ON THE TWIST-EQUIVALENCE OF CERTAIN QUADRATIC ALGEBRAS ASSOCIATED TO FINITE IRREDUCIBLE COXETER GROUPS

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During the course of my studies, I have mastered the basics of the theory of Hopf algebras, and in particular, learned of Drinfel'd's concept of a cocycle twist of a Hopf algebra, and of a module algebra over that Hopf algebra. A module algebra is is an algebra over a field upon which a Hopf algebra acts in a certain way. In particular, I came to focus upon how this concept works in the special case when the Hopf algebra is an algebra over a finite group. In this case, the module algebra is an algebra over the field on which the group acts by homomorphisms. This algebra may be twisted into another module algebra by means of a 2-cocycle on the group.

Having learned this, my attention was drawn to the work of Vendramin in [Ven12] in which he examined two module algebras over the symmetric group S_n, called Nichols algebras, defined using what are called rack cocycles. He showed that there is a cocycle twist that transforms one algebra into the other, i.e. that they are twist-equivalent. There are two quadratic algebras associated to the Nichols algebras, called $\mathcal{E}_n$ and Λ_n and first described in [FK99] and [Maj05], which are thought to be isomorphic to the Nichols algebras. It has for some years been conjectured, but not proven, that these two algebras are twist-equivalent. The most important result of this thesis is Theorem 4.7, which proves that $\mathcal{E}_n$ and Λ_n are indeed twist-equivalent.

Following this result I sought to see if analogous results could be obtained when considering other finite irreducible Coxeter groups than type A, which is what S_n is. To do this requires understanding of rack cocycles, and of the Schur multiplier of a group, which affects what kind of cocycle twisting is possible. I chose to focus on the case where the Coxeter group is a dihedral group since these groups are often fundamental to determining what happens for Coxeter groups of higher dimension. The last part of this work examines questions on whether the rack cocycles analogous to those that defined $\mathcal{E}_n$ and Λ_n are related to each other by cocycle twisting. The dihedral case, however, turns out to be less straightforward than was the case for Coxeter groups of type A, and it seems that there is scope for continuing research in this direction.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Preliminaries for Hopf algebras

The following chapter is an exposition of material needed to understand Hopf algebras.

We explain the tensor product of vector spaces and algebras over a field, by an explicit construction, and by universal property. We use this to redefine an algebra over a field, a module over that algebra, and related concepts. The resulting definitions can be written using commutative diagrams; by dualising those diagrams, we obtain the concepts of a coalgebra and a comodule. By combining the notions of algebra and coalgebra in a category-theoretically natural way, we obtain the definition of a bialgebra. We explain how this can be motivated by the idea of a monoidal category.

In the process of explaining these constructs, we define several concepts that will appear in this work, such as the group algebra (as a bialgebra), a group-graded vector space, and grouplike and primitive elements, as well as many coalgebra concepts which are dual analogues of corresponding concepts for algebras: coideal, cocommutativity, comodules etc. We also prove that the cocommutative elements of a bialgebra form a subalgebra, that the ideal generated by a subset of a bialgebra that forms a coideal is a biideal, and that the primitive elements of a bialgebra form a Lie algebra.

Throughout, we work over a fixed ground field k, which is often left entirely implicit. Rarely is anything assumed about k; any assumptions are stated explicitly.

We also adopt the convention that whenever we are discussing an algebraic structure of the form $\{\sum_{i \in I} x_i\}$, where I may be infinite, we are only considering the elements to have finitely many nonzero terms in their sums.

1.1 The tensor product

1.1.1 The definition of tensor product

The entirety of the subject of Hopf algebras depends on the notion of a tensor product. Expositions of this basic notion may be found in many sources – among the most frequently cited of the references, [Kas95] has a good explanation of the concept.

Definition 1.1. Let A and B be k-vector spaces. We define the **tensor product of A and B,** $A \otimes B$, by taking the free vector space with basis of pairs of points in A and B, and quotienting out by the following relations:

$$(a_1, b_1 + b_2) = (a_1, b_1) + (a_1, b_2),$$

$$(a_1 + a_2, b_1) = (a_1, b_1) + (a_2, b_1),$$

$$k_1(a_1, b_1) = (k_1 a_1, b_1) = (a_1, k_1 b_1)$$

for all $a_1, a_2 \in A, b_1, b_2 \in B, k_1 \in k$. The point in $A \otimes B$ represented by the pair (a, b) is traditionally denoted $a \otimes b$.

It is easier to understand the tensor product when you know the following: if X is a basis of A and Y is a basis of B, then $\{x \otimes y \mid x \in X, y \in Y\}$ is a basis of $A \otimes B$. This means, of course, that if A and B are both finite-dimensional, then $\dim(A \otimes B) = \dim(A) \times \dim(B)$, a formula that is used frequently in the literature, for instance, in [EG98].

1.1.2 Associativity, unitality and commutativity

It is also useful to know that there are several natural isomorphisms of tensor products, which it is an easy exercise to check using the above construction: for all k-vector spaces, A, B, C,

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C),$$

which means that the tensor product is *associative*, and we usually write $A \otimes B \otimes C$,

$$k \otimes A \cong A \cong A \otimes k, \tag{1.1}$$

where k denotes the field considered as a one-dimensional space over itself; it can be replaced by any one-dimensional space, and

$$A \otimes B \cong B \otimes A, \tag{1.2}$$

whose natural isomorphism map does not have a consistent name, but is usually denoted τ.

Furthermore, linear maps have a tensor product: specifically, given $f\colon A \to B$ and $g\colon C \to D$, there is $f \otimes g\colon A \otimes C \to B \otimes D$, defined on elements of the form $a \otimes c, a \in A, c \in C$ by

$$(f \otimes g)(a \otimes c) = f(a) \otimes g(c),$$

and extended by linearity.

1.1.3 The tensor product of algebras

You can also define a tensor product of algebras: if A and B are algebras, then their tensor product is defined on the vector space tensor product $A \otimes B$, with multiplication given by

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

for all $a, c \in A$, $b, d \in B$. The unit of the algebra is $1_A \otimes 1_B$.

1.1.4 The universal property of $\otimes$

The main motivation for the tensor product construction is its **universal property**, which characterises tensor products as a way of representing bilinear maps. This can be taken as an alternative definition of tensor products. Specifically, given vector spaces A and B, their tensor product $A \otimes B$ can be defined uniquely, up to isomorphism, by the following: there is a natural bilinear map $\varphi\colon A \times B \to A \otimes B$ such that for any bilinear map $f\colon A \times B \to C$, where C is any other space, there is a unique linear map $g\colon A \otimes B \to C$ for which f factors through the tensor product i.e. f can be expressed as $f = g \circ \varphi$.

This means that bilinear maps $A \times B \to C$ are in natural one-to-one correspondence with linear maps $A \otimes B \to C$, no matter what A, B and C are, so matters concerning bilinear maps can instead be described using tensor products, a fact which is well illustrated in the following section.

1.2 Algebras

1.2.1 Categorical definition of an algebra

Recall the definition of a (unital, associative) algebra over a field.

Definition 1.2. An **algebra over a field** k is a k-vector space, A, with a bilinear map $A \times A \to A$, called multiplication and represented by concatenation, and a special element $1 \in A$, called the unit. For all $a, b, c \in A$, the algebra must satisfy the following two axioms.

$$(ab)c = a(bc),$$

$$1a = a = a1.$$

The first is called the associativity law.

The following well-known example of an algebra will prove important later.

Example 1.3. Let G be any group. Then we define its **group algebra,** kG as follows. As a k-vector space, kG has a basis consisting of elements of the group:

$$kG = \{\sum_{g \in G} k_g g \mid k_g \in k, \forall g \in G\}.$$

For the elements of this basis, the multiplication map coincides with the product on the group, and the multiplication then extends uniquely to the rest of the algebra by virtue of the fact that it has to be bilinear. Under this construction, the unit of the algebra is the identity of the group.

One of the first things that is done when Hopf algebras are concerned, is to rewrite the definition of an algebra in terms of tensor product. The following exposition is to be found in [Swe69] et al.

Definition 1.4. An **algebra over** k is a k-vector space with a map $\nabla \colon A \otimes A \to A$, called multiplication, and a map $\eta \colon k \to A$, called the unit, satisfying the following two axioms:

$$\nabla \circ (\nabla \otimes id_A) = \nabla \circ (id_A \otimes \nabla)$$

and

$$\nabla \circ (\eta \otimes id_A) = id_A = \nabla \circ (id_A \otimes \eta).$$

The first axiom is the version of the associativity law appropriate to this definition. In the second axiom, we are implicitly precomposing by the isomorphism maps (1.1). This definition is a way of describing the algebra without referring to its elements: 1_A is still present – it's implicit as $\eta(1_k)$. One may still describe various features of the algebra like this: for instance, let $\nabla^{op} = \nabla \circ \tau$. Then $\nabla^{op} = \nabla$ means that for all $a, b \in A$, $\nabla(a \otimes b) = \nabla \circ \tau(a \otimes b) = \nabla(b \otimes a)$ or, since one may keep the old notation whilst acknowledging the new definition, that $ab = ba$ – in other words, A is commutative. This definition makes some things clearer; such as the tensor algebra, the free algebra over a vector space.

1.2.2 Some important algebras

Example 1.5. Let V be a k-vector space. The **tensor algebra over** V**,** $T(V)$ [Kas95], is defined as $\bigoplus_{n=0}^{\infty} V^{\otimes n}$ as a vector space. To define $\nabla \colon T(V) \otimes T(V) \to T(V)$, it is easiest to define it on its graded components. Thus $\nabla_{|V^{\otimes m} \otimes V^{\otimes n}} \colon V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes m+n}$ is given simply by "dropping brackets", and using the tensor product's natural associativity, which serves to make ∇ associative. The unit is simply the identification map from k to $V^{\otimes 0}$, which satisfies the unit axiom by virtue of relation (1.1).

Clearly, $T(V)$ is easier to understand in terms of Definition 1.4 than of Definition 1.2. This seems a natural point at which to mention another kind of algebra that will be important later.

Example 1.6. Let $\mathcal{L}$ be a Lie algebra, whose product is denoted $[\,,]$. Then the **universal enveloping algebra of** $\mathcal{L}$**,** $\mathcal{U}(\mathcal{L})$ [Kas95], may be defined as

$$\mathcal{U}(\mathcal{L}) = \frac{T(\mathcal{L})}{\langle x \otimes y - y \otimes x - [x, y] \rangle} \ ,$$

as x, y range over $\mathcal{L}$.

This also helps us to see k as an algebra over itself, with $\nabla \colon k \otimes k \to k$ being the natural isomorphism and η the identity map.

1.2.3 Modules

We usually think of an A-module as a vector space V on which A acts linearly, but if we rephrase our interpretation of the axioms, we may also define an A-module as a vector space V and a bilinear action $\triangleright\colon A \times V \to V$ which is multiplicative: $(ab)\triangleright v = a\triangleright(b\triangleright v)$ and $1\triangleright v = v$ for all $a, b \in A, v \in V$. We may rewrite this in terms of the tensor product.

Definition 1.7. An A**-module** [Maj95] is a vector space V and an action $\triangleright\colon A\otimes V \to V$ satisfying

$$\triangleright \circ (id_A \otimes \triangleright) = \triangleright \circ (\nabla \otimes id_V)$$

and

$$\triangleright \circ (\eta \otimes id_V) = id_V$$

(once again, using (1.1) implicitly).

The notation comes from [Maj95] as well.

The homomorphisms can also be expressed in terms of the tensor product definition. An **algebra homomorphism** [DNR01] is a linear map $f\colon A \to B$ satisfying

$$f \circ \nabla_A = \nabla_B \circ (f \otimes f) \quad \text{and} \quad f \circ \eta_A = \eta_B.$$

A **module homomorphism** is a linear map $f\colon V \to W$ for which

$$f \circ \triangleright_V = \triangleright_W \circ (id_A \otimes f).$$

An **ideal** can be expressed this way too [DNR01]: we want to say that, for an ideal I, $\nabla(I \otimes A) \subseteq I$ and $\nabla(A \otimes I) \subseteq I$; we combine these by thinking like a homological algebraist and writing

$$\nabla(I \otimes A + A \otimes I) \subseteq I.$$

Since we have explained algebras in terms of maps, it is natural to consider what happens when we reverse the direction of the maps. That is what we consider in the next section.

1.3 Coalgebras

1.3.1 Definition of a coalgebra

Definition 1.8. A **coalgebra** [Swe69] is a k-vector space C, with linear maps $\Delta\colon C \to C\otimes C$, called comultiplication, and $\epsilon\colon C \to k$, called the counit, satisfying the following two axioms, the first of which is called coassociativity:

$$(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta$$

and

$$(\epsilon \otimes id_C) \circ \Delta = id_C = (id_C \otimes \epsilon) \circ \Delta.$$

In order to describe this structure, Sweedler invented a suitable standard notation for his book [Swe69]. Our notation is based on the variant of this used in [Maj95]. For any $x \in C$, we write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

The coassociativity law can be written as

$$\sum_{(x)} x_{(11)} \otimes x_{(12)} \otimes x_{(2)} = \sum_{(x)} x_{(1)} \otimes x_{(21)} \otimes x_{(22)},$$

which is usually just written

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}.$$

An inductive argument can be used to show that coassociativity implies that for any $n \in \mathbb{N}$, all maps $C \to C^{\otimes n}$ obtained by repeated comultiplication coincide, and so we might as well name all these maps Δ^n, and write

$$\Delta^n(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \ldots x_{(n)}.$$

The counit axiom becomes

$$\sum_{(x)} \epsilon(x_{(1)})x_{(2)} = \sum_{(x)} x_{(1)}\epsilon(x_{(2)}) = x.$$

To describe a **coalgebra homomorphism** [Kas95], we simply dualise the tensor product version of algebra homomorphism. A coalgebra homomorphism is a linear map $f\colon C \to D$ for which

$$\Delta_D \circ f = (f \otimes f) \circ \Delta_C \quad \text{and} \quad \epsilon_D \circ f = \epsilon_C.$$

In Sweedler notation,

$$\sum_{(f(x))} f(x)_{(1)} \otimes f(x)_{(2)} = \sum_{(x)} f(x_{(1)}) \otimes f(x_{(2)}) \quad \text{and} \quad \epsilon_D(f(x)) = \epsilon_C(x)$$

for all $x \in C$.

The property which kernels of homomorphisms satisfy turns out to be almost the dual of the one for algebras: a **coideal** of the coalgebra C is a subspace I with

$$\Delta(I) \subseteq I \otimes C + C \otimes I$$

and

$$\epsilon(I) = 0.$$

It can be shown [DNR01] that this is precisely what you need in order to be able to define a quotient coalgebra with respect to a subspace.

Note that the relation between coideal and subcoalgebra is the other way round from the analogous one for algebras: if I is a subcoalgebra contained within the kernel of ϵ, it satisfies $\Delta(I) \subseteq I \otimes I$, and so it must automatically be a coideal. However, there may be coideals of C which are not subcoalgebras.

Example 1.9. Again k forms a rather trivial coalgebra, with $\Delta\colon k \to k \otimes k$ being the isomorphism and ϵ the identity. Slightly more interestingly, there is a tensor product for coalgebras rather like that for algebras. For coalgebras C and D, the tensor product algebra $C \otimes D$ has comultiplication

$$\Delta(c \otimes d) = \sum_{(c)(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}$$

for all $c \in C$, $d \in D$. The counit is $\epsilon_C \otimes \epsilon_D$ [DNR01].

Example 1.10. [Kas95] The underlying space of kG has a natural coalgebra structure given by defining for every basis element $g \in G$,

$$\Delta(g) = g \otimes g \quad \text{and} \quad \epsilon(g) = 1.$$

Nonzero elements x in a coalgebra for which $\Delta(x) = x \otimes x$ are known as **grouplike** elements after this. They have various interesting properties, such as: it's easy to see that the homomorphic image of a grouplike element must be grouplike. Also, grouplikes are classic examples of cocommutative elements, as explained below.

A coalgebra is **cocommutative** [Swe69] if

$$\Delta^{op} = \tau \circ \Delta = \Delta.$$

An element x of the coalgebra is cocommutative if $\Delta^{op}(x) = \Delta(x)$, or

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(x)} x_{(2)} \otimes x_{(1)}.$$

1.3.2 Comodules

We also have the dual concept to that of modules.

Definition 1.11. A C**-comodule** [Maj95] is a vector space V with a **coaction map** $\rho\colon V \to C \otimes V$ satisfying

$$(\Delta \otimes id_V) \circ \rho = (id_C \otimes \rho) \circ \rho \quad \text{and} \quad (\epsilon \otimes id_V) \circ \rho = id_V.$$

We need a notation for this: there are several commonly-used variants; we shall use the one where we write

$$\rho(v) = \sum_{(v)} v_{(-1)} \otimes v_{(0)}.$$

In this notation, the two conditions are written

$$\sum_{(v)} (v_{(-1)})_{(1)} \otimes (v_{(-1)})_{(2)} \otimes v_{(0)} = \sum_{(v)} v_{(-1)} \otimes (v_{(0)})_{(-1)} \otimes (v_{(0)})_{(0)}$$

(this will usually be written $\sum_{(v)} v_{(-2)} \otimes v_{(-1)} \otimes v_{(0)}$) and

$$\sum_{(v)} \epsilon(v_{(-1)}) v_{(0)} = v.$$

The following definition provides an example of a comodule.

Definition 1.12. Let G be a group. A G**-graded vector space** [Maj95] is a k-vector space V equipped with a decomposition into a direct sum of subspaces, which are indexed by G. In other words, the space can be written as $V = \bigoplus_{g\in G} V_g$. The subspaces V_g are called the graded components of V, and an element of one of the graded components is called a homogeneous vector in V.

Proposition 1.13. *A kG-comodule is precisely the same thing as a G-graded vector space, with the coaction given by, for any homogeneous $v_g \in V_g$, $\rho(v_g) = g \otimes v_g$, and extending linearly.*

Proof. It is easy to check that a G-graded vector space, with ρ as described above, forms a kG-comodule. For the converse, let V be a kG-comodule, and for all $g \in G$, let V_g denote the subspace of V consisting of $x \in V$ such that $\rho(x) = g \otimes x$. Fix any $v \in V$, and consider $\rho(v)$. Note that, by using the properties of the tensor product, any $c \in kG \otimes V$ can be written in the form $c = \sum_{g \in G} g \otimes c_g$, and this expression is unique, by linear independence of all g in kG. Thus, without loss of generality, we may write $\rho(v) = \sum_{g \in G} g \otimes v_g$, where v_g is taken as notation for the moment. Now consider that

$$\begin{aligned} \sum_{g \in G} g \otimes \rho(v_g) &= \sum_{g \in G} \Delta(g) \otimes v_g \\ &= \sum_{g \in G} g \otimes g \otimes v_g. \end{aligned}$$

By linear independence of all the g, we may equate coefficients and conclude that $\rho(v_g) = g \otimes v_g$ for all $g \in G$, and so $v_g \in V_g$, as suggested by the notation. This means that for an arbitrary $v \in V$, we have a collection $\{v_g\}_{g \in G}$. But we also have

$$\begin{aligned} \sum_{g \in G} \epsilon(g) v_g &= \sum_{g \in G} 1 v_g \\ &= \sum_{g \in G} v_g \\ &= v. \end{aligned}$$

So we have expressed an arbitrary v as a sum of elements of the subspaces V_g, meaning they generate V. However, this sum is unique, for if $v = \sum_{g \in G} w_g$, with each $w_g \in V_g$, then

$$\begin{aligned} \sum_{g \in G} g \otimes v_g &= \rho(v) \\ &= \rho(\sum_{g \in G} w_g) \\ &= \sum_{g \in G} \rho(w_g) \\ &= \sum_{g \in G} g \otimes w_g \end{aligned}$$

which means, as pointed out above, that $w_g = v_g$ for all $g \in G$. Therefore, $V = \bigoplus_{g \in G} V_g$, and the coaction works as described. □

A **comodule homomorphism** is a map $f \colon V \to W$ satisfying $(id_C \otimes f) \circ \rho_V = \rho_W \circ f$ or

$$\sum_{(v)} v_{(-1)} \otimes f(v_{(0)}) = \sum_{f(v)} f(v)_{(-1)} \otimes f(v)_{(0)}.$$

The class of C-comodules with their homomorphisms forms a category [Kas95, Maj95].

1.4 Bialgebras

1.4.1 Definition of a bialgebra

Coalgebras are not frequently dealt with on their own, however. The structure of a coalgebra is usually found within the context of, at least, a bialgebra. Here is one possible motivation for bialgebras, and several of the algebraic structures that follow. Whereas a module over the algebraic structure is defined to be just a module over the underlying k-algebra, the rest of the structure does have an effect on the category of modules. A good part of [Kas95] is spent explaining this. In particular the modules over a bialgebra form a monoidal category.

Definition 1.14. A **monoidal category** [Kas95] is a category $\mathcal{C}$ with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product, an object $\mathbf{1}$, called the unit, and natural isomorphisms

$$ass\colon \otimes \circ(\otimes \times id_{\mathcal{C}}) \to \otimes \circ (id_{\mathcal{C}} \times \otimes),$$

$$l\colon \otimes \circ(\mathbf{1} \times id_{\mathcal{C}}) \to id_{\mathcal{C}}$$

and

$$r\colon \otimes \circ(id_{\mathcal{C}} \times \mathbf{1}) \to id_{\mathcal{C}}$$

satisfying the "pentagon axiom",

$$(id_A \otimes ass_{(B,C,D)}) \circ ass_{(A,B\otimes C,D)} \circ (ass_{(A,B,C)} \otimes id_D) = ass_{(A,B,C\otimes D)} \circ ass_{(A\otimes B,C,D)},$$

and the "triangle axiom",

$$(id_A \otimes l_B) \circ ass_{(A,\mathbf{1},B)} = r_A \otimes id_B,$$

for all objects A, B, C and D in $\mathcal{C}$.

Of course, the notation is chosen because the category of vector spaces over k, with the previously defined tensor product as $\otimes$ and the one-dimensional space as $\mathbf{1}$, form a monoidal category, as may easily be checked. There are many other examples, such as sets with cartesian product for $\otimes$ and a singleton set for $\mathbf{1}$, and abelian groups with direct sum and the trivial group as $\otimes$ and $\mathbf{1}$.

Definition 1.15. A **bialgebra** [Swe69] is a vector space B with a multiplication ∇ and a unit η making B an algebra, a comultiplication Δ and a counit ϵ making B a coalgebra, such that ∇ and η are coalgebra homomorphisms or equivalently, that Δ and ϵ are algebra homomorphisms.

That these are equivalent may be seen by writing

$$\Delta \circ \nabla = (\nabla \otimes \nabla) \circ (id_B \otimes \tau_{B,B} \otimes id_B) \circ (\Delta \otimes \Delta) \tag{1.3}$$

$$\Delta \circ \eta = \eta \otimes \eta \tag{1.4}$$

$$\epsilon \circ \nabla = \epsilon \otimes \epsilon \tag{1.5}$$

$$\epsilon \circ \eta = id_k \tag{1.6}$$

and noting that (1.3) and (1.5) mean that ∇ is a coalgebra homomorphism and (1.4) and (1.6) mean that η is a coalgebra homomorphism, whereas (1.3) and (1.4) mean that Δ is an algebra homomorphism and (1.5) and (1.6) mean that ϵ is an algebra homomorphism [Dol10, DNR01].

1.4.2 Categories of modules or comodules

Let B be a bialgebra. The important details that turn the category B-*Mod* into a monoidal category are that for any two B-modules, V and W, B acts on $V \otimes W$ by

$$b \triangleright (v \otimes w) = \sum_{(b)} b_{(1)} \triangleright v \otimes b_{(2)} \triangleright w$$

for all $b \in B$, $v \in V$ and $w \in W$, and that there is a natural B-Module structure on k given by

$$b \triangleright \alpha = \epsilon(b)\alpha$$

for all $\alpha \in k$ [Kas95].

You can do something similar with comodules, which are simply comodules over the underlying coalgebra structure. If V and W are B-comodules, there is a comodule structure on $V \otimes W$ given by

$$\rho(v \otimes w) = \sum_{(v,w)} v_{(-1)} w_{(-1)} \otimes v_{(0)} \otimes w_{(0)},$$

and on k by sending $\alpha \in k$ to $1 \otimes \alpha$. It turns out [Maj95] that this makes the category of comodules a monoidal category as well, but we tend to concentrate on modules.

1.4.3 Basic properties of bialgebras

Proposition 1.16. *The cocommutative elements of a bialgebra form a subalgebra.*

Proof. It is easy to check that they are a subspace, and certainly 1 is cocommutative, for (1.4), written in Sweedler notation, is $\Delta(1) = 1 \otimes 1$. As for being closed under multiplication,

$$\begin{aligned}
\Delta_{op}(xy) &= \tau(\Delta(xy)) \\
&= \tau(\Delta(x)\Delta(y)) \\
&= \tau(\Delta(x))\tau(\Delta(y)) \\
&= \Delta_{op}(x)\Delta_{op}(y) \\
&= \Delta(x)\Delta(y) \\
&= \Delta(xy).
\end{aligned}$$

□

Definition 1.17. [Maj95] A **bialgebra homomorphism** is simply a linear map between bialgebras which is an algebra homomorphism and a coalgebra homomorphism. Likewise a **biideal** is a subspace which is both an ideal and a coideal. As one might expect, this is exactly what is needed to define a bialgebra on a quotient space. As it might not always be easy to find biideals, it is helpful to have easily verifiable propositions like the following:

Proposition 1.18. *Let J be a coideal of a bialgebra B. Then $\langle J \rangle$, the ideal generated by J, is a biideal.*

Proof. We only need to check that $\langle J \rangle$ is a coideal, and for that, let $x \in \langle J \rangle$. Now, x is a sum of monomials, each of which has at least one term in J. Applying Δ, the algebra homomorphism, to this, we find that $\Delta(x)$ can be written as a linear sum of elements of the form $\Delta(x_0)\Delta(x_1)\ldots\Delta(j)\ldots\Delta(x_z)$, where $j \in J$. We know that $\Delta(j) \in J \otimes B + B \otimes J$, whereas all the other terms are in $B \otimes B$. When these are all multiplied together, the resulting product must therefore be in $\langle J \rangle \otimes B + B \otimes \langle J \rangle$. This applies to all the monomials, and so $\Delta(x) \in \langle J \rangle \otimes B + B \otimes \langle J \rangle$, making $\langle J \rangle$ a coideal, as required. □

Example 1.19. $T(V)$ has a natural bialgebra structure on it [Kas95]. We know its algebra structure, we just need two algebra homomorphisms that give $T(V)$ a coalgebra structure. As $T(V)$ is the free algebra on V, we need only define them on V, so for all $v \in V$, let

$$\Delta(v) = v \otimes 1 + 1 \otimes v$$

(it is not hard to check that properties of the tensor product make this a linear map on V) and let $\epsilon(v) = 0$, so ϵ is the projection map onto the k component. Δ *is* coassociative:

$$\begin{aligned}(\Delta \otimes id)(\Delta(v)) &= (\Delta \otimes id)(v \otimes 1 + 1 \otimes v) \\ &= \Delta(v) \otimes 1 + \Delta(1) \otimes v \\ &= (v \otimes 1 + 1 \otimes v) \otimes 1 + 1 \otimes 1 \otimes v \\ &= v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v,\end{aligned}$$

and a simple computation shows that $(id \otimes \Delta)(\Delta(v))$ is the same. As for the counit, it cannot possibly be anything else, since we need

$$\begin{aligned}v &= \sum_{(v)} \epsilon(v_{(1)})v_{(2)} \\ &= \epsilon(v)1 + \epsilon(1)v \\ &= \epsilon(v)1 + 1_k v \\ &= \epsilon(v)1 + v\end{aligned}$$

so we must have $\epsilon(v) = 0$.

Elements with the property

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

are called **primitive** [Swe69]. These elements are of course cocommutative. We may denote the set of primitive elements of a bialgebra B by $\mathrm{prim}(B)$. Note that we have just shown that $\epsilon(x) = 0$ for any $x \in \mathrm{prim}(B)$. It is easy to see from the definition that any subspace of $\mathrm{prim}(B)$ is a coideal and, by Proposition 1.18, any ideal generated by primitives must be a biideal.

Proposition 1.20. *For a bialgebra B,* $\mathrm{prim}(B)$ *forms a Lie algebra under the commutator $[x, y] = xy - yx$.*

Proof. The whole bialgebra forms a Lie algebra under the commutator (as it is associative), so we need only check that $\mathrm{prim}(B)$ is a subalgebra of this. Indeed, it is easy to check that $\mathrm{prim}(B)$ is a subspace, and as for the commutator

$$\begin{aligned}
\Delta([x,y]) &= \Delta(xy-yx) \\
&= \Delta(x)\Delta(y)-\Delta(y)\Delta(x) \\
&= (x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y)-(y\otimes 1+1\otimes y)(x\otimes 1+1\otimes x) \\
&= xy\otimes 1+x\otimes y+y\otimes x+1\otimes xy-yx\otimes 1-y\otimes x-x\otimes y-1\otimes yx \\
&= (xy-yx)\otimes 1+1\otimes(xy-yx) \\
&= [x,y]\otimes 1+1\otimes[x,y].
\end{aligned}$$

□

That last computation shows that the embedding of V inside $T(V)$ is not the entirety of $\mathrm{prim}(T(V))$, but that the entire Lie subalgebra of $T(V)$ (considered as a Lie algebra under the commutator) generated by V consists of primitive elements. It can be shown [Kas95] that in characteristic zero this is in fact the entirety of $\mathrm{prim}(T(V))$, and moreover, that $\mathrm{prim}(T(V))$ is actually a free Lie algebra over V. We can sketch the proof of this: let $\mathfrak{F}(V)$ be a free Lie algebra over V and let $i\colon \mathfrak{F}(V)\to \mathrm{prim}(T(V))$ be the canonical homomorphism. It is surjective because every element of $\mathrm{prim}(T(V))$ consists of a sum of elements obtained from V by repeated applications of the commutator, each of which is a homomorphic image of a corresponding monomial in $\mathfrak{F}(V)$, with the Lie product replacing the commutator. It is injective because, if we let $l\in\mathfrak{F}(V)$ be nonzero, then $i(l)$ gives a nonzero polynomial in $\mathrm{prim}(T(V))$, which, by freeness of $T(V)$, must be a nonzero element of the algebra $\mathrm{prim}(T(V))$. Thus i gives us an isomorphism between $\mathrm{prim}(T(V))$ and a free Lie algebra, so $\mathrm{prim}(T(V))$ must be free.

Example 1.21. Looking at Examples 1.6 and 1.19, we find that the defining ideal of $\mathcal{U}(\mathcal{L})$, $\langle x\otimes y-y\otimes x-[x,y] \mid x,y\in\mathcal{L}\rangle$, is generated by elements which are primitive under the bialgebra structure of $T(\mathcal{L})$. Indeed, for all $x,y\in\mathcal{L}$, $[x,y]$ is primitive because it is in the generating space, which is defined to be primitive, and $x\otimes y-y\otimes x$ is primitive by Proposition 1.20, making $x\otimes y-y\otimes x-[x,y]$ primitive. This means that $\langle x\otimes y-y\otimes x-[x,y] \mid x,y\in\mathcal{L}\rangle$ is a biideal, and so the bialgebra structure of

$T(\mathcal{L})$ passes to the quotient, making $\mathcal{U}(\mathcal{L})$ a bialgebra.

Usually, however, we deal with bialgebras with a little extra structure.

Chapter 2

Hopf algebras

This chapter is mostly exposition. Along the way, a few proofs are given of common results that are rarely explained explicitly.

We define Hopf algebra, motivating it by monoidal categories with extra structure. We define Hopf ideals, and explain one way of constructing them. We prove that the grouplike elements of a Hopf algebra live up to their name – that is, they form a group – and hence they span a group Hopf algebra. We explain module algebras and module coalgebras, and use this to define a semidirect product of two Hopf algebras. Consequently, we can quote a theorem of Cartier, Kostant, Milnor and Moore, describing the structure of all cocommutative Hopf algebras where k is algebraically closed of characteristic zero, and give a few corollaries of this result. We go on to describe an important class of Hopf algebras, which can also be motivated by special monoidal categories; namely, quasitriangular Hopf algebras. Next we describe one of the most important concepts in this work: the Drinfel'd twist, for which we must also define 2-cocycles.

2.1 Definition and first principles

2.1.1 Rigid categories and Hopf algebras

Monoidal categories can have extra structure, as in the following definition. It is easy to verify that this is an abstraction of the concept of dual spaces in $k\text{-}vect_{f.d.}$.

Definition 2.1. A monoidal category is **rigid** [Kas95] if for each object V there is an object V^* and morphisms $b_V\colon \mathbf{1} \to V \otimes V^*$ and $d_V\colon V^* \otimes V \to \mathbf{1}$ satisfying

$$(id_V \otimes d_V)(b_V \otimes id_V) = id_V$$

and

$$(d_V \otimes id_{V^*})(id_{V^*} \otimes b_V) = id_{V^*}.$$

We may want to find if there is some extra structure one can have on a bialgebra that makes its category of modules rigid. It turns out that all that is needed is a way of making the dual space of a module into a module.

Definition 2.2. A **Hopf algebra** [Swe69] is a bialgebra H with a linear map $S\colon H \to H$, called the antipode, satisfying

$$\nabla \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon = \nabla \circ (id \otimes S) \circ \Delta$$

or

$$\sum_{(x)} S(x_{(1)})x_{(2)} = \epsilon(x)1 = \sum_{(x)} x_{(1)}S(x_{(2)})$$

for all $x \in H$.

Remark 2.3. It has a fairly tedious proof, but it is a basic fact of Hopf algebras [Kas95] that S is an algebra antiendomorphism, meaning that

$$S(xy) = S(y)S(x), \qquad S(1) = 1,$$

and a coalgebra antiendomorphism, meaning

$$(S \otimes S) \circ \Delta = \Delta^{op} \circ S \quad \text{and} \quad \epsilon \circ S = \epsilon.$$

A less tedious, and more routine proof, also explained in [Kas95], is that a linear map $S\colon H \to H$ is an antipode, provided it is an algebra antiendomorphism, and the antipode axiom is satisfied by a subset of H which generates H as an algebra.

Remark 2.4. If V is a module over H, then S turns V^* into an H-module by

$$(h \triangleright f)(x) = f(S(h) \triangleright x)$$

$\forall\, x \in V, h \in H$ and $f: V \to k$ [Kas95]. This is indeed an action, as if $j \in H$, then

$$\begin{aligned}(hj \triangleright f)(x) &= f(S(hj) \triangleright x)\\ &= f(S(j)S(h) \triangleright x)\\ &= f(S(j) \triangleright (S(h) \triangleright x))\\ &= (j \triangleright f)(S(h) \triangleright x)\\ &= (h \triangleright (j \triangleright f))(x).\end{aligned}$$

Definition 2.5. A **Hopf ideal** of a Hopf algebra is a biideal which is invariant under the antipode.

Its quotient space is a Hopf algebra under the induced operators [Maj95]. As with biideals, it is useful to be able to prove something is a Hopf ideal without explicitly checking every detail.

Proposition 2.6. *Let I be a coideal of a Hopf algebra H, that is invariant under the antipode, S. Then $\langle I \rangle$ is a Hopf ideal.*

Proof. We already know $\langle I \rangle$ is a biideal by Proposition 1.18, we only need to check that $\langle I \rangle$ is invariant under the antipode. Indeed, the proof of this is similar to that of Proposition 1.18: let $i \in \langle I \rangle$. Then i is a sum of monomials in H, each of which has a term in I. When we consider $S(i)$, we find that because S is an algebra antihomomorphism, $S(i)$ is also a sum of monomials, each of which has a term in I (the antipode of the term that was already in I), and thus $S(i) \in \langle I \rangle$, proving the proposition. □

2.1.2 Examples and basic properties

Example 2.7. [Kas95] Let G be a group, as in Example 1.3. There exists a Hopf algebra structure on the group algebra kG, with

$$\Delta(g) = g \otimes g \quad \epsilon(g) = 1 \quad S(g) = g^{-1}$$

for all $g \in G$. Note that this is also cocommutative, and that for any grouplike element g, $\epsilon(g) = 1$ because the counit axiom implies that $\epsilon(g)g = g$.

Proposition 2.8. *Let H be a Hopf algebra. Then $\mathcal{G}(H)$, the set of grouplike elements of H, forms a group under multiplication in H.*

Proof. It is easy to compute the product of two grouplikes using (1.3), and confirm that it, too, is grouplike. Furthermore, 1 is grouplike by (1.4). We want to show that $S(g)$ is the inverse of g. Firstly, $S(g)$ is grouplike because

$$\begin{aligned}\Delta(S(g)) &= (S\otimes S)(\Delta^{op}(g)) \\ &= (S\otimes S)(g\otimes g) \\ &= S(g)\otimes S(g).\end{aligned}$$

Finally, the antipode's axiom implies that

$$S(g)g = gS(g) = \epsilon(g)1 = 1.$$

□

Corollary 2.9. *The linear span of the grouplike elements, $k\mathcal{G}(H)$, is a sub-Hopf algebra of H isomorphic to the group algebra of $\mathcal{G}(H)$.*

Proof. Firstly, the set $\mathcal{G}(H)$ is linearly independent, as shown in [Swe69] (Prop. 3.2.1), so the subspace $k\mathcal{G}(H)$ has $\mathcal{G}(H)$ as a basis. Now we may verify the result by confirming that all five operations work as we expect. As long as they work that way on the basis, then they will extend to the whole of the space $k\mathcal{G}(H)$ as stated. Indeed, the multiplication works as alleged by the previous result, and $1 \in k\mathcal{G}(H)$ by definition. $\Delta(g) = g\otimes g\ \forall g \in \mathcal{G}(H)$ by definition, and extended linearly, the image of Δ must still be inside $k\mathcal{G}(H)\otimes k\mathcal{G}(H)$. The counit agrees by (2.7), and the previous result proves that $S(g) = g^{-1}$, $\forall\, g \in \mathcal{G}(H)$. □

Proposition 2.10. *If $x \in H$ is primitive, then $S(x) = -x$.*

Proof. We may compute

$$\begin{aligned}\sum_{(x)} x_{(1)}S(x_{(2)}) &= xS(1) + 1S(x) \\ &= x1 + S(x) \\ &= x + S(x).\end{aligned}$$

But this must equal $\epsilon(x)1 = 0\cdot 1 = 0$, so $x + S(x) = 0$ i.e. $S(x) = -x$. □

Example 2.11. By an echo of the preceding argument, we may turn $T(V)$ into a Hopf algebra by defining the antipode to be $S(v) = -v$, $\forall v \in V$, and extending S to $T(V)$ by it being an algebra antiendomorphism [Kas95]. By Remark 2.3, this defines an antipode on $T(V)$. This also serves to define a Hopf algebra structure on $\mathcal{U}(\mathcal{L})$, in the same manner as in Example 1.21. For the subspace of $\text{prim}(T(\mathcal{L}))$ spanned by the generators of the defining ideal of $\mathcal{U}(\mathcal{L})$, which must be a coideal, is invariant by the previous proposition. By Proposition 2.6, $\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathcal{L}\rangle$ is a Hopf ideal of $T(\mathcal{L})$, and so $\mathcal{U}(\mathcal{L})$ is also a Hopf algebra.

It seems natural to think that there should be an analogue of Corollary 2.9 for primitive elements; i.e. that I, the subalgebra generated by $\text{prim}(H)$, is a sub-Hopf algebra isomorphic to the universal enveloping algebra $\mathcal{U}(\text{prim}(H))$. Most of this may be easily verified. It is a sub-Hopf algebra: if $x \in I$, then x can be written as $\sum_{j=1}^{n} y_{1_j} y_{2_j} \dots y_{l_j} \dots y_{m_j}$, where $n \in \mathbb{N}$, $m_j \in \mathbb{N}$ for all $1 \le j \le n$, and each y_{l_j} is primitive. When we compute $\Delta(x)$, we find that

$$\begin{aligned}
\Delta(x) &= \Delta(\sum_{j=1}^{n} y_{1_j} y_{2_j} \dots y_{l_j} \dots y_{m_j}) \\
&= \sum_{j=1}^{n} \Delta(y_{1_j})\Delta(y_{2_j}) \dots \Delta(y_{l_j}) \dots \Delta(y_{m_j}) \\
&= \sum_{j=1}^{n} (y_{1_j} \otimes 1 + 1 \otimes y_{1_j})(y_{2_j} \otimes 1 + 1 \otimes y_{2_j}) \\
&\quad \dots (y_{l_j} \otimes 1 + 1 \otimes y_{l_j}) \dots (y_{m_j} \otimes 1 + 1 \otimes y_{m_j}) \\
&= \sum_{j=1}^{n} (\sum_{A \in \mathcal{P}\{y_{1_j} \dots y_{m_j}\}} (\prod_{y_t \in A} y_t \otimes \prod_{y_t \in A^c} y_t)).
\end{aligned}$$

Each of those expressions on either side of the tensor product symbol is a product of primitives and therefore in I, which means that the whole sum, which equals $\Delta(x)$, is in $I \otimes I$, making the comultiplication well-defined. Meanwhile, ϵ doesn't need to be checked, as any restriction to any subspace will still leave a well-defined map to k. As for the antipode,

$$\begin{aligned} S(x) &= S(\sum_{j=1}^{n} y_{1_j} y_{2_j} \dots y_{l_j} \dots y_{m_j}) \\ &= \sum_{j=1}^{n} S(y_{m_j}) S(y_{m_j - 1}) \dots S(y_{l_j}) \dots S(y_{1_j}) \\ &= \sum_{j=1}^{n} (-y_{m_j})(-y_{m_j - 1}) \dots (-y_{l_j}) \dots (-y_{1_j}) \\ &= \sum_{j=1}^{n} \pm y_{m_j} y_{m_j - 1} \dots y_{l_j} \dots y_{1_j}. \end{aligned}$$

This is obviously in I, so I is a sub-Hopf algebra.

Furthermore, it is easy to construct a natural surjective map $i\colon \mathcal{U}(\text{prim}(H)) \to I$. For any $x \in I$, that can be written as $\sum_{j=1}^{n} y_{1_j} y_{2_j} \dots y_{l_j} \dots y_{m_j}$, there is a corresponding $z \in \mathcal{U}(\text{prim}(H))$ that is written the same way. Let $i(z) = x$. It is easy to see that this is an algebra homomorphism, because it is in fact the commuting homomorphism that exists on account of the embedding of $\text{prim}(H)$ into both algebras, and the universal property of $\mathcal{U}(\text{prim}(H))$. By commuting, this map takes primitive elements to primitive elements. One may check that this last property, along with the fact that S is an algebra antihomomorphism and Δ, ϵ and i are all algebra homomorphisms, makes i a homomorphism of Hopf algebras.

The problem is that this map might turn out not to be an isomorphism, but only a quotient map, as in the following example.

Example 2.12. Suppose we are working over the field $\mathbb{F}_p$. Let us begin with the algebra $\frac{\mathbb{F}_p[x]}{(x^p)}$. We may put a Hopf algebra structure on $\frac{\mathbb{F}_p[x]}{(x^p)}$ by defining x to be primitive. This turns out to be well defined by the Freshman's Dream. In this case, the space of primitives is the space spanned by x, so the algebra generated by this is the whole of $\frac{\mathbb{F}_p[x]}{(x^p)}$. This is not, however, isomorphic to the universal enveloping algebra over the one-dimensional Lie algebra kx, which turns out to be just the polynomial algebra $\mathbb{F}_p[x]$.

2.2 H-module algebras and H-module coalgebras

2.2.1 Module algebras

When it comes to H-modules, it seems that, being simply modules over the underlying algebra, the module itself provides no information about the Hopf algebra structure, and that one must look at the category of H-modules to find out about this. But there is a kind of H-module with extra structure, which affords information about more than just the algebra structure of H. A good exposition and motivation of this, as well as of Hopf algebras themselves, can be found in [Ber85]. Here is a condensed explanation.

Definition 2.13. Let H be a Hopf algebra. An H**-module algebra** [Maj95, Kas95] is a k-algebra, A, such that A is an H-module (as a k-vector space), and the multiplication and unit maps of A are both H-module homomorphisms. In other words, that for all $h \in H$, $a, b \in A$,

$$h \triangleright (ab) = \sum_{(h)} (h_{(1)} \triangleright a)(h_{(2)} \triangleright b),$$

and

$$h \triangleright 1 = \epsilon(h)1.$$

As an example, if A is a kG-module algebra, then A is a kG-module – in other words, A is a k-vector space with a G-action. It is also a k-algebra, with the multiplication and unit maps being homomorphisms of group representations. But when those conditions are written down explicitly, it becomes apparent that this is equivalent to the action of G being an algebra homomorphism. Thus a kG-module algebra is precisely a k-algebra A on which G acts by endomorphisms.

2.2.2 Module coalgebras

There is also a kind of dual notion, namely:

Definition 2.14. An H**-module coalgebra** [Maj95] is a coalgebra C which is a module over H, such that the comultiplication and counit maps are module homomorphisms. Explicitly, this means that

$$\Delta(h \triangleright c) = \sum_{(h)(c)} h_{(1)} \triangleright c_{(1)} \otimes h_{(2)} \triangleright c_{(2)}$$

and

$$\epsilon(h \triangleright c) = \epsilon(h)\epsilon(c)$$

for all $c \in C$, $h \in H$.

One can check as an exercise that a kG-module coalgebra is precisely a k-coalgebra on which G acts by homomorphisms.

2.2.3 Semidirect products

It seems natural to try to combine the notions, and come up with a module bialgebra, or Hopf algebra. In fact, it turns out that the antipode is irrelevant, and we can come up with the following definition.

Definition 2.15. Given two Hopf algebras, A and B, we say that A **acts on** B [Kas95] when B is simultaneously a A-module algebra and a A-module coalgebra such that for all $a \in A$, $b \in B$,

$$\sum_{(a)} a_{(1)} \otimes a_{(2)} \triangleright b = \sum_{(a)} a_{(2)} \otimes a_{(1)} \triangleright b.$$

Note that this is always satisfied when A is cocommutative.

Now that we have a notion of one Hopf algebra acting on another, it makes sense to define a semidirect product, as with other algebraic structures. The definition given here is a special case of a construction described in [Kas95], the bicrossproduct, which we shan't be giving here.

Definition 2.16. Given two Hopf algebras, A and B, with A acting on B, we may define the **semidirect product of** A **and** B**,** $B \rtimes A$, on the space $B \otimes A$ as follows: for all $x, y \in A$, $z, w \in B$,

$$(z \otimes x)(w \otimes y) = \sum_{(x)} z(x_{(1)} \triangleright w) \otimes x_{(2)} y,$$

$$\Delta(z \otimes x) = \sum_{(z)(x)} (z_{(1)} \otimes x_{(1)}) \otimes (z_{(2)} \otimes x_{(2)}),$$

$$\epsilon(z \otimes x) = \epsilon(z)\epsilon(x),$$

$$S(z \otimes x) = \sum_{(x)} S_A(x_{(2)}) \triangleright S_B(z) \otimes S_A(x_{(1)}).$$

The unit is $1 \otimes 1$.

Example 2.17. Let G be a group, $\mathfrak{g}$ a Lie algebra, and let G act on $\mathfrak{g}$ by Lie algebra homomorphisms. As a vector space, $\mathfrak{g}$ is a G-module, and therefore also a kG-module, even when kG is regarded as a Hopf algebra, since a module over a Hopf algebra is just a module over the underlying algebra. We can extend the action of kG on $\mathfrak{g}$ to the whole of the Hopf algebra $\mathcal{U}(\mathfrak{g})$ by specifying that kG must act by algebra endomorphisms (here it matters that G acts by homomorphisms, because we need the action to preserve the ideal of $T(\mathfrak{g})$, $\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$). Looking at Definition 2.13, this turns $\mathcal{U}(\mathfrak{g})$ into not only a module, but also a module algebra. But it is also a module coalgebra, as for any $g \in G, x \in \mathfrak{g}$,

$$\begin{aligned} \Delta(g \triangleright x) &= 1 \otimes g \triangleright x + g \triangleright x \otimes 1 \\ &= g \triangleright 1 \otimes g \triangleright x + g \triangleright x \otimes g \triangleright 1 \end{aligned}$$

and it is not hard to check that this extends to the whole of kG and $\mathcal{U}(\mathfrak{g})$, and the counit's condition is easy to check as well. Since kG is cocommutative, this is all that we need for kG to act on $\mathcal{U}(\mathfrak{g})$ as Hopf algebras. Thus we may define the semidirect product $\mathcal{U}(\mathfrak{g}) \rtimes kG$ as above.

2.3 The Cartier-Kostant-Milnor-Moore theorem

This turns out to be quite an important kind of Hopf algebra – and all we needed for it to exist was a group acting on a Lie algebra. It isn't so hard to find such as that either, for it is a routine exercise to show that in any Hopf algebra H, $\mathcal{G}(H)$ acts homomorphically on $\mathrm{prim}(H)$. In any case where the subalgebra generated by $\mathrm{prim}(H)$ is isomorphic to $\mathcal{U}(\mathrm{prim}(H))$, the above construction allows us to find a Hopf subalgebra which is a semidirect product of the grouplikes acting on the primitives. And in a certain case, there is a fundamental theorem about this kind of Hopf algebra. It is a composite result, made by combining results from [Car62, Kos75, MM65, Swe69].

Theorem 2.18 (Cartier-Kostant-Milnor-Moore [And02]). *Let k be an algebraically closed field of characteristic zero, and let H be a cocommutative Hopf algebra over k. Then*

$$H = \mathcal{U}(\mathrm{prim}(H)) \rtimes k\mathcal{G}(H),$$

where $\mathcal{G}(H)$ acts on $\mathrm{prim}\, H$ *by conjugation.*

There is a corollary to this: the classical PBW theorem implies that the universal enveloping algebra of any Lie algebra that isn't zero dimensional must be an infinite dimensional algebra. So in the above case, if $\text{prim}(H)$ is not the trivial subspace, then H, which contains $\mathcal{U}(\text{prim}(H))$ as a subalgebra, must be infinite dimensional. Conversely, H can only be finite dimensional if $\text{prim}(H) = \{0\}$, in which case $\mathcal{U}(\text{prim}(H)) \cong k$, implying the following.

Corollary 2.19. *If k is algebraically closed, $\text{char}\, k = 0$, and H is cocommutative and finite-dimensional, then $H \cong kG$, for some finite group G.*

Furthermore, the issue mentioned at the end of the previous section is solved for this special case by the C-K-M-M theorem.

Corollary 2.20. *Suppose the ground field k has characteristic zero, and is algebraically closed. Then for H, a Hopf algebra over k, I, the sub-Hopf algebra generated by $\text{prim}(H)$, is isomorphic to $\mathcal{U}(\text{prim}(H))$.*

Proof. Since $\text{prim}(H)$ is cocommutative, by the proof of Proposition 1.16, I is cocommutative as well. Thus, by the Cartier-Kostant-Milnor-Moore theorem, $I = \mathcal{U}(\text{prim}(I)) \rtimes k\mathcal{G}(I) = \mathcal{U}(\text{prim}(H)) \rtimes k\mathcal{G}(I)$. But since I is generated by $\text{prim}(H)$, $I = \mathcal{U}(\text{prim}(H))$. □

This implies that, under the above hypotheses, if $\dim(\text{prim}(H)) > 0$, then H contains the infinite dimensional $\mathcal{U}(\text{prim}(H))$. Therefore if H is finite dimensional, then we must have $\text{prim}(H) = \{0\}$.

2.4 Quasitriangular Hopf algebras

2.4.1 Braided monoidal categories and quasitriangular Hopf algebras

So far we have seen only cocommutative Hopf algebras. It's time we looked a bit more widely. We describe an important class of Hopf algebras that Drinfel'd introduced in [Dri90], that may be motivated in a similar manner to that of bialgebras and Hopf algebras.

Definition 2.21. A **braided monoidal category** [Kas95] is a monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ with a natural isomorphism

$$c\colon \otimes \to \otimes \circ \tau$$

(where τ is the swap map $(A, B) \to (B, A)$) satisfying the two "hexagon axioms"

$$ass_{(B,C,A)} \circ c_{A,B\otimes C} \circ ass_{(A,B,C)} = (id_B \otimes c_{A,C}) \circ ass_{(B,A,C)} \circ (c_{A,B} \otimes id_C)$$

and

$$ass_{(C,A,B)}{}^{-1} \circ c_{A\otimes B,C} \circ ass_{(A,B,C)}{}^{-1} = (c_{A,C} \otimes id_B) \circ ass_{(A,C,B)}{}^{-1} \circ (id_A \otimes c_{B,C})$$

for all objects A, B and C.

Obviously, c corresponds with the map (1.2) for the category of k-vector spaces. It can be shown that k-*vect* is indeed braided. The name "braided" is used because it can be proven that the two hexagon axioms imply that c satisfies a condition similar to the braid relation. In fact, it turns out that the class of braids can itself be turned into a braided monoidal category [Kas95].

There is a certain class of Hopf algebras whose extra structure gives its module category the structure of a braided (rigid) monoidal category.

Firstly, some notation: if we have an element

$$X = \sum_{(i)} s_i \otimes t_i \in H \otimes H$$

then, to explain the convention by means of a few examples, $X_{21} \in H \otimes H$ denotes

$$\sum_{(i)} t_i \otimes s_i,$$

$X_{13} \in H \otimes H \otimes H$ denotes

$$\sum_{(i)} s_i \otimes 1 \otimes t_i,$$

$X_{43} \in H^{\otimes 5}$ denotes

$$\sum_{(i)} 1 \otimes 1 \otimes t_i \otimes s_i \otimes 1$$

etc.

Definition 2.22. A **quasitriangular** Hopf algebra [Maj95] is a Hopf algebra H with an invertible element

$$R \in H \otimes H,$$

often called the R-matrix, for which H is cocommutative up to conjugation by R:

$$\Delta^{op}(x) = R\Delta(x)R^{-1} \qquad \forall x \in H,$$

and which satisfies

$$(\Delta \otimes id_H)(R) = R_{13}R_{23} \quad \text{and} \quad (id_H \otimes \Delta)(R) = R_{13}R_{12}. \tag{2.1}$$

2.4.2 Properties of quasitriangular Hopf algebras

Proposition 2.23. *An R-matrix has the following properties:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

and

$$(\epsilon \otimes id)(R) = (id \otimes \epsilon)(R) = 1. \tag{2.2}$$

The first is known as the quantum Yang-Baxter equation.

Proof.

$$\begin{aligned} R_{12}R_{13}R_{23} &= (R \otimes 1)(\Delta \otimes id_H)(R) \\ &= (R \otimes 1)(R^{-1} \otimes 1)(\Delta^{op} \otimes id_H)(R)(R \otimes 1) \\ &= (\tau \otimes id_H)((\Delta \otimes id_H)(R))(R \otimes 1) \\ &= (\tau \otimes id_H)(R_{13}R_{23})R_{12} \\ &= R_{23}R_{13}R_{12} \end{aligned}$$

and

$$\begin{aligned} R &= (\epsilon \otimes id \otimes id)(\Delta \otimes id)(R) \\ &= (\epsilon \otimes id \otimes id)(R_{13}R_{23}) \\ &= (\epsilon \otimes id)(R)R. \end{aligned}$$

Postmultiplying by R^{-1}, we get $(\epsilon \otimes id)(R) = 1$. A similar argument shows that $(id \otimes \epsilon)(R) = 1$ [Kas95]. □

Remark 2.24. If H is quasitriangular, then the category of H-modules is braided because R defines a natural map

$$c_{V,W}\colon V \otimes W \to W \otimes V$$

for all modules V, W by

$$c_{V,W}(v \otimes w) = \tau(R \triangleright (v \otimes w)),$$

$\forall v \in V, w \in W$. It can be checked that the quantum Yang-Baxter equation is an "untwisted" version of the braid relation, and this makes H-*Mod* a braided monoidal category [Kas95].

We now define a generalisation of an R-matrix.

2.5 The Drinfel'd twist

2.5.1 The twist of a Hopf algebra's comultiplication

Definition 2.25. [Maj95] Let H be a Hopf algebra. A **2-cocycle** for H is an invertible element

$$\chi = \sum_{i=1}^{n} a_i \otimes b_i \in H \otimes H$$

such that,

$$(1 \otimes \chi)(id \otimes \Delta)(\chi) = (\chi \otimes 1)(\Delta \otimes id)(\chi).$$

We say that χ is **counital** if

$$\sum_{i=1}^{n} \epsilon(a_i) b_i = \sum_{i=1}^{n} a_i \epsilon(b_i) = 1.$$

An R-matrix is a 2-cocycle because the first relation becomes the quantum Yang-Baxter equation when equations (2.1) are substituted in and the second is a rewriting of (2.2).

Counital 2-cocycles are useful because they allow us to *twist* Hopf algebras, as follows.

Definition 2.26. Let H be a Hopf algebra, and let χ be a counital 2-cocycle for H. Then there is a Hopf algebra H_χ, called the **Drinfel'd twist of H by χ** [Maj95], defined on the same underlying space as H, with the same multiplication, unit and counit, comultiplication given by conjugation with χ i.e.

$$\Delta_\chi(x) = \chi\Delta(x)\chi^{-1},$$

and antipode given by conjugation with $U = \sum_{i=1}^{n} a_i S(b_i)$, that is,

$$S_\chi(x) = US(x)U^{-1}.$$

It turns out there is a corresponding twist for module algebras: If A is an H-module algebra, and χ is a counital 2-cocycle for H, then there is an algebra A_χ, which is a twisted version of A, defined on the same space as A with multiplication given by

$$\nabla_\chi(a \otimes b) = \nabla(\chi^{-1} \triangleright (a \otimes b))$$

for all a, b in the underlying space of A. The unit is the same. According to [Maj95], this algebra is naturally acted upon by H_χ, as a H_χ-module algebra.

2.5.2 The twist of a Hopf algebra's multiplication

There is also a dual to this: a **unital 2-cocycle *on*** H [Maj95] is a map $\chi\colon H \otimes H \to k$ with another map $\bar{\chi}\colon H \otimes H \to k$ satisfying

$$\begin{aligned} \sum_{(x,y)} \chi(x_{(1)} \otimes y_{(1)})\bar{\chi}(x_{(2)} \otimes y_{(2)}) &= \sum_{(x,y)} \bar{\chi}(x_{(1)} \otimes y_{(1)})\chi(x_{(2)} \otimes y_{(2)}) \\ &= \epsilon(x)\epsilon(y) \end{aligned} \tag{2.3}$$

for all $x, y \in H$, for which

$$\sum_{(y,z)} \chi(y_{(1)} \otimes z_{(1)})\chi(x \otimes y_{(2)}z_{(2)}) = \sum_{(x,y)} \chi(x_{(1)} \otimes y_{(1)})\chi(x_{(2)}y_{(2)} \otimes z)$$

for all $x, y, z \in H$, and which satisfies

$$\chi(x \otimes 1) = \chi(1 \otimes x) = \epsilon(x)$$

for any $x \in H$.

You can also twist a Hopf algebra via one of these unital 2-cocycles, only this time it is the multiplication that is twisted: $\nabla_\chi = (\chi \otimes \nabla \otimes \bar{\chi}) \circ (\Delta_{H\otimes H})^2$ or in Sweedler notation,

$$a \cdot_\chi b = \sum_{(a)(b)} \chi(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}\bar{\chi}(a_{(3)} \otimes b_{(3)}).$$

Again, you have to twist the antipode, and to do this, let

$$U = \chi \circ (id \otimes S) \circ \Delta.$$

It turns out [Maj95] there's a $\bar{U} \colon H \to k$ such that U and $\bar{U}$ satisfy (2.3) just like χ and $\bar{\chi}$. Then you define $S_\chi = (U \otimes S \otimes \bar{U}) \circ \Delta^2$, or, in Sweedler notation,

$$S_\chi(x) = \sum_{(x)} U(x_{(1)}) S(x_{(2)}) \bar{U}(x_{(3)}).$$

Chapter 3

Twists and modules

In this chapter we explain key ideas that we shall be using hereafter. We define Yetter-Drinfel'd modules, and the concept of algebras in a monoidal category, of which many previous notions are examples. Following this, we explain how to extend the Drinfel'd twist to a comodule algebra and to a Yetter-Drinfel'd module. Throughout, we apply these concepts to the case where H is a group algebra, because that is the case that we shall mainly be using. This means that we use a 2-cocycle on a group to twist an algebra graded by that group.

3.1 Yetter-Drinfel'd modules

One may wonder if there is a particularly natural example of a quasitriangular Hopf algebra. It turns out that for any finite-dimensional Hopf algebra H with invertible antipode, there is a construction by Drinfel'd called the **quantum double** [Kas95], which defines a quasitriangular Hopf algebra structure on the space $H^* \otimes H$. Of course, the category of modules over a quantum double will be braided. It turns out that one is often only interested in this module category. Since the construction of the quantum double is complicated to describe, and it seems to be inefficient to describe the algebra just to say that we are only interested in its modules, it is useful that there exists an equivalent condition to being a module over the quantum double.

Definition 3.1. Let H be a Hopf algebra. The k-vector space V is called a **Yetter-Drinfel'd module over** H (or ${}^H_H\mathcal{YD}$-module) [And02] iff V is both an H-module and an H-comodule such that

$$\rho(h \triangleright v) = \sum_{(h)(v)} h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \triangleright v_{(0)}$$

for all $h \in H, v \in V$.

According to [And02], given a finite dimensional H, V is a ${}^H_H\mathcal{YD}$-module iff V is a module over the quantum double of H. Furthermore, the class of Yetter-Drinfel'd modules over H forms a category, denoted ${}^H_H\mathcal{YD}$, with morphisms given by maps which are both module homomorphisms and comodule homomorphisms. This is a braided monoidal category, with braiding

$$c_{V,W} \colon V \otimes W \to W \otimes V$$

given by

$$c_{V,W}(v \otimes w) = \sum_{(v)} (v_{(-1)} \triangleright w) \otimes v_{(0)}$$

for all $v \in V, w \in W$.

Example 3.2. Let the Hopf algebra be kG, and let V be a ${}^{kG}_{kG}\mathcal{YD}$-module (it is conventional to write ${}^G_G\mathcal{YD}$ in the case of group algebras). Then V is a kG-comodule, meaning V is G-graded, and a kG-module, meaning G acts on V. The compatibility condition means that for any $v \in V_g$ and $h \in G$,

$$\rho(hv) = hgS(h) \otimes h \triangleright v = hgh^{-1} \otimes hv.$$

In other words, that

$$hv \in V_{hgh^{-1}}.$$

It is easy to verify that if this holds for all group elements and homogeneous vectors, then extending by linearity, the Yetter-Drinfel'd condition is satisfied for the whole kG and V. Therefore, a ${}^G_G\mathcal{YD}$-module is precisely a G-graded vector space V acted on by G such that for all $g, h \in G$,

$$hV_g \subseteq V_{hgh^{-1}}.$$

3.2 Algebras in monoidal categories

It turns out that the definition of algebra can be generalised yet further.

Definition 3.3. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category. An **algebra in** $\mathcal{C}$ [ML71] is an object in $\mathcal{C}$, A, with arrows

$$\nabla \colon A \otimes A \to A \quad \text{and} \quad \eta \colon \mathbf{1} \to A$$

satisfying

$$\nabla \circ (\eta \otimes id_A) = l_A,$$

$$\nabla \circ (id_A \otimes \eta) = r_A$$

and

$$\nabla \circ (\nabla \otimes id_A) = \nabla \circ (id_A \otimes \nabla) \circ ass_{(A,A,A)}.$$

It is easy to see that when $\mathcal{C}$ is *k-vect*, this becomes the tensor product definition of a k-algebra. Algebras are sometimes called monoids because in the category of sets and functions, that is precisely what an algebra is [ML71]. Many examples, however, are objects in a monoidal category which are also k-algebras, for which one must check that the multiplication and unit maps are morphisms in the category. In particular, note that an H-module algebra is simply an algebra in the category H*-Mod*. Similarly, since the comodules of a Hopf algebra also form a monoidal category, we may define an algebra here.

Definition 3.4. An **H-comodule algebra** [Maj95] is a k-algebra A which is also an H-comodule such that the algebra maps are both H-comodule homomorphisms. Specifically, that

$$\rho(ab) = \rho(a)\rho(b)$$

for all $a, b \in A$ and

$$\rho(1) = 1 \otimes 1.$$

As an example, consider the case of a kG-comodule algebra. We already know what a kG-comodule is, and that the category of kG-comodules is monoidal, but it might be helpful to explain this more explicitly.

Let G be a fixed group. We define the monoidal category G*-vect* [Kas95] as follows. Its objects are G-graded vector spaces and its morphisms are linear maps that preserve the grading. That is, the linear map $f \colon A \to B$ is graded iff

$$f(A_g) \subseteq B_g$$

for all $g \in G$. The tensor product is the usual tensor product with grading given by

$$(A \otimes B)_g := \bigoplus_{\{x,y \in G | xy = g\}} A_x \otimes B_y.$$

One may check that, with this definition, the tensor product of two graded maps is graded. The grading on the field k, considered as a one-dimensional vector space, is to assign the whole space to the graded component of the identity of the group; the other components will all be zero-dimensional. It is easy to check that, with these definitions, $ass_{(A,B,C)}$, l_A and r_A as defined for *k-vect* are all graded maps for all $A, B, C \in$ *G-vect*, making it a monoidal category with the same natural isomorphisms.

Evidently, an algebra in this category is a G-graded vector space A with graded maps $\nabla \colon A \otimes A \to A$ and $\eta \colon k \to A$, satisfying the associativity and unital axioms, which makes A a k-algebra. The fact that the multiplication and unit maps are graded means that

$$\eta(k) \subseteq A_1$$

and, more importantly, that

$$\nabla(A_g \otimes A_h) \subseteq A_{gh}.$$

In other words, an algebra in *G-vect* is the same thing as a G-graded algebra, defined as follows.

Definition 3.5. A **G-graded algebra** [Maj95] is a G-graded vector space, V, that is also a k-algebra such that

$$V_g V_h \subseteq V_{gh}$$

for all $g, h \in G$ and

$$1 \in V_1.$$

We should note that if $\mathcal{C}$ has countably infinite direct sums, then each object V of $\mathcal{C}$ will have a "tensor algebra", defined by analogy with the usual definition, which we may notate by $T(\ )$. This will still be an object of $\mathcal{C}$, but observe that $\mathbf{1}$ is embedded in $T(V)$, and that we may define a multiplication map $T(V) \otimes T(V) \to T(V)$ the usual way, componentwise. It is clear that this is associative, and that the embedding $\mathbf{1} \to T(V)$ defines a unital map, making $T(V)$ a $\mathcal{C}$-algebra. Moreover, $T(V)$ is still free: a map $f \colon V \to A$, where A is a $\mathcal{C}$-algebra, extends canonically to an algebra homomorphism from $T(V)$ to A, by defining f on the component V^n to be f^n. This means that a map $V \to W$ lifts to a map $T(V) \to T(W)$. It is easy to check that this is functorial, meaning that $T(\ )$ defines a functor $\mathcal{C} \to \mathcal{C}$-*alg*.

If A is a $\mathcal{YD}$-module, then A is an algebra if and only if A is a k-algebra whose multiplication and unit maps are homomorphisms of $\mathcal{YD}$-modules. But they are $\mathcal{YD}$-module homomorphisms precisely if they are both module and comodule homomorphisms. In other words, A is a $\mathcal{YD}$-module algebra precisely when A is a k-algebra, a $\mathcal{YD}$-module, a module algebra and a comodule algebra. Applied to the specific case of Example 3.2, a ${}^G_G\mathcal{YD}$-module algebra is a G-graded algebra, A, on which G acts by homomorphisms, such that for all $g, h \in G$,

$$hA_g \subseteq A_{hgh^{-1}}.$$

One can also dualise this to define a coalgebra in a monoidal category, which gives us objects like module coalgebras, which are coalgebras in H-*Mod* [Maj95, ML71]. Once again, we tend to focus more on algebras.

3.3 How to twist a $\mathcal{YD}$-module

Now that we have mentioned comodule algebras, it makes sense to define the twist of one. Let H be a Hopf algebra, and let $\chi\colon H \otimes H \to k$ be a unital 2-cocycle on H. We already have a twisted Hopf algebra, H^χ, defined dually to H_χ in Section 2.5. Let A be a comodule algebra over H. Then there exists [Maj95] a H^χ-comodule algebra A^χ which is identical to A as a H-comodule (note that this is the same thing as a H^χ-comodule, since the coalgebra structure is identical), but with twisted multiplication given by

$$\nabla^\chi(a \otimes b) = \sum_{(a,b)} \chi(a_{(-1)} \otimes b_{(-1)}) a_{(0)} b_{(0)}.$$

Given this, it seems natural to wonder if there are any other combinations that can be twisted. Preferably using cocycles on H rather than cocycles for H since, on the whole, we would prefer to be twisting multiplication. In particular, can Yetter-Drinfel'd modules, which are both modules and comodules, be twisted? Indeed, a way of doing this was discovered in [MO99]. As the coalgebra structure of H remains unchanged under twisting, it is unsurprising that the coaction remains untwisted, and that we only need to modify the action. Note, however, that we need a Yetter-Drinfel'd module and not just a module. Let V be a ${}_H^H\mathcal{YD}$-module. Then we may define V^χ, which is V with a ${}_{H^\chi}^{H^\chi}\mathcal{YD}$-module structure, by leaving the coaction as it is and defining a new action by

$$h \triangleright^\chi v = \sum_{(h,v,(h_{(2)} \triangleright v_{(0)}))} \chi(h_{(1)} \otimes v_{(-1)})(h_{(2)} \triangleright v_{(0)})_{(0)}\bar{\chi}((h_{(2)} \triangleright v_{(0)})_{(-1)} \otimes h_{(3)}).$$

One may wish to know if these ideas can be combined; if, by simultaneously twisting both action and multiplication as above, we can twist a Yetter-Drinfel'd module algebra. In fact, the result that [MO99] proved was that there is an isomorphism of monoidal categories between ${}_H^H\mathcal{YD}$ and ${}_{H^\chi}^{H^\chi}\mathcal{YD}$, defined by twisting one object into another. This means that an an algebra in one is an algebra in the other, and we only need to check what the multiplication is in the twisted algebra. It turns out that, due to the way the twisted monoidal product is defined, this comes exactly to the formula above, which means that, by combining the two previous concepts, we can twist a $\mathcal{YD}$-module algebra. Note that, although the coaction is unchanged, it is relevant as it is used to define the twisted action and multiplication. This result does not allow us to twist an arbitrary module algebra, we need it to be a $\mathcal{YD}$-module algebra for this to work.

Now we examine a special case of cocycle twisting, that in which the Hopf algebra is kG. This simplifies many aspects of the Drinfel'd twist. The following exposition is partly based on that found in [BB13].

Let $\chi\colon kG \otimes kG \to k$ be a unital 2-cocycle on kG. First note that since all conditions that will need to be satisfied are linear, we can do everything with respect to the basis and treat χ as a function from $G \times G$ to k. Next consider condition 2.3. We have

$$\epsilon(g)\epsilon(h) = 1$$

for all $g, h \in G$, and since kG is cocommutative, we only need to consider one of the remaining expressions. Thus condition 2.3 says that there exists another function $\bar{\chi}\colon G \times G \to k$ satisfying

$$\chi(g, h)\bar{\chi}(g, h) = 1$$

for all g and h. But one may observe that this will be true as long as $\chi(g, h)$ is always nonzero for any pair of arguments (in which case we of course have $\bar{\chi}(g, h) = \chi(g, h)^{-1}$).

So condition 2.3 simply tells us that χ is a map from $G \times G$ to $k^* = k \setminus \{0\}$. The other two conditions tell us that

$$\chi(g, hj)\chi(h, j) = \chi(g, h)\chi(gh, j)$$

and

$$\chi(1, g) = \chi(g, 1) = 1$$

for $g, h, j \in G$. It may be seen that the former of those two conditions is the condition for a 2-cocycle on a group G.

Now let χ satisfy the conditions, and let us try to compute kG^χ. We have

$$\begin{aligned} g \cdot^\chi h &= \chi(g, h)gh\bar{\chi}(g, h) \\ &= \chi(g, h)\bar{\chi}(g, h)gh \\ &= 1gh \\ &= gh \end{aligned}$$

and so the multiplication remains untwisted. Since the bialgebra structure is unchanged and the antipode of a Hopf algebra is unique, we know without needing to compute it directly that $S^\chi = S$ and so $kG^\chi \cong kG$. The Hopf algebra remains unaffected by χ, but some of the other structures may be twisted whilst still being modules, comodules or algebras over kG.

Let χ be a unital 2-cocycle on kG, and let

$$A = \bigoplus_{g \in G} A_g$$

be a G-graded k-algebra (i.e. a kG-comodule algebra). Then by twisting the multiplication but leaving the grading (coaction) unchanged, we obtain a new graded algebra by defining

$$x \cdot_\chi y = \chi(g, h)xy$$

for $x \in A_g, y \in A_h$ and extending to A by linearity. Furthermore, if V is a Yetter-Drinfel'd module over G, as in Example 3.2, then we have a twisted ${}^G_G\mathcal{YD}$-module, V^χ, with action given by

$$h \triangleright^\chi v = \chi(h, g)\bar{\chi}(hgh^{-1}, h)hv = \frac{\chi(h, g)}{\chi(hgh^{-1}, h)}hv$$

for $h \in G, v \in V_g$. And, as before, a $\mathcal{YD}$-module algebra A can be twisted in both action and multiplication to form A^χ, but this time the twisted algebra is still a Yetter-Drinfel'd module algebra over G.

Chapter 4

On $\mathcal{U}(\mathfrak{tr}_n)$, its quotient $\mathcal{E}_n$, and the twist of $\mathcal{E}_n$.

Here we begin to investigate topics which this thesis concerns.

We start by discussing the nth triangular universal enveloping algebra $\mathcal{U}(\mathfrak{tr}_n)$, introduced in [BEER06]. This algebra is known to be Koszul. The aim is to see if it is possible to apply the results of the previous chapter to construct a "twist" of $\mathcal{U}(\mathfrak{tr}_n)$, but it turns out that it is not graded by S_n.

Instead, we consider a quotient of this algebra, the Fomin-Kirillov algebra $\mathcal{E}_n$, introduced in [FK99]. In section 4.2, we explain how $\mathcal{E}_n$ can be defined using a cocycle over the transpositions in S_n, considered as a rack; this was shown in [MS99]. There is a conjecture that $\mathcal{E}_n$ is isomorphic to the Nichols algebra defined using the same rack cocycle.

In [Ven12], Vendramin shows that this Nichols algebra is twist-equivalent to a Nichols algebra defined using a different rack cocycle. Using this other rack cocycle, [MS99] applied the process used to construct $\mathcal{E}_n$, and produced Λ_n, the algebra defined by Majid in [Maj05]. We also explain this in section 4.2. It is also suspected that the Majid algebra is isomorphic to the corresponding Nichols algebra.

Naturally, it has been conjectured that $\mathcal{E}_n$ and Λ_n are twist-equivalent, but a full proof has not been written down before. This we do in section 4.3. A consequence of this result is the following.

Corollary 4.1. *The Fomin-Kirillov algebra $\mathcal{E}_n$ is isomorphic to a Nichols algebra if and only if the Majid algebra Λ_n is isomorphic to a Nichols algebra.*

4.1 The triangular universal enveloping algebra

I shall introduce the n**th triangular universal enveloping algebra**, $\mathcal{U}(\mathfrak{tr}_n)$, so called because it is the universal enveloping algebra of the nth triangular Lie algebra, defined in [BEER06].

Definition 4.2. For $n \in \mathbb{N}$, $\mathcal{U}(\mathfrak{tr}_n)$ is defined by generators r_{ij} for all $1 \leq i, j \leq n$, $i \neq j$, with relations

$$r_{ji} = -r_{ij}$$

$$r_{ij}r_{jk} - r_{jk}r_{ij} + r_{jk}r_{ki} - r_{ki}r_{jk} + r_{ki}r_{ij} - r_{ij}r_{ki} = 0$$

$$r_{ij}r_{kl} - r_{kl}r_{ij} = 0$$

for all distinct $1 \leq i, j, k, l \leq n$.

Obviously, $\mathcal{U}(\mathfrak{tr}_1)$ is not defined, and $\mathcal{U}(\mathfrak{tr}_2)$ is simply a polynomial algebra. After that, things become more interesting.

We might like it if $\mathcal{U}(\mathfrak{tr}_n)$ turned out to be a ${}^{S_n}_{S_n}\mathcal{YD}$-module algebra, for then we could twist it. It is not hard to see that, for any $n \in \mathbb{N}$, S_n acts on $\mathcal{U}(\mathfrak{tr}_n)$. Let $\sigma \in S_n$. We define the action of σ on $\mathcal{U}(\mathfrak{tr}_n)$ by specifying

$$\sigma(r_{ij}) = r_{\sigma(i)\sigma(j)}$$

for any generator, and extending to $\mathcal{U}(\mathfrak{tr}_n)$ by homomorphisms. This is indeed compatible with the relations, for we have

$$\begin{aligned}
\sigma(r_{ji}) &= r_{\sigma(j)\sigma(i)} \\
&= -r_{\sigma(i)\sigma(j)} \\
&= -\sigma(r_{ij}) \\
&= \sigma(-r_{ij}),
\end{aligned}$$

$$
\begin{aligned}
\sigma(r_{ij}r_{jk} - r_{jk}r_{ij} + r_{jk}r_{ki} - r_{ki}r_{jk} & \\
+r_{ki}r_{ij} - r_{ij}r_{ki}) &= \sigma(r_{ij})\sigma(r_{jk}) - \sigma(r_{jk})\sigma(r_{ij}) \\
&\quad + \sigma(r_{jk})\sigma(r_{ki}) - \sigma(r_{ki})\sigma(r_{jk}) \\
&\quad + \sigma(r_{ki})\sigma(r_{ij}) - \sigma(r_{ij})\sigma(r_{ki}) \\
&= r_{\sigma(i)\sigma(j)}r_{\sigma(j)\sigma(k)} - r_{\sigma(j)\sigma(k)}r_{\sigma(i)\sigma(j)} \\
&\quad + r_{\sigma(j)\sigma(k)}r_{\sigma(k)\sigma(i)} - r_{\sigma(k)\sigma(i)}r_{\sigma(j)\sigma(k)} \\
&\quad + r_{\sigma(k)\sigma(i)}r_{\sigma(i)\sigma(j)} - r_{\sigma(i)\sigma(j)}r_{\sigma(k)\sigma(i)} \\
&= 0 \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma(r_{ij}r_{kl} - r_{kl}r_{ij}) &= \sigma(r_{ij})\sigma(r_{kl}) - \sigma(r_{kl})\sigma(r_{ij}) \\
&= r_{\sigma(i)\sigma(j)}r_{\sigma(k)\sigma(l)} - r_{\sigma(k)\sigma(l)}r_{\sigma(i)\sigma(j)} \\
&= 0.
\end{aligned}
$$

That this is in fact an action easily follows from the definition, from the way that the action defines a permutation representation on the generators. Therefore, $\mathcal{U}(\mathfrak{tr}_n)$ is a kS_n module algebra.

One may hope that $\mathcal{U}(\mathfrak{tr}_n)$ is a kS_n comodule, i.e. that it is S_n-graded, with the grading given by defining each generator r_{ij} to be in the space $\mathcal{U}(\mathfrak{tr}_n)_{(ij)}$. The two classes of relations $r_{ji} = -r_{ij}$ and $r_{ij}r_{kl} - r_{kl}r_{ij}$ prove to be compatible with this putative grading, but the other class does not. Consider the relation

$$r_{ij}r_{jk} - r_{jk}r_{ij} + r_{jk}r_{ki} - r_{ki}r_{jk} + r_{ki}r_{ij} - r_{ij}r_{ki}$$

for fixed i, j, k (with the order fixed). Choosing to read from right to left, we find that $r_{ij}r_{jk}$, $r_{jk}r_{ki}$ and $r_{ki}r_{ij}$ are in $\mathcal{U}(\mathfrak{tr}_n)_{(ijk)}$ and $r_{jk}r_{ij}$, $r_{ki}r_{jk}$, and $r_{ij}r_{ki}$ are in $\mathcal{U}(\mathfrak{tr}_n)_{(ikj)}$. In order to be able to rewrite relations of this kind in such a way that they are compatible with the "grading", we would have to express the space defined by the relations using linear combinations of them which were in the graded components. Note that it would be useless to try to combine relations formed by a different set of three natural numbers from 1 to n, as they would be linearly independent. Our only chance is to consider different permutations of $\{i, j, k\}$. It may be seen that a cyclic permutation will result in the exact same relation, so the only thing left that could make a difference is to swap two of the indices, and it will not matter which. Choosing i and j, we get the relation

$$r_{ji}r_{ik} - r_{ik}r_{ji} + r_{ik}r_{kj} - r_{kj}r_{ik} + r_{kj}r_{ji} - r_{ji}r_{kj}.$$

To determine whether this can make any difference in the actual algebra, let us use the relation class $r_{ji} = -r_{ij}$ to swap round each pair of indices. The result is

$$r_{ij}r_{ki} - r_{ki}r_{ij} + r_{ki}r_{jk} - r_{jk}r_{ik} + r_{jk}r_{ij} - r_{ij}r_{jk},$$

which is the relation we started with, multiplied by -1. This makes one of the relations redundant, meaning that no linear combination of them can give a relation in $\mathcal{U}(\mathfrak{tr}_n)_{(ijk)}$ or $\mathcal{U}(\mathfrak{tr}_n)_{(ikj)}$, so $\mathcal{U}(\mathfrak{tr}_n)$ is not graded as we would like.

Because of this, it might make sense to consider instead an algebra defined the same way but replacing the relations

$$r_{ij}r_{jk} - r_{jk}r_{ij} + r_{jk}r_{ki} - r_{ki}r_{jk} + r_{ki}r_{ij} - r_{ij}r_{ki}$$

with

$$r_{ij}r_{jk} + r_{jk}r_{ki} + r_{ki}r_{ij} \quad \text{and} \quad r_{ij}r_{ki} + r_{jk}r_{ij} + r_{ki}r_{jk}.$$

For reasons that will become apparent, we shall add the extra relation ${r_{ij}}^2$ for all i and j, and call the resulting algebra $\mathcal{E}_n$. This is easily an S_n-graded algebra (the new relations ${r_{ij}}^2$ are all in the identity permutation's component), and still has the S_n-action, for

$$\begin{aligned} \sigma({r_{ij}}^2) &= \sigma(r_{ij})^2 \\ &= {r_{\sigma(i)\sigma(j)}}^2 \\ &= 0, \end{aligned}$$

and the other two relations can be proven to be invariant by splitting up (4.1). All we need is for the $\mathcal{YD}$-module condition to hold. Consider one of the generators of $\mathcal{E}_n$, r_{ij}. We have

$$r_{ij} \in (\mathcal{E}_n)_{(ij)}.$$

Also,

$$\sigma(r_{ij}) = r_{\sigma(i)\sigma(j)} \in (\mathcal{E}_n)_{(\sigma(i)\sigma(j))}.$$

However,

$$(\sigma(i)\sigma(j)) = \sigma(ij)\sigma^{-1},$$

so

$$\sigma(r_{ij}) \in (\mathcal{E}_n)_{\sigma(ij)\sigma^{-1}}.$$

Now, if $\theta \in S_n$, then any element of $(\mathcal{E}_n)_\theta$ is a k-linear combination of monomials of the form

$$r_{i_m j_m} \dots r_{i_2 j_2} r_{i_1 j_1},$$

where $(i_m j_m) \dots (i_2 j_2)(i_1 j_1) = \theta$. If the Yetter-Drinfel'd condition holds for all such monomials, then it holds for all $(\mathcal{E}_n)_\theta$, so let

$$r_{i_m j_m} \dots r_{i_2 j_2} r_{i_1 j_1} \in (\mathcal{E}_n)_\theta$$

be one of these monomials. Then

$$\begin{aligned}
\sigma(r_{i_m j_m} \dots r_{i_2 j_2} r_{i_1 j_1}) &= \sigma(r_{i_m j_m}) \dots \sigma(r_{i_2 j_2})\sigma(r_{i_1 j_1}) \\
&\in (\mathcal{E}_n)_{\sigma(i_m j_m)\sigma^{-1}} \dots (\mathcal{E}_n)_{\sigma(i_2 j_2)\sigma^{-1}} (\mathcal{E}_n)_{\sigma(i_1 j_1)\sigma^{-1}} \\
&= (\mathcal{E}_n)_{\sigma(i_m j_m)\sigma^{-1} \dots \sigma(i_2 j_2)\sigma^{-1}\sigma(i_1 j_1)\sigma^{-1}} \\
&= (\mathcal{E}_n)_{\sigma(i_m j_m) \dots (i_2 j_2)(i_1 j_1)\sigma^{-1}} \\
&= (\mathcal{E}_n)_{\sigma\theta\sigma^{-1}}.
\end{aligned}$$

This means that $\mathcal{E}_n$ is a Yetter-Drinfel'd module algebra over S_n, and so it may be twisted. We may later see if this twist can be lifted to a kind of twist of $\mathcal{U}(\mathfrak{tr}_n)$.

The reason we have introduced the relation ${r_{ij}}^2$ is because the algebra $\mathcal{E}_n$ has appeared in the literature before: it was introduced in [FK99] and has since become known as the Fomin-Kirillov algebra. Note that as we have expressed it, one of the relations is in fact redundant, for if

$$r_{ij} r_{jk} + r_{jk} r_{ki} + r_{ki} r_{ij} = 0$$

for all $1 \leq i, j, k \leq n$, then

$$\begin{aligned}
r_{ij} r_{jk} + r_{jk} r_{ki} + r_{ki} r_{ij} &= (-r_{ji})(-r_{kj}) + (-r_{kj})(-r_{ik}) + (-r_{ik})(-r_{ji}) \\
&= r_{ji} r_{kj} + r_{kj} r_{ik} + r_{ik} r_{ji} \\
&= 0,
\end{aligned}$$

for all $1 \leq i, j, k \leq n$. Up to a permutation of the indices, this is the other cyclic relation, so $\mathcal{E}_n$ is usually defined without it. This algebra has been twisted before

[Ven12], and its only nontrivial twist turns out to be an algebra introduced in [Maj05] as Λ_n, and defined on generators

$$\{s_{ij} \mid 1 \leq i, j \leq n, i \neq j\}$$

with relations

$$s_{ji} = s_{ij}, \quad {s_{ij}}^2, \quad s_{ij}s_{kl} + s_{kl}s_{ij} \;\text{ and }\; s_{ij}s_{jk} + s_{jk}s_{ki} + s_{ki}s_{ij}$$

for distinct i, j, k, l. Note that Λ_n also satisfies the relation

$$s_{ji}s_{kj} + s_{kj}s_{ik} + s_{ik}s_{ji},$$

by a similar argument to that which shows the relation is redundant in $\mathcal{E}_n$. In trying to come up with a possible twist of $\mathcal{U}(\mathfrak{tr}_n)$, whose quotient is Λ_n, it makes sense to drop the relation ${s_{ij}}^2$ and to try to combine the relations

$$s_{ij}s_{jk} + s_{jk}s_{ki} + s_{ki}s_{ij} \;\text{ and }\; s_{ji}s_{kj} + s_{kj}s_{ik} + s_{ik}s_{ji}.$$

Since the corresponding relation in $\mathcal{U}(\mathfrak{tr}_n)$ is the difference of those two, and the other two relations both have their signs reversed, it seems reasonable to guess that the appropriate relation should be the sum. Hence we conjecture that the algebra which ought to be regarded as a twist of $\mathcal{U}(\mathfrak{tr}_n)$ is given by the presentation

$$\begin{aligned}\langle s_{ij}, 1 \leq i, j \leq n, i \neq j \quad | \quad & s_{ji} = s_{ij}, \\ & s_{ij}s_{jk} + s_{jk}s_{ij} + s_{jk}s_{ki} + s_{ki}s_{jk} + s_{ki}s_{ij} + s_{ij}s_{ki} \\ & s_{ij}s_{kl} + s_{kl}s_{ij}\rangle.\end{aligned}$$

Observe that both this algebra and $\mathcal{U}(\mathfrak{tr}_n)$ can be presented as

$$\begin{aligned}\langle x_{ij}, 1 \leq i, j \leq n, i \neq j \quad | \quad & x_{ji} = \pm x_{ij}, \\ & [x_{ij}, x_{jk}]^{\pm} + [x_{jk}, x_{ki}]^{\pm} + [x_{ki}, x_{ij}]^{\pm} \\ & [x_{ij}, x_{kl}]^{\pm}\rangle\end{aligned}$$

for one choice of $+$ or $-$, where $[x, y]^{\pm}$ is interpreted as $xy \pm yx$. Furthermore, if $[\,,]^-$ is interpreted instead as the product in a Lie algebra, the result is precisely the Lie algebra $\mathfrak{tr}_\mathfrak{n}$, as defined in [BEER06]. This suggests that the new algebra can be regarded as a "universal enveloping algebra" over a twist of $\mathfrak{tr}_\mathfrak{n}$, which is a commutative, nonassociative algebra whose product can be represented using an associative algebra with product given by

$$[x, y]^+ = xy + yx.$$

4.2 An alternative description of $\mathcal{E}_n$ and Λ_n

Here is a way to define $\mathcal{E}_n$ and Λ_n, along the same lines as the definitions of the related Nichols algebras in [BB13]. Note that since the algebras have the relations $x_{ji} = \pm x_{ij}$, they can be defined by generators indexed by $\mathcal{T}_n$, the set of transpositions in S_n, and that is how it is done here.

Definition 4.3. A function $q\colon S_n \times \mathcal{T}_n \to k^\times$, where $k^\times$ is the multiplicative group formed by the nonzero elements of the field, satisfying

$$q(\rho\sigma, \tau) = q(\sigma, \tau)q(\rho, \sigma\tau\sigma^{-1}), \qquad \text{for all } \rho, \sigma \in S_n,\ \tau \in \mathcal{T}_n$$

is called a *rack cocycle.*

The rack cocycle condition is needed to ensure that the upcoming construction gives us an action [BB13].

Let $kX_{\mathcal{T}_n}$ denote the k-linear span of $\{x_\tau \mid \tau \in \mathcal{T}_n\}$. Observe that $kX_{\mathcal{T}_n}$ is an S_n-graded vector space, where each basis element x_τ is assigned to the graded component of degree τ. The following Lemma is straightforward.

Lemma 4.4. *Let $q\colon S_n \times \mathcal{T}_n \to k^\times$ be a rack cocycle.*

(1) The formula

$$\sigma \triangleright x_\tau = q(\sigma, \tau)x_{\sigma\tau\sigma^{-1}}, \qquad \sigma \in S_n, \quad \tau \in \mathcal{T}_n,$$

defines an action of the symmetric group S_n on the vector space $kX_{\mathcal{T}_n}$.

(2) This action, together with the S_n-grading on $kX_{\mathcal{T}_n}$, make $kX_{\mathcal{T}_n}$ an ${}^{S_n}_{S_n}\mathcal{YD}$-module structure. □

Call the resulting $\mathcal{YD}$-module X_q. Being a Yetter-Drinfel'd module over S_n, there is a braid map

$$c_q = c_{X_q, X_q}\colon X_q \otimes X_q \to X_q \otimes X_q,$$

which is in this case given by

$$c_q(x_\sigma \otimes x_\tau) = q(\sigma, \tau)\ x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma$$

for all transpositions σ and τ.

We are interested in two possible $\mathcal{YD}$-modules, $X_{q^\pm}$, for two rack cocycles $q^\pm$. If we let $\tau, (i,j) \in \mathcal{T}_n$, with $i < j$, then we specify that

$$q^+(\tau,(i,j)) = 1 \text{ if } \tau(i) < \tau(j) \qquad q^+(\tau,(i,j)) = -1 \text{ if } \tau(j) < \tau(i)$$

and that

$$q^-(\tau,(i,j)) = -1$$

for all $\tau, (i,j)$.

Remark 4.5. Note that we have only given the values of q^+ and q^- when the arguments are both transpositions. We may do this, as these maps extend uniquely to a function over $S_n \times \mathcal{T}_n$ that satisfies the cocycle condition of definition 4.3. Proof of this is deferred to section 5.2, where we can prove it in greater generality.

Claim 4.6. *We have the isomorphisms*

$$\mathcal{E}_n \cong T(X_{q^+})/\langle \ker(id_{(X_{q^+} \otimes X_{q^+})} + c_{q^+})\rangle$$

and

$$\Lambda_n \cong T(X_{q^-})/\langle \ker(id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-})\rangle.$$

Proof. We shall show that, for each of $q^\pm$, $\ker(id_{(X_{q^\pm} \otimes X_{q^\pm})} + c_{q^\pm})$ is precisely the space spanned by the relations that define $\mathcal{E}_n$, respectively Λ_n.

Let $x \in kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$. We see that

$$(id_{(X_{q^\pm} \otimes X_{q^\pm})} + c_{q^\pm})(x) = 0$$

if and only if

$$c_{q^\pm}(x) = -x.$$

Writing

$$x = \sum_{(\sigma,\tau) \in \mathcal{T}_n \times \mathcal{T}_n} k_{(\sigma,\tau)} x_\sigma \otimes x_\tau,$$

for some coefficients $k_{(\sigma,\tau)} \in k$, the previous equation becomes

$$\sum k_{(\sigma,\tau)} c_{q^\pm}(x_\sigma \otimes x_\tau) = \sum -k_{(\sigma,\tau)} x_\sigma \otimes x_\tau.$$

Therefore, $x \in \ker(id_{(X_{q^\pm} \otimes X_{q^\pm})} + c_{q^\pm})$ precisely when the coefficients that define x give

$$\sum k_{(\sigma,\tau)} q^\pm(\sigma,\tau) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma = \sum -k_{(\sigma,\tau)} x_\sigma \otimes x_\tau.$$

Since we want to equate coefficients, let us avoid confusion by renaming the variables on one side:

$$\sum k_{(\sigma,\tau)} q^{\pm}(\sigma,\tau) x_{\sigma\tau\sigma^{-1}} \otimes x_{\sigma} = \sum -k_{(\alpha,\beta)} x_{\alpha} \otimes x_{\beta}.$$

Equating coefficients when $\alpha = \sigma\tau\sigma^{-1}$, $\beta = \sigma$, we get

$$k_{(\sigma,\tau)} q^{\pm}(\sigma,\tau) = -k_{(\sigma\tau\sigma^{-1},\sigma)}$$

or

$$k_{(\sigma\tau\sigma^{-1},\sigma)} = -q^{\pm}(\sigma,\tau) k_{(\sigma,\tau)} \qquad \text{for all } \sigma, \tau \in \mathcal{T}_n.$$

To find out what this means with regard to the actual coefficients, there are three cases to consider: when σ and τ are the same, when σ and τ are disjoint, commuting transpositions, and when $\tau\sigma$ is a 3-cycle.

$\sigma = \tau$: When $\tau = \sigma$, we find that

$$-q^{\pm}(\sigma,\sigma) k_{(\sigma,\sigma)} = k_{(\sigma\sigma\sigma^{-1},\sigma)} = k_{(\sigma,\sigma)}$$

Since $-q^{\pm}(\sigma,\sigma) = 1$ whether we are talking about q^{+} or q^{-}, this means that there are no restrictions on the values of $k_{(\sigma,\sigma)}$ for any transposition σ.

$\tau\sigma = \sigma\tau$ but $\sigma \neq \tau$: In this case,

$$-q^{\pm}(\sigma,\tau) k_{(\sigma,\tau)} = k_{(\sigma\tau\sigma^{-1},\sigma)} = k_{(\tau,\sigma)}.$$

Note that $-q^{\pm}(\sigma,\tau) = \mp 1$, so $x \in \ker(id_{(X_{q^{\pm}} \otimes X_{q^{\pm}})} + c_{q^{\pm}})$ requires its coefficients to satisfy

$$k_{(\tau,\sigma)} = \mp k_{(\sigma,\tau)}$$

for all disjoint transpositions σ and τ.

$\tau \circ \sigma \neq \sigma \circ \tau$: Let us write $\sigma = (ij)$, $\tau = (jk)$, for some $1 \leq i, j, k \leq n$. Then we calculate:

$$\begin{aligned}\sigma\tau\sigma^{-1} &= (ij)(jk)(ij)\\ &= (ik),\end{aligned}$$

$$\begin{aligned}(\sigma\tau\sigma^{-1})\sigma(\sigma\tau\sigma^{-1})^{-1} &= (ik)(ij)(ik)\\ &= (jk)\\ &= \tau,\end{aligned}$$

$$\begin{aligned}\tau(\sigma\tau\sigma^{-1})\tau^{-1} &= (jk)(ik)(jk)\\ &= (ij)\\ &= \sigma.\end{aligned}$$

Therefore, writing the transpositions in disjoint cyclic notation, we find that

$$k_{((ik),(ij))} = -q^{\pm}((ij),(jk))k_{((ij),(jk))},$$

$$k_{((jk),(ik))} = -q^{\pm}((ik),(ij))k_{((ik),(ij))}$$

and

$$k_{((ij),(jk))} = -q^{\pm}((jk),(ik))k_{((jk),(ik))}.$$

In the case when we are dealing with the cocycle q^-, this gives us

$$\begin{aligned}k_{((ij),(jk))} &= k_{((ik),(ij))}\\ &= k_{((jk),(ik))}\\ (&= k_{((ij),(jk))}.)\end{aligned}$$

With q^+, note that for arbitrary $1 \leq a, b, c \leq n$, $q^+((ab),(bc)) = -1$ if and only if c is between a and b. This must mean that exactly one of the three expressions

$$\{-q^+((ij),(jk)), -q^+((ik),(ij)), -q^+((jk),(ik))\}$$

is 1 and the other two are -1. Since we are only trying to work out the relations of the coefficients relative to each other, we can assume at this point that j is between i

and k, giving us that

$$\begin{aligned} k_{((ij),(jk))} &= -k_{((ik),(ij))} \\ &= -k_{((jk),(ik))} \\ (&= k_{((ij),(jk))}.) \end{aligned}$$

Since these are the relations on the coefficients, we must have that

$$\ker(id_{(X_{q^\pm}\otimes X_{q^\pm})} + c_{q^\pm})$$

has the following basis:

- the elements $x_{(ij)} \otimes x_{(ij)}$ for all transpositions $(ij) \in \mathcal{T}_n$;
- the elements $x_{(ij)} \otimes x_{(kl)} \mp x_{(kl)} \otimes x_{(ij)}$ for all pairs $((ij), (kl))$ of disjoint transpositions such that $i < j$, $k < l$, $i < k$;
- $x_{(ij)} \otimes x_{(jk)} \mp x_{(jk)} \otimes x_{(ki)} \mp x_{(ki)} \otimes x_{(ij)}$ for all triples of distinct indices i, j, k such that j is between i and k.

Note that in the third case, one of the three terms is singled out by the condition that the index, j, that appears in both subscripts, $x_{(ij)}$ and $x_{(jk)}$, is between the other two indices, i and k.

It remains to observe that these bases correspond to the presentations of the algebras $\mathcal{E}_n$ and Λ_n given above. □

4.3 The algebras $\mathcal{E}_n$ and Λ_n are twist-equivalent

The goal of this section is to show that Λ_n is a Drinfeld twist of $\mathcal{E}_n$.

Theorem 4.7. *There exists a 2-cocycle χ, defined on $S_n \times S_n$, such that the Drinfel'd twist of the S_n-graded algebra $\mathcal{E}_n$ by χ is isomorphic as $\mathbb{Z}$-graded algebras to Λ_n.*

To prove the Theorem, we will need several auxiliary results.

Firstly, recall one of the main results of [Ven12], namely Theorem 3.8:

Proposition 4.8. *There exists a 2-cocycle $\chi\colon S_n \times S_n \to k^\times$ such that*

$$q^-(\sigma,\tau) = \frac{\chi(\sigma,\tau)}{\chi(\sigma\tau\sigma^{-1},\sigma)} q^+(\sigma,\tau)$$

for all $\sigma \in S_n$, $\tau \in \mathcal{T}_n$. Moreover, the 2-cocycle χ can be chosen in such a way that

$$\chi(\sigma, \tau) \in \{1, -1\}$$

for all σ, τ.

Let χ be a 2-cocycle given by Proposition 4.8. Let $P_\chi \colon kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n} \to kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$ be the linear map defined on the basis by

$$P_\chi(x_\sigma \otimes x_\tau) = \chi(\sigma, \tau)\ x_\sigma \otimes x_\tau.$$

Note that P_χ is an invertible map, since it acts diagonally in the basis

$$\{x_\sigma \otimes x_\tau \mid \sigma, \tau \in \mathcal{T}_n\},$$

and each diagonal entry $\chi(\sigma, \tau)$ is non-zero.

Lemma 4.9. *P_χ is an involution.*

Proposition 4.10. *As endomorphisms of $kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$,*

$$c_{q^+} = P_\chi c_{q^-} {P_\chi}^{-1}.$$

Proof. We may prove this on an arbitrary basis element $x_\sigma \otimes x_\tau$.

$$\begin{aligned}
P_\chi c_{q^-} {P_\chi}^{-1}(x_\sigma \otimes x_\tau) &= P_\chi c_{q^-}(\chi(\sigma, \tau)^{-1} x_\sigma \otimes x_\tau) \\
&= P_\chi(\chi(\sigma, \tau)^{-1} q^-(\sigma, \tau) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma) \\
&= \chi(\sigma, \tau)^{-1} q^-(\sigma, \tau) \chi(\sigma\tau\sigma^{-1}, \sigma) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma \\
&= q^+(\sigma, \tau) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma \text{ (Proposition 4.8)} \\
&= c_{q^+}(x_\sigma \otimes x_\tau)
\end{aligned}$$

□

Corollary 4.11. *As subspaces of $kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$,*

$$\ker(id_{(X_{q^+} \otimes X_{q^+})} + c_{q^+}) = P_\chi(\ker(id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-})).$$

Proof. It follows from Proposition 4.10 that, as linear maps from $kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$ to $kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$,

$$id_{(X_{q^+} \otimes X_{q^+})} + c_{q^+} = P_\chi(id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-}) P_\chi^{-1}.$$

Take ker of both sides and note that, due to invertibility of P_χ, the kernel of the right-hand side is the same as $\ker((id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-}) P_\chi^{-1})$, which is equal to the P_χ-image of $\ker(id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-})$. □

From now on, we denote by $(\mathcal{E}_n)^\chi$ the cocycle twist of the S_n-graded algebra $\mathcal{E}_n$ by the cocycle χ. Let $\cdot_\chi$ be the multiplication in the algebra $(\mathcal{E}_n)^\chi$. Recall that $\mathcal{E}_n$ is also a $\mathbb{Z}$-graded algebra,

$$\mathcal{E}_n = k \oplus kX_{\mathcal{T}_n} \oplus \dots,$$

and the two gradings are compatible with each other. Also, the components of degree 0 and 1 are unaffected by the twist of the multiplication, hence the new algebra $(\mathcal{E}_n)^\chi$ is again $\mathbb{Z}$-graded and is of the form

$$(\mathcal{E}_n)^\chi = k \oplus kX_{\mathcal{T}_n} \oplus \dots.$$

In the following Proposition, we will use two algebra maps:

- $T(kX_{\mathcal{T}_n}) \xrightarrow{p} (\mathcal{E}_n)^\chi$, which uniquely extends the map $kX_{\mathcal{T}_n} \to (\mathcal{E}_n)^\chi$ given by the embedding of $kX_{\mathcal{T}_n}$ as the degree 1 component in $(\mathcal{E}_n)^\chi$;
- $T(kX_{\mathcal{T}_n}) \xrightarrow{\pi} \Lambda_n$, which is the canonical quotient map onto the algebra Λ_n generated by the space $kX_{\mathcal{T}_n}$.

Proposition 4.12. *There exists a unique algebra map θ such that the diagram*

$$\begin{array}{ccc} T(kX_{\mathcal{T}_n}) & & \\ \downarrow{\scriptstyle \pi} & \searrow^{p} & \\ \Lambda_n & \xrightarrow[\theta]{} & (\mathcal{E}_n)^\chi \end{array}$$

commutes.

Proof. Note that Λ_n is generated by the subspace $kX_{\mathcal{T}_n}$, and so θ is defined by where it sends that subspace. Also note that there can be only one such possible θ that commutes, namely the one that takes each element of $kX_{\mathcal{T}_n}$ to the corresponding element of $kX_{\mathcal{T}_n} \subseteq (\mathcal{E}_n)^\chi$. We only need to prove that the map θ is well defined, that is, $\ker \pi \subseteq \ker p$.

Since $\ker p$ is a two-sided ideal, as well as $\ker \pi$, it is enough to check that $p(x) = 0$ for any $x \in kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$ that is a member of the generating set of $\ker \pi$ defined in the proof of claim 4.6. Specifically, that generating set is $\ker(id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-})$, so we need to show that $p(\ker(id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-})) = 0$.

Now let

$$\sum_{1 \le i \le l} k_i(x_{\sigma_i} \otimes x_{\tau_i}) \in \ker(id_{(X_{q^-} \otimes X_{q^-})} + c_{q^-}).$$

We calculate that

$$\begin{aligned} p(\sum_{1\leq i\leq l} k_i(x_{\sigma_i}\otimes x_{\tau_i})) &= \sum_{1\leq i\leq l} k_i(x_{\sigma_i}\cdot_\chi x_{\tau_i}) \\ &= \sum_{1\leq i\leq l} k_i\chi(\sigma_i,\tau_i)x_{\sigma_i}x_{\tau_i}, \end{aligned}$$

where the latter is regarded as an element of $\mathcal{E}_n$. Note that

$$\sum_{1\leq i\leq l} k_i\chi(\sigma_i,\tau_i)\ x_{\sigma_i}\otimes x_{\tau_i}\in P_\chi(\ker(id_{(X_{q^-}\otimes X_{q^-})}+c_{q^-}))$$

by definition of P_χ, and so

$$\sum_{1\leq i\leq l} k_i\chi(\sigma_i,\tau_i)x_{\sigma_i}x_{\tau_i}$$

is in the image of $P_\chi(\ker(id_{(X_{q^-}\otimes X_{q^-})}+c_{q^-}))$ under the natural quotient map from $T(kX_{\mathcal{T}_n})$ to $\mathcal{E}_n$. But, by corollary 4.11, the former is the same as $\ker(id_{(X_{q^+}\otimes X_{q^+})}+c_{q^+})$, which is in the kernel of that quotient map. Therefore,

$$\sum_{1\leq i\leq l} k_i\chi(\sigma_i,\tau_i)x_{\sigma_i}x_{\tau_i}=0.$$

Since $\mathcal{E}_n$ and $(\mathcal{E}_n)^\chi$ are defined on the same vector space, we may conclude that

$$p(\sum_{1\leq i\leq l} k_i(x_{\sigma_i}\otimes x_{\tau_i}))=0,$$

and we are done. □

To prove that θ is surjective, we use the following proposition.

Proposition 4.13. *The monomials in $X_{\mathcal{T}_n}$ form a spanning set of $(\mathcal{E}_n)^\chi$.*

Proof. This easily follows from the fact that the equivalent monomials span $\mathcal{E}_n$, and the monomials under the twisted multiplication are only scalar multiples of those under the untwisted multiplication, as we may check with a simple induction on the length of the monomials. It is trivially true for monomials of length one. Now suppose that every monomial in $(\mathcal{E}_n)^\chi$ of length l is a scalar multiple of the corresponding monomial in $\mathcal{E}_n$, that is, that

$$x_{\sigma_1}\cdot_\chi x_{\sigma_2}\cdot_\chi\cdots\cdot_\chi x_{\sigma_l}=\alpha\ x_{\sigma_1}x_{\sigma_2}\ldots x_{\sigma_l}$$

for some $\alpha\in k$. Let

$$x_{\sigma_1}\cdot_\chi x_{\sigma_2}\cdot_\chi\cdots\cdot_\chi x_{\sigma_l}\cdot_\chi x_{\sigma_{l+1}}$$

be a monomial in $(\mathcal{E}_n)^\chi$ of length $l+1$. Then, using the inductive hypothesis, we may calculate that

$$\begin{aligned}(x_{\sigma_1} \cdot_\chi x_{\sigma_2} \cdot_\chi \dots \cdot_\chi x_{\sigma_l}) \cdot_\chi x_{\sigma_{l+1}} &= \alpha(x_{\sigma_1} x_{\sigma_2} \dots x_{\sigma_l}) \cdot_\chi x_{\sigma_{l+1}} \\ &= \alpha\, \chi(\sigma_1\sigma_2 \dots \sigma_l, \sigma_{l+1}) x_{\sigma_1} x_{\sigma_2} \dots x_{\sigma_l} x_{\sigma_{l+1}}.\end{aligned}$$

Thus,

$$x_{\sigma_1} \cdot_\chi x_{\sigma_2} \cdot_\chi \dots \cdot_\chi x_{\sigma_l} \cdot_\chi x_{\sigma_{l+1}}$$

is a scalar multiple of

$$x_{\sigma_1} x_{\sigma_2} \dots x_{\sigma_l} x_{\sigma_{l+1}}$$

and the proof is complete. ☐

The Proposition immediately implies the following

Corollary 4.14. *The algebra $(\mathcal{E}_n)^\chi$ is generated by its degree 1 component, $kX_{\mathcal{T}_n}$.*

Given that θ restricted to $kX_{\mathcal{T}_n}$ is the identity map, we conclude that the image of θ contains a generating set of $(\mathcal{E}_n)^\chi$, hence the following

Corollary 4.15. *The map θ is surjective.*

Our next goal is to show that θ is not only surjective but bijective. We will achieve that by comparing the Hilbert series of the $\mathbb{Z}$-graded algebras Λ_n and $(\mathcal{E}_n)^\chi$.

For a $\mathbb{Z}$-graded algebra A, we denote

$$Hilb(A,t) = \sum_{k \geq 0} (\dim A^k) t^k.$$

Clearly, $Hilb(A,t)$ is an element of $\mathbb{Z}[[t]]$, the ring of formal power series with integer coefficients. If $g(t)$, $h(t) \in \mathbb{Z}[[t]]$, we write

$$g(t) \leq h(t)$$

if $h(t) - g(t)$ is a formal power series with non-negative coefficients.

Lemma 4.16. *$Hilb((\mathcal{E}_n)^\chi, t) \leq Hilb(\Lambda_n, t)$.*

Proof. Observe that Λ_n^k and $((\mathcal{E}_n)^\chi)^k$ are the images of $kX_{\mathcal{T}_n}^{\otimes k}$ under π and p respectively. Since $p = \theta \circ \pi$, we must have $\theta(\Lambda_n^k) = ((\mathcal{E}_n)^\chi)^k$. Hence $\dim((\mathcal{E}_n)^\chi)^k \leq \dim(\Lambda_n^k)$. ☐

To prove the converse, we use a similar construction, with Λ_n twisted instead of $\mathcal{E}_n$.

We define:

- Let $T(kX_{\mathcal{T}_n}) \xrightarrow{\bar{\pi}} (\Lambda_n)^\chi$ be the map defined by taking the embedding of $kX_{\mathcal{T}_n}$ as the degree 1 component in $(\Lambda_n)^\chi$, and extending it uniquely to an algebra homomorphism.
- Let $T(kX_{\mathcal{T}_n}) \xrightarrow{\bar{p}} \mathcal{E}_n$ be the canonical quotient map onto the algebra $\mathcal{E}_n$ generated by the space $kX_{\mathcal{T}_n}$.

The following is by analogy with proposition 4.12.

Proposition 4.17. *There exists a unique algebra map $\bar{\theta}$ such that the diagram*

$$\begin{array}{ccc} T(kX_{\mathcal{T}_n}) & & \\ \downarrow \bar{\pi} & \searrow^{\bar{p}} & \\ (\Lambda_n)^\chi & \xleftarrow[\bar{\theta}]{} & \mathcal{E}_n \end{array}$$

commutes.

Proof. Note that $\mathcal{E}_n$ is generated by the subspace $kX_{\mathcal{T}_n}$, and so $\bar{\theta}$ is defined by where it sends that subspace. Also note that there can be only one such possible $\bar{\theta}$ that commutes, namely the one that takes each element of $kX_{\mathcal{T}_n}$ to the corresponding element of $kX_{\mathcal{T}_n} \subseteq (\Lambda_n)^\chi$. We only need to prove that the map $\bar{\theta}$ is well defined, that is, $\ker \bar{p} \subseteq \ker \bar{\pi}$.

Since $\ker \bar{\pi}$ is a two-sided ideal, as well as $\ker \bar{p}$, it is enough to check that $\bar{\pi}(x) = 0$ for any $x \in kX_{\mathcal{T}_n} \otimes kX_{\mathcal{T}_n}$ that is a member of the generating set of $\ker \bar{p}$ defined in the proof of claim 4.6. Specifically, that generating set is $\ker(id_{(X_{q^+} \otimes X_{q^+})} + c_{q^+})$, so we need to show that $\bar{\pi}(\ker(id_{(X_{q^+} \otimes X_{q^+})} + c_{q^+})) = 0$.

Now let $\sum_{1 \le i \le l} k_i(x_{\sigma_i} \otimes x_{\tau_i}) \in \ker(id_{(X_{q^+} \otimes X_{q^+})} + c_{q^+})$. We calculate that

$$\bar{\pi}(\sum_{1 \le i \le l} k_i(x_{\sigma_i} \otimes x_{\tau_i})) = \sum_{1 \le i \le l} k_i(x_{\sigma_i} \cdot_\chi x_{\tau_i}) = \sum_{1 \le i \le l} k_i \chi(\sigma_i, \tau_i) x_{\sigma_i} x_{\tau_i},$$

where the latter is regarded as an element of Λ_n. Note that

$$\sum_{1 \le i \le l} k_i \chi(\sigma_i, \tau_i)\ x_{\sigma_i} \otimes x_{\tau_i} \in P_\chi(\ker(id_{(X_{q^+} \otimes X_{q^+})} + c_{q^+}))$$

by definition of P_χ, and so

$$\sum_{1\le i\le l} k_i\chi(\sigma_i,\tau_i)x_{\sigma_i}x_{\tau_i}$$

is in the image of $P_\chi(\ker(id_{(X_{q^+}\otimes X_{q^+})}+c_{q^+}))$ under the natural quotient map from $T(kX_{\mathcal{T}_n})$ to Λ_n. But, by corollary 4.11, the former is the same as $\ker(id_{(X_{q^-}\otimes X_{q^-})}+c_{q^-})$, which is in the kernel of that quotient map. Therefore,

$$\sum_{1\le i\le l} k_i\chi(\sigma_i,\tau_i)x_{\sigma_i}x_{\tau_i}=0.$$

Since Λ_n and $(\Lambda_n)^\chi$ are defined on the same vector space, we may conclude that

$$\bar{\pi}(\sum_{1\le i\le l} k_i(x_{\sigma_i}\otimes x_{\tau_i}))=0,$$

and we are done. □

And thus we may prove:

Lemma 4.18. *$Hilb((\Lambda_n)^\chi,t)\le Hilb(\mathcal{E}_n,t)$.*

Proof. Observe that $\mathcal{E}_n^k$ and $((\Lambda_n)^\chi)^k$ are the images of $kX_{\mathcal{T}_n}^{\otimes k}$ under $\bar{p}$ and $\bar{\pi}$ respectively. Since $\bar{\pi}=\bar{\theta}\circ\bar{p}$, we must have $\theta(\mathcal{E}_n^k)=((\Lambda_n)^\chi)^k$. Hence

$$\dim((\Lambda_n)^\chi)^k\le\dim(\mathcal{E}_n^k).$$

□

Since the Hilbert series is invariant under twisting, this corollary follows:

Corollary 4.19. *$Hilb((\mathcal{E}_n)^\chi,t)\ge Hilb(\Lambda_n,t)$.*

Together with lemma 4.16, this implies that the Hilbert series are identical, meaning that θ is an isomorphism. This proves the main theorem of this section.

Chapter 5

Generalising to Coxeter groups of other types

The previous section contained results concerning relations between algebras defined using rack cocycles on reflection groups of type A. We would like to see if we can come up with analogous results for other Coxeter groups. This section concerns rack cocycles over Coxeter groups and twists between them, the basics of which are explained in 5.1.

In section 5.2, we advance some conjectures as to how the results of Chapter 4 should be generalised, though as we admit, this may not be the full story. Nevertheless, it gives us an avenue of study. Section 5.3 is a brief explanation of central extensions, which we can use to find group cocycles with which we can twist rack cocycles and algebras defined using them.

We then start applying this to other Coxeter groups. We begin by considering the dihedral case, because it is likely that we can apply the results found here to Coxeter groups of higher dimension. Using the presentation found in [IY65] for a central extension of a dihedral group, we calculate a nontrivial group cocycle that we can use to twist rack cocycles on the dihedral group. Next, we calculate what the rack cocycles mentioned in section 5.2 would be in this case, and compare them with our conjectures made there.

5.1 On rack cocycles and the map T

In this section we explain rack cocycles and introduce a map T, which gives us a rack cocycle from a group cocycle.

Strictly speaking, what we are calling a rack cocycle isn't the same thing as a 2-cocycle in the rack cohomology, with the group regarded as a rack, as defined in [EG03]. As explained in [AFGV11], the condition which a function $q\colon G \times G \to k^\times$ must satisfy for all $g, h, j \in G$ in order to be a rack cocycle in this sense is

$$q(g, hjh^{-1})q(h, j) = q(g, j)q(ghg^{-1}, gjg^{-1}).$$

The condition described in Section 4.2 is a stronger property.

Proposition 5.1. *Let $q\colon G \times G \to k^\times$ have the property that*

$$q(gh, j) = q(g, hjh^{-1})q(h, j)$$

for all $g, h, j \in G$. Then q satisfies

$$q(g, hjh^{-1})q(h, j) = q(g, j)q(ghg^{-1}, gjg^{-1}).$$

Proof.

$$q(g, j)q(ghg^{-1}, gjg^{-1}) = q((ghg^{-1})g, j) = q(gh, j) = q(g, hjh^{-1})q(h, j).$$

□

To explain why we use the definition of rack cocycle that we are using, note that any function from $G \times G$ to $k^\times$ can also be thought of as a function from G to $\mathrm{Fun}(G, k^\times)$. If we translate our condition for a rack cocycle into these terms, then we find that $q\colon G \times G \to k^\times$ is a rack cocycle if and only if $q\colon G \to \mathrm{Fun}(G, k^\times)$ is a 1-cocycle in $C^1(G, \mathrm{Fun}(G, k^\times))$, where the (right) action of G on $\mathrm{Fun}(G, k^\times)$ is given by $(f \triangleleft g)(x) = f(gxg^{-1})$ for $x, g \in G$, $f \in \mathrm{Fun}(G, k^\times)$ [BB13].

The main feature of this section is a map T from the set of group cocycles, $Z^2(G \times G, k^\times)$ to the rack cocycles, $Z^1(G, \mathrm{Fun}(G, k^\times))$, where the latter are written as functions of two variables.

Definition 5.2. Let $\chi\colon G \times G \to k^\times$ be a 2-cocycle on G. We define $T\colon Z^2(G \times G, k^\times) \to Z^1(G, \mathrm{Fun}(G, k^\times))$ by

$$T(\chi)(g,h) = \frac{\chi(g,h)}{\chi(ghg^{-1}, g)}$$

for all g and $h \in G$.

This is indeed a well defined map.

Proposition 5.3. *As defined, $T(\chi)$ is a rack cocycle.*

Proof. Let g, h and j be elements of G. Then

$$\begin{aligned} T(\chi)(gh,j) &= \frac{\chi(gh,j)}{\chi(ghjh^{-1}g^{-1}, gh)} \\ &= \frac{\chi(gh,j)\chi(g,h)}{\chi(ghjh^{-1}g^{-1}, gh)\chi(g,h)} \\ &= \frac{\chi(g,hj)\chi(h,j)}{\chi(ghjh^{-1}g^{-1}, g)\chi(ghjh^{-1}g^{-1}g, h)} \\ &= \frac{\chi(g,hjh^{-1})\chi(h,j)}{\chi(ghjh^{-1}g^{-1}, g)\chi(hjh^{-1}, h)} \qquad (5.1) \\ &= T(\chi)(g, hjh^{-1})T(\chi)(h,j). \end{aligned}$$

The step at 5.1 uses the fact that

$$\chi(ghjh^{-1}g^{-1}g, h) = \chi(ghjh^{-1}, h) = \frac{\chi(g, hjh^{-1}h)\chi(hjh^{-1}, h)}{\chi(g, hjh^{-1})} = \frac{\chi(g, hj)\chi(hjh^{-1}, h)}{\chi(g, hjh^{-1})}.$$

□

In fact, the map T preserves coboundaries as well. One may check that a function $\varphi\colon G \times G \to k^\times$ is a coboundary in $C^1(G, \mathrm{Fun}(G, k^\times))$ if and only if there is a function $f\colon G \to k^\times$ such that, for all $g, h \in G$, $\varphi(g,h) = \frac{f(ghg^{-1})}{f(h)}$.

Proposition 5.4. *Let $\chi\colon G \times G \to k^\times$ be a group coboundary. Then $T(\chi)$ is a rack coboundary.*

Proof. We know that, for some $f\colon G \to k^\times$,

$$\chi(g,h) = \frac{f(g)f(h)}{f(gh)},$$

and we want to find a function $\phi\colon G \times G \to k^\times$ such that

$$T(\chi)(g,h) = \frac{\phi(ghg^{-1})}{\phi(h)}.$$

Indeed, if we define $\bar{f}$ by $\bar{f}(g) = f(g)^{-1}$ for all $g \in G$, then we find we may choose $\phi = \bar{f}$.

$$\begin{aligned} T(\chi)(g,h) &= \frac{\chi(g,h)}{\chi(ghg^{-1},g)} \\ &= \frac{f(g)f(h)f(ghg^{-1}g)}{f(gh)f(ghg^{-1})f(g)} \\ &= \frac{f(h)}{f(ghg^{-1})} \\ &= \frac{\bar{f}(ghg^{-1})}{\bar{f}(h)} \end{aligned}$$

□

It therefore follows that T induces a map of cohomologies $H^2(G \times G, k^\times) \to H^1(G, \mathrm{Fun}(G, k^\times))$.

The reason we are interested in T is because we can use T to twist rack cocycles, as in [BB13]. If q is a rack cocycle and χ a group cocycle, then we call the rack cocycle given by $T(\chi)q$, the pointwise multiplication of $T(\chi)$ and q, the twist of q by χ. It may be seen that proposition 4.8 is an example of this construction, in which q^+ is twisted to q^- by the cocycle from [Ven12], which we further used to twist $\mathcal{E}_n$ and Λ_n. Notice that the Fomin-Kirillov and Majid algebras, as well as the rack cocycles $q^\pm$, are defined in terms of symmetric groups and the transpositions, or in other words, in terms of Coxeter groups of type A and their reflections. This motivates us to see if we can generalise these results to other reflection groups.

5.2 Defining specific rack cocycles

Notice that in section 4.2, we only defined the rack cocycles $q^\pm$ when the right argument was a reflection, and the applications of those cocycles only required that. This shall continue to be the case for these generalisations.

One may recall that the values of q^+ and q^- were only given for when both the arguments were transpositions, and that we claimed in Remark 4.5 that this would uniquely define the cocycle for any group element in the left argument. Proof of this was deferred to here, for we may now prove a more general result applying to any reflection group.

Proposition 5.5. *Let G be a finite irreducible reflection group, R its reflections, and let q be a function from $R \times R$ to $k^\times$. Then there is at most one extension of q to the domain $G \times R$ such that q satisfies the rack cocycle condition.*

Proof. Consider $q(s_1 s_2 \dots s_n, \tau)$, where $s_1, s_2 \dots s_n$ and τ are all reflections. We show how to write $q(s_1 s_2 \dots s_n, \tau)$ in terms of a product of values of q whose arguments are all reflections. By the rack cocycle condition,

$$q(s_1 s_2 \dots s_n, \tau) = q(s_1 s_2 \dots s_{n-1}, s_n \tau (s_n)^{-1}) q(s_n, \tau).$$

Observe that reflections are invariant under conjugation, and therefore $s_n \tau (s_n)^{-1}$ is also a reflection. One can again apply the cocycle relation to obtain

$$\begin{aligned} q(s_1 s_2 \dots s_{n-1}, s_n \tau (s_n)^{-1}) q(s_n, \tau) &= q(s_1 s_2 \dots s_{n-2}, s_{n-1} s_n \tau (s_n)^{-1} (s_{n-1})^{-1}) \\ &\quad q(s_{n-1}, s_n \tau (s_n)^{-1}) q(s_n, \tau) \end{aligned}$$

and similarly keep going until we have written $q(s_1 s_2 \dots s_n, \tau)$ as required. □

Note that, given such a function defined only on the reflections, we would still need to check that it extended to a well defined rack cocycle.

In fact, we could specify only for simple reflections in the right argument. This is because, if we want to find $q(x, \sigma)$, where σ is a reflection, then σ is conjugate to some simple reflection. Therefore we may write

$$q(x, \sigma) = q(x, y s_\alpha y^{-1}),$$

for some y and some simple root α, and then calculate

$$q(x, y s_\alpha y^{-1}) = \frac{q(xy, s_\alpha)}{q(y, s_\alpha)}.$$

With that in mind, we make the following definitions (after [Baz06]), which are generalisations of q^+ and q^- as defined in section 4.2. That is, that the definitions there are special cases of these for Coxeter groups of type A, with the values only given for reflections. Here we describe the cocycles over the entire domain.

Definition 5.6. Let G be a finite irreducible reflection group, R its reflections. Then we define a rack cocycle $q^- \colon G \times R \to k^\times$ by specifying that

$$q^-(\sigma, \tau) = -1$$

for all pairs of reflections σ and τ.

Remark 5.7. This does extend to a well-defined cocycle defined over $G \times R$. If we want to compute $q^-(x, \sigma)$ for an arbitrary $x \in G$, then we write x as a product of reflections, and then rewrite $q^-(x, \sigma)$ as a product of terms which are specified in Definition 5.6 by applying the procedure described in the proof of proposition 5.5. This will give us a value of $q^-(x, \sigma) = \pm 1$, depending on whether x was written as a product of an even or an odd number of reflections. But since that is independent of how x is written in terms of reflections, this gives us a well-defined rack cocycle over $G \times R$.

Definition 5.8. Let G be a finite irreducible reflection group, R its reflections. Then we define a rack cocycle $q^+ : G \times R \to k^\times$ as follows. Let $x \in G$, and let s_α be a reflection, where $\alpha \in \Phi^+$, the positive roots. Then

$$q^+(x, s_\alpha) = \begin{cases} 1 & x(\alpha) \in \Phi^+ \\ -1 & x(\alpha) \in \Phi^-. \end{cases}$$

Theorem 5.9. *This function satisfies the rack cocycle condition.*

Proof. Let $x, y \in G$ and let s_α be a reflection. We wish to prove that

$$q^+(xy, s_\alpha) = q^+(x, y s_\alpha y^{-1}) q^+(y, s_\alpha).$$

We analyse this by cases.

- If $y(\alpha)$ is a positive root, then $q^+(y, s_\alpha) = 1$, so we hope to prove that

  $$q^+(xy, s_\alpha) = q^+(x, y s_\alpha y^{-1}).$$

  In this case,

  $$y s_\alpha y^{-1} = s_{y(\alpha)},$$

  so we want to prove that $xy(\alpha)$ and $x(y(\alpha))$ are either both positive roots or both negative roots. But they are clearly equal.

- If $y(\alpha)$ is a negative root, then $q^+(y, s_\alpha) = -1$, so we hope to prove that

  $$q^+(xy, s_\alpha) = -q^+(x, y s_\alpha y^{-1}).$$

  In this case,

  $$y s_\alpha y^{-1} = s_{-y(\alpha)},$$

so we want to prove that, of the roots $xy(\alpha)$ and $x(-y(\alpha))$, one is positive and the other is negative. But this is obviously true because

$$x(-y(\alpha)) = -xy(\alpha).$$

□

The conjecture is that there is some group cocycle χ such that

$$q^- = T(\chi)q^+,$$

just as was the case with Theorem 3.8 of [Ven12] for the special case of type A. This may possibly be true only up to coboundary. Another question that could be asked at this point is, are there other group cocycles over G such that twisting q^+ or q^- by these cocycles gives interesting rack cocycles? Possibly ones that can be used to define interesting algebras analogous to $\mathcal{E}_n$ and Λ_n, and to relate them by twists. Algebras that might be related to these appear in [Lau14], which concerns Koszulness properties of algebras which seem to be the analogues of $\mathcal{U}(\mathfrak{tr}_n)$ for Coxeter groups of type B and D. In any case, the Schur multipliers of all the finite reflection groups are listed in [IY65], which therefore provides a useful starting point.

5.3 Central extensions

As explained in [BB13], it has long been known that there is a natural correspondence between cocycles of a group G and sections of central extensions over G:

Definition 5.10. Let G be a group. A **central extension of** G is a group $\hat{G}$ with a quotient map $\bar{q}\colon \hat{G} \to G$ whose kernel Γ is in the centre of $\hat{G}$. A **section** is a set theoretic map $\rho\colon G \to \hat{G}$ such that

$$\bar{q} \circ \rho = id_G.$$

Fix a section ρ, and let $g, h \in G$. Consider $\rho(g)\rho(h)$ and $\rho(gh)$. By definition,

$$\bar{q}(\rho(g)\rho(h)) = \bar{q}(\rho(g))\bar{q}(\rho(h)) = gh = \bar{q}(\rho(gh)).$$

This means that there is some element of Γ, depending on g and h, which we will write $\chi(g, h)$, such that

$$\rho(g)\rho(h) = \chi(g, h)\rho(gh).$$

Remember that $\chi(g,h)$ is central. This χ defines a function from $G \times G$ to Γ.

Proposition 5.11. *This $\chi\colon G \times G \to \Gamma$ is a group cocycle.*

Proof. Let $g, h, j \in G$. Then (allowing the central elements to commute whenever convenient)

$$\begin{aligned}\chi(g,h)\chi(gh,j)\rho(ghj) &= \chi(g,h)\rho(gh)\rho(j)\\ &= \rho(g)\rho(h)\rho(j)\\ &= \chi(h,j)\rho(g)\rho(hj)\\ &= \chi(h,j)\chi(g,hj)\rho(ghj).\end{aligned}$$

The result follows by multiplying by $\rho(ghj)^{-1}$. □

This means that group cocycles can be described in terms of central extensions, and that is how the Schur multipliers of the reflection groups were described in [IY65], without giving cocycles directly. It turns out to be difficult to compute the cocycles themselves from this, but luckily we only need to find their image under the map T of definition 5.2.

Proposition 5.12. *Let ρ be a section of a central extension of G, and let χ be its corresponding group cocycle. Let $g, h \in G$. Then*

$$T(\chi)(g,h) = \rho(g)\rho(h)\rho(g)^{-1}\rho(ghg^{-1})^{-1}.$$

Proof. Note that $\rho(ghg^{-1})\rho(g) = \chi(ghg^{-1}, g)\rho(ghg^{-1}g) = \chi(ghg^{-1}, g)\rho(gh)$. Therefore,

$$\begin{aligned}T(\chi)(g,h) &= \chi(g,h)\chi(ghg^{-1},g)^{-1}\\ &= \chi(g,h)\rho(gh)(\chi(ghg^{-1},g)\rho(gh))^{-1}\\ &= \rho(g)\rho(h)(\rho(ghg^{-1})\rho(g))^{-1}\\ &= \rho(g)\rho(h)\rho(g)^{-1}\rho(ghg^{-1})^{-1}.\end{aligned}$$

□

5.4 The even dihedral case

To begin with, it makes sense to work out the dihedral case, because all the others have dihedral subgroups and the span of any two independent roots is a dihedral root system. However, it turns out [IY65] that a dihedral group with an odd number of reflections has no nontrivial cocycles on it, so we won't be able to try out our conjectures in that case. In the case of a dihedral group of order $2n$, where n is even, [IY65] found that there is exactly one nontrivial class of cocycles, and gave a central extension of $Dih(n)$ such that cocycles on $Dih(n)$ are given by sections of this central extension. The central extension $\widehat{Dih(n)}$ is given by generators t_1, t_2, α with relations

$$(t_1)^2 = (t_2)^2 = \alpha^2 = 1, \quad \alpha t_1 = s_1\alpha, \quad \alpha t_2 = t_2\alpha \text{ and } (t_1t_2)^n = \alpha.$$

The quotient map sends t_1 and t_2 to the simple roots s_1 and s_2, and sends α to 1, Γ in this case being $\{1, \alpha\}$. In order to find a nontrivial rack cocycle of the form $T(\chi)$, for some group cocycle χ, all we have to do is to choose the right section. Indeed, the most natural section to choose is

$$\rho(s_is_js_i\ldots s_i) = t_it_jt_i\ldots t_i,$$

where $s_is_js_i\ldots s_i$ is written in its shortest form. We now calculate $T(\chi)$ for this section.

Suppose we have reflections of the form $s_is_js_i\ldots s_i$, of lengths l and m. We use Proposition 5.12 to calculate

$$\begin{aligned} T(\chi)((s_is_j)^{\frac{l-1}{2}}s_i, (s_is_j)^{\frac{m-1}{2}}s_i) &= \rho((s_is_j)^{\frac{l-1}{2}}s_i)\rho((s_is_j)^{\frac{m-1}{2}}s_i)\rho((s_is_j)^{\frac{l-1}{2}}s_i) \\ &\quad \rho((s_is_j)^{\frac{l-1}{2}}s_i(s_is_j)^{\frac{m-1}{2}}s_i(s_is_j)^{\frac{l-1}{2}}s_i)^{-1} \\ &= (t_it_j)^{\frac{l-1}{2}}t_i(t_it_j)^{\frac{m-1}{2}}t_i(t_it_j)^{\frac{l-1}{2}}t_i \\ &\quad \rho((s_is_j)^{\frac{l-1}{2}}s_i(s_is_j)^{\frac{m-1}{2}}s_i(s_is_j)^{\frac{l-1}{2}}s_i)^{-1} \end{aligned}$$

Observe that the length of $(s_is_j)^{\frac{l-1}{2}}s_i(s_is_j)^{\frac{m-1}{2}}s_i(s_is_j)^{\frac{l-1}{2}}s_i$ is $|2l-m|$, so this can be written as

$$(t_it_j)^{\frac{|2l-m|-1}{2}}t_i\rho((s_is_j)^{\frac{|2l-m|-1}{2}}s_i)^{-1}.$$

If $|2l-m| < n$, then $(s_is_j)^{\frac{|2l-m|-1}{2}}s_i$ is in its shortest form and so

$$\rho((s_is_j)^{\frac{|2l-m|-1}{2}}s_i) = (t_it_j)^{\frac{|2l-m|-1}{2}}t_i,$$

which means that

$$T(\chi)((s_is_j)^{\frac{l-1}{2}}s_i,(s_is_j)^{\frac{m-1}{2}}s_i)=1.$$

If $|2l-m|>n$, then the actual length of the reflections is $2n-|2l-m|$, so we must rewrite

$$(t_it_j)^{\frac{|2l-m|-1}{2}}t_i\rho((s_is_j)^{\frac{|2l-m|-1}{2}}s_i)^{-1}=\alpha(t_jt_i)^{\frac{2n-|2l-m|-1}{2}}t_j\rho((s_js_i)^{\frac{2n-|2l-m|-1}{2}}s_j)^{-1}.$$

Now the reflections are in their shortest form, so we may write

$$\rho((s_js_i)^{\frac{2n-|2l-m|-1}{2}}s_j)=(t_jt_i)^{\frac{2n-|2l-m|-1}{2}}t_j$$

and so

$$T(\chi)((s_is_j)^{\frac{l-1}{2}}s_i,(s_is_j)^{\frac{m-1}{2}}s_i)=\alpha.$$

Now suppose we have reflections $s_is_js_i\dots s_i$ and $s_js_is_j\dots s_j$, with lengths l and m. In a similar manner, we find

$$\begin{aligned}T(\chi)((s_is_j)^{\frac{l-1}{2}}s_i,(s_js_i)^{\frac{m-1}{2}}s_j) &= (t_it_j)^{\frac{l-1}{2}}t_i(t_jt_i)^{\frac{m-1}{2}}t_j(t_it_j)^{\frac{l-1}{2}}t_i\\ &\quad \rho((s_is_j)^{\frac{l-1}{2}}s_i(s_js_i)^{\frac{m-1}{2}}s_j(s_is_j)^{\frac{l-1}{2}}s_i)^{-1}.\end{aligned}$$

The length of this expression is $2l+m$, so we rewrite it as

$$(t_it_j)^{\frac{2l+m-1}{2}}t_i\rho((s_is_j)^{\frac{2l+m-1}{2}}s_i)^{-1}.$$

If $2l+m<n$, then the reflections are already in their shortest form, and so by a similar argument to the one before,

$$T(\chi)((s_is_j)^{\frac{l-1}{2}}s_i,(s_js_i)^{\frac{m-1}{2}}s_j)=1.$$

Otherwise, we must once again use the relations and as before,

$$T(\chi)((s_is_j)^{\frac{l-1}{2}}s_i,(s_js_i)^{\frac{m-1}{2}}s_j)=\alpha.$$

Thus we have proven the following.

Theorem 5.13. *Let $\widehat{Dih(n)}$ be as described, and let*

$$\rho(s_is_js_i\dots s_i)=t_it_jt_i\dots t_i,$$

as above. Let $s_i s_j s_i \dots s_i$ and $s_p s_q s_p \dots s_p$ be two reflections in $Dih(n)$, of length $l < n$ and $m < n$ respectively. Then

$$T(\chi)(s_i s_j s_i \dots s_i, s_p s_q s_p \dots s_p) = \begin{cases} 1 & p = i,\ |2l - m| < n \\ \alpha & p = i,\ |2l - m| > n \\ 1 & p = j,\ \ 2l + m < n \\ \alpha & p = j,\ \ 2l + m > n. \end{cases}$$

We now confirm that this is not a coboundary.

Proposition 5.14. *There is no function $f\colon Dih(n) \to \Gamma$ such that*

$$T(\chi)(\sigma, \tau) = \frac{f(\sigma\tau\sigma^{-1})}{f(\tau)}$$

for all reflections σ and τ.

Proof. Suppose, for a contradiction, that there is such a function. Let σ be s_1, and let τ range over all the reflections of the form $s_1 s_2 s_1 \dots s_1$. The theorem gives us

$$T(\chi)(\sigma, \tau) = \frac{f(\sigma\tau\sigma^{-1})}{f(\tau)} = \frac{f(s_2 s_1 s_2 \dots s_2)}{f(s_1)} = 1,$$

where the length of the first reflection is less than $n - 1$, so in those cases we have $f(s_2 s_1 s_2 \dots s_2) = f(s_1)$. Now let τ range over the reflections of the form $s_2 s_1 s_2 \dots s_2$, of length less than $n - 1$. This gives us

$$T(\chi)(\sigma, \tau) = \frac{f(s_1 s_2 s_1 \dots s_1)}{f(s_1)} = 1,$$

where the length of the first reflection is no greater than n. Therefore, we have $f(s_1 s_2 s_1 \dots s_1) = f(s_1)$. Lastly, let τ be $(s_2 s_1)^{\frac{n-2}{2}} s_2$. Then, according to the theorem,

$$T(\chi)(\sigma, \tau) = \frac{(s_1 s_2)^{\frac{n}{2}} s_1}{f(s_1)} = \frac{(s_2 s_1)^{\frac{n-2}{2}} s_2}{f(s_1)} = \alpha,$$

which means that $f(s_1) = \alpha (s_2 s_1)^{\frac{n-2}{2}} s_2$. Therefore, for any reflection τ, $f(\tau) = f(s_1)$, except when $\tau = (s_2 s_1)^{\frac{n-2}{2}} s_2$.

Now repeat the same argument with $\sigma = s_2$, and all other cases of the two generating reflections being swapped round, and we have a contradiction. □

Note that we can identify $\{1, \alpha\}$ with $\{1, -1\}$, and thus we define q^+ and q^- in the case of $Dih(n)$. With regards to our conjecture that $q^- = T(\chi) q^+$, we will calculate $q^-(q^+)^{-1}$, and thus we must calculate q^+. To do so, we must describe a root system that defines $Dih(n)$.

Definition 5.15. We define a root system that gives $Dih(n)$. To do this, we identify $\mathbb{R}^2$ with $\mathbb{C}$, and will define the roots with imaginary part greater than zero to be the positive roots, and those with imaginary part less than zero to be the negative roots. The roots themselves are $\{e^{l\frac{\pi}{2n}\cdot i} \mid l \in 2\mathbb{Z}+1\}$.

We shall name the simple roots $\alpha_1 = e^{\frac{\pi}{2n}\cdot i}$ and $\alpha_2 = e^{(2n-1)\frac{\pi}{2n}\cdot i}$.

Lemma 5.16. *For all odd $0 \leq l \leq 2n$,*

$$s_{e^{l\frac{\pi}{2n}\cdot i}} = (s_{\alpha_1}s_{\alpha_2})^{\frac{l-1}{2}}s_{\alpha_1}.$$

Proof. In the other well known presentation of $Dih(n)$, all n reflections are expressed in the form r^js, where s is one of the reflections, $0 \leq j < n$, and r is rotation by $\frac{2\pi}{n}$. This indeed makes sense: let α be a root corresponding to s, and let t be the reflection given by α rotated by $m\frac{\pi}{n}$, for some $0 \leq m < n$. Then t is given by s, followed by a rotation of $m\frac{2\pi}{n}$ – in other words, $t = r^m s$.

One may take s to be any one of the reflections; in this case, we shall have

$$s = s_{\alpha_1} = s_{e^{\frac{\pi}{2n}\cdot i}}.$$

Now, the root

$$e^{l\frac{\pi}{2n}\cdot i} = e^{((l-1)\frac{\pi}{2n}+\frac{\pi}{2n})\cdot i}$$

is a rotation of α_1 by $(l-1)\frac{\pi}{2n} = \frac{l-1}{2}\frac{\pi}{n}$, which means that

$$s_{e^{l\frac{\pi}{2n}\cdot i}} = r^{\frac{l-1}{2}}s_{\alpha_1}.$$

But $r = s_{\alpha_1}s_{\alpha_2}$, and the result follows. □

Remark 5.17. A similar argument tells us that

$$s_{e^{(2n-l)\frac{\pi}{2n}\cdot i}} = (s_{\alpha_2}s_{\alpha_1})^{\frac{l-1}{2}}s_{\alpha_2}.$$

Corollary 5.18. *Let $l \in 2\mathbb{Z}+1$. If $0 < l < n$, then the length of $s_{e^{l\frac{\pi}{2n}\cdot i}}$ is l. If $n < l < 2n$, then the length of $s_{e^{l\frac{\pi}{2n}\cdot i}}$ is $2n-l$.*

Proof. This follows from the fact that in Lemma 5.16, for $0 < l < n$ the reflections $s_{e^{l\frac{\pi}{2n}\cdot i}}$ are expressed as a product of l simple reflections, and that in Remark 5.17, for $0 < l < n$ the reflections $s_{e^{(2n-l)\frac{\pi}{2n}\cdot i}}$ are expressed as a product of l simple reflections.

The second part of the corollary is obtained by a replacement of indices. Note that a reflection in $Dih(n)$ may be uniquely expressed as a product of less than n simple reflections, so we have indeed found the length. $\square$

This means that the length of a reflection is proportional to the acute angle its defining root makes with the real axis.

Lemma 5.19. *Let $e^{a\cdot i}$ and $e^{b\cdot i}$ be two points on the unit circle in $\mathbb{C}$, with $0 < a, b < \pi$. Then*

$$Im(s_{e^{a\cdot i}}(e^{b\cdot i})) < 0 \quad \text{if and only if} \quad 2a - \pi < b < 2a.$$

Proof. This is a matter of simple geometry. The points $e^{a\cdot i}$ and $e^{b\cdot i}$ are points in the unit circle in the upper half plane, and the reflection $s_{e^{a\cdot i}}$ is simply the reflection that sends $e^{a\cdot i}$ to $-e^{a\cdot i}$. The collection of points on the upper half of the unit semicircle that are sent to the lower half is therefore simply the preimage of the lower half under this reflection. In other words, it is the set of points on the side of the image of the real axis under $s_{e^{a\cdot i}}$ that includes $e^{a\cdot i}$ (since $e^{a\cdot i}$ itself is always sent to the lower half plane). The laws of geometry indicate that the line through the points 0 and $e^{a\cdot i}$ bisects the angle between the real axis and its image under $s_{e^{a\cdot i}}$. Therefore the boundaries of the half-circle which is sent to the lower half plane by $s_{e^{a\cdot i}}$ must be defined by $e^{2a\cdot i}$ and $-e^{2a\cdot i} = e^{(2a-\pi)\cdot i}$, and the result follows from this. $\square$

Thus we may calculate the following:

Theorem 5.20. *Let $s_i s_j s_i \ldots s_i$ and $s_p s_q s_p \ldots s_p$ be two reflections in $Dih(n)$, of length $l < n$ and $m < n$ respectively. Then*

$$q^{+}(s_i s_j s_i \ldots s_i, s_p s_q s_p \ldots s_p) = \begin{cases} 1 & p = i, \quad\quad 2l < m \\ -1 & p = i, \quad\quad 2l > m \\ 1 & p = j, \; 2l + m < 2n \\ -1 & p = j, \; 2l + m > 2n. \end{cases}$$

Proof. The following argument considers the cases $p = i$ and $p = j$. During the calculations, we can assume without loss of generality that $i = 1$. This is because of the automorphism of the Dynkin diagram, which corresponds with an automorphism of $Dih(n)$ given by exchanging the two simple roots and transposing the two simple reflections. When $i = 2$, we can just apply this automorphism by swapping the simple roots, calculate q^+ of this, and the result will be the same.

- Suppose $p = i$. Then $s_i s_j s_i \ldots s_i$ is the reflection given by root $e^{l\frac{\pi}{2n}\cdot i}$ and $s_p s_q s_p \ldots s_p$ is the reflection given by root $e^{m\frac{\pi}{2n}\cdot i}$ (Lemma 5.16). By Lemma 5.19, $q^+(s_{e^{l\frac{\pi}{2n}\cdot i}}, s_{e^{m\frac{\pi}{2n}\cdot i}}) = -1$ if and only if $2l\frac{\pi}{2n} - \pi < m\frac{\pi}{2n} < 2l\frac{\pi}{2n}$. Since $l < n$, the leftmost expression is redundant, and we are left with the result that $q^+(s_i s_j s_i \ldots s_i, s_p s_q s_p \ldots s_p) = -1$ iff $m < 2l$.

- Suppose $p = j$. Then, by Lemma 5.16 and Remark 5.17, $s_i s_j s_i \ldots s_i = s_{e^{l\frac{\pi}{2n}\cdot i}}$ and $s_p s_q s_p \ldots s_p = s_{e^{(2n-m)\frac{\pi}{2n}\cdot i}}$. By Lemma 5.19, $q^+(s_{e^{l\frac{\pi}{2n}\cdot i}}, s_{e^{(2n-m)\frac{\pi}{2n}\cdot i}}) = -1$ if and only if $2l\frac{\pi}{2n} - \pi < (2n-m)\frac{\pi}{2n} < 2l\frac{\pi}{2n}$. Again, $l < m$, the leftmost expression is less than zero and therefore irrelevant. We therefore have $q^+(s_i s_j s_i \ldots s_i, s_p s_q s_p \ldots s_p) = -1$ if and only if $2l + m > 2n$.

□

We may thus calculate $q^-(q^+)^{-1}$.

Corollary 5.21. *Let $s_i s_j s_i \ldots s_i$ and $s_p s_q s_p \ldots s_p$ be two reflections in $Dih(n)$, of length $l < n$ and $m < n$ respectively. Then*

$$q^-(q^+)^{-1}(s_i s_j s_i \ldots s_i, s_p s_q s_p \ldots s_p) = \begin{cases} -1 & p = i, \quad 2l < m \\ 1 & p = i, \quad 2l > m \\ -1 & p = j, \; 2l + m < 2n \\ 1 & p = j, \; 2l + m > 2n. \end{cases}$$

Thus we may see that $q^-(q^+)^{-1}$ is not equal to $T(\chi)$, as defined in theorem 5.13. It remains to be seen whether they are equal up to coboundary, as is the case with $Dih(3)$, the Coxeter group of type A_2.

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