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Asymptotic expansions for bivariate normal extremes

by

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Abstract: Nadarajah [Statistics and Probability Letters 103 (2015) 127-133] derived a complete asymptotic expansion for normal extremes. Here, we extend the expansion for bivariate normal extremes.

Keywords: Bell polynomials; Bivariate normal distribution; Expansions

1 Introduction

Let $\Phi(\cdot)$ denote the cumulative distribution function (cdf) of a standard normal random variable. It is well known that $\Phi(\cdot)$ belongs to the max domain of attraction of the Gumbel extreme value distribution, i.e.,

$$\Phi^n(u_n(x)) \rightarrow \exp(-\exp(-x)) \quad (1)$$

as $n \rightarrow \infty$ and for any $-\infty < x < +\infty$, where

$$u_n(x) = a_n x + b_n \quad (2)$$

and

$$a_n = (2 \log n)^{-1/2}, \quad b_n = a_n^{-1} - \frac{a_n c_n}{2} \quad (3)$$

for $n \geq 1$, where $c_n = \log \log n + \log(4\pi)$.

Nadarajah (2015) provided a complete asymptotic expansion for (1), In particular, an expansion was provided for

$$\Phi^n(u_n(x)) - \sum_{m=0}^n \frac{(-1)^m \exp(-mx)}{m!}$$

as $n \rightarrow \infty$, where u_n , a_n and b_n are given by (2)-(3).

The aim of this note is to extend Nadarajah (2015)'s work for the bivariate normal distribution given by the joint cdf

$$F(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^y \int_{-\infty}^x \exp\left(-\frac{u^2 + v^2 - 2\rho uv}{2(1-\rho^2)}\right) dudv$$

for $-\infty < x < +\infty$, $-\infty < y < +\infty$ and $-1 < \rho < 1$. It is well known that

$$F^n(u_n(x), u_n(y)) \rightarrow \exp(-\exp(-x) - \exp(-y)) \quad (4)$$

as $n \rightarrow \infty$, where u_n , a_n and b_n are given by (2)-(3).

To the best of our knowledge, we are aware of no studies on convergence aspects of (4). However, there have been studies on convergence aspects of the extremes of other multivariate distributions and multivariate processes, see Hashorva and Ji (2014a, 2014b), Hashorva and Kortschak (2014), Hashorva (2015), Hashorva and Ji (2015), Hashorva et al. (2015), Hashorva and Li (2015), Hashorva and Ji (2016), Hashorva and Ling (2016) and Hashorva et al. (2016).

Asymptotic expansions for (4) could have both practical and theoretical appeal given the wide applicability of the bivariate normal distribution. In a practical sense, they could lead to better approximations for the limit $\exp(-\exp(-x) - \exp(-y))$. Theoretically, such expansions can be used to derive expansions for the corresponding joint probability density function (pdf), moments, cumulants, quantiles, etc.

Complete asymptotic expansions for (4) are derived in Section 2. Technical lemmas for their proof are given and proved in Section 3. The expansions involve Bell polynomials. In-built routines for Bell polynomials are available in most computer algebra packages. For example, see BellY in Mathematica.

The following notation is used throughout this note: $\mathbf{0}_k$ denotes a $k \times 1$ vector of zeros; $\mathbf{1}_k$ denotes a $k \times 1$ vector of ones; ∞_k denotes a $k \times 1$ vector of infinities; $|\mathbf{a}| = a_1 + a_2 + \dots + a_k$ for a $k \times 1$ vector $\mathbf{a} = (a_1, a_2, \dots, a_k)'$; $\sum_{\mathbf{m}=\mathbf{0}_k}^{\infty_k}$ denotes the k fold summation $\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dots \sum_{m_k=0}^{\infty}$; $(a)_k = a(a+1) \dots (a+k-1)$ denotes the ascending factorial; $B_{r,k}(\mathbf{x})$ denotes the *partial exponential Bell polynomial* defined by

$$\left(\sum_{r=1}^{\infty} x_r t^r / r! \right)^k / k! = \sum_{r=k}^{\infty} B_{r,k}(\mathbf{x}) t^r / r!$$

for $\mathbf{x} = (x_1, x_2, \dots)$. This polynomial is tabled on page 307 of Comtet (1974) for $r \leq 12$.

Our results in Sections 2 and 3 can in principle be extended to any other joint cdf. We have illustrated our results for the bivariate normal distribution because of its universality.

2 Main results

Theorem 1 derives an asymptotic expansion for $F^n(u_n(x), u_n(y))$. Theorem 2 uses Theorem 1 to prove (4). Theorems 3 and 4 derive complete asymptotic expansions for

$$\Phi^n(u_n(x)) \Phi^n(u_n(y)) - \sum_{m_1=0}^n \sum_{m_2=0}^n \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp(-m_1 x - m_2 y)$$

and

$$F^n(u_n(x), u_n(y)) - \sum_{m_1=0}^n \sum_{m_2=0}^n \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp(-m_1 x - m_2 y),$$

respectively, where u_n , a_n and b_n are given by (2)-(3). Proofs are given for Theorems 1 and 2. These proofs involve the use of Lemmas 1 to 9 in Section 3. Theorems 3 and 4 are straightforward consequences of Theorems 1 and 2, so their proofs are not given.

Theorem 1 For u_n , a_n and b_n given by (2)-(3),

$$\begin{aligned}
F^n(u_n(x), u_n(y)) = & \sum_{m_1, p_1, q_1, i, \mathbf{i}}^{(1)} \sum_{m_2, p_2, q_2, j, \mathbf{j}}^{(1)} \sum_{k=0}^n \binom{n}{k} \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{m}_1, q_3, p_3, \ell, \ell}^{(2)} \sum_{\mathbf{m}_2, q_4, p_4, r, \mathbf{r}}^{(2)} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v}^{(3)} \\
& \cdot \lambda(m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}, \mathbf{m}_1, p_3, q_3, \ell, \ell, \mathbf{m}_2, p_4, q_4, r, \mathbf{r}, \mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y) \\
& \cdot c_n^{2i_2+i_3+2j_2+j_3+2\ell_2+\ell_3+2r_2+r_3} \\
& \cdot n^{-m_1-m_2-|\mathbf{m}_1|-|\mathbf{m}_2|-2k} \\
& \cdot a_n^{2|\mathbf{i}|-m_1+i+2|\mathbf{j}|-m_2+j+\ell+2|\ell|-|\mathbf{m}_1|+r+2|\mathbf{r}|-|\mathbf{m}_2|-2k-u-v} \\
& \cdot b_n^{-m_1-2q_1-i-m_2-2q_2-j-|\mathbf{m}_1|-2p_3-\ell-|\mathbf{m}_2|-2p_4-r-2|\mathbf{p}|-2|\mathbf{q}|-s_1-s_2+u+v}
\end{aligned} \tag{5}$$

as $n \rightarrow \infty$, where $\lambda = \gamma(m_1, p_1, q_1, i, \mathbf{i}, x) \gamma(m_2, p_2, q_2, j, \mathbf{j}, y) \delta(\mathbf{m}_1, p_3, q_3, \ell, \ell, x) \delta(\mathbf{m}_2, p_4, q_4, r, \mathbf{r}, y) \in (\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y)$ and $\gamma(m_1, p_1, q_1, i, \mathbf{i}, x), \gamma(m_2, p_2, q_2, j, \mathbf{j}, y), \delta(\mathbf{m}_1, p_3, q_3, \ell, \ell, x), \delta(\mathbf{m}_2, p_4, q_4, r, \mathbf{r}, y), \in (\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y)$, $\sum_{m_1, p_1, q_1, i, \mathbf{i}}^{(1)}, \sum_{m_2, p_2, q_2, j, \mathbf{j}}^{(1)}, \sum_{\mathbf{m}_1, q_3, p_3, \ell, \ell}^{(2)}, \sum_{\mathbf{m}_2, q_4, p_4, r, \mathbf{r}}^{(2)}, \sum_{\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v}^{(3)}$ are as defined in Lemmas 1 to 9.

Proof: By Balakrishnan and Lai (2009, page 493, equation (11.30)),

$$F(x, y) = \Phi(x)\Phi(y) + \phi(x)\phi(y) \sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(x) H_{j-1}(y),$$

where $\phi(\cdot)$ denotes the pdf of a standard normal random variable and $H_\nu(\cdot)$ denotes the Hermite polynomial of order ν . So, we can write

$$\begin{aligned}
& F^n(u_n(x), u_n(y)) \\
& = \left[\Phi(u_n(x)) \Phi(u_n(y)) + \phi(u_n(x)) \phi(u_n(y)) \sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(u_n(x)) H_{j-1}(u_n(y)) \right]^n \\
& = \Phi^n(u_n(x)) \Phi^n(u_n(y)) \\
& \quad \cdot \left[1 + \frac{\phi(u_n(x)) \phi(u_n(y))}{\Phi(u_n(x)) \Phi(u_n(y))} \sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(u_n(x)) H_{j-1}(u_n(y)) \right]^n \\
& = \Phi^n(u_n(x)) \Phi^n(u_n(y)) \sum_{k=0}^n \binom{n}{k} \left[\frac{\phi(u_n(x))}{\Phi(u_n(x))} \right]^k \left[\frac{\phi(u_n(y))}{\Phi(u_n(y))} \right]^k \\
& \quad \cdot \left[\sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(u_n(x)) H_{j-1}(u_n(y)) \right]^k.
\end{aligned}$$

$\Phi^n(u_n(x))$ and $\Phi^n(u_n(y))$ can be expanded using Lemma 7. $\left[\frac{\phi(u_n(x))}{\Phi(u_n(x))} \right]^k$ and $\left[\frac{\phi(u_n(y))}{\Phi(u_n(y))} \right]^k$ can be

expanded using Lemma 8. $\left[\sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(u_n(x)) H_{j-1}(u_n(y)) \right]^k$ can be expanded using Lemma 9.

The result follows. $\square$

The term in (5) corresponding to $k = 0$ is

$$\Phi^n(u_n(x)) \Phi^n(u_n(y)) = \sum_{m_1, p_1, q_1, i, \mathbf{i}} \sum_{m_2, p_2, q_2, j, \mathbf{j}} \mu(m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}, x, y)$$

$$c_n^{2i_2+i_3+2j_2+j_3} n^{-m_1-m_2} a_n^{2|\mathbf{i}|-m_1+i+2|\mathbf{j}|-m_2+j} b_n^{-m_1-2q_1-i-m_2-2q_2-j}, \quad (6)$$

where

$$\mu(m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}, x, y) = \gamma(m_1, p_1, q_1, i, \mathbf{i}, x) \gamma(m_2, p_2, q_2, j, \mathbf{j}, y).$$

The terms in (5) corresponding to $k > 0$ all approach zero because of the presence of $n^{-m_1-m_2-|\mathbf{m}_1|-|\mathbf{m}_2|-2k}$.

Theorem 2 shows that (6) approaches $\exp(-\exp(-x) - \exp(-y))$ as $n \rightarrow \infty$.

Theorem 2 For u_n , a_n and b_n given by (2)-(3),

$$\Phi^n(u_n(x)) \Phi^n(u_n(y)) \rightarrow \exp(-\exp(-x) - \exp(-y))$$

as $n \rightarrow \infty$.

Proof: The terms in (6) corresponding to $q_1 > 0$, $i > 0$, $\mathbf{i} \neq \mathbf{0}_3$, $q_2 > 0$, $j > 0$ or $\mathbf{j} \neq \mathbf{0}_3$ each approach zero as $n \rightarrow \infty$. So, we consider the terms corresponding to $q_1 = 0$, $i = 0$, $\mathbf{i} = \mathbf{0}_3$, $q_2 = 0$, $j = 0$ and $\mathbf{j} = \mathbf{0}_3$:

$$\begin{aligned} & \sum_{m_1=0}^n \sum_{m_2=0}^n (a_n b_n)^{-m_1-m_2} n^{-m_1-m_2} \binom{n}{m_1} \binom{n}{m_2} (-1)^{m_1+m_2} \exp(-m_1 x - m_2 y) \\ &= \sum_{m_1=0}^n \sum_{m_2=0}^n (a_n b_n)^{-m_1-m_2} \prod_{i=0}^{m_1-1} \left(1 - \frac{i}{n}\right) \prod_{j=0}^{m_2-1} \left(1 - \frac{j}{n}\right) \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp(-m_1 x - m_2 y). \end{aligned}$$

Since $a_n b_n \rightarrow 1$ as $n \rightarrow \infty$, the above limits to

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp(-m_1 x - m_2 y) = \exp(-\exp(-x) - \exp(-y)).$$

The proof is complete. $\square$

It is clear from the proof of Theorem 2 that the leading term in the expansion for $\Phi^n(u_n(x)) \Phi^n(u_n(y))$ is

$$\sum_{m_1=0}^n \sum_{m_2=0}^n \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp(-m_1 x - m_2 y). \quad (7)$$

Theorem 3 gives the complete expansion of $\Phi^n(u_n(x)) \Phi^n(u_n(y))$ excluding this leading term.

Theorem 3 Let $\omega_i(p)$ be defined by

$$\prod_{i=0}^p \left(1 - \frac{i}{n}\right) = 1 + \sum_{i=1}^p \omega_i(p) n^{-i}.$$

Then, for u_n , a_n and b_n given by (2)-(3), we can write

$$\begin{aligned} & \Phi^n(u_n(x)) \Phi^n(u_n(y)) - \sum_{m_1=0}^n \sum_{m_2=0}^n \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp(-m_1 x - m_2 y) \\ &= \sum_{m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}}^{(1)*} \mu(m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}, x, y) \end{aligned}$$

$$\begin{aligned}
& \cdot c_n^{2i_2+i_3+2j_2+j_3} n^{-m_1-m_2} a_n^{2|\mathbf{i}|-m_1+i+2|\mathbf{j}|-m_2+j} b_n^{-m_1-2q_1-i-m_2-2q_2-j} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \omega_{i_1} (m_1-1) n^{-i_1} + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_2-1} \omega_{i_1} (m_2-1) n^{-i_1} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{\infty} \binom{-m_1-m_2}{i_1} \left(-\frac{1}{2}\right)^{i_1} a_n^{2i_1} c_n^{i_1} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{m_2-1} \omega_{i_1} (m_1-1) \omega_{i_2} (m_2-1) n^{-i_1-i_2} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{\infty} \omega_{i_1} (m_1-1) \binom{-m_1-m_2}{i_2} \left(-\frac{1}{2}\right)^{i_2} a_n^{2i_2} n^{-i_1} c_n^{i_2} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_2-1} \sum_{i_2=1}^{\infty} \omega_{i_1} (m_2-1) \binom{-m_1-m_2}{i_2} \left(-\frac{1}{2}\right)^{i_2} a_n^{2i_2} n^{-i_1} c_n^{i_2} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{m_2-1} \sum_{i_3=1}^{\infty} \omega_{i_1} (m_1-1) \omega_{i_2} (m_2-1) \binom{-m_1-m_2}{i_3} \left(-\frac{1}{2}\right)^{i_3} a_n^{2i_3} n^{-i_1-i_2} c_n^{i_3},
\end{aligned}$$

where $\sum^{(1)*}$ denotes $\sum^{(1)}$ excluding $q_1 = 0, i = 0, \mathbf{i} = \mathbf{0}_3, q_2 = 0, j = 0, \mathbf{j} = \mathbf{0}_3$.

Since (7) is the leading term in the expansion for $\Phi^n(u_n(x)) \Phi^n(u_n(y))$, it is also the leading term in the expansion for $F^n(u_n(x), u_n(y))$. Theorem 4 gives the complete expansion of $F^n(u_n(x), u_n(y))$ excluding this leading term.

Theorem 4 For u_n , a_n and b_n given by (2)-(3),

$$\begin{aligned}
& F^n(u_n(x), u_n(y)) - \sum_{m_1=0}^n \sum_{m_2=0}^n \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \exp(-m_1 x - m_2 y) \\
& = \left(\frac{\pi}{2}\right)^k \sum_{m_1, p_1, q_1, i, \mathbf{i}}^{(1)} \sum_{m_2, p_2, q_2, j, \mathbf{j}}^{(1)} \sum_{k=1}^n \binom{n}{k} \sum_{\mathbf{m}_1, q_3, p_3, \ell, \ell}^{(2)} \sum_{\mathbf{m}_2, q_4, p_4, r, r}^{(2)} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v}^{(3)} \\
& \quad \cdot \lambda(m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}, \mathbf{m}_1, p_3, q_3, \ell, \ell, \mathbf{m}_2, p_4, q_4, r, r, \mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y) \\
& \quad \cdot c_n^{2i_2+i_3+2j_2+j_3+2\ell_2+\ell_3+2r_2+r_3} \\
& \quad \cdot n^{-m_1-m_2-|\mathbf{m}_1|-|\mathbf{m}_2|-2k} \\
& \quad \cdot a_n^{2|\mathbf{i}|-m_1+i+2|\mathbf{j}|-m_2+j+\ell+2|\ell|-|\mathbf{m}_1|+r+2|\mathbf{r}|-|\mathbf{m}_2|-2k-u-v} \\
& \quad \cdot b_n^{-m_1-2q_1-i-m_2-2q_2-j-|\mathbf{m}_1|-2p_3-\ell-|\mathbf{m}_2|-2p_4-r-2|\mathbf{p}|-2|\mathbf{q}|-s_1-s_2+u+v} \\
& + \sum_{m_1, p_1, q_1, i, \mathbf{i}}^{(1)*} \sum_{m_2, p_2, q_2, j, \mathbf{j}}^{(1)*} \mu(m_1, p_1, q_1, i, \mathbf{i}, m_2, p_2, q_2, j, \mathbf{j}, x, y) \\
& \quad \cdot c_n^{2i_2+i_3+2j_2+j_3} n^{-m_1-m_2} a_n^{2|\mathbf{i}|-m_1+i+2|\mathbf{j}|-m_2+j} b_n^{-m_1-2q_1-i-m_2-2q_2-j} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \omega_{i_1} (m_1-1) n^{-i_1} + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_2-1} \omega_{i_1} (m_2-1) n^{-i_1} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{\infty} \binom{-m_1-m_2}{i_1} \left(-\frac{1}{2}\right)^{i_1} a_n^{2i_1} c_n^{i_1}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{m_2-1} \omega_{i_1} (m_1 - 1) \omega_{i_2} (m_2 - 1) n^{-i_1-i_2} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{\infty} \omega_{i_1} (m_1 - 1) \binom{-m_1-m_2}{i_2} \left(-\frac{1}{2}\right)^{i_2} a_n^{2i_2} n^{-i_1} c_n^{i_2} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_2-1} \sum_{i_2=1}^{\infty} \omega_{i_1} (m_2 - 1) \binom{-m_1-m_2}{i_2} \left(-\frac{1}{2}\right)^{i_2} a_n^{2i_2} n^{-i_1} c_n^{i_2} \\
& + \sum_{m_1=0}^n \sum_{m_2=0}^n \sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{m_2-1} \sum_{i_3=1}^{\infty} \omega_{i_1} (m_1 - 1) \omega_{i_2} (m_2 - 1) \binom{-m_1-m_2}{i_3} \left(-\frac{1}{2}\right)^{i_3} a_n^{2i_3} n^{-i_1-i_2} c_n^{i_3}
\end{aligned}$$

as $n \rightarrow \infty$, where $\sum^{(1)*}$ denotes $\sum^{(1)}$ excluding $q_1 = 0, i = 0, \mathbf{i} = \mathbf{0}_3, q_2 = 0, j = 0, \mathbf{j} = \mathbf{0}_3, \lambda = \gamma(m_1, p_1, q_1, i, \mathbf{i}, x) \gamma(m_2, p_2, q_2, j, \mathbf{j}, y) \delta(\mathbf{m}_1, p_3, q_3, \ell, \ell, x) \delta(\mathbf{m}_2, p_4, q_4, r, \mathbf{r}, y) \in (\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y)$ and $\gamma(m_1, p_1, q_1, i, \mathbf{i}, x), \gamma(m_2, p_2, q_2, j, \mathbf{j}, y), \delta(\mathbf{m}_1, p_3, q_3, \ell, \ell, x), \delta(\mathbf{m}_2, p_4, q_4, r, \mathbf{r}, y), \in (\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v, x, y)$,
 $\sum_{m_1, p_1, q_1, i, \mathbf{i}}^{(1)}, \sum_{m_2, p_2, q_2, j, \mathbf{j}}^{(1)}, \sum_{\mathbf{m}_1, q_3, p_3, \ell, \ell}^{(2)}, \sum_{\mathbf{m}_2, q_4, p_4, r, \mathbf{r}}^{(2)}, \sum_{\mathbf{p}, \mathbf{q}, \mathbf{s}, u, v}^{(3)}$ are as defined in Lemmas 1 to 9.

3 Technical lemmas

The following lemmas are needed to prove the main results.

Lemma 1 For u_n, a_n and b_n given by (2)-(3),

$$\exp(-pu_n^2(x)) = \sum_{\mathbf{i}=\mathbf{0}_3}^{\infty_3} b(\mathbf{i}, p, x) \frac{c_n^{2i_2+i_3}}{n^{2p} a_n^{2p-2|\mathbf{i}|}},$$

where $\mathbf{i} = (i_1, i_2, i_3), p > 0, -\infty < x < +\infty$, and

$$b(\mathbf{i}, p, x) = (2\pi)^p \exp(-2px) \frac{(-1)^{i_1+i_2} p^{|\mathbf{i}|} x^{2i_1+i_3}}{2^{3i_2+i_3-|\mathbf{i}|} i_1! i_2! i_3!}.$$

Proof: Note that

$$\begin{aligned}
& \exp(-pu_n^2(x)) \\
& = \exp(-pa_n^2 x^2 - pb_n^2 - 2pa_n b_n x) \\
& = \exp\left(-\frac{px^2}{2 \log n}\right) \exp\left(-2p \log n - \frac{pc_n^2}{8 \log n} + pc_n\right) \exp\left(-2px + px \frac{c_n}{2 \log n}\right) \\
& = (4\pi)^p \exp(-2px) n^{-2p} (\log n)^p \exp\left(-\frac{px^2}{2 \log n}\right) \exp\left(-\frac{pc_n^2}{8 \log n}\right) \exp\left(px \frac{c_n}{2 \log n}\right).
\end{aligned}$$

The result now follows by using the series expansion for exponential. $\square$

Lemma 2 For u_n, a_n and b_n given by (2)-(3),

$$u_n^m(x) = \sum_{i=0}^m \binom{m}{i} x^i a_n^i b_n^{m-i},$$

where $m > 0$ is an integer and $-\infty < x < +\infty$.

Proof: Follows by binomial theorem. $\square$

Lemma 3 For u_n , a_n and b_n given by (2)-(3),

$$u_n^{-m}(x) = \sum_{i=0}^{\infty} \binom{-m}{i} x^i a_n^i b_n^{-m-i},$$

where $m > 0$ and $-\infty < x < +\infty$.

Proof: Follows by binomial theorem. $\square$

Lemma 4 As $x \rightarrow \infty$,

$$\Phi^n(x) = \sum_{m=0}^n \sum_{p=0}^m \sum_{q=p}^{\infty} c(m, p, q) x^{-m-2q} \exp\left(-\frac{mx^2}{2}\right),$$

where

$$c(m, p, q) = \binom{n}{m} \binom{m}{p} \frac{(-1)^{m+q} 2^{q-\frac{m}{2}} p!}{\pi^{\frac{m}{2}} q!} B_{q,p}(\alpha),$$

where

$$\alpha = \left(\left(\frac{1}{2} \right)_1 1!, \left(\frac{1}{2} \right)_2 2!, \dots, \left(\frac{1}{2} \right)_{q-p+1} (q-p+1)! \right).$$

Proof: Using the fact

$$\Phi(x) = \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \right],$$

where $\operatorname{erf}(\cdot)$ denotes the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt,$$

we can write

$$\Phi^n(x) = \frac{1}{2^n} \left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \right]^n.$$

As $x \rightarrow \infty$,

$$\operatorname{erf}(x) = 1 - \frac{1}{\sqrt{\pi}x} \exp(-x^2) {}_2F_0\left(1, \frac{1}{2}; -\frac{1}{x^2}\right),$$

where ${}_2F_0(a, b; ; x)$ denotes a hypergeometric function defined by

$${}_2F_0(a, b; ; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k x^k}{k!}.$$

So, as $x \rightarrow \infty$, we can write

$$\begin{aligned}
\Phi^n(x) &= \frac{1}{2^n} \left[2 - \frac{\sqrt{2}}{\sqrt{\pi}x} \exp\left(-\frac{x^2}{2}\right) \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(-\frac{2}{x^2}\right)^k \right]^n \\
&= \left[1 - \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right) \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(-\frac{2}{x^2}\right)^k \right]^n \\
&= \sum_{m=0}^n \binom{n}{m} \frac{(-1)^m}{(2\pi)^{\frac{m}{2}}} x^{-m} \exp\left(-\frac{mx^2}{2}\right) \left[\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(-\frac{2}{x^2}\right)^k \right]^m \\
&= \sum_{m=0}^n \binom{n}{m} \frac{(-1)^m}{(2\pi)^{\frac{m}{2}}} x^{-m} \exp\left(-\frac{mx^2}{2}\right) \left[1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)_k \left(-\frac{2}{x^2}\right)^k \right]^m \\
&= \sum_{m=0}^n \binom{n}{m} \frac{(-1)^m}{(2\pi)^{\frac{m}{2}}} x^{-m} \exp\left(-\frac{mx^2}{2}\right) \sum_{p=0}^m \binom{m}{p} \left[\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)_k \left(-\frac{2}{x^2}\right)^k \right]^p \\
&= \sum_{m=0}^n \binom{n}{m} \frac{(-1)^m}{(2\pi)^{\frac{m}{2}}} x^{-m} \exp\left(-\frac{mx^2}{2}\right) \sum_{p=0}^m \binom{m}{p} p! \sum_{q=p}^{\infty} B_{q,p}(\boldsymbol{\alpha}) \frac{(-2)^q}{q!} x^{-2q},
\end{aligned}$$

where the last step follows by the definition of the Bell polynomial. The result follows by rearranging. $\square$

Lemma 5 As $x \rightarrow \infty$,

$$\left[\frac{\phi(x)}{\Phi(x)} \right]^k = \sum_{\mathbf{m}=\mathbf{0}_k}^{\infty_k} \sum_{q=0}^{|\mathbf{m}|} \sum_{p=q}^{\infty} \frac{d(\mathbf{m}, p, q)}{x^{|\mathbf{m}|+2p}} \exp\left(-\frac{(|\mathbf{m}|+k)x^2}{2}\right),$$

where $\mathbf{m} = (m_1, m_2, \dots, m_k)$ and

$$d(\mathbf{m}, p, q) = (2\pi)^{-\frac{|\mathbf{m}|+k}{2}} \binom{|\mathbf{m}|}{q} \frac{(-2)^p}{p!} B_{p,q}(\boldsymbol{\beta}),$$

where

$$\boldsymbol{\beta} = \left(\left(\frac{1}{2}\right)_1 1!, \left(\frac{1}{2}\right)_2 2!, \dots, \left(\frac{1}{2}\right)_{p-q+1} (p-q+1)! \right).$$

Proof: As in the proof of Lemma 4, we can write

$$\begin{aligned}
\frac{\phi(x)}{\Phi(x)} &= \sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^2}{2}\right) \left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \right]^{-1} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \left[1 - \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right) \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(-\frac{2}{x^2}\right)^k \right]^{-1} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} (2\pi)^{-\frac{m}{2}} x^{-m} \exp\left(-\frac{(m+1)x^2}{2}\right) \left[\sum_{p=0}^{\infty} \left(\frac{1}{2}\right)_p \left(-\frac{2}{x^2}\right)^p \right]^m.
\end{aligned}$$

Taking the k th power, we obtain

$$\begin{aligned}
\frac{\phi(x)}{\Phi(x)} &= (2\pi)^{-\frac{k}{2}} \left\{ \sum_{m=0}^{\infty} (2\pi)^{-\frac{m}{2}} x^{-m} \exp\left(-\frac{(m+1)x^2}{2}\right) \left[\sum_{p=0}^{\infty} \left(\frac{1}{2}\right)_p \left(-\frac{2}{x^2}\right)^p \right]^m \right\}^k \\
&= \sum_{\mathbf{m}=\mathbf{0}_k}^{\infty_k} (2\pi)^{-\frac{|\mathbf{m}|+k}{2}} x^{-|\mathbf{m}|} \exp\left(-\frac{(|\mathbf{m}|+k)x^2}{2}\right) \left[\sum_{p=0}^{\infty} \left(\frac{1}{2}\right)_p \left(-\frac{2}{x^2}\right)^p \right]^{|\mathbf{m}|} \\
&= \sum_{\mathbf{m}=\mathbf{0}_k}^{\infty_k} (2\pi)^{-\frac{|\mathbf{m}|+k}{2}} x^{-|\mathbf{m}|} \exp\left(-\frac{(|\mathbf{m}|+k)x^2}{2}\right) \left[1 + \sum_{p=1}^{\infty} \left(\frac{1}{2}\right)_p \left(-\frac{2}{x^2}\right)^p \right]^{|\mathbf{m}|} \\
&= \sum_{\mathbf{m}=\mathbf{0}_k}^{\infty_k} (2\pi)^{-\frac{|\mathbf{m}|+k}{2}} x^{-|\mathbf{m}|} \exp\left(-\frac{(|\mathbf{m}|+k)x^2}{2}\right) \sum_{q=0}^{|\mathbf{m}|} \binom{|\mathbf{m}|}{q} \sum_{p=q}^{\infty} B_{p,q}(\beta) \frac{(-2)^p}{x^{2p} p!},
\end{aligned}$$

where the last step follows by the definition of the Bell polynomial. The result follows by rearranging. $\square$

Lemma 6 *We have*

$$\left[\sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(x) H_{j-1}(y) \right]^k = \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{p}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{q}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{m}=\mathbf{0}_2}^{\mathbf{k}_2} f(\mathbf{p}, \mathbf{q}, \mathbf{m}) x^{2|\mathbf{p}|+m_1} y^{2|\mathbf{q}|+m_2},$$

where $\mathbf{m} = (m_1, m_2)$, $\mathbf{p} = (p_1, p_2, \dots, p_k)$, $\mathbf{q} = (q_1, q_2, \dots, q_k)$, and

$$f(\mathbf{p}, \mathbf{q}, \mathbf{m}) = \sum_{\mathbf{j}=\mathbf{1}_k}^{\infty_k} \frac{(2\rho)^{|\mathbf{j}|}}{j_1 \cdots j_k} e(\mathbf{j}, \mathbf{p}, \mathbf{q}, \mathbf{m}),$$

where $\mathbf{j} = (j_1, j_2, \dots, j_k)$, and $e(\mathbf{j}, \mathbf{p}, \mathbf{q}, \mathbf{m})$ satisfies

$$\begin{aligned}
&\prod_{i=1}^k \left[\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_1(j_i, p, q) x^{2p} y^{2q} + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_2(j_i, p, q) x^{2p} y^{2q+1} \right. \\
&\quad \left. + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_3(j_i, p, q) x^{2p+1} y^{2q} + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_4(j_i, p, q) x^{2p+1} y^{2q+1} \right] \\
&= \sum_{\mathbf{p}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{q}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{m}=\mathbf{0}_2}^{\mathbf{k}_2} e(\mathbf{j}, \mathbf{p}, \mathbf{q}, \mathbf{m}) x^{2|\mathbf{p}|+m_1} y^{2|\mathbf{q}|+m_2},
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1(j, p, q) &= \frac{\left(\frac{1-j}{2}\right)_p \left(\frac{1-j}{2}\right)_q}{\Gamma^2\left(\frac{2-j}{2}\right) \left(\frac{1}{2}\right)_p \left(\frac{1}{2}\right)_q 2^{p+q} p! q!}, \\
\alpha_2(j, p, q) &= -\frac{\left(\frac{1-j}{2}\right)_p \left(\frac{2-j}{2}\right)_q}{\Gamma\left(\frac{2-j}{2}\right) \Gamma\left(\frac{1-j}{2}\right) \left(\frac{1}{2}\right)_p \left(\frac{3}{2}\right)_q 2^{p+q+\frac{1}{2}} p! q!},
\end{aligned}$$

$$\alpha_3(j, p, q) = -\frac{\left(\frac{1-j}{2}\right)_q \left(\frac{2-j}{2}\right)_p}{\Gamma\left(\frac{2-j}{2}\right) \Gamma\left(\frac{1-j}{2}\right) \left(\frac{1}{2}\right)_q \left(\frac{3}{2}\right)_p 2^{p+q+\frac{1}{2}} p! q!},$$

$$\alpha_4(j, p, q) = \frac{\left(\frac{2-j}{2}\right)_p \left(\frac{2-j}{2}\right)_q}{\Gamma^2\left(\frac{1-j}{2}\right) \left(\frac{3}{2}\right)_p \left(\frac{3}{2}\right)_q 2^{p+q+1} p! q!}.$$

Proof: We can write

$$\left[\sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(x) H_{j-1}(y) \right]^k = \sum_{\mathbf{j}=1_k}^{\infty} \frac{\rho^{|\mathbf{j}|}}{j_1 \cdots j_k} \prod_{i=1}^k [H_{j_i-1}(x) H_{j_i-1}(y)]. \quad (8)$$

Using the fact

$$H_n(x) = 2^{\frac{n}{2}} \sqrt{\pi} \left[\frac{1}{\Gamma\left(\frac{1-n}{2}\right)} {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; \frac{x^2}{2}\right) - \frac{x}{\sqrt{2}\Gamma\left(-\frac{n}{2}\right)} {}_1F_1\left(\frac{1-n}{2}; \frac{3}{2}; \frac{x^2}{2}\right) \right],$$

where ${}_1F_1(a; b; x)$ denotes the confluent hypergeometric function defined by

$${}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(b)_k k!},$$

we can rewrite (8) as

$$\begin{aligned} & \left[\sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(x) H_{j-1}(y) \right]^k \\ &= \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{j}=1_k}^{\infty} \frac{(2\rho)^{|\mathbf{j}|}}{j_1 \cdots j_k} \\ & \quad \cdot \prod_{i=1}^k \left[\frac{1}{\Gamma\left(\frac{2-j_i}{2}\right)} {}_1F_1\left(-\frac{j_i-1}{2}; \frac{1}{2}; \frac{x^2}{2}\right) - \frac{x}{\sqrt{2}\Gamma\left(-\frac{j_i-1}{2}\right)} {}_1F_1\left(\frac{2-j_i}{2}; \frac{3}{2}; \frac{x^2}{2}\right) \right] \\ & \quad \cdot \prod_{i=1}^k \left[\frac{1}{\Gamma\left(\frac{2-j_i}{2}\right)} {}_1F_1\left(-\frac{j_i-1}{2}; \frac{1}{2}; \frac{y^2}{2}\right) - \frac{y}{\sqrt{2}\Gamma\left(-\frac{j_i-1}{2}\right)} {}_1F_1\left(\frac{2-j_i}{2}; \frac{3}{2}; \frac{y^2}{2}\right) \right] \\ &= \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{j}=1_k}^{\infty} \frac{(2\rho)^{|\mathbf{j}|}}{j_1 \cdots j_k} \prod_{i=1}^k \left\{ \frac{1}{\Gamma^2\left(\frac{2-j_i}{2}\right)} {}_1F_1\left(-\frac{j_i-1}{2}; \frac{1}{2}; \frac{x^2}{2}\right) {}_1F_1\left(-\frac{j_i-1}{2}; \frac{1}{2}; \frac{y^2}{2}\right) \right. \\ & \quad - \frac{1}{\sqrt{2}\Gamma\left(-\frac{j_i-1}{2}\right) \Gamma\left(\frac{2-j_i}{2}\right)} {}_1F_1\left(-\frac{j_i-1}{2}; \frac{1}{2}; \frac{x^2}{2}\right) {}_1F_1\left(\frac{2-j_i}{2}; \frac{3}{2}; \frac{y^2}{2}\right) \\ & \quad - \frac{x}{\sqrt{2}\Gamma\left(-\frac{j_i-1}{2}\right) \Gamma\left(\frac{2-j_i}{2}\right)} {}_1F_1\left(\frac{2-j_i}{2}; \frac{3}{2}; \frac{x^2}{2}\right) {}_1F_1\left(-\frac{j_i-1}{2}; \frac{1}{2}; \frac{y^2}{2}\right) \\ & \quad \left. + \frac{xy}{\Gamma^2\left(-\frac{j_i-1}{2}\right)} {}_1F_1\left(\frac{2-j_i}{2}; \frac{3}{2}; \frac{x^2}{2}\right) {}_1F_1\left(\frac{2-j_i}{2}; \frac{3}{2}; \frac{y^2}{2}\right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{j}=\mathbf{1}_k}^{\infty_k} \frac{(2\rho)^{|\mathbf{j}|}}{j_1 \cdots j_k} \prod_{i=1}^k \left\{ \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_1(j_i, p, q) x^{2p} y^{2q} + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_2(j_i, p, q) x^{2p} y^{2q+1} \right. \\
&\quad \left. + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_3(j_i, p, q) x^{2p+1} y^{2q} + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_4(j_i, p, q) x^{2p+1} y^{2q+1} \right\} \\
&= \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{j}=\mathbf{1}_k}^{\infty_k} \frac{(2\rho)^{|\mathbf{j}|}}{j_1 \cdots j_k} \sum_{\mathbf{p}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{q}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{m}=\mathbf{0}_2}^{\mathbf{k}_2} e(\mathbf{j}, \mathbf{p}, \mathbf{q}, \mathbf{m}) x^{2|\mathbf{p}|+m_1} y^{2|\mathbf{q}|+m_2} \\
&= \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{p}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{q}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{m}=\mathbf{0}_2}^{\mathbf{k}_2} \sum_{\mathbf{j}=\mathbf{1}_k}^{\infty_k} \frac{(2\rho)^{|\mathbf{j}|}}{j_1 \cdots j_k} e(\mathbf{j}, \mathbf{p}, \mathbf{q}, \mathbf{m}) x^{2|\mathbf{p}|+m_1} y^{2|\mathbf{q}|+m_2} \\
&= \left(\frac{\pi}{2}\right)^k \sum_{\mathbf{p}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{q}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{m}=\mathbf{0}_2}^{\mathbf{k}_2} f(\mathbf{p}, \mathbf{q}, \mathbf{m}) x^{2|\mathbf{p}|+m_1} y^{2|\mathbf{q}|+m_2}.
\end{aligned}$$

Hence, the result. $\square$

Lemma 7 For u_n , a_n and b_n given by (2)-(3),

$$\Phi^n(u_n(x)) = \sum_{m,p,q,i,\mathbf{i}}^{(1)} \gamma(m, p, q, i, \mathbf{i}, x) \frac{c_n^{2i_2+i_3}}{n^m a_n^{-2|\mathbf{i}|+m-i} b_n^{m+2q+i}}$$

as $n \rightarrow \infty$, where

$$\sum_{m,p,q,i,\mathbf{i}}^{(1)} := \sum_{m=0}^n \sum_{p=0}^m \sum_{q=p}^{\infty} \sum_{i=0}^{\infty} \sum_{\mathbf{i}=\mathbf{0}_3}^{\infty_3}$$

and

$$\gamma(m, p, q, i, \mathbf{i}, x) = x^i b\left(i, \frac{m}{2}, x\right) c(m, p, q) \binom{-m-2q}{i},$$

where $b\left(i, \frac{m}{2}, x\right)$ and $c(m, p, q)$ are as defined in Lemmas 1 and 4, respectively.

Proof: Follows by Lemmas 1, 3 and 4. $\square$

Lemma 8 For u_n , a_n and b_n given by (2)-(3),

$$\left[\frac{\phi(u_n(x))}{\Phi(u_n(x))} \right]^k = \sum_{\mathbf{m}, q, p, i, \mathbf{i}}^{(2)} \delta(\mathbf{m}, p, q, i, \mathbf{i}, x) \frac{c_n^{2i_2+i_3}}{n^{|\mathbf{m}|+k} a_n^{-i-2|\mathbf{i}|+|\mathbf{m}|+k} b_n^{|\mathbf{m}|+2p+i}}$$

as $n \rightarrow \infty$, where

$$\sum_{\mathbf{m}, q, p, i, \mathbf{i}}^{(2)} := \sum_{\mathbf{m}=\mathbf{0}_k}^{\infty_k} \sum_{q=0}^{|\mathbf{m}|} \sum_{p=q}^{\infty} \sum_{i=0}^{\infty} \sum_{\mathbf{i}=\mathbf{0}_3}^{\infty_3}$$

and

$$\delta(\mathbf{m}, p, q, i, \mathbf{i}, x) = x^i b\left(i, \frac{|\mathbf{m}|+k}{2}, x\right) d(\mathbf{m}, p, q) \binom{-|\mathbf{m}|-2p}{i},$$

where $b\left(i, \frac{|\mathbf{m}|+k}{2}, x\right)$ and $d(\mathbf{m}, p, q)$ are as defined in Lemmas 1 and 5, respectively.

Proof: Follows by Lemmas 1, 3 and 5. $\square$

Lemma 9 For u_n , a_n and b_n given by (2)-(3),

$$\left[\sum_{j=1}^{\infty} \frac{\rho^j}{j} H_{j-1}(u_n(x)) H_{j-1}(u_n(y)) \right]^k = \left(\frac{\pi}{2} \right)^k \sum_{\mathbf{p}, \mathbf{q}, \mathbf{m}, i, j}^{(3)} \epsilon(\mathbf{p}, \mathbf{q}, \mathbf{m}, i, j, x, y) a_n^{i+j} b_n^{2|\mathbf{p}|+2|\mathbf{q}|+m_1+m_2-i-j},$$

where

$$\sum_{\mathbf{p}, \mathbf{q}, \mathbf{m}, i, j}^{(3)} := \sum_{\mathbf{p}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{q}=\mathbf{0}_k}^{\infty_k} \sum_{\mathbf{m}=\mathbf{0}_2}^{\mathbf{k}_2} \sum_{i=0}^{2|\mathbf{p}|+m_1} \sum_{j=0}^{2|\mathbf{q}|+m_2}$$

and

$$\epsilon(\mathbf{p}, \mathbf{q}, \mathbf{m}, i, j, x, y) = x^i y^j f(\mathbf{p}, \mathbf{q}, \mathbf{m}) \binom{2|\mathbf{p}|+m_1}{i} \binom{2|\mathbf{q}|+m_2}{j},$$

where $f(\mathbf{p}, \mathbf{q}, \mathbf{m})$ is as defined in Lemma 6.

Proof: Follows by Lemmas 2 and 6. $\square$

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