Modelling foam displacement during improved oil recovery with the pressure-driven growth model

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Elizabeth Mas Hernández

School of Chemical Engineering and Analytical Science
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Nomenclature

\( \alpha \) Angle for direction of front displacement
\( \beta \) Direction of front movement in similarity solution
\( \gamma \) Surface tension
\( \lambda \) Mobility
\( \lambda_{rf} \) Gas relative mobility
\( n \) Normal vector in direction of front movement
\( t \) Tangent
\( \mu \) Viscosity
\( \phi \) Porosity
\( \rho \) Density
\( \tau \) Front width
\( \tau^* \) Characteristic width in model
\( \theta \) Angle through which tangent turns between front segments
\( \theta_{\text{sharp}} \) Specified finite turning angle between segments
\( \xi \) Horizontal distance from leading point at top to front
\( \zeta \) Viscous drag coefficient
\( A \) Normal cross-sectional area
\( a \) Power law index
\( A_b \) Bubble area
\( d \) Length of discretised front segment
\( g \) Gravitational acceleration
\( J \) Reservoir heterogeneity function
\( K \) Curvature
\( k \) Permeability
$k_s$ Amplitude of heterogeneity in permeability
$L$ Length of the sample
$M_{slump}$ Factor for relative change in mobility
$n_b$ Number of cell or bubble sides
$n_s$ Wave number of heterogeneity
$P$ Pressure
$P_r$ Relative increase in pressure
$P_{inj}$ Injection pressure
$Q$ Volumetric flow rate
$s$ Distance the front travels
$s_D$ Dimensionless distance the front has travelled
$S_{wf}$ Water saturation in gas bank
$t$ Time
$t_D$ Dimensionless time
$u_g$ Gas superficial velocity
$v$ Velocity
$v_x$ Velocity in $x$ direction
$v_z$ Velocity in $z$ direction
$x$ Horizontal position in reservoir
$X_D$ Dimensionless horizontal position
$Y_D$ Dimensionless upward vertical position
$z$ Vertical position in reservoir
$Z_D$ Dimensionless vertical position
Modelling foam displacement during improved oil recovery with the pressure-driven growth model
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Abstract

During oil production several processes are used for extracting oil from underground reservoirs at different stages of the production process. After exploiting so called primary recovery, that depends on the ‘natural’ depletion of the reservoir, other techniques are applied in subsequent stages. In tertiary recovery, foam can be injected and used as the displacing fluid. Foams have the capacity to provide a better percentage of recovery compared to other fluids because foams lower gas mobility, permitting a more uniform and efficient sweep of oil in the formation.

However foams are complex fluids and the study of their flow within porous systems, like oil reservoirs, is challenging. Therefore the aim of this work is to study the propagation of a foam front within reservoirs in the context of improved oil recovery. The perspective that is adopted here is to use a simple model for foam rheology known as pressure-driven growth, to describe the foam displacement process using numerical simulations and (in some cases) solving the system analytically. The pressure-driven growth model is a limiting case of the viscous froth model, where terms for surface tension and curvature are removed. Taking this particular limiting case has consequences for the numerical solution of the system as the governing equations become far less stable both physically and numerically.

An injection strategy called surfactant alternating gas is described by pressure-driven growth, where all resistance to motion in the advance of the foam is assumed to be focused on a region of wet small bubbles (the foam front) forming the interface of the water and gas phases. This front can be considered to be a one-dimensional curve. We then follow the propagation of the foam front over time, obtaining the front location and its shape.

For the case of a homogeneous reservoir with constant driving pressure, the foam front is expected to have a convex shape. However, the numerical solution of pressure-driven growth can admit the formation of concavities in the front shape. These prove to be difficult to handle numerically since they focus down into sharp concave corners. As a consequence, robust numerical schemes are needed, and such schemes can be derived informed by the analysis of asymptotic solutions for the process. In order to deal with concavities, a modification is applied to the velocity of concave corners, which is used to recover the expected convex shape for the entire foam front.

Other cases of interest arise in the scope of this study, where the development of concavities are expected due to the nature of the processes themselves, rather than being a mere numerical artifact. These are the case when there is surfactant slumping (i.e. downward migration of surfactant), the case when driving pressure is increased part-way through the process, and the case when the reservoir itself is heterogeneous. The pressure-driven growth model can be used in all these cases with the appropriate modifications to front velocities that each case requires within concavities, and spurious behaviour that would otherwise affect numerical results is thereby prevented.
Declaration

I declare that no portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Elizabeth Mas Hernández
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Chapter 1

Introduction

The objective of this thesis is to provide a deeper understanding and knowledge on the application of the pressure-driven growth model [1] for foam rheology within a porous medium to improve oil recovery resulting in increased production for the oil industry.

In the particular case, we focus on the theoretical description of the propagation of foam within a porous medium using this model.

In this Chapter we highlight the importance of this research and provide information that justifies and supports our aim in conducting it. Section 1.1 emphasizes the importance of this study. Background information about the process and methods are described briefly in section 1.2. Finally, section 1.3 gives an outline of the thesis.

1.1 Motivation

Over the years, there has been increasing global demand and consumption of energy, even when rates of growth in demand have been low lately compared to previous years [2–5]. Environmental concerns have driven the popularity and promotion of renewable energy sources. Nevertheless, the global energy market still has to rely on fossil fuels.
Despite the fact that the recent crisis in the energy sector, affecting oil prices, was triggered by the expanded capacity of oil and gas production in some markets [2], research that looks for better processes for the production in the oil and gas industry is still relevant nowadays.

With this incentive, a process intended to increase oil production is investigated from a theoretical perspective. Namely using foam as a displacing fluid during improved oil recovery.

In the extraction cycle of oil from a petroleum reservoir, the first stage that relies on the natural depletion of the well and the use of secondary methods (e.g. gas flooding) of extraction give 30 to 40 percent of the oil in situ [6, 7]. This means that after primary and secondary recovery there is still an important amount of product in the formation. The use of foams can add 20 % more of the available oil. Therefore, the use of foams is considered an efficient process [7].

Published oil and gas statistics [2, 4] also provide figures for production and prices. When considering this information and what a 20% more of product represents, the economic importance of producing such an amount of oil using improved oil recovery techniques is underlined.

On the other hand, modelling and simulation studies can provide valuable information for the design of efficient processes, in addition to laboratory-scale experiment and field trials that can be costly. However, reservoir simulators for the study of foam propagation inside underground reservoirs [8–12] to displace oil might require great computational capacity. We instead adopt a simple model for foam rheology to describe the process and proceed to develop our own program of the solution for the system.

The advantages of approaching the problem with a model that is simple is that it permits a more detailed analysis of the mathematical structure and the physics behind the model and the involved process. Despite the simplicity of the model [1, 13], the description of the system and its solution encounter other challenges. Therefore we aim to provide approaches that overcome these difficulties.
1.2 Overview

The oil and gas industry uses several fluids which are injected into natural oil reservoirs with the objective of increasing production [14, 15]. Injection of gas is a popular technique. However, the use of gas presents some challenges that lead to a low production efficiency [1, 14, 16]. Foam instead can provide better percentage of recovery in these processes [1, 17, 18].

The theoretical study of foams flowing in confined media can make use of different rheological models [19–23]. Among them the so called pressure-driven growth model has been chosen here [1, 13].

The pressure-driven growth model is suitable to describe the propagation of foam for a specific injection technique which involves the injection of surfactant followed by gas and this way produce foam in situ. Strictly speaking the oil phase is not taken into account, and only the interaction between gas and surfactant is considered. Also, the typical case to apply the model considers a homogeneous reservoir [1].

The model focuses on the foam front, a wet foam region which is the interface between the aqueous surfactant phase and the gas phase. The front is considered to have negligible thickness compared to the distance that it propagates and hence is regarded as a one-dimensional curve. For the solution of the system the front is discretised into segments, and the model is then solved numerically, which gives the shape of the foam front over time as it displaces through the reservoir. Convex front shapes are expected for a homogeneous reservoir. The characteristics of pressure-driven growth however entail some challenges for the numerical solution of the system, especially in the presence of concavities in the front shape.

Pressure-driven growth has already been used to model foam improved oil recovery processes [1, 24] but to date only partial results have been presented. A detailed mathematical analysis of the model behaviour has not yet been done, nor have there been attempts to propose better numerical schemes than can handle the challenging mathematical/numerical behaviour that the model can present.

Additional analytical treatments (e.g. long-time asymptotic behaviour) are also
pending. Obtaining analytical results is important because it provides a way to validate numerical results. Another reason why the analytical solution is relevant is because the area of the reservoir left unswept underneath the foam front can be calculated from it, providing grounds to judge the efficiency of the process.

Moreover, the long-time asymptotic solution explored [13] bypasses problems with numerical instabilities, and its analysis gives valuable information on the shape of the front in the limits of the solution domain (top and bottom region of front) that permits one developing better and more suitable numerical schemes for the solution of the system.

Once reliable solution methods are established, it is possible to use them in other cases of interest that are expected to arise naturally during the propagation of foam inside the rock, and for which the eventual development of concavities is anticipated.

The specific scenarios planned to be explored are when the effect of surfactant slumping (i.e. downward migration) is accounted for in the process, when the process requires a sudden increase in the driving pressure and when the medium is heterogeneous in terms of the permeability of the reservoir. Then we can adapt the pressure-driven growth model to make it suitable for all these different cases.

1.3 Structure

According to University guidelines, this thesis is submitted in alternative format where papers mainly comprise the results chapters. The thesis is structured as follows: Chapter 2 provides literature review about general characteristics of foams, foam in porous media, some models for flow of foams, and the use of foam in oil production processes. Chapter 3 introduces the main model (pressure-driven growth) employed throughout this thesis and gives details about its numerical and analytical solution. Chapter 4 presents the modelling and simulation of the foam front displacement in the scenario of surfactant slumping. Chapter 5 deals with the case of a sudden increase in injection pressure. The study of the foam propagation within heterogeneous formations is covered in Chapter 6. Overall conclusions are
stated in Chapter 7. Finally, an insight on future work that can be studied as complement to this project is described in Chapter 8.
Chapter 2

Literature review

Throughout the thesis we study via simulations the propagation of foam inside porous media, specifically, for its use during improved oil recovery processes where foam is injected inside underground reservoirs. It is therefore important to give some basic information related to models and techniques used to describe foam flow that sets a background for our study.

This Chapter is organised according to the following structure. First, some general information about foams is introduced in section 2.1. Section 2.2 presents foam properties. Section 2.3 extends the discussion on foam properties but focused on foams in porous media. Mathematical models used in the study of foam rheology are described in section 2.4. Section 2.5 gives information about oil recovery processes and the use of foam for this application. A theory for the study of foam mobility is presented in section 2.6. A brief description of the main numerical methods we use is presented in section 2.7. Finally, conclusions are offered in section 2.8.

2.1 Foams

A foam is a complex system of two phases, one dispersed gas phase and one continuous liquid phase [15], which attracts great interest in different areas due to its many potential applications [25–31].
Important characteristics of foams are those related to the interfacial properties. In fact, in the formation of foams, the presence of a surface active substance, a surfactant, is necessary; which, not only lowers the energy cost of creating foam, but also stabilizes the system [15,29,32,33].

Of particular interest for this project is the flow of foams. Therefore, transport and rheological properties of foams are important in this study. Furthermore, the rheological behaviour of foams is related to the interfacial properties mentioned previously [34,35].

The liquid and gas content can vary in a foam, resulting in the classification of foams. The two limits in the liquid and gas fraction are called dry foams and wet foams. A dry foam is one with little liquid content and high gas fraction. It is composed of thin films separating polyhedral bubbles, it is called ‘polyhedral foam’. Wet foams have a large percentage of liquid volume and low gas volume; the shape their bubbles adopt is spherical. They are also known as ‘spherical foam’ [15,26,29,32,36].

Foam can also be classified in terms of flow regime. This is closely related to the length scales of the bubble dimension and the size of the space confining the foam. The two regimes are macroflow and microflow [35]. In macroflow the bubble size is much smaller than the confining space. In this case, the foam is referred to as bulk. At the same time, a bulk foam can be classified in the spherical foam or polyhedral foam regime, as stated above. An example of this regime is the flow of foam through pipes. In the microflow regime however, the bubble diameter is comparable to the size of the confining area. Examples for this case are the flow of foam through capillary tubes and in porous media where the pore dimensions have similar length scales as the bubbles forming the foam. In this case it is also possible to have a dry or a wet foam [26,35]. However, in the wet regime bubbles will not be spherical owing to confinement by the pores.

According to these previous classifications, important characteristics of the foams will depend on the medium and their ratio of liquid to gas [29,33].

It is important to point out that the kind of foam of interest for this project is
one in a porous medium system with a comparatively low fraction of liquid. Before proceeding however, it is also important to describe some general characteristics of foams.

2.2 Foam properties

The surface tension is related to the contracting forces that are present on the surfaces of liquids [32]. Because foams are formed with liquid films, the surface tension of those films is an important property in the study of foams. This property is associated with the surface free energy per unit interfacial area [27], and this in turn is related to the total gas-solution interfacial area in a foam [17] and hence the bubble size [29,32].

In the formation of foams it is necessary to increase the surface area between the liquid and the gas; in order to achieve this, work has to be done [37]. Surfactants are used to decrease the surface tension of the original structure and aid in the formation of the increased interfacial area [15,17].

Once the foam has been formed and is stationary, it is more stable in a minimum energy configuration [29,38,39]. This means that foam films tend to adopt a shape that entails a minimum surface area [15,25,35]. So, the contraction of the surface is a spontaneous process that minimizes the surface area and reduces the surface free energy [32]. Foam flow can disrupt that minimum area configuration, but the rheology of foams and their rheological properties depend on the type of foam and the characteristics of the medium where they are found [26,29,35,40]. Transport properties relevant to foam rheology cover viscosity, shear modulus and yield stress [15,30,32,35].

Indeed foams present a wide range of rheological behaviour. Foam can be described as a non-Newtonian fluid [17,25,35]. Some foams can behave like pseudoplastic fluids, a Bingham plastic, as viscoplastic fluids, or can exhibit Herschel-Bulkley behaviour [17,20,28,29,32,41]. They can also present characteristics of a Newtonian fluid, or exhibit ‘slip at the wall’ [35,42].
As alluded to above, when an external strain is imposed on foams, an increase in the energy is caused. Therefore, the structure has to change in order to be stabilized again [25, 27, 32]. These changes depend on the original state of the structure and the characteristics of the imposed strain. In some cases the foam can recover its shape, in others, the foam rearranges irreversibly [25].

There are some changes that the foam structure experiences known as topological transformations where the foam films merge and then separate sharply [35]. These changes occur in order to relax the structure and recover a new stable configuration of the foam. This happens when a foam, considered to be in an initial state of equilibrium, is exposed to an applied strain, so the energy increases; then if a topological change occurs this will cause a reduction in the energy in order to reach a state of minimum energy again [15, 25, 28, 30, 43, 44].

Moreover, the stress imposed could cause the foam to begin flowing, but in general reaching a yield stress is necessary before sustained flow occurs [15, 32, 33]. A flowing foam can remain near equilibrium (with intermittent out of equilibrium events) or can be far from it with its films being deformed continuously, depending on the flow state and the characteristics of the strain [43].

Other factors related to the generation and rheology of foams are the capillary pressure and the capillary number. The capillary pressure is the pressure that exists across any curved fluid interface [17]. In the case of foams capillary pressure acts from the foam films to the so called Plateau borders, channels (with sharply curved surfaces) at which three films meet. Lower capillary pressure favours foam generation and stability, by contrast foam collapse is caused by higher capillary pressure [1].

Meanwhile, the capillary number represents the ratio of viscous forces to surface tension forces and it indicates the relation between the applied deformation rate and the foam relaxation rate [34, 43, 45]. The capillary number can be used to determine the transition between a slow and fast flowing foam state. Regarding the relation between capillary number and topological changes, as the capillary number increases, the shear required to produce a topological transformation increases. On the other hand, low values of capillary number exhibit periodic stress behaviour.
because the foam surface energy produced by topological transformations relaxes the energy that is added due to the shear [43].

Drainage, coarsening and collapse of foams are other interesting processes in the study of foams and there are numerous studies regarding those topics [15, 46–51].

Coarsening refers to the changes experienced by the bubbles forming the foam due to the pressure difference between them and specifically to the diffusion of gas through the films of the bubbles [29]. In such a way these bubbles shrink and eventually collapse [15]. Coarsening is affected by film thickness and the process of coarsening affects the radius of the bubbles forming the foam [46, 52].

Drainage is the flow of the liquid forming the bubble films caused by gravity from the films to the foam liquid channels or Plateau borders [52–55]. The process of drainage has been widely studied both with modelling and experimental work [49, 52, 53].

Foam collapse happens due to different factors. Among these factors coarsening and drainage play their part [33]. Another related aspect that causes collapse is film stability [35], because eventually the decrease in thickness and the rupture of films lead to foam collapse. Film stability in turn, is related to surface tension and the forces acting on the film [15, 51, 52].

The phenomena of coarsening, drainage and foam collapse are complicated. Their study in depth is not in the scope of the present thesis.

As mentioned previously, foam properties depend on the medium in which the foam is found. Therefore, some other properties need to be taken into account for foam in porous media, which are presented in the next section.

2.3 Foam in porous media

There is a wide range of applications involving the flow of fluids in porous systems. The rheology of fluids inside porous media is very complex and is affected by the characteristics of the medium [33]. In the case of oil reservoirs, because they
are basically permeable rock, it is important to be aware of their characteristics and properties [56]. Therefore, when a porous medium is considered, some other transport properties of the fluids will be important, such as permeability and mobility, which will be defined in detail below. In this conditions it is also important to take into account capillary pressure (refer to sections 2.2 and 2.6).

It has been already mentioned in section 2.1 that foams encountered in porous media have bubbles which size compares to the pores in the medium. Therefore, all mechanisms related to the flow of foams are affected by pore size. In addition, their special structure influences their flow within porous systems [35]. Accordingly, it is important to take into account the foam texture, the size of the bubbles forming the foam, in the study of the transport properties [17,26,33,57].

An important equation used in the description of flow through porous systems is Darcy’s law [33,58]. This is used mostly when a macroscopic system is considered (Section 2.1) [59,60], and it is used to represent the momentum balance [37,61].

The following equation describes Darcy’s law [60]:

\[ Q = -\frac{(kA/\mu)(\Delta P/L)}{2.1} \]

where \( Q \) is the volumetric flow rate, \( k \) is the permeability, \( A \) represents the normal cross-sectional area of the sample, \( L \) the length of the sample in the fluid direction, \( P \) is the ‘piezometric pressure’ and \( \mu \) the viscosity.

Darcy’s law can also be presented as:

\[ \nabla P = -\frac{Q}{kA\lambda}. \]  \hspace{1cm} (2.2)

Here, \( \nabla P \) is the pressure gradient, and \( \lambda \) is the mobility [12].

The term permeability can refer to the rock system or the fluids within it. Therefore, it is defined as the ability of the reservoir rock or the fluid inside the rock to permit flow [6]. The term relative permeability refers to the ratio between the permeability of a fluid (generally within a multiphase mixture) to the permeability
of the medium [60].

On the other hand mobility is defined as the reciprocal of an apparent viscosity [62], and it is an important indicator of the efficiency of foams in the sweep of fluids in porous media [1, 16].

Moreover, there are more medium and fluid characteristics to take into account for the flow of fluids in porous media [58, 60, 63]. However, here the scope is limited to the terms used later on, and only porosity and saturation are defined.

Porosity ($\phi$) refers to the fraction of total volume of the material that is occupied by the pores or voids. The fraction of the volume of a fluid phase in the medium or sample with respect to the total pore volume is its saturation (normally represented by $S$ with a subscript indicating the phase) [58, 60, 63].

More details of the flow of foams in porous media is given in Chapter 3. In the meantime, the next section presents information about models for foam rheology.

## 2.4 Models for foam rheology

It has been stated that the rheology of foams is complex. Therefore, different models have been proposed to study the flow of foams, which make simplifications in order to describe the foam behaviour. Some variables and considerations that might be taken into account in these models are pressure forces, film curvature, viscous effects, velocity, and surface forces, among others.

It is important to note that most of these models are intended for a system in two dimensions. This is because it is simpler to use two-dimensional systems in order to understand the rheology of complex systems like foams [27, 28, 41]. Also two-dimensional foam (hereafter 2D foam) models can be helpful to comprehend the dynamics of foams in three dimensions [23, 30].

In some of the studies using these models, a 2D foam is the experimental case of a single layer of bubbles confined between two glass plates where the separation between the plates is much smaller that the length of the plates and also smaller
than the sizes of the bubbles. A more fundamental reason for using two-dimensional systems is because 2D foams, as in the example described previously, often explicitly account for the drag from the confining plates. A foam in porous media likewise experiences drag from the confining medium, and hence can be considered as analogous to the 2D case [27, 41, 64].

Amongst the foam models that can be used in two dimensions, there is the Potts model [22], the Bubble model [20, 65], the vertex model [21] and quasi-static models [19, 66]. Additionally, other proposed models are the viscous froth model and its limiting cases: grain growth, soap froth, and the pressure-driven growth models [13, 23, 67, 68]. This section will focus on the viscous froth model and the models related to it.

Moreover, the mathematical implications of the viscous froth model and its simplifications when used in foam rheology are briefly mentioned in this section but the information is expanded later in section 3.3.

The viscous froth model (VFM) for foam rheology was introduced as part of the study of grain growth [67]. It works well for a 2D dry foam [23, 25, 41, 45] and the case of most interest is that of a fast-flow regime [43, 64].

Specifically, in the viscous froth model, the forces acting on the soap films include the surface tension/curvature forces, the pressure difference between bubbles and the viscous drag forces [23, 27].

The description of this model is based on the following assumptions: a group of bubbles of area $A_b$ separated by thin films (meeting at angles of $2\pi/3$ according to Plateau’s rules) of surface tension $\gamma$ is considered. Each point of the films has curvature $K$ whilst $\Delta P$ denotes the pressure difference between bubbles [69].

The equation governing this model arises from the balance between the forces acting on the soap films, previously mentioned, per unit length of the interface in the case of the 2D foam, and in the direction of the normal for each segment of film [23, 68, 69]:

$$\Delta P - \gamma K(s) = \zeta v(s)$$  \hspace{1cm} (2.3)
where \( \zeta \) is a drag coefficient and \( v \) represents the velocity of an element of film \( s \) [41, 64]. Figure 2.1 depicts the forces involved. There are cases when the drag relationship is not linear in velocity, then a power law index \( a \), should be considered affecting the velocity. Most of the models have considered the linear case however for computational convenience [45, 69].

The viscous froth model has been used to improve the description of bubble neighbour exchanges and their evolution when a small increment of strain is imposed [25, 69]. Another application of this model is to characterise foams that are subject to rapid strain, at a highly elongated state far from equilibrium [43]. Barry et al. [41] used the VFM to perform simulations of monodisperse and polydisperse foam samples in two dimensions subject to simple shear with the objective of studying the causes of ‘localisation of shear’ at a moving boundary. The VFM has also been used to simulate the behaviour of topological events such as the relaxation of a foam system after a film rupture, it has been applied to the example of a rapid foam flow around curved channels and to the motion of an isolated film [23, 27, 45].

The so called ideal grain growth model also emerged from the study in two dimensions of grain growth in metals [67]. This model assumes that the pressure difference between bubbles is insignificant. Therefore the motion of boundaries is driven by curvature and is retarded by some viscous drag. Then, equation (2.3) is simplified to [23, 67, 68]:

\[
v(s) = -\frac{\gamma}{\zeta} K(s).
\]

An equation of interest in this model is the Mullin’s law for grain growth, where the rate of grain growth is expressed as the change in the area and is a linear function
of the number of sides of the cell or bubble [23,67], which is expressed as:

\[ \frac{dA_b}{dt} = \frac{\pi \gamma}{3 \zeta} (n_b - 6) \]  

(2.5)

where \( A_b \) is the bubble area, \( t \) is the time, \( \gamma \) is the tension at the grain body, \( \zeta \) represents the viscous drag, and \( n_b \) is the number of sides.

Another model called the ideal soap froth model was proposed in the study of cellular patterns [70]. The drag forces accounted for in the viscous froth model are negligible in this model, it being possible to describe the behaviour quasistatically with the Laplace law [41]:

\[ \Delta P = \gamma K. \]  

(2.6)

The assumptions in this model are that the Plateau’s rules are respected and the curvatures add up to zero at the vertex.

Equation (2.6) can describe the system when it is considered in equilibrium (not flowing). For the cases of slow flow and when the viscous drag is considered zero the Laplace law is recovered (from the VFM) [27, 41, 45, 69]. The model does not however preclude slow motion of the films driven by gas diffusion across them, and hence can be used to describe diffusive coarsening.

In this case the bubble area change follows the von Neumann’s law, where \( k \) is the permeability constant for gas transport across the film [23,67]:

\[ \frac{dA_b}{dt} = \frac{\pi}{3} k \gamma (n_b - 6) . \]  

(2.7)

Actually the viscous froth model was first proposed as the bridge between soap froth and grain growth [23], in order that Mullin’s and von Neumann’s equations could be related [67, 70].

A third simplification of VFM however results in the pressure-driven growth model, which discards the term for surface tension. Therefore pressure difference is
the driving force for the motion \([13, 23]\). The model uses the following equation:

\[
v(s) = \frac{\Delta P}{\zeta}.
\]  

(2.8)

This model is relatively uncommon in literature, however there is a good potential for using it.

The viscous froth model, being a more general expression of the different forces acting on the bubble films, could describe foam rheology better, however its numerical solution is also more complicated \([43]\). An approach of the solution of VFM is its implementation in the Surface Evolver software \([45, 69]\). Furthermore, some specific cases permit simplifying the solution.

Regarding the mathematical implications, it has been established that the viscous froth model entails a diffusive character \([25, 43]\). In the case of confined bubbles, physically, it describes diffusion of curvature along the films as a result of imposing shear causing rearrangements to dissipate the energy of the foam structure \([43]\).

This diffusive character is strongly related to the term involving surface tension and curvature, in such a way that soap froth and grain growth still exhibit this character. Mathematically the system can be described with parabolic differential equations \([71]\). However, pressure driven growth acquires a hyperbolic character instead of parabolic, which complicates the solution of the system. It has also been suggested that pressure driven growth is equivalent to the eikonal equation used in optics \([72, 73]\). In turn this is related to the method of characteristics and fractional-flow theory for foam mobility \([74, 75]\).

In this present study the pressure-driven growth model is used and will be further described throughout this thesis. Prior to this, background on foam improved oil recovery is presented in the following section and more context on fractional-flow is given in section 2.6.
2.5 Foam improved oil recovery

Foams have an important application in the petroleum and gas industries such as in drilling wells, sand clean-out of wells, sealing of formations to control ground water movement, and in enhanced oil recovery [14, 15, 33, 76, 77]. The latter is the application of interest in this research.

Foams are used as displacing fluids for oil recovery processes because they have characteristics that allow them to control the permeability and mobility of gases and liquids which permits achieving a more uniform displacement [17, 33, 35, 76].

The production of oil from underground formations is possible due to the pressure gradient that is induced between the reservoir and production wells, in such a way that the oil can flow out of the rock formations. Therefore, it is important to maintain this pressure difference to drive the oil towards production. The displacement is influenced by the conditions and characteristics of the reservoir (e.g. permeability) and the oil properties. The process itself affects the pressure over time, in such a way that production drops. It is therefore necessary to use methods that keep the desired conditions for production as long as possible [78, 79].

The different processes in the production of oil from petroleum reservoirs can be classified as primary, secondary and tertiary recovery. The primary recovery relies on the oil flow caused by the pressure differences between the formation and a production well, without the use of external fluids to maintain the reservoir pressure. This is also known as natural depletion. Secondary recovery consists of the injection of a liquid or a gas to drive oil through the reservoir to a production well after the decompression of the well has occurred. Even after using these processes, the percentage of oil recovered from the formation is low. For this reason, tertiary recovery methods, also known as enhanced or improved oil recovery, have emerged. As well as secondary processes, enhanced recovery also involves flooding with another phase in order to keep the pressure necessary for production and displace oil through the formation. Through these techniques the percentage of recovery is improved, after the early stages of production [6, 14, 15, 80].
The injection of substances into oil wells is carried out using the facilities provided for production. The structures, consisting of the casing and the tubing, among other elements, have perforations along their length in the production formation layer [78]. In such a way that it is possible to assume that the entry of the displacing fluids is even along the depth of the production structure.

Gas has been used as a driving fluid in secondary and tertiary oil recovery. It has been reported that in theory gas flooding processes could have a 100% efficiency of recovery [8,10,81]. However, there are many problems and limitations in the use of gas for this purpose [1,14]. Some problems that arise when gas is used are because of reservoir heterogeneity, low gas density and high gas mobility which cause fingering and gravity override, where the gas tends to rise and sweep only the upper portion of reservoir. This does not permit one to raise the percentage of recovered oil [1,14,16].

The use of foams can counteract some of the problems mentioned above, because foams lower the gas mobility [1,14,17,18,81,82]. As a result, the displacing medium is diverted from well-swept areas of the reservoir into parts of the formation that were not swept or just partially swept and in this way the oil displacement efficiency is improved [1,8,14,18,76,82]. This has been proved in a vast number of studies that include laboratory experiments resembling the processes conditions and also in field applications [7,83,84].

Figure 2.2 shows a graphic representation of the injection of gas and foam into a formation and illustrates the contrast of using these fluids for the shape of the boundaries between the displaced and displacement phases. As mentioned above it is expected that the displacement using foam would be more uniform.

In oil recovery applications, foams are formed with an aqueous surfactant (e.g. alpha olefin sulfonates with different carbon chain lengths, alkyl benzene sulfonate, dodecylbenzene sulfonate, sodium and amino oxyethylene sulfates, lauryl betaine [83,86–89]) solution as the continuous phase and with gases (i.e. carbon dioxide, steam, hydrocarbon gas or nitrogen) as the dispersed phase [26,33].

The physical properties of foams depend on different factors. Many works present data for foam viscosity [9,33,62,90–92] due to its relation to gas mobility reduction.
(foam viscosity being calculated using the gas viscosity). In addition it is also common to report viscosity for the aqueous and gas phase separately [9, 62, 91]. Apaydin and Kovscek [91] also report surface tension and limiting capillary pressure in terms of surfactant concentration. Just to mention an example, Ma et al [9] report viscosities for the aqueous and gas phases as modelling parameters of 1 mPa s and 0.02 mPa s, respectively; when an internal olefin sulfonate (IOS 1518) is employed as surfactant.

One method of placing foams in reservoirs for improved oil recovery is surfactant-alternating-gas (SAG) injection. In this method surfactant is injected alternately with gas. This methods is further described in section 3.1. In the context of this method an idealized model to predict the structure and evolution of a foam front has been suggested [1].

It is important to bear in mind that this model is also based on so called fractional-flow theory. Fractional-flow simulations give information about the SAG process having a spreading wave with low mobility at the gas front and high mobility near the injection well. Therefore the main assumption of the model is that all resistance to movement occurs in the low mobility zone. Some more details on fractional-flow analysis are covered in the following section.
2.6 Fractional-flow theory

Numerous studies about the foam propagation inside oil reservoirs rely on information provided by fractional-flow theory [1, 8, 10, 11, 93–95]. Studies based on this theory on the one hand, and, on the other hand, research performed with complex foam simulators are often used to validate each other. Simulations corroborate results obtained via this analysis [8, 11] and vice versa, analytical results benchmark numerical results [10, 11].

Fractional-flow theory is relevant in the context of this project because assumptions for the model adopted for this study are based on results obtained with this theory. The topic is vast and not trivial. However, details about it are not in the scope of our study. Therefore, only an overview is provided in order to set some basics for the information to be presented later.

Essential information, useful for this project, is about foam mobility that is determined from fractional-flow analysis. There are a wide range of models for foam mobility [11]. Among these there are empirical methods (e.g. the Fisher model [96]), the ‘population balance’ method and similar approaches [26], and models based on data provided by Persoff et al. [97], as the ‘limiting capillary pressure’ model [81]. They emerged trying to explain the mechanisms that govern foam mobility, which are complex as they depend on the foam texture [11]. However some of these models leave foam texture out. In addition, trying to apply them in field scale simulations is difficult. Still, it is possible to portray these complex processes using models which are simple [11,93].

Fractional-flow analysis employs experimental data obtained from coreflood tests for foam. In combination with a model for foam stability, it permits one to make conclusions precisely about foam mobility. The ‘limiting capillary pressure’ model, is a popular method used for this purpose [81,93,98].

The fractional-flow method consists in the construction of fractional-flow curves which are graphs of water fractional flow versus water saturation, at some specific permeability of the medium. Then, so called time-distance diagrams are derived.
from fractional-flow curves. These are dimensionless diagrams that indicate velocities of the injection fluids and their relative mobility at the moment of injection and for the different phases present in the reservoir [98]. Equations to describe the process are related to curves in such a way that velocities are determined from the slope of curves and mass balances are also indicated graphically [93].

In turn, the ‘limiting capillary pressure’ model is based on the effects that capillary pressure have on foam texture and hence foam mobility [11, 81, 93]. The model is quite simple compared to previous proposals [93]. In porous medium, capillary pressure is the difference in pressure between gas and aqueous phases [11]. As already mentioned in section 2.2, capillary pressure plays an important role in foam generation and affects coalescence [1, 11].

An important implication of this model is that it is the mobility of the gas (rather than that of the liquid) that is affected by foam texture. This implies lots of simplifications for these kind of studies [81, 93]. Then, it is possible to apply fractional-flow theory (i.e. develop fractional-flow curves) to determine the influence of capillary pressure on mobility [81, 93].

At the end it is possible to make conclusions about the efficiency of certain processes, for example in the case of the propagation of fluids inside reservoirs for field applications, better injection strategies are chosen by determining the conditions necessary to divert the flow of foam (or gas) from high permeability layers to low permeability [1, 98]. This is important, having in mind that this is the aim when using foam (i.e. obtaining a more uniform displacement) [81].

For the SAG process, fractional-flow methods predict that the foam collapses in a thin shock front with low mobility and it is possible to estimate the velocity of this front [11, 93]. Based on this information, Shan and Rossen [1] developed a model for foam propagation.

Fractional-flow theory can be considered as an application of the method of characteristics for foam processes. Therefore, it is also related to the eikonal equation and the pressure-driven growth model, as pointed out in section 2.4.
2.7 Numerical methods

In view of studying the phenomena of interest via modelling and simulations, it is necessary to deal with numerical solutions for the systems. Therefore, this section gives a general description of the numerical methods used.

Our main concern is the solution of differential equations. The system for the model described in detail in Chapter 3 is defined by partial differential equations (PDE). With appropriate assumptions (i.e. via spatial discretisation) these are simplified to ordinary differential equations (ODE).

A simple method for the solution of ODE is Euler’s method. This consists on calculating approximations of the solution using finite steps taking into account the conditions at the beginning of each step. A general formula for the method given by Press [71] is:

\[ y_{n+1} = y_n + hf(x_n, y_n) \] (2.9)

where \( y_{n+1} \) is the solution sought at the current step, \( y_n \) is at the previous step, \( h \) is the interval size. However, this is not a reliable method because is is not accurate and its instability adds to the complication of the system itself [71].

A better approximation can be achieved using Heun’s method, without adding complexity to the solution of the system. Actually, the first approximation in this methodology uses the same procedure as Euler’s (prediction), and a method for numerical integration is used to obtain the final value in the corresponding step (correction) [99]. A general formula for the method when the trapezoidal rule is used for integration is made up of the following equations:

\[ p_{n+1} = y_n + hf(x_n, y_n) \] (2.10)

\[ y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]. \] (2.11)

Regarding numerical integration, which in our system is required for computing areas of a reservoir that are swept by (and not swept by) foam, Simpson’s rule can be used. The approximation to an integral of the function \( f(x) \) for an interval \([a, b]\)
which is subdivided into $2M$ subintervals $[x_n, x_{n-1}]$, with $h = (b - a)/(2M)$ and $x_n = x_0 + bh$, is [99]:

$$
\int_a^b f(x) \approx \frac{h}{3} [f(a) + f(b)] + \frac{2h}{3} \sum_{n=1}^{M-1} f(x_{2n}) + \frac{4h}{3} \sum_{n=1}^{M} f(x_{2n-1}). \tag{2.12}
$$

Some details on how these methods for the solution of differential equations are used in the solution of the system are given further down in section 3.5.2.

## 2.8 Conclusions

Foams are complex fluids and study of how they flow is challenging. They are however promising for different applications. In this particular case we are interested in dry foams flowing through porous media for oil recovery processes.

When studying the flow of foams, a wide range of properties needs to be considered. However foam texture and pore size of the medium are of paramount importance as they affect the rheological properties. These characteristics in turn are related to surface area and strongly connected to the energy state of the foam.

Foam in porous media is commonly described with Darcy’s law with appropriate considerations for multiphase flow. In addition, there is a vast number of models aiming to describe foam rheology. A popular and promising model is the viscous froth. Special cases of viscous froth give place to three more models: grain growth, soap froth and pressure-driven growth. However, numerical solution of viscous froth is expensive, requiring a heavy computational load. Instead, we explore one of its limiting cases (pressure-driven growth) that has not had much attention in foam applications, and the solution is expected to be simpler.

Moreover, we give an overview of the importance of foam for petroleum industry. Foams have suitable characteristics for using them in improved oil recovery (e.g. they lower gas mobility and permit increasing the percentage of recovery). In this context the study and analysis of the flow of foams is important because it gives an insight in the design necessities of the production process.
A method to place foams inside the oil formation is through alternation of surfactant and gas. The model that can describe this process is pressure-driven growth. Considerable simplifications assumed for the model are derived from fractional-flow theory, specifically regarding mobility at the foam front as minimum compared to the surrounding aqueous and gas phases.

It is remarkable that pressure-driven growth is a valid model for the displacement of the low mobility foam front. This is despite the fact that in section 2.4 the model was developed as a special case of viscous froth for a single foam film, whereas the foam front in improved oil recovery applications is on a much larger scale and itself contains a multitude of bubbles.

As stated before, we take advantage of the characteristics of the pressure-driven growth model and use it to describe the propagation of a foam front within rock formations for oil recovery. Even when this application has been explored previously [1], there are open questions and a more detailed analysis of the process can be performed that gives insight into the physics of the process and its theoretical description.
Chapter 3

Modelling foam displacement for improved oil recovery

This Chapter describes the basics of the topic studied throughout this research project. These include the foam rheological model and numerical schemes used to describe the foam displacement. In addition the numerical schemes used in this model are applied in subsequent chapters. As these chapters are presented as published papers, details of the model are not described in the main text however are provided (along with complementary data) in the paper published in the Journal of Fluid Mechanics, 751:346-405, 2014. Authors: Paul Grassia, Elizabeth Mas-Hernández, Nima Shokri, Simon J. Cox, Gennady Mishuris, and William R. Rossen.

On the other hand, we also introduce the specific topic of foam displacement for a stratified reservoir, where some preliminary results are shown. The information given in this Chapter about the heterogeneous reservoir is to provide some alternative data to Chapter 6, which presents a manuscript on different aspects of the same topic. A paper about the heterogeneous anisotropic reservoir has been published in The European Physical Journal E, 39:42. Authors: Paul Grassia, Carlos Torres-Ulloa, Stefan Berres, Elizabeth Mas-Hernández, and Nima Shokri.

The structure of this Chapter is as follows: section 3.1 introduces the injection strategy for the process of interest. Section 3.2 shows the derivation of the
mathematical model we have adopted. Singularities that arise for the numerical solution of the model are discussed in section 3.3. An asymptotic solution for the model is explored in section 3.4. Section 3.5 presents the numerical algorithm adopted, where some improvements to the numerical scheme are explained, including a technique that results from the analysis of the asymptotic solution, and a method to regularise the system. Numerical and analytical results are also shown in this section. Section 3.6 gives an insight of particular cases where we intend to use pressure-driven growth. Furthermore, in 3.7 we present some results for one of these specific cases, the heterogeneous reservoir but accounting for anisotropy. Finally the Chapter is concluded in section 3.8.

3.1 Surfactant alternating gas

For improved oil recovery, the methods for placing foams into the reservoirs are the simultaneous injection of surfactant and gas, and the alternation of gas and surfactant known as surfactant-alternating-gas (SAG) [1,100]. The latter is preferred because this technique avoids gravity override and other complications to the process [1].

In the SAG process the foam is formed inside the reservoir from the interaction between surfactant and gas, introduced alternately with slugs. We focus specifically in the process where one large slug of each surfactant and gas are used. Then, this injection strategy generates an aqueous phase (the surfactant solution), followed by a wet foam region (the foam front), in turn followed by a gas phase (comparatively dry and with bubbles substantially larger than in the wet foam front).

In this process the foam front motion is driven by a difference between the injection pressure and the hydrostatic pressure. Hydrostatic pressure is a function of the reservoir depth. Thus the front speed, being proportional to this pressure difference, is larger at the top of the front and decreases further down.

Considering the SAG process, Shan and Rossen [1] proposed an idealised model to predict the displacement of a foam front, which is concluded from results obtained
with large scale reservoir simulator and using the so called fractional-flow theory (see section 2.6). With the objective to determine the optimal injection strategy, they present a series of simulations that provides information about water saturation in a reservoir. With these data they came to the conclusion that the SAG process originates a spreading wave with low mobility at the gas front and high mobility in the surrounding phases. This is one of the main assumptions of the idealised model presented in detail in the next section.

3.2 Model for foam displacement

The simplified mathematical description of the SAG process [1] was motivated by findings derived from fractional flow theory.

The basic premises of the model in its simplest form are the following:

- The reservoir is homogeneous
- The phases flowing are gas and water
- All resistance to flow is concentrated in a narrow zone at the displacement front,
- the mobility is infinite on both sides of the front compared to the mobility at the front itself,
- ahead of the front, water saturation is uniform and equal to unity and the pressure is governed by hydrostatics,
- behind the front in the gas bank, water saturation is constant, gas density is approximately zero, and the pressure is comparatively uniform.

Consequently, the pressure drop is focused on the front width thickness, which is narrow compared to the length of the reservoir. It is also assumed that the width of the front increases proportionately to the distance the front has travelled. However it is always small compared to the distance it displaces inside the reservoir, in such
a way that it is possible to depict the foam front as a one-dimensional curve in a
two-dimensional system. Figure 3.1 shows a graphic representation of the model.

The mathematical representation is as follows: Darcy’s law is applied to the low
mobility zone obtaining the superficial velocity of the gas \( u_g \) inside the low-mobility
zone:

\[
    u_g = -k \lambda_{rf} |\nabla P|
\]

where \( k \) represents the permeability, \( \lambda_{rf} \) represents the gas relative mobility, and
\( \nabla P \) is the pressure gradient in the low mobility zone.

However, taking into account that we are following multiphase flow inside porous
media, the interstitial velocity is deduced accounting for the porosity of the medium,
relative permeabilities or mobilities, and the volume fraction of the liquid.

Then, the interstitial velocity \( v \) of the displacement front is

\[
    v = -\frac{k \lambda_{rf} |\nabla P|}{(1 - S_{wf}) \phi}
\]

where \( S_{wf} \) is the water saturation at the gas bank and \( \phi \) is the porosity of the
reservoir.

The pressure gradient across the front is determined by the width of the front, \( \tau \)

\[
    |\nabla P| = -\frac{\Delta P}{\tau}
\]

where \( \Delta P \) is the well-to-well pressure at the top of the reservoir.

Meanwhile \( \tau \) is proportional to \( s \), the distance the front has travelled:

\[
    \tau = \tau^* s
\]

where \( \tau^* \) is a parameter of the model, being the width of the front when it has
travelled certain distance.

To monitor the movement of the front, it is necessary to track the motion of
individual points in the front at positions \([x(t), z(t)]\) (see Figure 3.1b, where \( x \) is
Figure 3.1: Schematic diagram for the SAG process (a) showing the liquid-filled and gas-filled zones, and the foam front. (b) showing variables considered in the idealised model.
measured horizontally and z vertically). The distance each point has moved since the beginning of the displacement is \( s(t) \). The velocities defined in equations (3.1) and (3.2) (superficial and interstitial respectively), the pressure gradient and the direction the front advances, are all perpendicular to the foam front. The direction of front movement is indicated with the angle \( \alpha \) as depicted in Figure 3.1b:

\[
\alpha = -\arctan\left( \frac{\partial x}{\partial z} \right)_t
\]  

(3.5)

where \((\partial x/\partial z)_t\) is the slope of the front at a given time.

The components \( x \) and \( z \) of the interstitial velocity of a material point on the front are:

\[
v_x = \frac{dx}{dt} = \frac{k\lambda_f}{(1 - S_{wf}) \phi \tau^* s} \Delta P \cos \alpha
\]  

(3.6)

\[
v_z = \frac{dz}{dt} = \frac{k\lambda_f}{(1 - S_{wf}) \phi \tau^* s} \Delta P \sin \alpha.
\]  

(3.7)

Remembering that the pressure gradient is the difference between the injection pressure \((P_{inj})\) and the pressure due to hydrostatics,

\[
\Delta P = P_{inj} - \Delta \rho g z
\]  

(3.8)

here \( \Delta \rho \) is the density difference between phases, \( g \) is the gravitational acceleration.

The speed \( v_s \) is given by:

\[
v_s = \frac{ds}{dt} = (v_x^2 + v_z^2)^{1/2}.
\]  

(3.9)

The initial and boundary conditions are

\[
x(z, 0) = 0
\]  

(3.10)

\[
s(z, 0) = 0
\]  

(3.11)

\[
\alpha(0, t) = 0.
\]  

(3.12)
The above equations can be re-expressed in dimensionless form. The dimensionless differential equations for the positions of individual points on the displacement front are:

\[
\frac{dX_D}{dt_D} = \frac{(1 - Z_D) \cos \alpha}{s_D} \tag{3.13}
\]

\[
\frac{dZ_D}{dt_D} = \frac{(1 - Z_D) \sin \alpha}{s_D} \tag{3.14}
\]

\[
\frac{ds_D}{dt_D} = \sqrt{\left(\frac{dX_D}{dt_D}\right)^2 + \left(\frac{dZ_D}{dt_D}\right)^2} \tag{3.15}
\]

where dimensionless parameters are defined as:

\[
\alpha \equiv -\arctan \left(\frac{\partial X_D}{\partial Z_D}\right)_{t_D} \tag{3.16}
\]

\[
X_D \equiv \frac{\Delta \rho g x}{P_{inj}} \tag{3.17}
\]

\[
Z_D \equiv \frac{\Delta \rho g z}{P_{inj}} \tag{3.18}
\]

\[
t_D \equiv \frac{k \lambda_r f \Delta \rho g}{(1 - S_{w_f}) \phi \tau_s} \tag{3.19}
\]

\[
s_D \equiv \frac{\Delta \rho g s}{P_{inj}}. \tag{3.20}
\]

The initial and boundary conditions are

\[
X_D(Z_D, 0) = 0 \tag{3.21}
\]

\[
s_D(Z_D, 0) = 0 \tag{3.22}
\]

\[
\alpha(0, t_D) = 0. \tag{3.23}
\]

The solution of this system results in a family of curves that are functions of \(t_D\) and describe the front displacement. Numerical results are presented later in section 3.5. Before, it is necessary to explain some singularities of the model (covered in the next section), and an analytical solution free of these artifacts (section 3.4).
3.3 Singularities for pressure-driven growth

As mentioned previously in section 2.4, pressure-driven growth and its governing equations (3.6)–(3.7) (in dimensional form) or (3.13)–(3.14) (in dimensionless form), correspond to a limit case of the viscous froth model where the surface tension/curvature term is neglected [13]. This has repercussions on the numerical solution of the system. The other two limiting cases of the viscous froth model are [67] the standard soap froth model (where the viscous term is eliminated) and the grain growth model (considers the pressure term negligible).

The pressure-driven growth model changes substantially the physics and the mathematics of the system taken under consideration. When surface tension is taken into account, this term tends to drive the system to lower energy configurations; because surface tension is associated with interfacial energy. Low energy configurations are achieved by rounding off sharp kinks in film shapes, which helps to reduce film lengths and their associated interfacial energy.

The surface tension/curvature term in equation (2.3) is a second derivative of point position with respect to space, whilst the drag term ($\zeta v$) is a first derivative of position with respect to time. Therefore, the VFM is diffusive and can be expressed as a diffusion equation for curvature [101]. Diffusion of curvature strives to make curvature uniform with respect to space along a film, the situation presented in using the standard soap froth model.

When surface tension is omitted, the diffusive character of the equation is lost. Loss of diffusive character in a system of partial differential equations (in this case the partial differential equations governing point position with respect to arc length along the film and time) has serious consequences. The equations are said to become hyperbolic instead of parabolic [102], and the equations can admit solutions that are not smooth. Instead kinks or shocks can be present, where films change direction suddenly and/or curvature diverges.

When seeking numerical solutions considerable care is needed to identify those kinks/shocks or similar divergences which are a real part of the solution from any
spurious divergences which are an artifact of the numerical scheme. On the other hand, it is possible to obtain asymptotic solutions which are free of these numerical difficulties. This is presented in the following section.

3.4 Asymptotic solution for the foam displacement

The challenges faced when seeking the numerical solution of a process described with pressure-driven growth are an incentive to explore asymptotic solutions, valid for longer times, which are not affected by numerical instabilities. Another importance of the analytical solution is that it can be used to validate numerics (results from both methods are compared later in section 3.5.3). Furthermore, the analysis of this solution gives insight into the front behaviour at the top and bottom of the domain solution [13].

To obtain the solution as outlined in [13] we have redefined the direction of the vertical axis ($Z_D$) and used a complementary angle to that indicating the direction of the front motion; here the axis is indicated by $Y_D$ (measured upward, whilst $z$ and $Z_D$ were measured downward) and the used angle is $\beta$, as shown in the schematic in Figure 3.2.

Then, the equations describing the advance of a front material point, which are equivalent to the components $x$ and $z$ of the front velocity defined previously for the foam displacement and given in dimensionless form by (3.13)–(3.14), are:

\[
\frac{dX_D}{dt_D} = \frac{Y_D \sin \beta}{s_D} \quad (3.24)
\]
\[
\frac{dY_D}{dt_D} = -\frac{Y_D \cos \beta}{s_D}. \quad (3.25)
\]

At long times $t_D \gg 1$, almost all material points have displaced through a distance $s_D$, roughly equivalent to the displacement of the point at the top, which
Figure 3.2: Sketch for the motion of the front at large time, where the distance the point at the top has advanced is $\sqrt{2t_D}$.

is $s_D \equiv \sqrt{2t_D}$ (the deduction of this approximation is given in detail later in section 3.5.2). Therefore, equations (3.24)–(3.25) simplify to:

\[
\frac{dX_D}{dt_D} \approx \frac{Y_D \sin \beta}{\sqrt{2t_D}} \quad (3.26)
\]

\[
\frac{dY_D}{dt_D} \approx -\frac{Y_D \cos \beta}{\sqrt{2t_D}} \quad (3.27)
\]

where the denominator does not depend on the point position.

The similarity solution that describes the shape of the front we look for is

\[
Y_D = Y_D(\xi) \equiv Y_D(X_D - \sqrt{2t_D}) \quad (3.28)
\]

where $\xi$ is a similarity variable that measures horizontal distance from the top of the advancing front.

Moving the material point from $(X_D, Y_D)$ at time $t_D$, to $(X_D + dX_D, Y_D + dY_D)$ at time $t_D + dt_D$, the similarity variable will displace by

\[
\xi + d\xi = X_D + \frac{Y_D \sin \beta}{\sqrt{2t_D}} dt_D - \sqrt{2t_D} - \frac{1}{\sqrt{2t_D}} dt_D \quad (3.29)
\]
from here,
\[
d\xi = \frac{Y_D \sin \beta - 1}{\sqrt{2t_D}} dt_D. \tag{3.30}
\]

The self-similar shape of the front is defined by the ratio of the displacements \(dY_D\) to \(d\xi\):
\[
\frac{dY_D}{d\xi} = \frac{Y_D \cos \beta}{1 - Y_D \sin \beta}. \tag{3.31}
\]

An equivalent expression after some algebra and using
\[
\cos \beta = \frac{1}{\sqrt{1 + (dY_D/d\xi)^2}} \tag{3.32}
\]
\[
\sin \beta = \frac{dY_D/d\xi}{\sqrt{1 + (dY_D/d\xi)^2}} \tag{3.33}
\]
is given:
\[
\frac{dY_D}{d\xi} = \frac{Y_D}{\sqrt{1 - Y_D^2}}. \tag{3.34}
\]

Solving this equation the similarity variable is found:
\[
\xi = \sqrt{1 - Y_D^2} + \frac{1}{2} \log \left( \frac{1 - \sqrt{1 - Y_D^2}}{1 + \sqrt{1 - Y_D^2}} \right). \tag{3.35}
\]

Equation (3.35) in terms of our original vertical axis \(Z_D\) is simply
\[
\xi = \sqrt{1 - (1 - Z_D)^2} + \frac{1}{2} \log \left( \frac{1 - \sqrt{1 - (1 - Z_D)^2}}{1 + \sqrt{1 - (1 - Z_D)^2}} \right). \tag{3.36}
\]

Then, using (3.36) and the definition of \(\xi\) given by (3.28) it is possible to plot the foam front shape given values for \(Z_D\) and \(t_D\). Figure 3.3 shows a plot with the curve obtained in this fashion for time \(t_D = 30\).

The next section explains the algorithm for the numerical solution of the system, which comprises a method deduced form the analysis of equation (3.35).
Figure 3.3: Asymptotic solution for the front shape at $t_D = 30$.

### 3.5 Numerical algorithm

In this section we present the numerical algorithm to solve the system of differential equations that describe the foam front motion, equations (3.13)–(3.23). In addition, we point out some modifications to improve the performance of the computer code, compared to the work presented previously [1]. These improvements include techniques derived from the analysis of the asymptotic solution that have implications for the numerical schemes.

Therefore, the information in this section is presented according to the following structure: 3.5.1 introduces the issue of numerical stability of the system. Next, detailed information on how to implement the solution in the computer program and some improvements to the Shan and Rossen algorithm are shown in 3.5.2. After that 3.5.3 presents the aforementioned techniques to deal with one of the singularities of the model. Finally 3.5.4 explores another method to handle the particular behaviour of the propagation caused by the singular nature of the model.
3.5.1 Numerical stability

As mentioned previously, the system as it is proposed is potentially very unstable. In order to define a suitable time step in terms of the space step used to discretise the front, it is necessary to use the Courant condition. This is possible because the equations in the system are similar to the general flux-conservative equation found in initial value problems, which has the form [71]:

\[
\frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{F}(\mathbf{u})}{\partial S}
\]  

(3.37)

where \( \mathbf{u} \) and \( \mathbf{F} \) are vectors, \( t \) is time and \( S \) is a spatial parameter (e.g. arc length along front defining the spatial discretisation of the front shape).

The prototypical example of the general flux-conservative equation when applied for an scalar \( u \) is considered advected with some velocity \( v \).

\[
\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial S}.
\]  

(3.38)

It is necessary to consider the von Neumann stability analysis for a numerical scheme such as the Lax method to obtain:

\[
\frac{|v| \Delta t}{\Delta S} \leq 1
\]  

(3.39)

known as the Courant condition, from which a suitable time step can be chosen.

In our dimensionless system of equations at times in the order of unity, velocities are likewise order unity. At early times however, velocities are much larger, so very tiny time steps may be required to solve the equations for any reasonable spatial resolution.

The method followed to obtain the numerical solution of the system is given after.
3.5.2 Implementation of numerical solution in a computer program

In this section we present the algorithm used for the solution of the differential equations describing the foam front displacement.

The model of interest, proposed by Shan and Rossen [1], assumes that all resistance to flow arises from a low mobility region near the foam front. The extent of the low mobility region however is very small at early times, and in fact Shan and Rossen’s model associates zero drag to it in the limit as time $t \to 0$. This situation is reflected in the equations (3.13)–(3.14) having $s_D$ in the denominator and the initial condition assigning it zero. Clearly however the overall system still has a finite drag arising from finite size regions (albeit with comparative high mobility ahead of and behind the foam). It is therefore sensible to set the foam front a small finite drag (instead of zero drag). A small value $s_{D_0}$ different from zero is reassigned to $X_D$ and $s_D$ as initial conditions,

$$X_D(Z_D, 0) = s_D(Z_D, 0) = s_{D_0}$$

for example: at $t_D = 0$, $X_D = s_{D_0} = 0.001$ (in dimensionless units). This also prevents the derivatives approaching $\infty$ as $s_D \to 0$.

The $X_D$ value at the top of the reservoir for time different from zero is derived as follows: if $\alpha = 0$ at $Z_D = 0$, given by the boundary condition (3.23), then equations (3.13)–(3.14) simplify to

$$\frac{dX_D}{dt_D} = \frac{1}{s_D}$$

$$\frac{dZ_D}{dt_D} = 0.$$  

Therefore:

$$\frac{ds_D}{dt_D} = \frac{dX_D}{dt_D} = \frac{1}{s_D}.$$  

from which

$$X_D = s_D = \sqrt{2t_D + s_{D_0}^2}.$$
Equation (3.44) approximates to

$$X_D = s_D \approx \sqrt{2t_D}$$

(3.45)

provided $2t_D \gg s_D^2$. After one time step, $t_D = \Delta t$ assumed to be no larger than order of $(\Delta S s_{D0})$ where $\Delta S$ is spatial resolution along the front, the value of $2t_D$ will already exceed $s_D^2$ provided $\Delta S \geq s_D$, i.e. if $s_D \leq \Delta S$. It makes sense that the minimum thickness of the front ($s_{D0}$) is comparable with the spatial resolution $\Delta x$. Therefore, equation (3.45) becomes true to a good approximation after just a few time steps.

At first, the system was solved assigning a fixed number of $Z_D$ values from 0 to 1, as well as the time step (according to equation (3.39)). Therefore the initial conditions, given by equation (3.40), permit one to calculate the time derivatives at $t_D = 0$. For the next time step, the boundary condition, equation (3.23), and that from equation (3.44) (or (3.45)) define $X_D$, $Z_D$ and $s_D$ at the top; then, as a first approximation, Euler’s method [99] can be used to calculate $X_D$, $Z_D$ and $s_D$ at the corresponding time for the rest of the $Z_D$ intervals (see also section 2.7):

$$X_D(i,n) = X_D(i,n-1) + \left( \frac{dX_D}{dt_D} \right)_{(i,n-1)} \cdot \Delta t$$

(3.46)

where the subscripts $i$ and $n$ indicate the space and time step respectively, and $\Delta t$ is the duration of the time step ($\Delta t = t_{Dn} - t_{Dn-1}$). Analogous equations are used for $Z_D(i,n)$ and $s_D(i,n)$.

Having the points $X_D$, and $Z_D$ it is possible to calculate $\alpha$ with

$$\alpha(i,n) = -\arctan \left( \frac{X_D(i,n) - X_D(i-1,n)}{Z_D(i,n) - Z_D(i-1,n)} \right).$$

(3.47)

After that, the calculations for the time derivatives ($dX_D/dt_D$), ($dZ_D/dt_D$), and ($ds_D/dt_D$) can be performed at this time step. The same procedure is repeated for further time steps.

Implementing the solution in a computer program (specifically in Matlab) we are
able to obtain curves for the foam front displacement similar to the data presented by Shan and Rossen [1]. These data for the foam front propagation has only been obtained by Shan and Rossen for comparatively short times ($t_D \approx 0.45$) and in consequence comparatively short horizontal distance ($X_D < 1$). The data themselves are not reproduced here as they are already available in the literature.

Afterwards, the algorithm was modified to improve the calculations. First of all, the solution of the differential equations was carried out using Heun’s method (already described in section 2.7). This permits a better approximation to the $X_D$ and $Z_D$ values, because Euler’s method used earlier is not so accurate. The error in Euler’s method accumulates with each time step, which can be mitigated taking a smaller time step but resulting in the need of more iterations. Heun’s method achieves better accuracy with fewer steps.

Secondly the calculation of the angle $\alpha$ was changed, because the former method gives the front orientation angle for the interval between points $i-1$ and $i$ instead of the junction at point $i$. Angles at the junctions can be obtained via a weighted average using $\alpha$ on the adjacent intervals and the lengths of those intervals.

Even after these modifications the solution of the system still has more room for improvement. Additional changes to the algorithm are made based on the analysis of the formula for the asymptotic solution presented in the section to follow.

3.5.3 Numerical regridding routine

The results for the foam displacement presented by Shan and Rossen [1], which we were able to replicate, only show data for dimensionless time that reach a value close to $X_D = 1$. However, the structure of a geological oil reservoir is such that it is likely to be many times longer than it is deep. Thus the horizontal distance over which we flood with foam needs to be larger that the depth (here unity) to which foam penetrates.

We are seeking to obtain the foam front shape for horizontal position $X_D \gg 1$, and in consequence for longer times, in such a way that we can compare numerics to
the formulae obtained by the asymptotic analysis (see section 3.4). The algorithm, as described in the previous section does not at present allow us to achieve this (reasons are explained below). Therefore we need to apply further modifications.

Solving the system describing the motion of the foam front (for longer time that the obtained so far), without any modification encounters many challenges. In principle, in accordance with the physics of the process, the shape of the front indicates that the speed in the top part of the reservoir is greater than everywhere else. This causes the front to orient in such a way that all material points (away from the top boundary) have a downward component to their motion (in addition to a horizontal component). Keeping a fixed number of material points, resolution is lost near the top boundary as the points move downward (in the numerical simulation, a straight and continually growing longer segment of the front then develops at the top).

From the boundary condition, equation (3.23), the foam front is meant to meet the top boundary at right angles. In addition, from the analysis of the asymptotic solution given by equation (3.35) in the limit $Z_D \to 0$, it can be deduced that curvature is not finite but diverges with a mild singularity. The analysis to reach this conclusion is not covered here as it is extensively explained in [13].

Shan and Rossen’s data [1], even with an extremely fine numerical grid, show that over time a very sharp curvature develops near the top boundary. Only if sufficient data points are available near the top boundary will the correct angle at the top boundary be attained (and even the data in the cited reference show a very slight variance from that).

Indeed, our preliminary results for the front displacement appear to deviate further and further from the right angle as time proceeds. In order to overcome this situation, new points are added. The regridding routine is not trivial but has to respect the boundary condition and the foam front shape in this region with the aforementioned mild singularity in curvature [13].

As a consequence, a suitable formula for the placement of a new point $(X_{\text{new}}, Z_{\text{new}})$ close the top of the reservoir, between a point at the top boundary $(X_0, Z_0)$
and the point below \((X_1, Z_1)\), and defining its vertical position \(Z_{new}\) is the following:

\[
X_{new} = X_0 - \left( \frac{Z_0 - Z_{new}}{Z_0 - Z_1} \right)^{3/2} (X_0 - X_1).
\] (3.48)

This equation gives a location between those resulting from the use of a straight line and a parabola to join points \((X_0, Z_0)\) and \((X_1, Z_1)\). The straight line violates the condition \(\alpha = 0\) at the top. The parabola, instead, respects that condition. The problem with adjusting via a parabola is that finite curvature at the top boundary is assumed, but this is not actually the case asymptotically for the curve (similarity solution) describing the front (section 3.4).

Incorporating equation (3.48) into the algorithm it is now possible to obtain results for larger times and compare them to the asymptotic solution. Figure 3.4 presents this comparison for dimensionless time equal to 20 and 30. Numerical results are obtained discretising the front initially with 50 material points, time step used is \(1 \times 10^{-5}\), and \(s_{D_0} = 0.001\). It can be seen from the graph that as the time is larger the numerical and analytical curves move closer together. So, the program is giving the expected results.

Figure 3.4 shows that numerical solutions near the bottom of the front are below the analytical solutions. Such behaviour is expected [13] since the numerical
solutions always reach $Z_D = 1$ (the point at which injection pressure balances hydrostatic head) at a finite distance $\sqrt{2t_D}$ behind the leading edge of the foam front, whereas in the analytical solution this depth is only reached arbitrarily far behind the foam front.

On the other hand, the points near the bottom of the curve move closer to $Z_D = 1$ with every iteration in the time increment, in such way that the accumulated points in this region do not contribute with important information (i.e. the front is virtually horizontal here). Also, the continual addition of points just enlarges the size of the calculation in the computer program. Therefore it is worthwhile to eliminate some points at the bottom part of the system (near $Z_D = 1$). The orientation of the front in this region can be specified as being close to $\pi/2$. Therefore, we stablish that points are removed if their corresponding angle $\alpha$ is near this value.

### 3.5.4 System instabilities

The system has the feature of being potentially very unstable, not just numerically (section 3.5.1), but also due to the physical character of the governing equations (see section 3.3). In order to regularise the system it is possible to account for a physical or a numerical diffusivity within the system, these options are explored in [13]: they do stabilise the governing equations, but tend to be numerically expensive. However, another less expensive choice is to use a non-diffusive modification, which is presented as follows.

Regarding numerical instabilities, it is necessary to choose a small time step at the beginning of the iterations. For later times it is possible to increase the time step. This follows from the Courant condition (see section 3.5.1), as velocities fall over time.

The physical character of the equations can also lead to instabilities that could subsequently manifest themselves numerically. It is in the nature of the governing ‘pressure-driven growth’ equations that convex regions tend to expand smoothly, whereas concave regions tend to focus down into ever sharper curves (see Figure
3.5), eventually forming a cusp.

A cusp once formed represents a ‘discontinuous shock’ (in this case a discontinuity of the tangent vector), and strategies to deal with and propagate such shocks (if present in a solution) are important parts of any numerical scheme.

The shapes of the foam front represented by Figure 3.4 are of course convex ones, so in principle cusps should never arise. Certain parts of the front are however only very weakly curved. If some numerical error (round off error and/or truncation error in the integration scheme) were to cause just one data point to fall slightly behind its neighbours then a very weak local concavity would be introduced that would develop into a cusp over time.

Moreover, the evolution of cusps over time, if they are not addressed, causes the formation of spurious loops (regions where points of the front cross over each other). This is because [13] the points towards the back at the cusp are left behind but the points ahead converge towards one another and eventually cross over one another. This is problematic because the numerical scheme treats the material inside the loop as having a higher pressure than material outside, and hence inflates the loop. To suppress the formation of loops, points at the back can have their velocities corrected to avoid them falling behind. Devising a robust numerical scheme that can give a good representation of the overall curve shape without local errors is important.

One way to handle the presence of concavities in the front shape is by applying a correction in the velocity calculations (which are reflected in the time derivatives, equations (3.13) and (3.14) in the original system) when certain criteria on the degree of concavity are met. This aims to speed up the displacement of points in the concave regions to catch up with those in the convex region and in this way
Figure 3.6: Sketch of vectors for curvature calculations where $\mathbf{t}_1$ and $\mathbf{t}_2$ are tangents to an edge, $\mathbf{t}$ is a tangent at a point, $\mathbf{n}$ is a normal at a point, and $\theta$ is the angle through which the front tangent turns between adjacent intervals.

return towards a convex shape for the entire foam front.

The first requirement in such a scheme is to distinguish between (non-problematic) convex regions and (potentially problematic) concave ones. This is achieved via examination of the film curvature.

If we have two adjacent segments of the front of length $d_1$, $d_2$, with tangents $\mathbf{t}_1$, $\mathbf{t}_2$ respectively, and $\theta$ being the angle through which the tangent turns as shown schematically in Figure 3.6; curvature ($K$) can be computed as:

$$K = \frac{(\mathbf{t}_2 - \mathbf{t}_1)}{rac{1}{2}d_2 + rac{1}{2}d_1} \cdot -\mathbf{n}$$

(3.49)

where $\mathbf{n}$ is the normal vector in the direction of the movement of the front. According to this definition, points with $K > 0$ are convex but those with $K < 0$ are concave.

In turn, $\theta$ is defined as

$$\theta = \arccos(\mathbf{t}_2 \cdot \mathbf{t}_1).$$

(3.50)

We can then redefine the velocity rule [13] for any concave points ($K < 0$):

$$\frac{dX_D}{dt_D} = \frac{(1 - Z_D) \cos \alpha}{s_D \cos(\theta/2)}$$

(3.51)

$$\frac{dZ_D}{dt_D} = \frac{(1 - Z_D) \sin \alpha}{s_D \cos(\theta/2)}$$

(3.52)

where $\theta/2$ is the angle through which the tangent has turned between the midpoint of adjacent elements 1 and 2.
The significance of the $1/\cos(\theta/2)$ factor is that it is the ratio between apparent velocity of a geometric kink and the true velocity of material points on the front. If between time $t$ and $t + dt$ material points advance by $v dt$, then material points immediately adjacent to the kink (with a distance $v dt \tan(\theta/2)$) will be consumed, whilst the kink appears to advance by $v dt / \cos(\theta/2)$ (see sketch in Figure 3.7).

If concavities only arise as a result of round off/truncation errors placing a film point slightly behind its neighbours, it is clear that $\theta$ will be a very small angle, making $1/\cos(\theta/2)$ be just very slightly in excess of unity.

However, the slight speed up that we give to the erroneously concave point should be sufficient to stop it from falling any further behind its convex neighbours and so will stop the slight concavity from evolving into a true cusp/kink.

There may however be special solutions for which we wish to permit a slight concavity to focus down into something sharper (i.e. the concave regions may be part of the physical solution of the problem, like in the cases introduced in the following section and subsequent chapters, not a mere numerical artifact as in the case described here).

In that case we may want to impose the $1/\cos(\theta/2)$ adjustment to the velocities only where the computed curvature is both negative and of sufficiently large magnitude, to the extent that we consider the $x$ versus $z$ numerical data to be the discretised representation of a true cusp. The question then arises of how large
a curvature is permitted before the adjustment should be applied. Suggestions of how to address this question are given below.

Although we represent our foam front by an infinitesimal thin curve, in reality it corresponds to a finite thickness low mobility region, \( \tau \), given by equation (3.4). An object of thickness \( \tau \) cannot have a curvature measured along its centreline of greater than \( 2/\tau \). Hence the maximum permitted curvature at which we could declare our foam front to form a cusp is \( 2/(\tau^*s_D) \). However this implies that the maximum permitted change in angle between adjacent front segments becomes sensitive to discretisation. Moreover very short front segments might be required becoming expensive in computer terms. The requirement for short segment follows because, for order unity times and hence order unity values of \( s_D \), in order to be able to access the maximum curvature, front elements must be smaller than order \( \tau^* \), with \( \tau^* \) already being a small parameter.

Therefore, another choice to apply the \( 1/\cos(\theta/2) \) factor to ‘correct’ velocities for concavities is to select a permitted maximum curvature based on the magnitude of the turning angle \( \theta \), where a small finite angle \( (\theta_{\text{sharp}}) \) is set and compared to it. Then we can choose a value for this parameter that does not depend on how the front is discretised.

To test these modifications, concavities are deliberately generated in the original shape of the front at initial time \( (t_D = 0) \). A version of the program without any speed up imposed in the neighbourhood of concavities is run to compare to the modified program.

The changes to provoke the kink on the ‘original’ data, given by the initial condition in equation (3.40), consist in assigning a different value of \( X_D \) (and \( s_D \)) at certain \( Z_D \) value that could typically be chosen an order of magnitude larger than \( s_{D_0} \) (previously \( s_{D_0} \) was chosen as 0.001). The kink is produced at the interval that divides the domain of \( Z_D \) values in half (for example \( Z_D(25) \) for a space step that considers 50 intervals in the vertical axis, and with a time step for subsequent evolution chosen here as \( \Delta t = 1 \times 10^{-5} \)).

If the value assigned to \( s_{D_0} \) is very small we can choose \( X_D(25) = s_D(25) > s_{D_0} \);
but this leads to a convex defect instead of a concave one. Still, convex defects have concavities above and below them in such a way that the curve as a whole is not uniformly convex. Therefore we can follow the effect of this convex kink on the propagation of the foam front.

As an illustration of the method we have chosen $X_D(25) = s_D(25) = 0.04$ (the initial shape of the front with these values is shown in Figure 3.8). With this value the program that does not apply the correction factor for the speed of concavities gives a distorted front shape, see Figure 3.9a. On the other hand, the implementation of corrections in the velocity equations permits recovering a convex shape as shown in Figure 3.9b. In both cases the simulations use 50 material points, with time step $1 \times 10^{-5}$ and the parameter $\theta_{\text{sharp}} = 2\pi/9$ to switch (3.51)–(3.52) on.

In the above we introduced concavities numerically by deliberately introducing an error into the shape of the curve. Nonetheless, there are other cases, not solely arising from numerical error, where concavities are present. Such cases are liquid/surfactant slumping, increase of the injection pressure and heterogeneous reservoir [13,103,104]. The following section presents a brief description of these cases.
Figure 3.9: Front displacements when a convexity surrounded by two concavities is introduced at initial time using $X_D = s_D = 0.04$ for $Z_D = 0.5$, (a) showing spurious loops in the front shapes, obtained without implementing modifications to speed of concavities in the program. (b) showing entire convex curves obtained with a program that applies velocity corrections. The space step in the vertical axis is given by 50 intervals, the time step used is $1 \times 10^{-5}$. 

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3.6 Scenarios for potential application of pressure-driven growth

In the previous section we presented an example on how the numerical schemes enable us to obtain convex front shapes when a concavity is deliberately introduced at the beginning of iterations. However there are particular cases where concavities arise as consequence of the process’s nature. Here we briefly introduce those and Chapters 4–6 present the publications that study them in more detail.

There are three main problems of interest: slumping (described in section 3.6.1), raising the injection pressure (section 3.6.2), stratified reservoir (section 3.6.3).

3.6.1 Slumping

For slumping, changes on the foam texture over time are taken into account. Liquid is heavier than gas, so that in any foam front there is a tendency for gas to migrate upwards and liquid to migrate downwards. In consequence, foam at the top of the front becomes drier and hence less stable, making films burst. Although slumping primarily takes effect due to liquid drainage in those parts of the foam which are a significant distance behind the foam front (and hence is not captured by the pressure-driven growth model, which focusses dissipation solely on the foam front itself), there is another more subtle slumping effect which can be captured by pressure-driven growth, namely surfactant slumping. Surfactant solution tends to be denser than aqueous liquid without surfactant, and so surfactant migrates downward, leaving less surfactant near the top. Once this migration has happened, and once gas arrives, the bubble films formed in this region of lower surfactant concentration near the top are more susceptible to bursting than the films of the bubbles lower down. Since most of the resistance to foam flow arises from the films, a region with bigger bubbles and fewer films will be more mobile than the underlying foam at higher surfactant concentration lower down.

We model this by supposing that the standard governing equations (3.13)–(3.14)
apply up to some characteristic slump time \( t_{\text{slump}} \) after which equations become

\[
\frac{dX_D}{dt_D} = \frac{M_{\text{slump}}(1 - Z_D) \cos \alpha}{s_D} \tag{3.53}
\]

\[
\frac{dZ_D}{dt_D} = \frac{M_{\text{slump}}(1 - Z_D) \sin \alpha}{s_D} \tag{3.54}
\]

where \( M_{\text{slump}} > 1 \) is a factor for the relative change in the mobility at the front, which affects certain region from the top of the front until a defined vertical position \( Z_{\text{slump}} \).

As material in the more mobile region higher up will run ahead of material in the less mobile region lower down, there is a potential to develop concavities and/or kinks/cusps as a result. This topic is explored on more detail in Chapter 4 where the corresponding publication [103] is included.

### 3.6.2 Pressure increase

In the case where there is an increment in the injection pressure during the process, we would solve the standard equations (3.13)–(3.14) up to time \( t_P \) after which

\[
\frac{dX_D}{dt_D} = \frac{(1 + P_t - Z_D) \cos \alpha}{s_D} \tag{3.55}
\]

\[
\frac{dZ_D}{dt_D} = \frac{(1 + P_t - Z_D) \sin \alpha}{s_D} \tag{3.56}
\]

where pressure is increased by a factor \( 1 + P_t \), the value of \( P_t \) being the change in injection pressure relative to initial injection pressure.

In addition to initial conditions already discussed in Section 3.2, an initial condition needs to be provided for foam in the region \( 1 < Z_D < 1 + P_t \) at time \( t_P \). We could set \( s_D = X_D = s_{D0} \) at time \( t_P \), which introduces an immediate kink in the solution. Tracking the motion of the kink will be an essential part of the dynamics. Chapter 5 expands information on the case of foam displacement when there is a sudden increase in pressure.
3.6.3 Heterogeneity

In stratified reservoirs the rock layers laid down in them might be heterogeneous from layer to layer, where some might be more permeable than others.

We would solve

\[
\frac{dX_D}{dt_D} = \frac{(1 - Z_D) \cos \alpha}{s_D} J(Z_D) \tag{3.57}
\]

\[
\frac{dZ_D}{dt_D} = \frac{(1 - Z_D) \sin \alpha}{s_D} J(Z_D) \tag{3.58}
\]

where \( J(Z_D) \) is a reservoir heterogeneity function that we could define as:

\[
J = 1 + k_s \sin(2\pi n_s Z_D) \tag{3.59}
\]

where \( k_s \) is the amplitude of the heterogeneity in the permeability and \( n_s \) is the wave number of the heterogeneity. Alternatively (depending on exactly how the high and low permeability layers are arranged in space) we could assume

\[
J = 1 - k_s \sin(2\pi n_s Z_D). \tag{3.60}
\]

This will permit foam in higher permeability regions to run ahead of that in lower permeability regions, potentially developing concavities in the latter.

The case of the stratified or heterogeneous reservoir is approached in Chapter 6 and also in [105]. Recognising that stratification might produce not just heterogeneity in permeability, but also anisotropy in permeability, [105] focusses on the mathematical analysis of propagation of concavities for an anisotropic system. For this reason, in the following section we introduce relevant information regarding the anisotropic case (originally considered by [24]) which is important background for Chapter 6.
3.7 Foam propagation in a heterogeneous anisotropic reservoir

The previous section has presented an insight on the case of the foam front propagation within a stratified reservoir. In this section we introduce and present some background information that sets the context for the manuscript that considers the heterogeneous reservoir (Chapter 6). Namely, the numerical solution for the propagation of the front within a heterogeneous anisotropic reservoir, focussed on special strategies for the motion of concave corners. Anisotropy in this context is regarded as the variation in permeability with the orientation of the foam front. A different paper by Grassia and co-workers dealing only with this topic has been published in The European Physical Journal E [105].

As mentioned in 3.6.3, the shape for the foam front propagation for a stratified reservoir is expected to develop concave regions. We have pointed out in 3.5.4 that the propagation of concavities requires special rules for their speed and have deduced a ‘correction factor’ \(1/\cos\theta/2\) that is applied to avoid concavities developing into undesired loops.

However, the propagation of these concave corners in the anisotropic case requires the speed to be adjusted in a different fashion (the \(1/\cos(\theta/2)\) speed up rule does not apply anymore). Details on the deduction of a suitable rule to be used in this case are given in the aforementioned manuscript [105], along with the mathematical analysis that this ‘new’ rule entails.

The equations for pressure-driven growth, modified to describe heterogeneity and anisotropy, are given in Chapter 6 and also in [105]. They are not presented in this section. Likewise, the formulae for the ‘correct’ propagation of corners are presented in this paper.

Therefore, according to the information provided in [105], we have solved the system numerically for the foam front propagation and here we present some graphs with results. In this way it is possible to compare the foam front shape that result when no modification to concavities is applied (Figure 3.10a) and
using the corresponding factor for the anisotropic system (Figure 3.10b). Details on the parameters are given in [105], but briefly we chose heterogeneity $J = 1 - k_s \sin(2\pi n_s Z_D)$ with $k_s = 0.3$ and $n_s = 3$ and with anisotropic permeability (i.e. permeability in the vertical being smaller than that in the horizontal by a factor 0.1).

These results show that, as when using a $1/\cos(\theta/2)$ speed up, it is possible to shift the propagation of concavities and change the obtained shape of the foam front, preventing the formation of spurious loops.

Therefore we can still use the pressure-driven growth model for the description of foam propagation within a stratified reservoir with anisotropic permeability, with some suitable modifications to the numerical scheme.

### 3.8 Conclusions

The preferred injection strategy for foam improved oil recovery can be modelled with pressure-driven growth. This is a simplified model for which the assumptions are based on fractional-flow theory. The model focusses on a small region of foam with low mobility, the foam front, that is generated from the injection of a slug of gas after the injection of surfactant solution.

The process is described with a system of differential equations that represent the change of position in function of time, where the front speed is proportional to the pressure difference between injection and hydrostatics and inversely proportional to the distance the front advances into the reservoir. To solve the system numerically the foam front is discretised.

Despite the simplicity of the model its numerical solution can present difficulties (i.e. singularities) that occur caused by the nature of the simplified model (which lacks any stabilising term). Information about the singularity of the model and how this affect the behaviour of the numerical solution is explained. We find that pressure-driven growth is a simplification of the viscous froth model where the term that represents surface tension has been discarded from the latter. This makes the
Figure 3.10: Foam front displacement for the anisotropic system (a) when no correction factor for the speed of concave corners are applied [105], (b) applying a suitable factor for the speed of concavities [105]. Heterogeneity and anisotropy parameters are chosen as discussed in the text.
numerical solution of pressure-driven growth subject to the influence of numerical artifacts that do not represent the physics of the process. Accordingly, the numerical algorithm needs to account for these features and include techniques to deal with them.

It is therefore attractive to investigate asymptotic solutions as these are unaffected by numerical methods; and additional to the derivation of the model, we have also presented some modifications to the algorithm to solve the system taking into account information obtained via the asymptotic solutions. We have obtained analytical and numerical results for the system describing the foam front motion, showing an agreement between them.

The shape of the foam front when described with pressure-driven growth is prone to develop undesired concave regions. It is then necessary to propagate concave corners with an apparent velocity different from that for material points. We then apply a speed up factor to concavities, derived on geometric grounds that makes their displacement to follow that of actual material points.

Once we derive a reliable algorithm for pressure-driven growth (being able to address some of the difficulties inherent to the numerical solution) we then proceed to explore different cases where it can be applied. We have introduced these cases. In addition we have provided information about one of these cases but with a particularity that is not covered more extensively in this document.

With this we finish the introduction and presentation of those important topics and results that are not covered in the publications for subsequent chapters. This will help to clarify the information that is not included in detail in the remainder of the thesis but is already available in the literature.
Chapter 4

Foam improved oil recovery: Foam front displacement in the presence of slumping

This Chapter is a copy of the article published in Colloids and Surfaces A: Physicochemical and Engineering Aspects, 473:123–132, 2015. Authors: Elizabeth Mas-Hernández, Paul Grassia, and Nima Shokri. The paper presents a case for foam displacement, using the pressure-driven growth model, when so called surfactant slumping occurs and its effect on the shape of the foam front is taken into account in the process. Analytical results are also presented.

For the displacement of oil with foam the injection strategy used consists in injecting surfactant solution followed by gas [100]. In this way the foam is formed inside the reservoir. There is a region of wet foam, which is also the interface between the aqueous phase ahead of it and the gas (dry foam) phase behind it. The model used for this injection technique, presented in Chapter 3, focuses on the wet foam region or foam front, and entails mobility in this zone being comparatively lower than at the surrounding liquid and gas phases [1].

Surfactant slumping is the migration of surfactant driven by gravity with respect to liquid free of the surface active component. It influences foam mobility due to the effect that phenomena such as drainage, coarsening and collapse, have on the bubble
films, which in turn leads to an increase in mobility in the top region of the front [16]. Higher mobility at the top implies a faster front motion. Therefore, slumping affects the foam front shape in such a way that concavities and even spurious loops (section 3.5.4) can develop on it. In this paper we present, precisely, the evolution of the front shape influenced by surfactant slumping in the context of the pressure-driven growth model.

In order to keep the mathematical description of surfactant slumping simple we make several assumptions. Namely, we specify the time at which slumping occurs and the vertical location for the front region that is slumped, and account for the increase in mobility by adding a factor to the differential equations for velocity components of the front motion.

The numerical solution implies finding the location of points for the discretised front over time. Details are given on the specific equations and parameters required, the numerical schemes used (specially to handle concavities), and their implementation on the computer program. As explained previously (in section 3.5.4), concave corners present challenges when solving the system for pressure-driven growth numerically.

The results show the formation of a sharp concave corner immediately after slumping takes effect. However with the implementation of numerical schemes to handle concavities [13], this corner stays contained and the formation of loops is prevented, slightly influencing the front shape at longer times. Nevertheless, a convex kink persists on the front.

In addition to numerics, we also perform a long-time asymptotic analysis, similar to the procedure followed for the homogeneous reservoir [13]; which gives nearly equivalent formulae as previously with the minor difference that they include parameters accounting for surfactant slumping. However, this solution represents neither the characteristic of the convex kink, which persists at the edge of the slumped zone, nor the front shape further below. Hence, a solution matching the front below the slumped region is sought. So, both solutions (above and below the convex kink) are combined to obtain the analytical representation of the entire front.
In summary, we present one of the cases of interest, surfactant slumping, mentioned previously (section 3.6) and introduced by [13]. Pressure-driven growth is only briefly introduced as it has been explained in detail in the same sources. Numerical and analytical results are obtained for the system of interest.
Foam improved oil recovery: Foam front displacement in the presence of slumping

Elizabeth Mas-Hernández, Paul Grassia *, Nima Shokri
CEAS, The Mill, The University of Manchester, Oxford Road, Manchester M13 9PL, UK

HIGHLIGHTS
- Shape of foam front during foam improved oil recovery is modelled.
- Foam more mobile at top as dense surfactant solution slumps downwards.
- Foam front develops transient concavity but this migrates toward bottom of the front.
- At long times, foam front actually develops convex kink.
- Apparent horizontal propagation velocity is uniform across convex kink.

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ABSTRACT
Foam is often used in improved oil recovery processes to displace oil from an underground reservoir. During the process, the reservoir is flooded with surfactant, and then gas is injected to produce foam in situ, with the foam front advancing through the reservoir. Here the effect of surfactant slumping (downward movement of surfactant in relation to a lighter phase) upon the advance of a foam front is presented. Slumping which can be associated with foam drainage, coarsening and collapse, causes a rise in mobility of the foam front specifically near the top of the front. The description of a foam front displacement for an initially homogeneous foam mobility is therefore modified to account for slumping-induced inhomogeneities. Numerical solution for the front shape shows that, although slumping transiently produces a localized concave region on the otherwise convex front, this concavity has little effect on the long term front evolution. In fact in the long-time limit, a convex kink develops on the front: an analytical solution describing the convex kink agrees very well with the numerics.

1. Introduction
Surfactant and gas are often injected into underground oil reservoirs so as to produce foam that can subsequently displace oil and achieve improved oil recovery: under these circumstances engineers wish to predict how the foam front advances through the reservoir displacing the oil. In the calculations of foam front movement through an oil reservoir an idealised model for so called surfactant-alternating-gas (SAG) injection [1,2], is used. The model is known as ‘pressure-driven growth’ [3].

In what follows we introduce the ‘pressure-driven growth’ model with a minimum of detail as it is already well discussed in literature [1–3]. However the appendix provides additional detail about the model for readers who require it.

During the surfactant-alternating-gas (SAG) process a foam bank advances into an oil reservoir that has already been flooded
with surfactant in aqueous solution (the so called liquid bank). The pressure driving the foam is the difference between the injection pressure and the hydrostatic pressure (the latter of course varying with depth). The pressure-driven growth model assumes (with some justification from so called fractional flow theory [1]) that most of the resistance to the foam motion arises from a region of wet foam located right at the foam front, and moreover this resistance grows according to the distance that the front has displaced (because the wet foam region gradually thickens over time). Balancing the driving pressure force with the resistance leads to a prediction for the speed of material points on the foam front, the direction of motion being normal to the front.

Mathematically the model consists of a system of differential equations for the motion of the material points, the equations being solved numerically [2], in our case with an algorithm programmed in Matlab.

The system of equations can be conveniently written in dimensionless form, using scales identified by [1] (see also Appendix A). The dimensionless equations describing the system (1)-(3) are shown as follows:

\[
\frac{dX_0}{dt} = \frac{(1 - Z_0) \cos \alpha}{s_0} \quad (1) \\
\frac{dZ_0}{dt} = \frac{(1 - Z_0) \sin \alpha}{s_0} \quad (2) \\
\frac{dz_d}{dt} = \sqrt{\left(\frac{dX_0}{dt}\right)^2 + \left(\frac{dZ_0}{dt}\right)^2} \quad (3)
\]

where \(X_0\) gives horizontal position in a rectangular reservoir, \(Z_0\) the vertical position (measured downwards from the top), \(t\) the time, \(s_0\) the distance the front travels, and \(\alpha\) the angle giving the front orientation as depicted in Fig. 1 (specifically \(\tan \alpha = -\frac{dZ_0}{dX_0}\), implying that Eqs. (1)-(3) are partial differential equations in space and time, rather than ordinary differential equations in time only). Note that \(Z_0 = 1\) corresponds to the point at which hydrostatic pressure balances injection pressure: the front cannot advance to depths beyond \(Z_0 = 1\).

The initial and boundary conditions are:

\[
X_0(Z_0, 0) = 0 \quad (4) \\
Z_0(Z_0, 0) = 0 \quad (5) \\
\alpha(0, t_0) = 0. \quad (6)
\]

In the implementation of the initial conditions in the computer program, those given by Eqs. (4) and (5) are changed to \(X_0(Z_0, 0) = \tilde{\alpha}(Z_0, 0) = 0\) where \(\tilde{\alpha}\) is a small parameter (typically chosen here to be \(\tilde{\alpha} = 0.001\)) to avoid having infinite values of \(dX_0/dt\) and \(dZ_0/dt\) at \(t_0 = 0\). Starting from a vertical front, for a homogeneous medium (i.e. a homogeneous reservoir, with in addition, foam mobility being homogeneous along the front), the system tends to give a convex shape for the front at finite time: the top of the front advances more than parts lower down because the difference between injection pressure and hydrostatic pressure is greater at the top.

Boundary condition [6], which according to the definition of angle \(\alpha\) says that material instantaneously at the top of the reservoir is moving parallel to the top, implies also that the top of the front has unit speed at the particular time when it has displaced by unit distance. The volume swept by the front grows over time, albeit this rate of sweeping volume slows down as time proceeds.

Numerical solution for pressure-driven growth, i.e. discretising the front shape and computing material points, is known to present challenges [2,3]. In our numerical implementation it is possible to trace the front displacement until large times, and still have a fair representation of its shape, because a rule has been implemented for subdividing front segments whenever they become too long, which occurs particularly near the top of the reservoir. Such a rule is given by Eq. (7), where the subscript 0 indicates a point at the top of the reservoir, the subscript 1 refers to the next point below the top, and \(s\) indicates a new point between these two.

\[
X_{0s} = X_{00} - \left(\frac{Z_{0s} - Z_{00}}{Z_{00} - Z_{01}}\right)^{3/2} (X_{00} - X_{01}), \quad (7)
\]

This equation respects a known mild singularity [3] for the front curvature at the top boundary: specifically it has been shown [3] that in the limit of very small \(Z_0\) values, the amount that \(X_0\) falls behind the leading edge at the top of the reservoir is proportional to \(Z_0^{-1/2}\), corresponding to a curvature scaling proportional to \(Z_0^{-1/2}\).

Further details about this singularity and why it arises are given in Appendix A.

As already mentioned above, the foam front (at least for a homogeneous system) is expected to have a convex shape. This is somewhat fortunate, because concavities are known to be extremely problematic when implementing pressure-driven growth [3]. Nevertheless, even small concavities in the shape of the curve caused by numerical artifacts (for example, truncation and/or round-off error accumulation) or by the physical nature of the system (e.g. if the medium is not perfectly homogeneous) could lead to a completely distorted shape of the curve, as concave points tend to fall increasingly far behind their neighbours [3].

The evolution of concavities over time, if they are not addressed, causes the formation of spurious loops [3] (regions where points of the front cross over each other). Spurious loops are generated whenever the points towards the back of a concavity are left behind but the points further ahead converge towards each other and cross over another. Points towards the back of a concavity can have their velocities corrected to avoid them falling behind (the corrected version represents the physics that we want the pressure-driven growth model to capture [3], specifically the backs of the concavities constitute ‘shocks’ which have a different speed from front material points which constitute ‘characteristics’ of the governing partial differential equations). The velocity correction does however cause the length of the intervals towards the back of the concavity to decrease over time, in turn requiring implementation of a rule to eliminate short intervals. For these reasons, it is necessary that the algorithm be able to deal with both concavities and shrinking intervals.

Our way to handle the presence of concavities in the front shape is, as mentioned above, by applying a correction in the velocity calculations (the correction being reflected in the time derivatives, Eqs. (1) and (2) in the original system) when certain criteria on the degree of concavity are met. This aims to speed up the displacement
of points in the concave regions to catch up with those in the convex region and in this way return towards a convex shape for the entire foam front (or, at the very least, to keep the concavity localised and contained). The corrected velocities are [3]

\[
\begin{align*}
\frac{dx_0}{dt_1} &= \frac{(1 - Z_0) \cos \alpha - t_0 \cos \theta/2}{(1/2)d_1 + (1/2)d_2} = n \\
\frac{dx_0}{dt_0} &= \frac{(1 - Z_0) \sin \alpha - t_0 \cos \theta/2}{(1/2)d_1 + (1/2)d_2} = n
\end{align*}
\]

where \(d_1\) and \(d_2\) are the length of those segments, and \(n\) the normal vector in direction of the front movement; again see Fig. 2.

Meanwhile a small finite angle (\(\theta_{\text{blow}}\), the sharpest turning angle we permit) is set and compared to \(\theta\). A previous study gave consideration to how values of \(\theta_{\text{blow}}\) might be chosen, suggesting that any choice of \(\theta_{\text{blow}}\) significantly less than unity (but larger than the dimensionless length of discretised elements on the front) would have the desired effect of keeping concavities localised and contained, and moving them in a way that was compatible with the surrounding convex parts of the front. The criteria for the ‘IF’ statement become as follows: if \(K < 0\) and \(\theta > \theta_{\text{blow}}\), then we use Eqs. (8) and (9) in lieu of Eqs. (1) and (2). We assumed \(\theta_{\text{blow}} = \pi/18\) here.

The above formulae and criteria are implemented in a new version of the ‘pressure-driven growth’ program that is able to deal with concavities (i.e., keep them localised and contained) and evolve back towards a predominately convex shape for the displacement front.

As mentioned above, there are cases, not solely arising from numerical error, but rather from the physics of the problem itself, where concavities are present. Such cases include [3] a heterogeneous reservoir (as alluded to above), increase of the injection pressure, and also so called liquid/surfactant slumping. The following sections deal with the case of liquid/surfactant slumping and the two other cases will be addressed in future work.

The remainder of this paper is laid out as follows. The physics of slumping is discussed in Section 2, and a mathematical model for slumping within the context of pressure-driven growth is presented in Section 3. Results are given in Section 4, and these are used as the basis to derive a long-time asymptotic description of the front shape resulting during the slumping process in Section 5. Conclusions are offered in Section 6.

2. Slumping

In the process of interest here, injection of foam in an oil reservoir, slumping (i.e. downward migration of a heavier phase relative to a lighter one) can refer to liquid or surfactant; in both cases it takes place due to downward flow (whether of liquid relative to gas, or surfactant solution relative to surfactant-free liquid) but can have slightly different effects on the foam front itself, which are described in more detail below (along with the root causes).

The displacement of the foam front over time is affected by drainage and coarsening/collapse of the bubbles that make up the front. The effect these processes have on the bubble size is what ultimately leads to the increase of mobility in the topmost region of the reservoir. It is important to note that the description of drainage and coarsening/collapse here is not sophisticated; it is used mainly to explain the increase in mobility, an increase which we will represent empirically. It is this mobility increase which then affects the front displacement. There are different factors that contribute to this mobility increase which are explained below.

The liquid in the films tends to move downwards due to the effect of drainage driven by gravity, because it is heavier than gas [4–6]. On the other hand, the gas inside the bubbles tends to migrate upwards. As a consequence, the bubbles at the top become drier and as time passes they collapse [4,6,7], leaving behind a foam with coarser bubbles. Meanwhile surfactant solution moving downwards relative to surfactant-free liquid makes the foam films less stable (i.e. more liable to collapse) at the top, again producing coarser bubbles. Again the downwards migration of surfactant solution is gravity-driven, based on the density difference with surfactant-free liquid, even though the actual density difference involved will be far less than that between liquid and gas. Thus both liquid and surfactant slumping will be the cause of foam collapse. However the mechanisms are different: liquid slumping affects stability by making foam become drier at the top, whilst surfactant slumping will make foam less stable (even if the liquid fraction is unchanged) [3].

Previous work, including experiments and simulations [7–10], show that vertical arrangements of foams have bigger and drier bubbles at the top and, smaller and wetter bubbles at the bottom. These experiments were performed for two-dimensional foams of various levels of polydispersity; despite this, the results are loosely comparable to our case of a thin foam front in porous media. In
summary, the phenomena of drainage and coarsening/collapse affect the texture of foams (i.e., bubble diameter and distribution of small and big bubbles within the foam). This texture however then influences mobility.

For foam flow through porous media, fewer films implies less dissipation. Bigger and drier bubbles at the top result in an increase of the mobility [11, 12]. For the model used here (pressure-driven growth), the mobility at the front itself is considered low compared to the water bank ahead and the gas bank behind the front where mobility has a much greater value [1]. However, owing to slumping, the mobility in the region of the top part of the front will be slightly larger than the front mobility lower down.

As a consequence of the increase in mobility in the top region of the reservoir, which only happen once slumping onsets after finite time, the shape of the curve representing the foam front need not remain convex. It was postulated [3] that instead concave regions can start to develop as shown schematically in Fig. 3, due to material on the front higher up starting to move substantially faster than material lower down.

In actual fact, liquid slumping is expected to occur after the contact between the liquid and gas phases, which is when the foam front has already passed a given point in the reservoir. This is merely saying that liquid only drains from foam once the foam itself is formed. Accordingly, liquid slumping strictly speaking needs to take into account the change in mobility behind the foam front [3]. Therefore, the model studied here, pressure-driven growth, is not suitable in this case because it focuses on the mobility of the foam front itself. Since the mobility of the foam bank away from the foam front is already assumed in our model to be exceedingly high compared to that of the front itself, a further increase in foam bank mobility due to slumping does not impact on the model.

Surfactant slumping, on the other hand, can take place before the front arrives at some specified point of the reservoir (and so, when the front subsequently arrives at that point, affects the texture of the foam as it is being produced right at the front). The foam front mobilities are included in our pressure-driven growth model and surfactant slumping affects them. Hence pressure-driven growth seems more convenient for modelling surfactant slumping, so that is the situation we envisage here.

For the description of slumping at the front, the velocity formulae for material points need to reflect the above mentioned feature [the increase of mobility at certain vertical positions near the top of the front]. The next section presents the pertinent equations for the slumping case and explains the approach to solve them numerically.

3. Model for slumping

To model the effects of slumping on the front, the system can be defined by Eqs. (12) and (13) for the horizontal and vertical components of the velocity

\[ \frac{dX_0}{d\tau} = \frac{M_{\text{slump}}(1 - Z_D) \cos \alpha}{Z_0} \]  
\[ \frac{dZ_0}{d\tau} = \frac{M_{\text{slump}}(1 - Z_D) \sin \alpha}{Z_0} \]

where \( X_0, Z_0, t_0, \alpha \) and \( \alpha \) are defined previously, and \( M_{\text{slump}} \) is the relative increase in mobility (compared to the case with no slumping).

The increase in mobility due to slumping is not expected to be significant at early times and it will predominately affect the top region of the reservoir, therefore Eqs. (6) and (7) take effect after certain time \( t_{\text{slump}} \) and only for vertical positions \( Z_0 < Z_{\text{slump}} \). Where \( Z_{\text{slump}} \) is a certain location that we specify. Otherwise, the original equations describing the model for the foam front displacement, (12) and (13), are used, supplemented by Eqs. (8) and (9) in the event that concavities appear.

The increase in mobility will also affect the values of \( X_0 \) and \( Z_0 \) at the top of the reservoir (for \( Z_0 = 0 \)). Therefore, from the boundary condition, Eq. (6), substituted into Eqs. (12)-(13), these values are given by Eq. (14) for \( t_0 > t_{\text{slump}} \):

\[ X_0(0, t) = Z_0(0, t) = \sqrt{2M_{\text{slump}}(Z_D - t_{\text{slump}}) + 2t_{\text{slump}} + \frac{Z_0^2}{Z_D}} \]

The program has been modified in order to incorporate this feature. The other modifications to the code to solve the system for slumping mainly affect the time derivatives of front positions, and intend to depict the slumping occurring after a certain time and the concave region developed as a consequence of this. It is necessary to define some additional parameters such as the values of \( M_{\text{slump}}, t_{\text{slump}}, Z_{\text{slump}} \), in order to determine the modified front shape.

Specifically, the program has been modified via the construction of the conditional for the calculation of time derivatives \( (dX_0/d\tau, dZ_0/d\tau, \text{etc.}) \), and concavities are handled by the sign of curvature and the magnitude of the turning angle \( \theta \). If \( \theta < \theta_{\text{slump}} \), the code checks for negative values of curvature and the magnitude of the turning angle between intervals \( (K < 0 \text{ and } \theta > \theta_{\text{slump}}) \) and uses either Eqs. (12) and (13) for convex regions or (8) and (9) if necessary (for concave regions).

Regarding the implementation of Eq. (14) for the top boundary, an ‘if’ construction has also been used. As mentioned previously, this equation is applied for \( t_0 > t_{\text{slump}} \). On the other hand when, \( t_0 < t_{\text{slump}} \), the right hand side of Eq. (14) is replaced by \( \sqrt{2Z_0 + \frac{Z_0^2}{Z_D}} \), which again is compatible with Eq. (6).

The following section gives results for the numerical solution of the slumping case.

4. Results

This section presents the results of modelling slumping specifically in the case where the concavities that can develop as a consequence of it are propagated at a higher speed (as per Eqs. (8) and (9)) than adjacent convex regions, in addition to slumped regions moving faster (as per Eqs. (12) and (13)) than unslumped ones.

The front was discretised into 100 segments, with values of angle \( \alpha \) determined for each segment. Time evolution was achieved via a Heun method: this is a predictor-corrector method where an
estimate is made of the solution at the end of a time step, and then that estimate is refined based on rates of change averaged at the beginning and end of the step. We used a time step $1 \times 10^{-3}$. New segments were added to the discretised front, whenever segment length increased by a factor of 0.25. Likewise segments were removed whenever their length shrank down to 0.002. The values of $\phi_0$ and $\theta_0$ were 0.001 and $\pi/18$ as mentioned previously. In addition, the following parameters (chosen arbitrarily to illustrate the model for slumping) are used: $M_{\text{slump}} = 2.5$, $t_{\text{slump}} = 0.5$, and $Z_{\text{slump}} = 0.25$.

We emphasise that these slumping parameters are arbitrary choices to elucidate the typical behaviour of the model. There are modifications we could make to the slumping model which would arguably make it more realistic. For example the parameter $Z_{\text{slump}}$ could be taken to be a function of time $t_D$ with $Z_{\text{slump}}$ growing according to the value of $t_D - t_{\text{slump}}$. Likewise the parameter $M_{\text{slump}}$ could be taken to be a function of $Z_{\text{slump}} - Z_D$, the value of $M_{\text{slump}}$ growing with $Z_{\text{slump}} - Z_D$. In the present work however, we consider the simple model for which all our slumping parameters are treated as constants.

Fig. 4 shows the plot obtained where a concave region tends to develop transiently on the front. However this concave region has a different form from what was postulated in Fig. 3. Specifically it does not remain fixed at $Z_D = Z_{\text{slump}}$ but instead migrates downwards. Despite this downward migration, the concavity itself remains contained within a section of the front of rather limited extent. Eventually the concavity approaches the bottom of the front at $Z_D = 1$.

This concave region then has little long term impact on the overall front shape for at least two reasons. Firstly, when $Z_D$ is close to 1 velocities of points on the front are very small (note that velocities vanish identically when $Z_D$ is exactly 1), so the front is barely advancing there. Secondly, when the concave region is close to the bottom of the front, the local front orientation is actually quite close to horizontal.

The significance of the local near horizontal orientation, is that concave points are higher up than convex ones. Concave point then have an ‘extra’ effect helping them to catch up with convex ones: not only do they benefit from the $\cos(\theta/2)$ factor in the denominator of Eqs. (8) and (9), but also from the fact that $1 - Z_D$ for concave regions (albeit being much smaller than unity) is actually slightly larger than for convex ones.

At longer times the concavity has little impact on the front shape (as we have stated), and for the overwhelming majority of $Z_D$ values, the front is convex. The really notable feature of the concave shape observed for later times however, is that it has a kink at position $Z_D = Z_{\text{slump}}$. In other words, quite differently from the speculations offered in [3] about possible concave kinks, we actually see a convex one.

This is suggestive of a long-time asymptotic state limit in which the entire front both above and below location $Z_{\text{slump}}$ propagates with an apparent horizontal velocity which is uniform at all spatial locations. The corresponding analysis is presented in the next section.

5. Analytical solution for slumping

Here the analysis of the system of Eqs. (12) and (13) for slumping and their matching onto Eqs. (1) and (2) lower down that can be considered valid for long times is presented; this is the long-time asymptotic state (front shape) where the whole front propagates at the same apparent velocity.

5.1. Analytical solution in the slump zone

At a large time $t_D \gg 1$, the top of the foam front has displaced, according to Eq. (14):

$$X_0 \equiv X_D = \sqrt{2M_{\text{slump}} t_D}$$

(15)

where the value of $x_0$ and $z_{\text{slump}}$ in Eq. (14) are negligible for long times. The approximation we make at long times [3] is that all points on the front (except those at $Z_D$ exceedingly close to unity) have displaced through nearly the same distance, as given by Eq. (15).

The long-time asymptotic solution for the slumping case follows:

$$\begin{align*}
\frac{dx_0}{dt} &= \frac{M_{\text{slump}} (1 - Z_D) \cos \alpha}{\sqrt{2M_{\text{slump}} t_D}} \quad (16) \\
\frac{dz_0}{dt} &= \frac{M_{\text{slump}} (1 - Z_D) \sin \alpha}{\sqrt{2M_{\text{slump}} t_D}} \quad (17)
\end{align*}$$

Defining $Y_0$ as $Y_0 = 1 - Z_D$ and a complementary angle $\alpha_c$ to $\alpha$, as shown in Fig. 5, Eqs. (16) and (17) are expressed as:

$$\frac{dx_0}{dt} = \frac{M_{\text{slump}} Y_0 \sin \alpha_c}{\sqrt{2M_{\text{slump}} t_D}}$$

(18)

$$\frac{dy_0}{dt} = \frac{M_{\text{slump}} Y_0 \cos \alpha_c}{\sqrt{2M_{\text{slump}} t_D}}$$

(19)

The similarity variable is defined as $\xi_{\text{slump}}$, and measures the displacement between a point of the front and the leading edge at the top of the front:

$$\xi_{\text{slump}} = X_D - \sqrt{2M_{\text{slump}} t_D}$$

(20)

A solution for $\xi_{\text{slump}}$ as a function of $Y_0$ is required.
Following an analysis as in the case of a homogeneous system [3], the derivative \(\frac{dY_0}{d\xi_{\text{slump}}}\) is

\[
\frac{dY_0}{d\xi_{\text{slump}}} = \frac{Y_0 \cos \alpha}{1 - Y_0 \sin \alpha},
\]

which is a similar expression as for the homogeneous case \(\frac{dY_0}{d\xi}\), where \(\xi\) is the displacement between a point \(X_0\) and the leading edge of the front without slumping. Using Eqs. (22) and (23) for \(\cos \alpha_c\) and \(\sin \alpha_c\),

\[
\cos \alpha_c = \frac{1}{\sqrt{1 + (\frac{dY_0}{d\xi_{\text{slump}}})^2}},
\]

\[
\sin \alpha_c = \frac{\frac{dY_0}{d\xi_{\text{slump}}}}{\sqrt{1 + (\frac{dY_0}{d\xi_{\text{slump}}})^2}}.
\]

Eq. (21) can be expressed as:

\[
\frac{dY_0}{d\xi_{\text{slump}}} = \frac{Y_0}{1 - Y_0} \frac{1}{\sqrt{1 + (\frac{dY_0}{d\xi_{\text{slump}}})^2}},
\]

the integral of which is:

\[
-\xi_{\text{slump}} = -\sqrt{1 - Y_0^2} \log \frac{1}{Y_0^2} + \log \left( \sqrt{1 - Y_0^2} + 1 \right). \tag{25}
\]

The format of this equation emphasizes that \(\xi_{\text{slump}}\) depends on both \(Y_0\) and \(Y_0 < 1\).

The equation for \(\xi_{\text{slump}}\) in terms of \(X_0\) is:

\[
\xi_{\text{slump}} = \sqrt{1 - (1 - Y_0^2)^2} + \log(1 - Y_0) - \log \left( \sqrt{1 - (1 - Y_0^2)^2} + 1 \right). \tag{26}
\]

Therefore, the similarity variable in terms of \(Z_D\) (or \(Y_0\)) is the same as in the case for a homogeneous system. What is different here, from the previous case, is the definition of \(\xi_{\text{slump}}\) in terms of \(X_0\) and \(t^0\) and \(M_{\text{slump}}\), given by Eq. (20).

The front shape obtained with the asymptotic solution, given the interval for values of the coordinate \(X_0\) and the final time required \(t^0\), is calculated using Eqs. (26) and (20). Fig. 6 depicts analytical results, for both cases with slumping and without slumping, for \(t^0 = 50\) (with \(M_{\text{slump}} = 2.5\) for slumping).

The asymptotic solution has the same shape for the case of a homogeneous system, the difference in this case is the definition of \(\xi_{\text{slump}}\) which is affected by the increase in mobility, thereby making the displacement of the front greater for the slumping case, i.e. the \(X_0\) values are larger.

Fig. 7 compares the asymptotic solution to the numerical solution. The analytical calculations use \(M_{\text{slump}} = 2.5\); regarding the numerical results, the program is used with \(M_{\text{slump}} = 2.5\), as well as \(t_{\text{slump}} = 0.5\), and \(Z_{\text{slump}} = 0.25\). Both cases are for \(t^0 = 50\).

The numerical data seem to match the analytical solution in the region affected by the increase in mobility \((0 < Z_D < Z_{\text{slump}})\) but has a sharp turn away from the analytical solution immediately below this region. However, the curves do not exactly match even in the region \(Z_D < Z_{\text{slump}}\), as can be seen in the inset in Fig. 7 (at least for \(t^0 = 50\)); this is because the terms containing \(t_{\text{slump}}\) are not taken into account in the calculation of the analytical solution leading to a displacement that is slightly faster for this case. This mismatch is however really very small (and arises due to our approximation that all points on the front have displaced by the same uniform amount \(\sqrt{2M_{\text{slump}}C_0}\)).

The feature of the sharp turn in the numerical solution is important. The sharp turn reflects a jump in the mobility occurring at \(Z_{\text{slump}}\). The change in mobility requires a change in the tangent angle \((\alpha)\) to maintain the apparent horizontal propagation velocity of the front as a whole. It is this property of the solution, which we can use to derive a new analytical solution in the domain \(Z_D > Z_{\text{slump}}\), that then agrees well with the numerical data.

### 5.2. Analytical solution in the zone that is not slumped

The condition that the front propagates at a uniform apparent horizontal velocity implies that

\[
\frac{u_{\text{perp}}}{\cos \alpha} = \frac{dX_0}{dt}, \tag{27}
\]

where \(u_{\text{perp}}\) is the perpendicular front speed and the right hand side of Eq. (27) is the speed of the leading edge at the top of the reservoir which follows from Eq. (14). Note that Eq. (27) applies both inside and outside the slump zone.

In the slump zone (in terms of \(Y_0\))

\[
u_{\text{perp}} = \frac{Y_0 M_{\text{slump}}}{X_0}. \tag{28}
\]

Substituting the expression from Eq. (27) into (28) and using (15) for the value of \(X_0\), we can deduce

\[
Y_0 = \cos \alpha, \tag{29}
\]

which in terms of \(\alpha_c\) is

\[
Y_0 = \sin \alpha_c. \tag{30}
\]

Recall that, \(u_{\text{perp}}/\sin \alpha_c\) must be continuous at \(X_0 = Z_{\text{slump}}\); however for the slumping case \(u_{\text{perp}}\) falls by a factor \(1/M_{\text{slump}}\) at that point and also \(\sin \alpha_c\) must fall by the same factor.

In fact for \(Z_D > Z_{\text{slump}}\), we obtain

\[
\sin \alpha_c = \frac{Y_0}{M_{\text{slump}}}. \tag{31}
\]
Substituting the value of \( \sin \alpha_i \) from Eq. (23) into (31) and rearranging, a formula for \( \tfrac{d\ell_{\text{slump}}}{dt} \) is obtained (now valid for \( \ell_0 > \ell_{\text{slump}} \)):

\[
\frac{d\ell_{\text{slump}}}{dt} = \sqrt{\frac{M_{\text{slump}}^2}{\ell_0} - 1}
\]

the solution of which is

\[
\ell_{\text{slump}} = \ell_{\text{match}} - M_{\text{slump}} \left( \frac{\ell_{\text{match}} - \ell_Y}{M_{\text{match}}} \right) \log \left( \frac{M_{\text{slump}}}{M_{\text{match}}} \right) - \ell_{\text{match}} \log \left( \frac{1 - \frac{\ell_{\text{match}}^2}{M_{\text{match}}}}{1 - \frac{\ell_{\text{slump}}^2}{M_{\text{slump}}}} \right)
\]

where \( \ell_{\text{match}} \) is obtained through Eq. (26) with \( \ell_0 = \ell_{\text{slump}} \).

Fig. 8 presents the comparison between the numerical results, obtained for \( \ell_0 = 50 \) with \( M_{\text{slump}} = 2.5, t_{\text{slump}} = 0.5 \) and \( \ell_{\text{slump}} = 0.25 \); and the curve obtained from the combination of Eqs. (26) and (33) for the analytical solution with \( M_{\text{slump}} = 2.5 \) and \( \ell_{\text{slump}} = 0.25 \).

Now with the correct equation to describe the interval \( \ell_0 > \ell_{\text{slump}} \), the analytical results follow the shape of the numerical data very closely. At \( \ell_0 = \ell_{\text{slump}} \) the analytical curve also presents the expected sharp turn. Note also that for small \( \ell_0 \) values the analytical predictions are at smaller \( \ell_0 \) values, i.e. higher \( Y_0 \) values than the numerical ones: this is a manifestation of the numerical solution approaching \( \ell_0 > 1 \) at \( Y_0 = 0 \), but the analytical solution only doing so asymptotically as \( \ell_0 \to \infty \). Despite the small discrepancy with the asymptotic results for \( \ell_0 \) near unity, the numerical solutions make it clear that we are dealing with a segment of front that is both very near horizontal at \( \ell_0 > 1 \) and very near stationary.

Considering \( \ell_0 \) values higher up, note also that even points located in the region which is not slumped have displaced further than would be the case with no slumping whatsoever (see Fig. 6). This is because points in the region which is not slumped historically have spent part of their evolution in the slump zone, where they were more mobile. In the model as formulated, material points lose mobility immediately once they leave the slump zone. Had we formulated the model in such a way that material points, once they entered a slump zone, retained high mobilities for all times thereafter, the advance of the front would be greater still [more akin to the analytic curve in Fig. 7].

Observe that compared to the unslumped case, the results for slumping in Fig. 8 indicate that the front is less uniform with depth, i.e. a greater area of the reservoir underneath the foam front (measured back from the leading edge at the top) remains unwept. Nevertheless the reservoir sweep indicated by Eq. (33) is still relatively efficient: Eq. (33) indicates a \( Y_0 \) value decaying towards zero over a characteristic horizontal distance \( M_{\text{slump}} \) back from the leading edge of the front, which is much less than the distance \( \sqrt{2M_{\text{slump}}^2} \) that the leading edge itself advances for large \( \ell_0 \).

The last section offers conclusions about the results presented so far.

6. Conclusions

We have presented numerical and analytical results for an idealised model of foam front propagation in an oil reservoir in the case of slumping. Specifically slumping (of surfactant) is assumed to enhance coarsening and collapse of bubbles near the top of the reservoir which will cause an increase in mobility in that zone.

This process is described by a model based on the case presented by Shun and Rosens (1) (the pressure-driven growth model), taking into account the feature of change in mobility. First of all, numerical data were obtained with the modified equations describing the foam front advance. From these results, an interesting behaviour was observed (specifically a convex kink at the height to which surfactant slumps), leading to the long-time asymptotic analysis of this case from which two expressions have been obtained and matched at the kink to describe surfactant slumping analytically.

Our results have addressed just one out of several open research questions that arise with the pressure-driven growth model: other open questions (that have already been identified by [3]) include what happens if there is an increase in the injection pressure during the course of the injection process, and moreover what happens if the reservoir itself is heterogeneous. As we mentioned in Section 1, such cases will be addressed in future work.

It is worth remembering also that pressure-driven growth is itself an idealised model, in the sense that it assigns finite mobilities only to the foam front, with mobilities assumed to be infinite everywhere else. We have managed to model surfactant slumping using pressure-driven growth, but liquid slumping (a related phenomenon, which tends to occur quite some distance behind the foam front, rather than at the foam front itself) needs more sophisticated models, which must be able to consider variation of mobilities throughout the whole reservoir.

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Appendix A. Background to the pressure-driven growth model

Here we give the background to the pressure-driven growth model. Full details can be found in the original references [1–3].

The discussion is laid out as follows. Appendix A.1 describes the assumptions underlying the model. Next Appendix A.2 gives the governing equations in dimensional form (those in the main text having been given in dimensionless form). Singularities in and stability of the solutions are considered in Appendix A.3 and Appendix A.4 respectively.
A.1. Assumptions underlying the pressure-driven growth model

We consider the surfactant alternating gas process in which an oil reservoir is flooded first with liquid surfactant solution and then gas is injected to form foam in situ. A foam front develops at the boundary between the liquid and gas, and migrates forward displacing the liquid with the ultimate aim of increasing the amount of oil that is recovered. Thus ahead of the moving foam front we have a liquid bank whereas behind the foam front we have a gas bank (or more precisely a foam bank).

There are advantages [1] to using foam to displace oil from a reservoir compared to other recovery strategies (e.g. injecting gas alone without surfactant and hence without foam). Whereas gas alone, which has a very low viscosity, tends to be subject to fingering instabilities (i.e. it finds preferential flow paths through the reservoir leaving significant amounts of liquid in place) this tends not to happen with foam. The reason fingering is suppressed is because the presence of foam films greatly augments the resistance to gas flow through the porous medium of the reservoir. Films can for instance block off certain pores to gas flow. Moreover if gas manages to flow along a pore spanned by a film, then the film must move also, and such film motion is surprisingly dissipative [13,14].

The foam improved oil recovery process involves multiphase flow in a porous medium, which is more complicated than single phase flow [15]. Conventionally a single phase flowing in a porous medium is described via Darcy’s law, such that fluid fluxes are proportional to the driving pressure gradient, proportional to the medium permeability and proportional to the mobility of the fluid (the reciprocal of its viscosity). The interstitial velocity of fluid through the medium can be deduced given the flux, assuming the medium porosity is known. In the case of multiphase flow, the motion of the different phases (e.g. liquid and gas) can be described once again by Darcy’s law but with correction factors (relative permeabilities or relative mobilities) representing the fact that for a given pressure gradient the flux of any given phase is less in a multiphase system than in a single phase one. Again interstitial velocities of phases can be obtained from the fluxes, given the medium porosity and given the fraction of the pore space filled by each phase.

In the case of interest here, we have a very particular multiphase system, with a foam bank and a liquid bank separated by a foam front. The presence of foam reduces the relative mobility of gas both in the foam bank and at the foam front, without of course affecting the liquid bank [1]. However the relative mobility for gas at the foam front is much lower than further back in the foam bank [1]. The reason is that further back the foam is drier, and since the films tend to collapse as the foam dries out, the number density of foam films is highest at the front, but substantially lower further back in the foam bank. Since it is the foam films that restrict gas motion, the relative mobility at the front is far lower than anywhere else in the system.

Accordingly Shan and Rosson [1] supposed that the entire pressure drop along the flow occurring just across the foam front, instead of pressure drops being distributed over the entire medium. This driving pressure drop is the difference between the injection pressure of the gas, and the hydrostatic pressure in the reservoir (the latter dependent on depth). The top of the foam front advances faster than the rest of the front as there is no opposing hydrostatic pressure at the top. Sufficiently deep down in the reservoir the gas cannot advance at all, because a point is reached where the hydrostatic pressure exactly balances the injection pressure.

In the above discussion we have not said anything about the structure of the foam front merely treating it as the interface between the foam bank and the liquid bank. In reality the foam front (itself containing a very large number of bubbles and films) has a finite thickness (up to the order of hundreds of metres whilst the liquid bank and foam banks and the reservoir itself extend much further, especially in the horizontal direction, for several kilometres [3]). The pressure gradient within the front is then finite, being the aforementioned pressure drop divided by the front thickness, with the direction of the pressure gradient vector (and hence the direction of the flow it induces) being normal to the front.

The front thickness is not constant over time. Fluid flow theory (see [1,16,17]) indicates that the forward boundary of the foam front moves at slightly higher speed than the rear boundary. This means that the foam front spreads by an amount proportional to the distance it has propagated (albeit the coefficient of proportionality is typically much smaller than unity). The proportionality coefficient depends on exactly where the low mobility foam front is considered to join onto the comparatively higher mobility foam bank. This is the case to the assumed bursting/collapse process of the foam as it dries out.

A.2. Governing equations in dimensional form

Taking into account the discussion above, governing equations for pressure-driven growth can be obtained, which are dimensional analogues of Eqs. (1) and (2). The equations are

\[
\frac{d\alpha}{dt} = \frac{k_{\alpha}}{\left(1 - S_w \phi \right)} \frac{g(\alpha_{\text{max}} - \alpha)}{\cos \alpha} \times \frac{\sin \alpha}{(A.1)}
\]

\[
\text{and} \quad \frac{d\alpha}{dt} = \frac{k_{\alpha}}{\left(1 - S_w \phi \right)} \frac{g(\alpha_{\text{max}} - \alpha) \sin \alpha}{(A.2)}
\]

where \(\alpha\) is dimensional horizontal coordinate, \(Z\) is dimensional vertical coordinate (measured downwards), \(\alpha\) is dimensional distance that the front has travelled, and \(t\) is dimensional time. Meanwhile \(k\) is the reservoir permeability, \(s_r\) is the gas relative mobility at the foam front (relative mobility being the reciprocal of an effective viscosity), \(S_o\) is the volume fraction of liquid in the foam, \(\phi\) is the porosity of the reservoir, \(\rho\) is the density difference between the aqueous liquid region (ahead of the foam) and the gas within the foam, and \(g\) is gravity. Moreover \(\alpha_{\text{max}}\) is the maximum depth to which foam can penetrate, which for a pressure \(P_{\text{max}}\), driving the injection, is determined by \(\alpha_{\text{max}} = P_{\text{max}}[\rho g];\). Additionally \(\tau\) is the proportionality ratio between the thickness of the wet foam region and the distance that the front has travelled, which by assumption is a small parameter.

Distances are made dimensionless on the scale \(\alpha_{\text{max}}\) whilst times are made dimensionless on the scale \(\tau = \frac{1 - S_w \phi \alpha_{\text{max}}}{P_{\text{max}}} \frac{1}{\rho g}. \quad (A.3)\)

We can give an estimate of these scales as follows. With a driving pressure \(P_{\text{max}}\) of 2.4 ± 10^1 Pa given by [1], a density difference \(\varrho\) of order 1000 kg m^-3, and gravity \(g\) of 0.8 m s^-2, we obtain \(\alpha_{\text{max}} = 265\) m. Later work by [2] obtained \(\alpha_{\text{max}} = 2200\) m but for a substantially higher driving pressure. We are interested moreover in horizontal distance scales rather larger than these vertical ones. Meanwhile for a permeability of \(k\) of 2.95 × 10^-13 m, and a relative mobility of 243 Pa^-1 s^-1 (both values given by [1]), assuming a low liquid fraction \(S_o\) in the foam rather less than unity, adopting the value \(\alpha_{\text{max}} = 265\) m given here, and estimating \(\tau = \alpha_{\text{max}}/\alpha_{\text{max}}\) as in [3], we can compute \(\tau_{\text{scale}}\) as being on the order of 11 days.

A.3. Singularities in solutions of the pressure-driven growth model

Although the governing Eqs. (A.1) and (A.2) appear comparatively simple, they have been shown to be very challenging to solve numerically with certain numerical schemes proving susceptible to instabilities [2,3]. The underlying reasons for this have been studied by [3] who showed that the solutions of the equations themselves
admit singularities. The most severe type of singularity was found to occur when a front contained a concavity (as seen from the liquid bank). The concavity evolved over time in such a way as to focus down to a point. Once that happened, two otherwise convex sections of front would meet at concave corner. This represented a singularity because the tangent to the front underwent a jump (i.e. a change in direction) at the corner.

The situations of main interest studied by [3] actually involved convex shapes, rather than concave ones, making these ‘sharp corner’ singularities less prevalent. Those solutions still exhibited other types of singularities however, in particular at the top boundary of the foam front. Specifically if one moved a small (dimensionless) depth $Z_0$ vertically into the reservoir, the horizontal location of the front moved a distance on the order of $Z_0^{1/2}$ behind the horizontal location of the leading edge at the top of the front. The implication is that the curvature at the top of the front diverges near the top like $Z_0^{-1/2}$.

The detailed mathematical derivation of this singularity is given in [3], but the physical reason it appears can be explained as follows.

The top of the front must be vertical to ensure that the motion there is parallel to the top boundary of the reservoir. The curvature near the top of the front defines on purely geometric grounds, the horizontal displacement between a point slightly below the top of the front and the leading edge at the top.

There is however a second way to compute this horizontal displacement which needs to be compatible with the first. Material points near the top of the front move primarily horizontally, but also have slight tendency to migrate downwards. The more sharply the front curves near the top, i.e. the more quickly the normal to the front acquires a significant component along the vertical, the more rapid this downwards migration becomes. The horizontal velocity component of front material points falls as these points migrate downwards, primarily because the pressures driving these points are opposed by higher hydrostatic pressures. The horizontal displacement between a material point slightly below the top of the front and the leading edge at the top is then given by the deficit in horizontal velocity component integrated over the time that is required to migrate to a given depth.

Consider for example a hypothetical case for which the front is assumed to curve only extremely weakly near the top and thus is almost perfectly vertical. On geometric grounds, front points near the top then are displaced horizontally hardly at all with respect to the leading edge at the top. However front points also have exceedingly slow downward migration (as the vertical component of the front normal is extremely small). This however leads to a contradiction: front points below the top have taken an arbitrarily long time to migrate downwards to their current location and so, during this migration, should have fallen an arbitrarily long distance behind the leading edge. This is incompatible with the edge being almost perfectly vertical.

In fact the same argument applies for any finite curvature at the top. As long as curvature is finite, points arbitrarily close to the top can be made to start their vertical migration arbitrarily slowly. Hence points at given distance below the top still take an arbitrarily long time to migrate to their current location and so should have fallen an arbitrarily long distance behind the leading edge. This is incompatible with a finite curvature.

By introducing a weak singularity in curvature at the top boundary such that the horizontal displacement of the front relative to the leading edge grows like the $3/2$ power of depth, we ensure that material points released exactly at the top boundary can migrate a finite distance vertically away from the top boundary during a finite time. During this finite time, the cumulative amount that they fall behind the leading edge (owing to the deficit in their horizontal velocity component) exactly matches the assumed curve shape.

The fact that the singularity in curvature allows a point released from the top boundary to migrate vertically away from the top boundary comparatively rapidly (compared to a case with finite curvature) has implications for a numerical scheme [3]. It becomes necessary to subdivide the topmost numerical interval very often, and the way that subdivision is done should reflect the known singular variation of $X_0$ with $Z_0$ that applies there.

The above analysis applies to the case of a front which is infinitesimally thin. In case of a front of small but finite thickness the curvature singularity is relaxed: the front cannot curve more sharply than its finite thickness permits. Despite the finite curvature, the paradox of points at depth appearing to fall arbitrarily far behind the leading edge no longer arises: the velocity of the top of a finite thickness front becomes weakly sensitive to the curvature suppressing the otherwise problematic velocity deficit [3].

A.4. Stability of solutions of the pressure-driven growth model

The reason why the pressure-driven growth model is challenging to solve numerically, and likewise the reason why it admits the singular solutions referred to above, is that the model itself is ‘neutrally stable’, right on the borderline of turning unstable. The model per se does not assign any ‘energy cost’ to either sharp corners or diverging curvatures on fronts, which is why these are admitted within solutions.

Grassia et al. [3] have discussed stability of the model at length. The governing equations here are hyperbolic partial differential equations [18]. They are advection equations, such that disturbances to the front shapes tend in the first instance to be advected along, instead of either growing or decaying exponentially.

It is possible to stabilise the equations by adding a weak diffusive term, and Grassia et al. [3] did this. The resulting equations are then advection-diffusion equations: these are of parabolic type [18]. The diffusive term even has a physical interpretation [3]: the foam front is an object of small but finite thickness, and the diffusive term prevents it from being bent into a radius of curvature smaller than its thickness. What the diffusion term does is relax the concave corners produced by the original ‘non-diffusive’ advection equation replacing them by concave regions of large but finite curvature. It also relaxes the convex singularity (where curvature diverges) on the top boundary, again making it into a region of large but finite curvature. These are effectively ‘curvature boundary layers’ (a concept discussed by [19]) and they match onto regions of much lower curvature where diffusive terms are not required.

The parabolic advection-diffusion equations whilst definitely stable are stiff numerically [3]. One must resolve the dynamics on the fine scale of the curvature boundary layers, even though the results of interest are on the much larger scale of the entire reservoir. Grassia et al. [3] therefore explored solution methods for the original advection equations, extracting the foam front shapes of interest outside the curvature boundary layers, and keeping the curvature boundary layers contained or localised at isolated points without needing to resolve their structure.

As we are now back to dealing with hyperbolic partial differential equations, the solutions exhibit the mathematical features known to occur for that class of equations, namely characteristics (front material points correspond to characteristics), discontinuous shocks (concave corners are shocks) and fans (the convex curvature singularity at the top of the foam front is a fan) [3]. An important contribution of Grassia et al. [3] was to determine how to propagate the shocks (i.e. concavities), which turn out to have higher velocity than the characteristics (front material points). The sharper the corner, the faster it must propagate.

The equations are neutrally stable in the sense that a small disturbance to the shape of the foam front exhibits neither exponential
growth nor exponential decay. This does not imply that such disturbances do not evolve at all: in fact disturbances are advected along fronts and evolve as the front geometry evolves [3]. Disturbances in the fan region can spread along the front as the fan itself spreads out. Disturbances can also evolve into concave shocks themselves. Information is lost at the concave shocks since material points are consumed there. A disturbance located close by to a shock might well be consumed by a shock and thereby cease to exist.

The overall tendency however is for both material points and shocks to migrate over time to the bottom of the solution domain. Driving pressures are balanced by hydrostatic pressures at the bottom of the domain so that the front is horizontal and stationary there. All disturbances to the front eventually reach this ‘dead zone’ and do not impact thereafter on the front shape.

References

Chapter 5

Foam improved oil recovery: Modelling the effect of an increase in injection pressure

This Chapter is a copy of the article published in The European Physical Journal E: Soft Matter and Biological Physics, 38:67, 2015. Authors: Elizabeth Mas-Hernández, Paul Grassia, and Nima Shokri. The paper is about the theoretical study of a foam front displacement for improved oil recovery, using the pressure-driven growth model, when there is an abrupt increase in the injection pressure.

It has been introduced in previous chapters (1, 2) that foams are a good choice as the displacing fluid in tertiary oil recovery processes. Furthermore, the preferred injection strategy is introducing surfactant solution followed by gas (SAG); where a foam front is formed and it displaces the oil left inside the reservoir [1].

When trying to maintain the production rate during the SAG process, it might be necessary to adjust it (e.g. quickly increasing the injection pressure at a certain point). Otherwise, the speed of the front motion will decrease.

We have been studying modelling and simulation of the foam propagation for oil recovery processes. Specifically, we are able to follow the evolution of the foam
front shape over time. In this case, we also want to know what the consequences of this sudden increase in pressure are. It is important to bear in mind that this topic is part of the specific cases of interest within the processes of foam-improved oil recovery, already introduced in chapter 3 and in literature [13]. Accordingly, we intend to extend the scope of our base model (pressure-driven growth), applying it in the present case too.

The influence that the change in pressure is expected to have on the shape of the front is that it will lead to the development of concave regions. This is illustrated schematically in Figure 5.1. It has been widely explained (section 3.5.4) that concavities are difficult to manage within the pressure-driven growth model. However it has also been shown that it is possible to handle these with appropriate numerical schemes [13].

Therefore, the model is applied in the present case, keeping in mind the modifications it needs to maintain singularities (i.e. concavities) contained. The terms for additional parameters to describe the influence of a rise in pressure and how they are incorporated into the equations for this particular case are given. The outcomes from the numerical solution show the development of a concavity as a
result of the pressure raise but that migrates to the bottom of the front over time. Later on, the entire foam front shape evolves to a convex curve.

An asymptotic solution that describes the convex foam front at longer times is obtained, from which it is possible to calculate the area left unswept by foam. The expression that describes unswept area is given in [13] and later in this Chapter (equation (20) in the paper); which is a function of the similarity variable $\xi_p$, as depicted in Figure 5.2. Two cases are presented according to the depth to which the foam penetrates in the reservoir. Furthermore, it is possible to analyse how the area left unswept changes as a function of the parameter for the relative increase in pressure. Thus, unswept area is calculated with numerical integration and with analytical formulae.

Results from the computation of the area left unswept by the foam displacement show that more increase in pressure gives a better sweep efficiency (the unswept area is smaller).
Foam-improved oil recovery: Modelling the effect of an increase in injection pressure

Elizabeth Mas Hernández1, Paul Grassia2,3,*, and Nima Shokri1

1 CEAS, The Mill, University of Manchester, Oxford Rd, Manchester M13 9PL, UK
2 Chemical & Process Engng, University of Strathclyde, James Weir Bldg, 75 Montrose St, Glasgow G1 1XJ, UK
3 Ciencias Matemáticas y Físicas, Universidad Católica de Temuco, Rudecindo Ortega 02950, Temuco, Chile

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Abstract. A model, called pressure-driven growth, is analysed for propagation of a foam front through an oil reservoir during improved oil recovery using foam. Numerical simulations of the model predict, not only the distance over which the foam front propagates, but also the instantaneous front shape. A particular case is studied here in which the pressure used to drive the foam along is suddenly increased at a certain point in time. This transiently produces a concave front shape (seen from the domain ahead of the front): such concavities are known to be delicate to handle numerically. As time proceeds however, the front evolves back towards a convex shape, and this can be predicted by a long-time asymptotic analysis of the model.

The increase in driving pressure is shown to be beneficial to the improved oil recovery process, because it gives a more uniform sweep of the oil reservoir by the foam.

1 Background

When extracting oil from an underground reservoir, the amount of oil recovered can be increased by injecting a displacing fluid into the reservoir under pressure. Foam is known to have advantages over other choices of displacing fluid (e.g. gas injection or water injection) since foam tends to give a more even sweep/displacement [1–3]. Rather than direct injection of foam into an oil reservoir, improved oil recovery often uses the so-called surfactant solution, e.g. gas injection or water injection) since foam tends to give a more even sweep/displacement [1–3].

Typically, the thickness of this wet foam front is much less than the distance over which it propagates. Moreover most of the dissipative resistance to the foam motion is known to be contained in this wet foam region [2].

Shan and Rossen [2] proposed a simple idealised model for the foam motion where all the resistance is assumed to be contained in the wet foam region. This model has been termed “pressure-driven growth” [5] and it allows the prediction of how the shape of the wet foam front evolves over time [6]. The model (in its simplest form) assumes that an element of foam front moves in a direction that is normal to the front and at a speed that is proportional to the difference between the driving injection pressure and the background hydrostatic pressure in the reservoir, the latter increasing with depth.

This implies that the speed of the front decreases with increasing depth. The dissipative wet foam zone is also known [2,5,7] to spread out proportionally to the distance the front travels (albeit with a coefficient of proportionality much smaller than unity). This leads to further reductions in the speed of the foam front over time.

The model is most compactly described in dimensionless position and time variables (denoted in what follows via a subscript “D”). Specifically length scales are made dimensionless [2,5] by dividing through by $P_{\text{drive}}/(\Delta \rho g)$ where $P_{\text{drive}}$ is the pressure initially used to drive the foam along, $\Delta \rho$ is the density difference between the injected surfactant solution and the injected gas, and $g$ is the acceleration due to gravity. Meanwhile time scales are made dimensionless [2,5] by dividing through by the quantity $(1 - S_w) \phi P_{\text{drive}}/k \lambda_0 \Delta \rho^2 g^2$, where $S_w$ is the liquid fraction in the foam (generally considerably smaller than unity), $\phi$ is the reservoir porosity, $\tau$ is the ratio between the thickness of the wet foam front and the distance over which it propagates ($\tau$ is expected to be nearly constant [2]), $k$ is the reservoir permeability, and $\lambda_0$ is the relative mobility of the wet foam front. In the “typical” example presented by Shan and Rossen [2], 1 unit of dimensionless distance corresponds to around 300 m and 1 unit of dimensionless time corresponds to 14 days. In dimensionless form, the governing system of differential equations (1)–(5) for this process known as pressure-driven...
growth describes the evolution of the foam front inside a homogeneous and isotropic reservoir (the simplest case) during improved oil recovery [2, 5]:

\[
\frac{dX_D}{dt_D} = \frac{(1-Z_D)\cos\alpha}{s_D}, \tag{1}
\]

\[
\frac{dZ_D}{dt_D} = \frac{(1-Z_D)\sin\alpha}{s_D}, \tag{2}
\]

\[
ds_D/dt_D = \sqrt{(dX_D/dt_D)^2 + (dZ_D/dt_D)^2}, \tag{3}
\]

\[
X_D(Z_D,0) = s_D(Z_D,0) = s_{D_0}, \tag{4}
\]

\[
\alpha(0, t_D) = 0, \tag{5}
\]

which are actually equivalent to eqs. (2.10)–(2.11), (2.13) and (2.18) of [5]. Here, \(X_D\) is the horizontal position in a rectangular reservoir, \(Z_D\) the vertical position downward, \(t_D\) is the time, \(s_D\) the distance that a material point on the front travels, \(s_{D_0}\) is a value much smaller than unity that we chose as \(s_{D_0} = 0.001\) (which can be taken to represent the initial thickness of the wet foam front relative to its initial length), and \(\alpha\) the angle giving the orientation of the front normal with respect to the horizontal (which can be determined given the position of a material point on the front and its neighbours). Note that whereas eqs. (1)–(3) apply to front material points, specifying \(Z_D\) at any given \(t_D\) is sufficient to fix a material point, so that \(X_D\) and \(s_D\) can alternatively be written as functions of the two arguments \(Z_D\) and \(t_D\), and this is reflected in the notation used in eqs. (4), (5).

Equations (1), (2) are specific to the case of a unit driving pressure (in dimensionless units) corresponding to a front that penetrates to a unit depth below the surface: in dimensional units this corresponds to the depth that matches the injection pressure to the hydrostatic pressure in the system. Note that, by design, the driving pressure is chosen such that the penetration depth for the foam front is at least as large as the depth at which the oil is located. There can however [6] be gains in choosing a penetration depth substantially larger than that (corresponding then to the oil being located in just a small fraction of one dimensionless depth unit). One advantage of this is that (upon converting back to dimensional variables) the distance over which the front propagates is greater at any given time. Another advantage of choosing a larger penetration depth is that it reduces the amount of the so-called “gravity override” [2]. If the front is highly inclined across the region where the oil is located, then the leading edge of the front has run far ahead other parts of the front that are still sweeping oil: in effect some oil is being left behind. However if the driving pressure is higher and the front is close to vertical across the region where the oil is located, the sweep is more uniform and hence considered to be more efficient.

Numerical and analytical solutions for eqs. (1)–(5) give a front described by convex curves in a two-dimensional \(X_D\) vs. \(Z_D\) plot. This is fortunate because the equations governing pressure-driven growth are well behaved in the case of convex shapes, whereas to handle concave shapes [5], special strategies need to be put in place for solving the equations. Unfortunately there are a number of scenarios (generalisations of the simple homogeneous, isotropic, unit driving pressure case that is discussed above) for which the foam front will no longer be convex, instead concavities will appear [5, 8]. Amongst these is the case of a sudden rise in the driving pressure which is what is considered here.

The remainder of this study is laid out as follows. Section 2 introduces the case where the driving pressure is variable, presenting the equations that are used, along with explanation about how those equations are implemented in a computer program. Section 3 gives more detail on the parameter values used in the numerical solution and the results obtained for the foam front displacement. Section 4 presents an analytical solution for the foam front shape at long time, and quantifies how this shape changes in response to changing pressure. Finally, sect. 5 offers conclusions.

2 Increase in pressure

The whole idea of improved oil recovery processes is maintaining the pressure necessary for production in the reservoir. Unfortunately the production rate is not constant over the course of the process. In particular (as was mentioned earlier) the dissipative wet foam front spreads out over time [2, 5, 7], albeit still maintaining a thickness that is smaller than the distance over which the front propagates. If the driving pressure is held fixed as the dissipative wet foam front thickens, the front propagation speed must fall. A production engineer could respond to this falling rate by raising the driving pressure\(^1\). Thus, after some time during the displacement of a foam front, a higher pressure than the one used at the start of the operation might be required. The higher pressure may also have the additional benefit of reducing the amount of gravity override, bringing the foam front closer to vertical across the region where the oil is located.

Here we consider how this increase in the driving pressure will affect the shape of the foam front that has been described above. As has been postulated in [5], we expect that the pressure increase will introduce concavities on the curves as shown schematically in fig. 1. Such concavities are known to be problematic for implementation of the pressure-driven growth model and special techniques are required to handle them [5]. Basically the concavity (once it becomes sufficiently sharp) needs to be propagated at a higher velocity than surrounding parts of the front to avoid the appearance of spurious loops in the front shape.

The equations describing horizontal and vertical front velocity must moreover include a term for the relative pressure increase \(P_i\), that is imposed after some time that

\(^1\) Shan and Rosen [2] in fact studied some cases where pressure was continually increased over time to hold injection rate fixed—see figs. 24(a) and 25(a) in the cited reference.
Fig. 1. Schematic of foam front shape in coordinates $X_D$ vs. $Z_D$ with an increase in driving pressure. For a front material point, both $X_D$ and $Z_D$ are functions of time $t_D$. At a certain time $t_p$ pressure is raised so as to become $1 + P_i$ times the original pressure.

we specify as $t_p$. The modified equations are shown below

\[
\frac{dX_D}{dt_D} = \frac{(1 + P_i - Z_D) \cos \alpha}{s_D}, \quad \frac{dZ_D}{dt_D} = \frac{(1 + P_i - Z_D) \sin \alpha}{s_D},
\]

where the other variables have been described previously in sect. 1. The $X_D$ and $s_D$ values at the top of the front are given by the following eq. (8), which is obtained with the aid of the boundary condition, eq. (5),

\[
X_D(0, t_D) = s_D(0, t_D) = \sqrt{2(1 + P_i)(t_D - t_p) + 2t_p + s_D^2}.
\]

As depicted in fig. 1, the pressure increase will also affect the depth to which the foam front extends, for this reason it is necessary to give initial conditions for this region, $1 < Z_D < 1 + P_i$ (recall that the maximum vertical dimensionless distance has been unity up till now). Hence, initial conditions for this vertical interval will satisfy $X_D = s_D = s_{Dp}$ at $t_D = t_p$; and from there we will follow the advance of the front.

This case has been implemented numerically with Mat-

lab via a discretised representation of the foam front. Here we describe the changes that have been done to the computer code used originally for a constant driving pressure (the algorithm for the constant pressure case having been described already in literature [5]). First of all, it is necessary to specify the values of the new parameters $t_p$ and $P_i$. Then, new variables (i.e. points on the foam front) used in calculations are set for vertical positions $1 < Z_D < 1 + P_i$ at time $t_p$. Discrete positions of points $(X_D, Z_D)$ are used to obtain segment orientations (angle $\alpha$), and then it is possible to calculate velocities for each point.

Calculation of the speed of the front advance is per-

formed using a conditional construction in the following way: if $t_D < t_p$, eqs. (1), (2) are used; on the other hand, if $t_D \geq t_p$, the code checks for the sign of front curvature ($K$) and magnitude of turning angle ($\theta$) between adjacent segments and applies either eqs. (6), (7) for $K > 0$ and/or $\theta$ less than a specific value (specified here as $\theta_{\text{sharp}}$), or the following eqs. (9), (10) for $K < 0$ and $\theta > \theta_{\text{sharp}}$:

\[
\frac{dX_D}{dt_D} = \frac{(1 + P_i - Z_D) \cos \alpha}{s_D \cos \frac{\theta}{2}}, \quad \frac{dZ_D}{dt_D} = \frac{(1 + P_i - Z_D) \sin \alpha}{s_D \cos \frac{\theta}{2}},
\]

where the additional term in the denominator, compared to (6), (7), speeds up the displacement of concave regions correcting the velocity of points there. The reasons for needing to do this are explained at length in [5]; physically the reason is that front material points travel at different speed from “shocks”, i.e. corners or cusps (into which concave regions develop). Such “shocks” arise be-

cause the pressure-driven growth is a singular limit of a more general model called the “viscous froth”\(^3\) which is well known in the foam physics literature [9,10]. Note the the corner shown on fig. 1, which joins points originally in $Z_D < 1$ to points newly set in motion in $Z_D > 1$ will always be propagated via eqs. (9), (10) (at least for any reasonable choice of the value of $\theta_{\text{sharp}}$). Careful consider-

ation has been given in the literature regarding how to choose values of $\theta_{\text{sharp}}$, with any value chosen significantly smaller than unity being deemed suitable for keeping otherwise problematic concavities localised and contained [5].

The values of $X_D$ and $s_D$ at the top are calculated with a conditional construction also, where eq. (8) is used if $t_D \geq t_p$, but $X_D(0, t_D) = \sqrt{2t_D + s_D^2}$ applies if $t_D < t_p$.

The next section presents results obtained from these modifications to the program.

3 Results

For the numerical solution of the displacement of the foam for increase in the driving pressure at a certain $t_D = t_p$, a Heun method is used. The front is discretised by setting 200 intervals along the ZD axis, the time step used is cho-

\(^2\) Determining curvature and angle between segments is nec-

essary for handling concavities: see [5] for more details on this mat-

\(^3\) Realising the analogy between pressure-driven growth and viscous froth is non-trivial [5], as one model describes the ad-

vance of an entire foam front on a reservoir engineering scale

and the other describes individual foam films on a much smaller

scale.
curves at the top), short segments are removed when they decrease to a length of 0.002 (helping to avoid formation of spurious loops \([5]\)). \( \theta_{\text{spury}} = \frac{\pi}{18} \), \( t_D = 1 \) and \( P_i = 0.2 \): these values have been chosen arbitrarily for the purpose of illustrating the model.

The foam front obtained with the above-mentioned values is presented in fig. 2: it is observed that the pressure increase leads to formation of concavities. There is a clear concave corner or cusp, separating points that have been moving continuously since the start of the process from those that have only newly been set in motion at time \( t_p \). These results also confirm a suggestion by Grassia et al. \([5]\) that points in the proximity of the cusp (formed due to a sudden increase in pressure) could have negative curvature\(^4\). In fact, some points above and to the right of the cusp have historically displaced less distance than points further to the right. Since the thickness of (and hence the dissipation of) the wet foam front grows according to the displacement from the topmost point of the cusp (formed due to a sudden increase in pressure) could have negative curvature\(^4\). The reason why those negative curvature points appear is because points immediately to the right of the cusp have historically displaced less distance than points further to the right. Since the thickness of (and hence the dissipation of) the wet foam front grows according to the displacement, points immediately to the right of the cusp move faster than those further to the right (owing to the 1/\( s_D \) factor in eqs. (6), (7)); this then produces negative curvature (transiently). Subsequently, these negative regions eventually focus down into the cusp itself.

Figure 3 presents data for a longer \( t_D \) interval, where it is shown that after a while the concavity becomes less significant and indeed (after \( t_D \approx 3 \) in this case) the concavity is pushed virtually to the bottom of the front where the front is barely advancing any more, and so the shape is dominated by the convex part of the front higher up, which corroborates the findings presented in \([8]\).

As we have seen, over time concavities tend to be pushed right to the bottom of the solution domain, and the foam front is dominated by a convex front shape higher up.

It is possible to obtain an analytical solution for the (convex) foam front through an asymptotic analysis, which is presented in the following section.

4 Asymptotic analysis

Here we present the asymptotic analysis of the system of eqs. (6), (7) which gives the shape of the foam front for long times.

We define

\[
\xi_p = X_D - X_D(0, t_D), \tag{11}
\]

where \( \xi_p \) is the displacement from the topmost point of the front to an arbitrarily chosen point \( X_D \). The subscript “\( p \)” on \( \xi_p \) is to remind us that the solutions depend on driving pressure. By definition \( \xi_p \) is negative, since the topmost point displaces the farthest.

At long times, \( t_D \gg 1 \), the value for \( X_D(0, t_D) \) and \( s_D(0, t_D) \) given by eq. (8), can be simplified to

\[
X_D(0, t_D) = s_D(0, t_D) \approx \sqrt{2(1 + P_i) t_D}, \tag{12}
\]

where the terms \( t_p \) and \( s_D \), in eq. (8) are now negligible. Therefore, eqs. (6), (7) and (11) can be written as

\[
\frac{dX_D}{dt_D} = \frac{1 + P_i - Z_D}{\sqrt{2(1 + P_i) t_D}} \cos \alpha, \tag{13}
\]

\[
\frac{dZ_D}{dt_D} = \frac{1 + P_i - Z_D}{\sqrt{2(1 + P_i) t_D}} \sin \alpha, \tag{14}
\]

\[
\xi_p = X_D - \sqrt{2(1 + P_i) t_D}, \tag{15}
\]

where we have introduced an approximation \([5]\) that (for the purposes of computing velocities) at leading order, all points on the foam front displaced through the same distance \( s_D(0, t_D) \) as the topmost point.
which, in turn, gives
\[ -\frac{\xi_p}{1+P_i} = -\sqrt{1-(1-Z_D)/(1+P_i)} \]
\[ + \log((1+P_i)/(1+P_i-Z_D)) \]
\[ + \log\left(1 + \sqrt{1-(1-Z_D)/(1+P_i)}\right). \]
\[ (19) \]

Equation (19) is thus a rescaled version of eq. (1.9) in [5].

Now, given a range of \( Z_D \) values, the time \( t_D \) and the pressure increase \( P_i \), it is possible to obtain the corresponding \( X_D \) using eqs. (15) and (19). Results for analytical calculations are presented in fig. 4, where they are compared to numerical results, both for \( t_D = 20 \). Note that the asymptotic analytical shape is slightly higher up in the reservoir than the numerical one: by construction, the asymptotic shape only attains the lowermost point \( Z_D = 1 + P_i \) as \( X_D \to -\infty \), i.e. at horizontal locations that are arbitrarily far behind the leading edge at the top of the front.

4.1 Unswept area

In oil recovery applications, once foam begins to exit from a production well, one quantity of interest to the reservoir engineer is the “unswept area” [5], i.e. the amount of area underneath the foam front (measured from the leading edge at the top) where foam has not yet reached to displace liquid. Areas here are made dimensionless on the scale \( P_{\text{drive}}/(\Delta \rho g^2) \), where we recall that \( P_{\text{drive}} \) is the driving pressure, \( \Delta \rho \) is a density difference (between injected surfactant solution and injected gas) and \( g \) is the acceleration due to gravity. The lower the unswept area, the more efficient the reservoir sweep by the foam, i.e. gravity override is reduced.

We consider two distinct cases. In the first case (to be discussed further in sects. 4.2 and 4.3), we suppose that the driving pressure has initially been set at the level such that the maximum penetration depth of the foam front corresponds to the greatest depth at which oil is located: as mentioned previously, the initial driving pressure would not be set any lower than that. We then define the unswept area by \( A_u = \int_0^1 \xi_p dZ_D \), where (to permit a fair comparison between cases with different values of \( P_i \)) the integration proceeds only to the maximum front depth \( t \) prior to imposing a pressure increase. In other words, the integration does not extend all the way to the bottom of the foam front, as indeed the front now extends even further down than the depth to which the oil is located [6]. In the second case (discussed in more detail in sect. 4.4), we suppose that the driving pressure has initially been set such that the maximum penetration depth (even prior to any pressure increase) is a factor \( D_{\text{max}} \) times larger than the greatest depth at which oil is located, with the parameter \( D_{\text{max}} \gg 1 \). The consequent definition of \( A_u \) becomes \( A_u = \int_0^{D_{\text{max}}} \xi_p dZ_D \). Note the integration does not extend all the way to the bottom of the front, even before the pressure increase: it only extends down as far as the oil itself extends, i.e. 1/\( D_{\text{max}} \) dimensionless units.

In the first case it is possible to calculate the unswept area for a given pressure increase as\(^5\)

\[ A_u = \int_0^1 \xi_p dZ_D = (1+P_i)^2 \int_0^{1/(1+P_i)} -\xi_p dZ_{\text{mod}}. \]
\[ (20) \]

Using, for example, \( P_i = 0.2 \) (as in figs. 2, 3) and integrating numerically eq. (20) with Simpson’s rule (using 10 intervals\(^6\)), the unswept area is \( A_u = 0.534 \), which is a smaller value than for the constant pressure case, where the unswept area is \( \pi/4 \) (see [5]); hence the foam front

\(^5\) In the present work we only consider a 2-dimensional foam displacement. In the case of an axisymmetric displacement about a production well, the “unswept area” would be replaced by an “unswept volume”. This is readily calculated [5] by multiplying the “unswept area” by the perimeter of the leading edge of the foam front. This perimeter is given by \( 2 \tau \) times eq. (12).

\(^6\) Using more intervals, e.g. 20 or 30, shows a very close agreement to using 10 intervals.
The unswept area for this small $P_i$ increases, because a larger pressure increase will make the foam front at any given depth displace further, so the area left unswept will be smaller.

Furthermore, we can obtain asymptotic expressions for $A_u$ in the cases of small and large $P_i$ as explained below.

4.2 The small $P_i$ limit

If $P_i \ll 1$, the right-hand side of eq. (20) becomes

$$\left(1 + P_i\right)^2 \left(\int_0^1 -\xi_{\text{mod}} \, dZ_{\text{Dmod}} - \int_{1/(1 + P_i)}^1 -\xi_{\text{mod}} \, dZ_{\text{Dmod}}\right).$$

(21)

where the first integral in the bracket is \[5\]

$$\int_0^1 -\xi_{\text{mod}} \, dZ_{\text{Dmod}} = \frac{\pi}{4},$$

(22)

The integration range of the second integral remains close to one (since $P_i$ is assumed small). When $Z_{\text{Dmod}}$ is close to one (towards the bottom of the foam front) \[5\]

$$-\xi_{\text{mod}} \approx -1 + \log\left(2/(1 - Z_{\text{Dmod}})\right),$$

(23)

making the second integral in (21) become

$$\int_{1/(1 + P_i)}^1 -\xi_{\text{mod}} \, dZ_{\text{Dmod}} = \frac{P_i}{1 + P_i} \log\left(\frac{2(1 + P_i)}{P_i}\right).$$

(24)

The unswept area for this small $P_i$ case is then

$$A_u = \left(1 + P_i\right)^2 \left[\frac{\pi}{4} - \frac{P_i}{1 + P_i} \log\left(\frac{2(1 + P_i)}{P_i}\right)\right].$$

(25)

which in turn can be written as

$$A_u \approx \frac{\pi}{4} - P_i \log(1/P_i) + \text{corrections of order } P_i \text{ or smaller.}$$

(26)

From (26) it can be observed that the term $P_i \log(1/P_i)$ vanishes as $P_i \to 0$, but its derivative is infinite. This is why fig. 5 shows a very sharp decrease in $A_u$ as $P_i$ increases near $P_i = 0$.

4.3 The large $P_i$ limit

In the case of a large $P_i$ the integration range in eq. (20) includes only $Z_{\text{Dmod}}$ values much smaller than unity. In that domain \[5\],

$$\xi_{\text{mod}} \approx -\frac{2\sqrt{2}}{\pi}Z_{\text{Dmod}}^{3/2}.$$  

(27)

Substituting eq. (27) into the right-hand side of (20), integrating and evaluating the limits, an expression for $A_u$ is obtained for this case:

$$A_u \approx \frac{4\sqrt{2}}{15}(1 + P_i)^{-1/2}.$$  

(28)

Figure 6 compares the unswept area obtained from Simpson’s rule and analytical results using eqs. (25), (26) and (28). Here it can be observed that results from eqs. (25) and (26) match Simpson’s rule computations for values of $P_i$ close to $P_i = 0$, whereas predictions from eq. (28) are close to Simpson’s rule results for larger values of $P_i$.

This completes our discussion of the first case for which the penetration depth of the foam front (prior to any pressure increase) is chosen to match the depth to which the oil is located. The next section treats the second case, for
which the initial penetration depth of the foam front is $D_{\text{max}}$ times greater than the depth to which oil is located, with $D_{\text{max}} \gg 1$. The penetration depth following pressure increase then becomes larger still.

### 4.4 The large $D_{\text{max}}$ limit

As mentioned in sect. 4.1, in the case $D_{\text{max}} \gg 1$, we evaluate $A_u$ via

$$A_u = \int_0^{1/D_{\text{max}}^2} \left| \xi_p \right| \, dZ_D = (1 + P_i) \int_0^{1/(D_{\text{max}}^2(1 + P_i))} \left| \xi_{\text{mod}} \right| \, dZ_{D_{\text{max}}}. \quad (29)$$

Substituting for $\xi_{\text{mod}}$ from eq. (27) and integrating we obtain

$$A_u = \frac{4\sqrt{2}}{15} (1 + P_i)^{-1/2} D_{\text{max}}^{-5/2}. \quad (30)$$

Relative to the situation prior to pressure increase, $A_u$ has decreased by a factor $(1 + P_i)^{1/2}$ although it should be remembered that both before and after pressure increase, $A_u$ is actually very small (being of order $D_{\text{max}}^{-5/2}$ with $D_{\text{max}} \gg 1$).

### 5 Conclusions

We have presented numerical and analytical results, using the pressure-driven growth model, for a case with a sudden increase in the driving pressure during the process of foam-improved oil recovery. The rise in pressure causes the formation of concavities in the curves representing the foam front displacement. To obtain the correct foam front shape and avoid spurious numerical behaviour, these concavities need to be handled specially in our numerical schemes, which were adapted successfully from schemes we have implemented previously [5]. At sufficiently long times the concavities migrate right to the bottom of the solution domain, and a convex front shape is recovered everywhere else. A longtime asymptotic solution has been developed, from which it was possible to calculate the area left unswept by the foam front. The higher the driving pressure, the lower the unswept area, implying a more uniform foam front displacement.

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### References

Chapter 6

Modelling foam improved oil recovery within a heterogeneous reservoir

This chapter is a manuscript submitted to Colloids and Surfaces A: Physicochemical and Engineering Aspects. Authors: Elizabeth Mas-Hernández, Paul Grassia, and Nima Shokri. The manuscript presents modelling and simulation results for the propagation of foam inside heterogeneous reservoirs. A parametric study is presented to investigate how different levels of heterogeneity affect the shape of the foam front over time.

We have been studying the displacement of a foam front for improved oil recovery for the cases described in section 3.6. Previous chapters present results for the foam front motion for homogeneous reservoirs. Homogeneity is regarded as having a uniform permeability within the reservoir. For field applications however, a heterogeneous reservoir, where permeability varies, is more relevant.

The pressure-driven growth model for foam rheology is being employed to represent the process mathematically. However, the numerical solution of this model requires special techniques. The singularity of the model and how it tends to develop spurious behaviour has been explained (sections 3.3 and 3.5.4); specifically when the front shape presents concave regions either due to numerical artifacts or the physics
of the process. In this case it is necessary to modify the equations for the speed given by pressure-driven growth [13]. However it has also been shown that this behaviour can be handled [13].

For a stratified reservoir, convex but also concave regions will arise describing the front shape. Therefore, special methods need to be applied.

With the purpose of illustrating the model, variation in permeability is represented with a sinusoidal function, in turn depending on vertical position. This modification is easy to incorporate into the model but does result in the formation of concavities. With the aforementioned special methodology to deal with concavities, the pressure-driven growth model can be used for the case of the stratified reservoir.

Then, it becomes interesting to study how the variation of parameters affecting heterogeneity influence the numerical solution of the model and how results compare to the case of a homogeneous reservoir.

This is possible by calculating the root mean square displacement difference between homogeneous and heterogeneous fronts and then confirm its change with the amplitude of heterogeneity and its level of sensitivity if any, to varying the number of high and low permeability layers.

These calculations seek to quantify how far the heterogeneous front is from an analogous homogeneous curve. Root mean square displacement is calculated as follows (see Figure 6.1): the distance (along the normal to the homogeneous front: $d$) between heterogeneous and homogeneous fronts is obtained for each point in the homogeneous curve. Then, $d$ is squared and multiplied by the length of arc (in the homogeneous curve), corresponding to that point: $l$. The result for each point ($d^2 \times l$) is added together for all points and divided by the total length of the homogeneous front. Finally, the root square of this amount is taken.

It is also possible to follow the trajectory of some points that lead or lag the displacement in the heterogeneous case and compare them to homogeneous fronts. The vertical displacement of such points in heterogeneous systems is significantly weaker than the vertical displacement of front material points in the homogeneous
Anisotropic and heterogeneous systems are also covered. Some results on this topic have already been presented in [105]. Therefore simulations changing the parameters affecting heterogeneity are performed. In addition, an asymptotic solution of the heterogeneous anisotropic system in the limit of negligible vertical permeability is also presented.

Results show that it is possible to use a suitable adaptation of the pressure-driven growth model to describe the motion of a foam front when the reservoir is heterogeneous in terms of permeability. It is possible to deal with the complications that the presence of concavities can cause when solving the system numerically. Furthermore, it is possible to proceed with a parametric analysis of the case, showing dependence of the results upon the parameters that indicate the level of heterogeneity.
Modelling foam improved oil recovery
within a heterogeneous reservoir

Elizabeth Mas Hernández, Paul Grassia, Nima Shokri

Abstract

The displacement of foam within a heterogeneous reservoir during foam improved oil recovery is described with the pressure-driven growth model. The pressure-driven growth model has previously been used to study foam motion for homogeneous cases. Here, the foam model is modified in such a way that it includes terms for variable permeability. This model gives the evolution of the foam motion over time and the shape of the foam front, a wet foam zone between liquid-filled and gas-filled zones. The foam front shape for a heterogeneous or stratified reservoir develops concave and convex regions. For shapes such as these, the numerical solution of pressure-driven growth requires special numerical techniques, particularly in the case where concavities arise. We also present some analysis of the level of heterogeneity and how it affects the displacement.

Keywords: Improved Oil Recovery; Foam in Porous Media; Pressure-driven Growth; Heterogeneity; Anisotropy; Mathematical Modelling;

Highlights

- Pressure-driven growth model for foam improved oil recovery is considered
- Reservoir is heterogeneous (stratified) and possibly also anisotropic
- Heterogeneity produces convexities and concavities in the foam front
- Concavities focus into corners that propagate differently from the rest of the front
- Exceedingly anisotropic systems give sharply-curved concavities but not corners

*Corresponding author
1. Introduction

Foams are able to improve sweep efficiency during oil production processes from underground formations compared to other injection fluids [1]. However, many studies have focused on homogeneous formations even when in real fields heterogeneous conditions are found [2, 3]. A heterogeneous formation has variable permeability [4]. Therefore, the flow of fluids, and in this specific case, of foams is affected by this difference in permeability.

There are some laboratory and simulation studies on heterogeneous reservoirs which give insight into the advantage of using foam as a displacing fluid for oil recovery [2, 5–10]. They have found that foam is able to divert the gas flow from high permeability regions towards zones with lower permeability [2, 8–10].

Here we present a simulation study about the flow of foams within a heterogeneous reservoir. We have studied previously the displacement of foam within a homogeneous reservoir [11] (revisiting a model of Shan and Rossen [12]), and also two additional cases, one where the injection pressure is increased part way through the process and one taking into account the effect of surfactant slumping due to gravity [13, 14].

The mathematical description of the system for the homogeneous case uses a foam model known as pressure-driven growth [11, 12]. The model computes the advance of a foam front which forms the boundary between liquid ahead and foam behind. Motion of foam front is driven by pressure difference across the front, i.e. the difference between a driving injection pressure and the hydrostatic pressure in the liquid. The front speed falls as depth increases because the hydrostatic pressure rises. The front speed also falls the further the foam displaces: this is because most of the dissipation in the system occurs in a wet foam region where the foam meets the liquid, and this wet foam region thickens over time [11, 12, 15]. The two additional cases, mentioned above, use a suitable modification of the same model [13, 14].

Therefore, in a similar fashion, we propose some changes to the original pressure-driven growth model that will make it appropriate to describe the case of the het-
erogeneous reservoir.

The rest of this work is structured as follows: section 2 describes the changes we have applied to the pressure-driven growth model to make it suitable for a heterogeneous reservoir. Section 3 presents results of the numerical solution of the system highlighting some numerical implementation issues. Quantitative comparison between homogeneous and heterogeneous fronts is addressed in sections 4 and 5, which give insight into the behaviour of the heterogeneous displacement. We have also explored the case of a heterogeneous but anisotropic reservoir, this is shown in section 6. Finally, section 7 offers conclusions.

2. Pressure-driven growth for variable permeability

For the description of the heterogeneous case, the pressure-driven growth model\(^1\) has been modified in a simple way: basically this takes into account permeability variation.

We describe relative changes in permeability with the help of a sinusoidally varying function, given below, which represents the reservoir heterogeneity and is included within the equations describing the speed of the foam displacement.

Therefore the equations that apply in this case for horizontal and vertical components of velocity are\(^2\):

\[
\frac{dX_D}{dt} = \frac{(1 - Z_D) \cos \alpha}{s_D} J(Z_D) \tag{1}
\]

\[
\frac{dZ_D}{dt} = \frac{(1 - Z_D) \sin \alpha}{s_D} J(Z_D) \tag{2}
\]

where to illustrate the model \(J(Z_D)\) can be chosen to be one of the following functions:

\[
J(Z_D) = 1 + k_s \sin(2\pi n_s Z_D) \tag{3}
\]

\[
J(Z_D) = 1 - k_s \sin(2\pi n_s Z_D) \tag{4}
\]

\(^1\)Refer to [11, 12] for the mathematical description of the model for constant permeability.

\(^2\)These equations are in dimensionless form. Dimensional equations for the original system and their conversion to the dimensionless version are given in the Appendix.
Figure 1: Schematic for the foam front displacement in a two-dimensional heterogeneous reservoir as function of time. Here $s_D$ is the distance travelled by a material point on the front, and $\alpha$ is the angle between the normal to the front and the horizontal. Permeability differences affect the shape of foam front and are described using equation (3) or (4).

$X_D$ is the horizontal position of a material point in a rectangular reservoir, $Z_D$ the vertical position downwards, $t_D$ time, $s_D$ the distance the front travels, $\alpha$ the angle giving the orientation of the front normal with respect to the horizontal, $k_s$ the amplitude of the heterogeneity variation about the mean (a factor less than unity), and $n_s$ the number of low and high permeability layers (for simplicity taken to be an integer). The sketch in Figure 1 illustrates the system.

Contrary to the convex shape expected for displacement in a homogeneous reservoir, in the present case the foam front shape will develop alternate convex and concave regions. This is because when the sinusoidally-varying permeability function affects the velocity of the front motion, describing the propagation through the different layers with high and low permeability, velocity becomes respectively faster and slower than a system with unit permeability.

Owing to the nature of the pressure-driven growth model, the formation of concavities during the course of the evolution will lead to spurious numerical results, including predictions of spurious loops in the shape of the front [11].

In order to handle concave regions and avoid the formation of loops, velocity displacement is corrected as in [11]: concave regions are set to move with an apparent velocity $v_{app} = v / \cos(\theta/2)$ ($v$ is the front speed and $\theta$ is the angle through which the front turns between adjacent segments), when the angle $\theta$ exceeds a certain threshold given by $\theta_s$. In line with previous work [11] we take $\theta_s$ to be a constant significantly
smaller than unity specifically with the value $\pi/18$. Setting $v_{app} = v/\cos(\theta/2)$ will speed up the displacement of the points with the aim of catching up with points which have displaced further ahead. The reason why the apparent velocity $v_{app}$ takes the above mentioned form has been explained on physical and geometric grounds by [11]: concavities can focus down into short corners and these need to propagate at a higher speed than points nearby to prevent indefinite sharpening.

In the above, we have shown the main changes applied to the mathematical model for pressure-driven growth. The next section presents numerical results without and with implementation of velocity corrections for concavities. Details of numerical implementation issues within the system are also given.

3. Numerical results for displacement with variable permeability

When solving the system for pressure-driven growth taking into account variable permeabilities, as described by equations (1)–(2) with any given values for the parameters $k_s$ and $n_s$, the resulting front shape has smooth convex and concave regions at short times for either of the $J(Z_D)$ functions presented previously.

Later on, concavities focus down to sharp corners and (unless the velocity at each corner is corrected as alluded to above) these develop spurious loops [11]. Also, highly curved convex regions can be formed at the top [11]. These are associated with the boundary condition usually imposed at the top boundary, namely $\alpha = 0$. Material points near the top with $\alpha$ close to zero tend to migrate downwards only very slowly, but as they migrate, they also fall increasingly behind the leading edge of the front on the top boundary itself. This is what produces high curvatures. This results in material points migrating even further down and leaving only a few points at the top region of the foam front thereby giving a poor representation of its shape.

We present numerical results, first for the original system (without special features to handle the formation of loops and sharp convexities) in section 3.1, and then implementing modifications to deal with these issues (section 3.2).
Figure 2: Foam front shape for a stratified reservoir with $k_s = 0.3$ and $n_s = 3$ using equation (3) for $J(Z_D)$. For these data, no correction is applied to velocities in concave regions of the front.

3.1. Numerical results for original system

Figure 2 shows some results when the front is discretised using 500 points along the $Z_D$ axis, the time step is $1 \times 10^{-5}$, $k_s = 0.3$, and $n_s = 3$ using equation (3). The values for $k_s$ and $n_s$ are chosen arbitrarily for the purpose of illustrating the model. Also, the segment at the top of the front is subdivided if its length reaches a value equal to 0.01 [11]. In Figure 2 a spurious loop is clearly evident for $t_D = 0.2$ and even for $t_D = 0.14$ a tiny loop can be seen.

Similarly, Figure 3 shows results for the same parameters but using equation (4). In this case the loop is formed earlier: already by time $t_D = 0.1$ a very significant loop is in evidence³. We can see from these figures that, due to the nature of the model, the effect of high and low permeability layers on the foam front shape seems more evident closer to the top of the front.

Near the bottom (of the domain of solution), the concavity formed is weak and points in this region migrate downwards where the displacement of the front is null: as the front reorients over time, owing to the net driving pressure (i.e. injection

³In the case of a homogeneous reservoir presented by [11], and deliberately inserting a concavity into the otherwise convex shape, it takes an order of magnitude longer to develop into a problematic loop.
Figure 3: Foam front shape for a stratified reservoir with $k_s = 0.3$ and $n_s = 3$ using equation (4) for $J(Z_D)$. For these data, no correction is applied to velocities in concave regions of the front.

Pressure less hydrostatic pressure (i.e., decreasing with depth, material points tend to move downwards as well as to the right, so that any weak concavities that develop near the bottom of the front then migrate further and further down towards a region where the velocity is virtually nil.

Actually, this behaviour of the front displacement is what makes the concavity formed closer to the top, due to a local minimum permeability value, to evolve faster into a spurious loop than the next low permeability layer further down.

Figure 2 shows a concavity forming around about $Z_D = 0.3$, but above this (i.e., at smaller $Z_D$ values) there is a convexity around about $Z_D = 0.1$. This ‘local’ convexity has higher curvature than the corresponding convex front shape that arises in a system with homogeneous permeability [11]. As a result, material points in this convexity separate comparatively quickly from one another, and hence over time (and without adding extra material points) the foam front starts to look quite jagged.

Moreover it appears that the jagged foam front in Figure 2 fails to satisfy the boundary condition imposed at the top boundary (the front is supposed to meet that boundary at right angles). We can make the foam front less jagged by subdividing the intervals between material points, particularly those intervals in the neighbourhood of this convexity near $Z_D = 0.1$. That means in turn that intervals on these
local convexities need to be subdivided, just as in the homogeneous reservoir case we needed to subdivide the topmost interval\(^4\). Therefore, strategies to regrid the foam front are also needed to have a better representation of its shape. In concave regions, material points actually move closer together instead of separating, and this means that we have to remove material points when regridding instead of adding new ones.

The following section presents results when the implementation of speed up for concavities and front regridding are used.

3.2. Modifying velocity for concavities and regridding the front

To avoid the concavities focusing down to sharp corners and forming spurious loops, velocities must be modified for concave regions (using a speed up factor \(1/\cos(\theta/2)\) as explained in section 2). Also we have used a linear interpolation for regridding the front as a first approximation.

Figures 4–5 present foam front displacement, respectively for the two \(J(Z_D)\) functions, now modifying velocities for concave regions and implementing the front regridding. For the plots in these figures, 500 material points are used to discretise the front, time step is \(1 \times 10^{-5}\), \(k_s = 0.3\), \(n_s = 3\), the parameter \(\theta_s = \pi/18\) is used as the critical angle for switching on the velocity modification (this value of \(\theta_s\) is chosen arbitrarily but results are not sensitive to \(\theta_s\), as long as it is much smaller than \(\pi\) as explained by [11]), points between short spatial intervals (shorter than 0.002) are set to be consumed when implementing velocity corrections [11], regridding takes place when the front segments are longer than a specified value (0.02), and the top segment is subdivided when it becomes longer than 0.01.

The effect is to contain concavities, keeping them from developing into loops. In addition, having more points to describe the foam front ensures a less jagged shaped front, away from the concave corner of course. We are also able to obtain results for longer time values. Figure 6 shows results at comparatively long times when the

\(^4\)There is a special rule that is used to subdivide the topmost front segment which arises due to the very high curvatures that can occur on the top boundary: see [11] for details.
Figure 4: Foam front shape for a stratified reservoir correcting velocities on concavities, regridding the front, and using equation (3) for $J(Z_D)$.

Figure 5: Foam front shape for a stratified reservoir correcting velocities on concavities, regridding the front, and using equation (4) for $J(Z_D)$. 
Figure 6: Foam front shape for longer time and using equation (4) for $J(Z_D)$.

heterogeneity function is given by equation (4).

However, it is still not readily possible to reach longer times when describing permeability variation with equation (3), at least not when using only the modifications described above. Points below the top boundary displace further in the horizontal direction than the topmost point when using this equation, in consequence the segments in this region develop negative curvature and points start migrating above $z_D = 0$ and developing loops in this region. In such cases, we can impose a condition in the algorithm to discard points with vertical position $z_D < 0$. This restriction solves this problem of course. Furthermore, in such a case we cannot set the orientation of the front along the horizontal top boundary (i.e. we can no longer satisfy the boundary condition $\alpha = 0$ there). It is therefore necessary to calculate the horizontal location of the intersection of the front with $z_D = 0$, instead of specifying the trajectory of the front at $z_D = 0$ in advance as was done for an analogous homogeneous system in [11]. However we have not explored this situation in detail in the present study, leaving it instead for future study with more sophisticated numerical techniques [16].

We are interested in how results for the heterogeneous reservoir compare to the homogeneous one and in the effect of using different values for $k_s$ and $n_s$. To be able to measure this quantitatively, we perform calculations for the root mean square displacement between homogeneous and heterogeneous fronts, which is considered
in the following section.

4. Root mean square displacement between homogeneous and heterogeneous fronts

Root mean square displacement is obtained as follows. For each point in the homogeneous front the displacement (normal to this front) ahead of and behind it are followed until the intersection with the heterogeneous front is obtained. These displacements are squared and multiplied by their corresponding length of arc in the homogeneous curve, integrated, divided by total length of the curve for the homogeneous front and the square root of this amount is taken.

It is important to note that the number of material points is different in each front, heterogeneous versus homogeneous (the number of points for the heterogeneous front tends to decrease early on as points are consumed more rapidly than they are created; on the other hand the number of points for the homogeneous front does not quite so vary much). Point positions are also different on each front, in such a way that to calculate the displacement normal to the homogeneous curve it is necessary to interpolate between points in the heterogeneous front for any given point on the homogeneous front because there is no guarantee that the normals to the discrete material points at the homogeneous curve will intersect the discrete points representing the heterogeneous curve.

Figures 7–9 compare heterogeneous fronts to homogeneous ones for the function $J(Z_D)$ described by equation (4) for various $k_s$ and $n_s$. These plots are obtained discretising the homogeneous front with 100 points and the heterogeneous one with 200 points, the time step and criteria to correct velocities, and for regridding, are the same as before.

Table 1 gives values for root mean square displacement between homogeneous and heterogeneous fronts at different time values and varying parameter $k_s$.

Figure 10(a) shows a plot of root mean square displacement against time for the values in Table 1. In addition, we have also plotted the value for $t_D = 0$ when both fronts coincide. Figure 10(b) presents the data for the displacement versus $k_s$. 

11
Figure 7: Front for homogeneous and heterogeneous displacement ($k_s = 0.3, n_s = 3$).

Figure 8: Homogeneous and heterogeneous fronts ($k_s = 0.4, n_s = 3$).

Figure 9: Comparison between homogeneous and heterogeneous fronts ($k_s = 0.3, n_s = 4$).
Figure 10: Root mean square displacement (a) vs time, varying $k_s$ ($n_s = 3$), (b) vs $k_s$ for $t_D = 0.1$, $t_D = 0.2$, and $t_D = 0.3$. 
These results allow us to confirm assumptions about how the displacement between homogeneous and heterogeneous fronts changes with different values for amplitude of heterogeneity. Root mean square displacement increases with increase in $k_s$. The reason for this is that with the increase in amplitude of heterogeneity, the heterogeneous front tends to displace further away (ahead and behind) from the homogeneous one.

Regarding the parameter for the number of high and low permeability layers, Table 2 gives values for the root mean square displacement varying time and $n_s$. The change seems to be very small, so we have done the calculations varying the number of data points (using also 300 and 400 points) to see if the predicted root mean square displacements change with the discretisation of the front. This was not the case and the values on this table are the averages of the cases considered.

We have also plotted values from Table 2 in Figure 11, showing root mean square displacement against time in Figure 11(a) and against $n_s$ in Figure 11(b).

<table>
<thead>
<tr>
<th>$t_D$</th>
<th>$k_s = 0.3$</th>
<th>$k_s = 0.4$</th>
<th>$k_s = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_s = 3$</td>
<td>0.0299</td>
<td>0.0391</td>
<td>0.0504</td>
</tr>
<tr>
<td>$n_s = 4$</td>
<td>0.0403</td>
<td>0.0535</td>
<td>0.0700</td>
</tr>
<tr>
<td>$n_s = 5$</td>
<td>0.0467</td>
<td>0.0671</td>
<td>0.0866</td>
</tr>
</tbody>
</table>

Table 1: Root mean square displacement between homogeneous and heterogeneous fronts varying parameter $k_s$.

<table>
<thead>
<tr>
<th>$t_D$</th>
<th>$k_s = 0.3$</th>
<th>$k_s = 0.3$</th>
<th>$k_s = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_s = 3$</td>
<td>0.0298</td>
<td>0.0285</td>
<td>0.0277</td>
</tr>
<tr>
<td>$n_s = 4$</td>
<td>0.0400</td>
<td>0.0392</td>
<td>0.0395</td>
</tr>
<tr>
<td>$n_s = 5$</td>
<td>0.0472</td>
<td>0.0474</td>
<td>0.0479</td>
</tr>
</tbody>
</table>

Table 2: Root mean square displacement between homogeneous and heterogeneous fronts varying parameter $n_s$. Values were obtained for the averages changing number of data points for the discretisation of the heterogeneous front.
Figure 11: Root mean square displacement (a) vs time, varying $n_s$ ($k_s = 0.3$), (b) vs $n_s$ at times $t_D = 0.1$, $t_D = 0.2$, and $t_D = 0.3$. 
These results show that root mean square displacement does not depend upon \( n_s \), at least for the parameters and time values presented so far.

When comparing fronts between homogeneous and heterogeneous results, we are also interested in identifying points on the heterogeneous curve which are local maxima and minima of the displacement relative to the homogeneous one (normal to the homogeneous front). These maximum and minimum points presumably develop in regions where permeability is maximum or minimum, but potentially they could migrate as material points themselves migrate. This is covered in the next section.

5. Local maxima and minima on heterogeneous front

Observing the front for heterogeneous displacement in the previous figures, and specially when comparing them to the homogeneous ones (e.g. Figure 7), it seems that for \( n_s = 3 \) there are three local maxima ahead and three local minima behind on each curve along the normal to the homogeneous fronts, and that these points tend to migrate downwards over time.

Trying to prove this hypothesis, we have calculated where these points are located and the displacement (the distance between the heterogeneous point and the homogeneous front along the normal to the homogeneous front) to check how they change over time for the data presented previously in Figure 7 (at times \( t_D = 0.1, 0.2 \) and 0.3, with \( k_s = 0.3, n_s = 3 \)). Figure 12 shows similar data: segments normal to the homogeneous front for points which are local minima and maxima are also shown; and it is evident that there are 3 local minima and 3 local maxima alternately.

Figures 13 and 14 present graphs of vertical locations (\( Z_D \)) of the maximum and minimum points and the displacement between separating those points from the homogeneous curve versus time, respectively. At these time values, for all local minimum and maximum points \( Z_D \) increases but the rate of increase is comparatively low.

We have also explored downward migration of selected material points at early time in a homogeneous system. Comparing Figure 15 and Figure 13 we see that
Figure 12: Homogeneous and heterogeneous foam fronts at \( t_D = 0.1, 0.2 \) and 0.3 (with \( k_s = 0.3, \ n_s = 3 \)). Segments which are normal to the homogeneous curves and join heterogeneous points for local maxima and minima are also shown.

Figure 13: Vertical point position for local minima and maxima from the top (min 1) to the bottom (max 3) of the front at times \( t_D = 0.1, 0.2 \) and 0.3.
Figure 14: Distance between local minimum and maximum points and the homogeneous front. From the top (min 1) to the bottom (max 3) of the heterogeneous front at times $t_D = 0.1, 0.2$ and 0.3.

The downward displacement of the material points is substantially faster than that of the maxima and minima. This underlines that fact that the maxima and minima are geometric features of the heterogeneous front, but do not themselves correspond to trajectories of material points.

The foam displacement for a homogeneous reservoir but with anisotropic permeability has been studied by de Velde et al. [15]. Further results for anisotropic systems have been obtained by Grassia et al. [17] who specifically determined the speed of the concave corners relative to neighbouring material points (an anisotropic generalisation of the $1/\cos(\theta/2)$ speed up factor that was discussed previously). In the next section we also explore this case.

6. Anisotropic reservoir

For an anisotropic reservoir we consider that vertical and horizontal permeability differ from each other. Hence the ratio between permeabilities (denoted $k_v$) is no longer unity ($k_v \neq 1$).

Velocity components in a homogeneous but anisotropic reservoir are described with the following equations [15]:

$$\frac{dX_D}{dt_D} = \frac{(1 - Z_D)}{s_D} \frac{\cos \alpha}{\cos(\alpha - \beta)} \tag{5}$$
Figure 15: Vertical point position for points that would be local minima and maxima from the top (min 1) to the bottom (max 3) in a heterogeneous system at times \( t_D = 0.1, 0.2 \) and 0.3, set at initial time with a homogeneous displacement.

\[
\frac{dZ_D}{dt_D} = k_v (1 - Z_D) \frac{s_D}{s_D} \sin \alpha \cos(\alpha - \beta)
\]

where \( k_v \) is the ratio of vertical to horizontal permeability and \( \beta \) is the angle giving the direction (measured with respect to the horizontal) in which the front is moving for this case, as illustrated in Figure 16. The reason for the \( \cos(\alpha - \beta) \) term in the denominator is that the front thickness (which determines the amount of dissipation in the system [12]) when measured along the direction \( \beta \) grows proportionally to the distance travelled \( s_D \), whereas the pressure gradient across the front scales inversely with the lesser thickness measured along the direction \( \alpha \) (i.e. along the front normal).

For the heterogeneous case, equivalent equations to (5)–(6) are used but including a function \( J(Z_D) \) for heterogeneity [17], in the same way as for the isotropic case presented in section 2. Therefore, velocity components are:

\[
\frac{dX_D}{dt_D} = \frac{1 - Z_D}{s_D} \frac{\cos \alpha}{\cos(\alpha - \beta)} J(Z_D)
\]

\[
\frac{dZ_D}{dt_D} = k_v (1 - Z_D) \frac{s_D}{s_D} \sin \alpha \cos(\alpha - \beta) J(Z_D)
\]

Some preliminary numerical results for the foam front shape over time (with \( k_v = 0.1 \)) and some numerical analysis for the heterogeneous anisotropic reservoir are presented in [17]. However the focus there was mostly on checking that the algorithms employed were able to avoid the formation of spurious loops. A detailed
parametric study (varying heterogeneity parameter \( k_s \) and contrasting anisotropic systems with isotropic ones) is not included there. Therefore, in what follows, we have obtained results for various values of \( k_s \) (keeping \( n_s = 3 \)) comparing data for \( k_v = 0.1 \) to \( k_v = 1 \) (actually the latter is equivalent to an isotropic system).

Comparing fronts at \( k_v = 0.1 \) and \( k_v = 1 \), in Figure 17(a) for small \( k_s \) and short time the fronts look similar. Figure 17(b) shows that increasing \( k_s \) and at longer time, the fronts do not coincide any more, the anisotropic displacement (with \( k_v = 0.1 \)) maintains smooth concave and convex regions for longer and these lag behind corners developed using \( k_v = 1 \).

In addition, an earlier study dealing with the homogeneous anisotropic reservoir also presents an analytical solution for the front shape when the vertical permeability is identically zero [15]. Here we extend this approach and apply it to the heterogeneous reservoir.

When vertical permeability is zero, the system of differential equations to describe (heterogeneous anisotropic) foam displacement is simplified to:

\[
\frac{dX_D}{dt_D} = \frac{(1 - Z_D)}{s_D} J(Z_D) \tag{9}
\]

\[
\frac{dZ_D}{dt_D} = 0 \tag{10}
\]

\[
\frac{ds_D}{dt_D} = \frac{dX_D}{dt_D} \tag{11}
\]
Figure 17: Fronts for heterogeneous anisotropic displacement varying $k_v$. (a) $t_D = 0.1, k_s = 0.1$ (b) $t_D = 0.3, k_s = 0.5$. 
from which
\[
\frac{dX_D}{dt_D} = \frac{(1 - Z_D)}{X_D} J(Z_D) \tag{12}
\]
which can be solved analytically, giving 5:
\[
X_D = \sqrt{2} (1 - Z_D) J(Z_D) t_D. \tag{13}
\]

Using either equation (3) or (4) for \(J(Z_D)\), defining parameters \(k_s\) and \(n_s\), and setting the time and values for \(Z_D\) it is possible to calculate \(X_D\) with (13). One consequence of having zero vertical permeability is that material points no longer move vertically, and hence no longer move down from the top boundary even if the front meets it obliquely, which implies that it is no longer necessary to impose a condition \(\alpha = 0\) on the top boundary (contrast with the situation in [11]).

Figures 18–20 depict analytical results for both homogeneous and heterogeneous fronts in the limiting case of zero vertical permeability using equation (3) changing \(k_s\) and \(n_s\) for the heterogeneous front. Heterogeneous data are obtained using equation (13) and homogeneous points with \(X_D = \sqrt{2(1 - Z_D)t_D}\) (the analytical solution obtained by [15]).

We have also calculated root mean square displacements for the foam fronts presented in Figures 18–20, measuring displacement along the normal for points on the homogeneous front (as explained previously) but also measuring it along the horizontal (which is easy to determine since all motion is horizontal in the case when vertical permeability is identically zero). Results (although not reproduced here) show that root mean square displacement, either along the normal or the horizontal, increases with \(k_s\). Changes in the root mean square displacement with respect to \(n_s\) turn out to be very small; so we can conclude that this parameter does not influence root mean square displacement.

We have also compared numerical results for \(k_v = 0.1\) to the analytic solution with zero vertical permeability for parameters \(k_s = 0.1, 0.3, 0.4, 0.5\). Analytic

\[\text{Note that since } J(Z_D) \text{ is a smooth function here, equation (13) necessarily describes a smooth curve without any sharp corners. For sharp corners to develop, vertical permeability must be non-zero: material points must move both horizontally and vertically.}\]
Figure 18: Analytical data at $t_D = 8$ for homogeneous and heterogeneous ($k_s = 0.3, n_s = 3$) foam fronts when vertical permeability is zero.

Figure 19: Analytical data at $t_D = 8$ for homogeneous and heterogeneous ($k_s = 0.5, n_s = 3$) foam fronts when vertical permeability is zero.
results obtained with equation (13) seem to match closely with $k_v = 0.1$ numerics for any of the explored values of $k_s$ and $t_D$. This is consistent with the findings of [15] in the limit of homogeneous and isotropic systems. Figure 21 shows results for $t_D = 0.3$ and $k_s = 0.5$, which is analogous to what was considered in Figure 17(b).

7. Conclusions

We have shown that it is possible to use a simple adaptation of the pressure-driven growth model for a stratified reservoir. A sinusoidal function is used to describe permeability variation. The shape of the foam front in this case develops convex and concave regions. The computational algorithm prevents concavities from forming into spurious loops and are contained using a factor to correct speed of points in concave sections. At longer times we still obtain a good representation of the front shape for which convex regions are joined together at corners, which themselves become less sharp towards the bottom of the front. It is possible to compare quantitatively results for the heterogeneous case to data for the homogeneous reservoir. For this we calculated root mean square displacement, which increases with the amplitude of heterogeneity ($k_s$) and with time, and is not sensitive to the wavenumber ($n_s$). Points at the heterogeneous front which are local maxima and minima with respect to the distance from the homogeneous front move downwards...
for small values of $t_D$. The rate of downward migration of these maxima and minima is however much smaller than that of material points. We have obtained an analytical solution for a heterogeneous but anisotropic reservoir for the limit case of zero vertical permeability and compared these data to some numerical results and to a homogeneous anisotropic formula. Results show agreement between numerical and analytical front shapes, particularly when the level of anisotropy is high (i.e. when the ratio $k_v$ of vertical to horizontal permeability is small).

**Appendix**

The foam front displacement for a homogeneous reservoir has been described with a simplified model (pressure-driven growth) proposed by Shan and Rossen [12] that keeps track of material points $(x(t), z(t))$, where $x$ and $z$ are horizontal and vertical point positions, $t$ is time; and $s(t)$ is the distance that individual points travel. Dimensional equations for horizontal and vertical velocity components are:

$$\frac{dx}{dt} = k \lambda_{rf} |\nabla P| \cos \alpha \left(1 - S_{wf}\right) \phi$$

$$\frac{dz}{dt} = k \lambda_{rf} |\nabla P| \sin \alpha \left(1 - S_{wf}\right) \phi$$

with

$$|\nabla P| = \frac{\Delta P - \Delta \rho g z}{\tau}$$
\[ \alpha = \arctan \left( -\frac{\partial x}{\partial z} \right) \quad (17) \]

and

\[ \frac{ds}{dt} = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \quad (18) \]

where \( k \) is permeability, \( \lambda_{rf} \) is gas relative mobility in the foam front region, \( \nabla P \) is pressure gradient, \( \Delta P \) is injection pressure less hydrostatic, \( \alpha \) is front orientation, \( S_{wf} \) is water saturation in the foam front region, \( \phi \) is porosity, \( \Delta \rho \) is liquid to gas density difference, \( g \) is acceleration due to gravity, and \( \tau \) is the width of the front, assumed to be proportional to the distance the front has displaced.

Distances \( x, z \) and \( s \) are made dimensionless dividing by \( \Delta P/\Delta \rho g \). Time is made dimensionless using:

\[ t_D = \frac{k\lambda_{rf}\Delta \rho g}{(1 - S_{wf})\phi \tau^*} t \quad (19) \]

where \( \tau^* \) is a characteristic front width (when the front has displaced a distance equal to \( \Delta P/\Delta \rho g \)).

When equations (14) and (15) are non-dimensionalised based on these scales, and subsequently generalised to the case of heterogeneous permeability, equations (1) and (2) result.

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References


Chapter 7

Conclusions

We have studied the process of foam propagation inside porous media through a theoretical approach. In conclusion, a simplified model for foam rheology, the pressure-driven growth model, can be used for this purpose. In this way, we intend to provide useful information for design and operation of the process of foam improved oil recovery. In this Chapter the important remarks obtained from this research are summarised.

In order to increase efficiency of recovery during oil extraction processes several stages need to be performed. Tertiary recovery techniques in particular often use a rheologically complex driving fluid to displace the oil left inside the reservoir after other methods have been employed.

Foams are an excellent option as the driving fluid for improved oil recovery. However, further study was required for predicting how foam displaces in porous media, specially from a mathematical perspective, which has been successfully developed in this present study.

In particular, the displacement of a foam front when the surfactant-alternating-gas injection strategy is employed, has been analysed and presented in Chapter 3. The mathematical model used for this purpose is pressure-driven growth. The model follows the motion of material points that represent the discretised foam front which is a thin wet foam region and can be
considered as a one-dimensional curve.

We have presented numerical and analytical results for the solution of the model. The mathematical and asymptotic analysis permits corroborating the outcomes from numerics. Even though we have not yet developed our own set of experiments to test the model predictions in detail, the predictions broadly agree with results from previous studies which in turn have been validated by comparing them to results from reservoir simulators and experimental studies [1, 24, 92] and even used to illustrate an example in field applications [24].

An important outcome of the present study is being able to relate pressure-driven growth to the viscous froth model [23, 69], even when the scale of the processes they can be commonly used for are quite different. Pressure-driven growth is a limiting case where the surface tension term in the viscous froth model is discarded [23]. The implication of leaving surface tension out, which accounts for energy cost of creating new surface, is that the differential equations acquire a hyperbolic nature (instead of parabolic) and therefore admit numerical singularities like the development of sharp corners and concavities in a otherwise expected convex front shape. Moreover, pressure-driven growth can also be associated to the eikonal equation [13], which is well known to be hyperbolic in nature.

In contrast, the analytical solutions of pressure-driven growth are less problematic to handle, and their analysis permits determining the cause of such singularities, and suggest numerical techniques to deal with them. In particular, this analysis points out the need to regrid the front at the top and the numerical methodology to achieve it. Further refinements were applied to the algorithm such as the calculation of the front orientation with a weighted average, the application of Heun’s method to solve the system and the removal of short segments at the back of concavities.

In order to avoid the formation of concavities and loops the velocity for the propagation of concave corners uses the same equations for pressure-driven growth but adding a correction factor to speed up the motion of corners and recover an entire convex front. This technique was tested by deliberately introducing a concavity at
initial conditions and proved to be effective (see Section 3.5.4 and [13]).

In addition, three cases were introduced where concavities are inherent to the physics of the processes: surfactant slumping which was addressed in more detail in Chapter 4, increase in driving pressure in Chapter 5, and a stratified reservoir in Chapter 6.

Surfactant slumping contributes along with drainage and coarsening to the bubbles collapse at the front, specifically at the top region. In consequence, there is an increase in the mobility in the slump zone. In turn, a higher mobility at the top allows a faster displacement of the front in this section but leaving behind the front region that is not slumped.

A concavity develops immediately after the effect of slumping takes place, but this is prevented from evolving into a loop when it is propagated with a modified velocity and it does not impact on the foam front shape at long times. Indeed, in the long-time limit, it is actually observed that a convex kink persists. We have deduced analytical expressions in the long-time limit that describe numerical results accurately.

Maintaining the production rate might require an increase in the injection pressure at some point during the process. This augment in pressure permits the foam penetrating deeper into the reservoir than before applying this change, which also speeds up the advance of the front. As with surfactant slumping, the concavities formed on the front caused by the raise in pressure have little effect on the shape for longer time when they are propagated according to the rules determined previously.

In this situation it is also possible to obtain an expression analytically to describe the front advance for long time values and with this calculate the area left unswept. Furthermore, the dependence of the area left unswept on the factor for the relative increase in pressure has been analysed.

The pressure-driven growth model can also be used, with suitable modifications, to describe the foam displacement for stratified reservoirs. A simple implementation of permeability variation is achieved incorporating a sinusoidally varying function
in the equations for the velocity of displacement. In this instance more than one concave corner can develop on the foam front. The formation of spurious loops is prevented using ‘corrected’ velocities for the propagation of concavities in the case of isotropic heterogeneity. Then it is possible to compare results between homogeneous and heterogeneous fronts, exploring the effect of changing parameters for the level of heterogeneity.

Moreover, the foam front propagation within a heterogeneous but anisotropic reservoir is also simulated, but this case requires applying a different rule for the motion of concave corners [105]. In addition, an asymptotic formula for the case of null vertical permeability has been derived for the anisotropic case.

In summary, the pressure-driven growth model can be used to describe the propagation of a foam front for homogeneous and heterogeneous reservoirs, when the effects of surfactant slumping and increase in driving pressure are accounted in the process.

Furthermore, the developed models can be used as preliminary assessment tools for the oil recovery process, before the use of reservoir simulators, when the surfactant-alternating-gas injection strategy is employed. Due to the fact that better numerical schemes for the solution of the pressure-driven growth model have been provided, the model could be implemented using simpler algorithms and software. One of the advantages of using these models is that they are sufficiently quick to solve. Therefore, they can be used to make real-time predictions in the field. Another important contribution of this research is the description for the calculation of the area left unswept by foam.
Chapter 8

Future work

The scope of the present study can be expanded further and this Chapter presents recommended future work that would complement the research in this thesis.

In the particular case of surfactant slumping, the parameters for the relative increase in mobility and the location of the slumped zone were treated as constants. Therefore, this study can be expanded by setting these parameters as functions of the depth of the slumped zone and time, as has been stated previously (refer to Chapter 4).

Furthermore, the description of liquid slumping (i.e. downward liquid motion relative to gas, which occurs in the high mobility foam zone far behind the front) needs to employ a different foam model from pressure-driven growth that does not focus on the low mobility zone and takes into account the variation in mobility behind the foam front.

The study of the heterogeneous case can be expanded in many areas. First of all, the function describing the variation in permeability can be modified informed by the geology of the reservoir. In addition, the long-time asymptotic solution for the isotropic case could be determined. For the majority of the cases numerical results are compared to data obtained from the analytical solutions. Therefore, it is important to provide this solution also in the present case. Moreover, it could also be possible to compare our results to other studies dealing with heterogeneous
cases [94, 95].

Simple changes that can also be applied concern the use of better numerical methods that are already well established [71, 99]. For example, for the solution of the differential equations, a fourth order Runge-Kutta methodology can be incorporated in the computing codes to check the effect this has on the obtained results and how the numerical schemes to deal with numerical problems encountered before are influenced. Level-set methods to solve the eikonal equations underlying pressure-driven growth are also a promising avenue to explore [106, 107].

Results from the solution of the pressure-driven growth model can also be compared to data obtained through reservoir simulators. That would also work to benchmark results obtained with the simplified model.

The viscous froth model has been analysed and compared to results from pressure-driven growth for the base case (with homogeneous permeability, mobility and constant driving pressure for the whole process) [13]. Then, it is also possible to analyse what the implication of using the more complex model is when applying it to other cases of interest: surfactant slumping, increase in driving pressure and stratified reservoir.

Finally, another interesting case is the consideration of an oil phase present in the process of the foam displacement and the incorporation of models that take into account the influence of oil in the texture or quality of the foam used during the process.
References


