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TORSION IN FREE CENTRE-BY-NILPOTENT-BY-ABELIAN LIE RINGS OF RANK 2

RALPH STÖHR

ABSTRACT. For $c \geq 2$, the free centre-by-(nilpotent-of-class- $c-1$)-by abelian Lie ring on a set X is the quotient $L/[(L')^c, L]$ where L is the free Lie ring on X , and $(L')^c$ denotes the c th term of the lower central series of the derived ideal $L' = L^2$ of L . In this paper we give a complete description of the torsion subgroup of its additive group in the case where $|X| = 2$ and c is a prime number.

1. INTRODUCTION

For an integer $c \geq 2$, the free centre-by-(nilpotent-of-class $c - 1$)-by-abelian Lie ring on a set X is the quotient

$$(1.1) \quad L/[(L')^c, L]$$

where L is an (absolutely) free Lie ring on X and $(L')^c$ is the c th term of the lower central series of the derived ideal $L' = L^2$ of L . In this note we determine the torsion subgroup of the additive group of (1.1) in the case where L has rank 2, that is X is a set of two elements, and c is a prime number. It is easily observed that for any $c \geq 2$ and any X with $|X| \geq 2$ the torsion subgroup of (1.1) is contained in the central ideal $(L')^c/[(L')^c, L]$. If $c = p$, where p is a prime, then the ideal $(L')^p/[(L')^p, L]$ is a direct sum of a free abelian group and an infinite elementary abelian p -group. We exhibit an explicit $\mathbb{Z}_p$ -basis of the torsion part if $|X| = 2$. This is the main result of this paper, see Theorem 5.1 in Section 5. In the earlier paper [1] a complete description of the torsion subgroup of the additive group of the more reduced quotient

$$(1.2) \quad L/([(L')^c, L] + L''')$$

was obtained, again in the case of rank 2, but for arbitrary $c \geq 2$. Note that (1.2) coincides with (1.1) for $c = 2$ and $c = 3$, but not for $c \geq 4$. We make essential use of the results from [1].

The interest in torsion in relatively free Lie rings of the form (1.1) as well as their group-theoretic counterparts, the relatively free groups $F/[\gamma_c(F'), F]$, where F is a free group and $\gamma_c(F')$ is the c th term of the lower central series of the derived subgroup F' , has a long history. If $c = 2$, these turn into the free centre-by-metabelian Lie rings and the free centre-by-metabelian groups, respectively, and it was the latter in which Kanta Gupta [8] first discovered torsion elements, a major

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surprise at the time. Later torsion was detected in $F/[\gamma_c(F'), F]$ if c is a prime [15] and if $c = 4$ [16]. Quite surprising, though, it turned out that $F/[\gamma_c(F'), F]$ is torsion free if c is divisible two distinct primes [10]. As to Lie rings, early work in [12] on the free centre-by-metabelian Lie rings $L/[L'', L]$, and in particular on torsion in the additive group, turned out in need of some rectification, and this was eventually accomplished in [14] and [11]. For larger values of c , Drensky [7] proved that for any prime p the free Lie ring $L/[(L')^p, L]$ contains non-trivial multilinear elements of degree $2p + 1$ which have order p .

The paper is organized as follows. In Section 2 we introduce notation and assemble a number of preliminary results on the homogeneous components of free Lie rings. These are further examined in Section 3 where we derive our main result on Lie powers of free modules in prime degree, postponing, however, one key ingredient on Lie powers of prime degree p over fields of characteristic p to Section 4. In the final Section 5 we exploit the results of the previous sections to derive our main result.

2. NOTATION AND SOME PRELIMINARIES

We write maps on the right and use left-normed notation for Lie products. Let A be a free abelian group. By $L(A)$ we denote the free Lie ring on A . For a positive integer c , we let $L^c(A)$ denote the degree c homogeneous component of $L(A)$, that is the span of all left normed simple Lie products $[a_1, a_2, \dots, a_c]$ with $a_i \in A$. In particular, $L^1(A) = A$. The universal envelope of $L(A)$ can be identified with the tensor ring $T(A) = \bigoplus_{c \geq 0} T^c(A)$ where $T^0(A) = \mathbb{Z}$ and, for $c > 0$, $T^c(A) = \underbrace{A \otimes \dots \otimes A}_c$, the c th tensor power of A . By A^c we denote the c th symmetric power of A . The free metabelian Lie ring on A is the quotient of $L(A)$ by its second derived ideal: $M(A) = L(A)/L(A)''$. This too is a graded Lie ring and we let $M^c(A)$ denote its c th homogeneous component, that is $M^c(A) = L^c(A)/(L^c(A) \cap L(A)'')$. We call $L^c(A)$ and $M^c(A)$ the c th free Lie power and the c th free metabelian Lie power of A , respectively. It is well-known that if $\mathcal{A}$ is an ordered $\mathbb{Z}$ -basis of A , then the left normed simple Lie products $[b_1, b_2, \dots, b_c]$ with $b_i \in \mathcal{A}$ and $b_1 > b_2 \leq \dots \leq b_c$ form a $\mathbb{Z}$ -basis of $M^c(A)$ (see [2, Section 4.2.2]).

The canonical embedding of $L(A)$ into its universal envelope $T(A)$ induces in each degree c an embedding $\nu_c : L^c(A) \rightarrow T^c(A)$. By a well-known theorem of Wever (see [13, Chapter 5, Theorem 5.16]), the composite of this embedding with the natural projection $\rho_c : T^c(A) \rightarrow L^c(A)$ defined by $a_1 \otimes \dots \otimes a_c \mapsto [a_1, \dots, a_c]$ amounts to multiplication by c on $L^c(A)$:

$$(2.1) \quad L^c(A) \xrightarrow{\nu_c} T^c(A) \xrightarrow{\rho_c} L^c(A), \quad \nu_c \rho_c = c.$$

The definition of $M^c(A)$ gives rise to a short exact sequence

$$(2.2) \quad 0 \rightarrow B^c(A) \rightarrow L^c(A) \xrightarrow{\eta_c} M^c(A) \mapsto 0$$

where $B^c(A) = L^c(A) \cap L(A)''$ and η_c is the natural projection map. Moreover, for $c \geq 2$ the metabelian Lie power $M^c(A)$ fits into a short exact sequence

$$(2.3) \quad 0 \rightarrow M^c(A) \xrightarrow{\mu_c} A \otimes A^{c-1} \xrightarrow{\kappa_c} A^c \rightarrow 0$$

where the maps μ_c and κ_c are given by

$$[a_1, a_2, \dots, a_c] \mapsto a_1 \otimes (a_2 \circ \dots \circ a_c) - a_2 \otimes (a_1 \circ \dots \circ a_c)$$

and $a_1 \otimes (a_2 \circ \dots \circ a_c) \mapsto a_1 \circ a_2 \circ \dots \circ a_c$, respectively (see [9, Corollary 3.2]). Moreover, there is a map $\lambda_c : A \otimes A^{c-1} \rightarrow M^c(A)$, given by

$$a_1 \otimes (a_2 \circ \dots \circ a_c) \mapsto [a_1, a_2, a_3, \dots, a_c] + [a_1, a_3, a_2, \dots, a_c] + \dots + [a_1, a_c, a_2, \dots, a_{c-1}],$$

such that the composite of μ_c and λ_c amounts to multiplication by c on $M^c(A)$:

$$(2.4) \quad M^c(A) \xrightarrow{\mu_c} A \otimes A^{c-1} \xrightarrow{\lambda_c} M^c(A), \quad \mu_c \lambda_c = c$$

see [1, Section 3]. Finally, for $c \geq 2$ there is a map $\theta_c : M^c(A) \rightarrow L^c(A)$ given by

$$(2.5) \quad [a_1, \dots, a_c] \mapsto \frac{1}{c} \left(\sum_{\sigma} [a_1, a_{\pi(2)}, \dots, a_{\pi(c)}] - \sum_{\tau} [a_2, a_{\tau(1)}, \dots, a_{\tau(c)}] \right),$$

where the sums run over all permutations σ of $\{2, 3, \dots, n\}$ and all permutations τ of $\{1, 3, \dots, n\}$, respectively. Although we work over $\mathbb{Z}$, the factor $1/c$ in (2.5) makes sense as the expression on the right hand side can be written as a Lie polynomial with integer coefficients (see [4, Section 2]). This Lie polynomial has been calculated in [5, Proposition 7.3], but since it is rather involved, we prefer to use the compact form (2.5) in what follows. The composite of θ_c and the natural projection η_c as in (2.2) amounts to multiplication by $(c-2)!$ on $M^c(A)$:

$$(2.6) \quad M^c(A) \xrightarrow{\theta_c} L^c(A) \xrightarrow{\eta_c} M^c(A), \quad \theta_c \eta_c = (c-2)!$$

(see [4, Section 2]).

Now suppose that A carries the structure of a module for the polynomial ring $U = \mathbb{Z}[X]$ where X is a finite set of variables. Then all the objects introduced in this section such as Lie powers, symmetric powers etc. will be regarded as U -modules under the derivation action. For example, for $x \in X$, $a_i \in A$,

$$[a_1, a_2, \dots, a_c]x = \sum_{i=1}^c [a_1, \dots, a_i x, \dots, a_c],$$

Note that all the maps introduced in this section are compatible with the derivation action, that is, all these maps are, in fact, $\mathbb{Z}[X]$ -module homomorphisms. This will be used in what follows without further reference being given. The ring of integers $\mathbb{Z}$ will be regarded as a trivial U -module.

3. LIE POWERS OF FREE MODULES

In this section we retain the notation introduced in Section 2, but now we assume throughout that A is a *free* U -module for the polynomial ring $U = \mathbb{Z}(X)$. All homology groups in this section will be over the ground ring U . For brevity, if W is a U -module, the homology groups $H_k(U, W) = \text{Tor}_k^U(W, \mathbb{Z})$, $k \geq 0$, will be written as $H_k(W)$.

It is well known (see, for example, [14, Lemma 5.2]) that if A is a free U -module then both $T^c(A)$ and $A \otimes A^{c-1}$ with $c \geq 2$ are also free U -modules under the derivation action.

Lemma 3.1. *Let A be a free U -module, $c \geq 2$. Then*

- (i) *the tensor products $L^c(A) \otimes_U \mathbb{Z}$ and $M^c(A) \otimes_U \mathbb{Z}$ are direct sums of a free abelian group and a torsion group of exponent dividing c ,*
- (ii) *for $k \geq 1$ the homology groups $H_k(L^c(A))$ and $H_k(M^c(A))$ are torsion groups of exponent dividing c .*

Proof. By applying the homology functor to the maps in (2.1) we get that $H_k(\nu_c \rho_c)$ is multiplication by c on $H_k(L^c(A))$ for all $k \geq 0$. Since A is a free U -module, $H_0(T^c(A))$ is free abelian and $H_k(T^c(A)) = 0$ for $k \geq 1$. Then, for any $u \in H_k(L^c(A))$ with $k > 0$,

$$cu = uH_k(\nu_c \rho_c) = (uH_k(\nu_c))H_k(\rho_c) = 0H_k(\rho_c) = 0.$$

The same holds if $k = 0$ and $u \in \text{Ker}(H_0(\nu_c))$. Hence multiplication by c annihilates both the homology groups $H_k(L^c(A))$ for $k \geq 1$ and the kernel of the homomorphism $H_0(\nu_c)$. The image of this homomorphism is contained in the free abelian group $H_0(T^c(A))$, and hence itself free abelian. The results (i) and (ii) for $L^c(A)$ follow. The proof of (i) and (ii) for $M^c(A)$ are obtained by a similar argument using the maps in (2.4) instead of those in (2.1). $\square$

Now we consider Lie powers of prime degree. Let p be a prime. By applying the homology functor to the short exact sequence (2.2) we obtain the exact sequence

$$(3.1) \quad \cdots \rightarrow H_1(M^p(A)) \rightarrow B^p(A) \otimes_U \mathbb{Z} \rightarrow L^p(A) \otimes_U \mathbb{Z} \xrightarrow{\eta_p \otimes 1} M^p(A) \otimes_U \mathbb{Z} \rightarrow 0.$$

By Lemma 3.1., $L^p(A) \otimes_U \mathbb{Z}$ is a direct sum of a free abelian group and an elementary abelian p -group, and $H_1(M^p(A))$ is an elementary abelian p -group. Note that this does not exclude the possibility that these torsion subgroups are trivial. Now the exact sequence (3.1) yields that $B^p(A) \otimes_U \mathbb{Z}$ is a direct sum of a free abelian group and a (possibly trivial) p -group of exponent dividing p^2 . In fact, we will show that this group has actually no torsion, in other words, we will prove the following result.

Lemma 3.2. *Let A be a free U -module and p a prime. Then the tensor product $B^p(A) \otimes_U \mathbb{Z}$ is a free abelian group.*

Proof. Since we already know that $B^p(A) \otimes_U \mathbb{Z}$ is a direct sum of a free abelian group and a p -group of finite exponent, it is sufficient to show that no non-zero element in $B^p(A) \otimes_U \mathbb{Z}$ is annihilated by p . We use reduction modulo p , that is the short exact sequence

$$0 \rightarrow B^p(A) \xrightarrow{p} B^p(A) \rightarrow B^p(A) \otimes \mathbb{Z}_p \rightarrow 0$$

which, in its turn, gives rise to the exact sequence

$$(3.2) \quad \cdots \rightarrow H_1(B^p(A) \otimes \mathbb{Z}_p) \rightarrow B^p(A) \otimes_U \mathbb{Z} \xrightarrow{p} B^p(A) \otimes_U \mathbb{Z} \rightarrow B^p(A) \otimes_U \mathbb{Z}_p \rightarrow 0.$$

The Lemma will be proved once we show that the homology group on the left is zero, and this will certainly follow if we can verify that $B^p(A) \otimes \mathbb{Z}_p$, regarded as a

module for the polynomial ring $\mathbb{Z}_p[X]$, is projective. The proof of this fact will be given in the next section (see Corollary 4.2). This will then complete the proof of Lemma 3.2. $\square$

Now we have all the ingredients in place to prove the main result of this section. Recall the homomorphism $\theta_c : M^c(A) \rightarrow L^c(A)$ defined by (2.5).

Proposition 3.3. *Let A be a free U -module and p a prime. Then the torsion subgroups of $L^p(A) \otimes_U \mathbb{Z}$ and $M^p(A) \otimes_U \mathbb{Z}$ are isomorphic, and the homomorphism $\theta_p \otimes 1$ maps the latter isomorphically onto the former.*

Proof. By Lemma 3.1(i) both $L^p(A) \otimes_U \mathbb{Z}$ and $M^p(A) \otimes_U \mathbb{Z}$ are direct sums of a free abelian group and an elementary abelian p -group. Consider the maps in (2.6). By trivializing the U -action we obtain homomorphisms

$$M^p(A) \otimes_U \mathbb{Z} \xrightarrow{\theta_p \otimes 1} L^p(A) \otimes_U \mathbb{Z} \xrightarrow{\eta_p \otimes 1} M^p(A) \otimes_U \mathbb{Z}, \quad \theta_k \eta_k \otimes 1 = (p-2)!$$

So the restriction of the composite $\theta_k \eta_k \otimes 1$ to the torsion subgroup of the tensor product $M^p(A) \otimes_U \mathbb{Z}$, an elementary abelian p -group, is multiplication by $(p-2)!$, that is, it is an isomorphism. It follows that the homomorphism $\theta_p \otimes 1$ maps the torsion subgroup of $M^p(A) \otimes_U \mathbb{Z}$ isomorphically into the torsion subgroup of $L^p(A) \otimes_U \mathbb{Z}$. To prove the proposition, we need to verify that this map is also surjective, that is, the homomorphism $\theta_k \otimes 1$ maps the torsion subgroup of $M^p(A) \otimes_U \mathbb{Z}$ isomorphically onto the torsion subgroup of $L^p(A) \otimes_U \mathbb{Z}$. But this is true since otherwise the restriction of $\eta_p \otimes 1$ to the torsion subgroup of $L^p(A) \otimes_U \mathbb{Z}$ would have a non-trivial kernel. This, however, is not the case, as follows from the exactness of (3.1). Since $B^p(A) \otimes_U \mathbb{Z}$ is free abelian by Lemma 3.2, and $H_1(M^p(A))$ is torsion by Lemma 3.1(ii), there cannot be any torsion elements in the kernel of $\eta_p \otimes 1$. This proves the proposition. $\square$

In the next section we fill in the gap left in the proof of Lemma 3.2.

4. THE DEGREE p LIE POWER IN CHARACTERISTIC p

In this section V denotes a vector space over a field K of prime characteristic p . Moreover, we will assume that V is a module for the polynomial ring $K[X]$ where X is a finite set of indeterminates. Otherwise we will use all the notation introduced in Section 2, in particular, $L^p(V)$, $M^p(V)$ and $T^p(V)$ are the p th Lie, metabelian Lie, and tensor powers of V , respectively, $B^p(V) = L^p(V) \cap L(V)''$ is the kernel of the natural projection $L^p(V) \rightarrow M^p(V)$, and all of these will be regarded as $K[X]$ -modules under the derivation action. Recall that $L^p(V)$ may be regarded as a submodule of the tensor power $T^p(V)$, and hence $B^p(V)$ is also a submodule of $T^p(V)$.

Lemma 4.1. *The submodule $B^p(V)$ is a direct summand of the $K[X]$ -module $T^p(V)$.*

This result is essentially proved as Theorem 3.1 in [6], except that there the module V is assumed to be finite-dimensional. In what follows we reproduce the

proof from [6] with some minor amendments necessary to accommodate infinite dimensional modules.

Proof. For each $r \geq 1$ choose a basis $\mathcal{B}^{(r)}$ of $L^r(V)$ and let $\mathcal{B} = \bigcup_r \mathcal{B}^{(r)}$. Thus $\mathcal{B}$ is a basis of $L(V)$. For $b \in \mathcal{B}$, let $\deg(b)$ denote the degree of b , that is, $\deg(b) = r$ for $b \in \mathcal{B}^{(r)}$. Order $\mathcal{B}$ in any way subject to $b < b'$ whenever $\deg(b) < \deg(b')$. By the Poincaré–Birkhoff–Witt Theorem, $T^p(V)$ has a basis $\mathcal{C}$ consisting of all products of the form $b_1 \otimes b_2 \otimes \cdots \otimes b_k$ with $b_1, \dots, b_k \in \mathcal{B}$, $b_1 \leq b_2 \leq \cdots \leq b_k$ and $\deg(b_1) + \cdots + \deg(b_k) = p$. More specifically, any basis element $c \in \mathcal{C}$ has the form

$$(4.1) \quad c = b_1^{(1)} \otimes \cdots \otimes b_{k_1}^{(1)} \otimes b_1^{(2)} \otimes \cdots \otimes b_{k_2}^{(2)} \otimes \cdots \otimes b_1^{(p)} \otimes \cdots \otimes b_{k_p}^{(p)},$$

where $k_1, \dots, k_p$ are non-negative integers such that $k_1 + 2k_2 + \cdots + pk_p = p$ and where, for $i = 1, \dots, p$, we have $b_1^{(i)}, \dots, b_{k_i}^{(i)} \in \mathcal{B}^{(i)}$ and $b_1^{(i)} \leq \cdots \leq b_{k_i}^{(i)}$. We call the p -tuple $(k_1, \dots, k_p)$ the *type* of c and denote it by $\text{type}(c)$. Let Ω denote the set of all such types. We order Ω (lexicographically) by $(k_1, \dots, k_p) > (k'_1, \dots, k'_p)$ if for some $j \in \{1, \dots, p\}$ we have $k_i = k'_i$ for all $i < j$ but $k_j > k'_j$. Using this ordering, write $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ where $\omega_1 > \omega_2 > \cdots > \omega_m$. Thus $\omega_1 = (p, 0, \dots, 0)$ and $\omega_m = (0, \dots, 0, 1)$.

For $i = 1, \dots, m$, define $\mathcal{C}_i = \{c \in \mathcal{C} : \text{type}(c) = \omega_i\}$ and let X_i denote the subspace of $T^p(V)$ spanned by $\mathcal{C}_i \cup \mathcal{C}_{i+1} \cup \cdots \cup \mathcal{C}_m$. Also, write $X_{m+1} = 0$. Thus

$$T^p(V) = X_1 > X_2 > \cdots > X_m > X_{m+1} = 0.$$

Note that X_i/X_{i+1} has basis $\mathcal{C}_i$ modulo X_{i+1} . Furthermore, $\mathcal{C}_1$ consists of all products $b_1^{(1)} \otimes b_2^{(1)} \otimes \cdots \otimes b_p^{(1)}$ with $b_1^{(1)}, \dots, b_p^{(1)} \in \mathcal{B}^{(1)}$ and $b_1^{(1)} \leq \cdots \leq b_p^{(1)}$. Also, $\mathcal{C}_m = \mathcal{B}^{(p)}$ and $X_m = L^p(V)$.

Let $i \in \{1, \dots, m\}$ where $\omega_i = (k_1, \dots, k_p)$. For $c \in \mathcal{C}_i$ written as in (4.1) it is well known and easy to verify that the value of c modulo X_{i+1} is unchanged by any permutation of the factors $b_1^{(1)}, \dots, b_{k_p}^{(p)}$. In particular, for all $\pi_1 \in \text{Sym}(k_1), \dots, \pi_p \in \text{Sym}(k_p)$, we have

$$(4.2) \quad b_{\pi_1(1)}^{(1)} \otimes \cdots \otimes b_{\pi_1(k_1)}^{(1)} \otimes \cdots \otimes b_{\pi_p(1)}^{(p)} \otimes \cdots \otimes b_{\pi_p(k_p)}^{(p)} + X_{i+1} = c + X_{i+1}.$$

It follows easily that X_i is a $K[X]$ -submodule of $T^p(V)$. For c written as before let $\bar{c} \in S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V))$ be defined by

$$\bar{c} = (b_1^{(1)} \circ \cdots \circ b_{k_1}^{(1)}) \otimes (b_1^{(2)} \circ \cdots \circ b_{k_2}^{(2)}) \otimes \cdots \otimes (b_1^{(p)} \circ \cdots \circ b_{k_p}^{(p)}).$$

Clearly $\{\bar{c} : c \in \mathcal{C}_i\}$ is a basis of $S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V))$. Furthermore, it follows easily from (4.2) that the linear map given by $c + X_{i+1} \mapsto \bar{c}$ is a $K[X]$ -module isomorphism from X_i/X_{i+1} to $S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V))$. Thus

$$(4.3) \quad X_i/X_{i+1} \cong S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V)).$$

Suppose that $i \in \{2, \dots, m-1\}$ where $\omega_i = (k_1, \dots, k_p)$. Thus $k_1, \dots, k_p < p$ (and, in fact, $k_p = 0$). Let $\sigma_i : S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V)) \rightarrow T^p(V)$ be the linear map defined on the basis $\{\bar{c} : c \in \mathcal{C}_i\}$ by

$$\sigma_i(\bar{c}) = \frac{1}{k_1! \cdots k_p!} \sum_{\substack{\pi_1 \in \text{Sym}(k_1), \dots, \\ \pi_p \in \text{Sym}(k_p)}} b_{\pi_1(1)}^{(1)} \otimes \cdots \otimes b_{\pi_1(k_1)}^{(1)} \otimes \cdots \otimes b_{\pi_p(1)}^{(p)} \otimes \cdots \otimes b_{\pi_p(k_p)}^{(p)},$$

where c is written as in (4.1). It is straightforward to verify that σ_i is a $K[X]$ -module homomorphism. It follows easily from (4.2) that $\sigma_i(\bar{c}) \in X_i$ and, indeed, $\sigma_i(\bar{c}) + X_{i+1} = c + X_{i+1}$ for all $c \in \mathcal{C}_i$. Thus the map σ_i is injective and we have $X_i = \text{Im}(\sigma_i) \oplus X_{i+1}$. Since $X_m = L^p(V)$ we therefore have

$$X_2 = \text{Im}(\sigma_2) \oplus \cdots \oplus \text{Im}(\sigma_{m-1}) \oplus L^p(V).$$

Let W be the submodule of X_2 given by

$$(4.4) \quad W = \text{Im}(\sigma_2) \oplus \cdots \oplus \text{Im}(\sigma_{m-1}) \oplus (L(V)'' \cap L^p(V)).$$

Consider the maps

$$\alpha : T^p(V) \rightarrow V \otimes V^{p-1}, \quad \beta : V \otimes V^{p-1} \rightarrow T^p(V)$$

defined by

$$a_1 \otimes a_2 \otimes \cdots \otimes a_p \mapsto a_1 \otimes (a_2 \circ \cdots \circ a_p)$$

and

$$a_1 \otimes (a_2 \circ \cdots \circ a_p) \mapsto \frac{1}{(p-1)!} \sum_{\pi} a_1 \otimes a_{\pi(2)} \otimes \cdots \otimes a_{\pi(p)},$$

where $a_i \in V$ and π runs over all permutations of $\{2, 3, \dots, p\}$. Then α is surjective and the composite $\beta\alpha$ is the identity map on $V \otimes V^{p-1}$. Hence we have a direct decomposition

$$T^p(V) = \text{Ker}(\alpha) \oplus \text{Im}(\beta) \cong \text{Ker}(\alpha) \oplus (V \otimes V^{p-1}).$$

We claim that $\text{Ker}(\alpha) = W$. It is easily seen that W is contained in $\text{Ker}(\alpha)$. To verify that we have actually equality, note that the elements

$$(4.5) \quad b_1^{(1)} \otimes b_2^{(1)} \otimes \cdots \otimes b_p^{(1)} \quad \text{with } b_1^{(1)}, \dots, b_p^{(1)} \in \mathcal{B}^{(1)} \text{ and } b_1^{(1)} \leq \cdots \leq b_p^{(1)}$$

form a basis of $T^p(V)$ modulo X_2 . Furthermore, the Lie products

$$(4.6) \quad [b_1^{(1)}, b_2^{(1)}, \dots, b_p^{(1)}] \quad \text{with } b_1^{(1)}, \dots, b_p^{(1)} \in \mathcal{B}^{(1)} \text{ and } b_1^{(1)} > b_2^{(1)} \leq \cdots \leq b_p^{(1)}$$

form a basis of $L^p(V)$ modulo $L^p(V) \cap L(V)''$ (see Section 2). It follows that the elements (4.5) together with the elements (4.6) form a basis of $T^p(V)$ modulo W . Moreover, the images of these elements under the map α form a basis of $V \otimes V^{p-1}$. Indeed, we have

$$(4.7) \quad (b_1^{(1)} \otimes b_2^{(1)} \otimes \cdots \otimes b_p^{(1)})\alpha = b_1^{(1)} \otimes (b_2^{(1)} \circ \cdots \circ b_p^{(1)})$$

and

$$(4.8) \quad ([b_1^{(1)}, b_2^{(1)}, \dots, b_p^{(1)}])\alpha = b_1^{(1)} \otimes (b_2^{(1)} \circ \cdots \circ b_p^{(1)}) - b_2^{(1)} \otimes (b_1^{(1)} \circ \cdots \circ b_p^{(1)}).$$

This can easily be seen from the short exact sequence (2.3) (with V instead of A). Indeed, the elements (4.7) are mapped by κ_p one-to-one onto the canonical basis of V^p , and the elements (4.8) are precisely the images of the canonical basis elements of $M^p(V)$ under the map μ_p . Consequently, we have the desired equality $\text{Ker}(\alpha) = W$, and so $T^p(V) = W \oplus \text{Im}(\beta)$. By (4.4), $L^p(V) \cap L(V)''$ is a direct summand of W . Thus $L^p(V) \cap L(V)''$ is a direct summand of $T^p(V)$ and we have Lemma 4.1. $\square$

Now the result we need to complete the proof of Proposition 3.3 follows easily.

Corollary 4.2. *If V is a free $K[X]$ -module, then $B^p(V)$ is a projective $K[X]$ -module.*

Proof. If V is a free $K[X]$ -module, then the tensor power $T^p(V)$ is also a free $K[X]$ -module, see [14, Lemma 5.2]). Since $B^p(V)$ is a direct summand of $T^p(V)$, it is projective. $\square$

5. THE MAIN RESULT

In this Section L denotes a free Lie ring of finite rank with free generating set X . Our aim is to determine the torsion subgroup of the additive group of the quotient (1.1). In view of the short exact sequence

$$0 \rightarrow (L')^c / [(L')^c, L] \rightarrow L / [(L')^c, L] \rightarrow L / (L')^c \rightarrow 0,$$

this is a free central extension of the free (nilpotent-of-class- c -1)-by-abelian Lie ring $L / (L')^c$. The additive structure of the latter is well-understood. Its underlying abelian group is free abelian [3]. Consequently, any torsion elements must be contained in the central quotient $(L')^c / [(L')^c, L]$, and it is this quotient we will focus on from now on. By the Shirshov-Witt Theorem, the derived ideal L' is itself a free Lie ring, namely, the free Lie ring on L' / L'' : $L' = L(L' / L'')$. This is a graded Lie ring and its degree c homogeneous component $L^c(L' / L'')$ is isomorphic to the lower central quotient

$$(5.1) \quad L^c(L' / L'') \cong (L')^c / (L')^{c+1}.$$

The adjoint representation induces on these lower central quotients the structure of an L / L' -module, and hence of a module for its universal envelope $U = U(L / L')$. The latter may be identified with the polynomial ring on X : $U = \mathbb{Z}[X]$. Thus (5.1) is actually a U -module isomorphism. In view of the canonical isomorphism

$$((L')^c / (L')^{c+1} \otimes_U \mathbb{Z} \cong (L')^c / [(L')^c, L],$$

trivializing the U -action on both sides of (4.1) gives an isomorphism

$$(5.2) \quad (L')^c / [(L')^c, L] \cong L^c(L' / L'') \otimes_U \mathbb{Z}.$$

We will exploit this isomorphism to investigate the additive structure of the quotient on the left hand side by examining the tensor product on the right hand side.

Suppose that L has rank 2 and, say, $X = \{x, y\}$. Then L' / L'' is a free cyclic module over the polynomial ring $U = \mathbb{Z}[x, y]$ with free generator $[y, x]$, see [1, Proof of Theorem 6.1]. If c is a prime, say $c = p$, then Proposition 3.3 applies to the tensor product on the right hand side of (5.2). Hence this tensor product is a direct sum of a free abelian group and an elementary abelian p -group. Moreover, the torsion subgroup is the image in $L^p(L' / L'') \otimes_U \mathbb{Z}$ of the torsion subgroup of $M^p(L' / L'') \otimes_U \mathbb{Z}$ under the map $\theta_p \otimes 1$. A complete description of the latter is given in [1, Corollary 6.2]. The elements

$$[[u, y], [u, x], \underbrace{u, \dots, u}_{p-2}] \otimes 1$$

where $u = [y, x, \underbrace{x, \dots, x}_s, \underbrace{y, \dots, y}_t]$ with $s, t \geq 0$ form a basis of this torsion subgroup as a $\mathbb{Z}_p$ -module. Applying $\theta_p \otimes 1$ to such a basis element gives

$$\frac{(p-2)!}{p} \sum_{i=0}^{p-1} ([u, y], \underbrace{u, \dots, u}_i, [u, x], \underbrace{u, \dots, u}_{p-2-i}) - ([u, x], \underbrace{u, \dots, u}_i, [u, y], \underbrace{u, \dots, u}_{p-2-i}) \otimes 1.$$

Since this is an element of order p we can drop the factor of $(p-2)!$ in the statement of our main result, which summarizes the discussion in this final section.

Theorem 5.1. *Let L be a free Lie ring of rank 2 with free generators x and y , and let p be a prime. Then the quotient $(L')^p / [(L')^p, L]$ is a direct sum of a free abelian group and an elementary abelian p -group. Modulo $[(L')^p, L]$ the elements*

$$\frac{1}{p} \sum_{i=0}^{p-1} ([u, y], \underbrace{u, \dots, u}_i, [u, x], \underbrace{u, \dots, u}_{p-2-i}) - ([u, x], \underbrace{u, \dots, u}_i, [u, y], \underbrace{u, \dots, u}_{p-2-i})$$

where $u = [y, x, \underbrace{x, \dots, x}_s, \underbrace{y, \dots, y}_t]$ with $s, t \geq 0$ form a basis of this torsion subgroup as a $\mathbb{Z}_p$ -module. $\square$

The legality of the factor $1/p$ in the statement of the theorem is explained in Section 2.

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