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1.

On products and mixed sums of gamma and beta random variables motivated by availability

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Abstract

Motivated by the notion of availability, the exact distribution of the sum of beta random variables is derived under the impact of the environment. A suitable approximation is presented when the exact distribution cannot be identified. Also presented are some characterizations for the gamma distribution.

Keywords: Availability; Beta distribution; Lifetime.

1 Introduction

The distribution of a linear combination of random variables arises in many applied problems. It has been extensively studied by different researchers. The distribution of linear combinations of the form $\alpha X + \beta Y$ is of interest in problems in automation, control, fuzzy sets, quality, reliability engineering, and other areas of computer science (see e.g. [15, 23] and the references therein). There has been a great interest in the study of the distributions of the linear combination $\alpha X + \beta Y$ when X and Y are random variables and belong to beta families, see e.g. [22], [2], [3] and [4]. Some systems are subjected to a wide variety of fluctuations and interruptions; varying weather conditions, learning development on repetitive operations, equipment breakdowns, management interference, and other external factors may impact the output.

In the systems mentioned above, it is quite possible that α and β could be random variables themselves. The distribution of the stochastic linear combination (see Definition 2.2 in [11]) is of interest in engineering. Measuring attributes in different fields in engineering may be necessary. Also, a suitable statistical analysis of these attributes can be useful in decision making. These attributes are measured usually in laboratories in ideal conditions, which can be different from the usual observations in practice. Lifetime, availability, vehicle speed, etc are all scientific problems investigated in ideal conditions. But in most papers the environmental conditions are ignored. As an example, survival analysis is of fundamental importance in quality technology, see e.g. [5], [7], [8], [20] and [6]. Usually, lifetimes measured in a laboratory are different observations by different users in the environment. Suppose that Y_i 's are the lifetimes measured in a laboratory and X_i (with $0 \leq X_i \leq 1$) are the random effects of the environment on the lifetimes, such that $X_i Y_i \leq Y_i$ holds, and so $\sum_i X_i Y_i$ is the total lifetime in the environment. Here, we find the distribution of the sum of beta random variables under environmental impact. The innovative and interesting

works of some authors (see [14] and Theorem 2.2) lead us to using some suitable approximations for distributions of these categories of random variables.

2 Total availability under environmental impact

The concept of availability ¹ and [1] motivate us to discuss the distribution of the sum of beta random variables in this section. The method of this section for computing the exact distribution is a continuation of the method of [26, Section 7]. Throughout, $(W_1, \dots, W_r)$ is called random coefficient vector of environmental effect in r -position. The following theorem can be proved straightforwardly.

Theorem 2.1. *Suppose $X_1, \dots, X_r$ are independent random variables having, respectively, the $Beta(n_1, m_1), \dots, Beta(n_r, m_r)$ distributions and that the random vector $W = \langle W_1, \dots, W_r \rangle$ has the Dirichlet $(\alpha_1, \dots, \alpha_r)$ distribution. Then the moments of $Z = \sum_{j=1}^r W_j X_j$ are*

$$E(Z^k) = \sum_{i_1 + \dots + i_r = k} k Beta \left(\sum_{j=1}^r \alpha_j, k \right) \prod_{j=1}^r \frac{Beta(n_j + m_j, \alpha_j + i_j)}{i_j Beta(n_j, i_j) Beta(m_j + n_j + i_j, \alpha_j)}.$$

In particular, the first moment of Z is

$$E(Z) = \frac{\sum_{i=1}^r \alpha_i n_i / (n_i + m_i)}{\sum_{i=1}^r \alpha_i}.$$

The mean, variance and entropy of Z for different values of parameters are given in Table 1.

¹See e.g. the ARES conference series: <https://www.ares-conference.eu/>

Table 1: The mean, variance, and entropy.

(n_1, m_1)	(n_2, m_2)	(n_3, m_3)	$E(Z)$	$E(Z^2)$	$Var(Z)$	entropy
(1,1)	(1,1)	(1,1)	0.5	0.3079	0.0579	-0.0716
(1,1)	(1,1)	(1,1)	0.5	0.291	0.041	-0.1998
(1,1)	(1,1)	(1,1)	0.5	0.287	0.037	-0.2417
(1,1)	(1,1)	(1,1)	0.5	0.283	0.033	-0.2993
(1,1)	(1,1)	(1,1)	0.5	0.28	0.030	-0.3349
(1,1)	(1,1)	(1,1)	0.5	0.283	0.033	-0.3005
(2,6)	(3,7)	(4,8)	0.303	0.105	0.013191	-0.7583
(2,6)	(3,7)	(4,8)	0.294	0.096	0.009564	-0.9103
(2,6)	(3,7)	(4,8)	0.3008	0.0991	0.008619	-20.8889
(2,6)	(3,7)	(4,8)	0.303	0.099	0.007191	-1.0443
(2,6)	(3,7)	(4,8)	0.301	0.0978	0.007199	-1.0821
(2,6)	(3,7)	(4,8)	0.3083	0.1022	0.007151	-1.059
(3,11)	(5,13)	(7,15)	0.5	0.307	0.057	-0.0716
(3,11)	(5,13)	(7,15)	0.27	0.078	0.0051	-1.18
(3,11)	(5,13)	(7,15)	0.278	0.082	0.004716	-1.2484
(3,11)	(5,13)	(7,15)	0.281	0.083	0.004039	-1.3318
(3,11)	(5,13)	(7,15)	0.279	0.081	0.003159	-1.3736
(3,11)	(5,13)	(7,15)	0.287	0.086	0.003631	-1.352
(4,16)	(7,19)	(10,22)	0.273	0.080	0.05471	-1.1631
(4,16)	(7,19)	(10,22)	0.260	0.072	0.0044	-1.338
(4,16)	(7,19)	(10,22)	0.269	0.075	0.002639	-1.4141
(4,16)	(7,19)	(10,22)	0.272	0.077	.003016	-1.5081
(4,16)	(7,19)	(10,22)	0.27	0.075	0.0021	-1.554
(4,16)	(7,19)	(10,22)	0.279	0.080	0.02159	-1.1106
(5,21)	(9,25)	(13,29)	0.268	0.076	0.004176	-1.5345
(5,21)	(9,25)	(13,29)	0.255	0.068	0.002975	-1.4463
(5,21)	(9,25)	(13,29)	0.264	0.072	0.002304	-1.5299
(5,21)	(9,25)	(13,29)	0.268	0.074	0.002176	-1.6342
(5,21)	(9,25)	(13,29)	0.265	0.072	0.001775	-1.6839
(5,21)	(9,25)	(13,29)	0.275	0.077	0.001375	-1.667

Table 2: Checking the robustness of the approximation for $r = 2$.

(n_1, m_1)	(n_2, m_2)	(α_1, α_2)	$E(Z)$	$Var(Z)$	p	q	p -value
(1,1)	(1,1)	(1,1)	0.5	0.055	1.75	1.75	0.89
(1,1)	(1,1)	(0.2,0.3)	0.5	0.07	1.28	1.28	0.90
(1,1)	(1,1)	(1.75,2)	0.5	0.05	1.97	1.97	0.38
(1,1)	(1,1)	(3,5)	0.5	0.04	2.07	2.07	0.67
(1,1)	(1,1)	(7,10)	0.5	0.04	2.26	2.26	0.21
(1,1)	(1,1)	(10,20)	0.5	0.04	2.13	2.13	0.35
(2,4)	(3,5)	(0.2,0.3)	0.36	0.3704	3.08	5.51	0.39
(2,4)	(3,5)	(1.75,2)	0.35	0.0175	4.31	7.82	0.54
(2,4)	(3,5)	(3,5)	0.35	0.0175	4.75	8.47	0.46
(2,4)	(3,5)	(7,10)	0.35	0.0175	5.05	9.07	0.12
(2,4)	(3,5)	(10,20)	0.36	0.0104	5.0	8.85	0.47
(3,7)	(5,9)	(0.2,0.3)	0.33	0.0111	4.76	9.49	0.94
(3,7)	(5,9)	(1.75,2)	0.33	0.0011	6.64	13.45	0.65
(3,7)	(5,9)	(3,5)	0.33	0.0111	7.42	14.69	0.28
(3,7)	(5,9)	(7,10)	0.33	0.0111	7.87	15.72	0.46
(3,7)	(5,9)	(10,20)	0.33	0.0111	7.89	15.44	0.80
(4,10)	(7,13)	(0.2,0.3)	0.32	0.0076	6.34	13.22	0.60
(4,10)	(7,13)	(1.75,2)	0.32	0.0066	8.91	18.94	0.38
(4,10)	(7,13)	(3,5)	0.32	0.0076	10.066	20.82	0.29
(4,10)	(7,13)	(7,10)	0.32	0.0076	10.67	22.31	0.22
(4,10)	(7,13)	(10,20)	0.32	0.0076	10.76	22.007	0.91
(5,13)	(9,17)	(0.2,0.3)	0.31	0.0139	7.83	16.75	0.83
(5,13)	(9,17)	(1.75,2)	0.31	0.0039	11.12	24.27	0.31
(5,13)	(9,17)	(3,5)	0.32	0.0056	12.66	26.86	0.53
(5,13)	(9,17)	(7,10)	0.32	0.0056	12.66	26.85	0.75
(5,13)	(9,17)	(10,20)	0.32	0.0066	13.63	28.53	0.97

Table 3: Checking the robustness of the approximation for $r = 3$.

(n_1, m_1)	(n_2, m_2)	(n_3, m_3)	$(\alpha_1, \alpha_2, \alpha_3)$	p	q	p -value
(1,1)	(1,1)	(1,1)	(0.2,0.3,0.4)	1.655	1.655	0.11
(1,1)	(1,1)	(1,1)	(1,1,1)	2.5	2.5	0.67
(1,1)	(1,1)	(1,1)	(1.25,1.75,2)	2.802	2.802	0.31
(1,1)	(1,1)	(1,1)	(3,5,6)	3.25	3.25	0.14
(1,1)	(1,1)	(1,1)	(7,10,12)	3.55	3.55	0.44
(1,1)	(1,1)	(1,1)	(10,20,30)	3.26	3.26	0.84
(2,6)	(3,7)	(4,8)	(0.2,0.3,0.4)	4.481	10.274	0.75
(2,6)	(3,7)	(4,8)	(1,1,1)	5.949	14.257	0.78
(2,6)	(3,7)	(4,8)	(1.25,1.75,2)	0.046	16.375	0.58
(2,6)	(3,7)	(4,8)	(3,5,6)	8.365	19.191	0.80
(2,6)	(3,7)	(4,8)	(7,10,12)	8.971	20.762	0.22
(2,6)	(3,7)	(4,8)	(10,20,30)	8.855	19.866	0.64
(3,11)	(5,13)	(7,15)	(0.2,0.3,0.4)	1.655	1.655	0.11
(3,11)	(5,13)	(7,15)	(1,1,1)	9.137	24.694	0.94
(3,11)	(5,13)	(7,15)	(1.25,1.75,2)	11.092	28.798	0.61
(3,11)	(5,13)	(7,15)	(3,5,9)	13.471	34.386	0.17
(3,11)	(5,13)	(7,15)	(7,10,12)	14.490	37.414	0.19
(3,11)	(5,13)	(7,15)	(10,20,30)	14.543	36.061	0.17
(4,16)	(7,19)	(10,22)	(0.2,0.3,0.4)	8.986	23.922	0.96
(4,16)	(7,19)	(10,22)	(1,1,1)	11.966	33.957	0.62
(4,16)	(7,19)	(10,22)	(1.25,1.75,2)	14.822	40.232	0.36
(4,16)	(7,19)	(10,22)	(3,5,6)	018.407	49.033	0.20
(4,16)	(7,19)	(10,22)	(7,10,12)	19.912	53.722	0.38
(4,16)	(7,19)	(10,22)	(10,20,30)	20.195	52.105	0.87
(5,21)	(9,25)	(13,29)	(0.2,0.3,0.4)	10.769	22.335	0.28
(5,21)	(9,25)	(13,29)	(1,1,1)	14.486	42.209	0.53
(5,21)	(9,25)	(13,29)	(1.25,1.75,2)	18.263	50.777	0.85
(5,21)	(9,25)	(13,29)	(3,5,6)	23.170	63.157	0.71
(5,21)	(9,25)	(13,29)	(7,10,12)	25.226	69.690	0.57
(5,21)	(9,25)	(13,29)	(10,20,30)	25.803	68.011	0.32

Corollary 2.1. Assume that the random variables $X_1, \dots, X_r$ are independent and have the common Beta(n, m) distribution on $(0, 1)$ and the random vector $W = \langle W_1, \dots, W_r \rangle$ has the Dirichlet $(\alpha_1, \dots, \alpha_r)$ distribution. Then the moments of $Z = \sum_{j=1}^r W_j X_j$ are

$$E(Z^k) = \sum_{i_1 + \dots + i_r = k} k \text{Beta} \left(\sum_{j=1}^r \alpha_j, k \right) \prod_{j=1}^r \frac{\text{Beta}(n + m, \alpha_j + i_j)}{i_j \text{Beta}(n, i_j) \text{Beta}(m + n + i_j, \alpha_j)}.$$

The first moment and variance of Z are

$$E(Z) = \frac{n}{n + m},$$

and

$$\text{Var}(Z) = \frac{nm}{(n + m)^2(n + m + 1)} \frac{\sum_{j=1}^r \alpha_j (\alpha_j + 1)}{\left(\sum_{j=1}^r \alpha_j \right) \left(\sum_{j=1}^r \alpha_j + 1 \right)}.$$

In particular, when $n = m$, we have

$$E(Z) = \frac{1}{2},$$

and

$$\text{Var}(Z) = \frac{1}{4(2n+1)} \frac{\sum_{j=1}^r \alpha_j (\alpha_j + 1)}{\left(\sum_{j=1}^r \alpha_j\right) \left(\sum_{j=1}^r \alpha_j + 1\right)}.$$

Theorem 2.2. Suppose that the independent random variables $X_1, \dots, X_r$ have the Beta $(n_1, m_1), \dots, \text{Beta}(n_r, m_r)$ distributions and that the random vector $W = \langle W_1, \dots, W_r \rangle$ has the Dirichlet $(n_1 + m_1, \dots, n_r + m_r)$ distribution. Then $Z = \sum_{j=1}^r W_j X_j$ has the Beta $(\sum_{j=1}^r n_j, \sum_{j=1}^r m_j)$ distribution.

We find the k th moment of Z as follows:

$$E(Z^k) = \sum_{i_1 + \dots + i_r = k} \frac{k!}{i_1! \dots i_r!} E(W_1^{i_1} \dots W_r^{i_r}) E(X_1^{i_1}) \dots E(X_r^{i_r}).$$

By using the Dirichlet distribution, we have

$$E(Z^k) = \sum_{i_1! \dots i_r!} \frac{k!}{i_1! \dots i_r!} \frac{\Gamma\left(\sum_{j=1}^r (n_j + m_j)\right) \prod_{j=1}^r \Gamma(n_j + m_j + i_j)}{\prod_{j=1}^r \Gamma(n_j + m_j) \Gamma\left(\sum_{j=1}^r (n_j + m_j + i_j)\right)} E(X_1^{i_1}) \dots E(X_r^{i_r}),$$

where $i_1 + \dots + i_r = k$. It is well known that, see [9],

$$E(X_j^{i_j}) = \frac{\Gamma(n_j + m_j) \Gamma(n_j + i_j)}{\Gamma(n_j) \Gamma(m_j + n_j + i_j)}$$

for $j = 1, \dots, r$. So,

$$\begin{aligned} E(Z^k) &= \sum_{i_1 + \dots + i_r = k} \frac{k!}{i_1! \dots i_r!} \frac{\Gamma\left(\sum_{j=1}^r (n_j + m_j)\right) \prod_{j=1}^r \Gamma(n_j + m_j + i_j)}{\prod_{j=1}^r \Gamma(n_j + m_j) \Gamma\left(\sum_{j=1}^r (n_j + m_j + i_j)\right)} \prod_{j=1}^r \frac{\Gamma(n_j + m_j) \Gamma(n_j + i_j)}{\Gamma(n_j) \Gamma(m_j + n_j + i_j)} \\ &= \frac{\Gamma\left(\sum_{j=1}^r (n_j + m_j)\right)}{\Gamma\left(\sum_{j=1}^r (n_j + m_j + i_j)\right)} \sum_{i_1 + \dots + i_r = k} \frac{k!}{i_1! \dots i_r!} \prod_{j=1}^r \frac{\Gamma(n_j + i_j)}{\Gamma(n_j)} \\ &= \frac{\Gamma\left(\sum_{j=1}^r (n_j + m_j)\right)}{\Gamma\left(\sum_{j=1}^r (n_j + m_j + i_j)\right)} \sum_{i_1 + \dots + i_r = k} \binom{k}{i_1, i_2, \dots, i_r} \frac{\Gamma(n_1 + i_1)}{\Gamma(n_1)} \dots \frac{\Gamma(n_r + i_r)}{\Gamma(n_r)}. \end{aligned}$$

By considering the fact that the sum of the Dirichlet-multinomial distribution on its support is equal to one (see [25]), we have

$$E(Z^k) = \frac{\Gamma\left(\sum_{j=1}^r (n_j + m_j)\right)}{\Gamma\left(k + \sum_{j=1}^r (n_j + m_j)\right)} \frac{\Gamma\left(\sum_{j=1}^r n_j + k\right)}{\Gamma\left(\sum_{j=1}^r n_j\right)},$$

which is the k th moment of the Beta $\left(\sum_{j=1}^r n_j, \sum_{j=1}^r m_j\right)$ distribution. Since Z is a bounded random variable, its distribution is uniquely determined by its moments (Carleman's Theorem, see e.g. [12]).

In some cases our identification is indeed a characterization. Characterizations are of interest in many areas, see [10].

Theorem 2.3. *Assume that the independent random variables $X_1, \dots, X_r$ have common distribution on $(0, 1)$ and the random vector $W = \langle W_1, \dots, W_r \rangle$ has the Dirichlet $(2\alpha, \dots, 2\alpha)$ distribution. Then $Z = \sum_{i=1}^n W_i X_i$ has a Beta $(r\alpha, r\alpha)$ distribution on $(0, 1)$ if and only if $X_1, \dots, X_r$ have the Beta (α, α) distribution.*

For the “if” part, put $n_j = \alpha, m_j = \alpha$ for $j = 1, \dots, r$ in Theorem 2.2.

For the “only if” part we prove by induction on m that

$$E(X_j^m) = \frac{\Gamma(m + \alpha)\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(2\alpha + m)}.$$

- For $m = 1$ we have

$$E(X_j) = \frac{1}{2}.$$

- Now assume that for $k = 1, \dots, m$, we have

$$E(X_j^k) = \frac{\Gamma(k + \alpha)\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(2\alpha + k)},$$

and prove the desired conclusion for $m + 1$.

We know that

$$E(Z^{m+1}) = \sum_{i_1 + \dots + i_r = m+1} \binom{m+1}{i_1, i_2, \dots, i_r} E(W_1^{i_1} \dots W_r^{i_r}) E(X_1^{i_1}) \dots E(X_r^{i_r})$$

and

$$\frac{\Gamma(m+1+r\alpha)\Gamma(2r\alpha)}{\Gamma(r\alpha)\Gamma(2r\alpha+m+1)} = \sum_{i_1 + \dots + i_r = m+1} \binom{m+1}{i_1, i_2, \dots, i_r} \left(\frac{\Gamma(2r\alpha)}{\Gamma(2r\alpha+m+1)} \prod_{j=1}^r \frac{\Gamma(2\alpha+i_j)}{\Gamma(2\alpha)} \right) \prod_{j=1}^r E(X_j^{i_j}).$$

So,

$$\frac{\Gamma(m+1+r\alpha)}{\Gamma(r\alpha)} = \sum_{i_1 + \dots + i_r = m+1} \binom{m+1}{i_1, i_2, \dots, i_r} \prod_{j=1}^r \left(\frac{\Gamma(2\alpha+i_j)}{\Gamma(2\alpha)} E(X_j^{i_j}) \right)$$

and

$$\sum_{\substack{i_1, \dots, i_r \leq m \\ i_1 + \dots + i_r = m+1}} \frac{(m+1)!}{i_1! \dots i_r!} \prod_{j=1}^r \frac{\Gamma(2\alpha+i_j)}{\Gamma(2\alpha)} \frac{\Gamma(2\alpha)\Gamma(\alpha+i_j)}{\Gamma(\alpha)\Gamma(2\alpha+i_j)} + r \frac{\Gamma(2\alpha+m+1)}{\Gamma(2\alpha)} E(X_i^{m+1}) = \frac{\Gamma(r\alpha+m+1)}{\Gamma(r\alpha)}.$$

Now, the proof is complete upon noting that

$$\left(\frac{\Gamma(r\alpha+m+1)}{\Gamma(r\alpha)} - r \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha)} \right) + r \frac{\Gamma(2\alpha+m+1)}{\Gamma(2\alpha)} E(X_i^{m+1}) = \frac{\Gamma(r\alpha+m+1)}{\Gamma(r\alpha)}.$$

Johnson and Kotz in the concluding remarks of their paper [16] mentioned that the “case in which the X ’s have standard uniform $(0, 1)$ distribution leads to an interesting family of symmetric distributions.” We identify it to be the celebrated beta distribution.

Example 2.1. For independent random variables $X_1, \dots, X_r$ with standard uniform $(0, 1)$ distributions and the random vector $W = \langle W_1, \dots, W_r \rangle$ with the Dirichlet $(2, \dots, 2)$ distribution, the random variable $Z = \sum_{j=1}^r W_j X_j$ has the Beta (r, r) distribution. To see this it suffices to put $n_j = m_j = 1$ for $j = 1, \dots, r$ in Theorem 2.2.

Example 2.2. Let $\{\alpha_n, \beta_n\}_{n=1}^\infty$ be a sequence of independent random variables and the random variables $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ be defined inductively by

$$\begin{pmatrix} X_n & 1 - X_n \\ Y_n & 1 - Y_n \end{pmatrix} = \begin{pmatrix} \alpha_n & 1 - \alpha_n \\ \beta_n & 1 - \beta_n \end{pmatrix} \begin{pmatrix} X_{n-1} & 1 - X_{n-1} \\ Y_{n-1} & 1 - Y_{n-1} \end{pmatrix}, \quad n = 2, 3, \dots$$

with $X_1 = \alpha_1$ and $Y_1 = \beta_1$. Van Assche [24] showed that the limiting distributions of X_n and Y_n exist and are equal. [24, 17] reached the following equation: $\lim Y_n = \alpha_1 Y^* + \beta_1 (1 - Y^*)$, where α_1, β_1, Y^* are all independent and α_1, β_1 have the Beta (a, a) distributions and Y^* and $\lim Y_n$ have identical distributions (note that Y^* depends on α_n ’s and β_n ’s for $n \neq 1$). So, his results hold in our framework as well.

Theorem 2.4. Suppose $X_1, \dots, X_r$ are independent random variables having the Beta $(\frac{1}{2} + \alpha_1, \frac{1}{2} + \alpha_1), \dots, \text{Beta}(\frac{1}{2} + \alpha_r, \frac{1}{2} + \alpha_r)$ distributions and that the random vector $W = \langle W_1, \dots, W_r \rangle$ has the Dirichlet $(\alpha_1, \dots, \alpha_r)$ distribution. Then $Z = \sum_{j=1}^r W_j X_j$ has the Dirichlet $(\frac{1}{2} + \sum_{i=1}^r \alpha_i, \dots, \frac{1}{2} + \sum_{i=1}^r \alpha_i)$ distribution.

Let Y_j ($j = 1, \dots, n$) be independent random variables independent from $X_1, \dots, X_n$ that have the Gamma(α_j) distribution. It can be seen, by some classical ways (e.g. $E(e^{t'T}) = [\Psi(t)]^{\sum_j \alpha_j}$) from [18, Table 2]), that the distribution of $T = \sum_j T_j = \sum_j Y_j X_j$ is the same as the distribution of T_j with the parameter $\sum_j \alpha_j$. We can also write T as

$$T = \sum_j Y_j X_j = \left(\sum_i Y_i \right) \left(\sum_j \frac{Y_j}{\sum_i Y_i} X_j \right)$$

and so we have $T = YZ$ in which Y has the gamma distribution with the parameter $\sum_j \alpha_j$, and T has the F distribution with the parameter $\sum_j \alpha_j$, and Y and Z are independent from each other. Of course, one can define $T' = Y'X'$ in such a way that $T' \stackrel{d}{=} T$, $\sum_j Y_j \stackrel{d}{=} Y'$ and $X' \sim \text{Dirichlet}(\frac{1}{2} + \sum_j \alpha_j)$. One can conclude that Z and Z' have identical distributions by calculating the general moments (s_1, s_2) of T and T' , i.e., $E(Z_1^{s_1} Z_2^{s_2}) = E\left(\left(X'_1\right)^{s_1} \left(X'_2\right)^{s_2}\right)$.

3 An approximation

In this section, we propose an approximation for the distribution of $Z = \sum_{j=1}^r W_j X_j$. By Theorem 2.1 and the fact that the support of Z lies in the interval $[0, 1]$, we are interested in approximating its distribution by a suitable member of the two-parameter Beta(p, q) family of distributions. The idea of approximating distributions having complicated formulas by the beta distribution is

very well established in the statistics literature, see e.g. [13], [14] and [19]. Following these authors, the choice of the beta parameters p and q is made using the method of moments. Equating the first two moments of Z with those of the beta distribution, we infer that

$$E(Z) = \frac{p}{p+q},$$

and

$$E(Z^2) = \frac{p(p+1)}{(p+q)(p+q+1)},$$

which we must solve simultaneously to find the beta parameters p and q . After some algebraic manipulation, we find the solutions as

$$p = \frac{E(Z) - E(Z^2)}{Var(Z)} E(Z)$$

and

$$q = \frac{E(Z) - E(Z^2)}{Var(Z)} [1 - E(Z)].$$

Note that the above solutions satisfy $p > 0$ and $q > 0$. The two moments $E(Z)$ and $E(Z^2)$ can be computed by Theorem 2.3.

The robustness of this approximation was checked for a range of values of the parameters. The following procedure based on simulation and the Kolmogorov-Smirnov test was used:

- step 1 – For the given parameters, compute the $E(Z)$ and $E(Z^2)$.
- step 2 – Obtain the corresponding estimates for (p, q) .
- step 3 – Simulate 1000 random numbers of $Z = \sum_{j=1}^r W_j X_j$ by simulating the X_i 's and W_i 's.
- step 4 – Perform the Kolmogorov-Smirnov test to see whether the sample in step 3 arises from the beta distribution with parameters p and q calculated in step 2.

The p -values of this test for the approximation are given in Tables 2 and 3. These tables show that approximation is robust for most of the chosen parameters. Similarly the distribution of T can be approximated.

As an example, suppose that W, X_1, X_2 are independent and all have the uniform $[0, 1]$ distribution. The probability density function (PDF) of Z is $f_Z(z) = -2(1-z)\log(1-z) - 2z\log z$, $0 < z < 1$. According to the first row of Table 2, $B(1.75, 1.75)$ will be a good approximation for Z . Figure 1 shows the exact and approximate PDFs of Z .

Figure 1: The exact and approximate PDFs of Z when W , X_1 and X_2 are uniform. The approximate distribution is $B(1.75, 1.75)$.

The error in Euclidean norm of this approximation is

$$\|f - B(1.75, 1.75)\|_2 = \left\{ \int_0^1 [f(z) - B(1.75, 1.75)]^2 dz \right\}^{\frac{1}{2}} = 0.0045.$$

4 Distribution of lifetime in the environment independent r -position

Here, we identify the distribution of sums of the random variables introduced in [21]. Let us recall that identifying this distribution by direct means can be very difficult.

Theorem 4.1. *Suppose $X_1, \dots, X_r$ are independent random variables with the $Beta(n_1, n - n_1), \dots, Beta(n_r, n - n_r)$ distributions. Suppose $Y_1, \dots, Y_r$ are independent random variables with the $Gamma(n, \frac{1}{\theta})$ distributions independent from X_i 's. Then the distribution of $T = \sum_{i=1}^r X_i Y_i$ is equal to*

$$f(t) = \frac{\theta^{-rn} \Gamma\left(r - \sum_{j=1}^r n_j\right)}{\Gamma(rn) B\left(\sum_{j=1}^r n_j, r - \sum_{j=1}^r n_j\right)} z^{rn-1} e^{-\frac{t}{\theta}} \Psi\left(r - \sum_{j=1}^r n_j, 1 + n - \sum_{j=1}^r n_j; \frac{t}{\theta}\right),$$

where

$$\Psi(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xu} u^{a-1} (1+u)^{b-a-1} du.$$

Let $Y_+ = \sum_{i=1}^r Y_i$ and write

$$T = Y_+ \sum_{i=1}^r X_i \frac{Y_i}{Y_+}.$$

It can be clearly seen that Y_+ and Y_i/Y_+ 's are independent from each other. On the other hand the vector $\langle Y_1/Y_+, \dots, Y_r/Y_+ \rangle$ has the Dirichlet $(n, \dots, n)$ distribution. So, T is distributed as the product of two independent random variables one of which has the gamma distribution and the other one has the beta distribution by Theorem 2.2. Hence, the theorem is proved by Theorem 1 of [21].

Corollary 4.1. *Suppose $X_1, \dots, X_r$ are independent random variables with the Beta $(n_1, n - n_1), \dots, \text{Beta}(n_r, n - n_r)$ distributions. Suppose $Y_1, \dots, Y_r$ are independent random variables with the Gamma $(n, \frac{1}{\theta})$ distributions and also independent from X_i 's. Then the distribution of $\bar{T}$ is equal to*

$$f_{\bar{T}}(t) = \frac{\theta^{-rn} \Gamma\left(r - \sum_{j=1}^r n_j\right)}{\Gamma(rn) B\left(\sum_{j=1}^r n_j, r - \sum_{j=1}^r n_j\right)} z^{rn-1} e^{-\frac{rt}{\theta}} \Psi\left(r - \sum_{j=1}^r n_j, 1 + n - \sum_{j=1}^r n_j; \frac{rt}{\theta}\right),$$

where

$$\Psi(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xu} u^{a-1} (1+u)^{b-a-1} du.$$

Theorem 4.2. *Suppose $X_1, \dots, X_r$ are independent and identical random variables distributed over $(0, 1)$ and $Y_1, \dots, Y_r$ are independent exponential random variables with the mean θ and also independent from X_j 's. Then $T = \sum_{j=1}^r X_j Y_j$ has the Gamma $(r\alpha, \frac{1}{\theta})$ distribution if and only if X_j 's have the Beta $(r\alpha, r(1 - \alpha))$ distributions.*

For the “if” part, it suffices to find the distribution of $X_j Y_j$.

For the “only if” part, we have $T = Y_+ \left(\sum_{i=1}^n \frac{Y_i}{Y_+} X_i \right)$, where $Y_+ = \sum_{i=1}^r Y_i$. Theorem 1 of [27] implies that $\sum_{i=1}^n \frac{Y_i}{Y_+} X_i$ should have the beta distribution; now Theorem 2.2 completes the proof.

Using the techniques of the above theorems we can give an alternative proof for the main theorem:

[A second proof for Theorem 2.2] Suppose that $\{X_i\}_{i=1}^n$ are independent random variables with the $\{\text{Beta}(\alpha_i, \beta_i)\}_{i=1}^n$ distributions, and $\{Y_i\}_{i=1}^n$ are independent gamma random variables with parameters $\{\alpha_i + \beta_i\}_{i=1}^n$; also X_i 's and Y_i 's are independent from each other. Let

$$Y_+ = \sum_{i=1}^n Y_i \quad \text{and} \quad T = Y_+ \left(\sum_{i=1}^n \frac{Y_i}{Y_+} X_i \right).$$

By Theorem 1 of [27], since Y_+ has the Gamma $(\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i)$ distribution and T has the Gamma $(\sum_{i=1}^n \alpha_i)$ distribution, $\sum_{i=1}^n \frac{Y_i}{Y_+} X_i$ has the Beta $(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i)$ distribution. The theorem follows by noting that $\left\langle \frac{Y_1}{Y_+}, \dots, \frac{Y_n}{Y_+} \right\rangle$ has the Dirichlet $(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n)$ distribution.

The following theorem can be proved similarly to Theorem 4.1.

Theorem 4.3. Suppose $X_1, \dots, X_r$ are independent random variables with the Beta $(n_1, m_1), \dots, \text{Beta}(n_r, m_r)$ distributions and $Y_1, \dots, Y_r$ are independent random variables with the Gamma $(\alpha_i, \frac{1}{\theta})$ distributions ($i = 1, \dots, r$) which are independent from X_i 's. Then we have $T \stackrel{d}{=} Y \mid Z$ for a gamma random variable Y .

Therefore, T can be obtained by using Theorems 4.3 and 4.1.

5 Conclusions

We have studied the distribution of the mixture random variable Z (and T) when the X_i 's have beta distributions. It is interesting to note that the distribution of Z can be the beta distribution, and the distribution of T can be that introduced in [21]. Both these distributions have been studied with applications in [21]. It seems, by the results obtained in this paper, the environment does not essentially change the distribution; only its parameters. It would be interesting to study the multivariate cases of these results.

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