W = 0.6. These are higher than the threshold values for the peak lower bound limit loads, 0.625 and 1.25 respectively.

The solutions in equation (5.11) are valid as long as the slip-line intersects the side of the plate. This condition can be met as long as H > (W – a) tan θ for the forward slip-lines (see Figure 5.4), and H > (W + a) tan θ for the reverse slip-lines (Figure 5.5). All the slip-lines and for both crack sizes, are at 45° (Figure 5.3), which is also the angle at which the upper bound equations give a minimum value. Hence the forward slip-line solutions are valid for H/W > (1 – a/W) and the reverse slip-line H/W > (1 + a/W).

As (1 + a/W) > (1 – a/W) for all 0 < a/W < 1, both the forward and reverse slip-lines intersect the sides of the plate provided the inequality (5.12) holds.
\[
\frac{H}{W} > \left(1 + \frac{a}{W}\right)
\]

(5.12)

Hence a plate with \( H/W > 2 \) may be regarded as a long plate for all crack sizes and biaxial ratios.

### 5.2.3 Comparison of upper bound, lower bound and FEA solutions for centre cracked plates

The lower and upper bound solutions to the limit load are shown together with the FEA output of limit load in Figure 5.6.

![Figure 5.6: Upper and lower bound estimates and FEA solutions of the limit load for centre cracked plates](image)

(a) \( a/W = 0.2 \) and (b) \( a/W = 0.6 \)

The exact solution found by both the lower and upper bound methods is matched by the FEA output for the lower values of \( B \). For values of \( B \) greater than the lower bound peak where \( B = 1/2(1 - a/W) \), the FEA output lies between the lower and upper bound estimates. As \( B \) continues to increase, the FEA output becomes closer to the upper bound estimate.

For the intermediate values of \( B \) around equibiaxiality (\( B = 1 \)), there is a considerable difference between the upper bound estimate and the FEA and lower bound estimates. In this region, the limit load is at its highest, and this pattern is repeated for all values of \( a/W \) (Figure 5.7).

The limit load increases as \( a/W \) decreases, approaching infinity at \( B = 1 \) for the uncracked plate where the loading is pure hydrostatic tension in-plane strain.
Thus for the long plates, the lower bound solutions are accurate for lower values of \( B \) and conservative for \( B > 1 \) and it is these solutions that are recommended for conservative estimation of the limit loads.

In R6, the plane strain von Mises solution for the limit load of centre cracked plates under uniaxial tension (IV.5.3.1-1) is shown here in equation (5.13).

\[
n_L = \frac{2}{\sqrt{3}} \left( 1 - \frac{a}{W} \right)
\]

where \( n_L \) is the normalised limit load.

Substituting \( B = 0 \) into equation (5.11) gives a solution identical to (5.13) and thus the solutions here are consistent with the advice in R6 for uniaxially loaded specimens.

### 5.2.4 Short plate lower bound limit loads

For shorter plates, the stress field at the cracked section cannot necessarily be assumed to redistribute to satisfy the boundary conditions at the top and bottom loaded plate surfaces. In order to model the stress field more accurately, the following are assumed:
• above the crack, a shear stress of 0 at the centre of the plate increasing linearly with horizontal distance to a peak value of $a \sigma_2 / H$ at the crack tip.

• above the uncracked ligament, a shear stress of $a \sigma_2 / H$ at the crack tip decreasing linearly to 0 at the sides of the plate.

• above the crack, a vertical stress of 0 at the cracked centre line, increasing linearly to $\sigma_2$ at the top and bottom surfaces.

• above the ligament, a vertical stress of $\sigma_2 / (1 - a/W)$ at the cracked centre line decreasing linearly to $\sigma_2$ at the top and bottom surfaces.

• a horizontal stress of $B \sigma_2$ throughout.

The lower bound limit load obtained using the von Mises criterion is given in equation (5.14). A more detailed derivation of this formula can be found in appendix B.

$$
(\sigma_2)_{lb} = \frac{2 \sigma_y}{\sqrt{3}} \min \left[ \frac{z}{\sqrt{(1 - Bz)^2 + 4 \left(\frac{a}{H}\right)^2}}, \frac{1}{\sqrt{B^2 + 4 \left(\frac{a}{H}\right)^2}} \right]
$$

(5.14)

where $z = (1 - a/W)$

The left function is obtained from the stress field above the ligament and the right function is from the stress field above the crack.

Equation (5.14) tends towards equation (5.4) when $a/H \to 0$ i.e. for long plates.

5.2.5 Short plate upper bound limit loads

The upper bound limit loads are found by again assuming a deformation pattern given by the equivalent plastic strain output of the FEA (Figure 5.8).

The procedure for determining the upper bound limit load is as for the long plates with two notable exceptions.

• Some of the slip-lines intersect the top and bottom of the plate rather than the sides.
• The angle of the slip-lines to the horizontal varies and increases with increasing B. At B = 1 the slip-lines are at 90° to the horizontal, i.e. vertical and where B > 1 the angles are > 90°.

![Figure 5.8: Equivalent plastic strain FEA output for short centre cracked plates H/W = 0.5, elastic-perfectly plastic material](image)

(a) Crack size a/W = 0.2 and (b) Crack size a/W = 0.6. Higher values of strain are represented by darker shading.

Equating work done with energy dissipated along the slip-lines, using the dimensions in Figure 5.9, gives equation (5.15).

\[
(\sigma_2)_L [W + a + \frac{H}{\tan \theta}] \dot{v} \sin \theta - B(\sigma_2)_L 2H \dot{v} \cos \theta
- (\sigma_2)_L [W - a - \frac{H}{\tan \theta}] \dot{v} \sin \theta = 2\dot{v} \frac{\sigma_y}{\sqrt{3}} \frac{H}{\sin \theta}
\]

which reduces to equation (5.16) — see appendix B for derivation.

\[
(\sigma_2)_L = \left(\frac{2\sigma_y}{\sqrt{3}} \frac{H/W}{a/W(1 - \cos 2\theta)} - H/W (B - 1) \sin 2\theta \right)
\]

When B = 1, the solution is at a minimum when \( \theta = 90^\circ \), giving an upper bound solution for the limit load. This complies with the patterns seen in Figure 5.8 where for B = 1 the slip-lines are vertical.
\[(\sigma_2)_L = \left(\frac{2\sigma_y}{\sqrt{3}}\right) \frac{H/W}{a/W} \text{ for } B = 1\]

\[(5.17)\]

For values of B not equal to 1, an improved upper bound limit load can be found by determining the angle \(\theta = \theta_{L(F)}\), where L denotes a limit load and (F) refers to the forward slip-lines, for which equation (5.16) is a minimum.

This minimum can be found by differentiating the denominator with respect to \(\theta\) and setting to zero.

\[\frac{a}{W} \sin 2\theta - \frac{H}{W(B - 1)} \cos 2\theta = 0\]

\[(5.18)\]

\[\tan 2\theta = \frac{H/W(B - 1)}{a/W}\]

\[(5.19)\]
\[ \theta = \theta^{L(F)} = \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{H/W(1 - B)}{a/W} \quad B < 1 \]
\[ \theta = \theta^{L(F)} = \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{H/W(B - 1)}{a/W} \quad B > 1 \]

Equation (5.18) only leads to a maximum value of the denominator of equation (5.16) when the second derivative of this denominator, shown in equation (5.21), is negative.

\[ \frac{a}{W} \cos 2\theta + \frac{H}{W(B - 1)} \sin 2\theta < 0 \]

This inequality only holds for values of \( B < 1 \), so only the first of the formulae in equation (5.20) applies. This complies with the slip-line pattern seen in Figure 5.8 where the forward slip-lines are those with \( B < 1 \).

For the reverse slip-lines, repeating the process using the geometry in Figure 5.10 leads to equation (5.22) which reduces to equation (5.23) — see appendix B for derivation.

\[ (\sigma_2)_L \left[ W + a - \frac{H}{\tan \theta} \right] \dot{\nu} \sin \theta - B (\sigma_2)_L 2 \dot{\nu} \cos \theta \]
\[ - (\sigma_2)_L \left[ W - a + \frac{H}{\tan \theta} \right] \dot{\nu} \sin \theta = 2 \dot{\nu} \frac{\sigma_y}{\sqrt{3}} \frac{H}{\sin \theta} \]

\[ (\sigma_2)_L = \left( \frac{2\sigma_y}{\sqrt{3}} \right) \frac{H/W}{a/W(1 - \cos 2\theta) + H/W(B - 1) \sin 2\theta} \]
Similarly to the forward slip-line solution, equation (5.23) is a minimum when \( B = 1 \) and \( \theta = 90^\circ \) giving the same solution (equation (5.17)).

An angle \( \theta = \theta^{L(R)} \), where \( L \) denotes a limit load and \( (R) \) refers to the reverse slip-lines, for which equation (5.23) is a minimum can again be found by differentiating the denominator with respect to \( \theta \) and setting to zero.

\[
a/W \sin 2\theta + H/W(B - 1) \cos 2\theta = 0
\]

(5.24)

\[
\tan 2\theta = -\frac{H/W(B - 1)}{a/W}
\]

(5.25)

\[
\begin{cases}
\theta = \theta^{L(R)} = \frac{1}{2} \tan^{-1} \frac{H/W(1-B)}{a/W} & \text{if } B < 1 \\
\theta = \theta^{L(R)} = \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{H/W(B - 1)}{a/W} & \text{if } B > 1
\end{cases}
\]

(5.26)
The second derivative of the denominator of equation (5.23) is shown in equation (5.27).

\[
a/W \cos 2\theta - H/W(B - 1) \sin 2\theta < 0
\]

(5.27)

This inequality holds for values of \(B > 1\), thus the second formula in equation (5.26) applies.

For both forward and reverse slip-lines, the values of \(\theta^{L(F)}\) and \(\theta^{L(R)}\) are given in equation (5.28).

\[
\begin{align*}
\theta^{L(F)} &= \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{H/W(1 - B)}{a/W} & B < 1 \\
\theta^{L(R)} &= \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{H/W(B - 1)}{a/W} & B > 1
\end{align*}
\]

(5.28)

Thus for all \(B\),

\[
\theta^L = \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{H/W|B - 1|}{a/W}
\]

(5.29)

Similarly combining equations (5.16) and (5.23) gives the solution in equation (5.30).

\[
(\sigma^2)_L = \left(\frac{2\sigma_y}{\sqrt{3}}\right) a/W(1 - \cos 2\theta^L) + \frac{H/W}{a/W(1 - \cos 2\theta^L) + H/W|B - 1| \sin 2\theta^L}
\]

(5.30)

where \(\theta\) is determined using equation (5.31).
\[ \tan 2\theta^L = -\frac{H/W|B - 1|}{a/W} \]  
\( (5.31) \)

Equations (5.30) and (5.31) can be combined to form equation (5.32), the derivation of which is detailed in appendix B.

\[ (\sigma_2)_L = \left(\frac{2\sigma_y}{\sqrt{3}}\right) \frac{H/W}{a/W} \left[ 1 + \sqrt{1 + \frac{(a/W)^2 + (B - 1)^2(H/W)^2}{(a/W)^2}} \right] \]  
\( (5.32) \)

Like the longer plates, there are limits to the validity of the short plate upper bound solutions of equations (5.15) to (5.32). These solutions are only valid when the slip-lines intersect the top and bottom surfaces of the plates.

Consider first the forward slip-lines solution as shown in Figure 5.9. For the slip-line to intersect with the corner,

\[ \tan \theta^{cr(F)} = \frac{H}{W - a} = \frac{H/W}{1 - a/W} \]  
\( (5.33) \)

where \( \theta^{cr(F)} \) is the value of the angle \( \theta \) for the intersection with the corner and the superscripts ‘cr’ and ‘F’ denote corner and forward respectively.

Inequality (5.34) must be satisfied for the forward slip-lines to intersect the top surface of the plate.

\[ \frac{\pi}{2} \geq \theta > \theta^{cr(F)} = \tan^{-1}\left[ \frac{H/W}{1 - a/W} \right] \]  
\( (5.34) \)

For values of \( \theta > \pi/2 \), the reverse slip-line solution applies.

From Figure 5.10, for the reverse slip-lines to meet the corner equation (5.35) applies.
\[
\tan \theta^{\text{cr}(R)} = \frac{H}{W + a} = \frac{H/W}{1 + a/W}
\]

(5.35)

where \(\theta^{\text{cr}(R)}\) is the value of the angle \(\theta\) for the intersection with the corner and the superscripts ‘cr’ and ‘R’ denote corner and reverse respectively.

Inequality (5.36) must be satisfied for the reverse slip-lines to intersect the top surface of the plate.

\[
\frac{\pi}{2} \geq \theta > \theta^{\text{cr}(R)} = \tan^{-1} \left[ \frac{H/W}{1 + a/W} \right]
\]

(5.36)

As \(\theta\) varies and is a function of \(H/W, a/W\) and \(B\) (equations (5.20), (5.26), (5.28) and (5.29), the inequalities (5.34) and (5.36) are more complex than those for the long plates. However, solutions can be found for the upper bound using equation (5.32) for any values of \(a/W\) and \(B\) that satisfy these inequalities for \(H/W\).

The upper bound solutions for the long plates, equation (5.11), are for slip-lines of 45° and are valid only for \(H \geq (1 - a/w)\) for the forward slip-lines where \(B < 1/[1 - (a/W)^2]\) and \(H \geq (1 + a/w)\) for the reverse slip-lines where \(B > 1/[1 - (a/W)^2]\). As the limit load for the shorter plates cannot increase with reducing plate length, equation (5.11) is an upper bound solution for all values of \(H/W\).

Likewise, where \(H/W\) does not satisfy the inequalities (5.34) and (5.36), this corresponds to the slip-lines intersecting with the sides or corners of the plate and again the solution in equation (5.11) will apply.

Thus the short plate upper bound solution will be the lowest value found using equations (5.11) and (5.32).

5.2.6 Comparison of upper bound, lower bound and FEA solutions for short plates

Comparison of the lower bound, upper bound and FEA limit load solutions for a plate with \(H/W = 0.5\) are shown in Figure 5.11.
For \( a/W = 0.2 \) (Figure 5.11 (a)) and \( B < 0.75 \), the short plate upper bound estimates are slightly below and within 3% of the FEA output, so could conservatively estimate the limit load. In fact, for all values of \( B < 0 \) the lower bound solution for \( H/W = 0.5 \) gives values about 1–2% higher than the FEA output and 1–3% higher than the upper bound solution. For \( B \geq 0.75 \), the upper bound values are greater than the FEA and the lower bound solution give the conservative estimates of the limit loads.

![Graphs](image)

Figure 5.11: Upper bound, lower bound and FEA limit load solutions for centre cracked plates, \( H/W = 0.5 \)

a) \( a/W = 0.2 \) and (b) \( a/W = 0.6 \)

For \( a/W = 0.6 \) (Figure 5.11 (b)), the lower bound estimate gives conservative estimates of the limit loads for all \( B \).

The higher values of the lower bound limit load could be an artefact of the assumptions relating to the shear and stress field at the beginning of section 5.2.4. A more complex analysis of these stress fields would be necessary to model the stresses accurately for all values of \( a/W, H/W \) and \( B \), particularly for low values of all three.

5.2.7 Bi-axial stress and plate length

The short plate solutions for \( a/W = 0.2 \) are shown for fixed values of \( B \) and varying \( H/W \) in Figure 5.12. These show the extent to which the short plate solutions can be applied and for which cases the long plate solution is the more appropriate.

- (a) to (c) show that the lower and upper bound solutions are identical for values of \( B > 0.625 \) (see 5.2.1) and dominated by equations (5.3) and (5.8) respectively.
• For values of $B > 1.042$ (see 5.2.2) the long plate upper bound is dominated by equation (5.10). For the values of $H/W$ that satisfy equation (5.34) for $B < 1$ and equation (5.36) for $B > 1$, i.e. where the slip-lines intersect the top and bottom of the surfaces, the short plate solution always gives a lower estimate for the limit load where $B > 1$ (g) and (h).

• For $B < 1$, in (b), (c) and (d), there are values of $H/W$ (0.85, 0.95 and 1.05 respectively) above which the short plate upper bound estimates are higher than the long plate upper bound estimates. In these cases, the inequality (5.34) is satisfied as the angle $\theta > \theta^{cr(F)}$ for the given values of $B$, $H/W$ and $a/W$. However, for this range of plate sizes, slip-lines with angle $\theta = 45^\circ$ lead to satisfaction of the inequality $H/W > (1 - a/W) \tan \theta$ (which is $H/W > 0.8$ for $a/W = 0.2$) for the forward slip-lines to intersect the sides of the long plates. In these cases, the long plate upper bound solution is the lower of the two.

### 5.3 Edge Cracked Plates

R6 Sections IV.1.5.4 and IV.1.5.5 contain-plane strain von Mises limit load solutions in for double and single edge cracked plates respectively under uniaxial tensile loading. These solutions are given as functions of $\beta$, the normalised crack size $a/W$.

\[
n_L = \frac{(\sigma_2)_L}{\sigma_y} = \gamma f(\beta)
\]

(5.37)

where $n_L$ is the normalised limit load, $\gamma = 2/\sqrt{3}$ and $f(\beta)$ is a function of $\beta$ representing the limit load solutions given.

The plane strain Mises solution in R6 Section IV.1.5.3 for uniaxially loaded centre cracked plates is given here in equation (5.38).
\[ n_L = \gamma (1 - \beta) \]

\[ \frac{(\sigma_2)_L}{\sigma_y} = \frac{2}{\sqrt{3}} (1 - \frac{a}{W}) \]

(5.38)

Thus in this case \( f(\beta) = 1 - \frac{a}{W}. \)

The lower bound solution found here using the von Mises criterion for the centre cracked plates (section 5.2.1) is given in equation (5.4). Thus in terms of a function of \( \beta, f(\beta) = 1 - \frac{a}{W}, \) equation (5.4) can be written as in equation (5.39).

\[ (\sigma_2)_b = \frac{2\sigma_y}{\sqrt{3}} \min \left[ \frac{f(\beta)}{|1 - Bf(\beta)|}, \frac{1}{|B|} \right] \]

(5.39)

Approximate lower bound solutions for biaxially loaded double and single edge cracked plates are here taken to be of the form of equation (5.39) with \( f(\beta) \) being the R6 solution for the uniaxially loaded plate.

The upper bound solutions for edge cracked plates have been found using the same method as for the centre cracked plates, using slip-line fields modelled on the deformation patterns found by FEA.

5.3.1 Lower bound limit loads for double edge cracked plates

The R6 plane strain Mises solution for the limit loads for double edge cracked plates (IV.1.5.4-3) is shown here in equation (5.40)

\[ n_L = \frac{(\sigma_2)_L}{\sigma_y} = \begin{cases} \gamma (1 - \beta) \left[ 1 + \ln \left( \frac{2 - \beta}{2(1 - \beta)} \right) \right] & \text{for } 0 \leq \beta \leq 0.884 \\ 2.57\gamma (1 - \beta) & \text{for } 0.884 < \beta < 1 \end{cases} \]

(5.40)

For this research, analyses are limited mainly to \( a/W = 0.2 \) and \( a/W = 0.6. \)

For \( 0 \leq a/W \leq 0.884, f(\beta) \) is defined by equation (5.41).
Figure 5.12: Upper and lower bound short and long plate solutions for varying plate lengths for fixed values of $B$

(a) $B = -1$, (b) $B = 0$, (c) $B = 0.5$, (d) $B = 0.7$, (e) $B = 1$, (f) Legend, (g) $B = 1.5$ and (h) $B = 2$
Applying the formula in equation (5.39) to equation (5.41) gives an approximate lower bound limit load solution for a double edge cracked plate with remote tensile stress $\sigma_2$ applied normal to the crack and a remote tensile or compressive stress $B\sigma_2$ applied parallel to the crack.

\[
f(\beta) = (1 - \beta) \left[ 1 + \ln \left( \frac{2 - \beta}{2(1 - \beta)} \right) \right]
\]

(5.41)

A double edge cracked plate was modelled for FEA using the same quarter plate as has been used for the centre cracked plate by changing the symmetry boundary conditions to reflect the different position of the symmetry plane.

The FEA failure load values are plotted together with the estimates of limit load from equation (5.42) in Figure 5.13.

\[
(\sigma_2)^{lb} = \frac{2\sigma_y}{\sqrt{3}} \min \left[ 1, \frac{(1 - \beta) \left[ 1 + \ln \left( \frac{2 - \beta}{2(1 - \beta)} \right) \right]}{1 - B(1 - \beta) \left[ 1 + \ln \left( \frac{2 - \beta}{2(1 - \beta)} \right) \right]} \left( 1 - \frac{1}{|B|} \right) \right]
\]

(5.42)

Figure 5.13: Lower bound estimate and FEA limit load solutions for double edge cracked plates

(a) $a/W = 0.2$ and (b) $a/W = 0.6$. The pattern is the same as for the centre cracked plates, where there are values of $B$ above $B = 0$ for which the biaxial stress is beneficial to the limit load, increasing to a peak at $B = 1$ then decreasing as $B$ increases.

For both plates there is agreement within 3% between the estimated lower bound solution and the FEA loads for compressive and low tensile values of...
parallel stress ($B < 0.5$ for $a/W$ and $B < 0.25$ for $a/W = 0.6$). For the values of $B$ between 0.5 and 1 ($a/W = 0.2$) and 0.25 and 1 ($a/W = 0.6$) the lower bound estimate gives a higher value than the FEA limit load. For values of $B > 1$, the estimate gives exact agreement with the FEA output.

Given the accuracy of equation (5.42) relative to the FEA, a more accurate lower bound limit load solution based on an equilibrium stress field has been not pursued in detail. However, a lower bound limit load solution using the lower bound stress field for the centre cracked plates is examined.

Better lower bound estimates for the double edge cracked plate can be found using the same stress field used for the centre cracked plates (equation (5.4)). These are illustrated in Figure 5.14.

![Figure 5.14: Lower bound estimate and FEA limit load solutions for double edge cracked plates with lower bound estimate for centre cracked plates (CCP)](image)

(a) $a/W = 0.2$ and (b) $a/W = 0.6$

For double edge cracked plates with $a/W = 0.2$, the lower bound centre cracked plate estimate is a lower bound estimate for values of $B < 0.625$ and is exact for $B > 0.625$ however it is still higher than the FEA value for $B = 0.625$.

For double edge cracked plates with $a/W = 0.6$, the centre cracked plate lower bound estimate is lower than the FEA for values of $B < 1.5$ and an exact estimate for $B \geq 1.5$, thus it is a good lower bound estimate for all $B$ although for lower values of $B$, equation (5.42) is closer

5.3.2 Lower bound limit loads for single edge cracked plates

The R6 plane strain Mises solution for the limit loads for single edge cracked plates (IV.1.5.5–4) is shown here in equations (5.43) and (5.44).
for $0 \leq \beta \leq 0.545$

$$n_L = \frac{(\sigma_2)_L}{\sigma_y} = \gamma[1 - \beta - 1.232\beta^2 + \beta^3]$$  \hfill (5.43)

and for $0.545 < \beta \leq 1$

$$n_L = \frac{(\sigma_2)_L}{\sigma_y} = 1.702\gamma \left[ \sqrt{(0.794 - (1 - \beta))^2 + 0.5876(1 - \beta)^2} ight]$$  

$$- (0.794 - (1 - \beta))$$  \hfill (5.44)

Combining equations (5.39) and (5.43) for $a/W = 0.2 < 0.545$ leads to equation (5.45).

$$\left(\frac{\sigma_2}{\sigma_y}\right)^b = \frac{2\sigma_y}{\sqrt{3}} \text{Min} \left[ \frac{1}{|B|^\prime} \left| \frac{1 - \beta - 1.232\beta^2 + \beta^3}{1 - B(1 - \beta - 1.232\beta^2 + \beta^3)} \right| \right]$$  \hfill (5.45)

Combining equation (5.39) and (5.44) for $a/W = 0.6 > 0.545$ leads to equation (5.46).

$$\left(\frac{\sigma_2}{\sigma_y}\right)^b = \frac{2\sigma_y}{\sqrt{3}} \text{Min} \left[ \frac{1}{|B|^\prime} \left| \frac{1.702\sqrt{((0.794 - z)^2 + 0.5876z^2)} - (0.794 - z)}{1 - B\left[1.702\sqrt{((0.794 - z)^2 + 0.5876z^2)} - (0.794 - z)\right]} \right| \right]$$

where $z = (1 - \beta) = (1 - a/W)$  \hfill (5.46)

The FEA failure load values are plotted together with the estimates of limit load from equations (5.45) and (5.46) in Figure 5.15 (a) and (b) respectively.
Like the centre cracked plates and the double edge cracked plates, the estimated values reach a peak and then reduce beyond this. For the single edge cracked plates however, for $a/W = 0.2$, Figure 5.15 (a), there is agreement within 3% between the lower bound estimate and the FEA results. For $a/W = 0.6$ the limit load estimate results are within 4% of the FEA for $B < 1$ but thereafter the estimated limit loads are higher and do not appear to follow the same pattern.

For both the single and double edge cracked plates, some of the estimates, particularly around equibiaxiality ($B = 1$), for the derived lower bound limit load are higher than the FEA output. This can be attributed to the fact that the equations (5.40) and (5.44) from R6 are based on Miller [36] who describes them as having been derived from slip-line fields, which generally give an upper bound estimate.

For the single edge cracked plates, further work needs to be carried out to establish equilibrium stress fields leading to lower bound estimates, considering the moments as well as the tensile stresses.

### 5.3.3 Upper bound limit load solutions for double edge cracked plates

The upper bound limit load solutions are based on the deformation patterns observed following the FEA of the double edge cracked plates (Figure 5.16).

These slip-lines follow similar patterns to those of the centre cracked plates (Figure 5.8), such that for lower values of $B$ the slip-lines go forward ahead of the crack and then switch to the reverse slip-line pattern for higher values of $B$. 
An upper bound limit load estimate is carried out using the same methodology as for the centre cracked plates (sections 5.2.2 and 5.2.5). The geometry and loading for the calculations are shown in Figure 5.17.

For the forward going slip-lines, equating work done with energy along the slip-lines is shown in equation (5.47).

\[
B \sigma_L \dot{v} \cos \theta [H - (2W - a) \tan \theta] + (\sigma_2)_L 2W \dot{v} \sin \theta \\
- B (\sigma_2)_L H \dot{v} \cos \theta = \frac{\dot{v}(2W - a)}{\cos \theta} \frac{\sigma}{\sqrt{3}}
\]

(5.47)

Simplifying equation (5.47) (see appendix B) leads to equation (5.48).

\[
(\sigma_2)_L = \frac{2 \sigma_y}{\sqrt{3} \sin 2\theta} \frac{(2W - a)}{(2W - B(2W - a))}
\]

(5.48)
The minimum value of this limit load occurs when $\theta = 45^\circ$ and $\sin 2\theta = 1$, giving equation (5.49).

\[
(\sigma_2)_L = \frac{2\sigma_y}{\sqrt{3}} \frac{(2 - a/W)}{[2 - B(2 - a/W)]}
\]  

(5.49)

For the reverse slip-lines, equating work done with energy along the slip-lines is shown in equation (5.50).

\[
B(\sigma_2)_L \dot{\nu} \cos \theta [a \tan \theta] = \frac{\dot{\nu}(a)}{\cos \theta} \frac{\sigma_y}{\sqrt{3}}
\]  

(5.50)

Simplifying equation (5.50) (see appendix B) leads to equation (5.51).
\[
(\sigma_2)_L = \frac{2}{B \sin 2\theta} \frac{\sigma_y}{\sqrt{3}}
\]

(5.51)

The minimum value of this limit load occurs when \(\theta = 45^\circ\) and \(\sin 2\theta = 1\), giving equation (5.52).

\[
(\sigma_2)_L = \frac{2\sigma_y}{\sqrt{3}} \frac{1}{B}
\]

(5.52)

The overall upper bound limit load solution for double edge cracked plates using slip-line analysis will be the lesser of the values given by equations (5.49) and (5.52).

\[
(\sigma_2)_{Lub} = \frac{2\sigma_y}{\sqrt{3}} \min \left[ \frac{(2 - a/W)}{|2 - B(2 - a/W)|} \frac{1}{|B|} \right]
\]

(5.53)

For values of \(B > 2/(4 - a/W)\), the lower value is the \(1/|B|\) function. Note that this value is independent of \(a/W\) and is identical to the lower bound estimates.

The upper and lower bound estimates for double edge cracked plates are shown in Figure 5.18.

The upper bound estimate is higher than the lower bound estimate for lower values of \(B\) for both crack sizes. It was shown in Figure 5.14 that the lower bound estimates for centre cracked plates could be used as a lower bound estimate for the double edge cracked plates. As the lower bound estimates found using equation (5.42) are higher than the FEA output, it is values found using these equations that are a closer upper bound estimate than those found using equation (5.53).

5.3.4 Upper bound limit loads for single edge cracked plates

The upper bound limit load solutions are based on the deformation patterns observed following the FEA of the single edge cracked plates (Figure 5.19).
For plates with crack size $a/W = 0.2$, the deformation patterns follow the same pattern as the centre cracked plate (Figure 5.3). As $a/W$ and $B$ increase, the deformation occurs along the arc of a circle, with the radius decreasing as $B$ increases, moving the centre of the circle to the right and towards the crack tip.
The general case where the circular slip-lines intersect the side of the plate are analysed to find an upper bound solution to the limit load for all the circular slip-line cases.

Consider a single edge cracked plate with width W, length 2H and crack depth a. Using a Cartesian co-ordinate system (x, y) with the origin at the left-hand corner of the half-plate, the corners of the half plate are at (0,0), (W, 0), (0, H) and (W, H). The slip-line is the arc of a circle centred at (X, −Y), where X > a, and Y > 0, with radius r and central angle φ (Figure 5.20 (b)). The parameters r and φ are defined in terms of X, Y, W and a, using equations (5.54) and (5.55).

\[ r^2 = (X - a)^2 + Y^2 \]  

(5.54)

\[ \phi = \cos^{-1}[(X - W)/r] - \cos^{-1}[(X - a)/r] \]  

(5.55)

The upper bound slip-line solution is based on a rotation of the half-plate above the slip-line with an angular velocity \( \dot{\omega} \). The vertical and horizontal velocities at the corners of the half-plate are as shown in Figure 5.20 (c).
(x = 0), mean velocity = \[ \frac{\dot{\omega}[Y + H] + \dot{\omega}[Y]}{2} = \dot{\omega} \left[ \frac{Y + H}{2} \right] \]  

(5.56)

The mean velocities along the top and sides of the plate above the slip-line are given in equations (5.56) to (5.58).

(x = W), mean velocity = \[ \dot{\omega} \left[ \frac{Y + H + \sqrt{r^2 - (X - W)^2}}{2} \right] \]  

(5.57)

(y = H), mean velocity = \[ \dot{\omega} \left[ \frac{X - W}{2} \right] \]  

(5.58)

The external work done is found by multiplying these velocities by the stresses given in Figure 5.20 (d) and by the length of the part of the plate that is rotating above the slip-line. Equating this external work with energy dissipated along the slip-line gives equation (5.59).

\[
B(\sigma_2)_{l}(Y + H - \sqrt{r^2 - (X - W)^2}) \dot{\omega} \left[ \frac{(Y + H + \sqrt{r^2 - (X - W)^2})}{2} \right] \\
+ (\sigma_2)_{l} W \dot{\omega} \left( \frac{X - W}{2} \right) - B(\sigma_2)_{l} H \dot{\omega} \left( \frac{Y + H}{2} \right) = \frac{\sigma_y}{\sqrt{3}} r \varphi \dot{\omega} \r

(5.59)

This simplifies down (see appendix B) to give an upper bound limit load solution in equation (5.60).

\[
(\sigma_2)_{lb}^{ub} = \frac{2\sigma_y r^2 \varphi}{\sqrt{3} \left[ B(W^2 - a^2 + 2aX - 2WX) + W(2X - W) \right]} 

(5.60)

where r and \( \varphi \) are as defined in equations (5.54) and (5.55).
This gives an upper bound estimate for the limit load in terms of $a$, $W$, $B$, $X$ and $Y$. For each combination of $a$, $W$ and $B$, values of $X$ and $Y$ can be found that minimise this limit load.

Minimum limit loads and the values of $X$ and $Y$ that lead to them have been found by systematically substituting values of $X$ and $Y$ into equation (5.60). $X$ is taken from a minimum of $X = (W + a)/2$ (the minimum value of $X$ for the arc to intersect the side of the plate) in increments of 0.01. $Y$ is taken from a minimum of $Y = 0$ in increments of 0.01. The limit loads found using this method are shown in Table 5.1.

Table 5.1: Circular slip-line upper bound limit load solutions normalised by $\sigma_y$ for various $a/W$ and $B$ for a single edge cracked plate

<table>
<thead>
<tr>
<th>$B$</th>
<th>$a/W = 0.2$</th>
<th>$a/W = 0.4$</th>
<th>$a/W = 0.5$</th>
<th>$a/W = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>0.354</td>
<td>0.304</td>
<td>0.261</td>
<td>0.189</td>
</tr>
<tr>
<td>−1</td>
<td>0.510</td>
<td>0.404</td>
<td>0.319</td>
<td>0.201</td>
</tr>
<tr>
<td>0</td>
<td>0.902</td>
<td>0.574</td>
<td>0.372</td>
<td>0.207</td>
</tr>
<tr>
<td>0.5</td>
<td>1.438</td>
<td>0.662</td>
<td>0.386</td>
<td>0.210</td>
</tr>
<tr>
<td>1</td>
<td>2.455</td>
<td>0.704</td>
<td>0.394</td>
<td>0.211</td>
</tr>
<tr>
<td>1.5</td>
<td>2.550</td>
<td>0.717</td>
<td>0.398</td>
<td>0.212</td>
</tr>
<tr>
<td>2</td>
<td>1.599</td>
<td>0.718</td>
<td>0.399</td>
<td>0.212</td>
</tr>
<tr>
<td>3</td>
<td>0.672</td>
<td>0.720</td>
<td>0.399</td>
<td>0.213</td>
</tr>
<tr>
<td>4</td>
<td>0.425</td>
<td>0.510</td>
<td>0.400</td>
<td>0.213</td>
</tr>
</tbody>
</table>

Upper bound solutions for $a/W = 0.2$ and 0.4 can be found using the minimum value found by equation (5.60) (in Table 5.1) and $\gamma/|B|$. This uses the reverse slip-line upper bound estimate for double edge cracked plates for higher values of $B$.

These solutions are illustrated in Figure 5.21. Although the slip-line pattern for $a/W = 0.2$ is not circular, the circular slip-line solution is a better fit (within 1%) to the FEA output for lower values of $B$ than the double edge cracked plate upper bound estimate (equation (5.49)). Similarly, the double edge cracked plate reverse slip-line upper bound estimate applied to $a/W = 0.5$, which does not have a reverse slip-line pattern, gives a better fit to the FEA output.
Figure 5.21: Upper bound estimates for single edge cracked plates
(a) a/W = 0.2, (b) a/W = 0.4, (c) a/W = 0.5 and (d) a/W = 0.6

In the specific case where the segment of the circle intersects the corner of
the plate (for example see Figure 5.19, a/W = 0.6, B = 1 and a/W = 0.5,
B = 1.5), the slip-line solutions can be derived from the parameters in
Figure 5.22 using the same method as for the above general circular slip-
line cases.

Equations (5.61) and (5.62) define r and φ in terms of X, W and a.

\[ X = \frac{W + a}{2} \]  \hspace{1cm} (5.61)

\[ W - a = 2r \sin(\varphi/2) \]

\[ \therefore r = \frac{W - a}{2 \sin(\varphi/2)} \]  \hspace{1cm} (5.62)
The mean velocities along the top and sides of the plate above the slip-line are given in equations (5.63) and (5.64)

\[(x = 0)\text{ and } (x = W), \text{ mean velocity } = \frac{\omega}{2} \left(Y + \frac{H}{2}\right) \tag{5.63}\]

\[(y = H), \text{ mean velocity } = \frac{\omega}{2} \left(W\right) \tag{5.64}\]

Equating this external work with energy dissipated along the slip-line leads to equation (5.65).

\[B(\sigma_2)LH\omega \left(\frac{Y + \frac{H}{2}}{2}\right) + (\sigma_2)LW\omega \left(\frac{W}{2}\right) - B(\sigma_2)LH\omega \left(Y + \frac{H}{2}\right) = \frac{\sqrt{3}}{r} \varphi \omega r \tag{5.65}\]

This simplifies to equation (5.66).
\[ (\sigma_2)_L = \frac{2\sigma_y r^2 \varphi}{\sqrt{3}} \frac{W^2}{r^2} \]  
(5.66)

Substituting the value of \( r \) from equation (5.62) gives equation (5.67).

\[ (\sigma_2)_L = \frac{2\sigma_y (W - a)^2 \varphi}{\sqrt{3}} \frac{1}{4W^2 \sin^2(\varphi/2)} \]  
(5.67)

A minimum value can be found by differentiating equation (5.67) and setting to zero.

\[
\frac{\sin(\varphi/2) - 2(\varphi/2) \cos(\varphi/2)}{\sin^3(\varphi/2)} = 0
\]
(5.68)

\[ (\varphi/2) = 1.1656 \]  
(5.69)

This gives an upper bound limit load estimate in terms of \( a/W \). Note that the solution in equation (5.70) is independent of \( B \).

\[ (\sigma_2)_{L}^{\text{ub}} = 1.38 \frac{\sigma_y (1 - a/W)^2}{\sqrt{3}} \frac{1}{(a/W)} \]  
(5.70)

These values of \( \varphi/2 \) and limit load are the same as those found by Joch et al. [73]. Their calculations used the same slip-line velocity method but for a uniaxially loaded welded joint. However, as the biaxial solution is independent of \( B \), the uniaxial solution should be the same as the biaxial solutions.

Values of the uniaxial limit load at \( B = 0 \), found using the circular slip-line equations for the corner intersection (equation (5.70)) and side intersection
(equation (5.60)) as well as the R6 equation (IV.1.5.5-4) (equations (5.43) and (5.44)) are shown in Table 5.2.

The limit load estimates using the side intersecting circular equation give the closest approximation to the FEA output at < 1% difference throughout. All the estimates converge as a/W increases.

Table 5.2: Limit load solutions normalised by σ₀ for various a/W for uniaxial loading B = 0

<table>
<thead>
<tr>
<th>a/W</th>
<th>FEA output</th>
<th>Circular slip-line — intersect at corner, [Eqn (5.70)]</th>
<th>Circular slip-line — intersect at side, [Eqn (5.60)]</th>
<th>R6 (IV.1.5.5-4), [Eqns (5.43) &amp; (5.44)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>6.454</td>
<td>1.034</td>
<td>1.026</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.898</td>
<td>2.550</td>
<td>0.902</td>
<td>0.876</td>
</tr>
<tr>
<td>0.3</td>
<td>1.301</td>
<td>0.752</td>
<td>0.711</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.565</td>
<td>0.717</td>
<td>0.574</td>
<td>0.539</td>
</tr>
<tr>
<td>0.5</td>
<td>0.373</td>
<td>0.398</td>
<td>0.372</td>
<td>0.366</td>
</tr>
<tr>
<td>0.6</td>
<td>0.210</td>
<td>0.212</td>
<td>0.207</td>
<td>0.207</td>
</tr>
<tr>
<td>0.7</td>
<td>0.102</td>
<td>0.102</td>
<td>0.100</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.040</td>
<td>0.040</td>
<td>0.038</td>
<td></td>
</tr>
</tbody>
</table>

5.3.5 Comparison of upper bound, lower bound and FEA solutions for single edge cracked plates

The upper and lower bound estimates for the limit loads of single edge cracked plates are compared with each other and the FEA output in Figure 5.23.

For a/W = 0.2, for values of B < 0 the upper bound estimate is slightly lower than the FEA — due to the use of the circular slip-line solution — they are within 1% of the FEA. The lower bound estimates are within 3% of the FEA output. As noted previously, for B = 0.5, both the upper and lower bound estimates are higher than the FEA output, thus a lower bound estimate is still to be established. For B > 1, the upper bound, lower bound and FEA output are an exact match (accurate to less than 0.1%).

Similarly, for a/W = 0.4, 0.5 and 0.6, the upper bound estimates are closer to the FEA than the lower bound and the lower bound estimates were less than the FEA only for B < 0. For B > 1, the upper bound, lower bound and FEA output are an exact match (accurate to within 0.1%).
Analysis of short plates with edge cracks has not been carried out for this research. It is considered likely that the outcomes of such analysis for double edge cracked plates would be similar to those found for centre cracked plates.

For single edge cracked plates, the slip lines are generally confined to a narrow region ahead of the crack (Figure 5.19) so that, other than for shallow cracks with negative values of B, the plate length effects would only become important for very short plates.

5.4 Limit Load Estimates for Cylinders

Equations for calculating the limit loads for cylinders with defects under tension, bending and pressure are given in R6 Sections IV.1.8 (circumferential defects) and IV.1.9 (axial defects).

This research concentrates on circumferential defects in cylinders under pressure and tension only (i.e. no bending is considered) in order to establish the effects of orthogonal biaxial stress that can be compared to
those above determined for through cracked plates. A representative cylinder is shown in Figure 5.24.

As well as the orientation of the defect (axial and circumferential), the defect can be classified according to whether it is semi-elliptical or of constant depth. This section concentrates on cracks with constant depth and in particular on cylinders with through-wall or fully circumferential cracks to compare with centre cracked and single edge cracked plates respectively.

![Diagram of cylinder with circumferential constant depth defect](image)

**Figure 5.24:** Cylinder with circumferential constant depth defect under combined tension and pressure

(a) Cylinder showing axes, (b) cross-section through cylinder showing dimensions and (c) part long-section at the end of the cylinder showing applied forces p and N

Internal or external pressure applied to a cylinder will give rise to a hoop (or circumferential) stress, and, if the ends are closed, an axial (or longitudinal) stress.

Note that for the through-wall crack the hoop stress is in the same plane as the crack while for a fully circumferential crack the hoop stress will be out-of-plane relative to the crack.

The hoop stresses $\sigma_h$ induced by an internal pressure $p$ for thin walled cylinders (where the internal radius $R_i$ is greater than ten times the wall thickness) and thick-walled cylinders (where the internal radius $R_i$ is less than or equal to ten times the wall thickness) are shown in equation (5.71)
\[
\begin{align*}
\sigma_h &= \frac{pR_m}{t} & \frac{R_i}{t} > 10 \\
\sigma_h &= \frac{p}{2} \left[ \frac{R_o^2 + 3R_i^2}{R_o^2 - R_i^2} \right] & \frac{R_i}{t} \leq 10
\end{align*}
\]  

(5.71)

where \( R_o, R_i \) and \( R_m \) are the inner, outer and mean radii respectively and \( t \) is the wall thickness.

The axial stresses \( \sigma_a \) induced by the internal pressure \( p \) for thin and thick-walled cylinders are shown in equation (5.72).

\[
\begin{align*}
\sigma_a &= \frac{pR_m}{2t} & \frac{R_i}{t} > 10 \\
\sigma_a &= \frac{pR_i^2}{R_o^2 - R_i^2} & \frac{R_i}{t} \leq 10
\end{align*}
\]  

(5.72)

The axial stresses \( \sigma_a \) due to end force \( N \) is given in equation (5.73).

\[
\sigma_a = \frac{N}{2\pi t R_m}
\]  

(5.73)

The biaxial stress ratio for cylinders with circumferential defects is defined here as the ratio of the hoop stress to the axial stress.

\[
\begin{align*}
B &= \frac{\sigma_h}{\sigma_a} = \frac{pR_m}{t} \left/ \left[ \frac{pR_m}{2t} + \frac{N}{2\pi t R_m} \right] \right. & \frac{R_i}{t} > 10 \\
B &= \frac{\sigma_h}{\sigma_a} = \frac{p}{2} \left[ \frac{R_o^2 + 3R_i^2}{R_o^2 - R_i^2} \right] \left/ \left[ \frac{pR_i^2}{R_o^2 - R_i^2} + \frac{N}{2\pi t R_m} \right] \right. & \frac{R_i}{t} \leq 10
\end{align*}
\]  

(5.74)
Note that where there is no additional end force, \( N = 0 \), the biaxial ratio \( B \) for the thin walled cylinder is exactly 2. This value increases with increasing thickness for the thick-walled cylinders. End force only will give a value of \( B = 0 \).

R6 equations (IV.1.8.1-1) to (IV.1.8.1-17) can be used to determine the normalised axial limit force due to internal pressure, \( n_p \), the normalised axial limit force due to end force, \( n_N \), and the normalised total axial limit axial force, \( n_L \). The results of these calculations are shown in Figure 5.25.

The pattern of the limit load, increasing until it approaches unity and decreasing thereafter, is again apparent for these theoretical cylinder limit loads. There is very little difference between the values for external and internal cracks. In addition, the effect of the thickness of the cylinders appears to be negligible with the thick-walled cylinders (a) and thin wall cylinders (b) showing almost identical values.

Comparisons of the cylinder limit loads with the plate lower bound limit loads have been plotted in Figure 5.26.

For both configurations of cylinder and plate defects, the plate limit loads are considerably higher (up to 2 times) than the cylinder loads for \( 0.5 < B < 1.5 \) so the plate limit loads would not provide a conservative estimate for cylinder limit loads.

There is however agreement between the single edge cracked plate and fully circumferentially cracked cylinder values for \( B \leq -0.5 \) and \( B > 2 \), even though the biaxial loading (hoop stress) of the latter is out-of-plane.
A comparison of the limit loads in R6 has been carried out by Li et al. [74] who showed that these limit loads can conservatively predict those found using FEA.

5.5 **Effect of the Influence of Biaxial Stress on Limit Load on Structural Integrity Assessments using the FAD**

As described briefly in chapters 1 and 3, the failure assessment diagram (FAD) is based on the location of a point with an x coordinate value of $L_r$ (proximity to plastic collapse) and y coordinate value of $K_r$ (proximity to elastic fracture) relative to a failure assessment curve. The curve is typically a function of $L_r$.

R6 has three main options of failure assessment curves, Option 1, 2 and 3. They vary in their suitability for materials with differing stress–strain relationships as well as in complexity and level of conservatism.

For $L_r < 1$, the R6 Option 1 curve is suitable for all materials. For values of $L_r \geq 1$, the Option 1 curve is suitable for materials whose stress-strain curve does not have a yield discontinuity. The material used throughout this chapter is elastic-perfectly plastic with no discontinuity thus the Option 1 curve can be used to assess the effects of the biaxial stress.

The equation for the R6 Option 1 curve is given in R6 Section I.6.1 and reproduced here in equation (5.75).
\[
\begin{cases}
K_r = f_1(L_r) = [1 + 0.5L_r^2]^{-1/2} [0.3 + 0.7e^{-0.6L_r^2}] & L_r < L_r^{\text{max}} \\
K_r = f_1(L_r) = 0 & L_r \geq L_r^{\text{max}}
\end{cases}
\]

(5.75)

where \(L_r^{\text{max}}\) is the cut-off value for \(L_r\) and is defined in R6 equation (1.6.15) as the uniaxial flow stress, \(\bar{\sigma} = (\sigma_u + \sigma_y)/2\), divided by the uniaxial 0.2% proof stress, \(\sigma_y\).

An elastic-perfectly plastic material has an ultimate tensile stress \(\sigma_u\) equal to the yield stress \(\sigma_y\) (and 0.2% proof stress). Thus the flow stress \(\bar{\sigma}\) will be equal to \(\sigma_y\) and \(L_r^{\text{max}} = \bar{\sigma}/\sigma_y = 1\).

The values of \(K_r = K/K_{\text{mat}} = \) (Stress intensity factor)/ (Material fracture toughness) have been determined using the R6 equation (Section IV.3.3.5) for the centre cracked plate value of \(K\), and a fixed value of \(K_{\text{mat}}\). The values of \(L_r = \) (Applied load)/(Plastic limit load) have been calculated using the applied tensile loading, \(\sigma_2\), and the lower bound limit loads, \((\sigma_2)_L^b\), found using equation (5.4). As \(K_{\text{mat}}\) is fixed, \((\sigma_2)_L^b\) constant for each value of \(B\) and \(K\) is directly proportional to applied loading, all the loading lines will be straight.

The loading line points have been plotted on an R6 Option 1 failure assessment curve for centre cracked plates with crack sizes \(a/W = 0.2\) and \(a/W = 0.6\) and for values of \(B\) varying from \(-1\) to +2 (Figure 5.27) for incrementally increasing values of applied loading.

A fixed value \(K_{\text{mat}} = 100\,\text{MPa}\sqrt{\text{m}}\) has been chosen, as for this value of \(K_{\text{mat}}\) the loading line for uniaxial loading (\(B = 0\)) crosses the curve at the corner where \(L_r\) reaches its maximum value, \(L_r^{\text{max}} = 1\).

This illustrates the effect of non-zero biaxial stress ratios such that for values of \(B\) for which the limit load is lower than the uniaxial (\(B = 0\)) limit load, the loading lines cross the curve at \(L_r = 1\), indicating failure by plastic collapse. The loading lines for values of \(B\) with a limit load higher than the uniaxial limit cross the curve for values of \(L_r < 1\) and indicate failure.
controlled by crack tip events (brittle or ductile) rather than simply by overall collapse.

A higher fixed $K_{\text{mat}}$ will have the effect of systematically decreasing all the gradients and a lower value will increase them. The effects of a variable $K_{\text{mat}}$, dependent on constraint and loading, are covered in the next chapter.

For each loading line, the value of $L_r$ at which the line crosses the curve is calculated and multiplied by the equation (5.4) limit load to give a value of applied loading. These values have been normalised by the value for uniaxial loading $B = 0$ and are plotted Figure 5.28.

Figure 5.27: FAD with R6 Option 1 curve showing loading lines for various values of biaxial stress ratio $B$

(a) $a/W = 0.2$ and (b) $a/W = 0.6$

(a) Higher value of fracture toughness $K_{\text{mat}} = 100 \text{ MPa}\sqrt{\text{m}}$ and (b) Lower value of fracture toughness $K_{\text{mat}} = 50 \text{ MPa}\sqrt{\text{m}}$

Figure 5.28 (a) shows that for values of $0 < B \leq 1.25$ ($a/W = 0.2$) and $0 < B \leq 2.5$ ($a/W = 0.6$) the biaxial stress increases the failure load by up to 40% at $B = 0.5$ ($a/W = 0.2$) and 45% when $B = 1.25$ ($a/W = 0.6$).

Conversely, when $B$ is outside these values, the failure load could be
overestimated by more than twice its value if using the uniaxial load, but these may be for unrealistic values of B. For B = 2 and a/W = 0.2, a failure load of 0.625 of the uniaxial failure load is obtained and so the uniaxial failure load would overestimate the biaxial failure load by 60%.

Similarly, Figure 5.28 (b), with a lower value of $K_{\text{mat}}$, shows the same pattern but with a lower impact on the failure loads, in this case the conservative underestimates of failure loads at around 14% and the overestimates in the order of 36%.

The effect of biaxial stress using the FAD can also be examined by determining the failure crack size for a number of loadings and biaxial ratios. Figure 5.29 shows the failure crack sizes for a centre cracked plate of elastic-perfectly plastic material with fracture toughness $K_{\text{mat}}/\sigma_y = 1/\sqrt{m}$ and $0.5/\sqrt{m}$, subject to remote tensile loading $\sigma_2$ with values 0.25 $\sigma_y$, 0.5 $\sigma_y$ and 0.75 $\sigma_y$ and biaxial stress with ratio B varying from −2 to +4.

The pattern is again the failure crack size increasing up to a value of $B \approx 1$ and thereafter decreasing. The lower $K_{\text{mat}}$ leads to lower failure crack sizes.

While the analyses in this chapter have been based on an elastic-perfectly plastic material, the development of the Option 1 FAC, Equation (5.75), was based on Option 2 curves for strain hardening materials. However, the Option 1 FAC is considered applicable for analysis of defects in components made of many materials and is used in this chapter in order to allow for the future extension of the analysis to include different materials.
An Option 2 material specific curve for elastic-perfectly plastic material is illustrated in Figure 3.10 and shows that there is agreement between the two curves for \(0 < L_r < 0.5\) and \(L_r = 1\).

For \(0.5 < L_r < 1\), the failure \(L_r\) and \(K_r\) — and by deduction the failure load and crack width — are higher using the Option 2 curve, so that use of Option 1 leads to conservative estimates.

However, the differences are generally small as use of the Option 2 FAC can at most increase the value of \(L_r\) at failure to 1, and hence the estimated failure load to the limit load. When failure is predicted to occur at \(L_r = 1\), the influence of the biaxial loading is simply given by its influence on the associated limit load.

As both the Option 1 and Option 2 curves are independent of biaxiality, the effects of biaxiality on estimated failure load or limiting defect size are similar whichever curve is used.

The effects of the Option 3 curve are discussed in the next chapter.

### 5.6 Summary

In section 5.2, upper and lower bound limit loads have been found, for long centre cracked plates in-plane strain subject to biaxial stress, as functions of crack size \(a/W\) and biaxial stress ratio \(B\). These have been compared to Abaqus FEA limit load output for plates with crack sizes \(a/W = 0.2\) and \(a/W = 0.6\).

For all \(B < 1/[2(1-a/W)]\), including negative \(B\) (compressive parallel stress), the values found using the three methods are identical and thus can be considered exact. For higher positive values of \(B\), while the upper bound solution is closer to the FEA values, the lower bound estimate (equation (5.4)) can be used as a conservative estimate for the limit load of biaxially loaded long centre cracked plates.

For short centre cracked plates, the solutions are more complex as they depend on the plate length \(a/H\) as well as \(a/W\) and \(B\). Some of the estimates of lower bound limit load are higher than the FEA output or upper bound estimates. In addition, a small number of cases occur where the short plate solution is higher than the equivalent long plate solution. Thus for all cases,
the conservative solution will be the minimum value of equations (5.4), (5.14) and (5.32).

In section 5.3, for double edge cracked plates, a conservative limit load estimate for most values of B can be made using the centre cracked plate lower bound equation (5.4).

For single edge cracked plates, the minimum value found using equations (5.45), (5.53) and (5.60), the lower bound, upper bound straight slip-line and upper bound circular slip-line estimates respectively, gives a conservative estimate of the limit load for most values of B.

For all the cracked plate geometries, and for all three methods, as B increases the limit load tends to \((2/\sqrt{3})(1/|B|)\) independently of crack size and plate or crack geometry.

In section 5.4, lower bound limit load solutions for cylinders with circumferential defects under internal pressure and axial stress have been found using the method and equations described in R6.

In section 5.5 the effects of biaxial stress on assessing the failure load and crack size using the FAD have been examined. For negative, low and high positive values of B, for the theoretical material and geometrical parameters examined, using uniaxial limit loads could overestimate the failure load by up to 60%. For values of B around equibiaxiality \(B = 1\), using uniaxial limit loads could underestimate the failure load by up to 36%. A similar pattern is found for failure crack size.

### 5.7 Conclusions

Upper bound, lower bound and finite element limit load analyses have been carried out for centre cracked, double edge cracked and single edge cracked plates in-plane strain and circumferentially cracked cylinders under a wide range of biaxial stress ratios and for various crack sizes and differing plate lengths.

These analyses have shown that:

- Exact limit load solutions can be derived for a range of biaxial stress ratios, in particular negative and high positive ratios.
• For most of the plate and cylinder geometries, the lower bound limit load solutions found here can be used where there is not an exact solution to estimate the limit load conservatively.

• The effects of adding biaxial stresses on the failure loads and failure crack sizes of centre cracked plates are significant and follow a non-monotonic pattern.
6 Fracture Parameters & Constraint

The elastic crack driving force (CDF) and material fracture toughness form the numerator and denominator respectively of the $K_r$ parameter for fracture assessment with a failure assessment diagram (FAD). The FAD essentially provides an estimate of the elastic-plastic crack driving force.

The crack driving force, also known as the energy release rate, is essentially a measure of the rate of change of energy released with the crack area (as opposed to time) during the loading of a structure or component. When this crack driving force reaches a critical value — the material fracture toughness — fracture is initiated.

Component geometry, crack geometry and loading can influence both the crack driving force and material fracture toughness. In an experimental procedure where a change to these influencing factors has led to fracture initiating for, say, a higher loading than without the change, it is not always immediately clear whether the gain is an effect of a reduction in the resultant crack driving force or an increase in the component’s resultant resistance to fracture. Likewise, a decrease in the initiation load could indicate either a higher resultant crack driving force or a decreased resistance to fracture.

This chapter looks at the influence of biaxial loading on both the crack driving force and on the material resistance to fracture.

6.1 Crack Driving Force

6.1.1 Stress intensity factor $K$

The stress intensity factor (SIF) quantifies the stress field ahead of the crack tip described in section 6.2.1 and equation (3.22). There is a different value of $K$ for each mode of loading (Figure 3.4). For this research, only mode I opening, with SIF $K_1$, has been considered.

The value of $K_1$ is dependent on the plate and crack geometry. It is not affected by the addition of a biaxial loading parallel to the crack plane where the principal loading is normal to the crack plane. The value of the biaxial $K_2$ is, according to Anderson [7], found using equation (6.1).
where \( \beta \) is the angle of the crack to the horizontal and \( B \) is the biaxial loading ratio.

For a plate or cylinder with a horizontal crack and principal loading applied normal to the crack and biaxial loading applied parallel to the crack, \( \beta = 0 \) thus biaxial \( K_I = K_I \). This has been confirmed for the centre cracked and edge cracked plates in the previous chapter where it has been used for validation of the finite element analysis (FEA) models.

There is a compendium of solutions of mode I SIF, \( K_I \), in R6 [3] Section IV.3. The mode I SIF, \( K_I \), is related to the energy release rate \( G \) by equation (6.2).

\[
G = \frac{K_I^2}{E'}
\]

(6.2)

where, for plane strain, \( E' = E/(1 - \nu^2) \) such that \( E \) is the Young’s modulus and \( \nu \) is Poisson’s ratio.

In linear elastic fracture mechanics (LEFM), and for use in the FAD, the elastic part of the \( J \)-integral, \( J_e \), is equivalent to the energy release rate \( G \).

For plane strain, equation (6.2) can be rewritten in terms of \( K_I \) as equation (6.3).

\[
K_I = \frac{\sqrt{J_e E}}{\sqrt{1 - \nu^2}}
\]

(6.3)
6.1.2 J-integral and Crack Driving Force (CDF)

6.1.2.1 Elastic-perfectly plastic centre cracked and double edge cracked plates

During the FEA carried out for the assessment of limit loads in chapter 5, the values of the J-integral during the loading have been recorded as additional output from the FEA and used here to determine the J-integral as a function of loading, biaxial stress ratio and crack size.

The results for elastic-perfectly plastic centre cracked and double edge cracked plates with crack sizes $a/W = 0.2$ and $a/W = 0.6$ are shown in Figure 6.1. The output for $J$ is recorded during loading. The resulting $J$ values have been plotted against the applied tensile loads $\sigma_2$, which have been normalised by the yield stress $\sigma_y$.

Figure 6.1 indicates that there is a dependence of $J$ on the biaxial stress ratio $B$. All four graphs show significant differences between the curves. The value of $\sigma_2/\sigma_y$ where $J$ changes to a steeper dependence varies in the same pattern as the limit load in each case. The value of $J$ at which this occurs increases with the increase in limit load.

To examine the effects of biaxial stress without the influence of limit load and $J$ as a function of this load, the values of $J$ and $\sigma_2$ have been normalised by elastic $J$, $J_e$, and limit load $(\sigma_2)_L$ respectively. In this case the limit load used is the one found during the FEA so that $\sigma_2/(\sigma_2)_L \leq 1$.

These normalised curves are shown in Figure 6.2. This shows that by eliminating the influence of limit loads, the J-integral and hence crack driving force has only a weak dependence on $B$ as the applied stress exceeds around 0.8 of that at the limit load.

6.1.2.2 Effect of the biaxial loading and J-integral on CDF and failure load

This normalisation is the one used for the R6 [3] Option 3 failure assessment curve (FAC) equation (I.6.14) shown here in equation (5.73).

\[
\begin{align*}
\begin{cases}
  f_3(L_r) = (J_e/L)^{1/2} & \text{if } L_r < L_r^{\text{max}} \\
  f_3(L_r) = 0 & \text{if } L_r \geq L_r^{\text{max}}
\end{cases}
\end{align*}
\]

(6.4)
For an elastic-perfectly plastic material, $L_r^{\text{max}} = 1$ (see chapter 5 section 5.5).

Figure 6.3 illustrates the J-integral output in the form of the R6 Option 3 FAC and shows that the dependence of the FAC on B could lead to some overly conservative and some overoptimistic results if the R6 Option 3 curve derived from uniaxial loading were used for cases of biaxial loading.

In order to illustrate the dependence of failure on B for elastic-perfectly plastic centre cracked plates, an FAD has been constructed using the R6 Option 3 FAC (Figure 6.4).
The Option 3 curves are those determined as for Figure 6.3 using the J-integral output values of the FEA for the elastic-perfectly plastic centre cracked plates as described in section 6.1.2.1.

The values of $K_r$ for the loading lines have been determined using the SIF during loading calculated using the equation in R6 Section IV.3.3.5, and a $K_{mat} = 100 \text{ MPa}\sqrt{m}$, chosen arbitrarily to give a clearer illustration of the effects being examined.

The values of $L_r$ for the loading lines have been determined using the loading and limit load output of the FEA, the latter of which is described in more detail in chapter 4.

Figure 6.4 shows these FAC for $B = 0$ and $B = 1.5$ from Figure 6.3 (a) and (b) as well as the loading lines constructed as described above. The estimated failure loads and elastic crack driving forces for each level of biaxial load ratio can be calculated using the loading line and FAC appropriate to that ratio.

Figure 6.2: Normalised J- integral FEA output for elastic-perfectly plastic biaxially loaded centre cracked and double edge cracked plates
(a) $a/W = 0.2$, centre cracked plate, (b) $a/W = 0.6$, centre cracked plate, (c) $a/W = 0.2$, double edge cracked plate and (d) $a/W = 0.6$, double edge cracked plate.
However, to see the effect of the biaxial loading on the estimated failure and CDF, the failure loads and CDF have been calculated using the uniaxial FAC with the biaxial limit loads and vice versa.

From Figure 6.4 it can be seen that the position of the loading line, and hence the limit load, has the major influence on the failure point where the loading line crosses the FAC. However, (a) also indicates that for the \( B = 1.5 \) loading line there is a significant difference in where the line crosses the FAC.

For each of the loading lines, the value of \( \text{L}_r \) at the points of intersection of the line with the FAC was multiplied by the FEA limit load to give an estimate of the applied loading at failure as a fraction of the yield stress. The values of the Option 3 FAC, \( f_3(L_r) \) (see equation (5.73)), were squared to give the elastic CDF at failure as a fraction of material fracture toughness \( J_{\text{mat}} \). These results are shown in Table 6.1 for \( a/W = 0.2 \) and in Table 6.2 for \( a/W = 0.6 \).
Figure 6.4: FAD using the R6 Option 3 curves based on $B = 0$ and $B = 1.5$ for loading lines using the limit loads for $B = 0$ and $B = 1.5$.

Based on FEA of elastic-perfectly plastic centre cracked plates with crack sizes (a) $a/W = 0.2$ and (b) $a/W = 0.6$

Table 6.1: Normalised failure loads and elastic CDFs using the Option 3 FAC, $a/W = 0.2$

<table>
<thead>
<tr>
<th>$a/W = 0.2$</th>
<th>Option 3 FAC $B = 0$</th>
<th>Option 3 FAC $B = 1.5$</th>
<th>Option 3 FAC $B = 0$</th>
<th>Option 3 FAC $B = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_2/\sigma_y$</td>
<td>$\sigma_2/\sigma_y$</td>
<td>$CDF/J_{mat}$</td>
<td>$CDF/J_{mat}$</td>
</tr>
<tr>
<td>LL $B = 0$</td>
<td>0.923</td>
<td>0.924</td>
<td>0.281</td>
<td>0.282</td>
</tr>
<tr>
<td>LL $B = 1.5$</td>
<td>1.348</td>
<td>1.251</td>
<td>0.599</td>
<td>0.516</td>
</tr>
</tbody>
</table>

Table 6.2: Normalised failure loads and elastic CDFs using the Option 3 FAC, $a/W = 0.6$

<table>
<thead>
<tr>
<th>$a/W = 0.6$</th>
<th>Option 3 FAC $B = 0$</th>
<th>Option 3 FAC $B = 1.5$</th>
<th>Option 3 FAC $B = 0$</th>
<th>Option 3 FAC $B = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_2/\sigma_y$</td>
<td>$\sigma_2/\sigma_y$</td>
<td>$CDF/J_{mat}$</td>
<td>$CDF/J_{mat}$</td>
</tr>
<tr>
<td>LL $B = 0$</td>
<td>0.460</td>
<td>0.462</td>
<td>0.338</td>
<td>-</td>
</tr>
<tr>
<td>LL $B = 1.5$</td>
<td>0.745</td>
<td>0.714</td>
<td>0.889</td>
<td>0.816</td>
</tr>
</tbody>
</table>

Clearly from the tables the biggest effects are between the two loading lines reflecting the significant influence of the biaxial stress ratio on limit load (see chapter 5 section 5.5) where using the line for $B = 0$ could underestimate the failure load by up to 40% ($a/W = 0.6$).

The influence of biaxial loading on $J$, hence the Option 3 FAC, is to either underestimate the failure load by less than 1% if using the $B = 0$ loading line and to overestimate by up to 7% when using the $B = 1.5$ loading line.
The figures for the effect on elastic CDF are similar but up to 13% overestimated. The loading line for \( B = 0 \) crosses the FAC for \( B = 1.5 \) when \( L_r = 1 \) and therefore the failure CDF is not applicable.

Thus, there is an effect of biaxial loading on \( J \) and this in turn affects the failure loading and CDF. However, these effects are generally lower than those of the effect on limit loads.

Note that the value of CDF = 0 for \( a/W = 0.6 \) and loading line \( B = 0 \) seems to indicate a greater percentage drop. This is an artefact of the FAC whereby in the \( B = 0 \) FAC, the value \( L_r \) is very close to 1 giving a low value of \( K_r \), and for the \( B = 1.5 \) FAC, \( L_r = 1 \) which always gives a \( K_r \) of 0.

6.2 Material Fracture Toughness and Constraint

Constraint refers to the extent to which plasticity is contained. High constraint generally refers to plasticity contained in a small region around the crack tip. Low constraint refers to situations in which plasticity spreads out extensively. The level of constraint depends on the material properties, geometry and loading of the component under analysis.

A relationship between constraint and material fracture toughness, determined by Ainsworth and O’Dowd [53], is shown here in equation (6.5).

\[
\begin{align*}
K_{\text{mat}}^c &= K_{\text{mat}} & \beta L_r > 0 \\
K_{\text{mat}}^c &= K_{\text{mat}}[1 + \alpha (-\beta L_r)^m] & \beta L_r < 0
\end{align*}
\]  

(6.5)

where \( K_{\text{mat}}^c \) is the constraint modified material fracture toughness, \( \beta \) is the normalised structural constraint parameter and \( \alpha \) and \( m \) are material constants.

6.2.1 Calculation of the constraint parameter \( \beta \)

There are a number of ways in which the constraint parameter \( \beta \) can be calculated. The most widely used parameters are the T-stress for linear elastic materials, derived from the Williams expansion [14], the Q parameter definition for elastic-plastic materials proposed by O’Dowd and
Shih [54] and the $A_2$ parameter for power law hardening materials described by Chao et al. [55].

This thesis concentrates on the T-stress and the Q parameter. There are solutions available for T-stress in R6. Q is not as straightforward and generally requires FEA to be carried out. This section looks at the extent to which elastic T-stress can be used to predict Q for elastic-perfectly plastic and power law hardening materials.

6.2.1.1 T-stress

The Williams expansion for opening stress fields near to the crack tip in linear elastic fracture mechanics (LEFM) can be expressed simply as an infinite sum of terms in $\theta$ and powers of $r$, where $(r, \theta)$ are polar coordinates relative to the crack tip.

$$
\sigma_{ij}(r, \theta) = \sum_{n=-\infty}^{\infty} A_n f_{ij}^n(\theta) r^n
$$

(6.6)

where $A_n$ are constants and $f_{ij}^n(\theta)$ are angular functions of $\theta$.

The first two terms of the expansion of equation (6.6) are functions of $r^{-1/2}$ and $r^0$ respectively, the latter thus being independent of $r$. The higher order terms are considered negligible for small values of $r$.

Thus equation (6.6) can be reduced to one with two terms, the first with a $1/\sqrt{r}$ singularity and the second a constant. Adapted from the works of Westergaard [15] and Irwin [13], equation (3.22) shows the crack tip opening stress field widely used in LEFM applications.

$$
\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) + T \delta_{ij}
$$

(6.7)

where $K_I$ is the mode 1 stress intensity factor and $\delta_{ij}$ is the Kronecker delta.
Thus the SIF $K_I$ characterises the first term of the crack tip opening stress field and the second term, the T-stress component, is only applicable when $i = j = 1$, i.e. for $\sigma_{11}$, the stress in the direction parallel to the crack.

The value of $T$ depends on the geometry and loading type such as bending or tension. A number of solutions for $T$ can be determined from the R6 Section IV.5 compendium of $\beta$ solutions where $\beta_T$ is the normalised constraint parameter and $T$ is found by multiplying $\beta_T$ by reference stress $\sigma_{ref}$.

$$T = \beta_T \sigma_{ref} = \beta_T L_r \sigma_y$$

(6.8)

where $L_r$ is the proximity to plastic collapse and $\sigma_y$ is the yield stress.

In the R6 compendium, for each geometry for which the value of $\beta_T$ is given, the reference stresses are also given. These are generally in terms of membrane stress $\sigma_m$, which is equivalent to the crack opening stress $\sigma_2$ used here.

For the specimens given in R6, $\beta_T$ is independent of magnitude of the applied loading and is a function of geometry and crack width $a/W$ only.

For biaxially loaded specimens, by the elastic superposition principle, the applied stress parallel to the crack can simply be added to the uniaxial T-stress [29].

Thus, for a specimen with crack width $a/W$, remotely applied crack opening stress $\sigma_2$ and a stress parallel to the crack $B\sigma_2$,

$$T = \sigma_2 [f(a/W) + B] = \beta_T \sigma_{ref} + B\sigma_2$$

(6.9)

Note that where superposition in the form of equation (6.9) is used to calculate $T$ for biaxial loading, the normalised constraint parameter $\beta_T$ and the value of reference stress $\sigma_{ref}$ used are always those for the uniaxial loading. However, it is possible to define a value of $\beta_T$ for the biaxial case
using the biaxial value of $T$ and the biaxial limit load and this is a function of $B$ as well as $a/W$.

For a centre cracked plate with $H/W > 1.5$ and $0 \leq a/W \leq 0.6$, subject to remote uniaxial crack opening stress $\sigma_2$, combining the compendium equations for $\beta_T$ and $\sigma_{ref}$, R6 (IV.5.4.1), gives equation (6.10).

\[
T = \frac{\sqrt{3}\sigma_2}{2(1-a/W)} \left[ -1.1547 + 1.1511\left(\frac{a}{W}\right) - 0.7826\left(\frac{a}{W}\right)^2 + 0.4751\left(\frac{a}{W}\right)^3 - 0.1761\left(\frac{a}{W}\right)^4 \right]
\]

(6.10)

Thus for a biaxially loaded centre cracked plate with $H/W = 2$ subject to remote uniaxial crack opening stress $\sigma_2$ and parallel remote stress $B\sigma_2$, combining equations (6.9) and (6.10) gives an estimate for the $T$-stress, equation (6.11).

\[
T = \sigma_2 \left[ \frac{\sqrt{3}}{2(1-a/W)} \left( -1.1547 + 1.1511\left(\frac{a}{W}\right) - 0.7826\left(\frac{a}{W}\right)^2 \right) + 0.4751\left(\frac{a}{W}\right)^3 - 0.1761\left(\frac{a}{W}\right)^4 \right] + B
\]

(6.11)

The values of the $T$-stresses found using equation (6.11) are plotted in Figure 6.5 for $a/W = 0.2$ and $a/W = 0.6$. Using the elastic material model as described in chapter 4, the $T$-stress output from the FEA is also plotted on these graphs.

Figure 6.5 (a) shows that for the plates with crack width $a/W = 0.2$, for $B < 1$, $T$ decreases with increasing loading. The gradients increase in value as $B$ increases. For values of $B > 1$, the gradient is positive and $T$ increases with increasing loading. This pattern is repeated in for $a/W = 0.6$ (Figure 6.5 (b)) with the threshold value of $B \approx 1.5$ for the transition from negative
to positive gradient. The same pattern is observed for double and single edge cracked plates.

![Graphs showing constraint during loading for an elastic centre cracked plate with H/W = 2 calculated using R6 equations with validated FEA model values.](image)

(a) a/W = 0.2 and (b) a/W = 0.6

6.2.1.2 Q parameter

In elastic-plastic fracture mechanics (EPFM), the near tip stress fields can be determined by the addition of the Q parameter as defined by O’Dowd and Shih [54] to the HRR (Hutchinson, Rice and Rosengren) stress field [50, 51].

Q can also be defined by and more easily determined using the expression in R6 Section III.7.5.2 relating it to the stress field under small-scale yielding (SSY), shown here in equation (6.12).

\[
\sigma_{ij} = \sigma_{ij}^{SSY} + Q \sigma_y \delta_{ij}
\]  

(6.12)

where \(\sigma_{ij}^{SSY}\) is the SSY stress field, \(\sigma_y\) is the material yield stress and \(\delta_{ij}\) the Kronecker delta.

The SSY stress field is determined using a modified boundary layer model (MBL). The MBL model consists of a semi-infinite crack in an infinite plate to generate the highest theoretical constraint near the crack tip. The FEA model used is a circular body with a crack extending from the centre towards the circumference. The loading is applied as displacement boundary conditions. More detailed explanations are given by Verstraete et
al. [56] and Ren et al. [49] and the specific modelling parameters for this research can be found in chapter 4 of this thesis.

The values of $\sigma_{ij}^{\text{ssy}}$ are calculated using the stress ahead of the crack tip, $\sigma_{22}$, normalised by the yield stress $\sigma_y$ at a distance $r = 2(J/\sigma_y)$ from the crack tip. For each specific plate geometry for which Q is required, the crack opening stress component $\sigma_{22}$ at $r = 2(J/\sigma_y)$ is extracted from its FEA output (as S22) and the SSY stress from the MBL model can be subtracted to determine Q using equation (6.13), which is essentially equation (6.12) for the crack opening stress $\sigma_{22}$.

$$Q = \frac{\sigma_{22} - \sigma_{22}^{\text{ssy}}}{\sigma_y}$$  \hspace{1cm} (6.13)

As well as an elastic-perfectly plastic material, MBL and plate FEA were carried out using the data for a pipe material that had been used for research on J-R and J-T curves by the Bhabha Atomic Research Centre (BARC) in Mumbai [75]. The data were supplied for investigations into T-stress solutions in pipes [76] and constraint solutions in plates and cylinders [77].

Values of Q during biaxial loading of centre cracked and single edge cracked plates with crack size $a/W = 0.2$ for both materials are shown in Figure 6.6.

All four graphs in Figure 6.6 show an increasing gradient of Q as B increases, similar to that shown for the T-stresses in Figure 6.5. For the centre cracked plate with $a/W = 0.2$, for an elastic-perfectly plastic material Figure 6.6 (a), the value of B for which the gradient and values of the Q parameter are 0 is $B = 1$ which is the same value as for the T-stresses in Figure 6.5 (a).

### 6.2.1.3 Using T to estimate Q

The advice in R6, in accordance with equation (III.7.4), is that when $L_r \leq 0.5$, Q can be estimated by $T/\sigma_y$, for $-0.5 < T/\sigma_y \leq 0$, and by $0.5 T/\sigma_y$ when $0 < T/\sigma_y < 0.5$. However, for biaxial loading, as T can be positive when B is sufficiently high, using $0.5 T/\sigma_y$ to estimate Q would give a positive Q.
and imply a higher constraint than for SSY due to the addition of the biaxial loading.

Models with the positive T-stress added using an extra displacement [56] showed that the influence of T on constraint Q is limited once T becomes positive. The area of more interest is that of low constraint (negative Q and hence negative $\beta$) where the effective fracture toughness is increased (equation (6.5)).

Thus in order to extend the advice in R6 to biaxially loaded plates and to loading higher than $L_r = 0.5$,

- For values of $L_r < 1$, i.e. $\sigma_2 < (\sigma_2)_t$, and $T < 0$, $Q$ is estimated by $T/\sigma_y$
- For values of $L_r = 1$, i.e. $\sigma_2 = (\sigma_2)_t$, and $T < 0$, $Q$ is estimated using the Prandtl slip-line stress field (referred to in Rice [35]) in place of the SSY reference field (equation (6.14)).
- For all values of $L_r$ and $T > 0$, $Q = 0$

The Prandtl slip-line stress field is given by equation (6.14)).

Figure 6.6: $Q$ calculated using FEA S22 output for MBL and plates, $H/W = 2$ and $a/W = 0.2$

(a) Centre cracked plate, elastic-perfectly plastic material, (b) Single edge cracked plate, elastic-perfectly plastic material, (c) Centre cracked plate, pipe material and (d) Single edge cracked plate, pipe material
\[ \sigma_{xx} = \pi \tau_y = \frac{\pi}{\sqrt{3}} \sigma_y \]

\[ \sigma_{yy} = (2 + \pi) \tau_y = \frac{(2 + \pi)}{\sqrt{3}} \sigma_y \]

(6.14)

where \( \tau_y \) is the yield shear stress and \( \sigma_{xx} \) and \( \sigma_{yy} \) are stresses in the directions parallel to and perpendicular to the crack respectively.

Thus \( Q \) can be estimated using equation (6.15).

\[
Q(T) = \begin{cases} 
\frac{T}{\sigma_y} & \text{if } T < 0 \text{ and } \sigma_2 < (\sigma_2)_L \\
\frac{(\sigma_2)_L}{\sigma_y(1 - a/W)} - \frac{2 + \pi}{\sqrt{3}} & \text{if } T < 0 \text{ and } \sigma_2 = (\sigma_2)_L \\
0 & \text{if } T \geq 0
\end{cases}
\]

(6.15)

Note that the second formula of equation (6.13) is applicable for a centre cracked plate where the stress field in the ligament ahead of the crack is assumed to be uniform at collapse. For other geometries, if the stress field at collapse ahead of the crack is known, the appropriate stress fields can be applied for these estimations of \( Q \).

Figure 6.7 shows the results of \( Q \) plotted against proximity to plastic collapse for centre cracked plates and single edge cracked plates with the results for both the elastic-perfectly plastic material (EPP) and the pipe material.

These graphs show that for \( B < 1 \), the modelled \( Q \) values decrease linearly with increasing proximity to plastic collapse until the load approaches the limit load near to collapse where \( Q \) declines sharply. The values of \( T \) and \( Q \) both increase in gradient as \( B \) increases towards 1.

When \( B > 1 \), for these centre cracked and single edge cracked plates, \( T \geq 0 \) thus the estimate for \( Q(T) \) from equation (6.15) is \( Q(T) = 0 \). The \( Q \) stresses for both the materials are slightly above zero.
For the plates and loadings shown here, the magnitude of $Q(T)$ is less than that of $Q$, thus $Q(T)$ gives a conservative estimate of $Q$. This is in contrast to the findings of O’Dowd et al. [29] who found that for the plates they assessed numerically, $a/W = 0.1$ and $0.5$, $n = 10$, $T$ was consistently lower than $Q_0$. Thus for negative values of $T$, the first formula in equation (6.15) would lead to estimates of $Q$ higher in magnitude than those found numerically, and the equation would not provide a conservative estimate.

There are a few cases where the second formula of equation (6.15) yields a value of $Q(T)$ that is higher in value than that found using the first formula, $T/\sigma_y$. For centre cracked plates these occur for some combinations of $a/W < 0.22$ and $0.375 < B < 0.65$. In these cases, $Q(T)$ at limit load can be estimated directly by $T/\sigma_y$ as for these values of $B$, $T < 0$ for all values of $a/W$.

The normalised constraint parameter $\beta$ is defined in equation (6.16).

$$\beta = \frac{Q}{L_r} = \frac{T}{\sigma_y L_r} = \frac{T(\sigma_2)_{L}}{\sigma_y \sigma_2}$$

(6.16)

As the limit load $(\sigma_2)_{L}$ is a function of yield stress $\sigma_y$ and $T$ is a function of the loading $\sigma_2$, $\beta$ can be calculated independently of the yield stress and the magnitude of the loading and is a function of geometry, crack size $a/W$ and, in the case of biaxial loading, the ratio $B$.

### 6.2.2 Effect of $\beta$ on the FAD

Equation (6.5) can now be calculated for various values of the material constants $\alpha$ and $m$. These parameters are dependent on a number of material factors and temperature. For the purpose of illustration in this chapter, values of 2.15 and 2.0 respectively have been used. These are the values that define a reasonable lower-bound curve [78].

The effect of the constraint $\beta$ and thus the constraint modified fracture toughness $K_{\text{mat}}^c$ on the FAD can be examined in two ways (R6 Sections III.7.4.1 and III.7.4.2):
Modify the failure assessment curve (FAC) by multiplying the function $f(L_e)$ by the value of $\frac{K_{matc}}{K_{mat}}$. As $K_{matc} \geq K_{mat}$ (see equation (6.5)), this modification will raise the curve thus extending the safe zone.

Modify the loading line by replacing the material fracture toughness $K_{mat}$ with the constraint modified material fracture toughness $K_{matc}$ in the denominator of $K_r$, the proximity to elastic fracture.
As the constraint parameter $\beta$ is dependent on biaxial loading ratio $B$, a different FAC would be needed for each value of $B$ if using the modified FAC method. Thus the second method has been used here in order to compare a number of different levels of biaxial loading on the same FAD.

Figure 6.8 shows the effect of the constraint and modified material fracture toughness for the centre cracked plate, as described in chapters 4 and 5, of an elastic-perfectly plastic material and crack width $a/W = 0.2$. Figure 6.8 (a) and (b) are for a higher value of fracture toughness, $K_{\text{mat}} = 100 \text{ MPa} \sqrt{m}$ and (c) and (d) for a lower value $K_{\text{mat}} = 50 \text{ MPa} \sqrt{m}$, Figure 6.8 (b) and (d) illustrate the effect of modifying the toughness.

![Figure 6.8: Option 1 FAC with loading lines for centre cracked plate, elastic-perfectly plastic material, $a/W = 0.2$](image)

(a) Constant material fracture toughness, $K_{\text{mat}} = 100 \text{ MPa} \sqrt{m}$, (b) Constraint modified material fracture toughness with $K_{\text{mat}} = 100 \text{ MPa} \sqrt{m}$ and $K_{\text{mat}}^C = 100[1 + 2.15(-B L_r)^2]$ in accordance with equation (6.5), (c) Constant material fracture toughness, $K_{\text{mat}} = 50 \text{ MPa} \sqrt{m}$ and (d) Constraint modified material fracture toughness with $K_{\text{mat}} = 50 \text{ MPa} \sqrt{m}$ and $K_{\text{mat}}^C = 50[1 + 2.15(-B L_r)^2]$ in accordance with equation (6.5)

The modification affects each loading line by bringing them further clockwise round the FAD. Where the constant $K_{\text{mat}}$ loading lines crossed the vertical section of the failure assessment curve at $L_r = 1$, a move clockwise...
does not change the value of \(L\) nor hence the estimated failure load. Furthermore, for higher values of \(B\), as \(T\) increases and becomes greater than zero, the predicted \(Q(T)\) and hence \(\beta\) is zero (equations (6.15) and (6.16)) leading to \(K_{\text{mat}}^c = K_{\text{mat}}\) for all values of \(\sigma_2\).

Thus the constraint modification affects only a small number of cases of biaxial loading.

Figure 6.9 shows the comparisons between estimated failure loads for a number of values of \(K_{\text{mat}}\) with the constraint modified \(K_{\text{mat}}^c\) for the cases examined in Figure 6.8 ((a) \(K_{\text{mat}} = 100\) MPa\(\sqrt{\text{m}}\) and (b) \(K_{\text{mat}} = 50\) MPa\(\sqrt{\text{m}}\)). For additional illustration, charts for (c) \(K_{\text{mat}} = 250\) MPa\(\sqrt{\text{m}}\) and (d) \(K_{\text{mat}} = 25\) MPa\(\sqrt{\text{m}}\) are shown.

![Figure 6.9](image)

**Figure 6.9: Comparison of failure loads using \(K_{\text{mat}}\) and \(K_{\text{mat}}^c\) for various value of \(K_{\text{mat}}\)**

(a) \(K_{\text{mat}} = 100\) MPa\(\sqrt{\text{m}}\), (b) \(K_{\text{mat}} = 50\) MPa\(\sqrt{\text{m}}\), (c) \(K_{\text{mat}} = 250\) MPa\(\sqrt{\text{m}}\) and (d) \(K_{\text{mat}} = 25\) MPa\(\sqrt{\text{m}}\)

For \(K_{\text{mat}} = 50\) MPa\(\sqrt{\text{m}}\), only \(B = 0\) and \(B = 0.5\) are affected. The values of the failure loads for these are increased by 31% and 61% respectively. For the
higher $K_{mat} = 100 \text{ MPa} \sqrt{m}$, only $B = 0.5$ is affected and the value of the estimated failure load is increased by 16%.

The higher value (c) $K_{mat} = 250 \text{ MPa} \sqrt{m}$ shows no effect of the constraint modified toughness on the failure load, which is to be expected since, at such high toughness, failure is controlled by plastic collapse. The lower value (d) $K_{mat} = 25 \text{ MPa} \sqrt{m}$ shows that for the constant $K_{mat}$, the biaxial loading ratio $B$ has little effect on the failure load while the constraint modified $K_{mat}^c$ affects the failure loads for $B = 0$ and $B = -1$. For $B = -1$ the failure load is increased by around 40%.

However, a considerable contribution to these effects is made by the variation in limit loads for various biaxial stress ratios (see chapter 5) and the constraint correction (equation (6.5)) rather than specifically the biaxial loading effects on constraint.

Figure 6.10 (a) and (b) show that for all values of $B < 2$ (for $a/W = 0.6$) and $B < 1$ (a/W = 0.2) for the centre cracked plate, the decrease in constraint increases the fracture toughness.

In order to illustrate the effects on constraint of biaxial loading only, the constraint modified fracture toughness is normalised by that for uniaxial loading ($B = 0$) in Figure 6.10 (c) and (d). Here it is clear that it is only for $B = -1$ and $B = -2$, i.e. for compressive biaxial stress, that the biaxial loading increases the constraint modified fracture toughness from that modified by uniaxial loading.

The higher increases in failure load for $B = 0.5$ than for $B = 0$ shown in Figure 6.8 and Figure 6.9 are due to the shape of the FAC. Both failure lines move to the right by a similar amount however as the $B = 0$ line is, before modification, closer to the cut-off at $L_r = 1$, in turn due to its lower limit load, it reaches the cut-off before any additional benefit from increasing $L_r$ could be realised as the maximum in this case is $L_r = 1$. 
Summary

This chapter has focused on examining and quantifying the effect of biaxial loading on the inputs to the $K_r$ parameter, proximity to linear elastic fracture mechanics (LEFM) failure, of the FAD. These input parameters are stress intensity factor (SIF) $K_I$, crack driving force (CDF) $J$, and fracture toughness $K_{\text{mat}}$.

In section 6.1, it has been shown that while biaxial loading has no effect on the mode I SIF, $K_I$ (section 6.1.1), it does have an influence on the CDF, $J$, as the magnitude of the remotely applied normal loading approaches the limit load (section 6.1.2). The resultant influence on the failure loading and elastic CDF are considerably smaller (at around 10%) than those found in chapter 5 as a result of the effect of the biaxial loading on the limit loads (at around 50%) and apply to a narrower range of biaxial load ratios.

Section 6.2 established that for the cases examined, T-stress can be calculated directly as a function of the geometry, crack width and biaxial...
loading ratio $B$ and can be used to conservatively estimate the $Q$ parameter from which $\beta$ can be calculated for input into equation (6.5) (section 6.2.1). As $\beta$ is non-zero and negative for lower values of $B$ only, and that for very low values of $B$ the lower limit loads mean that the FAD indicates failure due to plastic collapse, the effect of the increase in material fracture toughness dependent on $B$ occurs over a narrow range of values of $B$ (section 6.2.2). For centre cracked plates with $a/W = 0.2$, these effects have been shown to be as high as 61% for lower values of $K_{\text{mat}}$.

While for all values of biaxial loading that lead to lower constraint there is a beneficial increase in fracture toughness, it is only for compressive biaxial remote stress (where $B < 0$) that the addition of the biaxial loading is more beneficial than for uniaxial loading (where $B = 0$).

### 6.4 Conclusions

Stress intensity factors for centre cracked plates have been determined using R6 equations. $J$-integrals have been determined for centre cracked and double edge cracked plates in-plane strain using FEA. $T$-stress has been shown to provide a conservative estimate of $Q$ for centre cracked and single edge cracked plates.

These analyses have shown that:

- Biaxial loading has an influence on the $J$-integral as the magnitude of the loading approaches the limit load
- The effect of this influence on the failure load is relatively small and over a narrow range of $B$
- Biaxial loading, via its effect on $T$ and $Q$, can increase the fracture toughness of the specimen or component. This increase in toughness occurs over a narrow range of $B$ and can increase the failure loading by up to about 40%
- This increase in failure load is, however, less due to the biaxial loading’s effect on constraint and more due to the lower constraint of the centre cracked plate than that of the SSY model and the effects of biaxial loading on the limit load.
7 Displacement Controlled Loading

When considering biaxial effects, it is important be clear about the meaning of biaxial loading ratio. In chapters 5 and 6 this was defined by the ratio of applied stresses. However, loading may also be applied as displacements.

This chapter examines how the biaxial loading ratio when measured as a ratio of displacements differs from the biaxial ratio measured as a ratio of stresses.

The first section of the chapter covers the calculation of the applied displacements and the second section compares the results of analyses carried out using stress controlled loading (SCL) and displacement controlled loading (DCL).

The third section of the chapter examines the effect of biaxial loading on cruciform specimens using data and experimental results from the Bhabha Atomic Research Centre (BARC) in Mumbai and examines these results in the context of displacement controlled loading and the previous chapters’ work on limit loads and constraint.

7.1 Calculation of the Applied Displacements

The advice in R6 [3] Section III.14.3.1 is that for a material behaving elastically the displacement is directly proportional to the applied load.

The FEA for the analyses described in chapters 5 and 6 were carried on the elastic-perfectly plastic centre cracked plate model in-plane strain using stress controlled loading (SCL), as described in chapter 4. For each biaxial stress ratio B analysed, a remote tensile stress $\sigma_2$ was applied along the top edge of the quarter plate model and a biaxial stress of $\sigma_1 = B\sigma_2$ applied on the right side of the model (Figure 4.1).

Initially, the equivalent displacement controlled cases were modelled as shown in Figure 7.1, with applied displacements given in equation (7.1), assuming simple independent uniaxial loadings, in accordance with the elastic displacements being proportional to stress.
Figure 7.1: Quarter plate model with displacements applied as boundary conditions for DCL.

The dotted lines represent the symmetry boundary conditions of the quarter plate model.

\[
\begin{align*}
\delta_2 &= \frac{H\sigma_2}{E} \quad \text{Top edge} \\
\delta_1 &= \frac{W\sigma_2}{E} \quad \text{Right side}
\end{align*}
\]

(7.1)

where H is the length of the quarter plate, W the width, E is the Young’s modulus and \( \delta_1 \) and \( \delta_2 \) are the applied displacements in the directions parallel and perpendicular to the crack respectively.

Applying displacement controlled loading in these proportions led to different values for the resultant J and Q to those found using the equivalent stress controlled loading with \( \sigma_1 = B_0 \). Thus a theoretical examination of DCL is necessary.

There is comprehensive advice in R6 Section III.14.3 for calculating the theoretical displacements, summarised by equation (III.14.1) (shown here in equation (7.2)) for an elastic-plastic material.
\[\delta_{(i)} = \delta_{uc}^{(i)} + \delta_{e}^{(i)} + \delta_{pl}^{(i)} \]  

(7.2)

where \( \delta_{uc} \) is the uncracked body displacement, \( \delta_{e} \) is the elastic displacement due to the crack and \( \delta_{pl} \) is the plastic displacement due to the crack.

The uncracked displacement determined in the following calculations is the elastic component only. For the geometry and elastic-perfectly plastic material considered, the uncracked plastic displacement is zero.

7.1.1 Uncracked body displacement \( \delta_{uc} \)

Under plane strain conditions, the application of displacements proportional to stresses using equation (7.1) does not lead to equivalent analyses and results to those found using biaxial stresses with ratio B.

Consider an uncracked rectangular plate, size \( 2H \times 2W \times t \), under plane strain conditions and remotely applied biaxial stresses such that \( \sigma_1 = B\sigma_2 \), shown in Figure 7.2.

Hooke’s law can be written as in equation (7.3).

\[
\begin{align*}
\varepsilon_1 &= \frac{1}{E} \left[ \sigma_1 - \nu(\sigma_2 + \sigma_3) \right] \\
\varepsilon_2 &= \frac{1}{E} \left[ \sigma_2 - \nu(\sigma_1 + \sigma_3) \right] \\
\varepsilon_3 &= \frac{1}{E} \left[ \sigma_3 - \nu(\sigma_1 + \sigma_2) \right]
\end{align*}
\]  

(7.3)

where \( \sigma_1, \sigma_2, \sigma_3 \) and \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are the components of stress and strain in the \( x, y \) and \( z \) directions respectively, \( E \) is Young’s modulus and \( \nu \) is Poisson’s ratio.

Under plane strain conditions, where the strain in the through thickness direction is zero, the formula for \( \varepsilon_3 \) in equation (7.3) can be rearranged, taking into account that \( \sigma_1 = B\sigma_2 \) (by definition of B).
\[ \varepsilon_3 = \frac{1}{E} [\sigma_3 - \nu(\sigma_1 + \sigma_2)] = 0 \]

\[ \therefore \sigma_3 = \nu(B\sigma_2 + \sigma_2) = \nu\sigma_2(1 + B) \]

Equation (7.3), Hooke's law, can thus be defined by the two formulae in equation (7.5).

\[ \varepsilon_1 = \frac{1}{E} [B\sigma_2 - \nu(\sigma_2 + \nu\sigma_2[1 + B])] \]

\[ \varepsilon_2 = \frac{1}{E} [\sigma_2 - \nu(B\sigma_2 + \nu\sigma_2[1 + B])] \]

(7.5)

Thus the ratio of strain in the x direction to strain in the y direction of the plate is given in equation (7.6)
\[
\frac{\varepsilon_1}{\varepsilon_2} = \frac{B - v - v^2 - Bv^2}{1 - Bv - v^2 - Bv^2} = \frac{B(1 - v^2) - v(1 + v)}{(1 - v^2) - Bv(1 + v)}
\]

\[
\therefore \frac{\varepsilon_1}{\varepsilon_2} = \frac{B(1 - v) - v}{1 - v - Bv}
\]

(7.6)

The value of \(\varepsilon_1/\varepsilon_2\) given by equation (7.6) will tend to infinity when

\(B = (1 - v)/v\), which for a Poisson’s ratio \(v = 0.3\) gives a value of \(B = 2.333\).

For a plate of height \(H\) and width \(W\), the biaxial displacement ratio \(B’\) is defined as \(\delta_1/\delta_2\) and can be calculated using equation (7.7).

\[
B' = \frac{\delta_1}{\delta_2} = \frac{W\varepsilon_1}{H\varepsilon_2} = \frac{W[B(1 - v) - v]}{H[1 - v - Bv]}
\]

(7.7)

Thus, for materials with a non-zero Poisson’s ratio, there is a non-linear relationship between the biaxial stress ratio \(B\) and the biaxial displacement ratio \(B’\). Figure 7.3 shows this relationship as well as the asymptote at \(B = 2.333\) for a plate with \(H/W = 2\) and Poisson’s ratio \(v = 0.3\).

For modelling, \(\delta_1\) and \(\delta_2\) can be calculated using equation (7.8).

\[
\delta_1 = \frac{W\sigma_2}{E} (B - v - v^2 - Bv^2)
\]

\[
\delta_2 = \frac{H\sigma_2}{E} (1 - Bv - v^2 - Bv^2)
\]

(7.8)

The potential considerable difference between \(B\) and \(B’\) can also be illustrated by the following example.

Consider a plate with \(H = 1000\) mm, \(W = 500\) mm, \(E = 200\) GPa, \(v = 0.3\), \(\sigma_2 = 100\) MPa and \(\sigma_1 = 50\) MPa.

\(B = \sigma_1/\sigma_2 = 0.5\)

Using equation (7.8),
δ₁ = 0.01625 mm

δ₂ = 0.3575 mm

B' = δ₁/δ₂ = 0.045 << 0.5

7.1.2 Elastic cracked body displacement δₑ

The formula in R6 (III.14.3) for calculating the elastic cracked body displacement is given here in equation (7.9), using t rather than B for plate thickness to avoid confusion with the biaxial stress ratio.

\[ \delta_{e(i)} = \left( \frac{2t}{E'} \right) \int_{0}^{a} K k_{(i)} \, da \]  

(7.9)

where t is the plate thickness, E' is the plane strain Young's modulus equal to E/(1 – ν²), a is the half crack width, K is the total stress intensity factor (SIF) due to all loadings and k_{(i)} s the SIF per unit value of load (i) associated with the displacement δ_{(i)}.

The value of k_{(i)}, SIF per unit load, can be calculated from the mode I SIF using equation (7.10).
\[ k_{(i)} = \frac{K_I}{2Wt\sigma} \]  

(7.10)

where \( \sigma \) is the remotely applied tensile stress.

Thus, for a cracked component under remote tensile stress inducing mode 1 opening only, the displacement can be calculated by substituting the formula for \( k_{(i)} \) from equation (7.10) into equation (7.9), giving equation (7.11).

\[
\delta_{e(i)}^c = \frac{(2t/E')}{2Wt} \int_0^a \frac{K_I^2}{\sigma_2} \, da
= \frac{\sigma_2}{E'W} \int_0^a \left( \frac{K_I}{\sigma_2} \right)^2 \, da
\]  

(7.11)

where \( K_I \) is the mode I SIF

Note that as the biaxial stress does not affect the mode I SIF \( K_I \) (chapter 3 section 3.3.3), the elastic component of cracked body displacement is independent of the biaxial stress, thus the applied stress for calculating the elastic cracked component of displacement is \( \sigma_2 \) only.

For a centre cracked plate under remote tensile stress normal to the crack opening direction, the formula for mode I SIF is given R6 Section IV.3.3.5 and repeated here in equation (7.12).

\[
K_I = \sigma \sqrt{\pi a} \left[ 1 - 0.025 \left( \frac{a}{W} \right)^2 + 0.06 \left( \frac{a}{W} \right)^4 \right] \left[ \sec \left( \pi a / 2W \right) \right]^{1/2}
\]  

(7.12)

Substituting the formula for \( K_I \) in equation (7.12) into equation (7.11) gives equation (7.13).
\[ \delta_{e(i)} = \frac{\sigma_e^2}{E \bar{W}} \int_0^a \pi a \left\{ 1 - 0.025 \left( \frac{a}{\bar{W}} \right)^2 + 0.06 \left( \frac{a}{\bar{W}} \right)^4 \right\}^2 \left[ \sec \left( \frac{\pi a}{2 \bar{W}} \right) \right] da \]

(7.13)

To simplify the integration, the sec function can be expanded using the Maclaurin series given in equation (7.14).

\[ \sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots \text{ for } |x| < \frac{\pi}{2} \]

(7.14)

where \( E_{2n} \) are the Euler numbers, the first 6 of which are 1, 5, 61, 1385, 50 521 and 2 702 765. Substituting this Maclaurin series for \( \sec \left( \frac{\pi a}{2 \bar{W}} \right) \) into equation (7.13) gives equation (7.15) which can be integrated to give equation (7.16).

\[ \delta_{e(i)} = \frac{\sigma_e^2}{E \bar{W}} a^2 \left[ 1 - 0.025 \left( \frac{a}{\bar{W}} \right)^2 + 0.06 \left( \frac{a}{\bar{W}} \right)^4 \right] \left[ 1 + \frac{\left( \frac{\pi a}{2 \bar{W}} \right)^2}{2} + \frac{5 \left( \frac{\pi a}{2 \bar{W}} \right)^4}{24} + \frac{61 \left( \frac{\pi a}{2 \bar{W}} \right)^6}{720} + \frac{1385 \left( \frac{\pi a}{2 \bar{W}} \right)^8}{40320} + \ldots \right] da \]

(7.15)

\[ \delta_{e(i)} = \frac{\sigma_e^2}{E \bar{W}} a^2 \left[ 1.5708 + 0.9297 \left( \frac{a}{\bar{W}} \right)^2 + 0.6950 \left( \frac{a}{\bar{W}} \right)^4 + 0.5321 \left( \frac{a}{\bar{W}} \right)^6 \right. \]

\[ + 0.4280 \left( \frac{a}{\bar{W}} \right)^8 + 0.3570 \left( \frac{a}{\bar{W}} \right)^{10} + 0.3061 \left( \frac{a}{\bar{W}} \right)^{12} + \ldots \] \]

(7.16)

Thus the elastic cracked component of displacement, \( \delta_{e(i)} \), is linearly dependent on \( \sigma_e \). The dependence on \( a/\bar{W} \) is polynomial. As most of the higher order terms are negligibly small, the displacement could be modelled by a cubic equation, shown in Figure 7.4.
This displacement relates to $\sigma_2$ and $K_I$ and is a component of the displacement $\delta_2$ perpendicular to the crack. It does not affect the displacement $\delta_1$ parallel to the crack.

The relative magnitudes of the uncracked and elastic cracked displacements in the y direction for a centre cracked plate with $H/W = 2$ are shown in Figure 7.5. These values are independent of the magnitude of the applied loading.

Both Figure 7.5 (a) and (b) show that for increases in both crack width $a/W$ and biaxial load ratio $B$, the elastic cracked displacement increases in value.
relative to the uncracked displacement until B = 2, after which the uncracked displacement and therefore the ratio become negative.

This is due to the relationship between B and the uncracked displacement ratio B’ shown in Figure 7.3.

In terms of the relative magnitude of the uncracked and elastic cracked displacements, the value of $\delta_{e(i)}^c/\delta_{e(2)}^{uc} \geq 1$ for $a/W = 0.5$ when $1.75 \leq B \leq 2.25$ and for $a/W = 0.6$ when $1.5 \leq B \leq 2.25$.

In other words, for the lower values of crack width $a/W$, and for the higher values of $a/W$ with $B$ outside the ranges above, the uncracked displacement is higher in value than the elastic cracked displacement.

For higher crack widths (not shown) the ranges of $B$ for which the elastic cracked displacement is higher in value than the uncracked displacement is $1.25 \leq B \leq 2.25$ for $a/W = 0.7$, $0.5 \leq B \leq 2.25$ for $a/W = 0.8$ and $-0.5 \leq B \leq 2.25$ for $a/W = 0.9$.

### 7.1.3 Plastic cracked body displacement $\delta_{pl}^c$

R6 equation (II.14.5) gives the expression for plastic cracked body displacement shown here in equation (7.17).

$$\delta_{pl(i)}^c = (2t/E') \int_0^a \frac{Kk_{(i)} (1 - [f(L_r)]^2)}{[f(L_r)]^2} \left[ \frac{K^2 (df/dL_r)(dL_r/dP(i))}{[f(L_r)]^3} \right] da$$

(7.17)

where $f(L_r)$ is the failure assessment curve (FAC) and $P(i)$ is the applied unit loading.

For the R6 Option 1 FAC, R6 equation (I.6.3) is reproduced here in equation (7.18).

$$f_1(L_r) = (1 + 0.5L_r^2)^{-1/2} \left[ 0.3 + 0.7e^{(-0.6L_r)} \right]$$

(7.18)

where $L_r$ is the proximity to plastic collapse as described in chapter 5, section 5.5 and is calculated using equation (7.19).
\[
L_r = \frac{\sigma_2}{\sigma_{LL}} = \frac{\sigma_2}{\sigma_y \left( \sigma_{LL}/\sigma_y \right)}
\]

(7.19)

where \(\sigma_{LL}\) is the limit load and \(\sigma_y\) is the yield stress.

Here, the limit loads (in terms of a stress) are estimated using the lower bound limit load for centre cracked plates as described in chapter 5 section 5.2.1.

\[
\frac{\sigma_{LL}}{\sigma_y} = \left(\frac{\sigma_2}{\sigma_y}\right)_L \frac{2}{\sqrt{3}} \min \left[ \frac{1 - a/W}{1 - B(1 - a/W)}, \frac{1}{|B|} \right]
\]

(7.20)

The complexity of the integration in equation (7.17) leads to the use of numerical rather than direct integration. Each term of the integrand has been calculated for various values of \(a/W\), \(\sigma_2\) and \(B\) using equations (7.10), (7.12), and (7.18) – (7.20).

The value of the plastic cracked component of displacement can then be approximated by calculating the total area under the curve of the integral vs half normalised crack width \(a/W\) as \(a\) increases from 0 to the chosen value of \(a/W\).

The resulting plastic displacements \(\delta_{pl}^c\) are dependent on geometry, crack width \(a/W\) and proximity to plastic collapse \(L_r\). These dependencies are all non-linear, as shown in Figure 7.6.

Equation (7.17) for the plastic cracked component of displacement is based on the SIF, which is independent of the biaxial stresses. The dependence of the displacement on \(L_r\), which is itself dependent on the biaxial stress ratio \(B\), means that there will be a relationship between \(B\) and \(\delta_{pl}^c\). This relationship is also non-linear.
In Figure 7.6 (a), the displacement increases gently at first then rapidly as the stress approaches the limit load. The dependence of displacement on $a/W$, (b), follows a similar pattern and increases rapidly as the applied remote normal stress, $\sigma_2 = 0.5\sigma_y$, is nearer the limit load as the greater crack sizes have lower limit loads (all other parameters being equal).
Figure 7.7 shows a comparison between the uncracked body displacement, elastic displacement due to the crack and plastic displacement due to the crack as it varies with proximity to limit loading for a centre cracked plate with H/W = 2 and a/W = 0.6.

For the smaller B, Figure 7.7 (a), the uncracked displacement is greater than the elastic cracked displacement which is greater than the plastic cracked displacement until the loading nears the limit load after which the plastic cracked displacement increases rapidly. For the larger B, Figure 7.7 (b), the uncracked and elastic cracked displacement are nearly identical and again the plastic cracked displacement is lower until the limit load is approached.

7.2 Comparison of Displacement Controlled Loading and Stress Controlled Loading of Centre Cracked Plates

FEA were carried out on elastic-perfectly plastic centre cracked plates. The displacement boundary conditions were determined by adding the elastic uncracked displacement to the elastic cracked displacement (sections 7.1.1 and 7.1.2). For these analyses, the plastic displacements were not included in the applied initial displacement as they are not linearly dependent on loading.

The analyses were carried out for plates with H/W = 2 and a/W = 0.2 with biaxial stress ratios B = -1, B = 0, B = 0.5 and B = 1 whose corresponding
biaxial displacement ratios are \( B' = -0.489, B' = 0, B' = 0.044 \) and \( B' = 0.473 \) respectively.

The FEA output of stresses is displayed in Figure 7.8. The ratio of stresses \( S_{11}/S_{22} \) is the ratio of the average stress along the right edge in the direction parallel to the crack to the average stress along the top edge in the direction perpendicular to the crack. This is plotted against the normalised applied displacements along the top edge. The normalising value \( \delta_y \) is the value of displacement \( \delta_2 \) when the stress \( \sigma_2 = \sigma_y \).

![Figure 7.8: FEA output of average stress along the free edges of a centre cracked plate with a/W = 0.2 subject to displacement controlled loading](image)

The biaxial ratios labelled are the stress controlled loading equivalent ratios.

Figure 7.8 shows that the correct stress ratios are reproduced at the edges for each of the DCL ratios throughout the FEA and for all values of \( B' \) and \( B \) tested. In the case where the applied displacement ratio \( B' = 0.044 \), there is a slight increase above \( B = 0.5 \) for values of \( \delta_2/\delta_y \) > 1.

This FEA output demonstrates the accuracy of the uncracked and elastic analyses in sections 7.1.1 and 7.1.2.

The outputs for the J-integrals of the non-negative \( B \) are shown in Figure 7.9. The figure shows that the J-integrals for the DCL now initially follow the same trajectory as their stress controlled counterparts. However, the SCL values of \( J \) increases rapidly as the loading approaches the limit load whereas the DCL values do not. For the DCL, the variation of J-integral with applied displacement at high displacements is approximately linear.
This is to be expected as, for the SCL of a theoretical elastic-perfectly plastic material in-plane strain, the strain at yield, and thus $J$, can increase indefinitely with no further increase in stress. For the DCL, the strain, and thus $J$, increase gradually in line with the applied displacement.

![Figure 7.9: J-integral FEA output for SCL and corresponding DCL](image)

The x-axis showing “time” indicates the steps through the FEA analysis. For the stress controlled analysis, it corresponds to the ratio of remotely applied normal stress $\sigma_2$ (which varies linearly with time from 0 to a multiple of yield stress) to the yield stress $\sigma_y$. For the DCL the time corresponds to the ratio of the displacement $\delta_2$ (also varying linearly with time) to the value of $\delta_2$ when $\sigma_2 = \sigma_y$.

The agreement of the J values at low loading between the SCL and DCL confirms the accuracy of the uncracked and elastic analyses in sections 7.1.1 and 7.1.2.

The other observed difference between SCL and DCL is that for the SCL, the distribution of displacement along the remote free edges of the plate varies with distance from the remote edge and vice versa for the DCL.

Figure 7.10 shows these variations for both SCL and DCL at the maximum applied loading at collapse.

In Figure 7.10, S11 and S22 are the FEA output values for stress fields in the x and y directions respectively and U1 and U2 are displacements in the x and y directions respectively. Note that at 500 mm from the remote edge — (a) and (c) — is the y direction centre line of the plate above the crack.
mouth. At 1000 mm — (b) and (d) — is the centre line of the plate across the crack plane.

Figure 7.10 (a) shows that there is a little variation in the opening stresses normal to the crack with the distance from the remote edge and (b) shows that there is considerably more variation in the parallel stresses Likewise (c) and (d) show small variations along the top edge and considerable variation of the displacements parallel to the crack direction along the right edge.

Figure 7.10: Variation in remote stress at maximum applied displacement controlled loading and remote displacement at maximum applied stress

(a) Stress along the top edge of the plate, (b) stress along the right edge of the plate, (c) displacement along the top edge of the plate and (d) displacement along the right edge of the plate.

In (d), the sudden increase in displacement for both $B = 0$ and $B = 0.5$ occur at the point where the $45^\circ$ slip-line (see chapter 5 section 5.2.2) meets the side of the plate.
7.3 Biaxial Loading Effects on Cruciform Specimens

In this section, cruciform tests carried out by the Bhabha Atomic Research Centre (BARC) in Mumbai are examined to see how biaxial loading influences fracture by plotting predicted failure loads on an R6 failure assessment diagram.

Although the applied loading of the plates appears to be stress controlled, it is in fact the central displacement that is the same for both arms of the cruciform specimen and hence the tests are discussed in the context of the methods set out in this chapter.

For cruciform specimens loaded in this way the biaxial load is applied out-of-plane relative to the crack plane.

7.3.1 Experimental data

The data have been reported by Pawar et al. [70]. Tests were performed in bending on two part-through cracked 20MnMoNi55 cruciform specimens at room temperature (CRRT10 and CRRT11) and four specimens at −70°C (CRSZ10, CRSZ11, CRSZ21-A and CRSZ21-B).

By changing the dimensions and support locations, differing biaxial loading ratios were obtained. Two specimens, CRRT10 and CRSZ10, had a biaxial load ratio \( B = 0 \), two specimens, CRRT11 and CRSZ11, were nominally equibiaxial \( B = 1 \), and two specimens, CRSZ21-A and CRSZ21-B, nominally had \( B = 2 \). Schematic diagrams are shown in Figure 7.11. The specimens have a part through crack with a straight crack front and a normalised crack depth, \( a/W \), of approximately 0.15 for the uniaxial specimens (\( B = 0 \)) and 0.26 for the biaxial specimens (\( B = 1, 2 \)).

The material properties of the 20MnMoNi55 — a reactor pressure vessel steel — were: Young’s modulus \( E = 210 \text{,}000 \text{ MPa} \), Poisson’s ratio \( \nu = 0.3 \); room temperature yield strength \( \sigma_y = 490 \text{ MPa} \), yield strength at −70°C \( \sigma_y = 490 \text{ MPa} \); elastic-plastic initiation fracture toughness at room temperature \( J_{IC} = 250 \text{ kJ/m}^2 \) and at −70°C, \( J_{IC} = 100 \text{ kJ/m}^2 \).
Each specimen was taken to a maximum load, then unloaded and the extent of any crack growth from the pre-existing defect was measured. Further details are contained in [70].

7.3.2 Analysis

In the absence of stress intensity factor and limit load solutions for the particular geometries tested, a method proposed by Zerbst et al. [79] has been adapted to provide the inputs to the failure assessment diagram from FEA results given in [70].

First, the figures of J versus load in [70] were digitised. The elastic $J_e$ was found by fitting $J$ as a quadratic function of load at lower loads to define the stress intensity factor for each specimen. The solutions were found to be very close to those found using the equations derived for cruciform specimens by Hohe et al. [69] although the geometries of the BARC tests are outside the range of application of the solutions in [69].

From $J_e$, values of $J/J_e$ were obtained from the J versus load finite element results for each specimen given in [70]. In [79], a reference load, $F_0$, is taken as the load for which $L_r = 1$, i.e., from the R6 Option 1 failure assessment curve, equation ((7.18), the load for which $K_r = 0.559$ or $J/J_e = 3.2$. However, the values of $J/J_e$ from the analyses in [70] do not exceed 3.05 and hence it was not possible to apply the method of [79] directly.

$F_0$ for each specimen was obtained by calculating, for the value of $J/J_e$ for the maximum finite element load, the corresponding value of $K_r$ and hence, from the R6 Option 1 failure assessment curve, the value of $L_r$. The load for
which $L_r = 1$, i.e. the limit load or reference load, $F_0$, for each specimen, is then the maximum finite element load divided by this value of $L_r$. Results are given in Table 7.1.

The limit load $F_0$ obtained by the method described above is subject to uncertainty when the value of $K_r$ is close to unity, since the R6 Option 1 failure assessment curve is only weakly dependent on $L_r$ in this region. However, at low $K_r$ the FAD assessments are not sensitive to the deduced values of $F_0$. Therefore, for example, although the estimate of $F_0$ for test CRSZ21-B appears high, this does not have a significant influence on the corresponding estimate of initiation load.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Max FE Load (kN)</th>
<th>$J/J_e$ at Max FE Load</th>
<th>$K_r$ at Max FE Load</th>
<th>$L_r$ at Max FE Load</th>
<th>Reference Load, $F_0$ (kN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CRRT10</td>
<td>2560</td>
<td>3.05</td>
<td>0.572</td>
<td>0.99</td>
<td>2586</td>
</tr>
<tr>
<td>CRSZ10</td>
<td>2493</td>
<td>1.37</td>
<td>0.856</td>
<td>0.70</td>
<td>3575</td>
</tr>
<tr>
<td>CRRT11</td>
<td>3576</td>
<td>2.23</td>
<td>0.670</td>
<td>0.91</td>
<td>3926</td>
</tr>
<tr>
<td>CRSZ11</td>
<td>3801</td>
<td>1.53</td>
<td>0.808</td>
<td>0.77</td>
<td>4955</td>
</tr>
<tr>
<td>CRSZ21-A</td>
<td>3011</td>
<td>1.24</td>
<td>0.898</td>
<td>0.61</td>
<td>4927</td>
</tr>
<tr>
<td>CRSZ21-B</td>
<td>3099</td>
<td>1.10</td>
<td>0.952</td>
<td>0.44</td>
<td>7016</td>
</tr>
</tbody>
</table>

The approach described above enables the FAD assessments to be performed as stress intensity factor and limit load solutions have been deduced. The $K_r$ vs $L_r$ loading lines for each specimen have been plotted on an R6 Option 1 FAD in Figure 7.12 using the initiation fracture toughness value given above.
The resulting predicted initiation loads are plotted against biaxial ratio $B$ in Figure 7.13.

The experimental loads and the estimated initiation loads in Figure 7.13 (b) do not vary monotonically with biaxiality. The fracture initiation loads for $B = 0$ and $B = 2$ are similar, while those for $B = 1$ are higher.

These effects are the same in terms of their qualitative dependence on $B$ as those in chapter 5 derived from upper and lower bound limit load analyses. Specifically, negative and high positive values of $B$ lead to increased plasticity and hence increased crack driving force for a given load normal to
the defect, relative to the uniaxial case, whereas $B = 1$ leads to a reduced crack driving force.

Therefore, the non-monotonic variations with $B$ for the experimental and predicted initiation loads for the cruciform specimens are consistent with the general trends of crack driving force with biaxiality.

For predicting fracture response, the material resistance to fracture is important as well as the crack driving force. The influence of biaxiality on fracture toughness may be considered by using the approach in R6 Section III.7 and here in chapter 6 section 6.2 where the constraint dependent fracture toughness, $K_{mat}^c$, is related to the normal fracture toughness, $K_{mat}$ by equation (6.5).

$$
\begin{align*}
K_{mat}^c &= K_{mat} \quad \beta L_r > 0 \\
K_{mat}^c &= K_{mat}[1+\alpha (-\beta L_r)^m] \quad \beta L_r < 0
\end{align*}
$$

(7.21)

where $\beta$ is a normalised measure of the constraint due to the loading and $\alpha$ and $m$ are material and temperature dependent constants.

The constraint parameter $\beta$ is defined here by the elastic $T$-stress (equation (7.22)) as it has been shown in chapter 6 that this is a reasonable approach for loads for which $L_r < 1$.

$$
\beta = \frac{T}{L_r \sigma_y}
$$

(7.22)

Thus the constraint depends on the value of $L_r$. At low $L_r$, constraint effects are smaller than at high $L_r$ as $\beta L_r$ is proportional to $T$ which is proportional to load.

For the cruciform tests at low temperature, the fracture toughness is relatively low and fracture initiation occurs at low values of $L_r (< 0.77)$. Therefore, constraint effects would be expected to be small and the effects
of biaxial loading would be expected to be dominated by the effects on crack driving force via the effects on \( F_0 \).

For the two cruciform tests at room temperature, a significant effect of biaxiality on limit load and predicted initiation load is predicted (45%) without any effect of constraint. The observed increase (12%) in maximum experimental load is rather smaller but no ductile crack growth was observed in the specimen with \( B = 1 \). Therefore, the results are not inconsistent with the predictions or with generally expected trends.

Finally, it is to be noted that the loading in the tests leads to a central displacement in the cruciform specimen. The loading in each arm of the specimen is then controlled by the relative stiffnesses of the two arms. The earlier analyses in this chapter show that these stiffnesses depend on crack size and the extent of plasticity. Therefore, apart from the uniaxial case where \( B = 0 \), the biaxiality ratios quoted from [70] can be regarded as approximate. However, the general trend of a higher failure load in the region of \( B = 1 \) for cases where significant plasticity occurs is expected to generally hold.

### 7.4 Summary

Section 7.1.1 demonstrated how the elastic displacements of the remote edges of an uncracked plate, in-plane strain and under applied remote biaxial stress, are not generally in the same proportion as the stresses. The biaxial stress ratio \( B \) and the biaxial displacement ratio \( B' \) can differ by a considerable margin. For the case shown, \( B \) is 11 times the value of \( B' \).

In section 7.1.2, for a cracked plate, additional elastic displacements are dependent on the plate and crack geometries and, as the displacement is dependent on SIF, the stress in the direction normal to crack opening, \( \sigma_2 \), only.

In section 7.1.3 values for plastic displacement were determined. They are dependent on plate and crack geometries and in this case, also on the biaxial stress as they are dependent on limit loads. However the displacements calculated are those in the normal direction only, \( \delta_2 \). The values of these displacements are much lower than uncracked and elastic cracked displacement until the loading stresses start to approach the limit load.
In section 7.2 FEA were carried out to illustrate DCL compared with SCL and the distribution of stresses along the remote edges when DCL is applied.

Section 7.3 examined the effect of biaxial loading on cruciform specimens and found that the main effects are on limit load and crack driving force and lesser effects on constraint and fracture toughness. The exact relationship between the biaxial loading ratio and these effects could not be established as the displacement controlled loading meant that the ratios quoted in [70] can only be regarded as approximate.

### 7.5 Conclusions

Values of uncracked elastic, cracked elastic and cracked plastic displacement and biaxial displacement ratio have been calculated for centre cracked plates under plane strain biaxial loading conditions. Plates under SCL and DCL have been compared.

- Biaxiality when measured as a ratio of displacements is not the same value as the ratio when measured as a ratio of stresses.

- A numerical relationship between B (biaxial stress ratio) and B’ (biaxial displacement ratio) can be established in the elastic region.

- The stress is non-uniform under displacement controlled loading and the displacements are non-uniform under stress controlled loading. Thus, there are still differences between displacement controlled and stress controlled loadings in the elastic regime even if the elastic relationship between B and B’ is used to match the biaxialities. When plasticity occurs, the stress biaxiality changes under fixed displacement biaxiality, and vice versa.

- These differences could have implications in the interpretation of some experimental research where the loading is displacement controlled.
Summary & Discussion

The application of the output of this research is to electricity generation and the role of nuclear power in the UK supply. The potential extension of the life of existing nuclear reactors requires the determination of the safety of the equipment and structures in the form of structural integrity assessments.

Fracture assessments used throughout the nuclear industry and beyond are generally carried out in accordance with procedures such as R6 [3], a defect assessment procedure developed in 1976 by the CEGB and currently managed and maintained by EDF Energy.

As the current assessment procedures generally contain little advice on biaxial loading, fracture assessments have been historically carried out under the assumption that an assessment based on the uniaxial loading only will be conservative.

The objective of this research has been to examine structures with defects subject to biaxial loading, with the aim of examining the influence of biaxial loading on the fracture assessments of these structures.

The assessment procedures are based on a failure assessment diagram (FAD). The FAD is a graph with the horizontal axis showing proximity to plastic collapse $L_r$ and the vertical axis showing proximity to elastic fracture $K_r$. The failure assessment curve (FAC) is a function of $L_r$. For each component considered, an assessment point $(L_r, K_r)$ is plotted on the graph and is considered safe if it lies within the area bounded by the two axes and the FAC.

The proximity to plastic collapse is measured by the ratio of the applied loading to the limit load. The proximity to elastic fracture is measured by the ratio of the stress intensity factor (SIF) to the material fracture toughness.

The effect of biaxial loading on assessments has been determined by its influence on limit loads, SIF, crack driving force (CDF), stress fields and fracture toughness as well as on their interactions and the overall effect on the position of the failure locus in the FAD.
8.1 Limit loads

Chapter 5 examined the effects of biaxial loading on the limit loads of centre cracked plates (CCP) both long and short edged, single and double edge cracked plates and cylinders.

Analytical solutions to the limit loads were derived using von Mises equilibrium stress fields for the lower bound solutions and slip-lines for the upper bound solutions. Numerical solutions were found using FEA.

Each of the three methods led to solutions, for all the geometries investigated, that varied with biaxial loading ratio $B$ in a non-monotonic pattern, similar to a bell curve, with a peak at around equibiaxial loading ($B = 1$).

For negative $B$ and high positive $B$, the upper bound, lower bound and FEA solutions derived for the CCP gave identical values and therefore exact solutions. As the biaxial loading approaches equibiaxiality, the solutions are more widely spread with the FEA output falling between the two analytical values.

In the literature, the investigations into the effects of biaxial loading on the limit loads of plates are based on von Mises yield criteria [29, 41] and FEA [29, 42], with only [41] looking at values of $B < 0$ and $B > 1$.

O’Dowd et al. [29] derived a lower bound solution for the CCP that is identical to the first part of the solution found in this research (equation (5.4)) using the stress above the ligament and a minimum for the lower values of $B$. They identified, and confirmed using FEA, that the formula is only valid for values of $B < \frac{1}{2}(1 - \frac{a}{W})$. For $\frac{a}{W} = 0.5$, this is at $B = 1$.

In the research presented here in chapter 5, a lower bound solution which also includes solutions for $B \geq \frac{1}{2}(1 - \frac{a}{W})$ has been derived, and confirmed using FEA. The solution, which in the case of CCP is based on the stress above the crack, is a multiple of $\frac{1}{|B|}$. This function is also part of the lower bound solutions for single and double edge cracked plates and the upper bound solution for double edge cracked plates for high positive values of $B$.

FEA for plates with surface cracks [42] showed that for values of biaxial loading ratios 0, 0.5 and 1, for a material with yield stress of 593 MPa the limit loads were, respectively, 557 MPa ($0.94 \sigma_y$), 680 MPa ($1.14 \sigma_y$) and 592
MPa (1.00 \sigma_y). These follow the same pattern as the CCP solutions as does the analytical solution presented in [41] for plates with extended surface cracks.

The large-scale tests on A533B steel plates [17] identified that for biaxial loading ratios of 0.5 and 1 there was a delay in the onset of plasticity under biaxial loading conditions when compared with uniaxial. This indicates higher limit loads for biaxial loading which partially supports the findings here. Full confirmation would be achieved by similar tests with higher biaxial ratios leading to an onset of plasticity at lower loads than for those required for a plate under equibiaxial loading conditions.

Solutions for the short CCP are more complex than the long plate solutions as they are dependent on the height of the plate H as well as the crack width a/W and biaxial ratio B. Exact solutions have been found where the lower bound, upper bound and FEA solutions are identical.

For the short plate and crack sizes examined, the upper bound estimates were an exact match to the FEA for all values. However, there are cases where for the same combination of B and a/W, the lower bound solution for the long plate gives a lower value of limit load than the short plate solution and therefore the former must apply.

The analysis of the single and double edge cracked plates also produced solutions where the upper bound estimates were a closer match to the FEA than the lower bound solutions. Some of the lower bound solutions were higher than the FEA, this was due to the R6 solutions on which they were based not being a strict lower bound.

Limit loads for cylinders are available in R6. The section considering cylinders has been included in this research to examine the relationship between the values of the biaxial loading ratio — the ratio of the hoop stress to the axial stress — and the limit loads. Again, the bell shape relationship peaking at around B = 1 still holds for all the crack widths, crack types and wall thicknesses examined.

The effect on the failure load and failure crack width for CCP, in applying the lower bound solutions of limit load to the FAD, is that the failure loads assessed using uniaxial loads could be unduly conservative and underestimated by up to 45%. Conversely, they could also be overestimated by up to 60%.
8.2 Fracture Parameters and Constraint

Basic FEA confirmed that the addition of biaxial loading had no effect on the SIF, in agreement with [16-18, 52].

FEA of centre cracked and double edge cracked plates showed that the CDF $J$ is influenced by biaxial loading, again in a non-monotonic fashion. For negative and high positive $B$, the value of $J$ is higher than that for uniaxially loaded plates under the same value of loading perpendicular to the crack. For values of $B$ close to equibiaxiality, $J$ is lower than the uniaxially loaded plates.

This is the inverse of the pattern in the limit loads. Lower limit loads lead to an earlier onset of plasticity and an increased CDF and vice versa.

The decrease in $J$ for biaxially loaded surface cracked plates was also found by Lei et al. [41]. The values of biaxial load ratio $\lambda_1$ applied in the FEA were $\lambda_1 = -1$, $\lambda_1 = 0$ and $\lambda_1 = 0.5$. Values of $J$ for a particular value of applied opening loading decreased as $\lambda_1$ increased, which is to be expected with the increase in limit load over these values of $\lambda_1$.

In Wright et al., [25], the biaxial loading ratio $\sigma_1 = 1.35 \sigma_2$ was applied in both the experimental large-scale plate tests and the accompanying FEA. In this case the $J$-integral values were higher for the biaxially loaded plates. This is unexpected as the limit loads for the biaxially loaded plates were theoretically higher than those for the uniaxially loaded plate.

This effect of the variation in $J$ with variation in biaxial load ratio is minimised by normalising the applied loading by the limit load for each value of $B$ and $J$ by $J_e$ the elastic $J$-integral. This minimising effect was also found by Kim et al. [39].

The above results suggest that the FAC, once limit load has been accounted for, is weakly dependent on biaxial loading. The effects on the FAC of the CDF under varying biaxial loads occur for a narrow range of $B$, and overestimate or underestimate the failure load by about 1%, considerably less than the effect of biaxiality on the limit loads.

The level of constraint around the crack tip can affect the fracture toughness of a component. This effect has been quantified by Ainsworth and O’Dowd [53]. The influence of biaxial loading on fracture toughness is
driven by the increase or decrease in constraint relative to the uniaxially loaded structure.

The calculation of the constraint involves the calculation of the constraint parameter \( \beta \), which is a function of the parameter \( Q \). \( Q \) can be determined using FEA or, as [29], [57] and section 6.2.1.3 in chapter 6 demonstrated, \( Q \) can be estimated using the T-stress. While there is little advice in R6 regarding the T-stress for biaxially loaded specimens, the additional T-stress due to biaxial loading can be calculated simply by superposition — adding the applied biaxial stress on to the T-stress determined for the uniaxially loaded specimen.

There is a narrow range of values of \( B \) that can affect the failure loads due to an improved fracture toughness. Low negative and high positive values of \( B \) lead to structures with lower limit loads which, for a sufficiently high fracture toughness, will tend to fail by plastic collapse and will not be affected by a higher fracture toughness.

High positive values of \( B \) lead to higher constraint and an estimate of zero for the \( Q \) stress and \( \beta \) and thus no effect on the fracture toughness. Where there is an effect, the failure load can increase by around 60% with the increased fracture toughness for a lower constraint under negative values of \( B \) (compression).

### 8.3 General Comments on Biaxial Loading and Failure

It can be seen that the most significant influence on the fracture assessment using the FAD of the effect of biaxial loading is its effect on the limit load. The other parameters, CDF and fracture toughness, are also affected by biaxial loading. In the case of the CDF the effect is largely due to the effect on limit loads, and in the case of fracture toughness the effect on failure load is only for a narrow range of \( B \).

Theoretically, the constraint modified fracture toughness for all positive values of \( B \) is less than or equal to that for \( B = 0 \) and thus the biaxial loading should lead to a decrease in fracture toughness.

In experimental practice, the effect of the addition of an in-plane biaxial load to plates is not consistent and has been shown to decrease crack
growth resistance [17], to increase fracture toughness or increase to a peak then decrease depending on the material [27], and to unexpectedly increase fracture resistance [25].

In cruciform specimens, the biaxial loading is generally out-of-plane and leads to an increase in constraint and reduction in fracture toughness [21, 23, 24].

The differences in fracture response could be due to the differing effects of biaxial loading on CDF and fracture toughness and depend on which dominates. Equibiaxial loading, for example, has been shown to lower CDF, via the increased limit load and decreased plasticity, but at the same time lower fracture toughness through increased constraint. This could lead to either a reduction or an increase in the load required to initiate crack growth depending on which effect is the stronger.

Otherwise the differences could be down to the type of fracture (brittle or ductile) or material (elastic or elastic-plastic) for which the effects of constraint on fracture toughness are different [80].

### 8.4 Displacement Controlled Loading

Significant differences were observed between FEA results when the displacement boundary conditions were applied to the model in the same proportions as the stresses. This led to a detailed analysis of the effects of biaxial displacement controlled loading.

Based on R6 [3] equations (III.14.1), (III.14.3) and (III.14.4), values for uncracked body displacement and elastic and plastic displacements due to the crack were determined for various values of crack width a/W and biaxial loading B.

These calculations show that the displacements are not in the same proportions as the stresses. For the uncracked displacement, the effect is due to the Poisson’s ratio, which for most steels is 0.3.

The elastic displacement due to the crack is based on the SIF, K, which for a CCP is a function of stress perpendicular to the crack, crack width a and normalised crack width a/W. It is thus independent of the biaxial loading parallel to the crack and its addition to the uncracked displacement
generally leads to the biaxial displacement ratio diverging further from the biaxial stress ratio.

The plastic displacement due to the crack is also dependent on the SIF and indirectly on biaxial load ratio $B$ via $L$, which is a function of the limit load and dependent on $B$. The plastic cracked displacements are less than the uncracked and elastic cracked displacements until the plate is close to yield at which point the plastic cracked displacement rises rapidly.

As the plastic cracked displacement is not proportional to loading, it is not straightforward to add it on to the uncracked and elastic cracked displacement when using FEA to verify the biaxial displacement formulae. However, the FEA carried out without the plastic cracked displacement show the applied displacements replicate the applied stresses from which they are calculated and the associated $J$-integrals for the biaxial stresses.

It was also observed that for a fixed applied stress in a stress controlled loading FEA, the displacements along the free surface of the plate, particularly the edge perpendicular to the crack, varied with distance from the crack plane. Likewise, for displacement controlled loading, the stress along the free surface varied with distance from the crack plane.

The main implications for these findings will be when interpreting outcomes of experimental procedures where the initial biaxial load ratios are deduced from the ratio of applied displacements and could thus be an inaccurate indicator of biaxial stress ratio.

### 8.5 Summary

This thesis has examined the effects of load biaxiality on fracture response through the effect of biaxial loading on four parameters; the elastic SIF, the limit load, the CDF via the FAC and the fracture toughness.

For stress controlled biaxial loading the SIF is a function of the loading normal to the crack only and is independent of the biaxial loading.

The limit load has a strong non-monotonic dependence on the biaxial stress ratio $B$, with the limit load increasing with increasing $B$ up to around $B = 1$ and thereafter decreasing with increasing $B$.

The dependence of the CDF, $J$, on biaxiality is defined largely by the dependence of limit load on biaxiality. It is related to SIF through the
dependence of the FAC on limit load. Thus \( J \) is dependent on the proximity to plastic collapse \( L_r \). As limit load increases, \( J \) at a given loading decreases. The reduction is greater for a greater level of plasticity.

The dependence of fracture toughness on in-plane biaxiality is through the effect of constraint on fracture toughness. Increasing biaxiality generally leads to increasing constraint and a reduction in fracture toughness. As biaxial loading increases, the T-stress — shown to estimate constraint parameter \( Q \) — increases. However, once the T-stress becomes \( \geq 0 \), the estimate of \( Q \) does not increase above 0 and there is little further effect of constraint nor further decrease in fracture toughness.

Thus the overall dependence of fracture load on biaxiality is complex. Increased biaxiality can lead to either a reduction or an increase in the load required to initiate crack growth depending on the level of plasticity and the competing effects of biaxiality on constraint and the limit load.

Differing effects are therefore predicted for different biaxial load ratio \( B \) depending on whether \( B \) is less than zero (compressive), less than 1 (equibiaxial loading) or greater than 1.

8.5.1 Negative biaxiality, \( B < 0 \)
For negative biaxial load ratios, with \( B < 0 \), the limit load is less than that for uniaxial loading (\( B = 0 \)) and reduces with increasingly negative \( B \). For large-scale plasticity with constant \( L_r = 1 \), the failure load is equal to the limit load and thus decreases with the increasingly negative \( B \).

At modest levels of plasticity, say with \( L_r < 0.5 \), the failure load increases with negative biaxiality due to the higher loss of constraint.

At intermediate levels of plasticity, the failure load can either increase or decrease with increasingly negative biaxiality due to the competing effects of the reduction in limit load and the increase in constraint loss.

8.5.2 Positive biaxiality, \( 0 < B < 1 \)
For positive biaxial load ratios with \( 0 < B < 1 \), the limit load is greater than that for uniaxial loading and increases with increasing \( B \). For large-scale plasticity with constant \( L_r = 1 \), the failure load is equal to the limit load and thus increases with increasing \( B \).
At modest levels of plasticity, the failure load decreases with increasing biaxiality due to the associated increase in constraint.

At intermediate levels of plasticity, the failure load can either increase or decrease with increasingly negative biaxiality due to the competing effects of the increasing limit load and the increase in constraint.

8.5.3 Positive biaxiality, $B > 1$

For positive biaxial load ratios with $B > 1$, the limit load decreases with increasing $B$, so at large-scale plasticity the failure load decreases with increasing biaxiality.

At modest levels of plasticity, the failure load also decreases with increasing biaxiality due to the associated increase in constraint.

The higher constraint levels associated with the higher biaxial loads are unlikely to lead to further reduction in fracture toughness. Thus for intermediate levels of plasticity the effect of the decrease in limit load will dominate and the failure load is likely to decrease with increasing biaxiality.

8.5.4 Displacement controlled loading

The above discussion has been in the context of stress controlled biaxial loading. However, the thesis has also examined displacement controlled biaxial loading. The displacement biaxiality ratio differs from the stress biaxiality ratio and depends on plate size, crack size and Poisson’s ratio.

It has been shown that the two ratios can be related in the elastic regime independently of load. However, for a uniform stress boundary condition in stress controlled loading, the associated boundary displacements are non-uniform and for uniform displacement boundary conditions, the associated boundary stresses are non-uniform.

Therefore the SIF based solutions may differ for the two different boundary conditions even for the same remote load. When plasticity occurs, it has been shown that the displacement biaxiality ratio may be related to the stress biaxiality ratio but the difference between the two ratios depends on the level of plasticity. This complicates the interpretation of experimental data where the biaxial load ratios are deduced from the ratio of applied displacements which can be a variable indicator of stress biaxial loading ratio.
9 Conclusions

The objective of this research was to investigate the effects of biaxial loading on fracture assessments with the aim of providing guidance that can be added to the R6 assessment procedure.

9.1 Effects of Biaxial Loading

This thesis has examined the effects of biaxial loading on stress intensity factor, limit load, crack driving force, constraint and fracture toughness. The analyses herein have demonstrated that the effects of biaxial loading are not always straightforward.

Each parameter has a different relationship with the biaxial load ratio B and most are dependent on the magnitude of B.

- Mode I stress intensity factor $K_I$ is independent of B.
- Limit load is non-monotonically dependent on B with a bell-shaped relationship peaking at approximately $B = 1$.
- Crack driving force $J$ is dependent on B however this dependence is weak once the effects have been normalised in terms of the ratio of applied load to the limit load. Then, values of B for which the limit load is higher generally relate to a lower crack driving force for a given load.
- Constraint increases with an increase in B and vice versa.
- The effective fracture toughness can increase with decreasing B and vice versa due to the effects of constraint loss. When B is sufficiently high that the constraint becomes positive, the higher constraint does not affect the fracture toughness. Thus, the effect of biaxial loading on fracture toughness, while numerically significant, is only applicable to a narrow range of values of B.

Each of these relationships has a different impact on the FAD and hence on estimated failure load, failure mode and failure crack size. The interactions between the effects on limit load, CDF and constraint also contribute to these determinations of failure.

- Displacement controlled loading leads to a ratio between the applied orthogonal displacement boundary conditions that is not the same as the
ratio of the equivalent stress boundary conditions. This can have implications when experimental loading is displacement controlled.

9.2 Solutions

- Limit load solutions have been found for elastic–perfectly plastic centre cracked plates, single edge cracked plates and double edge cracked plates for a wide range of values of biaxial ratio $B$ and crack width $a/W$.

- T-stress solutions for biaxially loaded plates can be determined by adding the biaxial stress onto the T-stress for the equivalent uniaxially loaded specimen.

- Estimations of constraint $Q$ for biaxially loaded specimens can be made using the T-stress for $T < 0$ and an estimate of 0 for $T ≥ 0$.

- A constraint modified fracture toughness can be calculated using the constraint parameters estimated using $T$ and $Q$ as above.

- Equations have been derived relating the biaxial ratio for displacement controlled loading, $B'$, to the biaxial ratio for stress controlled loading $B$.

9.3 Overall Implications for Industrial Application

The influences of biaxial loading are as described above, and solutions have been presented to quantify these. The greatest influence is that on the limit loads, and it is for these that the solutions here are most relevant and caution needs to be taken if uniaxial loading is assumed.

In particular, where the biaxial loading is negative (compressive) or, for most crack sizes and geometries, the biaxial stress ratio is greater than 1, the limit load is less than that for uniaxial loading and thus the assumption of uniaxial loading will be non-conservative. In practice this occurs, for example, in circumferentially cracked closed-ended cylinders under pressure where the hoop stress is twice the axial stress.

9.4 Further work

- Expand the analysis of the effect of $B$ on the J-integral and CDF to include more non-linear elastic and power law hardening materials.

- Refine some of the limit load estimates for single and double edge cracked plates where the estimates are not strictly lower bound.
• Continue the work on cylinders under multiple loadings to include comparisons of the limit load and constraint estimates with the equivalent centre cracked plate (for through-wall cracks) and single edge cracked plate (for fully circumferential cracks) solutions.

• Develop experimental procedures for testing the effects of biaxial loading which can separate the competing effects found in this research.

• Investigate the effects of biaxial loading on crack tip opening displacement (CTOD) and crack mouth opening displacement (CMOD).

• Examine the extent to which the results developed here can be used to determine the effects of biaxial loading on creep fracture assessments.
References


Appendix A  Validation of FEA models

Centre cracked plate

The results of the FEA of the centre cracked plate (CCP) subject to uniaxial tensile loading perpendicular to the crack tip were checked against stress field equations in the literature.

Westergaard [15] developed equations for describing stress and displacement fields ahead of a crack tip. Equations (A.1) and (A.2) are those for the stress parallel to and perpendicular to the crack respectively.

\[
\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right]
\]  
(A.1)

\[
\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right]
\]  
(A.2)

where \(K_I\) is the mode I stress intensity factor and \((r, \theta)\) are polar coordinates ahead of the crack tip.

Anderson [7] describes how these Westergaard equations are only applicable for short distances from the crack tip. Equations adapted for CCP, with reference to solutions published by Irwin [13], Sneddon [81], Williams [14] and Sih [82], are given in Example 2.7 [18] and shown here in equations (A.3) and (A.4).

\[
\sigma_{xx} = \frac{\sigma(a + r)}{\sqrt{2ar + r^2}} - \sigma
\]  
(A.3)

\[
\sigma_{yy} = \frac{\sigma(a + r)}{\sqrt{2ar + r^2}}
\]  
(A.4)
where \( a \) is the crack width and \( r \) is the linear polar coordinate.

Figure 9.1 shows output of stress versus distance from the crack tip, \( r \), for the CCP subject to uniaxial tensile loading perpendicular to the crack tip with crack widths \( a/W = 0.2 \) and \( a/W = 0.6 \). The three different curves are the FEA output and the two sets of stress field equations described above (equations (A.1)–(A.4)).

The stress intensity factor \( K_I \) used in equations (A.1) and (A.2) has been calculated using the R6 Section IV.3.3.5 compendium, and shown here in equation (A.5).

\[
K_I = \sigma \sqrt{\pi a} \left\{ 1 - 0.025 \left( \frac{a}{W} \right)^2 + 0.06 \left( \frac{a}{W} \right)^4 \right\} \left[ \sec \left( \frac{\pi a}{2W} \right) \right]^{1/2}
\]

where \( a \) is the absolute measure of the crack width and \( a/W \) the normalised crack width. The value of \( \sigma \) is the applied loading.

The \( T \)-stress, calculated using the R6 compendium Section IV.5.4.1 (shown here in equation (6.10)), has been added to the Westergaard equations and shown in Figure 9.1 (b) and (d) to more accurately reflect the stress in the direction parallel to the crack.

\[
T = \frac{\sqrt{3} \sigma_2}{2 \left( 1 - \frac{a}{W} \right)^2} \left\{ -1.1547 + 1.1511 \left( \frac{a}{W} \right) - 0.7826 \left( \frac{a}{W} \right)^2 + 0.4751 \left( \frac{a}{W} \right)^3 \right\} - 0.1761 \left( \frac{a}{W} \right)^4
\]

where \( \sigma_2 \) is the remotely applied tensile stress and \( a/W \) the normalised crack width. This is for the uniaxial \( (B = 0) \) case only.

The FEA curves in Figure 9.1 are a very good fit to the Anderson equation curves for both crack widths for the stress \( \sigma_{xx} \) parallel to the crack and for \( a/W = 0.2 \) in the case of the stress \( \sigma_{yy} \) perpendicular to the crack. For
\[ a/W = 0.6 \] in the case of the stress \( \sigma_{yy} \) perpendicular to the crack, the Westergaard equation is a better fit.

Figure 9.1: Stress ahead of the crack tip for a centre cracked plate comparing FEA output of stress to Anderson and Westergaard

(a) Stress in the direction perpendicular to the crack with \( a/W = 0.2 \), (b) Stress in the direction parallel to the crack with \( a/W = 0.2 \), (c) for stress in the direction perpendicular to the crack with \( a/W = 0.2 = 0.6 \) and (d) Stress in the direction parallel to the crack with \( a/W = 0.6 \).
Appendix B  Derivation of Formulae

Long plate forward slip-line upper bound solution

Equations (5.5) and (5.6) combined:

\[
B(\sigma_2)_L \dot{v} \cos \theta [H - (W - a) \tan \theta] + (\sigma_2)_L 2W \dot{v} \sin \theta
- B(\sigma_2)_L \dot{v} \cos \theta [H + (W - a) \tan \theta] = \frac{2\dot{v}(W - a)}{\cos \theta} \frac{\sigma_y}{\sqrt{3}}
\]  
(8.1)

\[
(\sigma_2)_L \left[ BH \cos \theta - BW \sin \theta + Ba \sin \theta + 2W \sin \theta 
- BH \cos \theta - BW \sin \theta + Ba \sin \theta \right] = \frac{2(W - a) \sigma_y}{\cos \theta \sqrt{3}}
\]
(8.2)

\[
2(\sigma_2)_L \sin \theta \left[ Ba - BW + W \right] = \frac{2(W - a) \sigma_y}{\cos \theta \sqrt{3}}
\]  
(8.3)

\[
(\sigma_2)_L = \frac{(W - a)}{\sin \theta \cos \theta \left[ Ba - BW + W \right] \sqrt{3}} \sigma_y
\]  
(8.4)

Using the double angle formula,

\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]  
(8.5)

\[
(\sigma_2)_L = \frac{(W - a)}{\sin 2\theta \left[ Ba - BW + W \right] \sqrt{3}} \frac{2\sigma_y}{\sqrt{3}}
\]  
(8.6)
\[
(\sigma_2)_L = \frac{(1 - a/W)}{\sin 2\theta} \frac{2\sigma_y}{\sqrt{3} [B a/W - B + 1]}
\]  
(B.7)

\[
(\sigma_2)_L = \frac{2\sigma_y}{\sqrt{3} \sin 2\theta} \frac{(1 - a/W)}{[1 - B (1 - a/W)]}
\]  
(B.8)

which is Equation (5.7).

**Long plate reverse slip-line upper bound solution**

Equation (5.9):

\[
B(\sigma_2)_L \dot{v} \cos \theta [H + (W + a) \tan \theta] - (\sigma_2)_L 2W \dot{v} \sin \theta
\]

\[
- B(\sigma_2)_L \dot{v} \cos \theta [H - (W + a) \tan \theta] = \frac{2\dot{v}(W + a) \sigma_y}{\cos \theta \sqrt{3}}
\]  
(B.9)

\[
(\sigma_2)_L [BH \cos \theta + BW \sin \theta + Ba \sin \theta] - 2W \sin \theta
\]

\[
- BH \cos \theta + BW \sin \theta + Ba \sin \theta] = \frac{2(W + a) \sigma_y}{\cos \theta \sqrt{3}}
\]  
(B.10)

\[
2(\sigma_2)_L \sin \theta [Ba + BW - W] = \frac{2(W + a) \sigma_y}{\cos \theta \sqrt{3}}
\]  
(B.11)

\[
(\sigma_2)_L = \frac{(W - a)}{\sin \theta \cos \theta [Ba + BW - W] \sqrt{3}} \sigma_y
\]  
(B.12)

Using the double angle formula,

\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]  
(B.13)
\[(\sigma_2)_L = \frac{(W - a)}{\sin 2\theta} \frac{2\sigma_y}{\sqrt{3}} \text{[Ba + BW - W] \sqrt{3}} \]  
(B.14)

\[(\sigma_2)_L = \frac{(1 - a/W)}{\sin 2\theta} \frac{2\sigma_y}{\sqrt{3}} \text{[aW/\sqrt{B} + B - 1]} \]  
(B.15)

\[(\sigma_2)_L = \frac{2\sigma_y}{\sqrt{3}} \frac{(1 - a/W)}{\sin 2\theta} \text{[B (1 + a/W) - 1]} \]  
(B.16)

which is Equation (5.10).

**Short plate lower bound limit load solution**

The short plate lower bound solution has been derived under the following assumptions regarding the distribution of stresses in the plate.

- A horizontal stress of $B\sigma_2$ throughout.
- Above the crack, a vertical stress of 0 at the cracked centre line, increasing linearly to $\sigma_2$ at the top and bottom surfaces.
- Above the ligament, a vertical stress of $\sigma_2/ (1 - a/W)$ at the cracked centre line decreasing linearly to $\sigma_2$ at the top and bottom surfaces.
- Above the crack, a shear stress of 0 at the centre of the plate increasing linearly with horizontal distance to a peak value of $a\sigma_2/H$ at the crack tip.
- Above the uncracked ligament, a shear stress of $a\sigma_2/H$ at the crack tip decreasing linearly to 0 at the sides of the plate.

Considering the points with the highest shear stress and using the von Mises equilibrium stress field, the solution can be deduced from the following.
\[ \sigma_{vm} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)} \]

(B.17)

At the crack tip, above the crack,

\[ \sigma_{11} = B\sigma_2 \]
\[ \sigma_{22} = 0 \]
\[ \sigma_{33} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) = \frac{B\sigma_2}{2} \]
\[ \sigma_{12} = \frac{a\sigma_2}{H} \]

(B.18)

\[ \sigma_{vm} = \frac{1}{\sqrt{2}} \sqrt{(B\sigma_2)^2 + \left(\frac{B\sigma_2}{2}\right)^2 + \left(\frac{B\sigma_2}{2}\right)^2 + 6\left(\frac{a\sigma_2}{H}\right)^2} \]

(B.19)

\[ \sigma_{vm} = \frac{\sigma_2}{\sqrt{2}} \sqrt{\frac{3B^2}{2} + 6\left(\frac{a}{H}\right)^2} = \sigma_y \]

(B.20)

\[ \sigma_2 = \frac{\sqrt{2}\sigma_y}{\sqrt{3} \left(\frac{\sqrt{B^2/2 + 2(a/H)^2} + 1}{\sqrt{B^2 + 4(a/H)^2}}\right)} = \frac{2\sigma_y}{\sqrt{3} \left(\frac{\sqrt{B^2/2 + 2(a/H)^2} + 1}{\sqrt{B^2 + 4(a/H)^2}}\right)} \]

(B.21)

At the crack tip, above the ligament,
\[ \sigma_{11} = B \sigma_2 \]

\[ \sigma_{22} = \frac{\sigma_2}{(1 - a/W)} \]

\[ \sigma_{33} = \frac{B \sigma_2}{2} + \frac{\sigma_2}{2(1 - a/W)} \]

\[ \sigma_{12} = \frac{a \sigma_2}{H} \]

(B.22)

\[ \sigma_{vm} = \frac{1}{\sqrt{2}} \sqrt{\left( B \sigma_2 - \frac{\sigma_2}{(1 - a/W)} \right)^2 + 2 \left( \frac{B \sigma_2}{2} - \frac{\sigma_2}{2(1 - a/W)} \right)^2 + 6 \left( \frac{a \sigma_2}{H} \right)^2} \]

(B.23)

\[ \sigma_{vm} = \frac{\sigma_2}{\sqrt{2}} \sqrt{\frac{3}{2} \left( B - \frac{1}{(1 - a/W)} \right)^2 + 6 \left( \frac{a}{H} \right)^2} = \sigma_y \]

(B.24)

\[ \sigma_2 = \frac{\sqrt{2} \sigma_y}{\sqrt{3}} \frac{1}{\sqrt{[B - 1/(1 - a/W)]^2 + 2(a/H)^2}} \]

\[ = \frac{2 \sigma_y}{\sqrt{3}} \frac{1}{\sqrt{[B - 1/(1 - a/W)]^2 + 4(a/H)^2}} \]

\[ = \frac{2 \sigma_y}{\sqrt{3}} \frac{(1 - a/W)}{\sqrt{[B(1 - a/W) - 1]^2 + 4(a/H)^2(1 - a/W)^2}} \]

\[ = \frac{2 \sigma_y}{\sqrt{3}} \frac{z}{\sqrt{[Bz - 1]^2 + 4(a/H)^2z^2}} \]

(B.25)
At the surface above the crack tip:

\[
\sigma_{11} = B\sigma_2
\]
\[
\sigma_{22} = \sigma_2
\]
\[
\sigma_{33} = \frac{B\sigma_2}{2} + \frac{\sigma_2}{2}
\]
\[
\sigma_{12} = \frac{a\sigma_2}{H}
\]

(B.26)

\[
\sigma\text{vm} = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{B\sigma_2 - \sigma_2}{2}\right)^2 + 2\left(\frac{B\sigma_2}{2} - \frac{\sigma_2}{2}\right)^2 + 6\left(\frac{a\sigma_2}{H}\right)^2}
\]

(B.27)

\[
\sigma\text{vm} = \frac{\sigma_2}{\sqrt{2}} \sqrt{\frac{3}{2} (B - 1)^2 + 6 \left(\frac{a}{H}\right)^2} = \sigma_y
\]

(B.28)

\[
\sigma_2 = \frac{\sqrt{2}\sigma_y}{\sqrt{3}} \frac{1}{\sqrt{(B - 1)^2/2 + 2(a/H)^2}}
\]

(B.29)

This is greater in value than both (B.21) and (B.25), so the solution is the minimum of (B.21) and (B.25), giving equation (5.14).

**Short plate forward slip-line upper bound limit load solution**

Equation (5.15):
Using the double angle formula,

\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]

\[
\cos 2\theta = 1 - 2 \sin^2 \theta
\]
\[ (\sigma_2)_L = \frac{\sigma_y}{\sqrt{3}} \frac{2H}{[a/(1 - \cos 2\theta) + H(1 - B) \sin 2\theta]} \] (B.37)

\[ (\sigma_2)_L = \frac{2\sigma_y}{\sqrt{3}} \frac{H/W}{[a/W(1 - \cos 2\theta) + H/W(1 - B) \sin 2\theta]} \] (B.38)

which is equation (5.16).

**Short plate reverse slip-line upper bound limit load solution**

Equation (5.22):

\[ (\sigma_2)_L[W + a - \frac{H}{\tan \theta}] \dot{\gamma} \sin \theta - B(\sigma_2)_L 2H \dot{\gamma} \cos \theta \]

\[ - (\sigma_2)_L[W - a + \frac{H}{\tan \theta}] \dot{\gamma} \sin \theta = 2\dot{\gamma} \frac{\sigma_y}{\sqrt{3}} \frac{H}{\sin \theta} \] (B.39)

\[ (\sigma_2)_L[W \sin \theta + a \sin \theta - H \cos \theta - 2BH \cos \theta - W \sin \theta \]

\[ + a \sin \theta - H \cos \theta] = \frac{2\sigma_y}{\sqrt{3}} \frac{H}{\sin \theta} \] (B.40)

\[ (\sigma_2)_L[a \sin \theta - H \cos \theta - BH \cos \theta] = \frac{\sigma_y}{\sqrt{3}} \frac{H}{\sin \theta} \] (B.41)

\[ (\sigma_2)_L = \frac{\sigma_y}{\sqrt{3}} \frac{H}{[a \sin \theta - H \cos \theta - BH \cos \theta] \sin \theta} \] (B.42)
Using the double angle formula,

\[ \sin 2\theta = 2 \sin \theta \cos \theta \]

\[ \cos 2\theta = 1 - 2 \sin^2 \theta \]  \hspace{1cm} (B.45)

\[
(\sigma_2)_{L} = \frac{\sigma_y}{\sqrt{3}} \frac{H}{[a(1 - \cos 2\theta) + H(B - 1)\sin 2\theta]} 
\]

\hspace{1cm} (B.46)

\[
(\sigma_2)_{L} = \frac{2\sigma_y}{\sqrt{3}} \frac{H}{[a/W(1 - \cos 2\theta) + H/W(B - 1)\sin 2\theta]} 
\]

\hspace{1cm} (B.47)

which is equation (5.23).

**Upper bound solution for short plates**

Equation (5.30):

\[
(\sigma_2)_{L} = \left( \frac{2\sigma_y}{\sqrt{3}} \right) \frac{H/W}{a/W(1 - \cos 2\theta^L) + H/W|B - 1|\sin 2\theta^L} 
\]

\hspace{1cm} (B.48)

where

Equation (5.31):
\[
\tan 2\theta = -\frac{H/W|B - 1|}{a/W}
\]  
(B.49)

Given that \(\tan 2\theta\) is negative and \(0 < \theta < 90^\circ\),

\[
\theta = \pi - \tan^{-1} \frac{H|B - 1|}{a}
\]  
(B.50)

The following was input into Mathematica [72],

\[
\text{Simplify} \left[ \frac{H}{a(1 - \cos[\pi - \text{ArcTan}[H*(B - 1)/a]])} + H*(B - 1)*\sin[\pi - \text{ArcTan}[H*(B - 1)/a]]) \right]
\]

(giving the formula,

\[
\text{Output} = \frac{H}{a \left[ 1 + \frac{a^2 + (B - 1)^2H^2}{a^2} \right]}
\]  
(B.52)

which is the same for \((B - 1)\) and \((1 - B)\).

Dividing through by \(W\) gives equation (5.32).

The solution is based on the double angle formulae,

\[
\sin 2\theta = 2\sin \theta \cos \theta
\]

\[
\cos 2\theta = 1 - 2\sin^2 \theta
\]

\[
\tan 2\theta = \frac{2\tan \theta}{1 - 2\tan^2 \theta}
\]  
(B.53)

and the Pythagorean identities,
\[
\sin^2 \theta + \cos^2 \theta = 1
\]

\[
\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta}
\]

\[
1 + \frac{1}{\tan^2 \theta} = \frac{1}{\sin^2 \theta}
\]

(B.54)

**Double edge cracked plate forward slip-line upper bound solution**

Equation (5.47):

\[
B(\sigma_2) L \dot{\nu} \cos \theta [H - (2W - a) \tan \theta] + (\sigma_2)_L 2W \dot{\nu} \sin \theta - B(\sigma_2)_L [H] \dot{\nu} \cos \theta = \frac{\nu(2W - a) \sigma}{\cos \theta \sqrt{3}}
\]

(B.55)

\[
(\sigma_2)_L [BH \cos \theta - 2BW \sin \theta + 2aB \sin \theta + 2W \sin \theta - BH \cos \theta]
\]

\[
= \frac{(2W - a) \sigma}{\cos \theta \sqrt{3}}
\]

(B.56)

\[
(\sigma_2)_L \sin \theta [aB - 2BW + 2W] = \frac{(2W - a) \sigma}{\cos \theta \sqrt{3}}
\]

(B.57)
\[(\sigma_2)_L = \frac{\sigma}{\sqrt{3}} \sin \theta \cos \theta \bigl[ aB - 2BW + 2W \bigr] \]

(B.58)

\[(\sigma_2)_L = \frac{2\sigma}{\sqrt{3}} \sin 2\theta \left( \frac{2W - a}{2W - B(2W + a)} \right) \]

(B.59)

\[(\sigma_2)_L = \frac{\sigma}{\sqrt{3}} \sin \theta \cos \theta \left( \frac{2 - a}{2a - BW + B} \right) \]

(B.60)

which is equation (5.48).

**Double edge cracked plate reverse slip-line upper bound solution**

Equation (5.50):

\[B(\sigma_2)_L \dot{v} \cos \theta \left[ a \tan \theta \right] = \frac{\dot{v}(a)}{\cos \theta} \frac{\sigma_y}{\sqrt{3}} \]

(B.61)

\[B(\sigma_2)_L a \sin \theta = \frac{a}{\cos \theta} \frac{\sigma_y}{\sqrt{3}} \]

(B.62)

\[\frac{1}{B \sin \theta \cos \theta} \frac{\sigma_y}{\sqrt{3}} \]

(B.63)
\[(\sigma_2)_L = \frac{2}{\text{Bsin} \ 2\theta \ \sqrt{3}} \frac{\sigma_y}{\sqrt{3}} \]

which is equation (5.51).

**Single edge cracked plate circular slip-line upper bound solution (general)**

Equation (5.59):

\[
B(\sigma_2)_L \left( Y + H - \sqrt{r^2 - (X - W)^2} \right) \left[ \left( Y + H + \sqrt{r^2 - (X - W)^2} \right) \right]
+ (\sigma_2)_L W\omega \left( X - \frac{W}{2} \right) - B(\sigma_2)_L H\omega \left( Y + \frac{H}{2} \right) = \frac{\sigma_y}{\sqrt{3}} r \varphi \omega r
\]

\[(\sigma_2)_L \left( \frac{[(Y + H)^2 - (r^2 - (X - W)^2)]}{2} + WX - \frac{W^2}{2} - BHY + B \frac{H^2}{2} \right)
= \frac{\sigma_y}{\sqrt{3}} r^2 \varphi \]

\[(\sigma_2)_L \left( \frac{[Y^2 + H^2 + 2YH - r^2 - X^2 + W^2 - 2WX]}{2} + WX - \frac{W^2}{2} - BH^2 \right)
- \frac{B H^2}{2} \right) = \frac{\sigma_y}{\sqrt{3}} r^2 \varphi \]

\[(\sigma_2)_L \left( \frac{[Y^2 - r^2 + X^2 + W^2 - 2WX]}{2} + WX - \frac{W^2}{2} \right) = \frac{\sigma_y}{\sqrt{3}} r^2 \varphi \]
\[(\sigma_2)^{\text{ub}}_L = \frac{2\sigma_y}{\sqrt{3}} \frac{r^2 \varphi}{[B(Y^2 - r^2 + X^2 + W^2 - 2XW) + W(2X - W)]} \]

(B.69)

Equation (5.54):

\[r^2 = (X - a)^2 + Y^2 \]

(B.70)

Substitute equation (B.70) into (B.69)

\[(\sigma_2)^{\text{ub}}_L = \frac{2\sigma_y}{\sqrt{3}} \frac{r^2 \varphi}{[B(W^2 - a^2 + 2aX - 2XW) + W(2X - W)]} \]

(B.71)

which is equation (5.60).

**Single edge cracked plate circular slip-line upper bound solution (circle intersects corner)**

Equation (5.65):

\[B(\sigma_2)_L H \omega \left( Y + \frac{H}{2} \right) + (\sigma_2)_L W \dot{\omega} \left( \frac{W}{2} \right) - B(\sigma_2)_L H \dot{\omega} \left( Y + \frac{H}{2} \right) = \frac{\sigma_y}{\sqrt{3}} \varphi \dot{r} \omega r \]

(B.72)

\[(\sigma_2)_L \left[ \frac{W^2}{2} \right] = \frac{\sigma_y}{\sqrt{3}} \varphi \dot{r} \omega r \]

(B.73)

\[(\sigma_2)_L = \frac{2\sigma_y r^2 \varphi}{\sqrt{3} W^2} \]

(B.74)

which is equation (5.66).