Horizontal Dynamic Stiffness and Interaction Factors of Inclined Piles

Jue Wang, Ph.D., College of Civil Engineering, Nanjing Tech University, Nanjing 211816, China; College of Mechanical & Electrical Engineering, Hohai University, Changzhou, China.
Ding Zhou*, Ph.D., College of Civil Engineering, Nanjing Tech University, Nanjing 211816, China.
Tianjian Ji, Ph.D., School of Mechanical, Aerospace and Civil Engineering, The University of Manchester, Manchester, M13 9PL, UK.
Shuguang Wang, Ph.D., College of Civil Engineering, Nanjing Tech University, Nanjing 211816, China.

Abstract:
A Timoshenko beam-on-Pasternak-foundation model (T-P) model is developed to estimate the horizontal dynamic impedance and interaction factors for inclined piles subjected to a horizontal harmonic load. The inclined pile with a fixed pile-to-cap connection is modeled as a Timoshenko beams embedded in the homogeneous Pasternak foundation. The reactions of the soil on the pile tip are simulated by three springs in parallel with corresponding dashpots. The differential equations of normal and axial vibrations of the inclined pile are solved by means of the initial parameter method. The model and the theoretical derivation are validated through comparisons with results from other theoretical models for some large-diameter end-bearing piles and inclined piles. The effects of shear deformations of the soil and inclined piles, during the vibration, are highlighted by comparing the results between the T-P model and the Euler beam–on–dynamic–Winkler–foundation (E-W) model. It is indicated that the T-P Model is more accurate for analyzing the dynamic performance of the inclined piles with small slenderness ratios and large inclined angles. The effects of the inclined angle, slenderness ratio and distance-diameter ratio on the dynamic impedance and interaction factors are studied by numerical examples. It is demonstrated that the increased angle of the pile results in an increase of horizontal impedance and decreases of interaction factors.

Keywords:
Soil-structure interaction; Inclined pile; Pile impedance; Interaction factor; Timoshenko beam; Pasternak foundation.

Introduction
Inclined piles were once considered to be detrimental for their seismic behavior (Deng et al. 2007;
and even recommended being avoided by some codes (AFPS 1990; CEN 2004) due to their poor performances in a series of earthquakes occurred in the 1990s. However, numerical (Gerolymos et al. 2008; Turan et al. 2013; Sadek and Isam 2004) and experimental studies (Okawa et al. 2002; Gotman 2013) showed the advantages of the inclined piles. The ultimate reasons of failures may be due to the incorrect design and construction techniques rather than the pile inclination itself. As inclined piles can transmit the applied horizontal loads partly through axial compression, they provide larger horizontal stiffness and bearing capacity than conventional vertical piles. Therefore, employing inclined piles in foundations for buildings, bridges and marginal wharfs is an efficient way to resist horizontal loads such as seismic waves, vehicle impacts and ocean waves (Gazetas and Mylonakis 1998; Padrón et al. 2010). Inclined angles commonly used in practice are between 5°- 15°, in addition to the less usual cases between 20° and 25°.

Padron et al. (2010, 2012, 2014) investigated dynamic impedances of both single inclined pile and group inclined piles, as well as the kinematic interaction factors between adjacent inclined piles, using a boundary element-finite element coupling model (BEM-FEM). In their study, the pile is modeled as an elastic compressible Euler-Bernoulli beam. Their investigation contributed an understanding of the effect of inclined piles on the dynamic impedances of deep foundations. Recently, they further studied the response of the structure supported by an inclined pile foundation (Medina et al. 2015). In addition to numerical methods, some simplified analytical models are also applied in the analysis of soil-pile dynamic interaction. Makris and Gazetas (1992) presented a Euler Beam-on-dynamic-Winkler-foundation (E-W) model to account for the impedance of a single vertical pile and interaction factors between adjacent vertical piles, which can be extended to evaluate the effect of dynamic pile groups using the superposition method (Kaynia and Kausel 1982). Mylonakis and Gazetas (1998, 1999) used the E-W model to determine the dynamic interaction factors for axial and lateral vibrations of vertical piles in layered soil by considering the presence of receiver piles. Subsequently, the E-W model has received a widespread application due to low computational complexity and reasonable agreement with the rigorous solutions (Liu et al. 2014; Sica et al. 2011). Gerolymos and Gazetas (2006) developed a multi-Winkler model to analyse the static and dynamic stiffness of rigid caisson foundations with small slenderness ratio. In recent years, some researchers have managed to extend the E-W model to the dynamics of inclined piles. Ghasemzadeh and Alibeikloo (2011) used the Euler beam-on-dynamic-Winkler-foundation (E-W)
model to analyse the interaction factor between two infinitely long inclined piles. As the limitation of
the theoretical assumptions, the E-W model only valid for those piles with large slenderness ratio
(Yokoyama, 1996; Wang et al., 2014a). Ghazavi et al. (2013) further considered the dynamic
interaction factor of adjacent inclined piles under oblique harmonic loads.

However, the E-W model neglects the shear deformations of both soil medium and the pile,
which is only valid for the piles with a large slenderness ratio (i.e. piles with large diameter and/or
short length). Actually, piles with small slenderness ratios are commonly used on the firm stratum to
support superstructures (Mylonakis 2001). Kampitsis et al. (2013) demonstrated the advantages of
Timoshenko theory on the kinematic and inertial interaction between soil and column-pile with a
small slenderness ratio. Recently, Wang et al. (2014a, 2014b) presented a Timoshenko
beam-on-Pasternak-foundation (T-P) model to determine the dynamic interaction factors between
vertical short piles with hinged-head connection in the multilayered soil medium. The parametric
studies confirmed that the T-P model subjected to high frequency excitation provided the improved
results for the dynamic interaction of piles with small slenderness ratio.

As an extension of previous works (Wang et al. 2014a, 2014b), the T-P model is further applied
to analyzing the impedance of single inclined pile and dynamic interaction factors between adjacent
inclined piles. The shear deformation and rotational inertia of inclined piles, as well as the shear
defformation of soil, have been simultaneously considered to overcome the limitation of conventional
E-W model (Ghasemzadeh and Aleikloo, 2011). Transformation between the local and global
coordinate systems is used in the analysis as the inclined piles can transmit the horizontal loads not
only through the bending deformation but also through the axial deformation. Therefore, the
horizontal and vertical interaction factors for inclined piles are coupled. The effects of slenderness
ratios, pile-soil modulus ratios, the inclined angles of single pile on adjacent piles have been studied
in detail.

**Model description**

Fig. 1 shows a cylindrical elastic pile embedded in a homogeneous half-space with an inclined angle
$\varphi_1$. The head of the pile is fixed to the cap. To conveniently evaluate the dynamic impedance atop the
head of the inclined pile, a horizontal harmonic excitation $H e^{i\omega t}$ in the global coordinate system ($x$-$z$)
is decomposed into an axial component \( Qe^{i\omega t} \) and a normal component \( Pe^{i\omega t} \) in a local coordinate system \((x'-z')\). Here, \( H, Q \) and \( P \) are the load amplitudes, \( \omega \) denotes the frequency of the external load, \( t \) represents the time and \( i = \sqrt{-1} \). The T-P model is introduced to analyse the normal vibration of the inclined pile, which is also a further extension of the previous work (Wang et al. 2014a). The Timoshenko beam theory is used to simulate the transverse vibration of the pile in which analyse the effect of shear deformation and the rotational inertia are considered. The Pasternak foundation is used to represent the elastic half-space. The normal reaction of soil against the pile-shaft is simulated by a system of uniformly distributed linear springs and dashpots, which are connected through an incompressible shear layer. It is confirmed by Poulos and Davis (1980) that load distribution along the inclined pile with angle less than 30° is reasonably assumed to be similar to that along the vertical pile. Therefore, the analytical expressions of springs, dashpots and the shear layer for the vertical pile, which based on some simplified physical models calibrated with results from numerical methods and experimental dates (Fwa et al. 1996; Gerolymos and Gazetas 2006; Wang et al. 2014), are introduced in the present model as follows:

\[
k_x \simeq \begin{cases} 
1.75(L/d)^{-0.13}E_s & \text{for } L/d < 10 \\
1.2E_s & \text{for } L/d \geq 10 \end{cases};  
\]

(1a)

\[
c_x \simeq 6a_0^{-1/4}\rho_s d V_s + 2\beta \omega \kappa_x / \omega;  
\]

(1b)

\[
g_x \simeq 0.5k_x.  
\]

(1c)

in which, \( E_s, G_s, \nu_s, \rho_s, \beta_s \) and \( V_s \) are the Young’s modulus, shear modulus, Poisson’s ratio, mass density, damping and shear wave velocity of the soil medium, respectively; \( L \) and \( d \) are the length and sectional diameter of the inclined pile, respectively. \( \omega = \omega_d/V_s \) is the dimensionless frequency.

As axial vibration of the pile does not induce the shear deformation and rotational inertia, the coefficients of springs and dashpots based on the Novak’s plain strain model (Novak and Aboul-Ella 1978) are given as follows:

\[
k_x \simeq 2\pi a_0 G_s \frac{J_1(a_0)J_0(a_0)+Y_1(a_0)Y_0(a_0)}{J_0^2(a_0)+Y_0^2(a_0)};  
\]

(2a)

\[
c_x \simeq G_s \left[ \frac{4}{J_0^2(a_0)+Y_0^2(a_0)} \right]^{1/2}.  
\]

(2b)

in which, \( J_0 \) and \( J_1 \) are the first kind of Bessel function of the zero-order and the first-order,
respectively; $Y_0$ and $Y_1$ are the second kind of Bessel function of the zero-order and the first-order, respectively.

It is well known that the mechanical properties at the pile head is unaffected by the boundary condition at the pile tip for the case with a large slenderness ratio $L/d$. However, different from the infinite long pile, the reaction of the soil on the pile tip should be considered when it has a small slenderness ratio ($L/d<10$) (Gerolymos and Gazetas 2006). The analytical expressions for normal, axial and rotational springs in parallel with dashpots at the pile tip are suggested by Novak and Sacks (1973) as follows:

$$
K_{\text{tip}}^x = \begin{bmatrix}
(4.3 + i 12.7 a_0) G_{s,\text{tip}} (d/2) & 0 \\
0 & (2.5 + i 0.43 a_0) G_{s,\text{tip}} (d/2)^3
\end{bmatrix};
$$

$$
K_{\text{tip}}^z = (7.5 + i 3.4) G_{s,\text{tip}} (d/2).
$$

Formulation

**Normal vibration of an inclined pile**

For simplifying the analysis, the displacement of the inclined pile subjected to horizontal harmonic excitation is decomposed into a normal component and an axial component in a local coordinate system. The horizontal impedance of the inclined pile can be finally obtained using the superposition method. As expected, the formulation in all steps are valid for vertical piles when $\phi_i = 0$.

By using the Hamilton principle and the Timoshenko beam theory, the governing differential equations for the translational and the rotational vibrations of a loaded pile are given by

$$
\rho_p A_p \frac{\partial^2 u_a(z',t)}{\partial t^2} = -\frac{\partial}{\partial z'} \left[ \kappa G_p A_p \left( \theta_a(z',t) - \frac{\partial u_a(z',t)}{\partial z'} \right) \right] - \left( k_s u_a(z',t) - g_s \frac{\partial^2 u_a(z',t)}{\partial z'^2} + c_s \frac{\partial u_a(z',t)}{\partial t} \right)
$$

$$
\rho_p I_p \frac{\partial^2 \theta_a(z',t)}{\partial t^2} = E_p I_p \frac{\partial^2 \theta_a(z',t)}{\partial z'^2} - \kappa G_p A_p \left( \theta_a(z',t) - \frac{\partial u_a(z',t)}{\partial z'} \right)
$$

where $\nu_p$, $\rho_p$, $E_p$ and $G_p$ are the Poisson's ratio, mass density, Young’s modulus, shear modulus of the pile, respectively; $A_p$, $I_p$ and $\kappa=6(1+\nu_p)/(7+6\nu_p)$ are the area of cross section, inertia moment and shear coefficient of the pile, respectively. $u_a$ and $\theta_a$, the normal displacement and bending rotation of the pile, should experience harmonic movements, $u_a(z',t) = U_a(z') e^{i\omega t}$ and $\theta_a(z',t) = \Theta_a(z') e^{i\omega t}$ as the system is subjected to a harmonic action. Applying the differentiation chain rule on Eq. (4) and
Eq. (5), the differential equations of normal vibration and the flexural rotational vibration can be expressed as

\[
\frac{d^4 U_a}{dz^4} + \frac{E_p I_p K_p - J_p g_s + S_p (J_p + g_s)}{E_p I_p (J_p + g_s)} \frac{d^2 U_a}{dz^2} - \frac{K_p (J_p - S_p)}{E_p I_p (J_p + g_s)} U_a = 0
\]

(6)

\[
\Theta_a = \frac{E_p I_p (J_p + g_s)}{J_p (J_p - S_p)} \frac{d^2 U_a}{dz^2} + \frac{E_p I_p K_p + J_p^2}{J_p (J_p - S_p)} \frac{d U_a}{dz}
\]

(7)

where \( J_p, S_p \) and \( K_p \) are defined as \( J_p = \kappa G_p A_p, \ S_p = \rho_p I_p \omega^2, \ K_p = \rho_p A_p \omega^2 - k_s - i \omega c_s \) for brevity. It is noted from Eqs. (6-7) that 1) when \( 1/J_p \rightarrow 0 \) and \( S_p \rightarrow 0 \), it simplifies to the equation of an Euler pile on elastic soil; 2) when \( g_s = 0 \), it reduces to the Winkler model. The general solution of Eq. (6) is

\[
U_a (z') = A_1 \cosh \alpha z' + B_1 \sinh \alpha z' + C_1 \cos \beta z' + D_1 \sin \beta z'
\]

(8)

where

\[
\alpha = \sqrt{\left[ \frac{E_p I_p K_p - J_p g_s + S_p (J_p + g_s)}{2E_p I_p (J_p + g_s)} \right]^2 + \frac{K_p (J_p - S_p)}{E_p I_p (J_p + g_s)} - \frac{E_p I_p K_p - J_p g_s + S_p (J_p + g_s)}{2E_p I_p (J_p + g_s)}}
\]

and

\[
\beta = \sqrt{\left[ \frac{E_p I_p K_p - J_p g_s + S_p (J_p + g_s)}{2E_p I_p (J_p + g_s)} \right]^2 + \frac{K_p (J_p - S_p)}{E_p I_p (J_p + g_s)}}
\]

\[ A_1, \ B_1, \ C_1, \ D_1 \] are unknown initial coefficients, which can be determined based on the known boundary conditions. For a linear elastic Timoshenko pile, the bending moment \( M_a(z') \) and shear force \( Q_a(z') \) of the pile are related to the displacement \( U_a(z') \) and rotation \( \Theta_a(z') \) as follows:

\[
Q_a (z') = J_p \left( \frac{dU_a(z')}{dz'} - \Theta_a(z') \right)
\]

(9)

\[
M_a (z') = E_p I_p \frac{d \Theta_a(z')}{dz'}
\]

(10)

Using the initial parameter method (Wang et al. 2014b) with consideration of Eqs. (7-10), the relationship of both deflections and internal forces between the pile head and the pile tip can be expressed by a transfer matrix \([T_A]\)

\[
[U_a(L), \ \Theta_a(L), \ Q_a(L), \ M_a(L)]^T = [T_A][U_a(0), \ \Theta_a(0), \ Q_a(0), \ M_a(0)]^T
\]

(11)

in which, \([T_A] = [a(L)][a(0)]^{-1}\). The algebraic expression for each element in \([a(z')]\) is given in
Appendix A. It is convenient to divide $[TA]$ into four $2 \times 2$ sub matrices as 

$$
\begin{bmatrix}
TA_{11} & TA_{12} \\
TA_{21} & TA_{22}
\end{bmatrix}
$$

for the following formula derivation. Substituting the tip boundary condition 

$$
\begin{bmatrix}
Q_a(L) \\
M_a(L)
\end{bmatrix} = \left[K^{z}_{\text{tip}}\right] \begin{bmatrix}
U_a(L) \\
\Theta_a(L)
\end{bmatrix}
$$

into Eq. (11), the force-displacement relationship in the normal direction at the pile head can be obtained

$$
\begin{bmatrix}
Q_a(0) \\
M_a(0)
\end{bmatrix} = [\mathcal{R}_a] \begin{bmatrix}
U_a(0) \\
\Theta_a(0)
\end{bmatrix}
$$

(12)

where the normal impedance matrix 

$$
[\mathcal{R}_a] = \left(K^{z}_{\text{tip}}[TA_{12}] - [TA_{22}]\right)^{-1} \left([TA_{21}] - K^{z}_{\text{tip}}[TA_{11}]\right).
$$

During the axial steady-state harmonic vibration, the axial displacement of the pile can be characterized by $w_a = W_a e^{i\alpha}$. As the axial vibration is not involved with the shear deformation, the dynamic equilibrium yields the following governing equation

$$
E_p A_p \frac{d^2W_a}{dz'^2} - \left(k_z + i\omega c_z - \rho_p A_p \omega^2\right) W_a = 0
$$

(13)

The general solution of above equation is

$$
W_a(z') = A_2 \cosh(\eta z') + B_2 \sinh(\eta z')
$$

(14)

with

$$
\eta = \sqrt{\frac{k_z + i\omega c_z - \rho_p A_p \omega^2}{E_p A_p}}.
$$

The axial force in the pile can be obtained from the differential relationship

$$
N_a(z') = E_p A_p \frac{dW_a}{dz'},
$$

The mechanical property between the pile head and tip can be expressed by the transfer matrix $[TB]$ as follows

$$
\begin{bmatrix}
W_a(L) \\
N_a(L)
\end{bmatrix} = [TB] \begin{bmatrix}
W_a(0) \\
N_a(0)
\end{bmatrix}
$$

(15)

in which, $[TB] = [tb(L)][tb(0)]^{-1}$. The algebraic expression for $[tb(z')]$ is given in Appendix B. Substituting the tip boundary condition $N_a(L) = K^{z}_{\text{tip}} W_a(L)$ into Eq. (15), the force-displacement relationship in the axial direction at the pile head can be obtained

$$
N_a(0) = \mathcal{R}_a W_a(0)
$$

(16)
where the axial impedance $\mathbf{R}_z = \frac{TB_{2,1} - K_{tip}^z TB_{1,1}}{K_{tip}^z TB_{1,2} - TB_{2,2}}$. Here, $TB_{i,j}$ represents the element in the $i$-th row and $j$-th column of matrix $[TB]$.

Considering the transformation of the coordinate system between the local and global axes of the inclined pile, the horizontal impedance of the inclined pile $\mathbf{R}_h$ at the pile head is given by:

$$\mathbf{R}_h = \frac{\mathbf{R}_z \mathbf{R}_{z',1,1}}{\mathbf{R}_z \cos^2(\varphi) + \mathbf{R}_{z',1,1} \sin^2(\varphi)}$$

(17)

where $\mathbf{R}_{z',1,1}$ is the element in the first row and first column of the matrix $[\mathbf{R}_{z'}]$. $\mathbf{R}_h$ is a complex value. The real part $K_h$ represents the dynamic stiffness, while the imaginary part $C_h$ represents the damping ratio.

**Dynamic interaction of adjacent inclined piles**

In addition to the loads transmitted from the pile cap, each inclined pile in a group experiences the action from both the normal and axial vibrations of surrounding piles. To better understand the dynamic interaction between adjacent inclined piles, a pair of inclined piles with inclined angles $\varphi_1$ and $\varphi_2$ to the vertical global axis respectively is considered as shown in Fig. 2. The T-P model is developed for evaluating the normal cross interaction between two piles. The normal-normal interaction factor $\lambda_{nn}$ and the axial-normal interaction factor $\lambda_{an}$ are defined as the ratios of the normal displacement at the head of the receiver pile, which is caused by the horizontal vibration of the source pile, to the normal and axil displacements at the head of the source pile respectively. The axial cross dynamic interaction, in which the shear effect of both soil and piles can be neglected (Ghasemzadeh and Alibeikloo 2011), is not studied in this paper as it has been properly solved by the classical E-W model. The derivations of $\lambda_{nn}$ and $\lambda_{an}$ are given in following sections.

The kinetic source pile could excite the surrounding soil and transfer the waves to receiver piles. Based on the coordinate transformation, the normal displacement of the soil around the receiver pile, induced by the normal vibration of the source pile can be obtained by the attenuation factor $f_s(s, \varphi)$:

$$U_s^{(1)} = U_s f_s(s, \varphi) \cos(\varphi_1 + \varphi_2)$$

(18)

where $\varphi$ is the angle between the center-line direction of two piles and the direction of the applied horizontal force, as shown in Fig. 2b. $s$ is the center distance between two piles.

According to the model developed by Dobry and Gazetas (1988), the attenuation factor
for an arbitrary \( \phi \) could be approximately evaluated by the values at \( \phi = 0 \) and \( \phi = \pi/2 \) as follows:

\[
f_x(s, \phi) = f_x(s, 0) \cos^2 \phi + f_x(s, \pi/2) \sin^2 \phi
\]

in which, \( f_x(s, 0) = \frac{d}{2s} \exp \left[ -\omega(\beta_i + i)(s - d/2) \right] \) and \( f_x(s, \pi/2) = \frac{d}{2s} \exp \left[ -\omega(\beta_i + i)(s - d/2) \right] \).

The vertical displacement field around the receiver pile also can be approximately evaluated by the factor \( f_x(s) \) as follows (Dobry and Gazetas 1988):

\[
f_z(s) = \sqrt{\frac{d}{2s}} \exp \left[ -\omega(\beta_i + 1) \right]
\]

Similarly, the normal displacement of the soil around the receiver pile induced by the axial vibration of the source pile can be obtained:

\[
U_s^{(2)} = W_s f_z(s) \sin(\varphi_1 + \varphi_2)
\]

**Normal-normal interaction factor**

Considering the dynamic interaction between the soil and the receiver pile, the normal dynamic equilibrium equation of the receiver pile, which is induced by the normal vibration of the source pile, is expressed as

\[
\frac{d^4 U_b^{(1)}}{dz^4} + \frac{E_p I_p K_p - J_p g_s + S_p (J_p + g_s)}{E_p I_p (J_p + g_s)} \frac{d^2 U_b^{(1)}}{dz^2} - \frac{K_p (J_p - S_p)}{E_p I_p (J_p + g_s)} U_b^{(1)} = f_{s1} U_s^{(1)} + f_{s2} \frac{d^2 U_s^{(1)}}{dz^2} + f_{s3} \frac{d^4 U_s^{(1)}}{dz^4}
\]

in which, \( f_{s1} = \frac{(J_p - S_p) (k_i + i\omega c_i)}{E_p I_p (J_p + g_s)} \); \( f_{s2} = \frac{E_p I_p (k_i + i\omega c_i) - S_p g_s}{E_p I_p (J_p + g_s)} \); \( f_{s3} = \frac{g_s}{J_p + g_s} \). \( U_b^{(1)} \) is the normal displacement of the receiver pile induced by the normal vibration of the source pile.

The complete solution for Eq. (22) is

\[
U_b^{(1)}(z') = A_3 \cosh \alpha z' + B_3 \sinh \alpha z' + C_3 \cos \beta z' + D_3 \sin \beta z'
\]

\[
z F_a \left( A_3 \sinh \alpha z' + B_3 \cosh \alpha z' \right) + z F_b \left( -C_3 \sin \beta z' + D_3 \cos \beta z' \right)
\]

where \( F_a = f_{s1} + f_{s3} \alpha^2 + f_{s2} \beta^2 \) and \( F_b = f_{s1} - f_{s2} \beta^2 + f_{s3} \beta^4 \).

The relationship between deflections and internal forces at the tip of the receiver pile can be written in a matrix form:
\[
\begin{bmatrix}
U_b^{(1)}(L) \\
\varrho_b^{(1)}(L) \\
Q_b^{(1)}(L) \\
M_b^{(1)}(L)
\end{bmatrix} = [TA]
\begin{bmatrix}
U_a^{(1)}(0) \\
\varrho_a^{(1)}(0) \\
Q_a^{(1)}(0) \\
M_a^{(1)}(0)
\end{bmatrix}
+ [TC]
\begin{bmatrix}
U_a(0) \\
\varrho_a(0) \\
Q_a(0) \\
M_a(0)
\end{bmatrix}
\] (24)

where \([TC] = -[ta(L)][ta(0)]^{-1}[tc(0)][ta(0)]^{-1} + [tc(L)][ta(0)]^{-1}\). The algebraic expression for \([tc(z')]\) is given in Appendix C. By substituting the boundary conditions at the tip of the receiver pile \(Q_b^{(1)}(L) = [K_{tip}^{x}][U_b^{(1)}(L)]\) into Eq. (23), we have

\[
[T1]\begin{bmatrix}
U_b^{(1)}(0) \\
\varrho_b^{(1)}(0)
\end{bmatrix} + [T2]\begin{bmatrix}
Q_b^{(1)}(0) \\
M_b^{(1)}(0)
\end{bmatrix} + [T3]\begin{bmatrix}
U_b^{(1)}(0) \\
\varrho_b^{(1)}(0)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (25)

in which, \([T1] = [K_{tip}^{x}][TA_{11}] - [TA_{21}]; \quad [T2] = [K_{tip}^{x}][TA_{12}] - [TA_{22}]; \quad [T3] = [K_{tip}^{x}][TC_{11}] - [TC_{21}] + \left([K_{tip}^{x}][TC_{12}] - [TC_{22}]\right)[\mathfrak{R}_x].\)

For the case of a fixed pile-to-cap connection, the boundary conditions at the head of the receiver pile are satisfied as \(Q_b^{(1)}(0) = 0\) and \(\varrho_b^{(1)}(0) = 0\). Submitting Eq. (12) and above boundary conditions into Eq. (25), the normal-normal interaction factor \(\lambda_{nn}\) is given by

\[
\lambda_{nn} = \frac{U_b^{(1)}(0)}{U_a(0)} = \frac{T3_{12}T2_{12} - T3_{11}T2_{22}}{T1_{11}T2_{12} - T1_{21}T2_{21}}
\] (26)

**Axial-normal interaction factor**

Considering the dynamic interaction between the soil and the receiver pile, the normal dynamic equilibrium equation of the receiver pile, which is induced by the axial vibration of the source pile, is expressed as:

\[
\frac{d^4U_b^{(2)}}{dz^4} + \frac{E_p I_p K_p - J_p g_s + S_p (J_p + g_s)}{E_p I_p (J_p + g_s)} \frac{d^2U_b^{(2)}}{dz^2} - \frac{K_p (J_p - S_p)}{E_p I_p (J_p + g_s)} U_b^{(2)} = f_{31} U_a^{(2)} + f_{32} \frac{dU_b^{(2)}}{dz^2} + f_{33} \frac{d^2U_b^{(2)}}{dz^4}
\] (27)

where \(f_{31} = \frac{(J_p - S_p)(k_s + i\omega c_s)}{E_p I_p (J_p + g_s)}\); \(f_{32} = \frac{E_p I_p (k_s + i\omega c_s) - S_p g_s}{E_p I_p (J_p + g_s)}\); \(f_{33} = \frac{g_s}{J_p + g_s}\). \(U_b^{(2)}\) is the normal displacement of the receiver pile induced by the axial vibration of the source pile.
The solution of equation (27) is:

\[ U_b^{(2)}(z') = A_4 \cosh \alpha z' + B_4 \sinh \alpha z' + C_4 \cos \beta z' + D_4 \sin \beta z' + F_c \left( A_5 \cosh \eta z' + B_5 \sinh \eta z' \right) \]  

in which, 

\[ F_c = \frac{f_{z1} + f_z \eta^2 + f_z \eta^4}{\eta^2 + \delta^2 \eta^2 - \lambda^2} f_z(s) \sin(\varphi_1 + \varphi_2). \]

The relationship of deflections and internal forces at the tip of the receiver pile can be written in the following matrix form

\[
\begin{bmatrix}
U_b^{(2)}(L) \\
\Theta_b^{(2)}(L) \\
Q_b^{(2)}(L) \\
M_b^{(2)}(L)
\end{bmatrix} = [T\!A] \begin{bmatrix}
U_b^{(2)}(0) \\
\Theta_b^{(2)}(0) \\
Q_b^{(2)}(0) \\
M_b^{(2)}(0)
\end{bmatrix} + [T\!D] \begin{bmatrix}
W_a(0) \\
N_a(0)
\end{bmatrix}
\]  

where \( [T\!D] = [td(L)][ib(0)]^{-1} - [ta(L)][ta(0)]^{-1} [td(0)][ib(0)]^{-1} \). The algebraic expression for \( [td(z')] \) is given in Appendix D. By applying the boundary conditions at the receiver pile tip

\[
\begin{bmatrix}
Q_b^{(2)}(L) \\
M_b^{(2)}(L)
\end{bmatrix} = [K_{\text{tip}}] \begin{bmatrix}
U_b^{(2)}(L) \\
\Theta_b^{(2)}(L)
\end{bmatrix}
\]  

to Eq. (29), We have

\[
[T\!1] \begin{bmatrix}
U_b^{(2)}(0) \\
\Theta_b^{(2)}(0)
\end{bmatrix} + [T\!2] \begin{bmatrix}
Q_b^{(2)}(0) \\
M_b^{(2)}(0)
\end{bmatrix} + [T\!4] \begin{bmatrix}
W_a(0) \\
N_a(0)
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

in which, \( [T\!4] = [K_{\text{tip}}] \begin{bmatrix}
TD_{1,1} & TD_{1,2} \\
TD_{2,1} & TD_{2,2}
\end{bmatrix} \left[ TD_{3,1} & TD_{3,2} \right] \). Substituting Eq. (16) and the boundary conditions at the head of the receiver pile, \( Q_b^{(2)}(0) = 0 \) and \( \Theta_b^{(2)}(0) = 0 \), into Eq. (30), the axial-normal interaction factor \( \lambda_{an} \) can be obtained as:

\[
\lambda_{an} = \frac{U_b^{(2)}(0)}{W_a(0)} = \frac{(T4_{2,1} + T4_{2,2} \Re) T2_{1,1} - (T4_{1,1} + T4_{1,2} \Re) T2_{2,1}}{T1_{1,1} T2_{2,2} - T1_{1,2} T2_{1,2}}
\]  

Comparison with Numerical models

Verification of T-P model for a single inclined pile

To verify the effectiveness of the present T-P model, the static stiffness and the horizontal impedance of the inclined pile are compared with those obtained by numerical models, as shown in
Figs. 3 and 4. The results given by Giannakou et al. (2006) are obtained by developing a simplified equation based on the 3-D FEM model. The results given by Padron et al. (2010) are obtained by using the FEM-BEM model. It can be seen from Figs. 3 and 4 that the analytical results obtained by the present T-P model are in agreement with those obtained by BEM-FEM model. The maximum difference is less than 15%. In addition, with the increasing of the inclined angle the static stiffness obtained from the estimating equations (Giannakou et al., 2006) gradually deviates from those obtained by the present model and the FEM-BEM model (Padron et al., 2010) as seen in Fig. 3.

**Verification of T-P model for adjacent inclined piles**

The normal-normal interaction factor and axial-normal interaction factor obtained from the T-P model are compared with those from Ghasemzadeh and Alibeikloo (2011) for different inclined angles ($\phi = 10^\circ, 20^\circ, 30^\circ$) in Figs. 5-6. Moreover, the comparisons for different distance-diameter ratios ($s/d = 3, 5, 10$) are given in Figs. 7-8. It can be seen that normal-normal interaction factors from the present T-P model is near to the results of W-E model from Ghasemzadeh and Alibeikloo (2011). A minor difference between them can be due to the shear effect of the system. However, the axial-normal interaction factors have a good agreement with those from the Ghasemzadeh and Alibeikloo’s model.

Ghazavi et al. (2013) based on the elasto-dynamic theory and Saitoh et al. (2016) based on the boundary elements-finite elements coupling method studied the horizontal interaction factor of a pair of inclined piles with $L/d = 15$, respectively. The present results obtained by the T-P model are compared with those obtained by Ghazavi et al. (2013) for $s/d = 5$, as well as with those obtained by Saitoh et al. (2016) for $s/d = 5\sqrt{2}$, as shown in Fig. 9. It can be seen from Fig. 9 that the present results agree with the Ghazavi et al.’s solutions well. However, there is an acceptable difference...
between the present solutions and the rigorous FE-BE solutions from Saitoh et al (2016) because the present model utilizes an attenuation function with approximately-assumed wave propagation given by Dobry and Gazetas (1988) to simulate the pile-to-pile interaction effect. If one wants to obtain a better accuracy of the interaction factors, the present attenuation function used for inclined piles should be further improved.

**Numerical examples and discussions**

**Shear effect of the inclined pile on the horizontal impedance**

Based on an improved plane-strain foundation model and the Euler beam theory, Mylonakis (2001) studied the stiffness at the head of a vertical large-diameter end-bearing pile for different pile-soil modulus ratios $E_p/E_s$ and different slenderness ratios $L/d$. A comparison of the results between the present model and Mylonakis’ model is shown in Table 1. The Young’s modulus in $[K_{tp}]$ is given a large enough value $E_{tp}=10E_p$ in the calculation to simulate the rigid base considered by Mylonakis (2001). As show in Table 1, the maximum relative error between the results from the two models is up to 30% for $L/d=5$ while it drops below 3.0% for $L/d=10$, due to the salient shear effect for the piles with small slenderness ratios. This example is also used to check the influence of the inclined angle on the horizontal stiffness. It can be seen from Table 1 that the horizontal stiffness at the pile head increases with the increase of the inclined angle. It is confirmed that the use of inclined piles is advantageous in supporting horizontal loadings.

Moreover, T-P model can be reduced to an Euler beam-on-Pasternak-foundation model (E-P) model when $1/J_p \rightarrow 0$ and $S_p \rightarrow 0$. Fig. 10 shows the stiffness variation with respect to slenderness ratio of the T-P model and the E-P model for $E_p/E_s=100$, 300 and 1000, respectively. It can be seen from Fig. 10 that the results from the E-P model approach to those from the T-P model as the
slenderness ratio \( L/d \) increases. It can be seen from Fig. 10 that the results from T-P model are smaller than those from E-P model, especially for low slenderness ratios because the E-P model ignores the effects of shear deformation and rotational inertia of piles.

*Shear effect of the soil on horizontal impedance*

Fig. 11 and Fig. 12 show the comparison of horizontal impedances obtained from the T-P model and the E-W model for different pile-soil elastic modulus ratios \( E_p/E_s \) and different inclined angles \( \varphi \). It can be seen from Fig. 11 that the results from T-P model are larger than those from the E-W model, especially for small pile-soil elastic modulus ratios. As a large slenderness ratio \( L/d=15 \) is considered in the calculation, this relative difference between the T-P model and the E-W model in Fig. 11 is mainly caused by the shear effect of soil. It can be seen from Fig. 12 that the dynamic stiffness of an inclined pile obtained from T-P model is also higher than that from the E-W model while the two models provide similar damping. In addition, it is indicated from Fig. 12 that the shear deformation of soil has a greater impact on piles with larger inclined angles by comparing the relative differences between two models for the three inclined angles \( \varphi=10^\circ, 20^\circ, 30^\circ \).

*Influence of the shear effects on interaction factors*

Now, we study a soil medium with average shear wave velocity \( V_s=200 m/s \) and the piles with diameter \( d=1 m \). A typical seismic excitation with dominant period \( T=0.15 s \) is considered in the following analysis. In such a case, \( a_0=0.2 \). Fig. 13 and Fig. 14 show the variations of normal-normal and axial-normal interaction factors with the slenderness ratio for different models. It can be seen from Fig. 13 that the normal-normal interaction factor obtained from the Timoshenko pile model is lower than those obtained from the Euler pile model for a small slenderness ratio. However, the difference decreases with the increase of the slenderness ratio. Fig. 14 indicates that the shear effect
of the inclined pile has almost no influence on axial-normal interaction factors. This is because that
the vibration of the source pile does not induce the shear deformation in the axial direction.
Moreover, with the increase of the slenderness ratio, the interaction factors approach to constant
values which are close to the results obtained by Ghasemzadeh and Alibeikloo (2011), as seen from
Fig. 13 and Fig. 14.

**Influence of the inclined angle on interaction factors**

The normal-normal interaction factor $\lambda_{nn}$ and axial-normal interaction factor $\lambda_{an}$, as functions of
inclined angles for a small slenderness ratios $L/d=5$ and a large slenderness ratio $L/d=25$, are
presented in Fig. 15 and Fig. 16, respectively. It is assumed that inclined angles for the source pile
and receiver pile are the same ($\varphi_1=\varphi_2$). It can be seen from Fig. 15 that $\lambda_{nn}$ decreases with the
increase of the inclined angle. In contrast, $\lambda_{an}$ increases with the increase of inclined angle as shown
in Fig. 16. Moreover, comparing the slopes of two curves in Fig. 15 (or in Fig. 16) shows that the
interaction factor of inclined pile with smaller slenderness ratio is more susceptible to the inclined
angle. As a result, choosing an optimum inclined angle in the pile group needs further considerations
when the slenderness ratio of inclined piles is small.

**Conclusion**

Closed-form expressions of dynamic impedance and interaction factors for inclined piles
embedded in homogeneous elastic half-space are developed based on the T-P model. The normal
reaction of the soil against the pile shaft is simulated by the Pasternak’s foundation modulus with
springs, dashpots and the shear layer. The reaction of the soil against the pile tip is simulated by three
springs in parallel with dashpots. The vibration of the inclined pile is analyzed by the Timoshenko
beam theory. The correctness of the derivation is verified by some comparative studies. The effects
of shear deformations on both soil and inclined piles are highlighted by comparing the results from the T-P model and the E-W model. The influence of the inclined angle on the horizontal impedance and interaction factors are presented by numerical parametric studies. The following main aspects can be emphasized:

- The relative difference of the horizontal dynamic stiffness between the T-P model and E-W model increases with the increase of the inclined angle and the decrease of the slenderness ration ($L/d$). Therefore, it is important to take the shear effect of both soil and piles into account for those piles with large inclined angle and small slenderness ratio.

- The inclined angle has a significant influence on both horizontal impedance for a single pile and interaction factors for adjacent piles. It is demonstrated that increasing the inclined angle results in increases of horizontal impedance and axial-normal interaction factor, while results in a decrease of normal-normal interaction factor. Therefore an optimum inclined angle for the inclined piles in a group needs to be studied.
Appendix A

The algebraic expression for \([ta(z')]\) is given as follows:

\[ [ta(z')] = [a_1, a_2, a_3, a_4]^T \]  \hspace{1cm} (32)

in which,

\[ \{a_1\} = \{\cosh(\alpha z') \sinh(\alpha z') \cos(\beta z') \sin(\beta z')\} \]  \hspace{1cm} (33)

\[ a_2 = \begin{bmatrix} (\Phi \alpha + \Upsilon \alpha^3) \sinh(\alpha z') \\ (\Phi \alpha + \Upsilon \alpha^3) \cosh(\alpha z') \\ (-\Phi \beta + \Upsilon \beta^3) \sin(\beta z') \\ (\Phi \beta - \Upsilon \beta^3) \cos(\beta z') \end{bmatrix} \]  \hspace{1cm} (34)

\[ \{a_3\} = \begin{bmatrix} (\Psi \alpha + J_p \alpha^3) \sinh(\alpha z') \\ (\Psi \alpha + J_p \alpha^3) \cosh(\alpha z') \\ (-\Psi \beta + J_p \beta^3) \sin(\beta z') \\ (\Psi \beta - J_p \beta^3) \cos(\beta z') \end{bmatrix} \]  \hspace{1cm} (35)

\[ a_4 = \begin{bmatrix} -(\Phi \alpha^2 + \Upsilon \alpha^4) E_p I_p \cosh(\alpha z') \\ -(\Phi \alpha^2 + \Upsilon \alpha^4) E_p I_p \sinh(\alpha z') \\ (\Phi \beta^2 - \Upsilon \beta^4) E_p I_p \cos(\beta z') \\ (\Phi \beta^2 - \Upsilon \beta^4) E_p I_p \sin(\beta z') \end{bmatrix} \]  \hspace{1cm} (36)

with \( \Phi = \frac{J_p^2 + K_p E_p I_p}{J_p (J_p - S_p)} \), \( \Psi = \frac{K_p E_p I_p + J_p S_p}{J_p - W_p} \), \( \Upsilon = \frac{E_p I_p (J_p + g_z)}{J_p (J_p - S_p)} \).

Appendix B

The algebraic expression for \([tb(z')]\) is given as follows:

\[ [tb(z')] = \begin{bmatrix} \cosh(\eta z') & \sinh(\eta z') \\ \eta E_p A_p \sinh(\eta z') & \eta E_p A_p \cosh(\eta z') \end{bmatrix} \]  \hspace{1cm} (37)

Appendix C

The algebraic expression for \([tc(z')]\) is given as follows:
\begin{align}
\mathbf{c}(z') &= \{c_1, c_2, c_3, c_4\}^T \\
in which, \\
\{c_1\} &= \{z'F_a \sinh(\alpha z') - z'F_b \sin(\beta z') - z'F_b \cos(\beta z')\} \\
\{c_2\} &= \begin{bmatrix}
F_a \left[ (\Phi + 3Y^2)z' \cosh(\alpha z') + \left( \Phi + 3Y^2 \right) \sinh(\alpha z') \right] \\
F_a \left[ (\Phi + 3Y^2)z' \cosh(\alpha z') + \left( \Phi + 3Y^2 \right) \sinh(\alpha z') \right] \\
F_a \left[ \beta \phi J_p \gamma \beta \gamma \zeta \cos(\beta z') + \left( \Phi - 3Y^2 \right) \sin(\beta z') \right] \\
F_a \left[ \beta \phi J_p \gamma \beta \gamma \zeta \cos(\beta z') + \left( \Phi - 3Y^2 \right) \sin(\beta z') \right]
\end{bmatrix} \\
\{c_3\} &= \begin{bmatrix}
F_a \left[ \alpha \phi J_p \gamma \alpha \gamma \zeta \cos(\alpha z') + \left( \Psi - 3J_p \gamma \beta \gamma \zeta \sin(\alpha z') \right] \\
F_a \left[ \alpha \phi J_p \gamma \alpha \gamma \zeta \cos(\alpha z') + \left( \Psi - 3J_p \gamma \beta \gamma \zeta \sin(\alpha z') \right] \\
F_a \left[ \beta \phi J_p \gamma \beta \gamma \zeta \cos(\beta z') + \left( \Psi - 3J_p \gamma \beta \gamma \zeta \sin(\beta z') \right] \\
F_a \left[ \beta \phi J_p \gamma \beta \gamma \zeta \cos(\beta z') + \left( \Psi - 3J_p \gamma \beta \gamma \zeta \sin(\beta z') \right]
\end{bmatrix} \\
\{c_4\} &= \begin{bmatrix}
F_a \left[ 2\alpha \phi J_p \gamma \alpha \gamma \zeta \cos(\alpha z') + \left( \Phi + 2^2 \gamma \zeta \sinh(\alpha z') \right] \\
F_a \left[ 2\alpha \phi J_p \gamma \alpha \gamma \zeta \cos(\alpha z') + \left( \Phi + 2^2 \gamma \zeta \sinh(\alpha z') \right] \\
F_a \left[ 2\beta \phi J_p \gamma \beta \gamma \zeta \cos(\beta z') + \left( \Phi - 2^2 \gamma \sin(\beta z') \right] \\
F_a \left[ 2\beta \phi J_p \gamma \beta \gamma \zeta \cos(\beta z') + \left( \Phi - 2^2 \gamma \sin(\beta z') \right]
\end{bmatrix}
\end{align}

\textbf{Appendix D}

The algebraic expression for $[\mathbf{d}(z')]$ is given as follows:

\begin{align}
[\mathbf{d}(z')] &= \begin{bmatrix}
F_c \cosh(\eta z') & F_c \sinh(\eta z') \\
F_c \left( \Phi \eta + \gamma \eta \zeta \right) \sinh(\eta z') & F_c \left( \Phi \eta + \gamma \eta \zeta \right) \cosh(\eta z') \\
F_c \left( \Psi \eta + J_p \gamma \eta \zeta \right) \sinh(\eta z') & F_c \left( \Psi \eta + J_p \gamma \eta \zeta \right) \cosh(\eta z') \\
-F_c \left( \Phi \eta^2 + \gamma \eta^4 \right) E_p I_p \cosh(\eta z') & -F_c \left( \Phi \eta^2 + \gamma \eta^4 \right) E_p I_p \sinh(\eta z')
\end{bmatrix}
\end{align}
Acknowledgments

The financial supports from the National Natural Science Foundation of China (51678302) and Fundamental Research Funds for the Central Universities (2016B15014) are greatly acknowledged. This work is also supported in part by the scholarship from China Scholarship Council (CSC) under the Grant No. 201408320149.
Notation list

The following symbols are used in this paper:

- \( A_p \) = cross-section area of the pile;
- \( a_0 \) = dimensionless frequency of the excitation;
- \( C_h \) = horizontal damping ratio at the pile head;
- \( c_x, c_z \) = normal and axial damping along the pile shaft;
- \( d \) = sectional diameter of the inclined pile;
- \( E_p, E_s \) = Young’s modulus of the pile and the soil along the pile shaft, respectively;
- \( G_s, G_{s, \text{tip}} \) = Shear modulus of the soil along the pile shaft and under the pile tip, respectively;
- \( g_s \) = shear stiffness of the soil along the pile shaft;
- \( I_p \) = inertia moment of the pile;
- \( K_h \) = horizontal dynamic stiffness at the pile head;
- \( K_{\text{tip}}^x \) = axial base impedance of the pile;
- \( K_{\text{tip}}^z \) = normal base impedance of the pile;
- \( k_x, k_z \) = normal and axial stiffness along the pile shaft;
- \( L \) = length of the inclined pile-shaft;
- \( M_a, M_b^{(j)} \) = bending moment of a source pile; rotation of a receiver pile induced by the vibration of the source pile in the \( j \) direction;
- \( N_a \) = axial force of a source pile;
- \( Q_a, Q_b^{(j)} \) = shear force of a source pile; rotation of a receiver pile induced by the vibration of the source pile in the \( j \) direction;
- \( s \) = center to center distance between two piles;
- \( U_a, U_b^{(j)} \) = normal displacement of a source pile; normal displacement of a receiver pile induced by the vibration of the source pile in \( j \) direction. \( j = 1 \) for normal direction, \( j = 2 \) for axial direction;
- \( V_s \) = shear wave velocity of the soil;
- \( W_a \) = axial displacement of a source pile;
- \( \phi \) = angle between the center-line direction of two piles;
\( \omega = \) frequency of the harmonic excitation;

\( \varphi_1, \varphi_2 = \) inclined angle of the source pile and the receiver pile, respectively;

\( \lambda_{nn}, \lambda_{an} = \) dynamic interaction factors in the normal-normal and axial-normal directions;

\( \nu_p, \nu_s = \) Poisson’s ratio of the pile and the soil, respectively;

\( \rho_p, \rho_s = \) mass density of the pile and the soil, respectively;

\( \beta_s = \) radiation damping of the soil;

\( \mathcal{R}_x = \) axial impedance matrix at the pile head;

\( \mathcal{R}_h = \) horizontal impedance at the pile head;

\[ \mathcal{R}_x \] = normal impedance matrix at the pile head;

\( \Theta_a, \Theta_b^{(j)} = \) rotation angle of a source pile; rotation of a receiver pile induced by the vibration of the source pile in the \( j \) direction;
References:


Fig. 1 T-P model for inclined pile-soil dynamic interaction
Fig. 2 Dynamic interaction between two adjacent inclined piles subjected to a horizontal harmonic force (a: Sectional view of adjacent inclined piles; b: Plane view of adjacent inclined piles)
Fig. 3 Comparison of T-P model and numerical models for static stiffness of inclined pile

\( (L/d=15, \rho_s/\rho_p=0.7, v_s=0.4, \beta_s=0.05) \)
Fig. 4 Comparison of T-P model and BEM-FEM model for horizontal impedance of inclined pile

\( \frac{E_p}{E_s} = 1000, \frac{L}{d} = 15, \frac{\rho_s}{\rho_p} = 0.7, v_s = 0.4, \beta_s = 0.05 \)
Fig. 5 Comparison of the normal-normal interaction factors for different inclined angles
\( \frac{E_p}{E_s} = 1000, \frac{L}{d} = 25, \frac{\rho_s}{\rho_p} = 0.7, v_s = 0.4, \beta_s = 0.05, \frac{s}{d} = 3 \)
Fig. 6 Comparison of the axial-normal interaction factors for different inclined angles
($E_p/E_s=1000, L/d=25, \rho_s/\rho_p=0.7, \nu_s=0.4, \beta_s=0.05, s/d=3$)
Fig. 7 Comparison of normal-normal interaction factors for different distance-diameter ratios

\( \frac{E_p}{E_s} = 1000, \frac{L}{d} = 25, \frac{\rho_s}{\rho_p} = 0.7, v_s = 0.4, \beta_s = 0.05, \phi_1 = \phi_2 = 30^\circ \)
Fig. 8 Comparison of axial-normal interaction factors for different distance-diameter ratios

\((E_p/E_s=1000, \text{L/d}=25, \rho_s/\rho_p=0.7, \nu_s=0.4, \beta_s=0.05, \varphi_1=\varphi_2=30^\circ)\)
Fig. 9 Comparison of normal-normal interaction factors for different models

($E_p/E_s = 1000$, $L/d = 15$, $\rho_s/\rho_p = 0.7$, $v_s = 0.4$, $\beta_s = 0.05$, $\phi_1 = \phi_2 = 20^\circ$)
Fig. 10 Variation of stiffness with slenderness ratio for different models

($\rho_s/\rho_p = 0.7, v_i = 0.4, \beta = 0.05, \varphi = 20^\circ$)
Fig. 11 Shear effect of soil on horizontal impedance of an inclined pile for different pile-soil elastic modulus ratios $E_p/E_s$ 
($L/d=15$, $\rho_s/\rho_p=0.7$, $v_s=0.4$, $\beta_s=0.05$, $\varphi=15^\circ$)
The dimensionless frequency $a_0$

**E-W model**

$\varphi=10^\circ$

$\varphi=20^\circ$

$\varphi=30^\circ$

**T-P model**

---

**Fig. 12** Shear effect of soil on the impedance of an inclined pile for different inclined angles

$(E_p/E_s=100, L/d=15, \rho_p/\rho_s=0.7, v_s=0.4, \beta_s=0.05)$
Fig. 13 Variation of normal-normal interaction factor with the slenderness ratio for different models

\((\frac{E_p}{E_s}=1000, \frac{\rho_s}{\rho_p}=0.7, \nu_s=0.4, \beta_s=0.05, s/d=3, a_0=0.2)\)
Fig. 14 Variation of axial-normal interaction factor with the slenderness ratio for different models

\(E_p/E_s=1000, \rho_s/\rho_p=0.7, v_s=0.4, \beta_s=0.05, s/d=3, a_0=0.2\)
Fig. 15 Variation of normal-normal interaction factor with inclined angle for different slenderness ratios

$\left( \frac{E_p}{E_s} = 1000, \frac{\rho_s}{\rho_p} = 0.7, v_s = 0.4, \beta_s = 0.05, s/d = 3, a_0 = 0.2 \right)$
Fig. 16 Variation of axial-normal interaction factor with inclined angle for different slenderness ratios

\(E_p/E_s=1000, \rho_s/\rho_p=0.7, v_s=0.4, \beta_s=0.05, s/d=3, a_0=0.2\)
**Table 1.** Effects of shear deformation and inclined angles on the normalized stiffness $K_h/(E_s d)$ at the head

<table>
<thead>
<tr>
<th>Inclined angle</th>
<th>$E_p/E_s=100$</th>
<th>$E_p/E_s=300$</th>
<th>$E_p/E_s=1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L/d=5$</td>
<td>$L/d=7$</td>
<td>$L/d=10$</td>
</tr>
<tr>
<td>$0^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mylonakis</td>
<td>2.85</td>
<td>2.59</td>
<td>2.35</td>
</tr>
<tr>
<td>Error</td>
<td>(8.97%)</td>
<td>(3.00%)</td>
<td>(2.28%)</td>
</tr>
<tr>
<td>$10^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$20^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$30^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: the calculation parameters: $v_s=0.40; v_p=0.30; \rho_s/\rho_p=0.8; \alpha_0=0.001.$