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Subsonic flow past three-dimensional localised heating elements in boundary layers

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Abstract. The problem of subsonic flow past three-dimensional micro-electro-mechanical-type (MEMS-type) heating elements placed on a flat surface, where the MEMS devices have hump-shaped surfaces, is investigated using the triple-deck theory. The compressible Navier-Stokes equations supplemented by the energy equation are considered in the limit when the Reynolds number is large. The dimensions of the MEMS devices considered are such that the flow perturbations are governed by the three-dimensional subsonic triple-deck equations formulated with aid of the method of matched expansions. The linear analysis of these equations is presented and our results provide an insight into how the MEMS heating elements may be used to positively control the local flow properties.

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1. Introduction

Boundary-layer separation from the surface of a rigid body continues to attract attention not least because the occurrence of separation can have a significant effect on the flow properties. In their review of the aerodynamics of unmanned small vehicles, Mueller and DeLaurier [2003] note that for Reynolds numbers in the range 7×10^5 - 2×10^6 controlling and eliminating separation is particularly important in maintaining laminar flow over the airfoil surface. The influence of separation on boundary layer behaviour in ‘low’ Reynolds number aerodynamics is also reviewed in Mueller et al. [1983], and Sondergaard [2013] where many key papers on experimental data and related applications are also cited. Research on the use of passive as well as active methods to control separation and reduce drag is of considerable interest as mentioned in Marks et al. [2012] where they were able to demonstrate a 33% reduction in the drag coefficient with active control by using a control strategy based on measuring surface stress with a thin stress sensitive film applied to the airfoil surface.

In the current work our focus is on trying to gain an insight into how surface mounted, thermally heated humps can alter the wall shear and thereby potentially inhibit or reduce separation. The main motivation is driven by a need to obtain a greater mathematical understanding of how micro-electro-mechanical-type (MEMS-type) heating elements placed on a flat surface operate and what parameters and factors influence the behaviour of the wall shear. The mathematical explanation of the phenomena of separation lies in the theory of the boundary layer, in particular triple-deck theory as discovered simultaneously by Stewartson and Williams [1969], Neiland [1969], and Messiter [1970]. Whilst the classical Prandtl [1904] boundary layer approximation is known to fail at separation when a boundary layer encounters adverse pressure gradients, the triple-deck theory is able to give a mathematically consistent account of a separated flow as described in the reviews by Stewartson [1974], Stewartson [1981], and Smith [1982] for instance. Moreover the triple-deck theory is able to account for the nonlinear response caused by large induced pressure gradients in many different contexts in both external as well as internal flow configurations, see Smith [1982]. In a series of papers Koroteev and Lipatov [2009, 2012, 2013], Lipatov [2006] have discussed how triple-deck theory may be used to describe the interaction between a heated surface-mounted element and an oncoming two-dimensional boundary layer flow with application to MEMS-type devices, for both subsonic and supersonic flows. The main finding in these studies is that a heated surface-mounted element is able to generate strong temperature perturbations which can be beneficial in allowing a boundary layer to avoid separation, in a similar manner to the behaviour of perturbations induced by suction and other related mechanisms. In conjunction with an active control strategy, this may prove useful in controlling separation.

Whilst the majority of the results obtained to date have been for flat plate heated elements, Aljohani and Gajjar [2017] have extended the Koroteev and Lipatov [2012] analysis and results to the flow past heated humps placed on an otherwise flat surface.

The results of Koroteev and Lipatov [2012] are recovered by taking the hump height to be zero. The results in Aljohani and Gajjar [2017] show that heated humps do also significantly enhance the favourable wall shear reduction properties as compared to the thermally heated flat plate elements. In addition by changing the extent of the heated region relative to the extent of the hump, smoother wall shear profiles can be obtained in contrast to the flat plate elements.

In our work here we extend previous work by looking at a two-dimensional subsonic boundary layer flow encountering a thermally heated three-dimensional hump. The dimensions of the hump are chosen such that triple-deck theory can be used. The problem discussed here is a generalisation of the one studied by Smith et al. [1977] but without the heating. In Koroteev and Lipatov [2009] the equations for the supersonic flow past flat three-dimensional elements were written down, but not investigated further. Our work extends the Koroteev and Lipatov [2009] analysis to subsonic flow and to the case when the elements are not flat. We also present new analytical solutions for linear theory when the thermal perturbations and hump height can be taken to be small and these results generalise the results of Smith et al. [1977] to subsonic flow over humps with heating/cooling.

The plan of the paper is as follows. In section 2 we consider the equations and asymptotic expansions leading to the fundamental triple-deck problem. The linear theory is presented in section 3 followed by results and discussion in section 4. Our conclusions are presented in section 5.

2. Problem formulation

Consider the subsonic viscous flow of a perfect gas past a semi-infinite flat plate on which there is a small heated hump. The Reynolds numbers Re defined by $Re = \rho_\infty U_\infty L / \mu_\infty$ is taken to be large. Here ρ_∞ , U_∞ and μ_∞ are the density, the stream-wise velocity, and the dynamic viscosity coefficient, respectively, in the undisturbed flow above the surface where the heated section is located, and L is the distance from the leading edge to the heating element. We use the following dimensionless variables

$$(x, y, z) = \frac{(x^*, y^*, z^*)}{L}, \quad (u, v, w) = \frac{(u^*, v^*, w^*)}{U_\infty},$$

$$T = \frac{T^* \mathcal{R}}{U_\infty^2}, \quad p = \frac{p^*}{\rho_\infty U_\infty^2}, \quad \mu = \frac{\mu^*}{\mu_\infty}, \quad \rho = \frac{\rho^*}{\rho_\infty},$$

where (x, y, z) are the cartesian coordinates, (u, v, w) the corresponding velocities, p is the pressure, ρ the density, T the temperature, μ the non-dimensional viscosity. The starred quantities denote dimensional terms and the subscript ∞ free-stream values. Also $\mathcal{R}$ is the gas constant.

The continuity, Navier-Stokes, energy equations and equation of state are given by:

$$u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$\begin{aligned} \rho(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}) = & -\frac{\partial p}{\partial x} + \frac{1}{Re} \frac{\partial}{\partial x} (\mu [\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} (\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})]) + \\ & \frac{1}{Re} \frac{\partial}{\partial y} (\mu [\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}]) + \frac{1}{Re} \frac{\partial}{\partial z} (\mu [\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}]), \end{aligned} \quad (2)$$

$$\begin{aligned} \rho(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}) = & -\frac{\partial p}{\partial y} + \frac{1}{Re} \{ \frac{\partial}{\partial y} (\mu [\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} (\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z})]) \} + \\ & \frac{1}{Re} \frac{\partial}{\partial x} (\mu [\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}]) + \frac{1}{Re} \frac{\partial}{\partial z} (\mu [\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}]), \end{aligned} \quad (3)$$

$$\begin{aligned} \rho(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}) = & -\frac{\partial p}{\partial z} + \frac{1}{Re} \frac{\partial}{\partial z} (\mu [\frac{4}{3} \frac{\partial w}{\partial z} - \frac{2}{3} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})]) + \\ & \frac{1}{Re} \frac{\partial}{\partial x} (\mu [\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}]) + \frac{1}{Re} \frac{\partial}{\partial y} (\mu [\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}]), \end{aligned} \quad (4)$$

$$\begin{aligned} \rho(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}) = & \frac{\gamma - 1}{\gamma} (u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z}) \\ & \frac{1}{Re Pr} [\frac{\partial}{\partial x} (\mu \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} (\mu \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z} (\mu \frac{\partial T}{\partial z})] \end{aligned} \quad (5)$$

$$\begin{aligned} & + \frac{\gamma - 1}{\gamma} \frac{1}{Re} \{ \mu (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 + \mu (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2 + \mu (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})^2 \\ & + \mu \frac{4}{3} [(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) \frac{\partial u}{\partial x} + \mu (\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}) \frac{\partial v}{\partial y} + \mu (\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}) \frac{\partial w}{\partial z}] \}, \end{aligned} \quad (6)$$

$$p = \rho T. \quad (7)$$

Other non-dimensional constants and parameters appearing in the above equations and later are the Prandtl number Pr , the ratio of specific heats γ , and the freestream Mach number $M_\infty = U_\infty/c_\infty$ where $c_\infty = \sqrt{\gamma p_\infty/\rho_\infty}$.

Let $Re^{-\frac{1}{2}} = \varepsilon$, with $\varepsilon \ll 1$. Starting with the assumptions similar to those of Lipatov [2006] we assume that above the heating region a change of density occurs due to the release of energy where an increase in the temperature leads to a decrease in density. Therefore, boundary layer thickness changes due to the density change. As a result, Lipatov [2006] suggested that the heating element forms an effective hump where the problem becomes similar to flows with a physical hump. However, the difference between thermal and physical humps is that the thermal hump shape is not known in advance and is formed via the interaction between the oncoming flow and the heating element. We assume that the heating element has streamwise and spanwise lengths

of $O(L\varepsilon^{\frac{3}{4}})$ and height of $O(L\varepsilon^{\frac{5}{4}})$, and that the themally heated hump resides in the lower-deck where $y = O(\varepsilon^{\frac{5}{4}})$. A schematic sketch of the hump and the triple-deck is shown in figure 1. The motivation for choosing these scalings is discussed in Lipatov [2006] and Koroteev and Lipatov [2009]. Considering heating elements with spanwise length $b \gg L\varepsilon^{\frac{3}{4}}$ leads to a case of quasi-two-dimensional flow, while for $b \ll L\varepsilon^{\frac{3}{4}}$ a case of corner flow is developed, see Smith et al. [1977] and Roget et al. [1998]. The hump height chosen is one which provokes a nonlinear response in the lower deck likely to lead to separation. Larger hump heights imply a strongly separated flow and with a very different flow structure to that used here. The order of the pressure gradient term becomes comparable to the order of the dissipative and convective terms only when the streamwise dimension of the hump a and spanwise dimension b are given by

$$a \sim b \sim O(\varepsilon^{\frac{3}{4}}), \quad y \sim O(\varepsilon^{\frac{5}{4}}),$$

which are the viscous lower deck scales for subsonic flow.

The method of matched asymptotic expansions is used to obtain equations of the interaction problem with novel boundary conditions. The expansions are similar to those given in Lipatov [2006] for the flat plate case, but for completeness the details are given here. It is noted that whilst many of the scaling and other details given below are hinted at in previous work, such as Lipatov [2006], (and also Stewartson [1974] for the two-dimensional case), there are some differences and for this reason we have chosen to present the full details for the Dorodnitsyn-Howarth and Prandtl transformation together with the normalisations leading to the triple-deck problem considered in the linear approximation.

2.1. Main deck

Firstly in the main deck which is the main part of the boundary layer the variables and expansions for the flow quantities are given by:

$$x = 1 + \varepsilon^{\frac{3}{4}}X, \quad y = \varepsilon y_2, \quad z = \varepsilon^{\frac{3}{4}}Z,$$

and

$$u = U_B(1, y_2) + \varepsilon^{\frac{1}{4}}u_{21} + \dots, \tag{8a}$$

$$v = \varepsilon^{\frac{1}{2}}v_{21} + \dots, \tag{8b}$$

$$w = \varepsilon^{\frac{1}{2}}w_{21} + \dots, \tag{8c}$$

$$p = (\gamma M_\infty^2)^{-1} + \varepsilon^{\frac{1}{2}}p_2 + \dots, \tag{8d}$$

$$\rho = \rho_B(1, y_2) + \varepsilon^{\frac{1}{4}}\rho_{21} + \dots, \tag{8e}$$

$$T = T_B(1, y_2) + \varepsilon^{\frac{1}{4}}T_{21} + \dots \tag{8f}$$

The suffix B in the above denotes the oncoming two-dimensional basic Blasius boundary layer flow. The properties of the basic flow that we will be using below are:

$$U_B(1, y_2) \rightarrow 1 + O(\exp), \quad \rho_B(1, y_2) \rightarrow 1 + O(\exp),$$

Figure 1: (a) Schematic of the flow showing the hump-shaped element and the oncoming boundary layer. (b) A side view of the configuration showing the triple-deck.

$$T_B(1, y_2) \rightarrow (\gamma M_\infty^2)^{-1} + O(\exp), \quad \text{as } y_2 \rightarrow \infty.$$

Also

$$\begin{aligned} U_B(1, y_2) &\rightarrow \lambda y_2 + O(y_2^2) \dots, \quad \text{as } y_2 \rightarrow 0, \\ \rho_B &\rightarrow \rho_{B0} + O(y_2), \quad T_B(1, y_2) \rightarrow T_{B0} + O(y_2) \quad \text{as } y_2 \rightarrow 0. \end{aligned}$$

Here the notation $O(\exp)$ denotes exponentially small terms in the limit $y_2 \rightarrow \infty$ and also $\lambda = \frac{\partial U_B}{\partial y_2}(1, 0)$. We will further assume that $\rho_{B0} = \rho_B(1, 0) = 1$, and $T_{B0} = T_B(1, 0) = 1$, so that the heating is confined to be in the vicinity of the element. Note however that the temperature perturbation is of $O(1)$ in the lower-deck.

Substitution of the expansions (8) into the Navier-Stokes equations and solving yields the solutions:

$$u_{21} = A(X, Z) \frac{du_B}{dy_2}, \quad v_{21} = -\frac{\partial A}{\partial X} U_B, \quad w_{21} = \frac{D(X, Z)}{U_B \rho_B}, \quad (9)$$

$$T_{21} = A \frac{dT_B}{dy_2}, \quad \rho_{21} = A \frac{d\rho_B}{dy_2}, \quad p = P(X, Z). \quad (10)$$

Here $A(X, Z)$, $D(X, Z)$ and $P(X, Z)$ are unknown functions.

2.2. Upper-deck

In the upper-deck $y = \varepsilon^{\frac{3}{4}} y_1$ and the expansions take the form:

$$\begin{aligned} u &= 1 + \varepsilon^{\frac{1}{2}} u_1 + \dots, \\ v &= \varepsilon^{\frac{1}{2}} v_1 + \dots, \\ w &= \varepsilon^{\frac{1}{2}} w_1 + \dots, \\ p &= (\gamma M_\infty^2)^{-1} + \varepsilon^{\frac{1}{2}} p_1 + \dots, \\ \rho &= 1 + \varepsilon^{\frac{1}{2}} \rho_1 + \dots, \\ T &= (\gamma M_\infty^2)^{-1} + \varepsilon^{\frac{1}{2}} T_1 + \dots \end{aligned}$$

After substituting the above expansions into the Navier-Stokes equations it is found that the pressure p_1 satisfies the equation

$$(1 - M_\infty^2) \frac{\partial^2 p_1}{\partial X^2} + \frac{\partial^2 p_1}{\partial y_1^2} + \frac{\partial^2 p_1}{\partial Z^2} = 0 \quad (11)$$

with

$$\begin{aligned} p_1(X, y_1 = 0, Z) &= P(X, Z), \\ \frac{\partial p_1}{\partial y_1}(X, y_1 = 0, Z) &= \frac{\partial^2 A}{\partial X^2}, \quad \frac{\partial p_1}{\partial Z}(X, y_1 = 0, Z) = -\frac{\partial D}{\partial X} \end{aligned}$$

arising from matching. We also require $p_1(X, y_1, Z) \rightarrow 0$ as $X^2 + y_1^2 + Z^2 \rightarrow \infty$.

The equation for p_1 and the boundary conditions leads to the subsonic interaction law

$$P(X, Z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 A(\zeta, \alpha) / \partial \zeta^2}{\sqrt{(X - \zeta)^2 + (Z - \alpha)^2}} d\zeta d\alpha.$$

2.3. Lower-deck

In the lower-deck with $y = \varepsilon^{\frac{5}{4}} y_3$ the expansions are:

$$\begin{aligned} u &= \varepsilon^{\frac{1}{4}} u_3 + \dots, \\ v &= \varepsilon^{\frac{3}{4}} v_3 + \dots, \\ w &= \varepsilon^{\frac{1}{4}} w_3 + \dots, \\ p &= (\gamma M_\infty^2)^{-1} + \varepsilon^{\frac{1}{2}} p_3 + \dots, \\ \rho &= \rho_3 + \dots, \\ T &= T_3 + \dots \end{aligned}$$

Substituting these expansions into (1)-(7), we obtain

$$\frac{\partial(u_3\rho_3)}{\partial X} + \frac{\partial(v_3\rho_3)}{\partial y_3} + \frac{\partial(w_3\rho_3)}{\partial Z} = 0 \quad (12)$$

$$\rho_3(u_3\frac{\partial u_3}{\partial X} + v_3\frac{\partial u_3}{\partial y_3} + w_3\frac{\partial u_3}{\partial Z}) = -\frac{\partial p_3}{\partial X} + \frac{\partial}{\partial y_3}(\mu\frac{\partial u_3}{\partial y_3}) \quad (13)$$

$$\rho_3(u_3\frac{\partial w_3}{\partial X} + v_3\frac{\partial w_3}{\partial y_3} + w_3\frac{\partial w_3}{\partial Z}) = -\frac{\partial p_3}{\partial Z} + \frac{\partial}{\partial y_3}(\mu\frac{\partial w_3}{\partial y_3}) \quad (14)$$

$$\frac{\partial p_3}{\partial y_3} = 0 \quad (15)$$

$$\rho_3(u_3\frac{\partial T_3}{\partial X} + v_3\frac{\partial T_3}{\partial y_3} + w_3\frac{\partial T_3}{\partial Z}) = \frac{\partial}{\partial y_3}(\frac{\mu}{Pr}\frac{\partial T_3}{\partial y_3}) \quad (16)$$

$$\rho_3 T_3 = 1. \quad (17)$$

To match with the main deck solutions we require that as $y_3 \rightarrow \infty$,

$$u_3 \rightarrow \lambda(A(X, Z) + y_3), \quad T_3 \rightarrow T_{B0} \quad \text{and} \quad w_3 \rightarrow \frac{D(X, Z)}{\lambda\rho_{B0}y_3}.$$

Also as $X \rightarrow -\infty$

$$u_3 \rightarrow \lambda y_3, \quad A(X, Z) \rightarrow 0, \quad D(X, Z) \rightarrow 0.$$

The additional boundary conditions on the hump-shaped wall are

$$u_3 = v_3 = w_3 = 0 \quad \text{and} \quad T_3 = T_w(X, Z), \quad \text{on} \quad y_3 = G(X, Z)$$

and T_w is some prescribed wall temperature profile. The function $G(X, Z)$ denotes the reduced hump shape.

2.4. Scalings and Dorodnitsyn-Howarth transformation

The equations may be reduced to a more convenient form with the use of the Dorodnitsyn-Howarth transformation $(X, y_3, Z) \rightarrow (\bar{X}, \bar{Y}, \bar{Z})$ as follows. Let $\bar{X} = X$, $\bar{Z} = Z$,

$$\bar{Y} = \int_0^{y_3} \rho_3(X, y) dy, \quad (18)$$

and

$$\rho_3 v_3 = \bar{V} - u_3 \frac{\partial Y}{\partial X} - w_3 \frac{\partial Y}{\partial Z}. \quad (19)$$

Next noting that

$$\frac{\partial \rho_3}{\partial \bar{X}} + \frac{\partial \bar{Y}}{\partial \bar{X}} \frac{\partial \rho_3}{\partial \bar{Y}} - \rho_3 \frac{\partial}{\partial \bar{Y}} \left(\frac{\partial \bar{Y}}{\partial \bar{X}} \right) = \frac{\partial \rho_3}{\partial \bar{X}} - \frac{\partial}{\partial y_3} \left(\frac{\partial \bar{Y}}{\partial \bar{X}} \right) = 0,$$

and

$$\frac{\partial \rho_3}{\partial \bar{Z}} + \frac{\partial \bar{Y}}{\partial \bar{Z}} \frac{\partial \rho_3}{\partial \bar{Y}} - \rho_3 \frac{\partial}{\partial \bar{Y}} \left(\frac{\partial \bar{Y}}{\partial \bar{Z}} \right) = \frac{\partial \rho_3}{\partial \bar{Z}} - \frac{\partial}{\partial y_3} \left(\frac{\partial \bar{Y}}{\partial \bar{Z}} \right) = 0,$$

the continuity equation simplifies to

$$\frac{\partial u_3}{\partial \bar{X}} + \frac{\partial \bar{V}}{\partial \bar{Y}} + \frac{\partial w_3}{\partial \bar{Z}} = 0. \quad (20)$$

We set $T_{B0} = 1$ and use the Chapman viscosity law with $\mu = T_3$. This together with the Dorodnitsyn transformation means that the momentum equations can be written as

$$(u_3 \frac{\partial u_3}{\partial \bar{X}} + \bar{V} \frac{\partial u_3}{\partial \bar{Y}} + w_3 \frac{\partial u_3}{\partial \bar{Z}}) = -T_3 \frac{\partial p_3}{\partial \bar{X}} + \frac{\partial}{\partial \bar{Y}} \left(\frac{\partial u_3}{\partial \bar{Y}} \right), \quad (21)$$

$$(u_3 \frac{\partial w_3}{\partial \bar{X}} + \bar{V} \frac{\partial w_3}{\partial \bar{Y}} + w_3 \frac{\partial w_3}{\partial \bar{Z}}) = -T_3 \frac{\partial p_3}{\partial \bar{Z}} + \frac{\partial}{\partial \bar{Y}} \left(\frac{\partial w_3}{\partial \bar{Y}} \right), \quad (22)$$

$$\frac{\partial p_3}{\partial \bar{Y}} = 0, \quad (23)$$

$$(u_3 \frac{\partial T_3}{\partial \bar{X}} + \bar{V} \frac{\partial T_3}{\partial \bar{Y}} + w_3 \frac{\partial T_3}{\partial \bar{Z}}) = \frac{\partial}{\partial \bar{Y}} \left(\frac{1}{Pr} \frac{\partial T_3}{\partial \bar{Y}} \right). \quad (24)$$

Notice that the boundary conditions are now applied on $\bar{Y} = H(\bar{X}, \bar{Z})$ where

$$H(\bar{X}, \bar{Z}) = \int_0^{G(\bar{X}, \bar{Z})} \rho_3(\bar{X}, Y, \bar{Z}) dY. \quad (25)$$

A further Prandtl transformation with

$$y_g = \bar{Y} - H, \quad v_g = \bar{V} - u_3 \frac{\partial H}{\partial \bar{X}} - w_3 \frac{\partial H}{\partial \bar{Z}}$$

and normalization with the scalings

$$(\bar{X}, \bar{Z}) = \beta^{-\frac{3}{8}} \lambda^{-\frac{5}{4}} (x_l, \beta^{-\frac{1}{3}} z_l), \quad y_g = \beta^{-\frac{1}{8}} \lambda^{-\frac{3}{4}} y_l,$$

$$(u_3, w_3) = \lambda^{\frac{1}{4}} \beta^{-\frac{1}{8}} (u_l, \beta^{-\frac{1}{2}} w_l), \quad v_g = \lambda^{\frac{3}{4}} \beta^{\frac{1}{8}} v_l,$$

$$p_3 = \lambda^{\frac{1}{2}} \beta^{-\frac{1}{4}} p_l, \quad T_3 = T_l, \quad \rho_3 = \rho_l,$$

$$A = \lambda^{-\frac{3}{4}} \beta^{-\frac{1}{8}} A_l, \quad D = \lambda^{\frac{1}{2}} \beta^{-\frac{1}{4}} C^{\frac{1}{4}} D_l$$

with $\beta = \sqrt{|1 - M_\infty^2|}$, reduces the equations (21)-(24) to the form:

$$\frac{\partial u_l}{\partial x_l} + \frac{\partial v_l}{\partial y_l} + \frac{\partial w_l}{\partial z_l} = 0, \quad (26)$$

$$u_l \frac{\partial u_l}{\partial x_l} + v_l \frac{\partial u_l}{\partial y_l} + w_l \frac{\partial u_l}{\partial z_l} + T_l \frac{\partial p_l}{\partial x_l} = \frac{\partial^2 u_l}{\partial y_l^2}, \quad (27)$$

$$u_l \frac{\partial w_l}{\partial x_l} + v_l \frac{\partial w_l}{\partial y_l} + w_l \frac{\partial w_l}{\partial z_l} + T_l \frac{\partial p_l}{\partial z_l} = \frac{\partial^2 w_l}{\partial y_l^2} \quad (28)$$

$$u_l \frac{\partial T_l}{\partial x_l} + v_l \frac{\partial T_l}{\partial y_l} + w_l \frac{\partial T_l}{\partial z_l} = \frac{1}{Pr} \frac{\partial^2 T_l}{\partial y_l^2}. \quad (29)$$

The no-slip conditions and wall conditions are now applied on $y_l = 0$ giving

$$u_l(x_l, 0, z_l) = v_l(x_l, 0, z_l) = w_l(x_l, 0, z_l) = 0 \quad (30)$$

and the prescribed wall temperature

$$T_l(x_l, 0, z_l) = T_w(x_l, z_l). \quad (31)$$

The boundary conditions at a significant distance from the surface ($y_l \rightarrow \infty$) are presented as

$$u_l \rightarrow y_l + A_l(x_l, z_l) + H(x_l, z_l), \quad (32)$$

$$w_l \rightarrow \frac{D_l(x_l, z_l)}{y_l + H(x_l, z_l) + \int_0^\infty (T_l - 1) d\eta}, \quad (33)$$

$$\mathcal{K}(x_l, z_l) = A_l(x_l, z_l) + \int_0^\infty (1 - T_l) d\eta, \quad (34)$$

$$T_l(x_l, y_l, z_l) \rightarrow 1 \text{ as } y_l \rightarrow \infty, \quad (35)$$

with

$$A_l(-\infty) \rightarrow 0, \quad (36)$$

and the interaction condition is

$$p_l(x_l, z_l) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 \mathcal{K}(\zeta, \alpha) / \partial \zeta^2}{\sqrt{(x_l - \zeta)^2 + (z_l - \alpha)^2}} d\zeta d\alpha. \quad (37)$$

The function $\mathcal{K}(x_l, z_l)$ is the total displacement of stream lines and $A_l(x_l, z_l)$ the displacement of stream lines caused by the viscosity of the fluid, $T_w(x_l, z_l)$ represents the prescribed wall temperature and $H(x_l, z_l)$ is the integral of surface element profile shape as given by (25). From the solution given in equation (9) it can be seen that the function $D(X, Z)$ gives the spanwise velocity just above the boundary layer region.

Hence we have formulated the three-dimensional problem of flow past a heating element placed on a flat surface where the element has a humped surface. The next step is to study this problem in the linear approximation which is presented below.

3. Linear theory

The equations (26)-(37) have a solution

$$u_l = y_l, \quad T_l = 1, \quad v_l = w_l = p_l = A_l = D_l = 0$$

for the case when there is no heating with $T_w = 1$ and no hump $H = 0$. The problem with $T_w = T_l = 1$ and H non-zero reduces to the problem studied in Smith et al. [1977]. We will be interested in the case with localised heating $T_w \neq 1$ everywhere, and non-zero hump height. For $O(1)$ variations of the temperature the nonlinear problem needs to be solved numerically. However, for small variations of the temperature and hump height, we may study the problem analytically in the linear approximation considered in this section. Consider small variations of the temperature i.e. $T_w - 1 = O(\sigma)$ with $\sigma \ll 1$. We may linearise the equations (26)-(37) related to the undisturbed boundary layer profile by expanding the flow variables as follows:

$$u_l = y_l + \sigma \tilde{U} + O(\sigma^2), \quad v_l = \sigma \tilde{V} + O(\sigma^2), \quad w_l = \sigma \tilde{W} + O(\sigma^2), \quad (38)$$

$$T_l = 1 + \sigma \tilde{T} + O(\sigma^2), \quad T_w = 1 + \sigma \tilde{f}(x_l, z_l) + O(\sigma^2), \quad (39)$$

$$p_l = \sigma \tilde{P} + O(\sigma^2), \quad A_l = \sigma \tilde{A} + O(\sigma^2), \quad (40)$$

$$\mathcal{K} = \sigma \tilde{\mathcal{K}} + O(\sigma^2) \quad H = \sigma \tilde{H}(x_l, z_l) + O(\sigma^2), \quad \text{and} \quad D_l = \sigma \tilde{D} + O(\sigma^2). \quad (41)$$

as $\sigma \rightarrow 0$. Then, we substitute the expansions, (38)-(41), into the equations and boundary conditions (26)-(37) collect the terms with the same powers of σ which gives at leading order the following problem:

$$\frac{\partial \tilde{U}}{\partial x_l} + \frac{\partial \tilde{V}}{\partial y_l} + \frac{\partial \tilde{W}}{\partial z_l} = 0, \quad (42)$$

$$y_l \frac{\partial \tilde{U}}{\partial x_l} + \tilde{V} + \frac{\partial \tilde{P}}{\partial x_l} = \frac{\partial^2 \tilde{U}}{\partial y_l^2}, \quad (43)$$

$$y_l \frac{\partial \tilde{W}}{\partial x_l} + \frac{\partial \tilde{P}}{\partial z_l} = \frac{\partial^2 \tilde{U}}{\partial y_l^2}, \quad (44)$$

$$y_l \frac{\partial \tilde{T}}{\partial x_l} = \frac{1}{Pr} \frac{\partial^2 \tilde{T}}{\partial y_l^2} \quad (45)$$

with boundary conditions

$$\tilde{U}(x_l, 0, z_l) = \tilde{V}(x_l, 0, z_l) = \tilde{W}(x_l, 0, z_l) = 0, \quad \tilde{T}(x_l, 0, z_l) = \tilde{f}(x_l, z_l), \quad (46)$$

$$\tilde{U} \rightarrow \tilde{A}(x_l, z_l) + \tilde{H}(x_l, z_l), \quad \tilde{\mathcal{K}}(x_l, z_l) = \tilde{A}(x_l, z_l) - \int_0^\infty \tilde{T}(x_l, \eta, z_l) d\eta, \quad (47)$$

$$\tilde{T}(x_l, y_l, z_l) \rightarrow 0 \quad \text{as} \quad y_l \rightarrow \infty, \quad (48)$$

$$\tilde{A}(x_l, z_l) \rightarrow 0 \quad \text{as} \quad x_l \rightarrow -\infty, \quad (49)$$

and the pressure-displacement condition given by

$$\tilde{P}(x_l, z_l) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 \tilde{\mathcal{K}}(\zeta, \alpha) / \partial \zeta^2}{\sqrt{(x_l - \zeta)^2 + (z_l - \alpha)^2}} d\zeta d\alpha. \quad (50)$$

Also, we may find the function $\tilde{D}$ using

$$\frac{\partial \tilde{D}}{\partial x_l} = -\frac{\partial \tilde{P}}{\partial z_l}. \quad (51)$$

We will use Fourier Transforms to solve the above equations. The Fourier Transform of $g(x_l, z_l)$ is defined by

$$\hat{g}(\omega, l) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_l, z_l) e^{-i\omega x_l} e^{-ilz_l} dx_l dz_l. \quad (52)$$

We apply double Fourier transformation with respect to x_l and z_l to equations (42)-(45), to get

$$i\omega \hat{U} + \frac{\partial \hat{V}}{\partial y_l} + il\hat{W} = 0, \quad (53)$$

$$i\omega y_l \hat{U} + \hat{V} + i\omega \hat{P} = \frac{\partial^2 \hat{U}}{\partial y_l^2}, \quad (54)$$

$$i\omega y_l \hat{W} + il\hat{P} = \frac{\partial^2 \hat{W}}{\partial y_l^2} \quad (55)$$

$$i\omega Pr y_l \hat{T} = \frac{\partial^2 \hat{T}}{\partial y_l^2}. \quad (56)$$

The solution of the equation (56) using the transform of the condition on the wall, is given by

$$\hat{T} = \frac{\hat{f}(\omega, l)}{\text{Ai}(0)} \text{Ai}(Pr^{\frac{1}{3}} Y) = \frac{\hat{f}(\omega, l)}{\text{Ai}(0)} \text{Ai}((i\omega Pr)^{\frac{1}{3}} y_l). \quad (57)$$

where $Y = (i\omega)^{\frac{1}{3}} y_l$, and we have taken a branch cut along the positive imaginary axis for the function $(i\omega)^{\frac{1}{3}}$. Next the equation (55) in terms of Y is

$$\frac{\partial^2 \hat{W}}{\partial Y^2} - Y\hat{W} = (i\omega)^{-\frac{2}{3}} il\hat{P}, \quad (58)$$

and its solution follows from Smith et al. [1977] which can be expressed as

$$\hat{W} = (i\omega)^{-\frac{2}{3}} il\hat{P} \left[-\frac{\sqrt{2}}{3} \int_0^{\infty} \sin\left(\frac{\zeta^3}{3} + \zeta Y - \frac{\pi}{6}\right) d\zeta \right] = (i\omega)^{-\frac{2}{3}} il\hat{P} \Phi(Y),$$

where

$$\Phi(Y) = -\frac{\sqrt{2}}{3} \int_0^{\infty} \sin\left(\frac{\zeta^3}{3} + \zeta Y - \frac{\pi}{6}\right) d\zeta$$

satisfies

$$\frac{\partial^2 \Phi}{\partial Y^2} - Y\Phi = 1$$

with $\Phi'(0) = -\frac{1}{3\text{Ai}(0)}$ and $\Phi''(0) = 1$. We note also that $\Phi(0) = 0$, and $\Phi(\infty) = 0$.

Now, we consider equations (53) and (54). We differentiate (54) with respect to y_l and use (53) to obtain

$$\frac{\partial^2 \hat{S}}{\partial Y^2} - Y\hat{S} = \frac{l^2}{(i\omega)^{\frac{5}{3}}} \hat{P}\Phi(Y) \quad (59)$$

where $\frac{\partial \hat{U}}{\partial y_l} = \hat{S}$. The solution is given by

$$\frac{\partial \hat{U}}{\partial Y}(\omega, Y, l) = \frac{l^2}{(i\omega)^{\frac{5}{3}}} \hat{P}\Phi'(Y) + R(\omega, l)\text{Ai}(Y), \quad (60)$$

for some unknown function $R(\omega, l)$. To find $R(\omega, l)$, we use the linearised x_l -momentum equation and the interaction condition. Using (54) at $y_l = 0$, we obtain

$$\frac{\partial \tilde{P}}{\partial x_l} = \frac{\partial^2 \tilde{U}}{\partial y_l^2}(x_l, 0, z_l). \quad (61)$$

Next applying the Fourier transform to equation (61) and differentiating (60) with respect to Y gives

$$(i\omega)^{\frac{1}{3}} \hat{P} = \frac{l^2}{(i\omega)^{\frac{5}{3}}} \hat{P} + R(\omega, l)\text{Ai}'(0). \quad (62)$$

Substituting (62) into (61) yields

$$\begin{aligned} (i\omega)^{-\frac{2}{3}} i\omega \hat{P} &= \frac{l^2}{(i\omega)^{\frac{5}{3}}} \hat{P} + R(\omega, l)\text{Ai}'(0) \\ \Leftrightarrow R(\omega, l) &= \frac{(i\omega)^{\frac{1}{3}} - (i\omega)^{-\frac{5}{3}} l^2}{\text{Ai}'(0)} \hat{P}. \end{aligned} \quad (63)$$

A Fourier transform of the interaction condition (50) gives

$$\hat{P} = \frac{\omega^2}{(\omega^2 + l^2)^{\frac{1}{2}}} \hat{\mathcal{K}}. \quad (64)$$

Considering the equations (47), (63) and (60) gives

$$\hat{U} - \hat{H}(\omega, l) = \frac{(\omega^2 + l^2)^{\frac{1}{2}}}{\omega^2} \hat{P} + \int_0^\infty \hat{T} d\eta.$$

Hence

$$\hat{P}(\omega, l) = \frac{(i\omega)^2 [\hat{H}(\omega, l) + (i\omega Pr)^{-\frac{1}{3}} \Lambda \hat{f}(\omega, l)]}{(\omega^2 + l^2)^{\frac{1}{2}} [1 + \Theta(i\omega)^{\frac{1}{3}} (\omega^2 + l^2)^{\frac{1}{2}}]} \quad (65)$$

where $\Theta = [-3\text{Ai}'(0)]^{-1}$ and $\Lambda = [3\text{Ai}(0)]^{-1}$. The transformed surface shear stress in the z_l -direction can be obtained from equation (59) and is represented as

$$\frac{\partial \hat{W}}{\partial y_l}(\omega, 0, l) = -\frac{il(i\omega)^{\frac{5}{3}} [\hat{H}(\omega, l) + (i\omega Pr)^{-\frac{1}{3}} \Lambda \hat{f}(\omega, l)]}{3\text{Ai}(0)(\omega^2 + l^2)^{\frac{1}{2}} [1 + \Theta(i\omega)^{\frac{1}{3}} (\omega^2 + l^2)^{\frac{1}{2}}]} \quad (66)$$

while the transformed surface shear stress in the x_l -direction can be deduced from equations (60),(63) and (65) which give

$$\frac{\partial \hat{U}}{\partial y_l}(\omega, 0, l) = \frac{3\text{Ai}(0)(i\omega)^{\frac{2}{3}}[\hat{H}(\omega, l) + (i\omega Pr)^{-\frac{1}{3}}\Lambda \hat{f}(\omega, l)]}{(\omega^2 + l^2)^{\frac{1}{2}}[1 + \Theta(i\omega)^{\frac{1}{3}}(\omega^2 + l^2)^{\frac{1}{2}}]} \left[\frac{\omega^2 + l^2}{\Theta^{-1}} - \frac{l^2}{9\text{Ai}^2(0)} \right]. \quad (67)$$

Finally, the function $\hat{D}$ is given by, using equation (51),

$$\hat{D} = -\frac{(il)(i\omega)[h\hat{H}(\omega, l) + (i\omega Pr)^{-\frac{1}{3}}\Lambda \hat{f}(\omega, l)]}{(\omega^2 + l^2)^{\frac{1}{2}}[1 + \Theta(i\omega)^{\frac{1}{3}}(\omega^2 + l^2)^{\frac{1}{2}}]}. \quad (68)$$

As a special case, the expressions (65)-(68) reduce to those in Smith et al. [1977] when there are no variations of the wall temperature and taking $\hat{f}(\omega, l) = 0$.

4. Results and discussion

The expressions (65)-(68) were inverted numerically using the Fast Fourier Transform (FFT). To validate the results we have compared the pressure perturbation in the two dimensional subsonic flow problem presented by Koroteev and Lipatov [2012], with our computed pressure on the centerline $z_l = 0$. This is shown in figure 2(a) where we have taken

$$\tilde{f}(x_l, z_l) = \begin{cases} \Delta T & |x_l| \leq 7, |z_l| \leq 7 \\ 0 & \text{elsewhere} \end{cases},$$

with $\Delta T = 0.2$ and $\tilde{H} = 0$. In figure 2(a) the normalised pressure (by dividing by ΔT to compare with $\tilde{P}$ from linear theory) of Koroteev and Lipatov [2012] is shown.

Another check for the three-dimensional results is provided by comparing the pressure distribution along $z_l = 0$ in the non-linear three dimensional subsonic flow problem as computed by Bodonyi and Duck [1988], figure 1c in their paper. This is reproduced as shown in figure 2(b) where we have taken

$$\tilde{H}(x_l, z_l) = 2.8 * \exp(-x_l^2 - z_l^2)$$

and $\tilde{f} = 0$. Excellent agreement can be observed in both figures 2(a) and 2(b).

Various grid size and other checks were carried out. Below we present our most accurate solutions where we have used 2048 Fourier points in ω with $-60 < \omega < 60$ and 2048 Fourier points in l with $-60 < l < 60$. The results are in qualitative agreement with the case of flow past an unheated hump as in work of Smith et al. [1977] who take a different hump shape from that used here.

Comparison of our results for the pressure distributions of the three dimensional linear problem and the two dimensional case considered by Aljohani and Gajjar [2017], shows some differences especially in the minimum value of the pressure. For this comparison we have taken two different spanwise sizes of lengths of the element. In the first instance the length of the element in the z_l -direction is taken to be much longer

Figure 2: Comparisons with previous work results for (a) pressure on the symmetry line $z_l = 0$ with $\Delta T = 0.2$ and $\tilde{H} = 0$ and taking $Pr = 1$, and (b) pressure along the plane $z_l = 0$ where $\tilde{H} = 2.8 * \exp(-x_l^2 - z_l^2)$ and $\Delta T = 0$.

than that in the x_l -direction. This is a quasi two-dimensional hump. The hump shape for this is chosen as

$$\tilde{H}(x_l, z_l) = h \exp(-0.25x_l^2 - 0.01z_l^2)$$

and the wall temperature function is

$$\tilde{f}(x_l, z_l) = \begin{cases} \Delta T & |x_l| \leq 5, |z_l| \leq 25 \\ 0 & \text{elsewhere} \end{cases}.$$

In the second situation we consider an element with square cross-section by taking the same lengths in the z_l and x_l -directions. Here, the hump shape is chosen as

$$\tilde{H}(x_l, z_l) = h \exp(-0.25x_l^2 - 0.25z_l^2)$$

and the wall temperature function is

$$\tilde{f}(x_l, z_l) = \begin{cases} \Delta T & |x_l| \leq 5, |z_l| \leq 5 \\ 0 & \text{elsewhere} \end{cases}.$$

With $\Delta T = 0.4$ and $h = 0.5$, results for the quasi two-dimensional hump are shown in figure 3(a). This figure shows moderate differences between the pressure distributions for the three dimensional problem and the two dimensional one. On the other hand for the hump with square cross-section the results presented in figure 3(b) show that there are more noticeable differences between the results of the three and two dimensional problems. In particular, it can be noticed that the pressure has a much lower minimum value in the three dimensional solution as compared to the two dimensional solution.

Figure 3: Pressure distributions along $z_l = 0$ versus the solution of the two-dimensional problem where $\Delta T = 0.4$, $h = 0.5$, and $Pr = 0.72$ for (a) a long hump in the z_l direction and (b) hump of equal dimensions in the z_l and x_l -directions.

In the remainder of the results we have taken the Prandtl number $Pr = 0.72$ and the wall temperature distribution to be given by

$$\tilde{f}(x_l, z_l) = \begin{cases} \Delta T & |x_l| \leq 0.5 \text{ \& } |z_l| \leq 0.5 \\ 0 & \text{elsewhere} \end{cases}.$$

The element surface shape is also described by

$$\tilde{H}(x_l, z_l) = h \exp(-5x_l^2 - 5z_l^2),$$

with h a parameter.

Results below are shown for three cases: case A with $\Delta T = 0.2$, $h = 0$ which describes the flat plate case with wall heating, while for case B, $\Delta T = 0.2$, $h = 0.5$ and this is the thermally heated hump. For comparison we also consider case C with $\Delta T = 0$, $h = 0.5$ which is unheated hump-shaped element. Contours of the pressure distributions for the cases A, B, and C are shown in figures 4(a), 4(b), and 4(c), respectively. The pressure distribution along the centerline $z_l = 0$ is presented in figure 5(a) for case A and case B. It can be seen that as the flow approaches the element, the pressure perturbation increases. However, on reaching the upstream edge of the element the pressure begins to decrease rapidly over the element. Thereafter, on the downstream edge of the element, the pressure starts to rise sharply and tends to vanish downstream. Similar behaviour of the pressure perturbations appear on sides of the element away from the centerline. The main differences between the case A and the case B can be seen in three aspects. Firstly, the minimum value of the pressure is at a smaller value with the presence of the hump as compared to the flat-plate case A. Secondly, on reaching the element, the maximum value of the pressure in the case B is larger than in case A. Finally, the zero isoline of the pressure on the leeward side of the

element is reached earlier in the case B than for case A. If we compare cases B and C, in figures 4(b),4(c) we see that the heated hump generates much larger drops and rises in the pressure in the vicinity of the element. Away from the element, the extent of the negative pressure region is increased with heating.

Concerning the axial shear distribution $\tilde{\tau}_x = \frac{\partial \tilde{U}}{\partial y_l}(x_l, 0, z_l)$, we present contours for the three cases in figures 6(a), 6(b) and 6(c) respectively. The axial shear distribution for cases A and B, along the centre-line $z_l = 0$ shown in figure 5(b) shows that the maximum value is found on reaching the upstream edge of the element while in the case B the maximum value is found in a neighbourhood of the centre of the element. With the heated hump, the shear rises substantially over the element, and there is also a larger drop at the rear of the element. Both figures 5(a), 5(b) show that with the hump shaped heated element, the rise and drop in the pressure and shear coinciding with the start and end locations of the heating element, is not as abrupt as seen in the flat-plate case. This was also noticed in the results of Aljohani and Gajjar [2017].

Also, for all the cases the corridor effect, appearing in Smith et al. [1977], is confirmed. However, its appearance for the case A is established at approximately ($x_l = 6$) much further than its location ($x_l = 3.8$) for the case B. On the other hand for the unheated hump, the corridor effect starts much earlier. In cases B and C, the shear $\tilde{\tau}_x$ drops to a minimum value on the leeward side of the element and this suggests that the flow separation may occur if the hump height h or the amount of heating ΔT are sufficiently large enough to provoke it. Flat plate heated elements are better in this respect.

Finally, contours of the cross-flow shear distribution $\tilde{\tau}_z = \frac{\partial \tilde{W}}{\partial y_l}(x_l, 0, z_l)$ and the function $\tilde{D}(x_l, z_l)$, which gives an indication of behaviour of the outer edge spanwise velocity component w_l for $z_l \gg 1$, are displayed in figures 7 and figure 8 respectively. Both functions vanish along the centreline $z_l = 0$ due to the symmetry of the element. Both the figures 7 and figure 8 show that the flow is pushed away from the front part of the element. There is a change of sign of both functions over the rear portion which suggests that the fluid is drawn in towards the centerline. This would indicate vortex formation behind the element. Near the element centre, $\tilde{\tau}_z$ changes sign which results in a negative minimum on the downstream edge of the heated element. Downstream, away from the element positive values of the cross-flow shear appear once again. This suggests a sink-like flow behaviour at the rear part of the element. Regarding $\tilde{D}$, it is positive upstream and over the heated zone. However, the sign is reversed near the downstream edge of the heated region and thereafter. This would lead to very similar four-part picture of the secondary flow as sketched in Smith et al. [1977] for the unheated hump. Comparing the heated and unheated humps, one may conclude that the larger peaks and troughs in the heated case, would suggest much stronger secondary flows arising from heating.

(a)

(b)

(c)

Figure 4: Contours of the pressure distribution with $Pr = 0.72$, and
 (a) $\Delta T = 0.2, h = 0$, (b) $\Delta T = 0.2, h = 0.5$, (c) $\Delta T = 0, h = 0.5$. The dashed lines denote negative contours, the solid line is the zero contour and contour levels are in steps of 0.01.

Figure 5: (a) Pressure distribution along $z_l = 0$ with $Pr = 0.72$ and $\Delta T = 0.2$ and $h = 0$ and $h = 0.5$. (b) The perturbed axial shear distribution $\bar{\tau}_x(x_l, 0)$ along the centerline with $Pr = 0.72$ and $\Delta T = 0.2$ and $h = 0$ and $h = 0.5$.

5. Conclusion

The linear problem of subsonic flow past a heating element in three dimensions has been analysed and results obtained for various cases. The results show the effects of using different heights of the hump and element temperatures on the flow perturbations. The results of the linear theory generally are in agreement with results of previous studies of flows past unheated humps such as that treated by Smith et al. [1977] including the occurrence of the 'corridor effect' and vortex formation behind the hump. In comparing results for the heated flat-plate with those for the heated hump, the heated hump smooths out the abrupt rise and drop in the shear and pressure distributions when the fluid enters and leaves the heated region. This is very similar to the results obtained by Aljohani and Gajjar [2017] in their investigations of the two-dimensional problem with hump-shaped elements. Also a hump shaped element promotes much larger drops and rises in the minimum and maximum pressure and shear values as compared to the heated flat plate element, or the unheated hump shaped element.

The linear theory is not adequate to describe the non-linear effects such as the flow separation that might be expected with bigger humps or stronger temperature perturbations. For this a numerical method to solve the non-linear problem would be useful to extend the analysis even further.

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(a)

(b)

(c)

Figure 6: Contours of axial shear $\tilde{\tau}_x(x_l, 0, z_l)$ with (a) $\Delta T = 0.2, h = 0$ and (b) $\Delta T = 0.2, h = 0.5$, and (c) $\Delta T = 0, h = 0.5$. Contour levels are in steps of 0.05.

(a)

(b)

(c)

Figure 7: Contours of cross-flow shear $\tilde{\tau}_z(x_l, 0, z_l)$ with (a) $\Delta T = 0.2, h = 0$ and (b) $\Delta T = 0.2, h = 0.5$, and (c) $\Delta T = 0, h = 0.5$. Contour levels are in steps of 0.01.

(a)

(b)

(c)

Figure 8: Contours of the function $\tilde{D}(x_l, z_l)$ with (a) $\Delta T = 0.2, h = 0$ and (b) $\Delta T = 0.2, h = 0.5$, and (c) $\Delta T = 0, h = 0.5$. Contour levels are in increments of 0.01.

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