STOCHASTIC HEAT EQUATIONS WITH MARKOVIAN SWITCHING

A THESSES SUBMITTED TO THE UNIVERSITY OF MANCHESTER
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN THE FACULTY OF SCIENCES AND ENGINEERING

2016

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Word count: 8028
Abstract

This Thesis consists of three parts. In the first part we recall some background theory that will be used throughout the thesis. In the second part, we studied the existence and uniqueness of solutions of the stochastic heat equations with Markovian switching. In the third part, we investigate the properties of solutions, such as Feller property, strong Feller property, and stability.
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Acknowledgements

I would like to express my sincere gratitude to my supervisor Professor Tusheng Zhang at University of Manchester. He has provided me a lot of encouragement and help not only in my study but also in my daily life. He showed me the beauty of stochastic analysis and led me to the research field. I learned a lot from him, not only the professional knowledge but also the attitude towards research. His advice will continue to bring me a far-reaching impact on my research.

Furthermore, I especially thank my parents for their permanent love, understand and support for me. I am so grateful to my husband, Xuan Wang. He unconditionally loves and supports me to finish my work and gives me a lot of advice and encouragement in life and study.

I thank all the staff and students in School of Mathematics and other friends I met in Manchester. They shared my enjoyments and sorrows and made my life here the most memorable.
Notations

Through this thesis, I numbered equations, lemmas, theorems, etc., separately per chapter.

I close all the proof with the symbol □. I give all the references by number enclosed with square brackets.

Table of Symbols

- a.s. /a.e. almost surely/almost everywhere
- s.t. such that
- w.r.t. with respect to
- Ω the whole probability space
- $\mathbb{S}$ the finite state space of a Markov chain
- $\mathbb{R}$ the set of all real number
- $\mathbb{R}^+$ the set of all non-negative real number
- $\mathcal{H}/\mathcal{K}$ the real separable Hilbert space
- $\mathbb{R}^d$ the d-dimensional Euclidean space
- $\theta$ the Markov Chain \{ $\theta(t)$: $t \geq 0$ \}
- $f: A \mapsto B$ the mapping $f$ from $A$ to $B$
- $A \subset B$ $A$ is subset of $B$
- $\mathbb{P}(\cdot), E(\cdot)$ probability, expectation
- $Q$ the trace class nonnegative operator on a Hilbert space $\mathcal{K}$
- $Q^{\frac{1}{2}}$ the Hilbert-Schmidt operator on $\mathcal{K}$
\( L^p(K; H) \)  
the space of all the p-integrable mappings from \( K \) to \( H \).

\( \mathcal{L}(K, H) \)  
the space of all bounded linear operators

\( \mathcal{L}_2^0 \)  
the space of all Hilbert-Schmidt operators

\( C(X; Y) \)  
the set of all the continuous mappings from \( X \) to \( Y \)

\( C_b^k(X) \)  
the set of all the bounded functions with continuous derivatives up to order \( k \) on \( X \)

\( C_0^\infty \)  
the set of infinitely continuously differentiable functions with compact support

\( \mathcal{P}(H \times \mathbb{S}) \)  
the space of probability measures on \( H \times \mathbb{S} \)

\( \mathbb{L} \)  
the family of mappings \( f: H \times \mathbb{S} \rightarrow \mathbb{R} \) satisfying  
\[ |f(u, r) - f(v, r')| \leq \|u - v\|_H + |r - r'| \text{ and } |f(\cdot, \cdot)| \leq 1 \]

\( \text{Sup} \)  
supermum

\( \text{Inf} \)  
inimum
Chapter 1: Introduction

1.1 Background

Stochastic modelling plays a key role in sciences and industries. In particular, if stochastic modelling systems have two parts, one takes values continuously and the other takes discrete values, they are called hybrid systems. Hybrid systems have been applied to model many practical systems. For instance, hybrid systems have been used to model the effect of federal housing removal policies on the stabilization of the housing sector in macroeconomics (Kazangey and Sworder, 1971). They have also become a basic framework to solve control-related issues in Battle Management Command, Control and Communications (BM/C3) systems (Athans, 1987). Other areas include electric power systems (Willsky and Rogers, 1979), solar thermal central receiver systems (Sworder and Robinson, 1973), target tracking, fault tolerant control and manufacturing processes (Mariton, 1990). Stochastic differential equations with Markovian switching have a number of important applications, such as population dynamics, financial modelling, stochastic stabilization and stochastic neural networks (Mao and Yuan, 2006). The limitation of previous studies only considers the hybrid systems in finite dimensions.
However, Da Prato and Zabczyk (1992) introduced the stochastic equations in infinite dimensions, did not consider any Markov chain. They studied the existence and uniqueness of solutions of general stochastic evolution equations in Hilbert space, and investigated the qualitative properties of solutions. There is no previous research on stochastic equations with Markovian switching in infinite dimensions. This thesis contributes to the existing literature by investigating the existence, uniqueness results and properties of the solutions of stochastic heat equations with Markovian switching (SHE-MS in short). We consider the heat conduction over thin wire $x \in D$ with a constant thermal diffusivity $\alpha > 0, k > 0$. Let $X(x,t)$ denote the temperature distribution at point $x$ and time $t > 0$ due to a spatially dependent white noise. Let $\theta(t)$ denote a continuous-time Markov chain taking values on a finite state space $\mathbb{S} = \{1,2,\ldots,N\}$. Suppose both ends of the wire are maintained at the freezing temperature. Then, given an initial temperature $h \in H = L^2(D)$, the temperature field $X(x,t)$ is governed by the initial-boundary value problem for SHE-MS:

$$\begin{align*}
\frac{\partial X(x,t)}{\partial t} &= (\kappa \Delta - \alpha)X(x, t) + f\left(X(x, t), t, \theta(t)\right) + \sigma\left(X(x, t), t, \theta(t)\right)\dot{W}(x, t); \\
X|_{\partial D} &= 0; \\
X(x, 0) &= h(x) \in H; \theta(0) = i \in \mathbb{S},
\end{align*}$$

(1.1.1)

Here the state vector has two components $X(\cdot, t)$ and $\theta(t)$: the first one is in general referred to as the state while the second is regarded as the mode. In its operation, the system will switch from one mode to another in a random way, and the switching between the modes is governed by a Markov chain.
This thesis is organised in four chapters to study the SHE-MS (1.1.1). In chapter 1, we introduce the background and state useful definitions and lemmas.

In chapter 2, we study the existence and uniqueness of solutions of SHE-MS (1.1.1). The system (1.1.1) is treated as nonlinear stochastic evolution equations in $L^2(D)$. The existence and uniqueness of mild solutions is usually associated with the semigroup approach, which is treated extensively in the book by Prato and Zabczyk (1992).

In chapter 3, we investigate Feller property and strong Feller property of Markov process $(X(t),\theta(t))$. This process is generated by $X(\cdot)$ is a continuous component taking values in $H = L^2(D)$ and $\theta(\cdot)$ is a jump component taking values in finite state space $\mathcal{S}$. In particular, the Markov process $\left( X^{h,i}(t), \theta^i(t) \right)$ is said to be Feller if the function $V(t,h,i) = E\psi \left( X^{h,i}(t), \theta^i(t) \right)$, for any $t > 0$, is continuous with respect to the initial data $(h,i) \in H \times \mathcal{S}$, and for any bounded and continuous function $\psi(\cdot; r), r \in \mathcal{S}$. Similarly, the Markov process $\left( X^{h,i}(t), \theta^i(t) \right)$ is said to be strong Feller if the function $V(t,h,i) = E\psi \left( X^{h,i}(t), \theta^i(t) \right)$, for any $t > 0$, is continuous with respect to the initial data $(h,i) \in H \times \mathcal{S}$, and for any bounded and measurable function $\psi(\cdot; r), r \in \mathcal{S}$.

In chapter 4, we study the asymptotic stability in distribution of SHE-MS. In particular, we introduce an approximating system and construct an appropriate metric between transition probability functions, to give sufficient conditions for stability in distribution of Markov process $(X(t),\theta(t))$. 
1.2 Preliminaries

In this section, we recall some background materials which will be used in the following chapters.

1.2.1 Hilbert Space Valued Wiener Processes

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). Without loss of generality, assume the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfies the usual conditions (i.e. it is increasing and right continuous while \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). Let \(H\) and \(K\) be two separable Hilbert spaces, with the inner products by \(<\cdot, \cdot>_H, <\cdot, \cdot>_K\) and norms by \(\|\cdot\|_H, \|\cdot\|_K\), respectively. Let \(Q\) be a trace class nonnegative operator on a Hilbert space \(K\). Let \(\{\lambda_n\}, n \in N\) be the eigenvalues of \(Q\) and \(\{e_n\}, n \in N\) be the corresponding eigenvectors. That is,

\[ Q e_n = \lambda_n e_n, \quad n = 1, 2, 3, \ldots \]

**Definition 1.2.1** (Prato and Zabczyk, 1992): A \(K\)-valued Wiener process \(W(t), t \geq 0\) is called a \(Q\)-Wiener process if

(i) \(W(0) = 0\),

(ii) \(W\) has continuous trajectories,

(iii) \(W\) has independent increments,

(iv) \(W(t) - W(s) \sim \mathcal{N}(0, (t - s)Q), t \geq s \geq 0\).
**Theorem 1.2.2** (Prato and Zabczyk, 1992, [Proposition 4.1, p.87]): Assume that $W$ is a $Q$-Wiener process, with $\text{tr}Q < +\infty$. Then the following statements hold.

(i) \( W \) is a Gaussian process on $K$ and

\[
E < W(t), g >= 0, E < W(t), g > < W(s), h > = (t \wedge s) < Qg, h >,
\]

\[\forall g, h \in K,\]

(ii) For arbitrary $t \geq 0$, $W$ is represented by

\[
W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t)e_j, t \geq 0,
\]

where \( \{\beta_j(t), j = 1,2,\ldots\} \) is a sequence of independent, identically distributed standard Brownian motions on probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), given by

\[
\beta_j(t) = \frac{1}{\sqrt{\lambda_j}} < W(t), e_j >, j = 1,2,\ldots
\]

To define stochastic integrals with respect to the $Q$-Wiener process $W(t), t \geq 0$, we introduce the subspace $K_0 = Q^{1/2}(K)$ of $K$ which, endowed with the inner product

\[
< u, v >_{K_0} = < Q^{-1/2}u, Q^{-1/2}v >_K, u, v \in K_0
\]

is a Hilbert space. Let \( \mathcal{L}_2^0 = \mathcal{L}_2(K_0, H) \) denote the space of all Hilbert-Schmidt operators from $K_0$ into $H$. The space \( \mathcal{L}_2^0 \) is also a separable Hilbert space, equipped with the norm

\[
\| \phi \|_{\mathcal{L}_2^0}^2 = \text{tr} \left( \phi Q^{1/2} \left( \phi Q^{1/2} \right)^* \right) \text{ for any } \phi \in \mathcal{L}_2^0.
\]

Clearly, for any bounded operators $\phi \in \mathcal{L}(K, H)$, this norm reduces to \( \| \phi \|_{\mathcal{L}_2^0}^2 = \text{tr}(\phi Q \phi^*) \).
Let $\phi: (0, \infty) \to L^0_2$ be a predictable, $\mathcal{F}_t$-adapted process such that for any $t \geq 0$,

$$\int_0^t E\|\phi(s)\|_{L^2_0}^2 \, ds < \infty.$$  

Then we can define an $H$-valued stochastic integral

$$\int_0^t \phi(s) \, dW(s), \quad t \geq 0,$$

which is a continuous square integrable martingale. For that construction, refer to Prato and Zabczyk (1992) p.90-96.

1.2.2 Continuous-time Markov Chain

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Without loss of generality, assume the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\theta(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, ..., N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{\theta(t + \delta) = j | \theta(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\delta + o(\delta) & \text{if } i = j \end{cases} \quad (1.2.1)$$

for $\delta > 0$. Here $\gamma_{ij} > 0$, it is the transition rate from $i$ to $j$, if $i \neq j$ while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$.

Note that the continuous-time Markov chain $\theta(t)$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ can be represented as a stochastic integral with respect to a Poisson random measure. Indeed, let
\( \Delta_{ij} > 0 \) be consecutive, left closed, right open intervals of the real line each having length \( \gamma_{ij} > 0 \). Define a function \( g: \mathbb{S} \times \mathbb{R} \rightarrow \mathbb{R} \) by

\[
g(i, y) = \begin{cases} j - i & \text{if } y \in \Delta_{ij} \\ 0 & \text{otherwise.} \end{cases}
\]

Then

\[
d\theta(t) = \int_{\mathbb{R}} g(\theta(t -), y)\nu(dt, dy),
\]

with initial value \( \theta(0) = i \), where \( \nu(dt, dy) \) is a Poisson random measure with intensity \( dt \times \mu(dy) \), in which \( \mu \) is the Lebesgue measure on \( \mathbb{R} \). In this thesis, we always assume that the Poisson random measure \( \nu(\cdot, \cdot) \) is independent of the \( Q \)-Wiener process \( W(\cdot) \), and the compensated Poisson random measure \( \tilde{\nu}(dt, dy) = \nu(dt, dy) - dt \times \mu(dy) \) is a martingale measure. For more information see (Ghosh et al. 1997) and (Skorohod, 1989).

### 1.2.3 Useful Lemmas

In this subsection, we list some lemmas which will be used frequently in the following chapters.

**Theorem 1.2.3 (Contraction Mapping Principle)** (p.119 Hutson, V., Pym, J.S. 1980) Let \( \mathcal{X} \) be a Banach space of random variable \( \xi(\omega) \) with norm \( \| \cdot \|_{\mathcal{X}} \). Let \( \mathcal{Y} \) be a closed subset of \( \mathcal{X} \). Suppose that the map \( \phi: \mathcal{Y} \rightarrow \mathcal{Y} \) is well defined. If there is a constant \( \rho \in [0,1) \) such that
\[ \| \phi(\xi) - \phi(\eta) \|_\mathcal{X} \leq \rho \| \xi - \eta \|_\mathcal{X}, \]

for any \( \xi, \eta \in \mathcal{Y} \), then there exists a unique solution \( \xi^* \in \mathcal{Y} \) of the equation \( \xi = \phi(\xi), \xi \in \mathcal{Y} \) which is the fixed point of the map \( \phi \). \( \square \)

**Lemma 1.2.4 (Gronwall Inequality) (p.36 Hale, J.K. 1969)** If \( \alpha \geq 0 \) is a constant, \( \beta(t) \geq 0 \), and \( \theta(t) \) are real continuous functions on \([a, b] \subset \mathbb{R}\) such that

\[ \theta(t) \leq \alpha + \int_a^t \beta(s)\theta(s)\,ds, \quad a \leq t \leq b, \]

then

\[ \theta(t) \leq \alpha \exp\left\{ \int_a^t \beta(s)\,ds \right\}, \quad a \leq t \leq b. \]
Chapter 2: Existence and Uniqueness of Solutions of SHE-MS

2.1 Introduction

In this chapter, we study the existence and uniqueness of solutions of stochastic heat equations with Markovian switching (SHE-MS in short). In particular, SHE-MS is treated as a nonlinear stochastic evolution equation in $L^2(D)$, the existence and uniqueness of mild solutions is associated with the semigroup approach, which is treated extensively in the book by Da Prato and Zabczyk, (1992).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\mathcal{F}_t_{t \geq 0}$. Without loss of generality, assume the filtration $\mathcal{F}_t_{t \geq 0}$ satisfies the usual conditions (i.e. it is right continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $D \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial D$. Denote $L^2(D) = H = K$ with inner products $\langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle_K$ and norms $\| \cdot \|_H, \| \cdot \|_K$, respectively. For any integer $k \geq 0$, let $H^k$ denote the $L^2$-Sobolev space of order $k$ which consists of order $L^2$-functions $\varphi$ with $k$-time generalized partial derivatives with norm
\[ \| \varphi \|_k = \left\{ \sum_{j=0}^k \sum_{i=1}^d \int_D \left| \frac{\partial^j \varphi(x)}{\partial x_i^j} \right|^2 dx \right\}^{1/2}. \]

In this thesis, we consider the following system of SHE-MS,

\[
\begin{cases} 
\frac{\partial X(x,t)}{\partial t} = (\kappa \Delta - \alpha)X(x,t) + f(X(x,t),t,\theta(t)) + \sigma(X(x,t),t,\theta(t))W(x,t); \\
x \in D; t \in (0,T); \\
X|_{\partial D} = 0; \\
X(x,0) = h(x) \in H; \theta(0) = i \in \mathbb{S},
\end{cases}
\]  

(2.1.1)

where \( \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \) is the Laplacian operator; \( \kappa, \alpha \) are positive constants; the non-linear terms \( f(u,t,r) \) and \( \sigma(u,t,r) \) satisfy properly formulated Lipschitz and linear growth conditions; \( W(x,t) \) be a spatially dependent white noise, by convention, is the formal time derivative \( \frac{\partial}{\partial t} W(x,t) \) of the Wiener random field \( W(x,t) \). Let \( W(\cdot, t) \) be an \( K \)-valued Wiener process with mean zero and covariance operator \( Q, \text{tr} Q < \infty \); initial value \( h(x) \) is a given function in \( H = L^2(D) \), and initial value \( i \in \mathbb{S} = \{1,2,..,N\} \), both of them are \( \mathcal{F}_0 \)-measurable; and let \( \theta(t), t \geq 0 \) be a right continuous Markov chain on probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) taking values in a finite space \( \mathbb{S} = \{1,2,..,N\} \) with generator \( \Gamma = (\gamma_{ij})_{N \times N} \) given by

\[
\mathbb{P}\{\theta(t + \delta) = j | \theta(t) = i\} = \begin{cases} 
\gamma_{ij} \delta + o(\delta) & \text{if } i \neq j \\\n1 + \gamma_{ii} \delta + o(\delta) & \text{if } i = j
\end{cases}
\]  

(2.1.2)

for \( \delta > 0 \). Here \( \gamma_{ij} > 0 \), it is the transition rate from \( i \) to \( j \), if \( i \neq j \) while \( \gamma_{ii} = -\sum_{i \neq j} \gamma_{ij} \).

The organization of this chapter is as follows: we state basic definitions in section 2.2. We establish the existence and uniqueness of solutions of equation (2.1.1) in section 2.3.
2.2 Preliminaries

In this section, we consider the SHE-MS (2.1.1) as the following stochastic evolution equation

\[ dX(t) = [AX(t) + F(X(t), t, \theta(t))]dt + \Sigma(X(t), t, \theta(t))dW(t), \quad t \geq 0, \]

\[ X(0) = h \in H, \theta(0) = i \in \mathbb{S}. \tag{2.2.1} \]

By convention, we write \( X(t) = X(\cdot, t), \quad F(X(t), t, \theta(t)) = f(X(\cdot, t), t, \theta(t)), \)

\( \Sigma(X(t), t, \theta(t)) = \sigma(X(\cdot, t), t, \theta(t)) \) and \( dW(t) = W(\cdot, dt). \) Here \( A = (\kappa \Delta - \alpha) \) for \( \kappa > 0, \alpha > 0; \)

\( A: D(A) \subset H \to H \) generates a contraction semigroup \( G(t), t \geq 0, \) on \( H = L^2(D), \) which satisfies the properties: \( \|G(t)h\| \leq \|h\|, \) for \( h \in H \) and \( (A\phi, \phi) \leq 0 \) for \( \phi \in D(A); W(t) \) be a \( Q \)-Wiener process in \( K = L^2(D) \) with \( trQ < \infty; \) initial values \( h \in H = L^2(D), \) and \( \theta(0) = i \in \mathbb{S} \) are \( \mathcal{F}_0 \)-measurable. Throughout this chapter, for the existence and uniqueness of the mild solutions, we shall impose the following assumptions:

**Hypothesis 2.2.1**

(H1) **(Lipschitz condition)** \( F(u, t, r) \) and \( \Sigma(u, t, r) \) are predictable random fields. There exists a constant \( K_1 > 0, \) such that

\[ \|F(u, t, r) - F(v, t, r)\|_H^2 + \|\Sigma(u, t, r) - \Sigma(v, t, r)\|_{L^2}^2 \leq K_1\|u - v\|_H^2, \]

for any \( u, v \in H, t \in [0, T], r \in \mathbb{S}. \)

(H2) **(Linear growth condition)** There exists a constant \( K_2 > 0, \) such that
\[ \|F(u, t, r)\|_H^2 + \| \Sigma(u, t, r) \|_{L_2^0}^2 \leq K_2 (1 + \|u\|_H^2). \]

for any \((u, t, r) \in H \times [0, T] \times \mathbb{S}.

**Definition 2.2.2** A \(H\)-valued stochastic process \(X(t), t \in [0, T]\), is called a mild solution of (2.2.1) if

(i) \(X(t)\) is adapted to \(\mathcal{F}_t\) and continuous in \(t\);

(ii) \(X(t)\) is measurable and \(\int_0^T \|X(t)\|_H^2 dt < \infty\);

(iii) For any \(t \in [0, T]\), equation

\[
X(t) = G(t)h + \int_0^t G(t-s)F(X(s), s, \theta(s))ds \\
+ \int_0^t G(t-s)\Sigma(X(s), s, \theta(s))dW(s). \tag{2.2.2}
\]

In order to prove existence, uniqueness theorem, we need some results.

**Lemma 2.2.3** (Prato and Zabczyk, 1992) For any \(p \geq 1\), and for arbitrary \(L_2^0\)-valued predictable process \(\phi(t), t \in [0,T]\), we have

\[
E \sup_{0 \leq s \leq t} \left( \int_0^s \phi(u)dW(u) \right)^{2p} \leq C_p E \left( \int_0^t \|\phi(s)\|_{L_2^0}^2 ds \right)^p,
\]

where \(C_p = (p(2p-1))^{\frac{p}{2p}}(\frac{2p}{2p-1})^{2p} \). □

**Theorem 2.2.4** (Chow, 2007) Suppose that \(G(t)\) be a contraction semigroup on \(H\) and for arbitrary \(L_2^0\)-valued predictable process \(\phi(t), t \in [0,T]\) satisfies the condition:

\[
E \int_0^T [tr(\phi(s)Q\phi^*(s))]^p ds < \infty,
\]
for $p \geq 1$. Then there exists constant $C_p > 0$ such that, for any $t \in [0, T],$

$$E \sup_{0 \leq s \leq t} \left\| \int_0^t G(t - s) \phi(s) dW(s) \right\|_{L^p_H}^2 \leq C_p E \left[ \int_0^t \| \phi(s) \|_{L^2_\mathcal{F}}^2 ds \right]^p,$$

where $\| \phi(s) \|_{L^2_\mathcal{F}}^2 = \text{tr} (\phi(s) Q \phi^*(s))$. □

### 2.3 Existence and Uniqueness of Mild Solutions

The following theorem shows that the existence and uniqueness of mild solutions of SHE-MS (2.2.1).

**Theorem 2.3.1 (Existence and Uniqueness Theorem)** Assume that hypothesis 2.2.1 holds, and let $h$ be a $\mathcal{F}_0$-measurable such that $E \| h \|^{2p} < \infty$ for $p \geq 1$. Then SHE-MS (2.2.1) has a unique (mild) solution $X(t), t \in [0, T]$ which is a continuous adapted process in $H$ such that $X \in L^{2p}(\Omega; C([0, T]; H))$ satisfying

$$E \sup_{0 \leq t \leq T} \| X(t) \|^{2p} \leq b_2 \{ 1 + E \| h \|^{2p} \}, \quad (2.3.1)$$

for some constant $b_2 > 0$, depending on $T, p, K_2$.

**Proof:** **Step 1. Uniqueness.** Suppose that $X(t)$ and $Y(t)$ are both mild solutions satisfying the integral equation (2.2.2). Let $C = \sup_{0 \leq t \leq T} \| G(t) \|_{L(H)}$. By making use of theorem 2.2.4, the holder inequality and Lipschitz condition (H1), we consider
$$E\|X(t) - Y(t)\|_H^2 = E\left\| \int_0^t G(t - s)\left[ F(X(s), s, \theta(s)) - F(Y(s), s, \theta(s)) \right] ds \right\|_H^2$$

$$+ \int_0^t G(t - s)\left[ \Sigma(X(s), s, \theta(s)) - \Sigma(Y(s), s, \theta(s)) \right] dW(s) \right\|_H^2$$

$$\leq 2\{C^2E\left\| \int_0^t \left[ F(X(s), s, \theta(s)) - F(Y(s), s, \theta(s)) \right] ds \right\|_H^2$$

$$+ 4E \int_0^t \left\| \Sigma(X(s), s, \theta(s)) - \Sigma(Y(s), s, \theta(s)) \right\|_{L_2}^2 ds \right\}$$

$$\leq 2\{C^2TE\int_0^t \left\| F(X(s), s, \theta(s)) - F(Y(s), s, \theta(s)) \right\|_H^2 ds$$

$$+ 4E \int_0^t \left\| \Sigma(X(s), s, \theta(s)) - \Sigma(Y(s), s, \theta(s)) \right\|_{L_2}^2 ds \right\}$$

$$\leq 2K_1(C^2T + 4) \int_0^t E\|X(s) - Y(s)\|_H^2 ds.$$
Then $\mathcal{H}_{p,T}$ is a Banach space under the norm:

$$\|X\|_{p,T} = \left\{ E \sup_{0 \leq t \leq T} \| X(t) \|^{2p} \right\}^{1/2p}. \tag{2.3.3}$$

Define an operator $\Lambda: \mathcal{H}_{p,T} \rightarrow \mathcal{H}_{p,T}$ as follows:

$$\Lambda(t)X = G(t)h + \int_{0}^{t} G(t-s)F(X(s), s, \theta(s))ds$$

$$+ \int_{0}^{t} G(t-s)\Sigma(X(s), s, \theta(s))dW(s), \tag{2.3.4}$$

for $t \in [0, T], X \in \mathcal{H}_{p,T}$.

To show the operator $\Lambda$ is well defined, let $I_1(t) = G(t)h;

$$I_2(t) = \int_{0}^{t} G(t-s)F(X(s), s, \theta(s))ds; I_3(t) = \int_{0}^{t} G(t-s)\Sigma(X(s), s, \theta(s))dW(s).$$

So that $\Lambda(t)X = I_1(t) + I_2(t) + I_3(t)$. Then

$$\|\Lambda X\|_{p,T} = \left\{ E \sup_{0 \leq t \leq T} \| \Lambda(t)X \|^{2p} \right\}^{1/2p}$$

$$= \left\{ E \sup_{0 \leq t \leq T} \| I_1(t) + I_2(t) + I_3(t) \|^{2p} \right\}^{1/2p}. \tag{2.3.5}$$

We shall estimate (2.3.5) separately. First, we have

$$E \sup_{0 \leq t \leq T} \| I_1(t) \|^{2p} = E \sup_{0 \leq t \leq T} \| G(t)h \|^{2p} \leq C_2^p E \| h \|^{2p} \leq C_1 E \| h \|^{2p}. \tag{2.3.6}$$

By making use of the Hölder inequality and the linear growth condition (H2), we have

$$E \sup_{0 \leq t \leq T} \| I_2(t) \|^{2p} = E \sup_{0 \leq t \leq T} \left\| \int_{0}^{t} G(t-s)F(X(s), s, \theta(s))ds \right\|^{2p}$$
\[
\leq C^{2p} E \left\| \int_0^T F(X(s), s, \theta(s)) ds \right\|^{2p}
\]
\[
\leq C^{2p} T^{2p-1} E \int_0^T \| F(X(s), s, \theta(s)) \|^{2p} ds
\]
\[
\leq C^{2p} T^{2p-1} K_2^p E \int_0^T (1 + \| X(s) \|^{2p}) ds
\]
\[
\leq C^{2p} T^{2p-1} K_2^p 2^{p-1} E \int_0^T (1 + \| X(s) \|^{2p}) ds
\]
\[
\leq C^{2p} T^{2p} K_2^p 2^{p-1} [1 + \| X \|^{2p}_{p,T}]
\]
\[
\leq C_2 (1 + \| X \|^{2p}_{p,T}), \quad (2.3.7)
\]

Where the positive constant \( C_2 \), depending on \( p, T, K_2 \).

Similarly, by theorem 2.2.4, the Hölder inequality and the linear growth condition (H2), we have

\[
E \sup_{0 \leq s \leq T} \| I_3(t) \|^{2p} = E \sup_{0 \leq s \leq T} \left\| \int_0^t G(t - s) \Sigma(X(s), s, \theta(s)) dW(s) \right\|^{2p}
\]
\[
\leq C_p E \left[ \int_0^T \| \Sigma(X(s), s, \theta(s)) \|_{L_2^2} ds \right]^p
\]
\[
\leq K_2^p T^{p-1} C_p E \int_0^T (1 + \| X(s) \|^{2p}) ds
\]
\[
\leq K_2^p 2^{p-1} C_p 2^{p-1} E \int_0^T (1 + \| X(s) \|^{2p}) ds
\]
\[
\leq K_2^p 2^{p-1} T^p C_p (1 + \| X \|^{2p}_{p,T})
\]
\[ \leq C_3(1 + \|X\|_{p,T}^{2p}), \quad (2.3.8) \]

where the positive constant \(C_3\), depending on \(p, T, K_2\).

Therefore, substitute (2.3.6) (2.3.7) (2.3.8) into (2.3.5), we have

\[
\|AX\|_{p,T}^{2p} = \sup_{0 \leq t \leq T} \|A(t)X\|^{2p} \\
\leq 3^{2p-1} \{C_1 E \|h\|^{2p} + C_2 \left(1 + \|X\|_{p,T}^{2p}\right) + C_3 \left(1 + \|X\|_{p,T}^{2p}\right)\} \\
\leq b_1 \left(1 + E \|h\|^{2p} + \|X\|_{p,T}^{2p}\right),
\]

where \(b_1 > 0\) is constant depending on \(T, K_2, p\). It is shown that the map \(A: \mathcal{H}_{p,T} \to \mathcal{H}_{p,T}\) is well-defined as asserted.

**Step 3.** We will show that \(A\) is a contraction mapping for a small \(T = T_1\). To this end, for \(X, X' \in \mathcal{H}_{p,T}\), we define

\[
J_1(t) = \int_0^t G(t - s)\left[F(X(s), s, \theta(s)) - F(X'(s), s, \theta(s))\right]ds,
\]

\[
J_2(t) = \int_0^t G(t - s)\left[\Sigma(X(s), s, \theta(s)) - \Sigma(X'(s), s, \theta(s))\right]dW(s).
\]

Then

\[
\|AX - AX'\|_{p,T}^{2p} = \sup_{0 \leq t \leq T} \|AX(t) - AX'(t)\|^{2p} \\
= \sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s)\left[F(X(s), s, \theta(s)) - F(X'(s), s, \theta(s))\right]ds \\
+ \int_0^t G(t - s)\left[\Sigma(X(s), s, \theta(s)) - \Sigma(X'(s), s, \theta(s))\right]dW(s) \right\|^{2p}
\]
\[ \begin{align*}
= E\sup_{0 \leq t \leq T} \| J_1(t) + J_2(t) \|^{2p} \\
\leq 2^{2p-1} (\| J_1 \|_{p,T}^{2p} + \| J_2 \|_{p,T}^{2p}). & \tag{2.3.9}
\end{align*} \]

Then, using the Hölder inequality, and the Lipschitz condition (H1), we get

\[ \begin{align*}
\| J_1 \|_{p,T}^{2p} &= E\sup_{0 \leq t \leq T} \| \int_0^t G(t-s) [F(X(s), s, \theta(s)) - F(X'(s), s, \theta(s))] ds \|^{2p} \\
&\leq C^p E \| \int_0^T [F(X(s), s, \theta(s)) - F(X'(s), s, \theta(s))] ds \|^{2p} \\
&\leq C^p T^{2p-1} E \int_0^T \| F(X(s), s, \theta(s)) - F(X'(s), s, \theta(s)) \|^{2p} ds \\
&\leq C^p T^{2p-1} K_1^p E \int_0^T \| X(s) - X'(s) \|^{2p} ds \\
&\leq C^p T^{2p} K_1^p \| X - X' \|_{p,T}^{2p}. & \tag{2.3.10}
\end{align*} \]

and using the Hölder inequality, the Lipschitz condition (H1) and theorem 2.2.4, we have

\[ \begin{align*}
\| J_2 \|_{p,T}^{2p} &= E\sup_{0 \leq t \leq T} \| \int_0^t G(t-s) \left[ \Sigma(X(s), s, \theta(s)) - \Sigma(X'(s), s, \theta(s)) \right] dW(s) \|^{2p} \\
&\leq C_p E \left[ \int_0^T \| \Sigma(X(s), s, \theta(s)) - \Sigma(X'(s), s, \theta(s)) \|_{L^2}^2 ds \right]^p \\
&\leq C_p T^{p-1} E \int_0^T \| \Sigma(X(s), s, \theta(s)) - \Sigma(X'(s), s, \theta(s)) \|_{L^2}^{2p} ds \\
&\leq C_p T^{p-1} K_1^p E \int_0^T \| X(s) - X'(s) \|^{2p} ds \\
&\leq C_p T^p K_1^p \| X - X' \|_{p,T}^{2p}, & \tag{2.3.11}
\end{align*} \]
Therefore, substitute (2.3.10) and (2.3.11) into (2.3.9), we have

\[ \|AX - AX'\|_{p,T}^{2p} = E\text{sup}_{0 \leq t \leq T} \| \Lambda(t)X - \Lambda(t)X' \|_{p,T}^{2p} \]

\[ \leq 2^{2p-1}(\|J_1\|_{p,T}^{2p} + \|J_2\|_{p,T}^{2p}) \]

\[ \leq 2^{2p-1}(C^{2p}T^{2p}K_1^p + C_pT^pK_1^p) \|X - X'\|_{p,T}^{2p} \]

\[ \leq 2^{2p}K_1^p(C^{2p} + C_p)T^{2p} \|X - X'\|_{p,T}^{2p}, \]

or

\[ \|AX - AX'\|_{p,T} \leq \rho(T)\|X - X'\|_{p,T}, \]

where \( \rho(T) = 2\sqrt{K_1}(C^{2p} + C_p)^{1/2}T \). Let \( T = T_1 \) be sufficiently small so that \( \rho(T_1) < 1 \).

Then \( \Lambda \) is a Lipschitz-continuous contraction mapping in \( H_{p,T} \) which has a unique fixed point \( X \). This element \( X(t) \) is the unique mild solution of (2.2.1). For \( T > T_1 \), we can extend the solution by continuation, from \( T_1 \) to \( T_2 \) and so on. This completes the existence and uniqueness proof.

**Step 4.** We verify the inequality (2.3.1). We use the linear growth condition (H2), theorem 2.2.4, and the Hölder inequality,

\[ E\text{sup}_{0 \leq t \leq T} \|X(t)\|^{2p} \leq 3^{2p-1}[E\text{sup}_{0 \leq t \leq T} \|G(t)h\|^{2p} \]

\[ + E\text{sup}_{0 \leq t \leq T} \| \int_0^t G(t - s)F(X(s), s, \theta(s))ds\|^{2p} \]

\[ + E\text{sup}_{0 \leq t \leq T} \| \int_0^t G(t - s)\Sigma(X(s), s, \theta(s))dW(s)\|^{2p} \]
\[
\leq 3^{2p-1} \{ C^{2p} E \| h \|^{2p} + C^{2p} E \left[ \| \int_0^T F(s, s, \theta(s)) ds \|^2 \right]^p
\]
\[
+ C_p E \left[ \int_0^T \| \Sigma(s, s, \theta(s)) \|_{L_2}^2 ds \right]^p \}
\]
\[
\leq 3^{2p-1} \{ C^{2p} E \| h \|^{2p} + C^{2p} T^p E \left[ \| F(s, s, \theta(s)) \|^2 ds \right]^p
\]
\[
+ C_p E \left[ \int_0^T \| \Sigma(s, s, \theta(s)) \|_{L_2}^2 ds \right]^p \}
\]
\[
\leq 3^{2p-1} \{ C^{2p} E \| h \|^{2p} + C^{2p} T^p K_2^p E \left[ \int_0^T (1 + \| X(s) \|^2) ds \right]^p
\]
\[
+ C_p K_2^p E \left[ \int_0^T (1 + \| X(s) \|^2) ds \right]^p \}
\]
\[
\leq 3^{2p-1} \{ C^{2p} E \| h \|^{2p} + (C^{2p} T^p K_2^p + C_p K_2^p) E \left[ \int_0^T (1 + \| X(s) \|^2) ds \right]^p \}
\]
\[
\leq 3^{2p-1} \{ C^{2p} E \| h \|^{2p} + 2^{p-1} K_2^p (C^{2p} T^p + C_p) [T^p + E \left[ \int_0^T \| X(s) \|^2 ds \right]^p] \}
\]
\[
\leq 3^{2p-1} \{ C^{2p} E \| h \|^{2p} + 2^{p-1} K_2^p (C^{2p} T^p + C_p) [T^p + T^{p-1} E \left[ \int_0^T \| X(s) \|^2 ds \right]] \}
\]
\[
\leq 3^{2p-1} \{ C^{2p} E \| h \|^{2p} + 2^{p-1} K_2^p (C^{2p} T^p + C_p) T^p \left[ 1 + T^{\frac{p-1}{p}} E \left[ \int_0^T \| X(s) \|^2 ds \right] \right] \}
\]
\[
\leq C_4 \left\{ 1 + E \| h \|^{2p} + T^{\frac{p-1}{p}} \int_0^T E \sup_{0 \leq s \leq t} \| X(s) \|^{2p} dt \right\},
\]
which, by the Gronwall inequality, implies the inequality (2.3.1). The constant $b_2$ depends on $T, p, K_2$. \qed
Chapter 3: Feller Property and Strong Feller Property

3.1 Introduction

In this chapter, we study the Markov processes \( (X(t), \theta(t)) \). This process is a two-component Markov process such that \( X(\cdot) \) is a continuous component taking values in \( H = L^2(D) \) and \( \theta(\cdot) \) is a jump component taking values in a finite state \( S \). In particular, we mention that the pioneering works are on Feller and strong Feller properties for diffusions by W. Feller and E.B. Dynkin, the study of such properties for stochastic processes has drawn much attention. For instance, the strong feller property for diffusions on Hilbert spaces has been treated by Peszat and Zabczyk (1995). The Feller property for a particular class of regime-switching diffusions has been established by Ghosh, Arapostathis, and Marcus (1993). Later, strong Feller property for regime-switching diffusions has been announced by Ghosh, Arapostathis, and Marcus (1997). The Feller property for stochastic differential equations with Markovian switching has been studied by Mao, and Yuan (2006). In this chapter, we aim to study the properties of solutions to stochastic heat
equations with Markovian switching (SHE-MS), such as Feller property and strong Feller property.

The organization of this chapter is as follows: We state the basic definition in section 3.2. In section 3.3 and section 3.4, we devote to study the Feller property and strong Feller property.

3.2 Preliminaries

In this section, we study \((X(t), \theta(t))\) is a Markov process. To prove the following theorem, we can consider the theorem 3.27 in Mao and Yuan (2006). Denote by \(\mathcal{B}_b(H)\) be the space of all real bounded and measurable functions in \(H\). Denote by \(C_b(H)\) be the set of all uniformly continuous and bounded functions in \(H\), and by \(C^k_b(H)\) be the set of all \(k\) times continuously differentiable functions with their first \(k\) derivatives bounded.

**Theorem 3.2.1** Let \(X(t) \in H = L^2(D)\) be a solution of the equation

\[
dX(t) = [AX(t) + F(X(t), t, \theta(t))]dt + \Sigma(X(t), t, \theta(t))dW(t), t \geq 0; X(0) = h, (3.2.1)
\]

whose coefficients satisfy the conditions of the existence and uniqueness theorem 2.3.1. Then \((X(t), \theta(t))\) is a Markov process whose transition probability is defined by

\[
P((s,h,i); t, A \times \{j\}) = \mathbb{P}\left\{X^{h,i}_s(t) \in A \times \{j\}\right\}, \quad (3.2.2)
\]
for \((h,i) \in H \times S\), \(A \in \mathcal{B}(H)\) and \(j \in S\), where \(X^{h,i}_s(t)\) be a solution of the problem (3.2.1) on \(t \geq s\) with initial data \(X(s) = h, \theta(s) = i \in S\), both of initial data is \(\mathcal{F}_s\) - measurable. That is

\[
X^{h,i}_s(t) = G(t - s)h + \int_s^t G(t - u)F(X^{h,i}_s(u), u, \theta^{i}_s(u))du \\
+ \int_s^t G(t - u)\Sigma(x^{h,i}_s(u), u, \theta^{i}_s(u))dW(u), \tag{3.2.3}
\]

on \(t \geq s\), where \(\theta^{i}_s(t)\) stands for the Markov chain on \(t \geq s\) starting from state \(i\) at time \(t = s\). □

### 3.3 Feller Property

In this section, we consider the Feller property of the Markov process \((X(t), \theta(t))\) in \(H \times S\), it can be stated as follows: for any \(i \in S, \delta > 0\) and any bounded continuous function \(\psi: \mathbb{R}^d \to \mathbb{R}\), the mapping

\[
(h, s) \rightarrow \sum_{j \in S} \int_{\mathbb{R}^d} \psi(y) P(s, (h,i); s + \delta, dy \times \{j\}) = E\psi(X^{h,i}_s(s + \delta))
\]

is continuous.

**Lemma 3.3.1** Assume that hypothesis 2.2.1 holds. For \((h,i) \in H \times S\), and \(0 \leq s \leq t < \infty\), let \(X^{h,i}_s(t)\) and \(\theta^{i}_s(t)\) are defined in Theorem 3.2.1. Then for any \(T > 0\),
\[ E \| X_s^{h,i}(t) - X_u^{q,i}(t) \|_H^2 \leq 5C \| (G(u - s)h - q) \|_2^2 + 5CK_2(u - s)^2 + 20K_2(u - s) \]
\[ + (40CK_2T^2 + 160K_2T)(C'(u - s) + o(u - s)) \] 
\[ + b_2(1 + E \| h \|_2^2) \] \exp(10CT^2K_1 + 40K_1T),

if \( 0 \leq s < u < t \leq T \), \( h \) is \( \mathcal{F}_s \) measurable, \( q \) is \( \mathcal{F}_u \) measurable and \( E \| h \|_2 < \infty \) and \( E \| q \|_2 < \infty \) where \( C, C' \) are positive constants.

Proof: For \( 0 \leq s \leq u \leq t \leq T \), \( p \geq 1 \), we get

\[ E \| X_s^{h,i}(t) - X_u^{q,i}(t) \|_H^2 = E \left\| G(t - s)h - G(t - u)q + \int_s^t G(t - v)F \left( X_s^{h,i}(v), \theta_s^i(v) \right) dv \right\|_H^2 
- \int_u^t G(t - v)F \left( X_u^{q,i}(v), \theta_u^i(v) \right) dv 
+ \int_s^t G(t - v) \Sigma \left( X_s^{h,i}(v), \theta_s^i(v) \right) dW(v) 
- \int_u^t G(t - v) \Sigma \left( X_u^{q,i}(v), \theta_u^i(v) \right) dW(v) \|_H^2 
= E \left\| G(t - s)h - G(t - u)q + \int_s^t G(t - v)F \left( X_s^{h,i}(v), \theta_s^i(v) \right) dv \right\|_H^2 
- \int_u^t G(t - v)F \left( X_s^{h,i}(v), \theta_s^i(v) \right) dv 
+ \int_u^t G(t - v) \left[ F \left( X_s^{h,i}(v), \theta_s^i(v) \right) - F \left( X_u^{q,i}(v), \theta_u^i(v) \right) \right] dv 
+ \int_s^t G(t - v) \Sigma \left( X_s^{h,i}(v), \theta_s^i(v) \right) dW(v) \]
\[-\int_u^t G(t-v) \Sigma \left(X_s^{h,i}(v), v, \theta^{i}_s(v)\right) dW(v)\]

\[+ \int_u^t G(t-v) \left[ \Sigma \left(X_s^{q,i}(v), v, \theta^{i}_s(v)\right) - \Sigma \left(X_u^{q,i}(v), v, \theta^{i}_u(v)\right)\right] dW(v)\]

\[\leq 5\{E\|G(t-s)h - G(t-u)q\|_H^2 + E \left\| \int_s^u G(t-v)F \left(X_s^{h,i}(v), v, \theta^{i}_s(v)\right) dW(v) \right\|_H^2\]

\[+ E \left\| \int_s^u G(t-v) \Sigma \left(X_s^{h,i}(v), v, \theta^{i}_s(v)\right) dW(v) \right\|_H^2\]

\[+ E \left\| \int_u^t G(t-v) \left[ F \left(X_s^{q,i}(v), v, \theta^{i}_s(v)\right) - F \left(X_u^{q,i}(v), v, \theta^{i}_u(v)\right)\right] dW(v) \right\|_H^2\]

\[+ E \left\| \int_u^t G(t-v) \left[ \Sigma \left(X_s^{q,i}(v), v, \theta^{i}_s(v)\right) - \Sigma \left(X_u^{q,i}(v), v, \theta^{i}_u(v)\right)\right] dW(v) \right\|_H^2\}.

\[(3.3.2)\]

Using the Hölder inequality, the linear growth condition (H2), and theorem 2.3.1, then there exists a positive constant $b_2$,
\[ E \left\| \int_{s}^{u} G(t - v) F \left( X_{s}^{h,i}(v), v, \theta_{s}^{i}(v) \right) \, dv \right\|_{H}^{2} \leq C(u - s) E \int_{s}^{u} \left\| F \left( X_{s}^{h,i}(v), v, \theta_{s}^{i}(v) \right) \right\|_{H}^{2} \, dv \]
\[ \leq C(u - s) K_{2} E \int_{s}^{u} \left( 1 + \left\| X_{s}^{h,i}(v) \right\|_{H}^{2} \right) \, dv \]
\[ \leq C(u - s)^{2} K_{2} \left( 1 + b_{2} \left( 1 + E\| h \|^{2} \right) \right), \quad (3.3.3) \]

where \( \| G(t) \|_{L(H)}^{2} \leq C \).

Using theorem 2.2.4, the linear growth condition (H2), and theorem 2.3.1, we have
\[ E \left\| \int_{s}^{u} G(t - v) \Sigma \left( X_{s}^{h,i}(v), v, \theta_{s}^{i}(v) \right) \, dW(v) \right\|_{H}^{2} \leq 4 E \int_{s}^{u} \left\| \Sigma \left( X_{s}^{h,i}(v), v, \theta_{s}^{i}(v) \right) \right\|_{L_{2}}^{2} \, dv \]
\[ \leq 4 K_{2} E \int_{s}^{u} \left( 1 + \left\| X_{s}^{h,i}(v) \right\|_{H}^{2} \right) \, dv \]
\[ \leq 4 K_{2} (u - s) \left( 1 + b_{2} \left( 1 + E\| h \|^{2} \right) \right). \quad (3.3.4) \]

Using the Hölder inequality and the Lipschitz condition (H1), we have
\[ E \left\| \int_{u}^{t} G(t - v) \left[ F \left( X_{s}^{h,i}(v), v, \theta_{s}^{i}(v) \right) - F \left( X_{u}^{q,i}(v), v, \theta_{u}^{i}(v) \right) \right] \, dv \right\|_{H}^{2} \leq C(t - u) E \int_{u}^{t} \left\| F \left( X_{s}^{h,i}(v), v, \theta_{s}^{i}(v) \right) - F \left( X_{u}^{q,i}(v), v, \theta_{u}^{i}(v) \right) \right\|_{H}^{2} \, dv \]
\[ \leq 2C(t - u) \left\{ E \int_{u}^{t} \left\| F \left( X_{s}^{h,i}(v), v, \theta_{s}^{i}(v) \right) - F \left( X_{u}^{q,i}(v), v, \theta_{u}^{i}(v) \right) \right\|_{H}^{2} \, dv \right\} \]
\[ + E \int_{u}^{t} \left\| F \left( X_{u}^{q,i}(v), \theta_{s}^{i}(v) \right) - F \left( X_{u}^{q,i}(v), \theta_{u}^{i}(v) \right) \right\|_{H}^{2} \, dv \} \]
\[
\leq 2CT \left\{ K_1 \int_u^t E \left\| X^{h,i}_s(v) - X^{q,i}_u(v) \right\|_H^2 dv + E \int_u^t \left\| F \left( X^{q,i}_u(v), \theta^i_s(v) \right) - F \left( X^{q,i}_u(v), \theta^i_u(v) \right) \right\|_H^2 dv \right\}, (3.3.5)
\]

Note that \( X^{q,i}_u(v) \) and the indicator function \( I_{(\theta^i_s(v) \neq \theta^i_u(v))} \) are conditional independent with respect to the \( \sigma \)–algebra generated by \( \theta^i_s(v) \). We compute, using the linear growth condition (H2), that

\[
E \int_u^t \left\| F \left( X^{q,i}_u(v), \theta^i_s(v) \right) - F \left( X^{q,i}_u(v), \theta^i_u(v) \right) \right\|_H^2 dv
\]

\[
\leq 2E \int_u^t \left[ \left\| F \left( X^{q,i}_u(v), \theta^i_s(v) \right) \right\|_H^2 + \left\| F \left( X^{q,i}_u(v), \theta^i_u(v) \right) \right\|_H^2 \right] I_{(\theta^i_s(v) \neq \theta^i_u(v))} dv
\]

\[
\leq 4K_2 E \int_u^t \left( 1 + \left\| X^{q,i}_u(v) \right\|_H^2 \right) I_{(\theta^i_s(v) \neq \theta^i_u(v))} dv
\]

\[
\leq 4K_2 E \int_u^t \left[ \left( 1 + \left\| X^{q,i}_u(v) \right\|_H^2 \right) \right] I_{(\theta^i_s(v) \neq \theta^i_u(v))} \left| \theta^i_s(v) \right| dv
\]

\[
\leq 4K_2 \int_u^t E \left[ \left( 1 + \left\| X^{q,i}_u(v) \right\|_H^2 \right) \right] \left| \theta^i_s(v) \right| dv
\]

But, by the Markov property,

\[
E \left[ I_{(\theta^i_s(v) \neq \theta^i_u(v))} \left| \theta^i_s(v) \right| \right] = \sum_{j \in \mathbb{S}} I_{\left[ \theta^i_s(v) = j \right]} P \{ \theta^i_u(v) \neq j \mid \theta^i_s(v) = j \}
\]

\[
= \sum_{j \in \mathbb{S}} I_{\left[ \theta^i_s(v) = j \right]} \sum_{j \neq k} (\gamma_{jk}(u-s) + o(u-s))
\]

\[
\leq \max_{i \in \mathbb{S}} (-\gamma_{ii})(u-s) + o(u-s)
\]
\[ \leq C'(u - s) + o(u - s), \quad (3.3.6) \]

then, applying theorem 2.3.1, there exists a positive constant \( b_2 \),

\[
E \int_u^t \left\| F \left( X_u^{q,i}(v), v, \theta_u^i(v) \right) - F \left( X_u^{q,i}(v), v, \theta_s^i(v) \right) \right\|_H^2 dv \\
\leq 4K_2 T \left[ 1 + b_2 (1 + E\|h\|^2) \right] [C'(u - s) + o(u - s)]. \quad (3.3.7)
\]

Hence (3.3.5) becomes

\[
E \left\| \int_u^t G(t - v) \left[ F \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) - F \left( X_u^{q,i}(v), v, \theta_u^i(v) \right) \right] \right\|_H^2 dv \\
\leq 2CT \left\{ 4K_2 T \left[ 1 + b_2 (1 + E\|h\|^2) \right] [C'(u - s) + o(u - s)] \right\} \\
+ K_1 \int_u^t E \left\| X_s^{h,i}(v) - X_u^{q,i}(v) \right\|_H^2 dv. \quad (3.3.8)
\]

Then, using theorem 2.2.4 we can show

\[
E \left\| \int_u^t G(t - v) \left[ \Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) - \Sigma \left( X_u^{q,i}(v), v, \theta_u^i(v) \right) \right] dW(v) \right\|_H^2 \\
\leq 4E \int_u^t \Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) - \Sigma \left( X_u^{q,i}(v), v, \theta_u^i(v) \right) \right\|_{\mathcal{L}_2^2}^2 dv \\
\leq 8 \left\{ E \int_u^t \Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) - \Sigma \left( X_u^{q,i}(v), v, \theta_u^i(v) \right) \right\|_{\mathcal{L}_2^2}^2 dv \\
+ E \int_u^t \Sigma \left( X_u^{q,i}(v), v, \theta_u^i(v) \right) - \Sigma \left( X_u^{q,i}(v), v, \theta_u^i(v) \right) \right\|_{\mathcal{L}_2^2}^2 dv \right\}, \quad (3.3.9)
\]

Using the Lipschitz condition (H1), we have
\[ E \int_u^t \| \Sigma \left( X_s^{h,i}(v), v, \theta_s^{i}(v) \right) - \Sigma \left( X_u^{q,i}(v), v, \theta_u^{i}(v) \right) \|_{L_2^2}^2 \, dv \]
\[ \leq K_1 \int_u^t E \| X_s^{h,i}(v) - X_u^{q,i}(v) \|_{H}^2 \, dv. \quad (3.3.10) \]

We use the similar way as (3.3.6)-(3.3.8) to get

\[ E \int_u^t \| \Sigma \left( X_u^{q,i}(v), v, \theta_u^{i}(v) \right) - \Sigma \left( X_u^{q,i}(v), v, \theta_u^{i}(v) \right) \|_{L_2^2}^2 \, dv \]
\[ \leq 4K_2 T \left[ 1 + b_2 (1 + E\|h\|^2) \right] \left[ C'(u - s) + o(u - s) \right]. \quad (3.3.11) \]

Hence, substitute (3.3.11) and (3.3.10) into (3.3.9), we have

\[ E \left\| \int_u^t G(t - v) \left[ \Sigma \left( X_s^{h,i}(v), v, \theta_s^{i}(v) \right) - \Sigma \left( X_u^{q,i}(v), v, \theta_u^{i}(v) \right) \right] \, dW(v) \right\|_{H}^2 \]
\[ \leq 8 \left\{ 4K_2 T \left[ 1 + b_2 (1 + E\|h\|^2) \right] \left[ C'(u - s) + o(u - s) \right] \right\} \]
\[ + K_1 \int_u^t E \| X_s^{h,i}(v) - X_u^{q,i}(v) \|_{H}^2 \, dv \}. \quad (3.3.12) \]

Using the properties of the semigroup of bounded linear operators on \( H \), we have

\[ \| G(t - s)h - G(t - u)q \|_2^2 = \| G(t - u + u - s)h - G(t - u)q \|_2^2 \]
\[ = \| G(t - u) (G(u - s)h) - G(t - u)q \|_2^2 \]
\[ = \| G(t - u) (G(u - s)h - q) \|_2^2 \]
\[ \leq C \| (G(u - s)h - q) \|_2^2. \quad (3.3.13) \]

Therefore, substitute (3.3.3) (3.3.4) (3.3.8) (3.3.12), and (3.3.13) into (3.3.2), we have
\[ E\|X^h,i_s(t) - X^q,i_u(t)\|_H^2 \leq 5C \|\{G(u - s)h - q\}\|_H^2 + [5CK_2(u - s)^2 + 20K_2(u - s) \\
+ (40CK_2T^2 + 160K_2T)(C'(u - s) + o(u - s))]\frac{1}{1 + b_2(1 + E\|h\|^2)} \\
+ (10CK_1 + 40K_1) \left( \int_u^t E\|X^h,i_s(v) - X^q,i_u(v)\|_H^2 dv \right). \]

The required assertion finally follows from the Gronwall inequality. □

**Theorem 3.3.2 (Feller Property)** The Markov process \((X^h,i_s(t), \theta^i_s(t))\) satisfies the Feller property. i.e. if for any \(i \in \mathbb{S}, \delta > 0\) and any bounded continuous function \(\psi : \mathbb{R}^d \to \mathbb{R}\), the mapping 

\[ (h,s) \to \sum_{j \in \mathbb{S}} \int_{\mathbb{R}^d} \psi(y) P(s,(h,i); s + \delta, dy \times \{j\}) = E\psi \left( X^h,i_s(s + \delta) \right), \]

is continuous.

Proof: For any bounded continuous function \(\psi : \mathbb{R}^d \to \mathbb{R}\) and for any fixed \(i \in \mathbb{S}, \delta > 0\). Note that for \(s \leq u \leq t \leq T\),

\[ E\psi \left( X^h,i_s(s + \delta) \right) - E\psi \left( X^q,i_u(u + \delta) \right) \]

\[ = E\psi \left( X^h,i_s(s + \delta) \right) - E\psi \left( X^h,i_s(u + \delta) \right) + E\psi \left( X^h,i_s(u + \delta) \right) \]

\[ - E\psi \left( X^q,i_u(u + \delta) \right). \]

By the lemma 3.3.1 and dominated convergence theorem

\[ E\psi \left( X^h,i_s(u + \delta) \right) - E\psi \left( X^q,i_u(u + \delta) \right) \to 0 \quad \text{as} \ (q,u) \to (h,s). \]
Then using the Hölder inequality, theorem 2.2.4 and the linear growth condition (H2) to consider

\[ E\left\| X_s^{h,i}(t) - X_s^{h,i}(u) \right\|^2_H \]

\[ = E \left\| G(t - s)h - G(u - s)h + \int_s^t G(t - v)F \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dv \right. \]

\[ - \int_s^u G(t - v)F \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dv \]

\[ + \int_s^t G(t - v)\Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dW(v) \]

\[ - \int_s^u G(t - v)\Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dW(v) \right\|^2_H \]

\[ = E \left\| G(t - s)h - G(u - s)h + \int_u^t G(t - v)F \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dv \right. \]

\[ + \int_u^t G(t - v)\Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dW(v) \right\|^2_H \]

\[ \leq 3\left\{ E\left\| G(t - s)h - G(u - s)h \right\|^2_H \right. \]

\[ + \left\| E \int_u^t G(t - v)F \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dv \right\|^2_H \]

\[ + \left\| E \int_u^t G(t - v)\Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) dW(v) \right\|^2_H \}

\[ \leq 3\left\{ CE\left\| G(t - u)h - h \right\|^2_H \right. \]

\[ \left. + C(t - u)E \int_u^t \left\| F \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) \right\|^2_H dv \right. \]
\[ +4E \int_u^t \| \Sigma \left( X_s^{h,i}(v), v, \theta_s^i(v) \right) \|_{L_2^2}^2 dv \]

\[ \leq 3 \left\{ CE \| G(t-u)h - h \|_H^2 + [C(t-u)K_2 + 4K_2] E \int_u^t \left( 1 + \| X_s^{h,i}(v) \|_H^2 \right) dv \right\} \]

Then using theorem 2.3.1 to get

\[ E \| X_s^{h,i}(t) - X_s^{h,i}(u) \|_H^2 \leq 3CE \| G(t-u)h - h \|_H^2 + 3K_2C(t-u)^2 \]

\[ +12K_2(t-u)(1 + b_2(1 + E \| h \|_2^2)) \]

From the above inequality, we obtain \( X_s^{h,i}(s+\delta) \) converges to \( X_s^{h,i}(u+\delta) \) as \( u \to s \).

Then using the dominated convergence theorem, we get

\[ E\psi \left( X_s^{h,i}(s+\delta) \right) - E\psi \left( X_s^{h,i}(u+\delta) \right) \to 0 \quad \text{as} \ u \to s. \]

Therefore,

\[ E\psi \left( X_s^{h,i}(s+\delta) \right) \to E\psi \left( X_u^{q,i}(u+\delta) \right) \quad \text{as} \ (q,u) \to (h,s). \]

In other words, \( E\psi \left( X_s^{h,i}(s+\delta) \right) \) as a function of \((h,s)\) is a continuous and that is Feller property. \( \Box \)

### 3.4 Strong Feller Property

Let \( (X(t), \theta(t)) \) be a Markov process given by the solution of the following equations:
\[ dX(t) = [AX(t) + F(t, X(t), \theta(t))]dt + \Sigma(t, X(t), \theta(t))dW(t), \quad X(0) = h \in H, (3.4.1) \]

and

\[ d\theta(t) = \int_{\mathbb{R}} g(\theta(t^{-}), y)v(dt, dy), \quad \theta(0) = i \in \mathbb{S}, \quad (3.4.2) \]

where \( A = (\kappa \Delta - \alpha) \) for \( \kappa > 0, \alpha > 0 \). \( A \) generates a contraction semigroup \( G(t) \) for \( t \geq 0 \) on \( H = L^2(D); \quad W(t) \) be a \( Q \)-Wiener process in \( K = L^2(D) \) with \( trQ < \infty \); \( F: [0, T] \times H \times \mathbb{S} \to H; \Sigma: [0, T] \times H \times \mathbb{S} \to L^0_2 \) are \( \mathcal{F}_t \)-adapted random fields; \( \theta(t) \) be a right-continuous Markov chain on finite state space \( \mathbb{S} \); initial value \( h \in H = L^2(D), \) and \( i \in \mathbb{S} \) are \( \mathcal{F}_0 \)-measurable;

For our purpose, we use the Itô formula (Ikeda and Watanabe, 1989), which will play an important role for strong Feller property analysis. Denote by \( C^{1,2}([0, T] \times H \times \mathbb{S}; \mathbb{R}) \) be the space of all non-negative functions \( V(t, u, r) \) on \( [0, T] \times H \times \mathbb{S} \) which are continuously twice differentiable in \( u \) and once in \( t \). If \( V \in C^{1,2}([0, T] \times H \times \mathbb{S}; \mathbb{R}) \) define an operator \( \mathcal{L} \) on \( \mathbb{R} \) by

\[
\mathcal{L}V(t, u, r) = \frac{1}{2} tr\left[ \Sigma^*(t, u, r)D_{uu}V(t, u, r)Q\Sigma(t, u, r) \right] + < D_uV(t, u, r), Au + F(t, u, r) > + \sum_{j=1}^{N} \gamma_{ij}V(t, h, j), \quad for all u \in D(A), r \in \mathbb{S}, \quad (3.4.3)
\]

where \( \sum_{j \in \mathbb{S}} \gamma_{ij} V(t, h, j) = \sum_{j \in \mathbb{S}, i \neq j} \gamma_{ij} \left[ V(t, h, j) - V(t, h, i) \right], i \in \mathbb{S}; \Gamma = (\gamma_{ij})_{N \times N} \) be an \( N \times N \) matrix satisfying (i) \( 0 \leq -\gamma_{ii} < \infty \), for all \( i \); (ii) \( \gamma_{ij} \geq 0 \) for all \( i \neq j \); (iii) \( \sum_{j \in \mathbb{N}} \gamma_{ij} = 0, \) for all \( i \). For convenience, we recall two kinds of solutions to (3.4.1).
Definition 3.4.1 (Ichikawa, 1982) A stochastic process \( X(t), t \in [0, T] \) is called a strong solution of (3.4.1) if

(i) \( X(t) \) is \( \mathcal{F}_t \)-adapted and continuous in \( t \);

(ii) \( X(t) \in D(A), \int_0^T \| AX(t) \|_H \, dt < \infty \);

(iii) for any \( t \in [0, T] \), equation

\[
X(t) = h + \int_0^t [AX(s) + F(s, X(s), \theta(s))] \, ds + \int_0^t \Sigma(s, X(s), \theta(s)) \, dW(s) \tag{3.4.4}
\]

In general, this concept is rather strong, and a weaker one described below is more appropriate for practical purposes.

Definition 3.4.2 A \( H \)-valued stochastic process \( X(t), t \in [0, T] \) is called a mild solution of (3.4.1) if

(i) \( X(t) \) is \( \mathcal{F}_t \)-adapted and continuous in \( t \);

(ii) \( X(t) \) is measurable and \( \int_0^T \| X(t) \|^2_H \, dt < \infty \);

(iii) for any \( t \in [0, T] \), equation

\[
X(t) = G(t)h + \int_0^t G(t - s)F(s, X(s), \theta(s)) \, ds + \int_0^t G(t - s)\Sigma(s, X(s), \theta(s)) \, dW(s) \tag{3.4.5}
\]

Following Ichikawa (1983), we introduce the approximating system of (3.4.1),

\[
dX_n(t) = \left[ AX_n(t) + R(n)F(t, X_n(t), \theta(t)) \right] \, dt + R(n)\Sigma(t, X_n(t), \theta(t)) \, dW(t), t \geq 0,
\]

\[
X_n(0) = R(n)h \in D(A), \theta(0) = i \in S \tag{3.4.6}
\]
where \( n \in \rho(A) \), the resolvent set of \( A \), and \( R(n) = nR(n,A) \), \( R(n, A) \) is the resolvent of \( A \).

Similar to operator \( \mathcal{L} \) defined in (3.4.3), the operator \( \mathcal{L}_n \) associated with (3.4.6), for any \( u \in D(A), r \in \mathbb{S} \), can be defined by

\[
\mathcal{L}_n V(t, u, r) = V_t(t, u, r) + \frac{1}{2} \text{tr} \left[ \left( R(n)\Sigma(t, u, r) \right)^T V_{uu}(t, u, r) Q R(n)\Sigma(t, u, r) \right] + < V_u(t, u, r), Au + R(n)F(t, u, r) > + \sum_{j=1}^{N} \gamma_{ij} V(t, R(n)h, j). (3.4.7)
\]

The following two lemmas, we can consider proposition 2.3 and lemma 3.1 in Ichikawa, 1982.

**Lemma 3.4.3** Suppose that

(a) \( h \in D(A) \),

(b) \( |AG(t - s)F(s, u, r)| \leq g_1(t - s)|u|, g_1 \in L^1(0, T), \)

(c) \( |AG(t - s)\Sigma(s, u, r)| \leq g_2(t - s)|u|, g_2 \in L^2(0, T), \)

then a mild solution \( X(t) \) is also a strong solution.

Proof: By the above conditions we have

\[
\int_0^T \int_0^t \left| AG(t - v)F(v, X(v), \theta(v)) \right| dv dt < \infty,
\]

\[
\int_0^T \int_0^t \left| AG(t - v)\Sigma(v, X(v), \theta(v)) \right| dv dt < \infty.
\]

Thus by Fubini’s theorem we have
\[
\int_0^t \int_0^s AG(s-v)F(v, X(v), \theta(v))dvds = \int_0^t \int_0^t AG(s-v)F(v, X(v), \theta(v))dvdv
\]

\[
= \int_0^t G(t-v)F(v, X(v), \theta(v))dv - \int_0^t F(v, X(v), \theta(v))dv,
\]

and we also have

\[
\int_0^t \int_0^s AG(s-v)\Sigma(v, X(v), \theta(v))dW(v)ds
\]

\[
= \int_{v}^{t} AG(s-v)\Sigma(v, X(v), \theta(v))dvdW(v)
\]

\[
= \int_0^t G(t-v)\Sigma(v, X(v), \theta(v))dW(v) - \int_0^t \Sigma(v, X(v), \theta(v))dW(v).
\]

Hence \(AX(t)\) is integrable with probability one, and

\[
\int_0^t AX(s)ds = G(t)h - h + \int_0^t G(t-s)F(s, X(s), \theta(s))ds - \int_0^t F(s, X(s), \theta(s))ds
\]

\[
+ \int_0^t G(t-s)\Sigma(s, X(s), \theta(s))dW(s) - \int_0^t \Sigma(s, X(s), \theta(s))dW(s)
\]

\[
= X(t) - h - \int_0^t F(s, X(s), \theta(s))ds - \int_0^t \Sigma(s, X(s), \theta(s))dW(s).
\]

Thus \(X(t)\) satisfies (3.4.4). \(\square\)

**Lemma 3.4.4** The approximating system (3.4.6) has a unique strong solution \(X_n(t)\) which lies in \(C([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H))\) for all \(T \geq 0\). Moreover, \(X_n(t)\) converges to the mild solution \(X(t)\) to (3.4.1) in \(C([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H))\) as \(n \to \infty\).
Proof: To prove the first part, by theorem 2.3.1, there exists a unique mild solution of the approximating system (3.4.6),

\[ X_n(t) = g(t)R(n)h + \int_0^t g(t-s)R(n)F(s, X_n(s), \theta(s))ds \]

\[ + \int_0^t g(t-s)R(n)\Sigma(s, X_n(s), \theta(s))dW(s). \]

By lemma 3.4.3, the mild solution \( X_n(t) \), \( t \geq 0 \) is the strong solution.

To prove the second part, we consider

\[ X(t) - X_n(t) = g(t)[h - R(n)h] \]

\[ + \int_0^t g(t-s)[F(s, X(s), \theta(s)) - R(n)F(s, X_n(s), \theta(s))]ds \]

\[ + \int_0^t g(t-s)[\Sigma(s, X(s), \theta(s)) - R(n)\Sigma(s, X_n(s), \theta(s))]dW(s) \]

\[ = g(t)[(I - R(n))h] + \int_0^t g(t-s)R(n)[F(s, X(s), \theta(s)) - F(s, X_n(s), \theta(s))]ds \]

\[ + \int_0^t g(t-s)(I - R(n))F(s, X(s), \theta(s))ds \]

\[ + \int_0^t g(t-s)R(n)[\Sigma(s, X(s), \theta(s)) - \Sigma(s, X_n(s), \theta(s))]dW(s) \]

\[ + \int_0^t g(t-s)(I - R(n))\Sigma(s, X(s), \theta(s))dW(s). \]

Since \( \|g(t)\|_{L(H)} \leq C \), for all \( t \in [0, T] \) and \( |R(n)| \leq 2 \), for large \( n \), we have
\[ E\|X(t) - X_n(t)\|_H^2 \leq 5\{E\|G(t)(I - R(n))h\|_H^2 \]

\[ + E \left\| \int_0^t G(t - s)R(n)[F(s,X(s),\theta(s)) - F(s,X_n(s),\theta(s))]ds \right\|_H^2 \]

\[ + E \left\| \int_0^t G(t - s)R(n)[\Sigma(s,X(s),\theta(s)) - \Sigma(s,X_n(s),\theta(s))]dW(s) \right\|_H^2 \]

\[ + E \left\| \int_0^t G(t - s)(I - R(n))F(s,X(s),\theta(s))ds \right\|_H^2 \]

\[ + E \left\| \int_0^t G(t - s)(I - R(n))\Sigma(s,X(s),\theta(s))dW(s) \right\|_H^2 \}. \]

Using the Lipschitz condition (H1), we have

\[ E \left\| \int_0^t G(t - s)R(n)[F(s,X(s),\theta(s)) - F(s,X_n(s),\theta(s))]ds \right\|_H^2 \]

\[ \leq 4C^2 tE \int_0^t \|F(s,X(s),\theta(s)) - F(s,X_n(s),\theta(s))\|_H^2 ds \]

\[ \leq 4C^2K_1 t \int_0^t E\|X(s) - X_n(s)\|_H^2 ds, \]

and

\[ E \left\| \int_0^t G(t - s)R(n)[\Sigma(s,X(s),\theta(s)) - \Sigma(s,X_n(s),\theta(s))]dW(s) \right\|_H^2 \]

\[ \leq 16E \int_0^t \|\Sigma(s,X(s),\theta(s)) - \Sigma(s,X_n(s),\theta(s))\|_{L^2}^2 ds \]

\[ \leq 16K_1 \int_0^t E\|X(s) - X_n(s)\|_H^2 ds, \]
We estimate the following terms,

\[ E \| G(t) [(I - R(n)) h] \|^2_H \leq C^2 E \| (I - R(n)) h \|^2_H \to 0 \text{ as } n \to \infty, \]

and

\[ E \left\| \int_0^t G(t-s) (I - R(n)) F(s, X(s), \theta(s)) \, ds \right\|_H^2 \]
\[ \leq C^2 t E \int_0^t \| (I - R(n)) F(s, X(s), \theta(s)) \|^2_H \, ds \to 0 \text{ as } n \to \infty, \]

\[ E \left\| \int_0^t G(t-s) (I - R(n)) \Sigma(s, X(s), \theta(s)) \, dW(s) \right\|_H^2 \]
\[ \leq 4 E \int_0^t \| (I - R(n)) \Sigma(s, X(s), \theta(s)) \|^2_{L^2} \, ds \to 0 \text{ as } n \to \infty, \]

by the dominated convergence theorem. Thus we can write

\[ E \| X(t) - X_n(t) \|^2_H \leq c \int_0^t E \| X(t) - X_n(t) \|^2_H \, ds + \varepsilon(n), \]

where \( c = 20 K_1 (C^2 t + 4) \) and \( \lim_{n \to \infty} \varepsilon(n) = 0 \). By Gronwall’s inequality we have

\[ E \| X(t) - X_n(t) \|^2_H \leq \varepsilon(n) e^{ct} \to 0 \text{ as } n \to \infty. \]

□

Now, we start to study the strong Feller property.

**Definition 3.4.5** (Dynkin, 1965) *We say that the process \((X(t), \theta(t))\) satisfies the strong Feller property if the function \( V(t, h, i) = E \psi \left( X^{h,i}(t), \theta^i(t) \right) \) is continuous with respect*
to the initial data \((h, i) \in H \times \mathbb{S}\), for any \(t \geq 0\) and any bounded and measurable function \(\psi(\cdot, \cdot) : H \times \mathbb{S} \to \mathbb{R}\).

**Hypotheses 3.4.6** \(F, \Sigma\) is Lipschitz, Gateaux differentiable on \(H\), respectively, and the Gateaux derivative \(D_h F\) and \(D_h \Sigma\) are bounded and continuous.

**Lemma 3.4.7** (Prato and Zabczyk, 1992) Assume that hypotheses 2.2.1 and hypotheses 3.4.6 hold, then for \(t \geq 0, h \in H, i \in \mathbb{S}, \zeta \in H\), the process \(X^{h,i}(t)\) has mean square directional derivative \(D_h X^{h,i}(t) \zeta\) at \(h\) and in the direction of \(\zeta\). Moreover, for each fixed \(\zeta \in H\), \(Y(t) = D_h X^{h,i}(t) \zeta\) is the unique mild solution of the equation

\[
dY(t) = \left[AY(t) + D_h F \left( t, X^{h,i}(t), \theta^i(t) \right) Y(t) \right] dt + D_h \Sigma \left( t, X^{h,i}(t), \theta^i(t) \right) Y(t) dW(t),
\]

\(Y(0) = \zeta,\)

satisfying

\[
E \sup_{0 < t < T} \|Y(t)\|^2 < \infty.
\]

In the following lemma, we should apply the Itô lemma (Ikeda and Watanabe, 1989) which is good to use the properties of \(Q\)-Wiener process \(W(\cdot)\), and Poisson random measure \(\nu(\cdot, \cdot)\). In this thesis, we always assume that the Poisson random measure \(\nu(\cdot, \cdot)\) is independent of the \(Q\)-Wiener process \(W(\cdot)\), and the compensated Poisson random measure \(\tilde{\nu}(ds, dy) = \nu(ds, dy) - ds \mu(dy)\) being a martingale measure.

**Lemma 3.4.8** Assume that Hypotheses 2.2.1 holds, and \(\psi(\cdot, r) \in C_b(H), r \in \mathbb{S}\). If for each \(r \in \mathbb{S}\), a function \(V(\cdot, r) \in C^{1,2}([0, T] \times H \times \mathbb{S}; \mathbb{R})\) is bounded and satisfies
\[
|D_uV(t,u,r)| + |D_{uu}V(t,u,r)| \leq \alpha(1 + |u|^{\beta}), \quad (3.4.8)
\]

for any \( t > 0, u \in D(A) \), where \( \alpha, \beta \) are some positive constants, and

\[
\begin{align*}
\frac{\partial V}{\partial t} &= \mathcal{L}V \\
V(0,u,r) &= \psi(u,r)
\end{align*}
\]

(3.4.9)

then \( V(t,h,i) = E\psi(X^{h,i}(t),\theta^i(t)) \).

Proof: For fixed \( t > 0 \), we apply the generalized Itô lemma (Ikeda and Watanabe, 1989) to the process \( V(t-s,X^{h,i}(s),\theta^i(s)) \), and take expectation on the both sides to obtain

\[
EV\left(0,X^{h,i}(t),\theta^i(t)\right) = EV\left(t,X^{h,i}(0),\theta^i(0)\right) + E\int_0^t \left[ \frac{\partial}{\partial s} + \mathcal{L} \right] V(t-s,X^{h,i}(s),\theta^i(s)) ds + EM_1(t) + EM_2(t), \quad (3.4.10)
\]

where

\[
M_1(t) = \int_0^t <D_uV(t-s,X^{h,i}(s),\theta^i(s)),\Sigma(s,X^{h,i}(s),\theta^i(s))dW(s)>,
\]

\[
M_2(t) = \int_0^t \int_{\mathbb{R}} [V(t-s,X^{h,i}(s),\theta^i(0) + g(\theta^i(s-),y))] dsdy,
\]

\[
-\int_{\mathbb{R}} [V(t-s,X^{h,i}(s),\theta^i(s))]|\bar{\nu}(ds,dy),
\]

with \( \bar{\nu}(ds,dy) = \nu(ds,dy) - ds\mu(dy) \) being a martingale measure. By the linear growth condition and (3.4.8) imply that \( M_1(t) \) is a mean-zero martingale, while the boundedness of \( V \) implies that \( M_2(t) \) is a mean-zero martingale.

Note that \( \left(X^{h,i}(0),\theta^i(0)\right) = (h,i) \in H \times \mathcal{S}, \) and by (3.4.9),
\[
V \left(0, X^{h,i}(t), \theta^i(t) \right) = \psi \left(X^{h,i}(t), \theta^i(t) \right).
\]

Therefore, we have

\[
V(t, h, i) = E\psi \left(X^{h,i}(t), \theta^i(t) \right). \quad \Box
\]

Note that the semigroup \(P_t, t \geq 0\) associated with the equation (3.4.1) is defined by

\[
P_t\psi(h, i) = E\psi \left(X^{h,i}(t), \theta^i(t) \right), \text{for } \psi(\cdot, r) \in B_b(H), r \in \mathbb{S},
\]

where \(X^{h,i}(t)\) is the solution of the equation (3.4.1) with initial data \(X^{h,i}(0) = h\).

**Lemma 3.4.9** Assume that Hypotheses 2.2.1 and 3.4.6 hold, and \(\psi(\cdot, r) \in C^2_b(H), r \in \mathbb{S} \). If for each \(r \in \mathbb{S}\), a function \(V(\cdot, r) \in C^{1,2}([0,T] \times H \times \mathbb{S}; \mathbb{R})\) is bounded, satisfies (3.4.8) and (3.4.9), then

\[
\psi \left(X^{h,i}(t), \theta^i(t) \right) = V(t, h, i)
\]

\[
+ \int_0^t < D_u V \left(t - s, X^{h,i}(s), \theta^i(s) \right), \Sigma(s, X^{h,i}(s), \theta^i(s)) dW(s) > + \int_0^t \int_{\mathbb{R}} \left[V \left(t - s, X^{h,i}(s), \theta^i(0) + g(\theta^i(s), y) \right) - V \left(t - s, X^{h,i}(s), \theta^i(s) \right) \right] \tilde{v}(ds, dy). \tag{3.4.11}
\]

Proof: For each \(n\), let \(X^{h,i}_n(t) \in D(A)\) be the strong solution of the approximating system (3.4.6), and \(V(t, u, r)\) satisfies the Kolmogorov equation

\[
\begin{cases}
\frac{\partial}{\partial t} V(t, u, r) = \frac{1}{2} \text{tr} \left[ (R(n) \Sigma(t, u, r))^{+} D_{uu} V(t, u, r) QR(n) \Sigma(t, u, r) \right] \\
\quad + < D_u V(t, u, r), Au + R(n) F(t, u, r) > + \sum_{j=1}^{N} \gamma_{ij} V(t, R(n) h, j), \\
V(0, u, r) = \psi(u, r), \quad \text{for any } (u, r) \in D(A) \times \mathbb{S}.
\end{cases} \tag{3.4.12}
\]
We will apply the Itô Lemma to the process $V(t - s, X^h,i(s), \theta^i(s)), s \in [0, t]$. Taking into account equation (3.4.12) we obtain

$$\psi(X^h,i(t), \theta^i(t)) = V(t, R(n)h, i)$$

$$+ \int_0^t < D_u V(t - s, X^h,i(s), \theta^i(s)), R(n)\Sigma(s, X^h,i(s), \theta^i(s)) dW(s) >$$

$$+ \int_0^t \int_{\mathbb{R}} [V(t - s, X^h,i(s), \theta^i(0) + g(\theta^i(s -), y))$$

$$- V(t - s, X^h,i(s), \theta^i(s))] \tilde{v}(\text{d}s, \text{d}y),$$

(3.4.13)

Then, letting $n \to \infty$ give the desired result. □

**Lemma 3.4.10** Assume that Hypothesis 3.4.6 holds, and set $V(t, h, i) = P_t \psi(h, i), t \geq 0, h \in H, i \in S$. For arbitrary $\psi(\cdot, r) \in C^2_b(H), r \in S$, then the directional derivatives $< D_h V(t, h, i), \zeta >$ is given by

$$< D_h V(t, h, i), \zeta >$$

$$= \frac{1}{t} E \left\{ \psi(X^h,i(t), \theta^i(t)) \int_0^t < \Sigma^{-1}(s, X^h,i(s), \theta^i(s)) D_h X^h,i(s) \zeta, dW(s) > \right\}.$$  (3.4.14)

Proof: Fix $\zeta \in H$, set $V(t, h, i) = P_t \psi(h, i), t \geq 0, h \in H, i \in S$. Multiplying the both sides of (3.4.11) by the term

$$\int_0^t < \Sigma^{-1}(s, X^h,i(s), \theta^i(s)) D_h X^h,i(s) \zeta, dW(s) >,$$

where $\|\Sigma^{-1}(s, u, r)\| \leq K, K$ is positive constant, and taking expectation, we have
\[
E \left\{ \psi \left( X^{h,i}(t), \theta^i(t) \right) \int_0^t < \Sigma^{-1} \left( s, X^{h,i}(s), \theta^i(s) \right) D_h X^{h,i}(s) \xi, dW(s) > \right\}
\]

\[
= E \int_0^t < \Sigma^* \left( s, X^{h,i}(s), \theta^i(s) \right) D_u V \left( t - s, X^{h,i}(s), \theta^i(s) \right), D_h X^{h,i}(s) \xi > ds
\]

\[
+ E \int_0^t \int_{\mathbb{R}} < \Sigma^{-1} \left( s, X^{h,i}(s), \theta^i(s) \right) D_h X^{h,i}(s) \zeta dW(s), \left( V \left( t - s, X^{h,i}(s), \theta^i(0) \right) + g(\theta^i(s -), y) \right) - V \left( t - s, X^{h,i}(s), \theta^i(s) \right) \bar{v}(ds, dy) >
\]

\[
= E \int_0^t < D_u V \left( t - s, X^{h,i}(s), \theta^i(s) \right), D_h X^{h,i}(s) \xi > ds
\]

\[
= \int_0^t < D_h \left( EV \left( t - s, X^{h,i}(s), \theta^i(s) \right) \right), \xi > ds
\]

\[
= \int_0^t < D_h \left( P_s P_{t-s} \psi(h, i) \right), \zeta > ds
\]

\[
= \int_0^t < D_h \left( P_t \psi(h, i) \right), \zeta > ds
\]

\[
= t < D_h V(t, h, i), \zeta >,
\]

which yield (3.4.14). In the term

\[
E \int_0^t \int_{\mathbb{R}} < \Sigma^{-1} \left( s, X^{h,i}(s), \theta^i(s) \right) D_h X^{h,i}(s) \zeta dW(s), \left( V \left( t - s, X^{h,i}(s), \theta^i(0) \right) + g(\theta^i(s -), y) \right) - V \left( t - s, X^{h,i}(s), \theta^i(s) \right) \bar{v}(ds, dy) > = 0,
\]

because the Poisson random measure \( \nu(\cdot, \cdot) \) is independent of the \( Q \)-Wiener process \( W(\cdot) \).
Theorem 3.4.11 Assume that hypothesis 3.4.6 holds. Then for any $T > 0$, there exists a constant $C_T > 0$, such that for all $\psi(\cdot, r) \in B_b(H), r \in S$ and $t \geq 0$,

$$|P_t \psi(h, i) - P_t \psi(q, i)| \leq \frac{C_T}{\sqrt{t}} \|\psi \left( X^{h,i}(t), \theta^i(t) \right) \|_{\infty} \|h - q\|_H; \text{ for } h, q \in H,$$

(3.4.15)

where $\|\psi \left( X^{h,i}(t), \theta^i(t) \right) \|_{\infty} = \sup_{h \in H} |\psi \left( X^{h,i}(t), \theta^i(t) \right)|$. In particular, $P_t, t \geq 0$ is strong Feller, i.e. $P_t$ maps $B_b(H)$ into $C_b(H)$ for $t \geq 0$.

Proof: According to lemma 7.1.5 (Prato and Zabczyk, 1996), it is sufficient to show that equation (3.4.15) holds for $\psi(\cdot, r) \in C^2_b(H), r \in S$. From lemma 3.4.10, we have

$$\|< D_h V(t, h, i), \zeta > \|^2 \leq \frac{1}{t^2} \|\psi \left( X^{h,i}(t), \theta^i(t) \right) \|_{\infty}^2 \left( s, X^{h,i}(s), \theta^i(s) \right) D_h X^{h,i} (s) \zeta \|^2_{\ell^2} ds$$

$$\leq \frac{K^2}{t^2} \|\psi \left( X^{h,i}(t), \theta^i(t) \right) \|_{\infty}^2 \left( s, X^{h,i}(s), \theta^i(s) \right) D_h X^{h,i} (s) \zeta \|^2_{\ell^2} ds. \quad (3.4.16)$$

Our goal is to evaluate the term

$$E \int_0^t \left\|D_h X^{h,i} (s) \zeta \right\|_{\ell^2}^2 ds.$$

To this end, write $Y(t) = D_h X^{h,i}(t) \zeta$. Lemma 3.4.7 gives

$$Y(t) = G(t) \zeta + \int_0^t G(t - s) D_h F \left( s, X^{h,i}(s), \theta^i(s) \right) Y(s) ds$$

$$+ \int_0^t G(t - s) D_h \Sigma \left( s, X^{h,i}(s), \theta^i(s) \right) Y(s) dW(s).$$

Then, using the Höld inequality, and theorem 2.2.4, we compute,
\[
E\|Y(t)\|^2 \leq 3\{E\|G(t)\|^2 + E\left\| \int_0^t G(t-s)D_h F\left(s, X^{h,i}(s), \theta^i(s)\right) Y(s) \, ds \right\|^2
\]

\[
+ E\left\| \int_0^t G(t-s)D_h \Sigma\left(s, X^{h,i}(s), \theta^i(s)\right) Y(s) \, dW(s) \right\|^2 \}
\]

\[
\leq 3\{C^2 E\|\zeta\|^2 + C^2 L^2 T E \int_0^T \|Y(s)\|^2 \, ds + 4L^2 E \int_0^T \|Y(s)\|^2 \, ds \}
\]

\[
\leq 3\{C^2 E\|\zeta\|^2 + (C^2 L^2 T + 4L^2) \int_0^T E\|Y(s)\|^2 \, ds \}
\]

where \(C = \text{sup}_{t\in[0,T]}\|G(t)\|_{L(H)}, L = \text{sup}_{t\in[0,T]}\|D_h F\|\).

By Gronwall inequality, there exists a constant \(C_1 > 0\), such that

\[
E\|Y(t)\|^2 \leq C_1 E\|\zeta\|^2,
\]

where \(C_1 = 3C^2 \exp(3C^2 L^2 T^2 + 12L^2 T)\).

Combining (3.4.16) with (3.4.17) gives

\[
\|D_h V(t, h, i), \zeta\|^2 \leq \frac{K^2}{t^2} \left\| \psi\left(X^{h,i}(t), \theta^i(t)\right) \right\|^2_{\infty} t C_1 \|\zeta\|^2
\]

\[
\leq \frac{K^2 C_1}{t} \left\| \psi\left(X^{h,i}(t), \theta^i(t)\right) \right\|^2_{\infty} \|\zeta\|^2,
\]

where \(C_1\) is the constant which appears in equation (3.4.17).

According to lemma 2.5 (Peszat and Zabczyk, 1995), we have

\[
|P_t \psi(h, i) - P_t \psi(q, i)| \leq \text{sup}_{z \in H}|D_z\left(P_t \psi(z, i)\right)(h - q) |
\]

\[
\leq \text{sup}_{z \in H}|D_z V(t, z, i)(h - q) |
\]
\[ \leq \frac{C_T}{\sqrt{t}} \| \psi \left( X^{\theta^i}(t), \theta^i(t) \right) \|_\infty \| h - q \|_H \]

where \( C_T = K\sqrt{C_1} \). This implies equation (3.4.15) completing the proof. \( \square \)
Chapter 4: Asymptotic Stability in Distribution of SHE-MS

4.1 Introduction

Recently the theory of stability of stochastic partial differential equations in separable Hilbert space has attracted a great deal of attention. For example, the almost sure stability and the mean square stability has been studied in Caraballo and Real (1994), Caraballo and Liu (1999), Chow (1982), Govindan (2005), Haussmann (1978), Ichikawa (1982), Liu and Mandrekar (1997), and Taniguchi (2007), Bao, Hou and Yuan (2010) to name a few. Moreover, for the finite dimensional case, the stability of stochastic differential equations with Markovian switching has been discussed by many authors. For example, Basak et al. (1996), Ji and Chizeck (1990), Mariton (1990), Mao (1999), Shaikhet (1996), Yuan and Mao (2003), to name a few.

In this chapter, we investigate the stability of stochastic heat equations with Markovian switching. It is useful to know whether or not the solution will converge weakly to some
distribution. Such convergence is called the stability in distribution and the limit distribution is known as a stationary distribution.

The key contribution of our work is, by introducing from Ichikawa (1983) an approximating system and constructing an appropriate metric between transition probability functions of Markov process \( X(t), \theta(t) \), to give sufficient conditions for stability in distribution. The rest of this chapter is organized as follows. In section 2, we state basic definitions. In section 3, several lemmas which lay the foundation for stability analysis are presented and the main result is derived by constructing a suitable metric between transition probability functions of Markov process \( X(t), \theta(t) \). In order to demonstrate the application of our theory, one example is established in the last section.

### 4.2 Preliminaries

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space with filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). Recall that \( \theta(t), t \geq 0 \) be a right-continuous Markov chain on the probability space taking values in a finite space \( S = \{1, 2, \ldots, N\} \) with generator \( \Gamma = (\gamma_{ij})_{N \times N} \) given by

\[
\mathbb{P}\{\theta(t + \delta) = j | \theta(t) = i\} = \begin{cases} 
\gamma_{ij}\delta + o(\delta) & i \neq j \\
1 + \gamma_{ii}\delta + o(\delta) & i = j
\end{cases}
\text{ for } \delta > 0.
\]

Here \( \Gamma = (\gamma_{ij})_{N \times N} \) be an \( N \times N \) matrix satisfying (i) \( 0 \leq -\gamma_{ii} < \infty \), for all \( i \); (ii) \( \gamma_{ij} \geq 0 \) for all \( i \neq j \); (iii) \( \sum_{j \in N} \gamma_{ij} = 0 \), for all \( i \);
Recall the nonlinear stochastic evolution equation (2.2.1) in chapter 2

\[ dX(t) = [AX(t) + F(X(t), t, \theta(t))]dt + \Sigma(X(t), t, \theta(t))dW(t), t \geq 0 \]

\[ X(0) = h \in H, \theta(0) = i \in \mathcal{S}, \quad (4.2.1) \]

where \( A = (\kappa \Delta - \alpha) \) for \( \kappa > 0, \alpha > 0 \). \( A \) generates a contraction semigroup \( G(t) \), for \( t \geq 0 \) on \( H = L^2(D) \); \( W(t) \) be a \( Q \)-Wiener process in \( K = L^2(D) \) with \( trQ < \infty \); \( F: H \times [0,T] \times \mathcal{S} \to H; \Sigma: H \times [0,T] \times \mathcal{S} \to \mathcal{L}^2 \) are \( \mathcal{F}_t \)-adapted random fields; initial value \( h \in H = L^2(D) \), and \( \theta(0) = i \in \mathcal{S} \) are \( \mathcal{F}_0 \)-measurable;

Let \( (X(t), \theta(t)) \) be a two-component Markov process such that \( X(\cdot) \) is a continuous component taking values in \( H = L^2(D) \) and \( \theta(\cdot) \) is a jump component taking values in finite state space \( \mathcal{S} \). Recall the process \( (X(t), \theta(t)) \) has a generator \( \mathcal{L} \) given as follows. If \( V \in C^{1,2}([H \times [0,T] \times \mathcal{S}; \mathbb{R}) \), then

\[ \mathcal{L}V(u, t, r) = V_t(u, t, r) + \frac{1}{2} tr[\Sigma^*(u, t, r)V_{uu}(u, t, r)Q\Sigma(u, t, r)] \]

\[ + < V_u(u, t, r), Au + F(u, t, r) > + \sum_{j=1}^{N} \gamma_{ij} V(h, t, j), \text{ for all } u \in D(A), r \in \mathcal{S}, (4.2.2) \]

where \( \sum_{j \in \mathcal{S}} \gamma_{ij} V(h, t, j) = \sum_{j \in \mathcal{S}, i \neq j} \gamma_{ij} [V(h, t, j) - V(h, t, i)] \). Recall the approximating system of (4.2.1),

\[ dX_n(t) = [AX_n(t) + R(n)F(X_n(t), t, \theta(t))]dt + R(n)\Sigma(X_n(t), t, \theta(t))dW(t), t \geq 0 \]

\[ X_n(0) = R(n)h \in D(A), \theta(0) = i \in \mathcal{S}, \quad (4.2.3) \]
where \( n \in \rho(A) \), the resolvent set of \( A \), and \( R(n) = nR(n,A) \), \( R(n,A) \) is the resolvent of \( A \).

Similar to operator \( \mathcal{L} \) defined in (4.2.2), the operator \( \mathcal{L}_n \) associated with (4.2.3), for any \( u \in D(A), r \in \mathcal{S} \), can be defined by

\[
\mathcal{L}_n V(u, t, r) = V_t(u, t, r) + \frac{1}{2} tr\left[\left(R(n)\Sigma(u, t, r)\right)^* V_{uu}(u, t, r)QR(n)\Sigma(u, t, r)\right]
\]

\[
+ < V_u(u, t, r), Au + R(n)F(u, t, r) > + \sum_{j=1}^{N} \gamma_{ij} V(R(n)h, t, j).
\]

(4.2.4)

Let \( y(t) \) denote the \( H \times \mathcal{S} \)-valued Markov process \( (X(t), \theta(t)) \). Let \( P(t, h, i, dy \times \{j\}) \) denote the transition probability of the Markov process \( y(t) \). Let \( \mathbb{P}(t, h, i, A \times B) \) denote the probability of event \( \{y(t) \in A \times B\} \) given initial condition \( y(0) = (h, i) \in H \times \mathcal{S} \), i.e.

\[
\mathbb{P}(t, h, i, A \times B) = \sum_{j \in B} \int_A P(t, h, i, dy \times \{j\}).
\]

**Definition 4.2.1** The process \( y(t) \) is said to be asymptotically stable in distribution if there exists a probability measure \( \pi(\cdot \times \cdot) \) on \( H \times \mathcal{S} \) such that the transition probability \( P(t, h, i, dy \times \{j\}) \) of \( y(t) \) converges weakly to \( \pi(dy \times \{j\}) \) as \( t \to \infty \) for every \( (h, i) \in H \times \mathcal{S} \). Equations (4.2.1) is said to be asymptotically stable in distribution if \( y(t) \) is asymptotically stable in distribution.  \( \square \)
4.3 Asymptotic Stability in Distribution

In this part, we will establish some sufficient criteria on the asymptotic stability in distribution for the process \( y(t) = (X(t), \theta(t)) \) of the problem (4.2.1). To highlight the initial values, we denote \( X^{h,i}(t) \) by the solution of equation (4.2.1) with initial data \( X(0) = h \in H = L^2(D), \theta(0) = i \in S \), let \( \theta^i(t) \) be the right-continuous Markov chain starting from state \( i \in S \) at \( t = 0 \).

**Lemma 4.3.1** Assume that hypothesis 2.2.1 holds and there exists a function \( V \in C^{1,2}(H \times [0,T] \times S; \mathbb{R}) \) such that for any \( u \in H, t \geq 0, r \in S \),

\[
\begin{align*}
    c \| u \|^2_H & \geq V(u, t, r) + \| u \|_H \| V_u(u, t, r) \|_H + \| u \|_H^2 \| V_{uu}(u, t, r) \|, \\
    c_1 \| u \|^2_H & \leq V(u, t, r), \quad (4.3.1)
\end{align*}
\]

where \( c \) and \( c_1 \) are certain positive constants. Moreover, assume that there are constants \( \alpha_1 \) and \( \beta \in \mathbb{R}^+ \) satisfying

\[
\mathcal{L}V(u, t, r) \leq -\alpha_1 V(u, t, r) + \beta, \quad u \in D(A). \quad (4.3.2)
\]

Then, for any \( (h,i) \in H \times S \) and \( \varepsilon > 0 \), there exists a constant \( M > 0 \) such that for any \( t \geq 0 \),

\[
\mathbb{P}\{\| X(t) \|_H \geq M \} < \varepsilon. \quad (4.3.3)
\]

Proof: Applying the Itô formula to the function \( V(u, t, r) = e^{\alpha_1 t}V(u, t, r) \), \( X_n(t) \) be the strong solution of approximating system (4.2.3), for any \( t \geq 0 \),
\[ E \left( e^{\alpha t}V(X_n(t), t, \theta(t)) \right) \]
\[ = EV(R(n)h, 0, i) + E \int_0^t e^{\alpha s}[\alpha_1 V(X_n(s), s, \theta(s)) + \mathcal{L}_n V(X_n(s), s, \theta(s))] ds \]
\[ = EV(R(n)h, 0, i) + E \int_0^t e^{\alpha s}[\alpha_1 V(X_n(s), s, \theta(s)) + \mathcal{L} V(X_n(s), s, \theta(s))] ds \]
\[ + E \int_0^t e^{\alpha s}[\mathcal{L}_n V(X_n(s), s, \theta(s)) - \mathcal{L} V(X_n(s), s, \theta(s))] ds \]
\[ = EV(R(n)h, 0, i) + E \int_0^t e^{\alpha s}[\alpha_1 V(X_n(s), s, \theta(s)) + \mathcal{L} V(X_n(s), s, \theta(s))] ds \]
\[ + E \int_0^t e^{\alpha s} < V_u(X_n(s), s, \theta(s)), (R(n) - I)F(X_n(s), s, \theta(s)) > ds \]
\[ + \frac{1}{2} E \int_0^t e^{\alpha s} tr \left[ \left( R(n)\Sigma(X_n(s), s, \theta(s)) \right)^* V_{uu}(X_n(s), s, \theta(s)) QR(n)\Sigma(X_n(s), s, \theta(s)) \right] ds \]
\[ - \frac{1}{2} E \int_0^t e^{\alpha s} tr \left[ \Sigma^*(X_n(s), s, \theta(s)) V_{uu}(X_n(s), s, \theta(s)) Q\Sigma(X_n(s), s, \theta(s)) \right] ds, \]
\[ + E \int_0^t e^{\alpha s} \left[ \sum_{j=1}^N y_{ij}V(R(n)h, s, j) - \sum_{j=1}^N y_{ij}V(h, s, j) \right] ds. \quad (4.3.4) \]

By (4.3.2), it follows that
\[ E \left( e^{\alpha t}V(X_n(t), t, \theta(t)) \right) \leq EV(R(n)h, 0, i) + \frac{\beta}{\alpha_1} (e^{\alpha t} - 1) \]
\[ + E \int_0^t e^{\alpha s} < V_u(X_n(s), s, \theta(s)), (R(n) - I)F(X_n(s), s, \theta(s)) > ds \]
\[ + \frac{1}{2} E \int_0^t e^{\alpha s} tr \left[ \left( R(n)\Sigma(X_n(s), s, \theta(s)) \right)^* V_{uu}(X_n(s), s, \theta(s)) QR(n)\Sigma(X_n(s), s, \theta(s)) \right] ds \]
\[-\frac{1}{2}E \int_0^t e^{\alpha_1 s} \text{tr} \left[ \Sigma^*(X_n(s), s, \theta(s))V_{nu}(X_n(s), s, \theta(s))Q \Sigma(X_n(s), s, \theta(s)) \right] ds \]

\[+ E \int_0^t e^{\alpha_1 s} \left[ \sum_{j=1}^N \gamma_{ij} (V(R(n)h, s, j) - V(h, s, j)) \right] ds. \]

Using Lemma 3.4.4, (4.3.1), and the uniform integrability, we obtain

\[E \left( e^{\alpha_1 t} V(X(t), t, \theta(t)) \right) \leq EV(h, 0, i) + \frac{\beta}{\alpha_1} (e^{\alpha_1 t} - 1), \]

and set \(V(h, 0, i) = V(h, i)\), and together with (4.3.1), give that

\[E \|X(t)\|_H^2 \leq \frac{EV(h, i) + \beta (e^{\alpha_1 t} - 1)}{c_1 e^{\alpha_1 t}} \]

\[\leq \frac{\alpha_1 EV(h, i) + \beta (e^{\alpha_1 t} - 1)}{c_1 \alpha_1 e^{\alpha_1 t}} \]

\[\leq \frac{\alpha_1 e^{-\alpha_1 t} EV(h, i) + \beta e^{\alpha_1 t} e^{-\alpha_1 t}}{c_1 \alpha_1} \]

\[\leq \frac{\alpha_1 EV(h, i) + \beta}{c_1 \alpha_1}. \]

Now, for any \(M > 0\), by the Chebyshev inequality

\[\mathbb{P}\{\|X(t)\|_H \geq M\} \leq \frac{E\|X(t)\|_H^2}{M^2}. \]

Then, the required assertion (4.3.3) follows. \(\Box\)

In what follows we consider the difference between two mild solutions of (4.2.1) starting from different initial data; namely, for any \(t \geq 0\),
\[ X^{h,i}(t) - X^{q,i}(t) = G(t)h - G(t)q \]

\[
+ \int_0^t G(t - s) \left[ F(X^{h,i}(s), s, \theta(s)) - F(X^{q,i}(s), s, \theta(s)) \right] ds
\]

\[
+ \int_0^t G(t - s) \left[ \Sigma(X^{h,i}(s), s, \theta(s)) - \Sigma(X^{q,i}(s), s, \theta(s)) \right] dW(s). \quad (4.3.5)
\]

Now, for \( t \geq 0 \) we introduce an approximating system in correspondence with (4.3.5):

\[
d[X^{h,i}_n(t) - X^{q,i}_n(t)] = \left[ A \left(X^{h,i}_n(t) - X^{q,i}_n(t)\right) 
+ R(n) \left( F(X^{h,i}_n(t), t, \theta(t)) - F(X^{q,i}_n(t), t, \theta(t)) \right) \right] dt
\]

\[
+ R(n) \left[ \Sigma(X^{h,i}_n(t), t, \theta(t)) - \Sigma(X^{q,i}_n(t), t, \theta(t)) \right] dW(t),
\]

\[
X^{h,i}_n(0) - X^{q,i}_n(0) = R(n)(h - q) \in D(A), \quad (4.3.6)
\]

where \( n \in \rho(A) \), the resolvent set of \( A \), and \( R(n) = nR(n, A), R(n, A) \) is the resolvent of \( A \).

For given \( U \in C^{1,2}(H \times [0, T] \times \mathbb{S}; \mathbb{R}) \), define an operator \( J_n \) associated with (4.3.6) for any \( u, v \in D(A), r \in \mathbb{S} \) by

\[
J_n U(u - v, t, r) = U_t(u - v, t, r)
\]

\[
+ \frac{1}{2} tr \left[ \left( R(n)(\Sigma(u, t, r) - \Sigma(v, t, r)) \right)^* U_{uu}(u - v, t, r)QR(n)(\Sigma(u, t, r) - \Sigma(v, t, r)) \right] 
\]

\[
+ \langle U_u(u - v, t, r), A(u - v) + R(n)(F(u, t, r) - F(v, t, r)) \rangle 
\]

\[
+ \sum_{j=1}^N y_{ij} U(R(n)(h - q), t, j). \quad (4.3.7)
\]
Lemma 4.3.2 Assume that hypothesis 2.2.1 holds, and there exists a function \( U \in C^2(H \times [0,T] \times \mathbb{S}; \mathbb{R}) \) such that for any \( u, v \in H, r \in \mathbb{S}, t \geq 0, \)

\[
d \|u - v\|_H^2 \geq U(u - v, t, r) + \|u - v\|_H \|U_u(u - v, t, r)\|_H + \|u - v\|_H^2 \|U_{uu}(u - v, t, r)\|
\]

\[
c_2 \|u - v\|_H^2 \leq U(u - v, t, r).
\]

(4.3.8)

where \( d \) and \( c_2 \) are certain positive constants. Moreover, assume that there is a constant \( \alpha_2 > 0 \) such that for any \( u, v \in D(A), r \in \mathbb{S}, t \geq 0, \)

\[
JU(u - v, t, r) \leq -\alpha_2 U(u - v, t, r).
\]

(4.3.9)

Then, for any \( \varepsilon > 0 \) and any bounded subset \( \mathcal{K} \) of \( H \), there exists a \( T = T(\varepsilon, \mathcal{K}) > 0 \) such that for any \( t \geq T, \)

\[
\mathbb{P}\left\{ \|X^{h,i}(t) - X^{q,i}(t)\|_H < \varepsilon \right\} \geq 1 - \varepsilon,
\]

(4.3.10)

whenever \( h, q \in \mathcal{K}, i \in \mathbb{S}. \)

Proof: It is easy to see from (4.3.8) that \( U(0, t, \theta(t)) = 0. \) For any \( \varepsilon \in (0, 1), \) by the continuity of \( U, \) we then can choose \( \alpha \in (0, \varepsilon) \) sufficiently small such that

\[
\sup_{\|u - v\|_H \leq \alpha, t \geq 0, r \in \mathbb{S}} U(u - v, t, r) \leq \frac{\varepsilon}{c_2 \varepsilon^2}.
\]

(4.3.11)

Denote by \( X^{h,i}(t) \) and \( X^{q,i}(t) \) two different mild solutions of (4.2.1) starting from initial data \( (h, i) \) and \( (q, i) \), respectively. Let \( \mathcal{K} \) be any bounded subset of \( H \) and fix any \( h, q \in \mathcal{K}, i \in \mathbb{S}. \) For \( \beta > \alpha, \) we define two stopping times as follows:
\[ \tau_\alpha = \inf \left\{ t \geq 0 : \| X^{h,i}(t) - X^{q,i}(t) \|_H \leq \alpha \right\}, \]

\[ \tau_\beta = \inf \left\{ t \geq 0 : \| X^{h,i}(t) - X^{q,i}(t) \|_H \geq \beta \right\}. \]

Set \( t_\beta = \tau_\beta \wedge t \). Using the Itô formula on the function \( U(u - \nu, t, r) \) and strong solution \( X_n^{h,i}(t) - X_n^{q,i}(t) \) to (4.3.6), we compute by (4.3.9)

\[ EU \left( X_n^{h,i}(t_\beta) - X_n^{q,i}(t_\beta), t_\beta, \theta(t_\beta) \right) \]

\[ = EU(R(n)(h - q), i) + E \int_0^{t_\beta} J_n U(X_n^{h,i}(s) - X_n^{q,i}(s), s, \theta(s))ds \]

\[ = EU(R(n)(h - q), i) + E \int_0^{t_\beta} JU(X_n^{h,i}(s) - X_n^{q,i}(s), s, \theta(s))ds \]

\[ + E \int_0^{t_\beta} [ J_n U \left( X_n^{h,i}(s) - X_n^{q,i}(s), s, \theta(s) \right) - JU(X_n^{h,i}(s) - X_n^{q,i}(s), s, \theta(s))]ds \]

\[ \leq EU(R(n)(h - q), i) - \alpha_2 E \int_0^{t_\beta} U(X_n^{h,i}(s) - X_n^{q,i}(s), s, \theta(s)) ds \]

\[ + E \int_0^{t_\beta} < U_u \left( X_n^{h,i}(s) - X_n^{q,i}(s), s, \theta(s) \right), (R(n) - I)(F \left( X_n^{h,i}(s), s, \theta(s) \right) \]

\[ - F \left( X_n^{q,i}(s), s, \theta(s) \right) > ds \]

\[ + \frac{1}{2} E \int_0^{t_\beta} \text{tr} \left[ \left( R(n) \left( \Sigma \left( X_n^{h,i}(s), s, \theta(s) \right) - \Sigma \left( X_n^{q,i}(s), s, \theta(s) \right) \right) \right)^* U_{uu} \left( X_n^{h,i}(s) \right) \]

\[ - X_n^{q,i}(s), s, \theta(s) \right) QR(n)(\Sigma \left( X_n^{h,i}(s), s, \theta(s) \right) - \Sigma \left( X_n^{q,i}(s), s, \theta(s) \right))] ds \]

\[ - \frac{1}{2} E \int_0^{t_\beta} \text{tr} \left[ \left( \Sigma \left( X_n^{h,i}(s), s, \theta(s) \right) - \Sigma \left( X_n^{q,i}(s), s, \theta(s) \right) \right)^* U_{uu} \left( X_n^{h,i}(s) \right) \]

\[ - X_n^{q,i}(s), s, \theta(s) \right) Q \left( \Sigma \left( X_n^{h,i}(s), s, \theta(s) \right) - \Sigma \left( X_n^{q,i}(s), s, \theta(s) \right) \right)] ds \]
\[ + E \int_0^{t_\beta} \left[ \sum_{j=1}^N \gamma_{ij} V(R(n)(h - q), s, j) - \sum_{j=1}^N \gamma_{ij} V(h - q, s, j) \right] ds. \]

Next, using Lemma 3.4.4 (4.3.8) and the uniform integrability, we have

\[ EU \left( X^{h,i}(t_\beta) - X^{q,i}(t_\beta), t_\beta, \theta(t_\beta) \right) \]

\[ \leq EU(h - q, i) - \alpha_2 E \int_0^{t_\beta} U \left( X^{h,i}(s) - X^{q,i}(s), s, \theta(s) \right) ds. \tag{4.3.12} \]

By (4.3.8), it directly follows that

\[ c_2 E \left[ \left\| X^{h,i}(t_\beta) - X^{q,i}(t_\beta) \right\|^2_H I_{\{t_\beta \leq t\}} \right] \leq EU(h - q, i), \]

which together with the definition of \( t_\beta \), gives that

\[ P\{ t_\beta \leq t \} \leq \frac{EU(h - q, i)}{c_2 \beta^2}. \]

Noting that \( EU(h - q, i) \) is bounded when \((h, q, i) \in \mathcal{K} \times \mathcal{K} \times S\), this implies that there exists a \( \beta = \beta(\mathcal{K}, \epsilon) > 0 \) such that

\[ P\{ t_\beta < \infty \} \leq \frac{\epsilon}{4}. \tag{4.3.13} \]

Fix \( \beta \) and let \( t_\alpha = t_\alpha \wedge t_\beta \wedge t \). In the same way as (4.3.12) was done, we can obtain from (4.3.8) that

\[ EU \left( X^{h,i}(t_\alpha) - X^{q,i}(t_\alpha), t_\alpha, \theta(t_\alpha) \right) \]

\[ \leq EU(h - q, i) - \alpha_2 E \int_0^{t_\alpha} U \left( X^{h,i}(s) - X^{q,i}(s), s, \theta(s) \right) ds \]

\[ \leq EU(h - q, i) - c_2 \alpha_2 E \int_0^{t_\alpha} \left\| X^{h,i}(s) - X^{q,i}(s) \right\|^2_H ds \]
\[ \leq EU(h - q, i) - c_2 \alpha_2 \alpha^2 E(\tau_{\alpha} \land \tau_{\beta} \land t). \]

So

\[ \mathbb{P}\{\tau_{\alpha} \land \tau_{\beta} \geq t\} \leq \frac{EU(h - q, i)}{c_2 \alpha_2 \alpha^2 t}, \]

which furthermore implies that for given \( \varepsilon \in (0, 1) \) there exists a constant \( T = T(K, \varepsilon) > 0 \) such that

\[ \mathbb{P}\{\tau_{\alpha} \land \tau_{\beta} \leq T\} > 1 - \frac{\varepsilon}{4}. \quad (4.3.14) \]

By (4.3.13) and (4.3.14), we have

\[ 1 - \frac{\varepsilon}{4} < \mathbb{P}\{\tau_{\alpha} \land \tau_{\beta} \leq T\} \leq \mathbb{P}\{\tau_{\alpha} \leq T\} + \mathbb{P}\{\tau_{\beta} < \infty\} \leq \mathbb{P}\{\tau_{\alpha} \leq T\} + \frac{\varepsilon}{4}, \]

which yields

\[ \mathbb{P}\{\tau_{\alpha} \leq T\} \geq 1 - \frac{\varepsilon}{2}. \quad (4.3.15) \]

Now, define the stopping time

\[ \sigma = \inf \left\{ t \geq \tau_{\alpha} \land T : \|X^{h, i}(t) - X^{q, i}(t)\|_H \geq \varepsilon \right\}. \]

Let \( t \geq T \) and compute

\[ c_2 \varepsilon^2 \mathbb{P}\{\tau_{\alpha} \leq T, \sigma \leq t\} \leq EI_{\{\tau_{\alpha} \leq T, \sigma \leq t\}} U \left( X^{h, i}(\sigma \land t) - X^{q, i}(\sigma \land t), \sigma \land t, \theta(\sigma \land t) \right) \]

\[ \leq EI_{\{\tau_{\alpha} \leq T\}} U \left( X^{h, i}(\tau_{\alpha} \land t) - X^{q, i}(\tau_{\alpha} \land t), \tau_{\alpha} \land t, \theta(\tau_{\alpha} \land t) \right) \]

\[ \leq EI_{\{\tau_{\alpha} \leq T\}} U \left( X^{h, i}(\tau_{\alpha}) - X^{q, i}(\tau_{\alpha}), \tau_{\alpha}, \theta(\tau_{\alpha}) \right) \]

\[ \leq \mathbb{P}\{\tau_{\alpha} \leq T\} \sup_{\|u - v\|_H \leq \alpha, t \geq 0, r \in S} U(u - v, t, r). \]
This, together with (4.3.11) and (4.3.15), gives
\[ \mathbb{P}\{ \tau_\alpha \leq T, \sigma \leq t \} < \frac{\varepsilon}{2}. \] (4.3.16)

While, by (4.3.15) and (4.3.16)
\[ \mathbb{P}\{ \sigma \leq t \} \leq \mathbb{P}\{ \tau_\alpha \leq T, \sigma \leq t \} + \mathbb{P}\{ \tau_\alpha > T \} < \varepsilon. \]

Letting \( t \to \infty \), we have
\[ \mathbb{P}\{ \sigma < \infty \} < \varepsilon. \]

This implies that for any \( h, q \in \mathcal{K}, i \in \mathbb{S} \), we must have that for \( t \geq T \)
\[ \mathbb{P}\left\{ \| X^{h,i}(t) - X^{q,i}(t) \|_H < \varepsilon \right\} \geq 1 - \varepsilon, \]
as required. \( \square \)

The following theorem is the main result in this chapter.

**Theorem 4.3.3** Assume that hypothesis 2.2.1 holds. If solutions of the equation (4.2.1) satisfy properties (4.3.3) and (4.3.10), then the equation (4.2.1) is asymptotically stable in distribution.

To prove this theorem we need to introduce more notations. Let \( \mathcal{P}(H \times \mathbb{S}) \) denote all probability measures on \( H \times \mathbb{S} \). For \( \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(H \times \mathbb{S}) \) define the metric
\[ d_L(\mathbb{P}_1, \mathbb{P}_2) = \sup_{f \in \mathcal{L}} \left| \sum_{r=1}^N \int_H f(u,r) \mathbb{P}_1(du,r) - \sum_{r=1}^N \int_H f(u,r) \mathbb{P}_2(du,r) \right|, \] (4.3.17)
and
\[ \mathbb{L} = \{ f : H \times S \to \mathbb{R} : |f(u,r) - f(v,r')| \leq \|u - v\|_H + |r - r'| \text{ and } |f(\cdot,\cdot)| \leq 1 \}. \]  

(4.3.18)

Let us now present two lemmas. In the following lemmas we should assume that the Markov chain is ergodic. i.e. the transition function \( P(t,h,i,dy \times \{j\}) \) is called ergodic if there is a probability measure \( \pi(\cdot,\cdot) \), such that \( P(t,h,i,dy \times \{j\}) \to \pi(dy \times \{j\}) \), as \( t \to \infty \), for every \( (h,i) \in H \times S \). (Anderson, 1991)

**Lemma 4.3.4** Assume that hypothesis 2.2.1 holds, and solutions of the equation (4.2.1) satisfy the properties (4.3.3) and (4.3.10). Then, for any bounded subset \( \mathcal{K} \) of \( H \),

\[
\lim_{t \to \infty} d_E( P(t,h,i,\cdot), P(t,q,j,\cdot) ) = 0, \tag{4.3.19}
\]

uniformly in \( h, q \in \mathcal{K} \) and \( i, j \in S \).

Proof: For any pair of \( i, j \in S \), define the stopping time

\[
\beta_{ij} = \inf\{ t \geq 0 : \theta^i(t) = \theta^j(t) \}. \tag{4.3.20}
\]

Recall that \( \theta^i(t) \) is the Markov chain starting from state \( i \in S \) at \( t = 0 \) and the ergodic of the Markov chain, \( \beta_{ij} < \infty \) a.s. (Anderson, 1991). So, for any \( \epsilon > 0 \), there exists a positive number \( T \) such that

\[
\mathbb{P}\{ \beta_{ij} \leq T \} > 1 - \frac{\epsilon}{8} \ \forall \ i, j \in S. \tag{4.3.21}
\]

For such \( T \), there is a sufficiently large \( M > 0 \) for

\[
\mathbb{P}(\Omega_{h,i}) > 1 - \frac{\epsilon}{16} \ \forall (h,i) \in \mathcal{K} \times S, \tag{4.3.22}
\]
where $\Omega_{h,i} = \left\{ \|X^{h,i}(t)\|_H \leq M \ \forall \ t \in [0,T] \right\}$.

Now, fix any $h, q \in \mathcal{K}$ and $i, j \in \mathcal{S}$. Let $I_G$ denote the indicator function for set $G$ and set $\Omega_1 = \Omega_{h,i} \cap \Omega_{q,j}$. For any $f \in \mathcal{L}$ and $t \geq T$, compute

\[
\left| Ef \left( X^{h,i}(t), \theta^i(t) \right) - Ef \left( X^{q,j}(t), \theta^j(t) \right) \right|
\]

\[
\leq 2\mathbb{P}\{\beta_{ij} > T\} + E \left( I_{[\beta_{ij} \leq T]} \left| f \left( X^{h,i}(t), \theta^i(t) \right) - f \left( X^{q,j}(t), \theta^j(t) \right) \right| \right)
\]

\[
\leq \frac{\varepsilon}{4} + E \left[ I_{[\beta_{ij} \leq T]} \left| f \left( X^{h,i}(t), \theta^i(t) \right) - Ef \left( X^{q,j}(t), \theta^j(t) \right) \right| \left| \mathcal{F}_{\beta_{ij}} \right| \right]
\]

\[
\leq \frac{\varepsilon}{4} + E \left[ I_{[\beta_{ij} \leq T]} \left| f \left( X^{h,i}(t), \theta^i(t) \right) - Ef \left( X^{q,j}(t), \theta^j(t) \right) \right| \left| \mathcal{F}_{\beta_{ij}} \right| \right]
\]

\[
\leq \frac{\varepsilon}{4} + E \left[ I_{[\beta_{ij} \leq T]} \left( 2\zeta \right) \left\| X^{g,k}(t - \beta_{ij}) - X^{p,k}(t - \beta_{ij}) \right\|_H \right]
\]

\[
\leq \frac{\varepsilon}{4} + 2\mathbb{P}(\Omega - \Omega_1) + E \left[ I_{\Omega_1 \cap [\beta_{ij} \leq T]} \left( 2\zeta \right) \left\| X^{g,k}(t - \beta_{ij}) - X^{p,k}(t - \beta_{ij}) \right\|_H \right]. \tag{4.3.23}
\]

where $g = X^{h,i}(\beta_{ij}), p = X^{q,j}(\beta_{ij})$ and $k = \theta^i(\beta_{ij}) = \theta^j(\beta_{ij})$. Note that given $\omega \in \Omega_1 \cap \{\beta_{ij} \leq T\}$, $\|g\|_H \vee \|p\|_H \leq M$. So, by lemma 4.3.2, there exists a constant $T_1$ such that

\[
E \left( 2\zeta \left\| X^{g,k}(t - \beta_{ij}) - X^{p,k}(t - \beta_{ij}) \right\|_H \right) < \frac{\varepsilon}{2}, \quad \forall t \geq T + T_1. \tag{4.3.24}
\]

It therefore follows from (4.3.22) - (4.3.24) that

\[
\left| Ef \left( X^{h,i}(t), \theta^i(t) \right) - Ef \left( X^{q,j}(t), \theta^j(t) \right) \right| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall t \geq T + T_1.
\]

Since $f$ is arbitrary, we must have that

\[
\sup_{f \in \mathcal{L}} \left| Ef \left( X^{h,i}(t), \theta^i(t) \right) - Ef \left( X^{q,j}(t), \theta^j(t) \right) \right| \leq \varepsilon, \quad \forall t \geq T + T_1.
\]

Namely,
for all \( h, q \in \mathcal{K} \) and \( i, j \in \mathbb{S} \).

\[ d_L(P(t+s,h,i,\cdot),P(t,h,i,\cdot)) \leq \varepsilon, \quad \forall t \geq T, s > 0. \]

This is equivalent to

\[
\sup_{f \in \mathcal{L}} \left| Ef \left( X^{h,i}(t+s), \theta^i(t+s) \right) - Ef \left( X^{h,i}(t), \theta^i(t) \right) \right| \leq \varepsilon, \quad \forall t \geq T, s > 0.
\]

(4.3.25)

By lemma 4.3.1, there exists a bounded subset \( \mathcal{K} \) of \( H \) such that for any \( \varepsilon > 0 \),

\[
\mathbb{P}(s,h,i,\mathcal{K} \times \mathbb{S}) \geq 1 - \frac{\varepsilon}{4}, \quad \forall s \geq 0 \quad (4.3.26)
\]

For any \( f \in \mathbb{L} \) and \( t, s > 0 \), compute

\[
\left| Ef \left( X^{h,i}(t+s), \theta^i(t+s) \right) - Ef \left( X^{h,i}(t), \theta^i(t) \right) \right| = \left| E \left[ Ef \left( X^{h,i}(t+s), \theta^i(t+s) \right) \big| \mathcal{F}_s \right] - Ef \left( X^{h,i}(t), \theta^i(t) \right) \right|
\]

\[
= \left| \sum_{l=1}^{N} \int_{\mathcal{H}} Ef\left( X^{z,l}(t), \theta^i(t) \right) P(s,h,i, dz \times \{l\}) - Ef \left( X^{h,i}(t), \theta^i(t) \right) \right|
\]
\[
\leq \sum_{l=1}^{N} \int_H \left| EF(X_{z,l}(t), \theta^l(t)) - EF(X_{h,l}(t), \theta^l(t)) \right| P(s, h, i, dz \times \{l\}) \\
= \sum_{l=1}^{N} \int_{\mathcal{K}} \left| EF(X_{z,l}(t), \theta^l(t)) - EF(X_{h,l}(t), \theta^l(t)) \right| P(s, h, i, dz \times \{l\}) \\
+ \sum_{l=1}^{N} \int_{H\setminus\mathcal{K}} \left| EF(X_{z,l}(t), \theta^l(t)) - EF(X_{h,l}(t), \theta^l(t)) \right| P(s, h, i, dz \times \{l\}) \\
\leq \sum_{l=1}^{N} \int_{\mathcal{K}} \left| EF(X_{z,l}(t), \theta^l(t)) - EF(X_{h,l}(t), \theta^l(t)) \right| P(s, h, i, dz \times \{l\}) + \frac{\varepsilon}{2},
\]

(4.3.27)

where (4.3.26) has been used in the last inequality. By lemma 4.3.4, there exists a \( T > 0 \) such that for the given \( \varepsilon > 0 \),

\[
sup_{f \in \mathcal{L}} \left| EF(X_{z,l}(t), \theta^l(t)) - EF(X_{h,l}(t), \theta^l(t)) \right| < \frac{\varepsilon}{2}, \forall t \geq T, \quad (4.3.28)
\]

whenever \((z, l) \in \mathcal{K} \times \mathcal{S}\). Substituting (4.3.28) into (4.3.27) yields

\[
\left| EF(X_{h,i}(t+s), \theta^i(t+s)) - EF(X_{h,i}(t), \theta^i(t)) \right| < \varepsilon, \quad \forall t \geq T, s > 0.
\]

Since \( f \) is arbitrary, the desired inequality (4.3.25) must hold. \( \square \)

Now, we prove our main result Theorem 4.3.3.

**Proof of Theorem 4.3.3:** By the definition 4.2.1, it is sufficient to show that there exists a probability measure \( \pi(\cdot \times \cdot) \in \mathcal{P}(H \times \mathcal{S}) \) such that for any \((h, i) \in H \times \mathcal{S}\), the transition probabilities \( \{P(t, h, i, \cdot; \cdot): t \geq 0\} \) converge weakly to \( \pi(\cdot \times \cdot) \). Recalling the well-known
fact that the weak convergence of probability measures is a metric concept (Ikeda and Watanabe, 1981, Proposition 2.5), we need to show that for any \((h, i) \in H \times S\),

\[
\lim_{t \to \infty} d_{L}(P(t, h, i; \cdot, \cdot), \pi(\cdot; \cdot)) = 0. \tag{4.3.30}
\]

By lemma 4.3.5, \(\{P(t, 0,1; \cdot; \cdot): t \geq 0\}\) is Cauchy in the space \(\mathcal{P}(H \times S)\) with metric \(d_{L}\). Since \(\mathcal{P}(H \times S)\) is a complete metric space under \(d_{L}\) (Chen, 1992, Theorem 5.4), there is a unique probability measure \(\pi(\cdot; \cdot) \in \mathcal{P}(H \times S)\) such that

\[
\lim_{t \to \infty} d_{L}(P(t, 0,1; \cdot; \cdot), \pi(\cdot; \cdot)) = 0.
\]

Moreover, for any \((h, i) \in H \times S\), by lemma 4.3.4,

\[
\lim_{t \to \infty} d_{L}(P(t, h, i; \cdot, \cdot), \pi(\cdot; \cdot)) \leq \lim_{t \to \infty} d_{L}(P(t, h, i; \cdot, \cdot), P(t, 0,1; \cdot, \cdot))
\]

\[
+ \lim_{t \to \infty} d_{L}(P(t, 0,1; \cdot; \cdot), \pi(\cdot; \cdot)) = 0,
\]

as required. \(\Box\)

### 4.4 Example

In this section, in order to demonstrate the application of our theory, one example is constructed.

**Example 4.4.1** Consider the following stochastic heat equations with Markovian switching
\[ dy(x,t) = \frac{\partial^2}{\partial x^2} y(x,t) dt + \sigma f(y(x,t), \theta(t)) dW(t), \quad t \geq 0, \quad 0 < x < 1; \]

\[ y(0, t) = y(1, t) = 0, t \geq 0; y(x, 0) = y_0(x), \theta(0) = \theta_0, \quad 0 \leq x \leq 1, \]

(4.4.1)

where \( W(t), t \geq 0 \), is a real standard Brownian motion, \( \theta(t), t \geq 0 \), is a right continuous Markov chain on finite state space \( \mathbb{S} \) with generator \( \Gamma = (\gamma_{ij})_{N \times N} \) given by

\[
P(\theta(t + \delta) = j | \theta(t) = i) = \begin{cases} \gamma_{ij} \delta + o(\delta) & \text{if } i \neq j \\ 1 + \gamma_{ii} \delta + o(\delta) & \text{if } i = j \end{cases} \text{ for } \delta > 0.
\]

Here, \( \Gamma = (\gamma_{ij})_{N \times N} \) be an \( N \times N \) matrix satisfying (i) \( 0 \leq -\gamma_{ii} < \infty \), for all \( i \); (ii) \( \gamma_{ij} \geq 0 \) for all \( i \neq j \); (iii) \( \sum_{j \in \mathbb{N}} \gamma_{ij} = 0 \), for all \( i \); \( \sigma \) is a real number and \( f \) is a real Lipschitz continuous function on \( L^2(0,1) \) satisfying for \( u, v \in L^2(0,1) \) and some positive constants \( c, k \),

\[
|f(u, r)|^2 \leq c(1 + \|u\|_{H^2}^2)
\]

\[
|f(u, r) - f(v, r)|^2 \leq k\|u - v\|_{H^2}^2.
\]

In this example, we take \( H = L^2(0,1) \) and \( A = \frac{\partial^2}{\partial x^2} \) with

\[
D(A) = \{ y \in H, y, y' are absolutely continuous with y', y'' \in H, y(0) = y(1) = 0 \}.
\]

Then, by Govindan (2005), let the function \( e_n = \sqrt{2/\pi} \sin(\pi nx), n \in \mathbb{N}, x \in [0,1] \), be orthonormal eigenvectors of \( A \), corresponding to the eigenvalues \( -\pi^2 n^2, n \in \mathbb{N} \). Then for any \( u \in D(A) \),

\[
Au = \sum_{n=1}^{\infty} \langle u, e_n \rangle Ae_n = \sum_{n=1}^{\infty} \langle u, e_n \rangle (-\pi^2 n^2) e_n,
\]
\[ < u, Au > = -\sum_{n=1}^{\infty} \pi^2 n^2 < u, e_n > < u, e_n > \]

\[ = -\sum_{n=1}^{\infty} \pi^2 n^2 < u, e_n >^2 \]

\[ \leq -\pi^2 \| u \|_{H^2}^2 . \]

For all \( u \in D(A) \), set \( V(u, t, r) = U(u, t, r) = \| u \|_{H^2}^2 \), \( \sum_{j=1}^{N} \gamma_{ij} = 0 \), for all \( i \).

\[ \mathcal{L} \| u \|_{H^2}^2 = 2 < u, Au > + \sigma^2 |f(u, r)|^2 + \sum_{j=1}^{N} \gamma_{ij} \| u \|_{H^2}^2 \]

\[ \leq -2\pi^2 \| u \|_{H^2}^2 + \sigma^2 c(1 + \| u \|_{H^2}^2) + \sum_{j=1}^{N} \gamma_{ij} \| u \|_{H^2}^2 \]

\[ \leq - \left( 2\pi^2 - \sigma^2 c - \sum_{j=1}^{N} \gamma_{ij} \right) \| u \|_{H^2}^2 + \sigma^2 c \]

\[ \leq -(2\pi^2 - \sigma^2 c) \| u \|_{H^2}^2 + \sigma^2 c , \]

and

\[ \mathcal{J} \| u - v \|_{H^2}^2 = 2 < u - v, A(u - v) > + \sigma^2 |f(u, r) - f(v, r)|^2 + \sum_{j=1}^{N} \gamma_{ij} \| u - v \|_{H^2}^2 \]

\[ \leq -2\pi^2 \| u - v \|_{H^2}^2 + \sigma^2 k \| u - v \|_{H^2}^2 + \sum_{j=1}^{N} \gamma_{ij} \| u - v \|_{H^2}^2 \]

\[ \leq - \left( 2\pi^2 - \sigma^2 k - \sum_{j=1}^{N} \gamma_{ij} \right) \| u - v \|_{H^2}^2 \]

\[ \leq -(2\pi^2 - \sigma^2 k) \| u - v \|_{H^2}^2 . \]
Therefore, if $2\pi^2 > \sigma^2 c$, and $2\pi^2 > \sigma^2 k$, it implies (4.3.3) and (4.3.10). Then we immediately deduce by theorem 4.3.3 that the mild solution process $y(x, t)$ of (4.4.1) is stable in distribution. □
Bibliography


