CONSENSUS CONTROL OF A CLASS OF NONLINEAR SYSTEMS

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This dissertation aims at solving the consensus control problem of multi-agent systems with Lipschitz nonlinearity. This depends on the design of the controller that enables each agent or subsystem in multi-agent systems with Lipschitz nonlinearity to reach consensus; using the understanding of the agents’ connection network from the knowledge of graph theory as well as the control system design strategy.

The objective is achieved by designing a type of distributed control, namely the consensus control, which manipulates the relative information of each agent in a multi-agent systems in order to arrive at a single solution. In addition, containment control is also developed to solve containment problem. It is an extension of consensus control via leader-follower configuration, aimed at having each agent contained by multiple leaders in a multi-agent systems with Lipschitz nonlinearity.

Four types of controllers are proposed - state-feedback consensus controller, observer-based consensus controller, state-feedback containment controller and observer-based containment controller; each provides the stability conditions based on Lyapunov stability analysis in time domain which enabled each agent or subsystem to reach consensus. The observer-based controllers are designed based on the consensus observer that is related to Luenberger observer. Linear Matrix Inequality (LMI) and Algebraic Riccati Equation (ARE) are utilized to obtain the solutions for the stability conditions.

The simulation results of the proposed controllers and observers have been carried out to prove their theoretical validity. Several practical examples of flexible robot arm simulations are included to further validate the theoretical aspects of the thesis.
Publications

Journal paper:

1. Leader-Follower Consensus Control of Lipschitz Nonlinear Systems by Output Feedback submitted to International Journal of System Science (published online).

2. Leader-Follower Containment Control of Lipschitz Nonlinear Systems, preparing for submission to IET Control Theory and Applications.

Conference paper:

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Ahmad Sadhiqin Mohd Isira
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Chapter 1

Introduction to Consensus Control

1.1 Brief Overview

Application of control systems to human life cannot be quantified, this varies in different automated devices such as cars, refrigerators, air-condition systems, etc. Advancement in technology, has made control system applications cheaper, simpler, and more cost effective, making the appliances more desirable, and affordable.

‘Control systems’ is an important topic in the engineering field. It is known as a system that consists of subsystems and process (plants) assembled to perform a specific task in order to get the desired output given a specific input [1]. At the lowest level it can be classified as centralized and decentralized control system. Decentralized control system, also known as distributed control system can be defined as component in each system contributing to the global, complex behaviour by acting on local information, unlike centralized control system where each component of the system depends on a central controller for its outcome.

In recent years there has been an increasing need of advanced techniques for control and optimization of complex large-scale dynamical systems. In line with increase in complexity there have been great technological improvement in the fields of communication networks and embedded computer technology. The natural approach to control such complex systems is by distributing the computational burden among different agents, through appropriate communication, cooperatively achieve system-wide
goals, in turn contributes to the development of distributed control system. Due to the lack of dependence on a central controller, the agent is working as part of cooperative control system which enables each agent or subsystem to have its own ‘brain’ which evaluates and processes the parameters to generate the required output. With the smaller and faster processing unit, it is possible to have a group of agents that make decisions on simple control instructions in large dynamical systems. Compared to the traditional control system, which requires a central controller for all processes, the distributed approach offers many advantages, such as lower operational costs, reduced system requirements, greater robustness, strong adaptivity and flexible scalability.

The distributed control system has its origins in distributed computing [2], management science [3], and statistical physics [4]. In distributed control, the consensus concept was from the area of computer science [5]. Meanwhile, the works in [6] and [7] among others were the basis of the pioneering activities in the field of control systems, where an asynchronous agreement concept was studied for distributed decision-making problems. Since then, consensus algorithms have been extensively studied under various information-flow constraints in [8–12].

A number of publications described the development of consensus control in the control research community. Initially, Vicsek [4] was influenced by the work by Reynolds [5] and proposed an algorithm that represented his work. Then Jadbabaie et.al [8] explained the observed behaviours of the graphic example of the models proposed in Vicsek [4] from the theoretical aspects by using the nearest neighbour rule despite the absence of centralized coordination together with each agents set of nearest neighbors change with time as the system evolves. The relationship between consensus control and graph theory was explored first by a number of publications. Analysis in Olfati-Saber et.al [10] connected the consensus control to algebraic graph theory [13], matrix theory [14] and control theory where they established a connection between the performance of a linear consensus protocol on a directed network and the Fiedler eigenvalue of the mirror graph of the information flow that is obtained via a mirror operation. Another publication from Fax et.al [9] also used tools from algebraic theory stated that a Nyquist criterion that uses the eigenvalues of the graph Laplacian
matrix to determine the effect of the communication topology on formation stability and a method for decentralized information exchange between vehicles which in the end leads to proving that a separation principle that decomposes formation stability into two components: Stability of the is achieved information flow for the given graph and stability of an individual vehicle for the given controller.

From graph theory, researchers tried to simplify the focus of control to information consensus. This can be grouped into three categories; first-order (single integrator), second-order (double integrator) and higher-order consensus. Detailed explanations of each group will be presented in the next chapter. Based on first-order consensus algorithm, Ren [15] proposed consensus algorithms with constant and time-varying reference state using graph theoretical tools, and extended to achieve relative state deviations among agents. By looking at consensus tracking problems, namely bounded control effort and directed switching interaction topologies, Ren [16] proposed the consensus tracking algorithms when the time-varying consensus reference state is available to a dynamically changing subgroup of the team under directed switching inter-vehicle interaction topologies. Sun et.al [17] studied consensus problems for continuous-time multi-agent systems in directed networks with dynamically changing topologies and nonuniform time-varying delays where they analyzed consensus problems in directed networks for dynamically changing topologies and nonuniform time-varying delays; intermittent communication and data packet dropout; and with dynamically changing topologies and nonuniform time-varying delays.

Ren et.al [18] extended the first-order protocols from literature by introducing second-order consensus protocols which considered the general case where information flow may be unidirectional due to sensors with limited fields of view or vehicles with directed, power-constrained communication links. Another publication, [19] studied synchronization of coupled second-order linear harmonic oscillators with local interaction and provide convergence conditions for directed fixed and switching network topologies by using tools from algebraic graph theory, matrix theory, and nonsmooth analysis. Xi et.al [20] studied of consensus problems for the second-order multi-agent systems with external disturbances, switching topology and communication time-delay
and gave a design criterion in terms of bilinear matrix inequality for the control protocol in the presence of disturbances. Yu et.al [21] considers a second-order consensus problem for multiagent systems with nonlinear dynamics and directed topologies where each agent is governed by both position and velocity consensus terms with a time-varying asymptotic velocity and defined new concept about the generalized algebraic connectivity for strongly connected networks and then extended to the strongly connected components of the connection network.

Second order consensus are extended to higher-order consensus. In one of the publications, Li et.al [22] introduced a distributed observer type consensus protocol based on relative output measurements is proposed where a new framework was introduced to address in a unified way the consensus of multiagent systems and the synchronization of complex networks. Li et.al [23] further proposed a multi-step consensus protocol design procedure which yields an unbounded consensus region and at the same time maintains a favourable decoupling property. Zhang et.al [24] showed that unbounded synchronization regions that achieve synchronization on arbitrary digraphs containing a spanning tree can be guaranteed by using linear quadratic regulator based optimal control and observer design methods at each node. Zhang et.al [25] proposes an impulsive consensus protocol and provide sufficient conditions are obtained for the states of follower agents converging to the state of leader asymptotically.

Attempts on using consensus control in nonlinear systems can be observed in a number of publications. For example, Liu et.al [26] presented the consensus problem via distributed nonlinear protocols for directed networks is investigated where its dynamical behaviors are described by ordinary differential equations (ODEs). Hui et.al [27] developed a thermodynamic framework for addressing consensus problems for nonlinear multiagent dynamical systems where they proposed the controller architectures predicated on system thermodynamic notions resulting in controller architectures involving the exchange of information between agents that guarantee the closed-loop dynamical network that was consistent with basic thermodynamic principles and presented in distributed nonlinear static and dynamic controller architectures for multiagent coordination. Xue et.al [28] introduced a new decentralised formation
strategy based on artificial potential functions (APF) in view of the complexity of the framework with switching coupling topology and non-linearity where a new concept of relative-position based formation stability is defined. Due to huge interest in consensus control over the years, there are a lot of references for consensus in the control research community. For more information, readers are encouraged to explore several surveys [29] [30] [31].

As part of distributed control system, the concept of consensus control is explained in Figure 1.1 [9], where the information between agents is utilized to produce an agreed output (consensus output) based on relative information of each agent as shown in Figure 1.2 [9]. The word ‘relative’ indicates the measurement that only related to each agent and not the exact measurement or position of the agent. Based on the intersection of system theory and graph theory, an agreement or consensus protocol can be obtained; that is governed by the interconnection topology and the initial condition for each agent. Hence, the concept of consensus control is based on the convergence of a group of agents or subsystems - known as multi-agent systems - in deciding the outcome of a certain operation or instruction. The agreement between agents can be realized by the consensus algorithm, which uses the relative information of each agent to generate a single outcome. This concept already exists in nature, for example among flock of birds as shown in Figure 1.3 and swarm of bees as shown in Figure 1.4. It is very interesting to see that flock of birds or swarm of bees are able to fly in a single formation while keeping the same distance between each other and avoiding collision by only looking at relative distance of other birds or bees.

Containment control can be seen as an extension of consensus control. Unlike consensus control, where multi-agent systems normally work with single leader; containment control involves multiple leaders that operate in a forest connection topology, and the outcome of such a control strategy is the containment of the followers by the leaders.

Our interest in this thesis is to utilize the relative output feedback information
from each agent in the case that not all agents’ states are measureable in a multi-agent system. This requires the application of observer. In our case, we are going to apply the observer that is based on Luenberger’s observer [32]. The information from
this observer, known as the observed signal is then utilized by the controller, hence the controller is known as the observer based controller. It is a standard type of controller and can be easily found in any control system textbook. When applied to a single linear system, the stability is best explained by the Separation Principle [33], where the stability of the observer does not affect the stability of the controller.

However, this does not apply to the nonlinear systems. Owing to the nonlinearity element, the stability of the system relies on the system stability of both observer and controller. Lyapunov stability theory can be used to analyze the system’s stability and provide the solution to the design of the controller and observer. To relate it to the multi-agent system, using the knowledge of graph theory and Lyapunov stability approach, the system analyzed and the observer based controller can be designed to make the system stable. A stable multi-agent system with consensus control strategy guarantees consensus outcome from each agent. This strategy can also be applied to the design of the consensus state feedback controller and consensus observer.

Hence, the main challenge of this thesis is to prove that the application of the observer-based controller in multi-agent systems is stable thus guarantee the consensus outcome from each agent. This requires careful study of graph theory that relates to the connections and communication structure of the agents’ network. On top of that, the common knowledge of control system in terms of stability such as Lyapunov...
stability theory needs to be utilized. From the theory, stability analysis are to be done with conditions of stability shown. Finally, simulations of the findings need to be presented.

This thesis aims to design the controllers that can solve the consensus and containment problem of multi-agent systems with Lipschitz nonlinearity. Four types of controllers are proposed - a state-feedback consensus controller, an observer-based consensus controller, a state-feedback containment controller and an observer-based containment controller that enable each agent to reach consensus. These controllers utilize the relative state and relative output information of each agent or subsystem. The observer-based consensus and containment controller are based on the design of the consensus and containment observers which are related to Luenberger observer. With careful study of Laplacian structure on the connection network of each agent or subsystem, the stability of the systems are analysed with Lyapunov stability analysis. Simulation examples are given to prove the validity of the theoretical results.

1.2 Outline of the thesis

Chapter 1 gives an overview of consensus control and the aim of the research. While Chapter 2 presents the mathematical preliminaries of consensus control, it covers the mathematical aspects and notations that are needed for consensus control to work. These include some points on graph theory, which also detail the adjacency matrix, Laplacian matrix and Kronecker product. This chapter also includes the fundamental of Lyapunov stability theorem, in relation to the design of controller and observer in a single linear system. The chapter then explains the mathematical description of the consensus control, which is divided into three categories: first-order (single integrator), second order (double integrators) and higher-order consensus with the recent publications that relate to each category. Publications related to Lipschitz nonlinear system in single system are also provided as the basis of the aim of this report. The method on getting the stability conditions for the system mentioned are briefly described. It is also noted that the analysis of this system is performed via Lyapunov
stability analysis. With these information, the Lipschitz nonlinearity is introduced to the multi-agent system and the stability conditions are listed. This is backed by a recent publication mentioned at the end of the Chapter 2.

Chapter 3 presents the first result of this thesis which is the solution to the consensus problem in multi-agent systems with Lipschitz nonlinearity. This relates to the proposed consensus controllers for the respective systems under a directed spanning tree connection topology. A leader-follower state-feedback consensus controller is proposed based on the study of graph theory and Lyapunov stability analysis. The stability of the system is then analysed and discussed in detail. Simulations are presented and discussed in relation to the stability of the system with the aforementioned controller. Next, an observer-based consensus controller is proposed. It is based on the design of a consensus observer that is related to Luenberger observer. Similar to the previous controller, Lyapunov stability analysis is also performed, with simulation examples presented as evidence of its effectiveness. Note that all solutions for the stability analysis are obtained with the Linear Matrix Inequality (LMI) and Algebraic Riccati Equation (ARE) standard routines. The concluding section summarizes the outline and concludes the outcome of Chapter 3.

Chapter 4 presents the next result of this thesis. One of the objectives of this thesis is to solve the containment problem of multi-agent systems with Lipschitz nonlinearity. Therefore, this chapter proposes two controllers - a state-feedback containment controller and an observer-based containment controller for the aforementioned system under directed spanning forest connection topology. Similar to the previous chapter, the observer-based containment controller is based on the consensus observer designed in relation to Luenberger observer. The differences between containment and consensus controllers are explained in detail, with respect to the structure of the Laplacian matrices together with the output of the system. Again, the stability of the system is analysed using a Lyapunov based approach. Simulation examples with detailed discussions are presented in the following section, in other words to prove that the controller is working accordingly. In this chapter, the solution of $P$ obtained from the LMI procedure could not guarantee that the containment is reached as shown in the
simulation section, hence the solution of $P$ is provided via ARE procedure.

*Chapter 5* displays the applications of consensus control. This is represented with the simulation of a group of flexible robot arms, which is modelled as the multi-agent systems with Lipschitz nonlinearity. The state-feedback controller, observer and an observer-based controller are introduced for the consensus control of the system. All simulation examples are performed and discussed in detail.

The thesis ends with the conclusion in *Chapter 6* which summarises all of the findings and outcomes of the previous chapters. It also provides a future feasibility research plan to be carried out using multi-agent system with respect to consensus and containment control.
Chapter 2

Background Studies of Consensus Control

2.1 Introduction

Consensus is defined as mutual agreement between every member in a group [5]. In the context of consensus control, each member is known as an agent or subsystem and the group is a multi-agent system. When consensus is reached, one single outcome is obtained which is obeyed by every member or every agent of the group. Therefore, in a distributed control system, consensus control forces every agent or subsystem to come out with a single output.

The concept of consensus was first introduced by Reynolds [5] in computer systems, by describing the flock of birds or other biological entities; by imitation, the flocking behaviour can be explained and simulated in computer systems. Meanwhile, Vicsek et.al. [4] investigated the biological nature of self-moving particles, by providing mathematical terms and limitations. Jadbaibaie et.al. [8] explored this further, by giving an analysis based on mutual agreement between each agent. Fax and Murray [9] started to include the usage of graph theory in consensus control, and provided several insights involving new definitions and terms; these also include the application of vehicle formation with respect to graph theory [34]. In [35] and [36], the mobile flocking agent issue is tackled by looking at the fixed and dynamic coordination of flocking agents without leaders. Following this work, and motivated by Reynolds [5], Olfati-Saber [37]
introduced free-space and constrained flocking algorithms, based on a non-leader arrangement of birds. Some works simplify the system by only looking at information consensus; this is divided into three categories as follows: first-order (single-integrator), second-order (double-integrator) and higher-order (more than double-integrator) consensus. The difference between the three is determined by the difference in algorithm, which will be explained later in this chapter.

2.2 Mathematical Preliminaries of Consensus Control

In order for consensus control to work, it is very important that the mathematical aspects of consensus to be investigated and understood. The concept of consensus itself is a well-known subject in the Computer Science field where mathematics are heavily involved [9]. Therefore, mathematical concept plays an important role in consensus control.

As explained in the Chapter 1, the process of designing consensus control for multi-agent systems usually involves certain steps; which include to identify the number of the agents, the derivation of graph based on the connections between agents, the conversion from graph to adjacency matrix, and the conversion from adjacency matrix to Laplacian matrix. Therefore, information involving graph theory that reflects the connection topology of multi-agent systems plays an important role for the control design. Hence this section will go through several important fundamental knowledge that is needed for consensus control. This relates to the knowledge of graph theory, where it involves on how to obtain a graph and translate it into adjacency matrix. Then, the adjacency matrix is converted to Laplacian matrix. Based on recent literatures, several conditions of Laplacian matrix are also listed later in the chapter. Laplacian matrix always involve the eigenvalues and eigenvectors together with Gersgorin theorem later in the design of consensus controller. Therefore, the explanation on those topics are also provided at the end of this section.
2.2.1 Graph Theory

In consensus control, algebraic graph theory is widely used to describe the networking topology between agents. The graph is based on the information transfer topology between agents, and it is in the shape of a directed or undirected topology. This section explains the fundamental concept of graph theory. It also explains graph topology arrangement of each agent in the group forming adjacency matrix. Then, the Laplacian matrix will be derived from the matrix to be used in consensus control.

The process of getting the Laplacian matrix from the communication graph is best explained by Figure 2.1. At first, each agent needs to be numbered and the links between agents have to be determined to form the communication graph. Then the adjacency matrix $\mathcal{Q}$ is derived from the graph and this in turn will produce the Laplacian matrix $\mathcal{L}$.

![Figure 2.1: The Process of Getting the Laplacian Matrix from the Communication Graph.](image)

**Graph**

A graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}$ and an edge $\mathcal{E}$, where an edge is an unordered pair of distinct vertices of $\mathcal{X}$ [13]. As shown in Figure 2.2, $xy$ is considered an edge when $x$ and $y$ are adjacent to each other. A vertex is incident with an edge if it is one of the two vertices of the edge. A graph is normally used to represent a system or a group of agents in the same domain, such as a group of robots that have a network connection between each other, whereby the adjacent vertices represent pairs of agents that are
linked in a communication network. Figure (2.3) shows two graphs of five vertices.

The directed graph is defined as a graph $\mathcal{X}$ that consists of a vertex set $\mathcal{V}(\mathcal{X})$ and an arc set $\mathcal{G}(\mathcal{X})$, where an arc - or directed edge - is an ordered pair of distinct vertices [13]. As shown in Figure (2.3) and Figure (2.4), it can be undirected or directed. If the vertex set is not given, the graph is also undefined, as shown in Figure (2.3) and Figure (2.4).
CHAPTER 2. BACKGROUND STUDIES OF CONSENSUS CONTROL

Going in depth about how to define the $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{E}$ of the graph $\mathcal{G}$, it can be seen that a graph $\mathcal{G}$ consists of a node set $\mathcal{V}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, in which an edge is represented by a pair of distinct nodes of $\mathcal{G}$: $(i, j) \in \mathcal{V}(\mathcal{X})$, where $i$ is the parent node, $j$ is the child node, and $j$ is neighbouring $i$.

A directed graph is said to be strongly connected if, for any pair of distinct nodes, there exists a path between them. In addition, a directed graph has a directed spanning tree if there exists a node called root, such that there exists a directed path from this node to every other node, as shown in Figure 2.5. In this report, all works on consensus control are based on a directed spanning tree communication topology.

![Directed Spanning Tree Connection Topology](image)

In addition, the directed spanning forest connection is an extension of the directed spanning tree connection topology. As shown in Figure 2.6, a single leader in the multi-agent group is added with one or more leaders represented by Agent 5 and Agent 6. Similar to the directed spanning tree topology, a directed spanning forest has strong connection if each leader’s node has a directed path to every other nodes. Hence, all works on containment control of multi-agent systems will be based on this topology.

2.2.2 Adjacency Matrix $\mathcal{Q}$

The adjacency matrix $\mathcal{Q}$ is derived from the graph $\mathcal{G}$ [13] mentioned in the previous section. For example, suppose that there are $N$ nodes in a graph: for this thesis, the adjacency matrix $\mathcal{Q} = (q_{ij} \in \mathbb{R}^{N \times N})$ is defined by $q_{ii} = 0$, $q_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and 0 otherwise. Figure 2.7 shows the graph $\mathcal{G}$ with the corresponding adjacency matrix $\mathcal{Q}$. 
Based on Figure 2.7, the method of producing the adjacency matrix is as follows. The arrow from agent 1 pointing to agent 2 shows the signal flow from agent 1 which give an outcome of 1 in the first column and second row of matrix $Q$. The signal from agent 5 to agent 1 and vise versa will give outcomes of 1 in column five and row one and column 1 and row 5 respectively. Agent 1 and agent 2 also receive information from agent 4, which is shown by having outcomes values of 1 in column four and row 1 and row 2. Agent 3 only receives information from agent 2, which results in an outcome of 1 in column 2 and row 3. The value 0 indicates that there is no information transfer.

If the connection is undirected, Figure 2.7 can be slightly modified to become Figure 2.8; instead of the line being represented by a line with arrows, it is replaced by line without arrows. The behaviour of these lines is the same as the line between
agent 1 and 5 in Figure 2.7, which means that the information transfer is bidirectional. This matrix is symmetrical, which indicates that the information transfer is two-way.

![Figure 2.8: Graph $\mathcal{G}$ with Undirected Connection Topology with Adjacency Matrix $\mathcal{Q}$.](image)

or bidirectional. Some publications highlight it as the special case of two directions of information from agents. Note that in Figure 2.8, agent 1 is connected to agent 2, 4 and 5. This results in a value of one in column one and row two, row four and row five in matrix $\mathcal{Q}$ respectively. Symmetrically, the value of one also exists in row one, column two, column four and column five. Similar outcome happens to agent 2 where outcome of 1 appears in column 2 row 3 and 4, and in row 2 column 3 and 4. It can be seen that the undirected connection topology is more complex compared to the directed topology. This outcome will affect the construction of the Laplacian matrix $\mathcal{L}$ in the next section.

### 2.2.3 Laplacian Matrix

The Laplacian matrix $\mathcal{L} = l_{ij} \in \mathbb{R}^{N \times N}$ is derived from the adjacency matrix $\mathcal{Q}$. It can be defined as

$$l_{ii} = \sum_{j \neq i} q_{ij}$$

$$l_{ij} = -q_{ij} \text{ for } i \neq j$$

(2.1)

where $q_{ij}$ is defined in Section 2.2.2. From the definition of the Laplacian matrix (2.1) and Figure 2.7, and adjacency matrix $\mathcal{Q}$, the Laplacian matrix $\mathcal{L}$ can be easily
obtained as
\[
\mathcal{L} = \begin{bmatrix}
2 & 0 & 0 & -1 & -1 \\
-1 & 2 & 0 & -1 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 1
\end{bmatrix}
\] (2.2)

The Laplacian matrix $\mathcal{L}$ in (2.2) is from the directed topology, based on Figure 2.7. Consequently, Laplacian matrix $\mathcal{L}$ for Figure 2.8 can be obtained as
\[
\mathcal{L} = \begin{bmatrix}
3 & -1 & 0 & -1 & -1 \\
-1 & 3 & -1 & -1 & 0 \\
0 & -1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 3 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{bmatrix}
\] (2.3)

It can be seen that the undirected connection topology is basically the special version of directed connection topology [34]. The Laplacian matrix obtained is symmetric, and each information transfer in the line in graph $\mathcal{G}$ in Figure 2.8 actually has the same behaviour as the information signal of agent 1 to 4 in Figure 2.7.

The Laplacian matrix $\mathcal{L}$ plays an important role in designing the consensus controller and observer later in this report. In short, several conditions need to be stated to provide a general understanding of the Laplacian matrix $\mathcal{L}$. These are the main conditions of the Laplacian matrix $\mathcal{L}$, which have important roles in designing the consensus controller [9] [38] as follows:

1. For an undirected graph, $\mathcal{L}$ is symmetric.

2. For a directed graph, $\mathcal{L}$ is not necessarily symmetric.

3. In both the undirected and directed cases, since $\mathcal{L}$ has zero row sums, 0 is an eigenvalue of $\mathcal{L}$ with the associated eigenvector $1 \triangleq [1, \ldots, 1]^T$, and $n \times 1$ column vector of ones.

4. $\mathcal{L}$ is diagonally-dominant, and has non-negative diagonal entries.

5. Gersgorin disc theorem states that, for an undirected graph, all of the non-zero eigenvalues of $\mathcal{L}$ are positive ($\mathcal{L}$ is positive semidefinite), whereas, for a directed
graph, all of the non-zero eigenvalues of $L$ have positive real parts. The details of this theorem are to be explained in the next section.

6. For an undirected graph, 0 is a simple eigenvalue of $L$ if and only if the undirected graph is connected.

7. For a directed graph, 0 is a simple eigenvalue of $L$ if the directed graph is strongly connected, although the converse does not hold true; for an undirected graph, let $\lambda_i(L)$ be the $i_{th}$ smallest eigenvalue of $L$ with $\lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_n(L)$ so that $\lambda_1(L) = 0$.

8. For an undirected graph, $\lambda_2(L)$ is the algebraic connectivity, which is positive if - and only if - the undirected graph is connected. The algebraic connectivity quantifies the convergence rate of consensus algorithms.

2.2.4 Eigenvalues and Eigenvectors

The knowledge of eigenvalues and eigenvectors are useful because the eigenvalues of Laplacian matrix is required for the design of consensus control. This section gives a clear explanation on the concept of eigenvalues and eigenvector [33]. For a square matrix $A = C^{n \times n}$, the eigenvalues of $A$ are defined as the solutions of the equation

$$\det(\lambda I - A)$$

where $I$ is defined as

$$I = \text{diag}(1) \in \mathbb{R}^{n \times n}$$

For the set of all the eigenvalues $\{\lambda_i\}_{i=1}^n$, if there is any non-zero vector $v_r \in C^n$ satisfying

$$Av_r = \lambda_i v_r$$

then the non-zero vector $v_r$ is called the right eigenvector of $A$.

Similarly, if a non-zero vector $v_l^T A = \lambda_l v_l^T$ satisfies

$$v_l^T A = \lambda_l v_l^T$$

(2.7)
then the non-zero vector \( v_l \) is called the left eigenvector of \( A \). Note that the left eigenvector can also be obtained from the equation \( A^T v_l = \lambda_i v_l \).

### 2.2.5 Gershgorin Circle Theorem

In mathematics, the Gershgorin circle theorem [9] is usually used to bound the eigenvalue of a square matrix i.e the Laplacian matrix. Define a square matrix \( A \in C^{n \times n} \), in which the elements of matrix \( A \) are \( a_{ij} \) for \( i, j = 1, 2, ..., n \). Let

\[
R_x = \sum_{i=1, i \neq j}^{n} a_{ij}
\]

which is the sum of the \( i^{th} \) rows of the matrix \( A \) except the element \( a_{ij} \). The Gershgorin circle theorem pointed out that the eigenvalues of matrix \( A \) will located in the closed discs, which are centred at \( a_{ij} \) with radius \( R_i \). Note that the closed discs centred at \( a_{ij} \) with radius \( R_i \) are also called Gershgorin discs.

### 2.2.6 Kronecker Product

In consensus control, Kronecker product is used to describe the connection of the multi-agent systems with respect to the Laplacian matrix structure. The Kronecker product [14] of matrices \( A \in \mathbb{R}^{m \times n}, \ B \in \mathbb{R}^{p \times q}, \ C \in \mathbb{R}^{r \times s} \) and \( D \in \mathbb{R}^{t \times u} \) is defined as

\[
A \otimes B = \begin{bmatrix}
    a_{11}B & \cdots & a_{1n}B \\
    \vdots & \ddots & \vdots \\
    a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}
\]

that satisfies

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD)
\]

\[
(A \otimes B)^T = A^T \otimes B^T
\]

\[
(A \otimes B) + (A \otimes C) = A \otimes (B + C)
\]
2.2.7 Lyapunov Stability Theorem, Linear State-feedback Controller and Linear Observer Designs

In this thesis, Lyapunov stability will be used to determine the stability of dynamic systems. Hence, it is wise to briefly go through some important aspects of Lyapunov stability, as shown in this section.

The Lyapunov stability is applied to both autonomous and non-autonomous systems, where the autonomous system is described by

\[ \dot{x} = f(x) \]  

(2.10)

where \( f : D \rightarrow \mathbb{R}^n \) is a nonlinear function map from a domain \( D \subset \mathbb{R}^n \). Here, notions and theorems related to Lyapunov stability of autonomous systems will be reviewed. If (2.10) has an equilibrium point at the origin; \( f(x) \) of equation (2.10) can be assumed to satisfy \( f(0) = 0 \). The equilibrium point \( x = 0 \) of (2.10) can have three types of stabilities: explained as follows:

**Definition 2.1** \([39]\) The equilibrium point \( x = 0 \) of equation (2.10) is

1. **stable** if, for each \( \varepsilon > 0 \), there is a \( \delta = \delta(\varepsilon) > 0 \) such that

\[ \|x(0)\| < \delta \implies \|x(t)\| < \varepsilon, \forall t \geq 0 \]  

(2.11)

2. **unstable** if it is not stable

3. **asymptotically stable** if it is stable and \( \delta \) can be chosen such that

\[ \|x(0)\| < \delta \implies \lim_{t \to \infty} x(t) = 0 \]  

(2.12)

Therefore, based on Definition 2.1, the Lyapunov stability theorem is stated as

**Theorem 2.2.1** \([39]\) Let \( x = 0 \) be an equilibrium point for equation (2.10) and \( D \subset \mathbb{R}^n \) be a domain containing \( x = 0 \). Let \( V : D \rightarrow \mathbb{R} \) be a continuously-differentiable function, such that

\[ V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \]

\[ \dot{V}(x) \leq 0 \text{ in } D \]  

(2.13)
then \( x = 0 \) is stable. Moreover, if
\[
\dot{V}(x) < 0 \quad \text{in } D - \{0\}
\]  
then \( x = 0 \) is asymptotically stable.

Note that the function \( V \) in Theorem 2.2.1 is known as the Lyapunov function [39]. Function \( V \) satisfies the condition (2.13) stated in Theorem 2.2.1, known as positive definite, while for \( V \geq 0 \) for \( x \neq 0 \) is said to be positive semi-definite. If there is a negative sign for the positive definite, or negative definite of \( V (\dot{V}(x)) \), it is known as negative definite, or negative semi-definite, respectively.

A dynamic system does not necessarily have a unique solution of the Lyapunov function - there can be many Lyapunov function candidates. Hence, there is no fixed way to find Lyapunov functions [39]. However, a chosen Lyapunov function sometimes cannot reflect the actual stability of a dynamical system. Let us consider an example from Khalil [39]:

**Example 2.1** Consider the pendulum equation with friction. This system is described by
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_1 &= a \sin(x_1) - bx_2
\end{align*}
\]  
(2.15)

In this example, the Lyapunov function candidate that represents the potential and kinetic energy of the pendulum in (2.15) is chosen as
\[
V(x) = a(1 - \cos x_1) + (1/2)x_2^2
\]

It can be shown that the Lyapunov function candidate’s derivative \( \dot{V}(x) \) is a negative semi-definite, while according to theorem 2.2.1, the system is stable. However, from the phase portrait, the origin of (2.15) is asymptotically stable for \( b > 0 \) [39]. To be more precise, another Lyapunov function candidate needs to be chosen so that \( \dot{V}(x) \) is negative definite. However, this can be avoided by using LaSalle’s theorem (Theorem 2.2.2) to support the chosen Lyapunov candidate function.
Theorem 2.2.2 \[39\] Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (2.10). Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\dot{V} \leq 0$ in $\Omega$. Let $E$ be the set of all points in $\Omega$ where $\dot{V}(x) = 0$. Let $M$ be the largest invariant set in $E$. Then every solution starting in $\Omega$ approaches $M$ as $t \rightarrow \infty$.

There are two other corollaries that support Lyapunov stability, apart from LaSalle’s Theorem 2.2.2 known as Barbashin and Krasovskii [39]. These corollaries can be described as

**Corollary 2.1** \[39\] Let $x = 0$ be an equilibrium point for (2.10). Let $V : D \rightarrow \mathbb{R}$ be a continuously-differentiable positive definite function on a domain $D$, containing the origin $x = 0$, such that $\dot{V}(x) \leq 0$ in $D$. Let $S = \{x \in D | \dot{V}(x) = 0\}$ and suppose that no solution can remain identically in $S$, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable.

**Corollary 2.2** \[39\] Let $x = 0$ be an equilibrium point for (2.10). Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable, radically unbounded, positive definite function such that $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Let $S = \{x \in \mathbb{R}^n | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in $S$, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable.

LaSalle’s theorem relax the negative definiteness requirement of Lyapunov theorem and extends the theorem’s direction into three: 1) It gives an estimate of the region of attraction, which is not necessary of the form of $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ where $\Omega$ can be any compact positively invariant set. 2) It can be used in cases where the system has an equilibrium set, rather than an isolated equilibrium point. 3) The function $V(x)$ does not have to be positive definite. Readers are encouraged to refer to Example 4.8 and Example 4.9 from Khalil et.al. [39] for detailed explanation.

**Linear state feedback design**

The controller design in this thesis is based on a state-feedback control strategy, hence it is necessary to briefly discuss the concept of the state-feedback controller as shown in Figure 2.9. Consider the $n$-dimensional single-variable state equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

(2.16)
where \( x \in \mathbb{R}^n \) is the state variable, \( u \in \mathbb{R}^p \) is the input, \( A, B, C \) are constant matrices with appropriate dimensions, and \( y \in \mathbb{R}^m \) is the output.

In order to design a controller for system (2.16), the system itself needs to be controllable [33]. This is defined by the controllability of the system where the definition can be obtained in any control system textbook [33].

In state-feedback, the input \( u \) given by

\[
    u = r - Kx = r - [k_1, k_2, \ldots, k_n]x = r - \sum_{i=1}^{n} k_ix_i
\]

(2.17)

where \( K \in \mathbb{R}^{p \times n} \) is the controller gain. If we denote \( r = 0 \), then system (2.16) is asymptotically stable, if the resultant matrix of \( A - BK \) has negative real parts (Hurwitz stable); these eigenvalues depend on the choice of controller gain \( K \).

**Figure 2.9: The Linear State Feedback Controller.**

**Linear observer design**

In practical applications, not all states are measurable. Hence, the linear observer design is very useful to recover the unmeasured states by states estimation from output feedback information. This information is important for the design of controllers for systems with nonlinearity element in later chapters.
From (2.16), the observer as shown in Figure 2.10 takes the form of

\[ \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y - C\hat{x}(t)) \]  \hspace{1cm} (2.18)

where \( L \in \mathbb{R}^{n \times p} \) is an observer gain. The error is represented as \( \tilde{x} = x - \hat{x} \). Hence, the error dynamics can be described as

\[ \dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t) \] \hspace{1cm} (2.19)

The equation (2.19) is asymptotically stable if the eigenvalues of the resultant matrix of \( A - LC \) have negative real parts or the matrix is said to be Hurwitz stable. These eigenvalues depend on the choice of observer gain \( L \). This type of linear observer (2.18) is known as Luenberger observer [32]. The existence of observer design (2.18) is guaranteed by the observability of the system which can be obtained in any control systems textbook [33].

**Observer-based Controller**

From the controller and observer explained in Section 2.2.7, an observer-based controller can be designed as shown in Figure 2.11. The input of the equation (2.16) is denoted as

\[ u = K\tilde{x}(t) \] \hspace{1cm} (2.20)

where \( K \in \mathbb{R}^{p \times n} \) is the feedback matrix with \( \tilde{x} \in \mathbb{R}^n \) is the observed signal from the observer (2.18).
With respect to the design of the controller and observer in Section (2.2.7) and denoting \( r = 0 \), the state-space equations can be rearranged as

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\tilde{x}}(t)
\end{bmatrix} =
\begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\tilde{x}(t)
\end{bmatrix}
\tag{2.21}
\]

where \( \tilde{x} = x - \hat{x} \). If matrices \( A, B, C \) are controllable and observable, with \( A - BK \) and \( A - LC \) both being Hurwitz-stable, the system is also stable.

The difference between the designed observer-based controller, and the state feedback controller in Section 2.2.7 is the input; the former utilized the observed signal from the observer whilst the latter used the states signal. This is useful for systems where it is impossible to get the measurement of every state, as long as the system is observable and controllable. With this controller, the system can be assured to be stable because the stability of the observer and controller do not affect each other, which is known as the Separation Principle [33]. The previous information about linear state-feedback and linear observer designs will be very useful for the development of consensus and containment controllers for multi-agent system with Lipschitz non-linearity in later chapters.

Figure 2.11: The Observer-based Controller based on Luenberger Observer.
2.3 The Mathematical Description of Consensus Control

Based on publications of consensus control, researchers simplified their analysis approaches when it comes to analysing the stability and convergence of the multi-agent systems. Unlike the single system, where normally the complex dynamics were included for the control design and stability analysis, only the information transfer between agents were considered without too much focus on the dynamics of the system. Hence, first-order (single integrator), second-order (double integrator) and higher-order multi-agent systems were introduced.

A single integrator (first-order) system is basically a linear system with one integrator. It only concentrates on the information transfer between agents, and is heavily influenced by the topology of the agents’ connection network. The dynamics of a single-integrator (first-order consensus) multi-agent system is represented by

\[
\dot{x}_i = u_i, \quad i = 1, \ldots, N
\]  

(2.22)

where for \(i = 0, \ldots, N\), \(x_i \in \mathbb{R}^n\) is the state vector of the subsystem and \(u_i \in \mathbb{R}^p\) is the input of the \(i^{th}\) subsystem. The general protocol for the single-integrator system is represented by

\[
u_i = K \sum_{j=1}^{N} l_{ij} (x_i - x_j)
\]  

(2.23)

where \(K \in \mathbb{R}^{p \times n}\) is the controller matrix that is to be determined, and \((x_i - x_j)\) shows that the input depends on the presence of relative information between state vector \(i\) of an agent and its neighbours \(j\).

The introduction of the protocol in equation (2.23) into (2.22) can be rewritten in matrix form as

\[
\dot{x} = -K \mathcal{L} x
\]  

(2.24)

where \(x = [x_1, \ldots, x_N]^T\) and \(\mathcal{L}\) is the Laplacian matrix defined in (2.1). The objective of the controller is for the agent to guarantee that \(x_i(t) \to x_j(t)\) for \(t \to \infty\). In literature, La Salles Invariant Principle [39] or Gersgorin circle [40] is used to show that all
states of (2.24) converge to the same value. A lot of publications have been reported
with respect to the single-integrator, such as in [9] [34] [15] [16] [41].

A double-integrator (second-order consensus) multi-agent system is represented by
\[
\dot{x}_i = v, \quad \dot{v}_i = u_i, \quad i = 1, \ldots, N
\]
(2.25)
where \(x_i \in \mathbb{R}^n\) is the position vector of the agent \(i\) and \(v_i \in \mathbb{R}^n\) is the velocity vector of
agent \(i\) for \(i = 1, \ldots, N\). Similar to the single-integrator system, the system dynamics
is ignored, and the consensus of the states depends on the topology of the agents’
network. The general second-order consensus protocol will be represented as
\[
u_i = K \sum_{j=1}^{N} l_{ij} ((x_i - x_j) + \alpha (v_i - v_j)), \quad i = 1, \ldots, N
\]
(2.26)
where \(K\) is the controller matrix that is to be determined later, and \(\alpha > 0\) a constant.
\((v_i - v_j)\) is the relative state derivative value for agent \(i\) and its neighbours. The
controller guarantees that \(x_i(t) \to x_j(t)\) and \(v_i(t) \to v_j(t)\) as \(t \to \infty\). By letting
\(x = [x_1, \ldots, x_N]^T\) and \(v = [v_1, \ldots, v_N]^T\), the system (2.25) with controller (2.26) can
be written as
\[
\begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix} = \Lambda 
\begin{bmatrix}
x \\
v
\end{bmatrix}
\]
(2.27)
where
\[
\Lambda = \begin{bmatrix}
0_{n \times n} & I_n \\
-K\mathcal{L} & -K\alpha \mathcal{L}
\end{bmatrix}
\]
Research related to second-order consensus has been reported in [18,20,42–46].

It is well documented that the consensus of multi-agent systems with single-integrator
kinematics often converge to a constant final value and consensus for double-integrator
dynamics might admit a dynamic final value. Hence, as an extension of the single-
integrator and double-integrator systems, the higher-order system is represented by
\[
\begin{aligned}
\dot{x}_i &= Ax_i + Bu_i, \quad i = 1, \ldots, N \\
y_i &= Cx_i
\end{aligned}
\]
(2.28)
where for \(i = 0, \ldots, N\), \(x_i \in \mathbb{R}^n\) is the state vector of the subsystem, \(u_i \in \mathbb{R}^p\) is the
input of the \(i\)th subsystem, and \(y_i \in \mathbb{R}^q\) is the measured output vector, \(A \in \mathbb{R}^{n \times n}\),
$B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{q \times n}$ are appropriate matrices.

Similarly, the purpose of the study is to discover whether a feedback control law for the consensus can be developed. With protocol (2.23) applied, in matrix form (2.28) can be rewritten as

$$\dot{x} = (I_N \otimes A - \mathcal{L} \otimes BK)x$$  \hspace{1cm} (2.29)

where $x = [x_1, \ldots, x_N]^T$ and $\otimes$ represents the Kronecker product [14]. Basically (2.29) is considered as the expansion of each agent linked by equation (2.23) in a network. In short, based on the structure of the system, stability can be achieved when $A - \lambda_i BK$ is Hurwitz [22] where $\lambda_i$ is the eigenvalue of $\mathcal{L}$ for $i = 1, \ldots, N$. The same publication [22] also reports that there exists a matrix $K$ for a consensus controller, based on the solution of $P = P^T > 0$ from LMI. A number of publications describe elements of this work, such as in [22, 23, 47, 48].

### 2.4 Lipschitz Nonlinear Systems (A Class of Nonlinear Systems)

In order to meet the aim of this thesis, Lipschitz nonlinearity needs to be introduced into the multi-agent systems. This information is vital and will be the basis of the consensus and containment control approach of this thesis. Therefore, several publications will be looked at to get enough information about the Lipschitz nonlinearity systems.

Previous publications for the Lipschitz nonlinearity systems mostly involved single systems. The use of nonlinear observer design known as Lipschitz observer was necessary. The importance of this observer design is that it is able to estimate the state based on state or output feedback, in order to recover the values of the unknown states due to the nonlinearity element in the system. This can be seen in a number of literatures such as [49–53] among others.
CHAPTER 2. BACKGROUND STUDIES OF CONSENSUS CONTROL

From literature, Thau [49] was the first to introduce the Lipschitz observer, based on Luenberger observer model. Rajamani [50] expanded the conditions of the observer where he proves that the observer is stable in certain conditions. In Aboky et.al [51], the relationship between the stability of the observer and the Algebraic Riccati Equation (ARE) was established, and a few points were corrected in [50]. Pagilla [52] introduced the conditions of stability for the observer, and an observer-based controller with ARE for a single system. Several advantages of ARE solution for observer are highlighted in Phanamchoeng et.al [53], and compared with solutions obtained with the Linear Matrix Inequality (LMI).

In general, Lipschitz nonlinear system can be expressed as

\[
\dot{x} = Ax + Bu + \phi(x) \quad (2.30)
\]

where \(x \in \mathbb{R}^n\) is the state and \(u \in \mathbb{R}^q\) is the input. \(\phi : \mathbb{R}^n \to \mathbb{R}^n\) is a Lipschitz nonlinear function with the Lipschitz constant \(\gamma\), i.e., for any two constant vectors \(a, b \in \mathbb{R}^n\), we have

\[
\|\phi(a) - \phi(b)\| \leq \gamma \|a - b\| \quad (2.31)
\]

It is clear that, in all of the publications previously mentioned, the design of a stable observer in the control system design is very important, and can be achieved by deriving the conditions of stability with the Lyapunov method. The Lipschitz observer as shown in Figure 2.12 is based on Luenberger observer, which can be expressed as

\[
\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) + \phi(\hat{x}) \quad (2.32)
\]

where \(L \in \mathbb{R}^{n \times p}\) and \(\hat{x} \in \mathbb{R}^n\) is the observed state. With error state \(\tilde{x} = x - \hat{x}\), the estimation error dynamics can be obtained as

\[
\dot{\tilde{x}} = (A - LC)\tilde{x} + \phi(x) - \phi(\hat{x}) \quad (2.33)
\]

where the nonlinear term \(\phi(x) - \phi(\hat{x})\) satisfies Lipschitz function defined in (2.31). In this design, the observer is said to be stable if the eigenvalues of the resultant matrix \(A - LC\) have negative real parts, or Hurwitz stable with \(\kappa > 0\) and \(\gamma > 0\).

It is easy to get a solution of \(P = P^T > 0\) for the observer \(L\) from a suitable Lyapunov candidate. For example, if there is no input for the system (2.30) due to
term $Bu$ cancelled, with the Lyapunov candidate $V = \hat{x}^T P \hat{x}$, where $\hat{x}$ is the error estimation states, it is possible to obtain Theorem 2.4.1 from Rajamani et.al [54].

**Theorem 2.4.1** [54] If a gain matrix $L$ can be chosen such that

\[(A - LC)^T P + P(A - LC) + \gamma^2 PP + I < 0 \tag{2.34}\]

for some positive definite symmetric matrix $P$, then this choice of $L$ leads to asymptotically-convergent estimates by the observer $L = P^{-1} C^T$ for state estimation of system (2.30).

Similar approaches have been reported in [51–53].

Even though the stability of the observer does not affect the stability of the system, the same approach can be used to obtain the stability condition for the system with the controller, or solution of $P = P^T > 0$ for controller gain matrix $K$ as reported in Pagilla et.al [52].

This approach will be used for stability analysis of the remaining work of this thesis. It will involve both the controller and observer, for both consensus and containment control of multi-agent systems with Lipschitz nonlinearity.
2.5 Application of Lipschitz nonlinearity in consensus control with Lyapunov stability analysis

As previously mentioned, one of the aims of this thesis is to solve consensus problem of multi-agent systems with Lipschitz nonlinearity. This requires the introduction of Lipschitz nonlinearity of a single system into multi-agent systems. After this process, the next objective is to analyse the effects of a Lipschitz nonlinearity in a consensus control system and the system stability. This section will explain the introduction of the Lipschitz nonlinearity into multi-agent systems as the basis of this research.

The nonlinear system defined in (2.30, pg.44) is for single system. This can easily be converted to the multi-agent system by having more of the same system (2.30). Hence, the multi-agent system with Lipschitz nonlinearity can be expressed as

\[ \dot{x}_i = Ax_i + Bu_i + \phi(x_i) \]  

(2.35)

where for \( i = 0, \ldots, N \), \( x_i \in \mathbb{R}^n \) is the state vector of the subsystem, \( u_i \in \mathbb{R}^p \) is the input of the \( i \)th subsystem, and \( y_i \in \mathbb{R}^q \) is the measured output vector, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times p} \), and \( C \in \mathbb{R}^{q \times n} \) are appropriate matrices. \( \phi(x_i) \) is defined in (2.31) for \( i = 1, \ldots, N \). The consensus controller (2.23, pg.41) is then introduced and the network dynamics of the system can be obtained as

\[ \dot{x} = (I_N \otimes A - \mathcal{L} \otimes BK)x + \Phi(x) \]  

(2.36)

where \( x = [x_1, \ldots, x_N]^T \) and \( \Phi(x) = [\phi(x_1), \ldots, \phi(x_N)]^T \) and \( \otimes \) is the Kronecker product [14].

Without the nonlinearity element \( \phi(x) \), the controller matrix \( K \) in (2.36) is selected so that \( A - \lambda_i BK \) is Hurwitz stable with \( \lambda_i \) represents the eigenvalues of Laplacian matrix \( \mathcal{L} \) for subsystems \( i = 1, \ldots, N \). For the correct value of \( \lambda_i \), each subsystem is stable and this guarantee each subsystem to reach consensus [22].

However, it is different with the multi-agent systems with Lipschitz nonlinearity. In order to prove the stability of this system, Lyapunov stability analysis needs to
be applied by selecting proper function candidates. One publication in particular presented the stability analysis of Lipschitz nonlinear systems with the application of state-feedback consensus controller [55]. It presented the consensus usability and stability of the designed state-feedback consensus controller with first-order protocol (2.23) by taking into account the Laplacian structure of the multi-agent network with distinct and multiple eigenvalues of Laplacian matrix ($\mathcal{L}$). This is shown in the following theorem.

**Theorem 2.5.1** [55] For a network-connected dynamic system (2.35) which satisfies the assumption that the eigenvalue of a Laplacian matrix at 0 is a single eigenvalue, the consensus control problem can be solved by the control system design (2.23), with a control gain $K = B^TP$, and the positive definite matrix $P$ specified in one of the following cases:

1. If the eigenvalues of the Laplacian matrix are distinct, i.e., $n_k = 1$ for $k = 1, \ldots, q$, the matrix $P$ satisfies
   \[ A^T P + PA - 2\alpha BB^TP + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n < 0 \] (2.37)
   with $\kappa$ being any positive real number, and $\alpha = \min\{\lambda_1, \ldots, \lambda_p, \alpha_{p+1}, \ldots, \alpha_q\}$.

2. If the Laplacian matrix have multiple eigenvalues, i.e., $n_k > 1$ for any $k \in \{1, \ldots, q\}$ the matrix $P$ satisfies
   \[ A^T P + PA - 2(\alpha - 1)BB^TP + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n < 0 \] (2.38)
   where $\gamma_0$ is obtained from the nonlinear element $\psi_i$ in Lemma 3.2 from Ding [55].
   Similarly, $n_k > 1$ is obtained from Lemma 3.1 [55] for the transformed Jordan matrix $J$ of Laplacian matrix ($\mathcal{L}$).

Here, the methodology used to get a stable system was based on Lyapunov stability theory, with the inclusion of the nonlinearity element $\phi(x_i)$. A solution of $P = P^T > 0$ could be obtained for the system to be asymptotically stable. Note that without the nonlinearity element $\phi(x_i)$, the stability of the the system can be achieved when $A - \lambda_i BK$ is Hurwitz stable for selected $\lambda_i$ that represents the eigenvalues of the
Laplacian matrix $L$ for subsystems $i = 1, \ldots, N$ [22]. When each subsystem is proven to be stable, it will influence the whole system to be stable, which in turn enable all subsystems to achieve consensus.

2.6 Conclusions

In this chapter, several mathematical aspects of consensus control have been explained such as graph theory and Lyapunov stability theory. In particular, the graph theory provided information about the connection topology between agents while the Lyapunov stability theory gave the steps for the stability conditions to be met to enable each agent to reach consensus. In other words, these knowledge are needed for the application of consensus control in multi-agent systems to be possible. The fundamental of linear state-feedback controller, observer and observer-based controller design have been introduced. They were then linked to the design of consensus control of multi-agent system and the stability conditions of the systems were presented.

Next, key results in the area of consensus control of multi-agent systems have been briefly explained. In short, these could be divided into three groups - first-order (single integrator), second-order (double integrators) and higher-order systems. The characteristics of each group were explained briefly. It was further pointed out that the higher-order system is the focus of this thesis.

While most of the results in consensus control were linear in nature, this thesis is dealing with consensus and containment problem of multi-agent systems with Lipschitz nonlinearity. This can be viewed as the major contribution of this report. Therefore, the chapter ended with Lipschitz nonlinearity in single systems introduced into multi-agent systems. The new system was also linked to the approaches made for stability analysis by referring to recent publications.

The next chapter will present the first result of the thesis which is the consensus control of multi-agent systems with Lipschitz nonlinearity.
Chapter 3

Consensus Control with Lipschitz Nonlinearity

3.1 Introduction

Consensus control is a part of distributed control systems. It is usually applied in a special communication network where it consists a number of agents or subsystems that agree to a single outcome. In other words, consensus is reached between agents or subsystem. This type of control design usually explores the communication structure (i.e. Laplacian matrix) of the network. Many applications, such as formation control, synchronization and others, are based on the consensus algorithms and it is getting more attention in the control systems research fields in recent decade.

There are numerous publications that relates to consensus control in various area of interests. Information consensus for first-order (single-integrator) systems has been investigated in the papers such as [15, 42, 56, 57], to name a few. It was then expanded to second order (double-integrator) consensus problem as shown in [18] and higher order consensus in [58, 59]. Some were analysed based on leader-follower configuration [60, 61]. Consensus for complex systems has also been studied as in [24], including the study of consensus problem for the systems with nonlinear terms [22, 62], with delay [63–65] or the linear systems with nonlinear consensus protocols [66]. Although some results on consensus control with nonlinearity have been stated in [67–69], most were restricted to local stability, or to certain connections of the network such as the
existence of a tree structure in the connection, or certain types of nonlinearities.

In relation to Lipschitz nonlinearity system, there were single systems with Lipschitz nonlinearity being reported in literature, namely [50–53, 70, 71] as previously presented in the previous chapter. Based on these results, several publications reported consensus problem of higher-order systems with Lipschitz nonlinearities which highlights the need of this thesis. In Li et.al [69], it was assumed that the left eigenvector corresponding to the eigenvalue of the Laplacian matrix at zero has no zero elements. Ding [55] investigated the global consensus control for a network-connected system in the presence of a Lipschitz nonlinearity with a fairly general assumption on the connection structure. Li.et.al [72] introduced a $H_{\infty}$ consensus protocol for Lipschitz multi-agent systems subjected to external disturbances under strongly connected balanced communication graph. Wen et.al [73] solved the consensus tracking problem without the assumption that the topology among followers is strongly connected and fixed.

The aim of this chapter is to design a consensus controller that will solve the consensus problem in multi-agent systems with Lipschitz nonlinearity. Two controllers - state-feedback and observer-based consensus controllers and a consensus observer are proposed which utilize the relative state and relative output information of each agent. The observer-based consensus controller is based on the consensus observer where it is based on Luenberger observer. Simulation examples will be provided to show the effectiveness of the controller designed.

Section 3.2 describes the dynamics of the multi-agent systems with Lipschitz nonlinearity and linked it the connection topology of the agents of subsystems in the multi-agent network.

Next, Section 3.3 includes the design of a controller using state-feedback of relative state information of neighbouring agents other than the leader. In Section 3.3.1 the stability condition of the controller is given in Theorem 3.3.1 with bounds obtained in Lemma 3.3.1.
CHAPTER 3. CONSENSUS CONTROL WITH LIPSCHITZ NONLINEARITY

Section 3.4 deals with the design of a consensus observer with output feedback. Based on Lyapunov stability analysis, several conditions are given by Theorem 3.4.1, with the bound given by Lemma 3.4.1.

Then, in Section 3.5, the controller in Section 3.3 is combined with the observer in Section 3.4. Several conditions of stability are given in Theorem 3.5.1, with the same bounds in Lemma 3.3.1 and Lemma 3.4.1.

All simulations are performed with respect to the outcome of the system, with the respective controller and observer each in its own section; this is to prove that all controllers and observers are working as they were designed to do, with output plots being presented. From this, several key points are raised and analysed in the discussion within the section. Finally, Section 3.6 concludes the findings of all sections above.

3.2 Problem Statement

Consider $N + 1$ nonlinear subsystems with identical dynamics, described as

$$
\dot{x}_i = Ax_i + \phi(x_i) + Bu_i \tag{3.1}
$$

$$
y_i = Cx_i \tag{3.2}
$$

where for $i = 0, \ldots, N$, $x_i \in \mathbb{R}^n$ is the state vector of the subsystem, $u_i \in \mathbb{R}^p$ is the input of the $i$th subsystem, and $y_i \in \mathbb{R}^q$ is the measured output vector, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{q \times n}$ are appropriate matrices, and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz nonlinear function, with the Lipschitz constant $\gamma$ as in (2.31, pg.44). Without loss of generality, a subsystem indexed by 0 is assigned as the leader with $u_0 = 0$, and the subsystems indexed by $i = 1, \ldots, N$ are referred to as followers.

As a revision, the connections between the subsystems are specified by a directed graph $\mathcal{G}$, which consists of a set of vertices denoted by $\mathcal{V}$, and a set of edges denoted by $\mathcal{E}$; each vertex represents a subsystem, and each edge represents a connection.
The adjacency matrix $Q$ associated with the graph $G$ is defined by elements $q_{ij} = 1$, otherwise $q_{ij} = 0$ when there is no connection. The Laplacian matrix $\mathcal{L} = \{l_{ij}\}$ is commonly defined as in (2.1, pg.31).

A few assumptions are needed

**Assumption 3.2.1** Triple $(A, B)$ is controllable.

**Assumption 3.2.2** A leader subsystem is fixed.

**Assumption 3.2.3** The communication network $G$ of the multi-agent systems contains a directed spanning tree with the leader as the root.

The leader has no neighbours, and the Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N \times N}$ as in (2.1) can be partitioned as

$$\mathcal{L} = \begin{bmatrix} 0 & 0_{1 \times N} \\ l_1 & \overline{\mathcal{L}} \end{bmatrix}$$

where $\overline{\mathcal{L}} \in \mathbb{R}^{N \times N}$ and $l_1 = [l_{10}, \ldots, l_{N0}]^T \in \mathbb{R}^{N \times 1}$. $\overline{\mathcal{L}}$ can be known as the Laplacian matrix of the followers.

**Lemma 3.2.1** (74) If the assumption 3.2.3 holds, then all eigenvalues of $\overline{\mathcal{L}}$ have positive real parts.

The main objective is to design the leader-follower consensus controller using measured output information of the neighbouring agents, such that the states of the follower subsystems in a group asymptotically track the state of the leader subsystem.

### 3.3 State-feedback Controller

The controller proposed using the relative state information takes the structure

$$u_i = K \sum_{j=0}^{N} l_{ij}x_j = K \sum_{j=1}^{N} l_{ij}(x_j - x_0)$$

where $K \in \mathbb{R}^{p \times n}$ is a constant control gain matrix to be designed later; $l_{i0} = -\sum_{j=1}^{N} l_{ij}$ is used in the calculation. This controller is based on the state-feedback controller used in [55]. The leader-follower consensus problem is said to be solved if

$$x_i(t) \rightarrow x_0(t), \ \forall i = 1, \ldots, N \text{ as } t \rightarrow \infty$$
CHAPTER 3. CONSENSUS CONTROL WITH LIPSCHITZ NONLINEARITY

We first introduce the disagreement between the leader and the follower subsystem’s state, denoted by

\[ \xi_i = x_i - x_0 \]  \hspace{1cm} (3.6)

With \( \xi = [\xi_1^T, \ldots, \xi_N^T]^T \), \( \Phi(x) = [\phi(x_1)^T, \ldots, \phi(x_N)^T]^T \) and \( \Phi(x_0) = 1 \otimes \phi(x_0) \), the compact form of the closed-loop dynamics of \( \xi \) is given by

\[ \dot{\xi} = (I_N \otimes A - \bar{\mathcal{L}} \otimes BK)\xi + \Phi(x) - \Phi(x_0) \]  \hspace{1cm} (3.7)

where \( \bar{\mathcal{L}} \) is defined in (3.3).

Let us introduce non-singular matrices \( T \in \mathbb{R}^{N \times N} \) and \( T^{-1} \in \mathbb{R}^{N \times N} \), such that

\[ T^{-1}\bar{\mathcal{L}}T = \bar{J} \]  \hspace{1cm} (3.8)

with \( \bar{J} \) being a block-diagonal matrix of real Jordan form

\[ \bar{J} = \begin{bmatrix} \bar{J}_1 \\ \bar{J}_2 \\ \vdots \\ \bar{J}_p \\ \bar{J}_{p+1} \\ \vdots \\ \bar{J}_q \end{bmatrix} \]  \hspace{1cm} (3.9)

where \( \bar{J}_k \in \mathbb{R}^{n_k} \) for \( k = 1, \ldots, p \) are the Jordan blocks for real eigenvalues \( \bar{\lambda}_k > 0 \) with the multiplicity \( n_k \) in the form

\[ \bar{J}_k = \begin{bmatrix} \bar{\lambda}_k & 1 \\ \bar{\lambda}_k & 1 \\ \vdots & \vdots \\ \bar{\lambda}_k & 1 \\ \bar{\lambda}_k \end{bmatrix} \]

and \( \bar{J}_k \in \mathbb{R}^{2n_k} \) for \( k = p + 1, \ldots, q \) are the Jordan blocks for conjugate eigenvalues \( \bar{\alpha}_k \pm j\bar{\beta}_k \), \( \bar{\alpha}_k > 0 \) and \( \bar{\beta}_k > 0 \) with multiplicity \( n_k \) in the form

\[ \bar{J}_k = \begin{bmatrix} \mu(\bar{\alpha}_k, \bar{\beta}_k) & I_2 \\ \mu(\bar{\alpha}_k, \bar{\beta}_k) & I_2 \\ \vdots & \vdots \\ \mu(\bar{\alpha}_k, \bar{\beta}_k) & I_2 \\ \mu(\bar{\alpha}_k, \bar{\beta}_k) \end{bmatrix} \]
with $I_2$ the identity matrix in $\mathbb{R}^{2 \times 2}$ and
\[
\mu(\bar{\alpha}_k, \bar{\beta}_k) = \begin{bmatrix} \bar{\alpha}_k & \bar{\beta}_k \\ -\bar{\beta}_k & \bar{\alpha}_k \end{bmatrix} \in \mathbb{R}^{2 \times 2} \quad (3.10)
\]

To exploit the structure of $\mathbf{L}$, two other transformations are introduced
\[
\eta = (T^{-1} \otimes I_n) \xi \quad (3.11)
\]
\[
\Psi(x, x_0) = (T^{-1} \otimes I_n)(\Phi(x) - \Phi(x_0)) \quad (3.12)
\]
Thus, the transformed closed-loop network dynamics is expressed as
\[
\dot{\eta} = (I_N \otimes A - \bar{J} \otimes BK)\eta + \Psi(x, x_0) \quad (3.13)
\]
where $\eta = [\eta_1^T, \ldots, \eta_N^T]^T$, $\Psi(x, x_0) = [\psi_1(x, x_0)^T, \ldots, \psi_N(x, x_0)^T]^T$ and $\bar{J}$ takes the Jordan matrix form of (4.13).

**Remark 3.3.1** The approach to obtain (3.11) is similar to the transformed dynamics of (2.36) as shown in Ding [55]. Hence the stability analysis would be obtained if $\Psi(x, x_0)$ is a Lipschitz function.

### 3.3.1 Stability Analysis

For the stability analysis, we need to establish a bound for the nonlinear function, in terms of the transformed $\eta$.

**Lemma 3.3.1** For a nonlinear element $\psi_i(x)$ of the nonlinear term $\Psi$ in the transformed closed-loop network dynamics (3.13), a bound can be established in terms of the state $\eta$, as shown by
\[
\|\psi_i(x, x_0)\| \leq \frac{\gamma_0}{\sqrt{N}} \|\eta\| \quad (3.14)
\]
with
\[
\gamma_0 = \gamma \lambda_\sigma(T^{-1})\lambda_\sigma(T)\sqrt{N} \quad (3.15)
\]
where $\lambda_\sigma(\cdot)$ denotes the maximum singular value of a matrix.

**Proof 3.3.1** The approach for the derivation of (3.14) and (3.15) is similar to Lemma 3.2 in [55]. From (3.12) we have
\[
\|\psi_i(x, x_0)\| \leq \|t_i \otimes I_n\|\|\Phi(x) - \Phi(x_0)\| \leq \lambda_\sigma(T^{-1})\gamma\|x - x_0\| = \lambda_\sigma(T^{-1})\gamma\|\xi\|
\]
where \( t_i \) denotes the \( i \)th row of \( T^{-1} \), and from (3.11) we have

\[
\| \xi \| \leq \| T \otimes I_n \| \| \eta \| \\
\leq \lambda_\sigma(T) \| \eta \|
\]

Then, we have

\[
\| \psi_i(x, x_0) \| \leq \gamma \lambda_\sigma(T^{-1}) \lambda_\sigma(T) \frac{\sqrt{N}}{\sqrt{N}} \| \eta \| \\
= \frac{\gamma_0 \sqrt{N}}{\sqrt{N}} \| \eta \|
\]

with \( \gamma_0 \) as in (3.15).

Throughout the thesis, \( \| \cdot \| \) denotes the Euclidean norm for vectors \( x \in \mathbb{R}^n \), defined by \( \| x \| = \sqrt{x^T x} \), and the induced norm corresponding to the vector Euclidean norm for matrices \( A \in \mathbb{R}^{m \times n} \), defined by \( \| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|} \). With the induced norm, the inequality \( \| \psi_i(x, x_0) \| \leq \| t_i \otimes I_n \| \| \Phi(x) - \Phi(x_0) \| \) holds. The bound derived in the Lemma 3.3.1 will be used for the control gain design in the following theorem.

**Theorem 3.3.1** For a network-connected nonlinear system (2.35) with the communication topology \( G \), satisfying Assumption 3.2.1 - 3.2.3, the consensus control design (3.4) with \( K = B^T P \) solves the leader-follower consensus problem if there exists a solution of \( P = P^T > 0 \) specified in one of the following two cases:

1. If the eigenvalues of the matrix \( \mathcal{L} \) are distinct, i.e., \( n_k = 1 \) for \( k = 1, \ldots, q \), the matrix \( P \) satisfies

\[
A^T P + PA - 2\bar{\alpha}PBB^TP + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n < 0 \tag{3.16}
\]

with \( \kappa \) being any positive real number and \( \bar{\alpha} = \min\{\bar{\lambda}_1, \ldots, \bar{\lambda}_p, \bar{\alpha}_{p+1}, \ldots, \bar{\alpha}_q\} \).

2. If the matrix \( \mathcal{L} \) has multiple eigenvalues, i.e., \( n_k > 1 \) for any \( k \in \{1, \ldots, q\} \), the matrix \( P \) satisfies

\[
A^T P + PA - 2(\bar{\alpha} - 1)PBB^TP + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n < 0 \tag{3.17}
\]

with \( \kappa \) being any positive real number.
Proof 3.3.2 Within each real Jordan block $\bar{J}_k$, for $k \leq p$ we have, for $i = N_{k-1} + 1, \ldots, N_k - 1$,

$$\dot{\eta}_i = (A - \bar{\lambda}_i BK)\eta_i - BK\eta_{i+1} + \psi_i(x)$$ (3.18)

and

$$\dot{\eta}_i = (A - \bar{\lambda}_i BK)\eta_i + \psi_i(x)$$

for $i = N_k$.

For the state variable associated with the Jordan blocks $J_k$ for $k > p$ corresponding to complex eigenvalues, we consider the dynamics of the state variables in pairs. For notational convenience, let

\[ i_1(j) = N_{k-1} + 2j - 1 \]
\[ i_2(j) = N_{k-1} + 2j \]

for $j = 1, \ldots, n_k/2$. The dynamics of $\eta_{i_1}$ and $\eta_{i_2}$ for $j = 1, \ldots, n_k/2 - 1$ are expressed by

$$\dot{\eta}_{i_1} = (A - \bar{\alpha}_k BK)\eta_{i_1} - \bar{\beta}_k BK\eta_{i_2} - BK\eta_{i_1+2} + \psi_{i_1}(x)$$

$$\dot{\eta}_{i_2} = (A - \bar{\alpha}_k BK)\eta_{i_2} + \bar{\beta}_k BK\eta_{i_1} - BK\eta_{i_2+2} + \psi_{i_2}(x)$$

and

$$\dot{\eta}_{i_1} = (A - \bar{\alpha}_k BK)\eta_{i_1} - \bar{\beta}_k BK\eta_{i_2} + \psi_{i_1}(x)$$

$$\dot{\eta}_{i_2} = (A - \bar{\alpha}_k BK)\eta_{i_2} + \bar{\beta}_k BK\eta_{i_1} + \psi_{i_2}(x)$$

for $j = n_k/2$.

Let $W_i = \eta_i^T P \eta_i$. Choose $V_k = \sum_{j=1}^{n_k} \sigma^{2(j-1)}W_j + N_{k-1}$ for $k = 1, \ldots, p$ and $V_k = \sum_{j=1}^{n_k/2} \sigma^{2(j-1)}(W_{i_1(j)} + W_{i_2(j)})$ for $k = p + 1, \ldots, q$, where $\sigma > 0$. We then consider the Lyapunov function $V = \sum_{i=1}^q V_k$; with $K = B^T P$, we have the following results.

Case 1. For distinct eigenvalues, we can obtain

$$\dot{V} \leq \sum_{i=1}^N \eta_i^T \left( A^T P + PA - 2\bar{\alpha}PBB^T P + \kappa PP + \frac{\gamma^2}{\kappa} I_n \right) \eta_i$$ (3.19)

The condition (3.16) guarantees $\dot{V} < 0$. 
Case 2. For multiple eigenvalues, we can obtain

$$\dot{V} \leq \sum_{i=1}^{N} \eta_{i}^{T} \left[ A^{T} P + PA - 2 \left( \bar{\alpha} - \frac{1}{\sigma} \right) PBB^{T} P + \kappa PP + \frac{\gamma_{0}^{2}}{\kappa} I_{n} \right] \eta_{i}.$$ \hspace{1cm} (3.20)

with $\sigma = 1$. The condition (3.17) guarantees $\dot{V} < 0$.

Hence, we can conclude that $\eta_{i}(t) \to 0, \forall i = 1, \ldots, N$ as $t \to \infty$. This completes the proof.

A few steps of the derivations of the proof have been omitted to space. Readers should refer to [55] for more details.

Remark 3.3.2 The solution of the state-feedback consensus controller in this section is simpler compared to [55]. Note that the state $x_{j} - x_{0}$ from controller (3.4) is used in the design of an observer-based controller which will be utilized for the observer based controller later.

3.3.2 Solution of Positive Definite $P$ with Linear Matrix Inequality (LMI)

The solution of $P$ can be obtained by the normal LMI (Linear Matrix Inequality) procedure [55]. The inequalities (3.16) and (3.17) can be converted into LMI using Schur Complements. Hence, we need the following lemma.

Lemma 3.3.2 The basic idea is as follows [75]: The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^{T} & R(x) \end{bmatrix} > 0$$ \hspace{1cm} (3.21)

where $Q(x) = Q(x)^{T}$, $R(x) = R(x)^{T}$, and $S(x)$ depends affinely on $x$; it is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^{T} > 0$$ \hspace{1cm} (3.22)
The set of nonlinear inequalities (3.16) and (3.17) can be represented as the LMI (3.21). Hence, when $Q = P^{-1}$ is multiplied to both sides of $P$ for Case 1 (3.16) and Case 2 (3.17), it is easy to obtain

$$
\begin{bmatrix}
QA^T + AQ - 2\beta BB^T & Q^T \\
Q & -\frac{\kappa}{\gamma^2} I_n 
\end{bmatrix} < 0
$$

with

$$
\beta = 2\bar{\alpha}
$$

for Case 1 and for Case 2

$$
\beta = 2 \left( \bar{\alpha} - \frac{1}{\sigma} \right)
$$

The solution of (3.23) can be obtained by applying the LMI instructions from the Matlab Control Toolbox.

### 3.3.3 Solution of Positive Definite $P$ with Algebraic Riccati Equation (ARE)

Alternatively, the solution of $P = P^T > 0$ can be obtained with ARE from the following lemma.

**Lemma 3.3.3** The basic idea is as follows [51]: To any ARE

$$
A^T P + PA + PRP + Q = 0 \quad (3.24)
$$

Hamiltonian matrix

$$
H = \begin{bmatrix}
A & R \\
-Q & -A^T
\end{bmatrix} \quad (3.25)
$$

is associated. A solution of $P = P^T > 0$ is available if $H$ has no eigenvalues which exist on the imaginary axis, if the pair $(A, R)$ is stabilizable, and if $R$ is sign-definite (i.e. semi-definite positive or semi-definite negative).

Hence from (3.16) and (3.17), by having $\delta_0 > 0$ we have

$$
A^T P + PA + P(kI_n - \beta BB^T)P + \frac{\gamma^2}{\kappa} I_n < 0
$$

$$
A^T P + PA + P(kI_n - \beta BB^T)P + \frac{\gamma^2}{\kappa} I_n = -\delta_0 I_n
$$
we can obtain
\[ A^T P + PA + P(kI_n - \beta BB^T)P + \left( \frac{\gamma^2}{\kappa} + \delta_0 \right) I_n = 0 \]  
(3.26)
and it is easy to obtain
\[ H_1 = \begin{bmatrix} A & kI_n - \beta BB^T \\ -\left( \frac{2\alpha}{\kappa} + \delta_0 \right) I_n & -A^T \end{bmatrix} \]  
(3.27)
If the eigenvalue of \( H_1 \) has no eigenvalues on the imaginary axis, if the pair \( (A,kI_n - \beta BB^T) \) is stabilizable, and if \( (kI_n - \beta BB^T) \geq 0 \) and \( \left( \frac{2\alpha}{\kappa} + \delta_0 \right) I_n > 0 \), it is possible to reach a solution of \( P = P^T > 0 \) for Case 1
\[ \beta = 2\bar{\alpha} \]
and for Case 2
\[ \beta = 2 \left( \bar{\alpha} - \frac{1}{\sigma} \right) \]
Hence the solution of \( P = P^T > 0 \) can be obtained by using the function 'care' from Matlab Control Toolbox.

Remark 3.3.3 \(-\delta_0 I_n < 0\) used is specifically for our case.

3.3.4 Simulation and Discussion

In this section, two examples are given to show some details of the state-feedback control scheme which has been designed. The first example will be based on the solution of \( P = P^T > 0 \) found using LMI, while the second example is from the solution of ARE. The main reason why this is done is due to some instances where the solution of \( P \) is obtained from LMI but consensus cannot be reached, and therefore requires a \( P \) solution from another procedure such as ARE.

Example 1

The system under consideration is a connection of four subsystems with a leader, where each of these is described by a second-order state-space model as
\[ \dot{x}_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 0 \\ \sin(Cx_i) \end{bmatrix} u_i \]  
(3.28)
and
\[ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i \]  \hspace{1cm} (3.29)

**Remark 3.3.4** System (3.28) is chosen due to its eigenvalues that are imaginary (no real part) in order to prove that the consensus is reached.

The connection of each subsystem can be described by the Figure 3.1.

![Directed Spanning Tree connection with Leader Follower Configuration](image)

Figure 3.1: Directed Spanning Tree connection with Leader Follower Configuration for Leader Subsystem 0, and Follower Subsystems 1, 2, 3, 4.

The state-feedback consensus controller (3.2) is applied and shown in Figure 2.9. The adjacency matrix \( Q \) for the connection of the followers is given by

\[
Q = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix} \hspace{1cm} (3.30)
\]

and the corresponding Laplacian matrix is obtained as

\[
\mathcal{L} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 \\
0 & -1 & 0 & 0 & 1
\end{bmatrix} \hspace{1cm} (3.31)
\]
where

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of $\mathbf{L}$ are given as \{0.2451, 1, 1.8774 \pm 0.7449j\} which are distinct.

The nonlinear function $0.05 \sin(C\mathbf{x}_i)$ in the dynamic system is globally Lipschitz. The values of the follower substates are set as $x_1 = [0.1, -0.1]^T$, $x_2 = [0.2, -0.2]^T$, $x_3 = [0.3, -0.3]^T$, $x_4 = [0.4; -0.4]$, and the leader as $x_0 = [0.5, -0.5]^T$. Based on matrix $\mathbf{L}$,
we have $\alpha = 0.2451$. From (3.28), we have

$$\phi(x_i) = \begin{bmatrix} \sin(Cx_i) \\ 0 \end{bmatrix}$$ (3.34)

From the LMI procedure, $P = P^T > 0$ is obtained as

$$P = \begin{bmatrix} 0.6443 & -0.0474 \\ -0.0474 & 0.6318 \end{bmatrix}$$ (3.35)

Similarly, the controller matrix $K$ is obtained as

$$K = \begin{bmatrix} -0.0474 & 0.6318 \end{bmatrix}$$ (3.36)

**Remark 3.3.5** In leader-follower configuration, the consensus controller (3.2) is not connected to the leader. However, the information of the leader is transferred to the followers via the Laplacian structure of the system.

**Figure 3.3:** Plot of leader follower configuration states 1 with the state-feedback consensus controller (3.4) for $\gamma = 0.05$.

It is obvious that, based on the structure of the Laplacian matrix $\mathcal{L}$, the eigenvalues contain no zero eigenvalues, due to the leader-follower connection topology. When the
Figure 3.4: Plot of leader-follower configuration states 2 with state-feedback consensus controller (3.4) for $\gamma = 0.05$.

Figure 3.5: Plot of leader-follower configuration states 1 with state-feedback consensus controller (3.4) for $\gamma = 0.7$. 
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Figure 3.6: Plot of leader-follower configuration states 2 with state-feedback consensus controller (3.4) for $\gamma = 0.7$.

Figure 3.7: Plot of leader-follower configuration states 1 with state-feedback consensus controller (3.4) for $\gamma = 10$. 
Simulation was performed, it can be seen that Figure 3.3 and Figure 3.4 could obtain consensus, where all subsystems were able to follow the leader with the controller (3.4) for $\gamma = 0.05$. With larger nonlinearity at $\gamma = 0.7$, the number of states increased due to the nonlinearity; however, consensus was reached more easily, as shown in Figure 3.5 and Figure 3.6. The states change trajectory when the $\gamma$ was increased, as shown in Figure 3.7 and Figure 3.8 with $\gamma = 10$. Nevertheless, consensus is reached in all cases.
Example 2

Note that with ARE, by selecting $\delta_0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$ for $n = 2$, and we can obtain the Hamiltonian matrix

$$H = \begin{bmatrix}
0 & 1.0000 & 1.0000 & 0 \\
-1.0000 & 0 & 0 & 1.0000 \\
0.5099 & 0 & 0 & 1.0000 \\
0 & 1.0001 & -1.0000 & 0
\end{bmatrix} \quad (3.37)$$

where there are no eigenvalues of $H$ that are on the imaginary axis, the corresponding controller gain $K$ and $P$ can be obtained as

$$K = \begin{bmatrix} -0.4934 & 5.0207 \end{bmatrix} \quad (3.38)$$

and

$$P = \begin{bmatrix} 5.2165 & -0.4934 \\
-0.4934 & 5.0207 \end{bmatrix} \quad (3.39)$$

where the plots of each subsystem are identical to the plots shown above, with consensus achieved by each subsystem.

The system used is general, therefore the nonlinear system included, in terms of Lipschitz stability and consensus reached, is also general. Since a directed topology with distinct eigenvalues of the Laplacian matrix $L$ is used, it should be stated that the condition achieved is somewhat restricted. For future work, it is recommended to explore the undirected topology, which is more complex.

### 3.3.5 Conclusion

In this section, a state feedback consensus controller has been proposed for a multi-agent systems with Lipschitz nonlinearity under directed spanning tree connection topology. It was observed that the follower subsystems were able to follow the leader subsystem with the state feedback controller (3.4). The conditions for asymptotic stability were also given in (4.29), for distinct eigenvalues and (4.30) for multiple
Figure 3.9: Plot of leader-follower configuration states 1 with state-feedback consensus controller (3.4) with ARE solution of $P$ for $\gamma = 0.05$.

Figure 3.10: Plot of leader-follower configuration states 2 with state-feedback consensus controller (3.4) with ARE solution of $P$ for $\gamma = 0.05$. 
eigenvalues based on Lyapunov stability analysis. The solutions for $P = P^T > 0$ was obtained from Linear Matrix Inequality (LMI) and Algebraic Riccati Equation (ARE) routines. From simulation examples, both solutions managed to give the same outcome, where for both cases, consensus was reached by each agent. These validated the theoretical results. These examples were only for for distinct eigenvalues. The results obtained is similar to Ding [55] albeit with slight modification due to the leader-follower configuration. However, it is useful for later section where an observer-based consensus controller is designed. A certain measure of nonlinearity was tested by changing the nonlinear constant value $\gamma$ where for all cases, consensus were reached.

### 3.4 Consensus Observer

An observer is useful in a system where it is difficult or impossible to obtain the exact values of the states. For a single system, Luenberger observer [32] is widely used - this compares the estimated and the exact value of all states. If the gain of the observer is properly selected, the states can be estimated quickly, and can thus produce the output signal required, which is used to estimate the values of the states. Therefore, the consensus observer in this report is essentially based on Luenberger observer. The objective of this section is to design a consensus observer that enables the estimated substates to track the actual substates - this will then be used in the observer-based consensus controller in the next section.

The following assumption is fundamental to this section:

**Assumption 3.4.1** The pair $(A, C)$ is observable.

The consensus observer for each subsystem is proposed as

$$\dot{x}_i = A\hat{x}_i + Bu_i + \phi(\hat{x}_i) + L \sum_{j=1}^{N} l_{ij} (y_j - C\hat{x}_j) \quad (3.40)$$

where $\hat{x}_i \in \mathbb{R}^n$ for $i = 1, \ldots, N$ is the estimated state of each subsystem (follower) and $L \in \mathbb{R}^{n \times q}$ is the observer gain matrix to be determined later.

The estimation error dynamics can be then derived as

$$\dot{\hat{x}}_i = A\hat{x}_i + \phi(x_i) - \phi(\hat{x}_i) - LC \sum_{j=1}^{N} l_{ij}\hat{x}_j \quad (3.41)$$
where $\tilde{x}_i = x_i - \hat{x}_i$. With $\tilde{x} = [\tilde{x}_1^T, \ldots, \tilde{x}_N^T]^T$, $\Phi(x) = [\phi(x_1)^T, \ldots, \phi(x_N)^T]^T$ and $\Phi(\hat{x}) = [\phi(\hat{x}_1)^T, \ldots, \phi(\hat{x}_N)^T]^T$, the estimation error dynamics (3.41) for each subsystem can be stacked into a compact form as

$$\dot{\tilde{x}} = (I_N \otimes A - \overline{L} \otimes LC)\tilde{x} + \Phi(x) - \Phi(\hat{x}) \quad (3.42)$$

where $\overline{L}$ is defined in (3.3). Likewise, we introduce the transformation

$$\tilde{\eta} = (T^{-1} \otimes I_n)\tilde{x} \quad (3.43)$$

with $T^{-1}\overline{L}T = \overline{J}$ defined in (3.9). Hence, the transformed dynamics of the consensus observer are given by

$$\dot{\tilde{\eta}} = (I_N \otimes A - \overline{J} \otimes LC)\tilde{\eta} + \overline{\Psi}(x, \hat{x}) \quad (3.45)$$

where $\tilde{\eta} = [\tilde{\eta}_1^T, \ldots, \tilde{\eta}_N^T]^T$, $\overline{\Psi}(x, \hat{x}) = [\overline{\psi}_1(x, \hat{x})^T, \ldots, \overline{\psi}_N(x, \hat{x})^T]^T$.

**Remark 3.4.1** In a similar way to the previous section, the estimation error dynamics (3.42) are now turned into a diagonally-dominant matrix, which facilitates the stability analysis.

**Remark 3.4.2** Unlike the traditional Luenberger observer, the proposed consensus observer in our paper only uses relative output information and estimated output information from neighbours over network connections, instead of absolute output and estimated output information. To be specific, the consensus observer (3.40) can be rewritten as

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + \phi(\hat{x}_i) + L \sum_{j=1}^N l_{ij}(y_j - C\hat{x}_j)$$

$$= A\hat{x}_i + Bu_i + \phi(\hat{x}_i) + L \sum_{j=1}^N l_{ij}y_j - L \sum_{j=1}^N l_{ij}C\hat{x}_j$$

$$= A\hat{x}_i + Bu_i + \phi(\hat{x}_i) + L \sum_{j=1}^N q_{ij}(y_j - y_i) - L \sum_{j=1}^N q_{ij}(C\hat{x}_j - C\hat{x}_i)$$

where we have used (2.1, pg.31), i.e., the definition of Laplacian matrix.
Remark 3.4.3 The performance of a single Luenberber observer and the consensus observer is similar since the stability of the consensus observer is represented by $A - \lambda JC$ where $\lambda$ is the eigenvalues of the Laplacian matrix $L$ and of the single observer represented by $A - JC$ to have negative real values. This can be shown by the Figure 3.11 where error output of the single system with single Luenberger’s observer represented by $e_1$ is compared to the error output of the multi-agent system with consensus observer.

Figure 3.11: Plot of the comparison between single observer error represented by $e_1$ and consensus observer error represented by $e_2 - e_5$.

3.4.1 Stability Analysis

Similar to the previous section, the bound of $\psi_i(x, \hat{x})$ in terms of $\eta$ is needed.

Lemma 3.4.1 For a nonlinear element $\psi_i$ of the nonlinear term $\overline{V}$ in the transformed closed-loop network dynamics (3.45), a bound can be established in terms of $\eta$, as shown by

$$\|\psi_i(x, \hat{x})\| \leq \frac{\gamma_0}{\sqrt{N}} \|\eta\|$$

(3.46)
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with

\[ \gamma_0 = \gamma \lambda_\sigma(T^{-1}) \lambda_\sigma(T) \sqrt{N} \]  \hspace{1cm} (3.47)

where \( \lambda_\sigma(\cdot) \) denotes the maximum singular value of a matrix.

**Proof 3.4.1** From (3.44), we have

\[ \| \bar{\psi}_i(x, \hat{x}) \| \leq \| t_i \otimes I_n \| \| \Phi(x) - \Phi(\hat{x}) \| \]  \hspace{1cm} (3.48)

where \( t_i \) is the \( i \)th row of \( T^{-1} \). And from (3.43), we have

\[ \| \tilde{x} \| \leq \| T \otimes I_n \| \| \tilde{\eta} \| \]  \hspace{1cm} (3.49)

Hence, similar to the proof in Lemma 3.3.1, the following can be obtained

\[ \| \bar{\psi}_i(x, \hat{x}) \| \leq \lambda_\sigma(T^{-1}) \gamma \| \tilde{x} \| \]

\[ \leq \lambda_\sigma(T^{-1}) \gamma \lambda_\sigma(T) \| \tilde{\eta} \| \]

\[ = \frac{\gamma_0}{\sqrt{N}} \| \tilde{\eta} \| \]

with \( \gamma_0 \) as in (3.47).

The bound obtained in the lemma above will be used for the consensus observer gain design in the following theorem.

**Theorem 3.4.1** For a network-connected dynamic system (2.35) with communication topology \( G \) satisfying Assumption 3.2.2 - Assumption 3.2.3 and Assumption 3.4.1, (3.40) with \( L = P^{-1}C^T \) is an asymptotically-stable observer for the system (2.35), if there exists a solution \( P = P^T > 0 \) specified in one of the following two cases:

1. If the eigenvalues of the matrix \( \overline{L} \) are distinct, i.e., \( n_k = 1 \) for \( k = 1, \ldots, q \) the matrix \( P = P^T > 0 \) satisfies

\[ A^T P + PA - 2\bar{\alpha}C^T C + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n < 0 \]  \hspace{1cm} (3.50)

with \( \kappa \) being any positive real number and \( \bar{\alpha} = \min\{\bar{\lambda}_1, \ldots, \bar{\lambda}_p, \bar{\alpha}_{p+1}, \ldots, \bar{\alpha}_q\} \).

2. If the matrix \( \overline{L} \) have multiple eigenvalues, i.e., \( n_k > 1 \) for any \( k \in \{1, \ldots, q\} \), the matrix \( P = P^T > 0 \) satisfies

\[ A^T P + PA - 2(\bar{\alpha} - 1)C^T C + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n < 0 \]  \hspace{1cm} (3.51)

with \( \kappa \) being any positive real number.
Proof 3.4.2  Similar to the proof of Theorem 3.4.1, for $k \leq p$ we have

$$\dot{\tilde{\eta}}_i = (A - \tilde{\lambda}_i LC)\tilde{\eta}_i - LC\tilde{\eta}_{i+1} + \tilde{\psi}_i(x) \quad (3.52)$$

with $i = N_{k-1} + 1, \ldots, N_k - 1$ and

$$\dot{\tilde{\eta}}_i = (A - \tilde{\lambda}_i LC)\tilde{\eta}_i + \tilde{\psi}_i(x)$$

for $i = N_k$.

For $k > p$, we have

$$\dot{\tilde{\eta}}_i = (A - \alpha_k LC)\tilde{\eta}_i - \beta_k LC\tilde{\eta}_{i+2} + \tilde{\psi}_i(x)$$

with $1, \ldots, n_k/2 - 1$ and

$$\dot{\tilde{\eta}}_i = (A - \alpha_k LC)\tilde{\eta}_i + \tilde{\psi}_i(x)$$

for $j = n_k/2$.

Let $W_i = \tilde{\eta}_i^T P \tilde{\eta}_i$. Choose $\bar{V}_k = \sum_{j=1}^{n_k} \sigma^{2(j-1)} W_{j+N_k-1}$ for $k = 1, \ldots, p$ and $\bar{V}_k = \sum_{j=1}^{n_k/2} \sigma^{2(j-1)} (W_{i_1(j)} + W_{i_2(j)})$ for $k = p + 1, \ldots, q$, where $\sigma > 0$. Then we consider the Lyapunov function $\bar{V} = \sum_{i=1}^q \bar{V}_k$. With $\sigma = 1$ and $L = P^{-1}C^T$, we have the following results.

Case 1. For the distinct eigenvalues, we can obtain

$$\dot{\bar{V}} \leq \sum_{i=1}^N \tilde{\eta}_i^T \left( A^T P + PA - 2\alpha C^T C + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n \right) \tilde{\eta}_i$$

Case 2. For multiple eigenvalues, we can obtain

$$\dot{\bar{V}} \leq \sum_{i=1}^N \tilde{\eta}_i^T \left[ A^T P + PA - 2(\alpha - 1) C^T C + \kappa PP + \frac{\gamma_0^2}{\kappa} I_n \right] \tilde{\eta}_i$$

Therefore, conditions (3.50) and (3.51) guarantee that $\dot{\bar{V}} < 0$ for both cases. Hence we conclude that $\tilde{\eta}_i(t) \to 0$, $\forall i = 1, \ldots, N$ as $t \to \infty$. This completes the proof.
3.4.2 Solution of Positive Definite $P$ from Linear Matrix Inequality (LMI)

The solution of $P$ can be obtained by the normal Linear Matrix Inequality (LMI) procedure [55]. Hence, according to Lemma 3.3.2 and by having $Q = P^{-1}$ multiplied at both sides of $P$ for Case 1 (3.19) and Case 2 (3.20) it is easy to obtain

\[
\begin{bmatrix}
A^T P + PA - \beta C^T C - \frac{\gamma_0^2}{\kappa} I_n & P \\
PT & \frac{1}{\kappa} I_n
\end{bmatrix} < 0
\]

(3.53)

\[
P = P^T > 0
\]

with

\[
\beta = 2\bar{\alpha}
\]

for both Case 1 and Case 2

\[
\beta = 2\left(\bar{\alpha} - \frac{1}{\sigma}\right)
\]

The solution of (3.53) can be obtained by applying the LMI instructions from the Matlab Control Toolbox.

3.4.3 Solution of Positive definite $P$ with Algebraic Riccati Equation (ARE)

Using the standard LMI procedure, it is difficult to get a solution for $P = P^T > 0$ that enables the estimated substates to track the substates. Similar to the state-feedback consensus controller section (pg.52), a solution with ARE is considered to be one of the alternative solutions for $P = P^T > 0$.

Hence from Lemma 3.3.3, (3.50) and (3.51), by having $\delta_0 > 0$ we have

\[
A^T P + PA + P(kI_n)P - \beta C^T C + \frac{\gamma_0^2}{\kappa} I_n < 0
\]

\[
A^T P + PA + P(kI_n)P - \beta C^T C + \frac{\gamma_0^2}{\kappa} I_n = -\delta_0
\]

we can obtain

\[
A^T P + PA + P(kI_n)P + \left( -\beta C^T C + \frac{\gamma_0^2}{\kappa} I_n + \delta_0 I_n \right) = 0
\]

(3.54)

and it is easy to obtain

\[
H_2 = \begin{bmatrix}
A & kI_n \\
-\left( -\beta C^T C + \frac{\gamma_0^2}{\kappa} I_n + \delta_0 I_n \right) & -A^T
\end{bmatrix}
\]

(3.55)
Therefore, based on Lemma 3.3.3, if matrix $H_2$ has no eigenvalues on the imaginary axis, if the pair $(A, kI_n)$ is stabilizable, and if $(kI_n) \geq 0$ and $\left( -\beta C^T C + \frac{\gamma^2}{\kappa} I_n + \delta_0 I_n \right) > 0$, it is possible to reach a solution of $P = P^T > 0$ for Case 1 as

$$\beta = 2\bar{\alpha}$$

, and for Case 2 as

$$\beta = 2 \left( \bar{\alpha} - \frac{1}{\sigma} \right)$$

The solution of $P = P^T > 0$ can be obtained by using the function 'care' from the Matlab Control Toolbox.

### 3.4.4 Simulation of the Consensus observer

**Example**

Using the same nonlinear system as in Section 3.3 (pg.52), we have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The control input $u_i = 0$ is used. Therefore, the observer gain matrix $L$ is obtained with LMI from the Matlab Control Toolbox as

$$L = \begin{bmatrix} 15.9464 \\ 9.5680 \end{bmatrix} \quad (3.56)$$

where $P = P^T$ is obtained as

$$P = \begin{bmatrix} 0.1568 & -0.1568 \\ -0.1568 & 0.2613 \end{bmatrix} \quad (3.57)$$

The values of the follower substates are set as $x_1 = [0.1, -0.1]^T$, $x_2 = [0.2, -0.2]^T$, $x_3 = [0.3, -0.3]^T$, and $x_4 = [0.4, -0.4]$. Similar to the previous section, the value of $\gamma$ is selected to be 0.05. However, the leader is not included since the leader subsystem does not give its information to other follower subsystems.

Based on Figure 3.12 and Figure 3.13, the estimated substates of subsystem 1 are able to track the substates denoted by $x_1$ and $x_1h$ for substate 1 and $x_2$ and $x_2h$ for substate 2 for $\gamma = 0.05$. The same applies to all substates of subsystems 2 – 4. Hence
the error signal \( e_1 = \hat{x}_1 \) for subsystem 1 also approaches zero as \( t \to \infty \) as shown in Figure 3.14.

Figure 3.12: Plot of follower subsystem 1 signal and observed signal for substates 1 with consensus observer (3.40) with solution of LMI for \( \gamma = 0.05 \).

Note that when the solution of \( P = P^T > 0 \) is obtained using ARE, with Hamiltonian matrix

\[
H_o = \begin{bmatrix}
0 & 1.0000 & 1.0000 & 0 \\
-1.0000 & 0 & 0 & 1.0000 \\
0.3902 & 0 & 0 & 1.0000 \\
0 & -0.1000 & -1.0000 & 0
\end{bmatrix}
\]  \hspace{1cm} (3.58)

where the eigenvalues of \( H_o \) are not on the imaginary axis, the corresponding observer gain \( L \) and \( P \) are obtained as

\[
L = \begin{bmatrix}
1.8514 \\
0.4976
\end{bmatrix}
\]  \hspace{1cm} (3.59)

and \( P = P^T \) is obtained as

\[
P = \begin{bmatrix}
0.6303 & -0.3355 \\
-0.3355 & 1.2484
\end{bmatrix}
\]  \hspace{1cm} (3.60)
Figure 3.13: Plot of follower subsystem 1 signal and observed signal for substates 2 with consensus observer (3.40) with solution of LMI for $\gamma = 0.05$

Figure 3.14: Plot of follower subsystem 1 error signal with consensus observer (3.40) with solution of LMI $\gamma = 0.05$
With the solution of ARE, the estimated substates are able to track the substates but at a slower rate. Based on Figure 3.15 and Figure 3.16, the estimated substates of subsystem 1 are able to track the substates denoted by $x_1$ and $x_{1h}$ for substate 1 and $x_2$ and $x_{2h}$ for substate 2 for $\gamma = 0.05$. The same applies to all substates of subsystems 2–4. Hence the error signal ($e_1 = \tilde{x}_1$) for subsystem 1 also approaches zero as $t \to \infty$ as shown in Figure 3.17. Hence for this case, LMI provides a better $P = P^T > 0$ solution than ARE.

![Figure 3.15: Plot of a follower subsystem 1 signal and observed signal for substates 1 with consensus observer (3.40) with $P$ solution of ARE for $\gamma = 0.05$.](image)

3.4.5 Conclusion

In this section, a consensus observer has been proposed. Based on the structure of the Laplacian matrix $\mathcal{L}$ in (2.1), the minimum eigenvalue of $\mathcal{L}$ is determined and the observer gain $L$ is selected from the stability analysis based on Lyapunov’s stability theory. The solution of $P = P^T > 0$ for the observer design was provided from Linear Matrix Inequality (LMI) and Algebraic Riccati Equation (ARE) standard routine. ARE was used as the alternative to LMI. Both solutions provided the same outcome. It was observed that the estimated substates were able to track the substates. This
Figure 3.16: Plot of follower subsystem 1 signal and observed signal for substates 2 with consensus observer (3.40) with $P$ solution of ARE for $\gamma = 0.05$.

Figure 3.17: Plot of follower subsystem 1 error signal with consensus observer (3.40) with $P$ solution of ARE for $\gamma = 0.05$. 
observer is going to be used in the controller that is going to be designed in the next section.

3.5 Observer-based Controller

In this section, the state feedback consensus controller design in Section (3.3) is combined with consensus observer design in Section (3.4) to design a leader-follower consensus controller with output feedback as shown in Figure 3.18. However, for nonlinear systems, the separation principle does not generally hold. Sufficient conditions are given in what follows to guarantee the state convergence of the followers to the state of the leader, by output feedback.

The output feedback consensus controller takes the following structure:

\[ u_i = -K(\hat{x}_i - x_0) \]  

where \( \hat{x}_i \in \mathbb{R}^n \) for \( i = 1, \ldots, N \) are the estimated states defined in (3.40) and the consensus controller gain \( K \in \mathbb{R}^{p \times n} \) is to be determined later. The input for the leader is set to be \( u_0 = 0 \).

**Remark 3.5.1** The controller utilizes the estimated information provided by the consensus observer for each of the subsystem. It should be noted that the consensus controller in (3.61) is an output feedback controller, since the consensus observer in (3.40) only depends on the outputs of the subsystems.

**Remark 3.5.2** The implementation of the consensus controller (3.61) requires the state of the leader to be accessed by the followers. In our case, this would not have been necessary if the systems are linear.

**Remark 3.5.3** Controller (3.61) can be seen as simpler than state-feedback consensus controller (3.2).

With (2.31, pg.44) and (3.61), system (3.1) can be written as

\[ \dot{x}_i = (A - BK)x_i + BKx_0 + BK\hat{x}_i + \phi(x_i) \]
and with (3.42), the augmented closed-loop network dynamics can be stacked into a compact form as

$$
\begin{bmatrix}
\dot{\xi} \\
\dot{\tilde{x}}
\end{bmatrix} = 
\begin{bmatrix}
I_N \otimes (A - BK) & I_N \otimes BK \\
0 & I_N \otimes A - \mathcal{L} \otimes LC
\end{bmatrix}
\begin{bmatrix}
\xi \\
\tilde{x}
\end{bmatrix} + 
\begin{bmatrix}
\Phi(x) - \Phi(x_0) \\
\Phi(x) - \Phi(\hat{x})
\end{bmatrix}
\tag{3.63}
$$

By using the same transformation (3.43) for $\tilde{\eta}$, we have the transformed network dynamics

$$
\begin{bmatrix}
\dot{\xi} \\
\dot{\tilde{\eta}}
\end{bmatrix} = 
\begin{bmatrix}
I_N \otimes (A - BK) & I_N \otimes BK \\
0 & I_N \otimes A - \mathcal{J} \otimes LC
\end{bmatrix}
\begin{bmatrix}
\xi \\
\tilde{\eta}
\end{bmatrix} + 
\begin{bmatrix}
\Phi(x) - \Phi(x_0) \\
\Psi(x, \hat{x})
\end{bmatrix}
\tag{3.64}
$$

### 3.5.1 Stability Analysis

From the stability analysis, the theorem below was obtained.

**Theorem 3.5.1** For a network-connected dynamic system (2.35) with the associated Laplacian matrix that satisfies Assumptions 3.2.1-3.2.3 and Assumption 3.4.1, the observer-based controller (3.61) with $K = B^T P_1$, and $L = P_2^{-1} C^T$ would solve the leader-follower consensus problem if there exist solutions $P_1 = P_1^T > 0$ and $P_2 = P_2^T > 0$ specified in one of the following cases:

1. If the eigenvalues of the matrix $\mathcal{L}$ are distinct, i.e., $n_k = 1$ for $k = 1, \ldots, q$, the matrices $P_1$ and $P_2$ satisfy

$$
A^T P_1 + P_1 A - P_1 B B^T P_1 + \kappa_1 P_1 P_1 + \frac{\gamma^2}{\kappa_1} I_n < 0
\tag{3.65}
$$

and

$$
A^T P_2 + P_2 A - 2\tilde{\alpha} C^T C + \kappa_2 P_2 P_2 + \frac{\gamma^2}{\kappa_2} I_n + P_1 B B^T P_1 < 0
\tag{3.66}
$$

with $\kappa_1$ and $\kappa_2$ being any positive real numbers, $\tilde{\alpha} = \min\{\bar{\lambda}_1, \ldots, \bar{\lambda}_p, \bar{\alpha}_{p+1}, \ldots, \bar{\alpha}_q\}$, $\gamma$ defined in (2.31) and $\gamma_0$ defined in (3.47).
2. If the eigenvalues of the matrix $\mathcal{L}$ are multiple eigenvalues, i.e., $n_k > 1$ for any $k \in \{1, \ldots, q\}$, the matrices $P_1$ and $P_2$ satisfy

$$A^T P_1 + P_1 A - P_1 B B^T P_1 + \kappa_1 P_1 P_1 + \frac{\gamma_1^2}{\kappa_1} I_n < 0 \quad (3.67)$$

and

$$A^T P_2 + P_2 A - 2(\bar{\alpha} - 1) C^T C + \kappa_2 P_2 P_2 + \frac{\gamma_2^2}{\kappa_2} I_n + P_1 B B^T P_1 < 0 \quad (3.68)$$

with $\kappa_1$ and $\kappa_2$ being any positive real numbers.
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Proof 3.5.1 Let \( V_1 = \sum_{i=1}^{N} \xi_i^T P_i \xi_i \) and taking derivative of \( V_1 \) along (3.64) yields

\[
\dot{V}_1 = \sum_{i=1}^{N} \left( (A - BK)\xi_i + BK\tilde{\eta}_i + \phi(x_i) - \phi(x_0) \right)^T P_i \xi_i \\
+ \sum_{i=1}^{N} \xi_i^T P_i [(A - BK)\xi_i + BK\tilde{\eta}_i + \phi(x_i) - \phi(x_0)] \\
= \sum_{i=1}^{N} \xi_i^T (A^T P_i + P_i A - 2P_i BK)\xi_i + 2\xi_i^T P_i BK\tilde{\eta}_i + \sum_{i=1}^{N} 2\xi_i^T P_i (\phi(x_i) - \phi(x_0)) \\
\leq \sum_{i=1}^{N} \xi_i^T (A^T P_i + P_i A - P_i BB^T P_i + \kappa_1 P_i P_i) \xi_i + \sum_{i=1}^{N} \tilde{\eta}_i^T P_i BB^T P_i \tilde{\eta}_i \\
+ \sum_{i=1}^{N} \frac{1}{\kappa_1} \|\phi(x_i) - \phi(x_0)\|^2 \\
\leq \sum_{i=1}^{N} \xi_i^T (A^T P_i + P_i A - P_i BB^T P_i + \kappa_1 P_i P_i + \frac{\gamma^2}{\kappa_1} I_n) \xi_i + \sum_{i=1}^{N} \tilde{\eta}_i^T P_i BB^T P_i \tilde{\eta}_i \\
(3.69)
\]

where \( K = B^T P_1 \). Note that we have used the inequality \( 2a^T b \leq \kappa a^T a + b^T b/\kappa \) for vectors \( a \) and \( b \) with the same dimension.

Let the Lyapunov function be \( V = V_1 + V_2 \) with \( V_2 = \bar{V} \). By Theorem 3.5.1, two cases are considered for stability analysis:

Case 1. If the eigenvalues of the matrix \( \Theta \) are distinct, we have

\[
\dot{V} \leq \sum_{i=1}^{N} \xi_i^T (A^T P_i + P_i A - P_i BB^T P_i + \kappa_1 P_i P_i + \frac{\gamma^2}{\kappa_1} I_n) \xi_i \\
+ \sum_{i=1}^{N} \tilde{\eta}_i^T \left( A^T P + PA - 2\alpha C^T C + \kappa PP + \frac{\gamma^2}{\kappa} I_n \right) \tilde{\eta}_i \\
+ \sum_{i=1}^{N} \tilde{\eta}_i^T P_i BB^T P_i \tilde{\eta}_i \\
= \sum_{i=1}^{N} \xi_i^T M_1 \xi_i + \sum_{i=1}^{N} \tilde{\eta}_i^T M_2 \tilde{\eta}_i \\
(3.70)
\]

where

\[
M_1 = A^T P_1 + P_1 A - P_1 BB^T P_1 + \kappa_1 P_i P_i + \frac{\gamma^2}{\kappa_1} I_n \\
M_2 = A^T P_2 + P_2 A - 2\alpha C^T C + \kappa_2 P_2 P_2 + \frac{\gamma^2}{\kappa_2} I_n + P_1 BB^T P_1
\]

Conditions (3.65) and (3.66) guarantee \( \dot{V} < 0 \) for \( \kappa_1 > 0, \kappa_2 > 0, \gamma \) defined in (2.31) and \( \gamma_0 \) defined in (3.47), which implies \( \xi_i(t) \to 0 \) and \( \tilde{\eta}_i(t) \to 0 \) as \( t \to \infty \) for all \( i \).
Case 2. Similarly, if the matrix $\mathbf{L}$ has multiple eigenvalues, we have

$$
\dot{V} \leq \sum_{i=1}^{N} \xi_i^T M_1 \xi_i + \sum_{i=1}^{N} \tilde{\eta}_i^T \overline{M}_2 \tilde{\eta}_i
$$

where

$$
\overline{M}_2 = A^T P_2 + P_2 A - 2(\bar{\alpha} - 1)C^T C + \kappa_2 P_2 P_2 + \frac{\gamma_0^2}{\kappa_2} I_n + P_1 B B^T P_1
$$

Conditions (3.67) and (3.68) guarantee $\dot{V} < 0$, which implies $\xi_i(t) \to 0$ and $\tilde{\eta}_i(t) \to 0$ as $t \to \infty$ for all $i$.

Hence, the leader-follower consensus is achieved. This completes the proof.

**Remark 3.5.4** The conditions derived in (3.65)-(3.68) can be formulated as linear matrix inequalities; using standard LMI routine a possible solution can be obtained and its feasibility can then be checked. Note that a possible solution $P_1$ of (3.65) can be calculated first, before the possible solution $P_2$ of (3.66) for the fixed $P_1$ can be obtained. One can follow the same manipulation for Case 2 in Theorem 3.5.1.

### 3.5.2 Solution of Positive Definite $P_1$ and $P_2$ with Linear Matrix Inequality (LMI)

The solution of $P_1$ and $P_2$ can be obtained by standard LMI (Linear Matrix Inequality) procedure from Lemma 3.3.2 (pg.57). First, we need to obtain a solution of $P_1$. From Lemma 3.3.2 (pg.57), and by having $Q_1 = P_1^{-1}$ multiplied to both sides of $P_1$ for $M_1$, we arrived at

$$
\begin{bmatrix}
Q_1 A^T + A Q_1 - B B^T & Q_1^T \\
Q_1 & -\frac{\gamma_0}{\kappa_2} I_n
\end{bmatrix} < 0
$$

(3.72)

$$
Q_1 = Q_1^T > 0
$$

Now that we have a solution of $P_1$, we can then obtain

$$
\begin{bmatrix}
A^T P_2 + P_2 A - \beta C^T C + \frac{\gamma_0^2}{\kappa_2} + P_1 B B^T P_1 & P_2 \\
P_2^T & -\frac{1}{\kappa_2} I_n
\end{bmatrix} < 0
$$

(3.73)

$$
P_2 = P_2^T > 0
$$

with, for Case 1

$$
\beta = 2\bar{\alpha}
$$
and for Case 2
\[ \beta = 2 \left( \bar{\alpha} - \frac{1}{\sigma} \right) \]

The solutions (3.72) and (3.73) can be obtained by applying the LMI instructions from the Matlab Control Toolbox.

3.5.3 Solution of Positive Definite \( P_1 \) and \( P_2 \) with Algebraic Riccati Equation (ARE)

Similar to the previous section, a solution with ARE is considered one of the alternative solutions of \( P = P^T > 0 \). Hence from \( M_1 \) and \( M_2 \), we have
\[ A^T P + PA + P(k_1 I_n - BB^T)P + \left( \frac{\gamma_0^2}{\kappa_1} I_n + \delta_0 I_n \right) = 0 \] (3.74)
and it is easy to obtain
\[ H_3 = \begin{bmatrix} A & k_1 I_n - BB^T \\ -\left( \frac{\gamma_0^2}{\kappa_1} I_n + \delta_0 I_n \right) & -A^T \end{bmatrix} \] (3.75)

Similarly, if the matrix of \( H_3 \) has no eigenvalues on the imaginary axis, if the pair \((A, k_1 I_n - BB^T)\) is stabilizable, and if \((k_1 I_n - BB^T) \geq 0 \) and \( \left( \frac{\gamma_0^2}{\kappa_1} I_n + \delta_0 I_n \right) > 0 \), it is possible to get a solution of \( P_1 = P_1^T > 0 \). Meanwhile for \( P_2 \), we have
\[ A^T P_2 + P_2 A + P_2(k_2 I_n)P + \left( -\beta C^T C + \frac{\gamma_0^2}{\kappa_2} I_n + \delta_0 I_n \right) = 0 \] (3.76)
and it is easy to obtain
\[ H_4 = \begin{bmatrix} A & k_2 I_n \\ -\left( -\beta C^T C + \frac{\gamma_0^2}{\kappa_2} I_n + \delta_0 I_n \right) & -A^T \end{bmatrix} \] (3.77)

If eigenvalue of \( H_4 \) has no eigenvalues on the imaginary axis, if the pair \((A, k_2 I_n)\) is stabilizable, and if \((k_2 I_n) \geq 0 \) and \( \left( -\beta C^T C + \frac{\gamma_0^2}{\kappa_2} I_n + \delta_0 I_n \right) > 0 \), it is possible to get a solution of \( P_2 = P_2^T > 0 \) for Case 1
\[ \beta = 2\bar{\alpha} \]
and for Case 2
\[ \beta = 2 \left( \bar{\alpha} - \frac{1}{\sigma} \right) \]

Hence the solution of \( P = P^T > 0 \) can be obtained by using the function 'care' from the Matlab Control Toolbox.
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3.5.4 Simulation of Observer-based Controller

In this section, a simulation is performed to demonstrate the effectiveness of the observer-based controller (3.61) as shown in Figure 3.19. Two examples are given; the first example deals with the solution obtained from LMI procedure, while the second one uses the ARE procedure. Similar to Section 3.3.4, the ARE procedure is only used when the solution of $P$ from the LMI procedure does not guarantee consensus in simulation.

Example 1

The Lipschitz constant is $\gamma = 1$ and the bound in Lemmas 3.3.1 and 3.4.1 can be calculated as $\gamma_0 = 0.0208$. With $\kappa_1 = 3.0, \kappa_2 = 1.0$, the control gain $K$ and the observer gain $L$ can be expressed as

$$K = \begin{bmatrix} -0.1196 & 0.2930 \end{bmatrix} \quad (3.78)$$

$$L = \begin{bmatrix} 3.1074 \\ -2.4868 \end{bmatrix} \quad (3.79)$$

where

$$P_1 = \begin{bmatrix} 0.5420 & -0.2392 \\ -0.2392 & 0.5859 \end{bmatrix} \quad (3.80)$$

$$P_2 = \begin{bmatrix} 0.5357 & 0.4684 \\ 0.4684 & 0.5852 \end{bmatrix} \quad (3.81)$$

Figure 3.20 and Figure 3.21 show the simulation plots for substates 1 and 2 of the leader, with substates 1 and 2 of all followers with controller (3.61), with respect to $\gamma = 1$. It can be seen that conditions (3.65) and (3.66) are sufficient for the control gain to achieve consensus control when there are nonlinearities in the system; however these conditions may be relatively conservative for a given Lipschitz nonlinear function in the control gain design. As shown in Figure 3.22 and 3.23, the same control gain can achieve consensus control for nonlinearities with a larger Lipschitz constant, $\gamma = 1.3611$. Notably, consensus can still be reached due to a stronger nonlinearity at $\gamma = 5$ as shown in Figure 3.24 and Figure 3.25. In addition, the subsystem trajectories shown in Figure 3.23 and 3.24 are different from those in Figure 3.20 and 3.21 due to
Example 2

With the same system in Example 1 and by having the solution of $P_1$ with ARE, where the Hamiltonian matrix can be expressed as

$$
H_1 = \begin{bmatrix}
0 & 1.0000 & 1.0000 & 0 \\
-1.0000 & 0 & 0 & 0 \\
-0.1025 & 0 & 0 & 1.0000 \\
0 & -0.1025 & -1.0000 & 0
\end{bmatrix}
$$

The eigenvalues of matrix $H_1$ are not on the imaginary axis and we can obtain the controller gain matrix

$$
K = \begin{bmatrix}
-0.8084 & 3.2269
\end{bmatrix}
$$
Figure 3.20: Plot of Leader substate 1 and follower substates 1 consensus controller with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 1$

Figure 3.21: Plot of Leader substate 2 and follower substates 2 consensus controller with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 1$
Figure 3.22: Plot of Leader substate 1 and follower substates 1 consensus controller with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 1.3611$.

Figure 3.23: Plot of Leader substate 2 and follower substates 2 consensus controller with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 1.3611$. 
Figure 3.24: Plot of Leader substate 1 and follower substates 1 consensus controller with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 5$.

Figure 3.25: Plot of Leader substate 2 and follower substates 2 consensus controller with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 5$. 
Similarly for the observer, where the Hamiltonian matrix is

\[
H_2 = \begin{bmatrix}
0 & 1.0000 & 1.0000 & 0 \\
-1.0000 & 0 & 0 & 1.0000 \\
0.3901 & 0 & 0 & 1.0000 \\
0 & -0.1001 & -1.0000 & 0
\end{bmatrix}
\]  

(3.84)

there are no eigenvalues of \( H_2 \) on the imaginary axis and we can obtain the observer gain

\[
L = \begin{bmatrix}
1.2531 \\
-0.0951
\end{bmatrix}
\]  

(3.85)

The plots with the ARE solutions give similar outcome as in the previous plots. However, using the solutions of matrix \( P_1 \) and \( P_2 \) with LMI, it is not always possible to achieve consensus from simulation when tested with different dynamics. Hence, the ARE procedure is used to produce a suitable matrix \( P_1 \) and \( P_2 \), which will be demonstrated in the next chapter.

### 3.5.5 Conclusion

In this section, an observer-based consensus controller has been proposed. The observer-based controller (3.61) utilized the functionality of the observer (3.40), which is useful when it is not possible to find the exact value of the states of the system. The stability conditions of the system were provided by Lyapunov stability analysis. The solution of \( P = P^T > 0 \) was provided from Linear Matrix Inequality (LMI) and Algebraic Riccati Equation (ARE) standard routine. It has been shown that the observer-based consensus controller (3.61) enabled each agent or subsystem to reach consensus under the influence of Lipschitz nonlinearity. The controller were tested with a few different values of nonlinearity represented by the change of \( \gamma \). In all cases, consensus was reached. However, the trajectory of the agents or subsystems changed as nonlinearity increased.
3.6 Conclusion of Chapter

In this chapter, a state-feedback consensus controller, a consensus observer, and an observer-based consensus controller were proposed. This chapter provided one of the main contributions of the thesis which is the design of consensus control for multi-agent systems with Lipschitz nonlinearity.

Several theories were provided that explained the conditions and restrictions of each controller and observer required for consensus to be reached based on Lyapunov stability analysis and the connection network. The solution for the stability analysis were obtained via LMI and ARE. The consensus observer enabled the estimated states to track the actual states of the agent or subsystem.

From simulation examples, all controllers were tested for certain values of Lipschitz nonlinearity by changing the value of Lipschitz constant $\gamma$. It was observed that consensus were reached by each agent or subsystem for each test. However, the increase of the nonlinearity constant affected the trajectory and output of each agent or subsystem. Therefore, the design objective of these controllers has been met - to enable each agent or subsystems in a nonlinear system with Lipschitz nonlinearity to reach consensus.
Chapter 4

Containment of Lipschitz Nonlinear Systems

4.1 Introduction

Distributed containment control is considered to be an extension of consensus control. Essentially, the concept is to contain the followers with the boundedness provided by the leaders. This is primarily derived from the multiple leader-follower configuration in consensus control. Ultimately, all followers will converge to the stationary convex hull formed by the stationary or dynamic leaders.

Previous studies have looked at various methods of solving containment problem. Through expansion of preliminary work in [76], Cao et.al [77] has proven that, when the leaders are stationary and all agents share a common inertial coordinate frame, necessary and sufficient conditions on the fixed or switching-directed network topology enable all followers to ultimately converge to the stationary convex hull formed by the stationary leaders, for arbitrary initial states in a space of any finite dimension. In particular, with a fixed directed network topology, the final states of the followers are constant. With a switching-directed network topology, the final states of the followers may change, depending on the switching graphs.

Dong et.al [78], investigated the output containment control problems for higher-order linear time-invariant swarm systems under directed interaction topologies, using
a dynamic output feedback approach.

The use of hybrid control with Model Predictive Control (MPC) technique is included in the design of the containment controller, and by considering constraints of inputs, the sensing manoeuvre is formulated in [79]. In addition, hybrid control enabled the followers to detect containment violations by simple computations based on the information of the leaders' positions. In another study, Ji et.al [80] exploited the theory of partial difference equations, and proposed hybrid control schemes based on stop-go rules for the leader-agents.

Hagshenas et.al [81] investigated the containment control problem for multi-agent systems, composed of non-identical agents governed by unknown nonlinear dynamics, parametrised by some continuous and uniformly-bound functions. On the other hand Lou et.al [82] and Zheng et.al [83] looked at the same problem where dynamics of the followers were identical.

Li et.al [84] propose two impulsive protocols for the NAMSs (Nonlinear Adaptive Multi-agent Systems) under the directed network topology, based on protocols in [85].

Conditions for containment control with a sliding-mode control algorithm are proposed by [86] for containment control, with an exponential leader span in finite time. Another containment control problem is related to exponential finite-time coordination, where in [87], the proposed protocol ensures that the boundary agents in the same strong component exponentially reach a consensus, and the internal agents exponentially converge to the convex hull spanned by the boundary agents in finite time.

From [88], the containment problem is viewed from the internal and boundary agents, with the pinning control strategy for second-order multi-agent systems with time-varying delays.

Liu et.al [89] investigated containment control problems with multiple stationary or dynamic leaders, under directed topologies for continuous-time and sampled-data
based protocols. In other work, Liu et.al [90] considered the same problem in the presence of transmission noise, and under both dynamically-switching and randomly-switching topologies.

Lou et.al [91] provided a target set of containment of second-order agents specified by multiple leaders, with the help of convex and stochastic analysis based on the containment of the first-order model from [92].

By manipulating the nature of Markovian switching, Li et.al [93] investigated the containment tracking problem for first-order agents affected by white noise.

With higher-order models, Li et.al. [94] analysed a linear-distributed containment controller based on relative state information between agents for continuous and discrete time under directed communication topologies; Ma et.al [95] then extended these findings with a solution of the Algebraic Riccati Equation (ARE).

In a different publication, Ma et.al [96] constructed two observation vectors namely angular velocity and input torque. Based upon these, two controllers are proposed for attitude containment control under a directed graph approach with multiple static and dynamic leaders, where they dealt with the finite-time attitude containment control under directed graphs.

Mei et.al [97] provided the analysis of the containment control problem without information of the neighbours’ velocities, based on the work of [98].

Meng et.al [99] proposed the containment controller using one-hop and two-hop neighbours’ information based on Euler-Lagrange model from work in [100].

Inspired by the swarming behaviour of silkworm moths, [101] investigated the containment of agents by multiple leaders via an undirected switching graph topology, with results proven for arbitrary state dimensions. Previously in [76], the same problem was solved for scalar dynamics only.
Peng et.al. [102] described a new predictor-based neural dynamic surface control (PNDSC) design approach; this was proposed in order to develop adaptive containment controllers for multiple autonomous underwater vehicles (AUV) in the presence of multiple dynamic leaders. The AUV were subject to model uncertainty and unknown ocean disturbances, where it is assumed that the information of leaders is only available to a few followers.

Rong et.al [103] studied the problem of distributed containment control for multi-agent systems with higher-order dynamics under directed network topologies, where the model is based on containment controllers, with both relative and absolute damping.

Containment problem with input delays is solved in [104] by proposing a distributed PD-type protocol based on information of the neighbours.

The containment control problem for first-order multi-agent systems, under heterogeneous unbounded communication delays, with an emphasis on the convergence rate analysis, was addressed in [105].

Motivated by effort in [106], the containment problem of first-order and second-order integral multi-agent systems with communication noise was investigated in [107].

In [108], adaptive containment control was considered for a class of multi-agent systems with multiple leaders containing parametric uncertainties.

Wang et.al [109] considered the distributed containment control problem of second-order multi-agent systems with inherent nonlinear dynamics. The authors proposed static and adaptive control protocols, where the containment control problem can be solved without any global information of the interaction topology among the followers.
Most of the previous containment control publications were based on linear systems. However, there are no publications that were related to containment problem of multi-agent systems with respect to general Lipschitz nonlinearity. Hence, motivated by Ding [55] and Li et.al. [67], this chapter considers the containment control problem in continuous time for linear system with Lipschitz nonlinearity under directed communication topology. The nonlinearity is restricted to Lipschitz, because this allows less restriction to its structure.

In this chapter, distributed containment controllers are proposed based on relative state-feedback and output feedback information of each subsystem. More specifically, the controllers will enable each subsystem to converge into the convex hull formed by the leaders’ subsystems, provided at least one leader subsystem exists with a directed path to the follower subsystem. Each subsystem’s stability is analysed using the real Jordan form framework, by utilizing Lyapunov stability functions in the real time domain. The framework is then used to explore the conditions on the nonlinearity for the global containment control. This framework analysis depends on the eigenstructure of the Laplacian matrix - containing either distinct or multiple eigenvalues. As long as each subsystem is detectable and stabilizable, it is possible to design a containment controller for a nonlinear system. Hence, the main contribution of this chapter is to solve the containment problem by providing a measure of nonlinearity which can be tolerated with global stability.

This chapter is about the containment of a number of followers by multiple leader subsystems. Two controllers are proposed, the state-feedback containment controller which is based on relative state information of the followers states, and the observer-based controller which depends on the relative output information of the system. A containment observer is proposed from the observer-based controller, . The stability of these controllers and observers with respect to the system are analysed and conditions of stability are provided using the Lyapunov stability method. The design of the controllers and observers is based on the solution of \( P \) from LMI and ARE, according to restrictions faced during simulation. Finally, the effectiveness of the controllers are then confirmed using simulations.
In Section 4.2, the nonlinear system (4.1) is defined with respect to the multi-agent system. A nonlinear element in the form of a Lipschitz nonlinearity is incorporated into the system; both the system and the nonlinearity are considered as general.

Next, Section 4.3 includes the design of a containment controller (4.4) using state-feedback of relative state information of neighbouring subsystems other than the leaders. In Section 4.3.1 the stability condition of the controller is given in Theorem 4.3.1 with bound obtained in Lemma 4.3.1. Simulation example is given at the end of the section which proves that the controller managed to contain the proposed state-feedback containment controller (4.4).

Section 4.4 deals with the design of a containment observer (4.48). Based on Lyapunov’s stability analysis, several conditions of the stability of the observer are given by Theorem 4.4.1, with the bound given by Lemma 4.4.1. This observer will be used in Section 4.5.

In Section 4.5, the controller in Section 4.3 is combined with the observer in Section 4.4. Several conditions of stability are given in Theorem 4.5.1, with the same bounds as in Lemma 4.3.1 and Lemma 4.4.1. A simulation example is given to prove that the containment controller (4.81) managed to contain the follower subsystems.

All the results in this chapter are concluded in Section 4.6; if the states of each of the nonlinear subsystem with Lipschitz nonlinearity (4.1) are known, a state-feedback containment controller (4.4) can be designed to enable multiple leader subsystems containing the follower subsystems. However, if the states of (4.1) are unknown, an observer-based controller (4.81) can be designed to enable the leader subsystems to contain the follower subsystems. Several simulation examples are given at the end of each section to prove the state-feedback and observer-based consensus controller designed managed to contain the follower subsystems.
4.2 Problem Statement

Consider \( N + 1 \) nonlinear subsystems with identical dynamics, described as

\[
\begin{align*}
\dot{x}_i &= Ax_i + \phi(x_i) + Bu_i \\
y_i &= Cx_i
\end{align*}
\]  

(4.1)  

(4.2)

where for \( i = 0, \ldots, N \), \( x_i \in \mathbb{R}^n \) is the state vector of the subsystem, \( u_i \in \mathbb{R}^p \) is the input of the \( i \)th subsystem, and \( y_i \in \mathbb{R}^q \) is the measured output vector, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times p} \), and \( C \in \mathbb{R}^{q \times n} \) are appropriate matrices, and \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) is a Lipschitz nonlinear function with the Lipschitz constant \( \gamma \), i.e. as in (2.31).

The connections between the subsystems are specified by a directed graph \( G \), which consists of a set of vertices denoted by \( V \), and a set of edges denoted by \( E \). A vertex represents a subsystem, and each edge represents a connection. Similar to Section 2.2.2, the adjacency matrix \( Q \) associated with the graph \( G \) is defined with elements \( q_{ij} = 1 \), if there is a connection between subsystem \( j \) to \( i \), and \( q_{ij} = 0 \) otherwise. The Laplacian matrix \( L = \{l_{ij}\} \) is defined as in Section 2.2.3.

A few assumptions are needed.

**Assumption 4.2.1** Matrices \((A, B)\) are controllable.

**Assumption 4.2.2** System (4.1) is stable.

**Assumption 4.2.3** The communication network \( G \) of the multi-agent systems contains a directed spanning forest with any of the leader has a directed path to the system.

Due to the leaders’ connections, the composition of \( L \) is arranged is such a way that

\[
L = \begin{bmatrix}
L_1 & L_2 \\
0_{(N-M) \times M} & 0_{(N-M) \times (N-M)}
\end{bmatrix}
\]  

(4.3)

with \( L_1 \in \mathbb{R}^{M \times M} \) and \( L_2 \in \mathbb{R}^{M \times (N-M)} \).

**Lemma 4.2.1** [67] Under Assumption 4.2.1, all eigenvalues of \( L_1 \) have positive real parts, each entry of \( L_1^{-1} L_2 \) is non-negative, and each row of \( L_1^{-1} L_2 \) has a sum equal to 1.
Definition 4.1 For nonlinear subsystems with multiple leaders, we define that there are $M$ followers, with $N < M$ and $N - M$ leaders. A subsystem is called a follower if it has at least one neighbour subsystem, while a leader has no neighbour subsystems. The follower subsystems are indexed by $1, \ldots, M$ and the leader subsystem is indexed by $M + 1, \ldots, N$ with control input to be zero. Note that the leader does not receive any information. For convenience, $\mathcal{R} \triangleq \{M + 1, \ldots, N\}$ and $\mathcal{F} \triangleq \{1, \ldots, M\}$ are used to denote the leader and follower set.

Definition 4.2 [77] Let $C$ be a set in a real vector space $V \subseteq \mathbb{R}^p$. The set $C$ is called convex if, for any $x$ and $y$ in $C$, the point $(1 - z)x + zy$ in $C$ for any $z \in [0, 1]$. The convex hull for a set of points $X = \{x_1, \ldots, x_q\}$ is minimal convex set containing all points in $X$, and defined as $\text{Co}\{x_j, j \in \mathcal{R}\}$.

The control objective is to design the distributed controllers that will solve the containment control problem in a nonlinear system (4.1), defined as follows:

Definition 4.3 The containment control problem is solved for the subsystems in (4.1), if the states of the followers - under a certain distributed nonlinear controller - asymptotically converge to the convex hull form by those of the leaders, defined in Definition 4.2.

4.3 State-feedback Containment Controller

Motivated by [47], the state-feedback containment consensus controller is proposed as

$$u_i = -K \sum_{j \in \mathcal{F} \cup \mathcal{R}} l_{ij} (x_i - x_j) \quad (4.4)$$

where $K \in \mathbb{R}^{p \times n}$ is a constant control gain matrix to be designed later. The containment consensus control problem is said to be solved if all followers always converged to the stationary convex hull $\text{Co}\{x_j, j \in \mathcal{R}\}$ as $t \to \infty$.

For the network dynamics, we have

$$\dot{x} = (I_N \otimes A - \mathcal{L} \otimes BK)x + \Phi(x) \quad (4.5)$$

where $\mathcal{L}$ is defined as (2.1) and (4.3), $\otimes$ is the Kronecker product, $x = [x_f \ x_l]^T$ where $x_f = [x_1^T, \ldots, x_M^T]^T$ and $x_l = [x_{M+1}^T, \ldots, x_N^T]^T$ together with $\Phi(x) = [\Phi(x_f) \ \Phi(x_l)]^T$. 
where $\Phi(x_f) = [\phi(x_1)^T, \ldots, \phi(x_M)^T]^T$ and $\Phi(x_l) = [\phi(x_{M+1})^T, \ldots, \phi(x_N)^T]^T$. Hence, we obtain a value for $x_f$ that satisfies the following dynamics:

$$
\dot{x}_f = (I_M \otimes A - \mathcal{L}_1 \otimes BK)x_f - (\mathcal{L}_2 \otimes BK)x_l + \Phi(x_f) \quad (4.6)
$$

Let $\xi_i = \sum_{j \in \mathcal{F} \cup \mathcal{R}} l_{ij}(x_i - x_j), i \in \mathcal{F}$ and we have

$$
\xi_f = (\mathcal{L}_1 \otimes I_m)x_f + (\mathcal{L}_2 \otimes I_m)x_l \quad (4.7)
$$

**Remark 4.3.1** Note that (4.7) is from

$$
\xi = (\mathcal{L} \otimes I_n)x \quad (4.8)
$$

or

$$
\begin{bmatrix}
\xi_f \\
\xi_l
\end{bmatrix} =
\begin{bmatrix}
\mathcal{L}_1 \otimes I_m & \mathcal{L}_2 \otimes I_m \\
0_{(N-M) \times M} & 0_{(N-M) \times (N-M)}
\end{bmatrix}
\begin{bmatrix}
x_f \\
x_l
\end{bmatrix} \quad (4.9)
$$

From (4.5) we can also have

$$
\dot{\xi} = (I_N \otimes A - \mathcal{L} \otimes BK)\xi + (\mathcal{L} \otimes I_n)\Phi(x) \quad (4.10)
$$

Considering the structure of $\mathcal{L}$ in (4.3), we can see from (4.5) and (4.6), that $\xi_f$ satisfies the following dynamics

$$
\begin{align*}
\dot{\xi}_f &= (\mathcal{L}_1 \otimes I_m)(I_M \otimes A - \mathcal{L}_1 \otimes BK)x_f \\
&\quad - (\mathcal{L}_1 \otimes I_m)(\mathcal{L}_2 \otimes BK)x_l + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) \\
&\quad + (\mathcal{L}_2 \otimes I_m)(I_{N-M} \otimes A)x_l + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \\
&= (I_M \otimes A - \mathcal{L}_2^2 \otimes BK)x_f - (\mathcal{L}_1 \mathcal{L}_2 \otimes BK)x_l + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) \\
&\quad + (\mathcal{L}_2 \otimes A)x_l + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \\
&= (I_M \otimes A - \mathcal{L}_2^2 \otimes BK)\left[\mathcal{L}_1^{-1} \otimes I_m \xi_f - (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_m)x_l\right] \\
&\quad - (\mathcal{L}_1 \mathcal{L}_2 \otimes BK)x_l + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) \\
&\quad + (\mathcal{L}_2 \otimes A)x_l + (\mathcal{L} \otimes I_m)\Phi(x_l) \\
&= (I_M \otimes A - \mathcal{L}_1 \otimes BK)\xi_f + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \\
&\quad - (\mathcal{L}_1 \mathcal{L}_2 \otimes I_p)x_l. \quad (4.11)
\end{align*}
$$

**Remark 4.3.2** Based on Theorem 3.1 stated by Cao et.al [77], without the nonlinearity term $\Phi(x_f)$ in (4.6), and if the topology is directed, the final positions of the followers are given by $-(\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_p)x_l$. This is related to the property of $\mathcal{L}_1^{-1} \mathcal{L}_2$, where
each row of $L_1^{-1}L_2$ has a sum equal to 1, which is stated in Lemma 4.2.1. Hence, incorporating nonlinear terms in (4.11), we can use the Lyapunov method to analyse its stability, and to find a suitable matrix $K$ for the control design. This can only be achieved by transforming (4.11) into a diagonally-dominant matrix.

Let us reintroduce nonsingular matrices $T \in \mathbb{R}^{N \times N}$ and $T^{-1} \in \mathbb{R}^{N \times N}$ such that

$$T^{-1}LT = J$$

with $J$ being a block-diagonal matrix of real Jordan form

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{bmatrix}$$

(4.13)

where $J_k \in \mathbb{R}^{n_k}$ for $k = 1, \ldots, p$ are the Jordan blocks for real eigenvalues $\lambda_k > 0$, with the multiplicity $n_k$ in the form

$$J_k = \begin{bmatrix} \lambda_k & 1 \\ & \lambda_k & 1 \\ & & \ddots & \ddots \\ & & & \lambda_k & 1 \end{bmatrix}$$

and $J_k \in \mathbb{R}^{2n_k}$ for $k = p + 1, \ldots, q$ are the Jordan blocks for conjugate eigenvalues $\alpha_k \pm j\beta_k$, $\alpha_k > 0$ and $\beta_k > 0$, with multiplicity $n_k$ in the form

$$J_k = \begin{bmatrix} \mu(\alpha_k, \beta_k) & I_2 \\ & \mu(\alpha_k, \beta_k) & I_2 \\ & & \ddots & \ddots \\ & & & \mu(\alpha_k, \beta_k) & I_2 \end{bmatrix}$$
with $I_2$ the identity matrix in $\mathbb{R}^{2 \times 2}$ and

$$
\mu(\alpha_k, \beta_k) = \begin{bmatrix}
\alpha_k & \beta_k \\
-\beta_k & \alpha_k
\end{bmatrix} \in \mathbb{R}^{2 \times 2}
$$

(4.14)

**Remark 4.3.3** In order to analyze the stability of system (4.6), (4.11) needs to be transformed by manipulating the structure of $L$. However, there is no direct transformation of (4.11). Therefore the required transformation is from (4.10).

In other words, from (4.10), we have

$$
\begin{bmatrix}
\dot{\xi}_f \\
\dot{\xi}_l
\end{bmatrix} = \begin{bmatrix}
I_M \otimes A & 0 \\
0 & I_{N-M} \otimes A
\end{bmatrix}\begin{bmatrix}
\xi_f \\
\xi_l
\end{bmatrix}
- \begin{bmatrix}
\mathcal{L}_1 \otimes BK & \mathcal{L}_2 \otimes BK \\
0_{(N-M)\times M} \otimes 0_{M\times M} & 0_{(N-M)\times (N-M)} \otimes 0_{M\times M}
\end{bmatrix}\begin{bmatrix}
\xi_f \\
\xi_l
\end{bmatrix}
+ \begin{bmatrix}
\mathcal{L}_1 \otimes I_m & \mathcal{L}_2 \otimes I_m \\
0_{(N-M)\times M} \otimes I_m & 0_{(N-M)\times (N-M)} \otimes I_m
\end{bmatrix}\begin{bmatrix}
\Phi(x_f) \\
\Phi(x_l)
\end{bmatrix}
$$

(4.15)

where we obtain

$$
\dot{\xi}_f = (I_M \otimes A - \mathcal{L}_1 \otimes BK)\xi_f - (\mathcal{L}_2 \otimes BK)\xi_l
+ (\mathcal{L}_1 \otimes I_m)\Phi(x_f) + (\mathcal{L}_2 \otimes I_m)\Phi(x_l)
$$

(4.16)

and

$$
\dot{\xi}_l = (I_{N-M} \otimes A)\xi_l
$$

(4.17)

By introducing transformations

$$
\eta = (T^{-1} \otimes I_n)\xi
$$

(4.18)

and

$$
\Psi(x) = (T^{-1} \otimes I_n)(\mathcal{L} \otimes I_n)\Phi(x)
$$

(4.19)

we obtain

$$
\begin{bmatrix}
\dot{\eta}_f \\
\dot{\eta}_l
\end{bmatrix} = \begin{bmatrix}
I_M \otimes A & 0 \\
0 & I_{N-M} \otimes A
\end{bmatrix}\begin{bmatrix}
\eta_f \\
\eta_l
\end{bmatrix}
- \begin{bmatrix}
J_f \otimes BK & 0 \\
0 & 0_{(N-M)\times (N-M)} \otimes 0_{M\times M}
\end{bmatrix}\begin{bmatrix}
\eta_f \\
\eta_l
\end{bmatrix}
+ \begin{bmatrix}
\Psi(x_f) \\
\Psi(x_l)
\end{bmatrix}
$$

(4.20)
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where

\[
\begin{bmatrix}
\Psi(x_f) \\
\Psi(x_l)
\end{bmatrix} = (T^{-1} \otimes I_n) \begin{bmatrix}
\mathcal{L}_1 \otimes I_m & \mathcal{L}_2 \otimes I_m \\
0_{(N-M) \times M} \otimes I_m & 0_{(N-M) \times (N-M)} \otimes I_m
\end{bmatrix} \begin{bmatrix}
\Phi(x_f) \\
\Phi(x_l)
\end{bmatrix}
\] (4.21)

and

\[
(T^{-1} \otimes I_n) = \begin{bmatrix}
U^{-1} \otimes I_m & 0 \\
0 & 0_{(N-M) \times (N-M)} \otimes I_m
\end{bmatrix}
\] (4.22)

From (4.20) we obtain

\[
\dot{\eta}_f = (I_M \otimes A - J_f \otimes BK) \eta_f + \Psi(x_f)
\] (4.23)

and

\[
\dot{\eta}_l = (I_{N-M} \otimes A) \eta_l + \Psi(x_l)
\] (4.24)

where \(\eta = [\eta_f \eta_l]^T\) and \(\Psi(x) = [\Psi(x_f) \Psi(x_l)]^T\).

Remark 4.3.4 The dynamics (4.6) are now turned into a diagonally-dominant matrix (4.23), which is utilized in the stability analysis.

Note that under Assumption 4.2.3, it follows from Lemma 4.2.1 that all the eigenvalues of \(\mathcal{L}_1\) have positive real parts. Let \(U \in \mathbb{R}^{M \times M}\) be a nonsingular matrix such that \(U^{-1}\mathcal{L}_1 U = J_f\), with \(\lambda_1, \ldots, M\) as its diagonal entries. Hence, in order to further manipulate the special characteristic of \(\mathcal{L}_1\), (4.18) and (4.19) with respect to \(\eta_f\) and \(\Psi(x_f)\) are derived as

\[
\eta_f = (U^{-1} \otimes I_m) \xi_f
\] (4.25)

and

\[
\Psi(x_f) = (U^{-1} \otimes I_m)(\mathcal{L}_1 \otimes I_m) \Phi(x_f)
\] (4.26)

(4.25) and (4.26) are used for the stability analysis.

4.3.1 Stability Analysis

For stability analysis, a bound need to be established for the transformed function \(\eta_f\).

Lemma 4.3.1 For nonlinear element \(\psi_i(x_f)\) of the nonlinear term \(\Psi(x_f)\) in the transformed closed loop network dynamics (4.23), a bound can be established in terms of the state \(\eta_f\) as shown by

\[
\|\psi_i(x_f)\| \leq \frac{\varrho_0}{\sqrt{N}} \eta_f
\] (4.27)
with
\[ \varrho_0 = \gamma \lambda_\sigma(U^{-1}) \lambda_\sigma(L_1) \lambda_\sigma(L_1^{-1}) \lambda_\sigma(U) \sqrt{N} \]
(4.28)
where \( \lambda_\sigma(\cdot) \) denotes the maximum singular value of a matrix.

Proof 4.3.1 The approach of the derivation of (4.27) and (4.28) is similar to the bound provided in the consensus chapter. From (4.19) we have
\[ \| \psi_i(x_f) \| \leq \| o_i \otimes I_n \| \| l_{ij} \otimes I_n \| \| \Phi(x_f) \| \]
\[ \leq \lambda_\sigma(U^{-1}) \lambda_\sigma(L_1) \gamma \| x_f \| \]
\[ \leq \lambda_\sigma(U^{-1}) \lambda_\sigma(L_1) \gamma \| L_1^{-1} \otimes I_n \| \| \xi_f \| \]
\[ \leq \lambda_\sigma(U^{-1}) \lambda_\sigma(L_1) \gamma \lambda_\sigma(L_1^{-1}) \| \xi_f \| \]
where \( o_i \) denotes the \( i \)th row of \( U^{-1} \) and \( l_{ij} \) denotes the \( i \)th row of \( L_1 \). From (4.18) we have
\[ \| \xi_f \| \leq \| \wedge_i \otimes I_n \| \| \eta_f \| \]
\[ \leq \lambda_\sigma(U) \| \eta_f \| \]
where \( \wedge_i \) is the \( i \)th row of \( U \). Then, it is easy to obtain
\[ \| \psi_i(x_f) \| \leq \gamma \lambda_\sigma(U^{-1}) \lambda_\sigma(L_1) \lambda_\sigma(L_1^{-1}) \lambda_\sigma(U) \sqrt{N} \| \eta_f \| \]
\[ \leq \frac{\varrho_0}{\sqrt{N}} \| \eta_f \| \]
\( \| \cdot \| \) denotes the Euclidean norm for vectors \( x \in \mathbb{R}^n \), defined by \( \| x \| = \sqrt{x^T x} \), and the induced norm corresponding to the vector Euclidean norm for matrices \( A \in \mathbb{R}^{m \times n} \), defined by \( \| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|} \). With the induced norm, the inequality \( \| \psi_i(x, x_0) \| \leq \| t_i \otimes I_n \| \| \Phi(x) - \Phi(x_0) \| \) holds. The bound in the lemma above will be used for the control gain design in the following theorem.

Theorem 4.3.1 For a network-connected nonlinear system (4.1) with the communication topology \( \mathcal{G} \), satisfying Lemma 4.2.1 and Assumption 4.2.1 to Assumption 4.2.3, the distributed control design (4.4) with \( K = B^T P \) solves the containment control problem if there exist a solution of \( P = P^T > 0 \) specified in one of the following two cases:
1. If the eigenvalues of the $L_1$ are distinct, i.e., $n_k = 1$ for $k = 1, \ldots, q$, the matrix $P$ satisfies

$$A^T P + PA - 2\alpha PBB^T P + \kappa PP + \frac{\theta_0}{\kappa} I_n < 0$$

with $\kappa$ being any positive real number and $\alpha = \min\{\lambda_1, \ldots, \lambda_p, \alpha_{p+1}, \ldots, \alpha_n\}$.

2. If matrix $L_1$ has multiple eigenvalues, i.e. $n_k > 1$ for any $k \in \{1, \ldots, q\}$, the matrix $P$ satisfies

$$A^T P + PA - 2(\alpha - 1)PBB^T P + \kappa PP + \frac{\theta_0}{\kappa} I_n$$

with $\kappa$ being any positive real number.

**Proof 4.3.2** Notice that Jordan block $J_f$ takes the form of (4.13), where within each real Jordan block $J_k$, for $k \leq p$, we have $i = N_{k-1} + 1, \ldots, N_k - 1$,

$$\dot{\eta}_{f_i} = (A - \lambda_i BK) \eta_{f_i} - BK \eta_{f_{i+1}} + \psi_i(x_f)$$

and

$$\dot{\eta}_{f_i} = (A - \lambda_i BK) \eta_{f_i} + \psi_i(x_f)$$

for $i = N_k$.

For the state variable associated with the Jordan blocks $J_k$ for $k > p$, corresponding to complex eigenvalues, we consider the dynamics of the state variables in pairs. For notational convenience, let

$$i_1(j) = N_{k-1} + 2j - 1$$

$$i_2(j) = N_{k-1} + 2j$$

for $j = 1, \ldots, n_k/2$. The dynamics of $\eta_{f_{i_1}}$ and $\eta_{f_{i_2}}$ for $j = 1, \ldots, n_k/2 - 1$ are expressed by

$$\dot{\eta}_{f_{i_1}} = (A - \alpha_k BK) \eta_{f_{i_1}} - \beta_k BK \eta_{f_{i_2}} - BK \eta_{f_{i_1+2}} + \psi_{i_1}(x_f)$$

$$\dot{\eta}_{f_{i_2}} = (A - \alpha_k BK) \eta_{f_{i_2}} + \beta_k BK \eta_{f_{i_1}} - BK \eta_{f_{i_2+2}} + \psi_{i_2}(x_f)$$

and

$$\dot{\eta}_{f_{i_1}} = (A - \alpha_k BK) \eta_{f_{i_1}} - \beta_k BK \eta_{f_{i_2}} + \psi_{i_1}(x_f)$$

$$\dot{\eta}_{f_{i_2}} = (A - \alpha_k BK) \eta_{f_{i_2}} + \beta_k BK \eta_{f_{i_1}} + \psi_{i_2}(x_f)$$
for $j = n_k/2$.

Let $W_i = \eta_i^TP\eta_i$. Choose $V_k = \sum_{j=1}^{n_k} \sigma^{2(j-1)}W_{j+n_k-1}$ for $k = 1, \ldots, p$ and $V_k = \sum_{j=1}^{n_k/2} \sigma^{2(j-1)}(W_{i(j)} + W_{i(j)})$ for $k = p + 1, \ldots, q$, where $\sigma > 0$. Then we consider the Lyapunov function $V = \sum_{k=1}^{n} V_k$. With $K = B^TP$, we have the following results:

Case 1. For the distinct eigenvalues, we can obtain

$$\dot{V} \leq \sum_{i=1}^{M} \eta_i^T \left( A^TP + PA - 2\alpha PBB^TP + \kappa PP + \frac{\rho_0}{\kappa} I_n \right) \eta_i. \quad (4.32)$$

The condition (4.29) guarantees $\dot{V} < 0$.

Case 2. For multiple eigenvalues, we can obtain

$$\dot{V} \leq \sum_{i=1}^{M} \eta_i^T \left[ A^TP + PA - 2\left(\alpha - \frac{1}{\sigma}\right) PBB^TP + \kappa PP + \frac{\rho_0}{\kappa} I_n \right] \eta_i. \quad (4.33)$$

with $\sigma = 1$. The condition (4.30) guarantees $\dot{V} < 0$. Hence we conclude that $\eta_i \to 0$ as $t \to \infty$, $\forall i = 1, \ldots, N$ as $t \to \infty$. Therefore, together with Lemma 4.2.1 and Assumption 4.2.3, containment is achieved. This completes the proof.

Remark 4.3.5 The conditions derived in (4.29)-(4.30) can be formulated as linear matrix inequalities [55]. It is difficult to get a possible solution from LMI routines, due to the nature of the containment connection topology, i.e. the number of connection that are working is more complex when compared to consensus connection topology. Hence, a solution from the Algebraic Riccati Equation (ARE) is obtained instead, and its feasibility can be checked.

4.3.2 Simulation and Discussion

In this section, an example is given to illustrate details of the state-feedback containment controller designed. The system under consideration is a connection of six subsystems with three leader subsystems, where each of them is described by the state space model as

$$\dot{x}_i = Ax_i + Bu_i + \phi(x_i) \quad (4.34)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (4.35)$$
and the nonlinear term is represented as

$$
\phi(x_i) = \begin{bmatrix}
0.05 \cos(Cx_i) \\
0
\end{bmatrix}
$$

(4.36)

Without the nonlinearity, the result is similar to [94]. The state-space dynamics are modified to make the system nonlinear, by having a class of nonlinearity injected into it in Lipschitz form.

![Figure 4.1: Containment configuration between the leaders 7 – 9 and the followers 1 – 6; dotted lines indicate that at least one leader is connected to the follower at one time.](image)

From Figure 4.1, Assumption 4.2.3 is obeyed, and the Laplacian matrix is obtained as

$$
\mathcal{L} =
\begin{bmatrix}
3 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(4.37)
where

\[
\mathcal{L}_1 = \begin{bmatrix}
3 & 0 & 0 & -1 & -1 & -1 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
-1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\] (4.38)

It can be verified that matrix pair \((A, B)\) is controllable. Based on Figure 4.1, a direct connection topology is applied, then the eigenvalues of \(\mathcal{L}_1\) are given as \(\{0.8213, 1, 2, 2.3329 \pm 0.6708j, 3.5129\}\) which are distinct - it can be seen that Lemma 4.2.1 is satisfied.

The Jordan matrix \(J\) corresponding to \(\mathcal{L}\) can be expressed as

\[
J = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.8213 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.3329 & 0.6708 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.6708 & 2.3329 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3.5129 & 0 \\
\end{bmatrix}
\] (4.39)

where

\[
J_f = \begin{bmatrix}
0.8213 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.3329 & 0.6708 & 0 \\
0 & 0 & 0 & -0.6708 & 2.3329 & 0 \\
0 & 0 & 0 & 0 & 3.5129 & 0 \\
\end{bmatrix}
\] (4.40)

The nonlinear function \(0.05 \sin(Cx_i)\) in the dynamic system is globally Lipschitz. The
values of the followers are set as

\[
\begin{align*}
x_1 &= [0.1; -0.1] \\
x_2 &= [0.2; -0.2] \\
x_3 &= [0.3; -0.3] \\
x_4 &= [0.4; -0.4] \\
x_5 &= [0.5; -0.5] \\
x_6 &= [0.6; -0.6]
\end{align*}
\] (4.41)

The leader values are set as

\[
\begin{align*}
x_7 &= [0.7; -0.7] \\
x_8 &= [0.8; -0.8] \\
x_9 &= [0.9; -0.9]
\end{align*}
\] (4.42)

Based on matrix \( \mathcal{L}_1 \), we have \( \alpha = 0.8213 \). From (4.34), we have

\[
\phi(x_i) = \begin{bmatrix}
\sin(Cx_i) \\
0_{N-1}
\end{bmatrix}
\] (4.43)

The Lipschitz constant is \( \gamma = 0.05 \), and the bound in Lemma 4.3.1 can be calculated as \( \varrho_0 = 7.5545 \times 10^{-7} \) and \( \kappa \) is set at 10. It is difficult to obtain the solution of \( P = P^T > 0 \) from the Linear Matrix Inequality (LMI) routine due to the the small value of \( \varrho_0 \). Hence, the solution of \( P \) is obtained from Algebraic Riccati Equation (ARE), and can be obtained as

\[
P = \begin{bmatrix}
0.0999 & 0.0054 \\
0.0054 & 0.1201
\end{bmatrix}
\] (4.44)

The controller gain \( K = B^T P \) is obtained from

\[
K = \begin{bmatrix}
0.0054 & 0.1201
\end{bmatrix}
\] (4.45)

It can be shown that the pair matrix \( (A,B) \) is controllable, and that the solution of \( P \) satisfies Lemma 3.3.3 (pg.58). Hence we can obtain the Hamiltonian matrix

\[
H = \begin{bmatrix}
0 & 1.0000 & -10.0000 & 0 \\
0 & 0 & 0 & -8.3574 \\
-1.0000 & 0 & 0 & 0 \\
0 & -1.0000 & -1.0000 & 0
\end{bmatrix}
\] (4.46)
Figure 4.2: Plot of containment of followers’ substates 1 (normal lines) with respect to the leaders (dotted lines), under influence of Lipschitz nonlinearity at $\gamma = 0.05$.

Figure 4.3: Plot of containment of followers’ substates 2 (normal lines) with respect to the leaders (dotted lines), under influence of Lipschitz nonlinearity at $\gamma = 0.05$. 
which provides no eigenvalues on the imaginary axis, with $-3.0632 + 0.4524i$, $-3.0632 - 0.4524i$, $3.0632 + 0.4524i$, and $3.0632 - 0.4524i$.

\[
\begin{bmatrix}
-0.2122 + 0.4472i \\
-0.2122 - 0.4472i \\
\end{bmatrix}
\]

where it can be seen that the real part of the eigenvalues are at the left-hand-side of the plane, and considered to be Hurwitz-stable, with $\lambda_i = 0.8231$ as the minimum.
eigenvalue of the Laplacian matrix $L$.

Figure 4.5: Plot of containment of followers’ substates 1 (normal lines) with respect to the leaders (dotted lines), under influence of Lipschitz nonlinearity at $\gamma = 0.07$.

When nonlinearities are present in the system, the conditions (4.29) and (4.30) are sufficient for the control gain to achieve containment control. However, these conditions can be conservative for a given Lipschitz nonlinear function in the control gain designed. When the nonlinearity constant $\gamma$ is increased to 0.07, the containment controller was able to achieve containment for the first substates of the followers, as shown in Figure 4.5. On the other hand, containment was not achieved for the second substates of the followers. The followers’ signal went slightly above the bound set by the leaders, as shown in Figure 4.6. For $\gamma = 0$, containment has not been realized for substates 2 of each subsystem but it remained stable, and oscillated within the bound of the leaders as $t \to \infty$. 
Figure 4.6: Plot of containment of followers’ substates 2 (normal lines) with respect to the leaders (dotted lines), under influence of Lipschitz nonlinearity at $\gamma = 0.07$.

Figure 4.7: Plot of containment of followers’ substates 1 (normal lines) with respect to the leaders (dotted lines), under influence of Lipschitz nonlinearity at $\gamma = 0.2$. 
4.3.3 Conclusion

In this section, a state-feedback containment consensus controller was proposed for multi-agent systems with Lipschitz nonlinearity, in networks subject to a directed spanning forest topology. The controller managed to contain the subsystems’ states to be within the boundary of the leaders’ states. This was achieved with certain measure of nonlinearity, where it can be considered as conservative but sufficient for the system to achieve containment.

4.4 Containment Observer

An observer is useful when states measurements are unobtainable, and need to be estimated. As an example, the Luenberger Observer takes the value of the error between the actual states and the observed states. This is done by comparing the output of each system and the actual states and giving a high gain value to the observer matrix. Similar to the description in Chapter 3, an observer is used to estimate the
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value of the states which is then applied in the control strategy - described in the next section. Before we proceed, Assumption 4.4.1 is added to Assumption 4.2.1 and to Assumption 4.2.3.

Assumption 4.4.1 Matrices \((A, C)\) is observable.

For a nonlinear system (4.1), an observer for each subsystem is proposed as

\[
\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + L \sum_{j \in \mathcal{F} \cup \mathcal{R}} l_{ij} C(x_j - \hat{x}_j) \tag{4.48}
\]

where \(\hat{x}_i \in \mathbb{R}^n\) for \(i = 1, \ldots, N\) is the estimated state of each subsystem (follower), and \(L \in \mathbb{R}^{n \times q}\) is the observer matrix to be determined later. \(j \in \mathcal{F} \cup \mathcal{R}\) means the number of \(j\) that corresponds to the followers.

The estimation error dynamics are obtained as

\[
\dot{\tilde{x}}_i = A\tilde{x}_i + \phi(x_i) - \phi(\hat{x}_i) - L \sum_{j \in \mathcal{F} \cup \mathcal{R}} l_{ij} C\tilde{x}_j \tag{4.49}
\]

where \(\tilde{x}_j = x_j - \hat{x}_j\). The network representation of (4.49) is denoted as

\[
\dot{\tilde{x}} = (I_N \otimes A - \mathcal{L} \otimes LC) \tilde{x} + \Phi(x) - \Phi(\hat{x}) \tag{4.50}
\]

Next a transformation is introduced as

\[
\tilde{\xi} = (\mathcal{L} \otimes I_n) \tilde{x} \tag{4.51}
\]

where \(\tilde{x} = [\tilde{x}_1^T, \ldots, \tilde{x}_N^T]^T\) and \(\tilde{\xi} = [\tilde{\xi}_1^T, \ldots, \tilde{\xi}_N^T]^T\). Note that, with \(\tilde{\xi} = [\tilde{\xi}_f^T, \tilde{\xi}_l^T]^T\) where \(\tilde{\xi}_f^T = [\tilde{\xi}_1^T, \ldots, \tilde{\xi}_M^T]^T\) and \(\tilde{\xi}_l^T = [\tilde{\xi}_{M+1}^T, \ldots, \tilde{\xi}_N^T]^T\), and with Laplacian \(\mathcal{L}\), which is partitioned as

\[
\mathcal{L} = \begin{bmatrix}
\mathcal{L}_1 & \mathcal{L}_2 \\
0_{(N-M) \times M} & 0_{(N-M) \times (N-M)}
\end{bmatrix} \tag{4.52}
\]

we have

\[
\dot{\tilde{\xi}}_f = (\mathcal{L}_1 \otimes I_m) \tilde{x}_f + (\mathcal{L}_2 \otimes I_m) \tilde{x}_l \tag{4.53}
\]

From (4.50) and (4.51), we have

\[
\dot{\tilde{\xi}} = (I_N \otimes A - \mathcal{L} \otimes LC) \tilde{\xi} + (\mathcal{L} \otimes I_n) (\Phi(x) - \Phi(\hat{x})) \tag{4.54}
\]
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Considering the structure of $\mathcal{L}$ in (4.52), we can see from (4.53) and (4.54), that $\dot{\xi}_f$ satisfies the following dynamics

$$\dot{\xi}_f = (\mathcal{L}_1 \otimes I_m) (I_M \otimes A - \mathcal{L}_1 \otimes LC) \dot{x}_f - (\mathcal{L}_1 \otimes I_m)(\mathcal{L}_2 \otimes LC) \dot{x}_l + (\mathcal{L}_1 \otimes I_m)(\Phi(x_f) - \Phi(\dot{x}_f))$$

$$+ (\mathcal{L}_2 \otimes I_m)(I_{N-M} \otimes A) \dot{x}_l + (\mathcal{L}_2 \otimes I_m)(\Phi(x_l) - \Phi(\dot{x}_l))$$

$$= (I_M \otimes A - \mathcal{L}_1^2 \otimes LC) \dot{x}_f - (\mathcal{L}_1 \mathcal{L}_2 \otimes LC) \dot{x}_l + (\mathcal{L}_1 \otimes I_m)(\Phi(x_f) - \Phi(\dot{x}_f))$$

$$+ (\mathcal{L}_2 \otimes A) \dot{x}_l + (\mathcal{L} \otimes I_m)(\Phi(x_l) - \Phi(\dot{x}_l))$$

$$= (I_M \otimes A - \mathcal{L}_1 \otimes LC) \dot{x}_f + (\mathcal{L}_1 \otimes I_m)(\Phi(x_f) - \Phi(\dot{x}_f))$$

$$+ (\mathcal{L}_2 \otimes I_m)(\Phi(x_l) - \Phi(\dot{x}_l))$$

(4.55)

Remark 4.4.1 Another transformation is required to analyze the stability of system (4.55). This is done by manipulating the structure of $\mathcal{L}$. Similar to the previous section, there is no direct transformation of (4.55). Hence, the transformation of system (4.55) has to be from (4.54).

In other words, we have

$$\begin{bmatrix} \dot{\xi}_f \\ \dot{\xi}_l \end{bmatrix} = \begin{bmatrix} I_M \otimes A & 0 \\ 0 & I_{N-M} \otimes A \end{bmatrix} \begin{bmatrix} \xi_f \\ \xi_l \end{bmatrix}$$

$$- \begin{bmatrix} \mathcal{L}_1 \otimes LC & \mathcal{L}_2 \otimes LC \\ 0_{(N-M) \times M} \otimes 0_{M \times M} & 0_{(N-M) \times (N-M)} \otimes 0_{M \times M} \end{bmatrix} \begin{bmatrix} \dot{\xi}_f \\ \dot{\xi}_l \end{bmatrix}$$

$$+ \begin{bmatrix} \mathcal{L}_1 \otimes I_m & \mathcal{L}_2 \otimes I_m \\ 0_{(N-M) \times M} \otimes I_m & 0_{(N-M) \times (N-M)} \otimes I_m \end{bmatrix} \begin{bmatrix} \Phi(x_f) - \Phi(\dot{x}_f) \\ \Phi(x_l) - \Phi(\dot{x}_l) \end{bmatrix}$$

(4.56)

From above, we can obtain

$$\dot{\xi}_f = (I_M \otimes A - \mathcal{L}_1 \otimes LC) \dot{\xi}_f - (\mathcal{L}_2 \otimes LC) \dot{\xi}_l$$

$$+ (\mathcal{L}_1 \otimes I_m) (\Phi(x_f) - \Phi(\dot{x}_f)) + (\mathcal{L}_2 \otimes I_m) (\Phi(x_l) - \Phi(\dot{x}_l))$$

(4.57)

and

$$\dot{\xi}_l = (I_{N-M} \otimes A) \dot{\xi}_l$$

(4.58)
Another transformation is then introduced as
\[
\tilde{\eta} = (T^{-1} \otimes I_n) \tilde{\xi}
\] (4.59)
and for the nonlinear terms
\[
\Psi(x, \hat{x}) = (T^{-1} \otimes I_n)(\Phi(x) - \Phi(\hat{x}))
\] (4.60)
we have
\[
\begin{bmatrix}
\dot{\tilde{\eta}}_f \\
\dot{\tilde{\eta}}_l
\end{bmatrix} =
\begin{bmatrix}
I_M \otimes A & 0 \\
0 & I_{N-M} \otimes A
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_f \\
\tilde{\eta}_l
\end{bmatrix}
- \begin{bmatrix}
J_f \otimes LC & 0 \\
0 & 0_{(N-M) \times (N-M)} \otimes 0_{M \times M}
\end{bmatrix}
\begin{bmatrix}
\tilde{\eta}_f \\
\tilde{\eta}_l
\end{bmatrix}
+ \begin{bmatrix}
\Psi(x_f, \hat{x}_f) \\
\Psi(x_l, \hat{x}_l)
\end{bmatrix}
\] (4.61)
where
\[
\begin{bmatrix}
\Psi(x_f, \hat{x}_f) \\
\Psi(x_l, \hat{x}_l)
\end{bmatrix} =
(T^{-1} \otimes I_n)
\begin{bmatrix}
\mathcal{L}_1 \otimes I_m & \mathcal{L}_2 \otimes I_m \\
0_{(N-M) \times M} \otimes I_m & 0_{(N-M) \times (N-M)} \otimes I_m
\end{bmatrix}
\begin{bmatrix}
\Phi(x_f) - \Phi(\hat{x}_f) \\
\Phi(x_l) - \Phi(\hat{x}_l)
\end{bmatrix}
\] (4.62)
and
\[
(T^{-1} \otimes I_n) = \begin{bmatrix}
U^{-1} \otimes I_m & 0_{M \times (N-M)} \\
0_{(N-M) \times M} & 0_{(N-M) \times (N-M)} \otimes I_m
\end{bmatrix}
\] (4.63)
Hence we obtain
\[
\dot{\tilde{\eta}}_f = (I_M \otimes A - J_f \otimes LC) \tilde{\eta}_f + \Psi(x_f, \hat{x}_f)
\] (4.64)
and
\[
\dot{\tilde{\eta}}_l = (I_{N-M} \otimes A) \tilde{\eta}_l + \Psi(x_l, \hat{x}_l)
\] (4.65)
where
\[
J = \begin{bmatrix}
J_f & 0_{M \times (N-M)} \\
0_{(N-M) \times M} & 0_{(N-M) \times (N-M)}
\end{bmatrix}
\] (4.66)
The estimation error dynamics (4.64) is a diagonally-dominant matrix, which will facilitate the stability analysis. Through further identification of matrix $J_f$ from $J$, 

it is possible to get the eigenvalues later which will be needed for the design of the observer gain $L$.

Note that under Assumption 4.2.3, it follows from Lemma 4.2.1 (pg.98) that all of the eigenvalues of $L_1$ have positive real parts. Let $U \in \mathbb{R}^{M \times M}$ be a nonsingular matrix $U^{-1}L_1U = J_f$, with $\lambda_i = 1, \ldots, M$ as its diagonal entries. Hence, in order to further manipulate the special characteristic of $L_1$, it is possible to obtain (4.18) and (4.19) with respect to $\tilde{\eta}_f$ and $\Psi(x_f, \hat{x}_f)$ as

\[
\tilde{\eta}_f = (U^{-1} \otimes I_m)\tilde{\xi}_f
\]  

and

\[
\Psi(x_f, \hat{x}_f) = (U^{-1} \otimes I_m)(L_1 \otimes I_m)[\Phi(x_f) - \Phi(\hat{x}_f)]
\]  

Then (4.67) and (4.68) are used for the stability analysis.

### 4.4.1 Stability analysis

**Lemma 4.4.1** For nonlinear element $\psi_i(x_f) - \psi_i(\hat{x}_f)$ of the nonlinear term $\Psi(x_f, \hat{x}_f)$ in the transformed closed loop network dynamics (4.64), a bound can be established in terms of the state $\tilde{\eta}_f$, as shown by

\[
\|\psi_i(x_f) - \psi_i(\hat{x}_f)\| \leq \frac{\varrho_0}{\sqrt{N}}\tilde{\eta}_f
\]  

with

\[
\varrho_0 = \gamma \lambda_\sigma(U^{-1})\lambda_\sigma(L_1)\lambda_\sigma(L_1^{-1})\lambda_\sigma(U)\sqrt{N}
\]

where $\lambda_\sigma(\cdot)$ denotes the singular value of a matrix.

**Proof 4.4.1** The approach of the derivation of (4.69) and (4.70) is similar to the previous section. From (4.68) we have

\[
\|\psi_i(x_f) - \psi_i(\hat{x}_f)\| \leq \|a_i \otimes I_n\|\|I_{ij} \otimes I_n\|\|\Phi(x_f) - \Phi(\hat{x}_f)\|
\]

\[
\leq \lambda_\sigma(U^{-1})\lambda_\sigma(L_1)\gamma\|x_f - \hat{x}_f\|
\]

\[
\leq \lambda_\sigma(U^{-1})\lambda_\sigma(L_1)\gamma\|L_1^{-1} \otimes I_m\|\|\tilde{\xi}_f\|
\]

\[
\leq \lambda_\sigma(U^{-1})\lambda_\sigma(L_1)\gamma\lambda_\sigma(L_1^{-1})\|\tilde{\xi}_f\|
\]
where \( o_i \) denotes the \( i \)th row of \( U^{-1} \) and \( l_{ij} \) denotes the \( i \)th row of \( L_1 \). From (4.67) we have

\[
\|\tilde{\xi}_f\| \leq \|\landa_i \otimes I_n\|\|\tilde{\eta}_f\|
\leq \lambda_{\sigma}(U)\|\tilde{\eta}_f\|
\]

where \( \landa_i \) is the \( i \)th row of \( U \). Then, it is easy to get

\[
\|\psi_i(x_f) - \psi_i(\hat{x}_f)\| \leq \frac{\gamma\lambda_{\sigma}(U^{-1})\lambda_{\sigma}(L_1)\lambda_{\sigma}(L_1^{-1})\lambda_{\sigma}(U)\sqrt{N}}{\sqrt{N}}\|\tilde{\eta}_f\|
\leq \frac{\varrho_0}{\sqrt{N}}\|\tilde{\eta}_f\|
\]

The bound in the lemma above will be used for the observer gain design in the following theorem.

**Theorem 4.4.1** For a network-connected dynamic system (4.1) with communication topology \( G \), satisfying Assumptions with \( L = P^{-1}C^T \), an asymptotically-stable observer exists for the follower states \( x_f \) of the system (4.1), if there exists a solution \( P = P^T > 0 \) specified in the following of the two cases:

1. If the eigenvalues of the matrix \( L_1 \) are distinct, i.e., \( n_k = 1 \) for \( k = 1, \ldots, M \) the matrix \( P = P^T > 0 \) satisfies

\[
A^T P + PA - 2\alpha C^T C + \kappa PP + \frac{\varrho_0^2}{\kappa} I_n < 0 \quad (4.71)
\]

with \( \kappa \) being any positive real number, and \( \alpha = \min\{\lambda_1, \ldots, \lambda_M, \alpha_{M+1}, \ldots, \alpha_N\} \).

2. If the matrix \( L_1 \) has multiple eigenvalues, i.e., \( n_k > 1 \) for any \( k \in \{1, \ldots, M\} \), the matrix \( P = P^T > 0 \) satisfies

\[
A^T P + PA - 2(\alpha - 1)C^T C + \kappa PP + \frac{\varrho_0^2}{\kappa} I_n < 0 \quad (4.72)
\]

with \( \kappa \) being any positive real number.

**Proof 4.4.2**

\[
\dot{\tilde{\eta}}_f = (A - \lambda_i LC)\tilde{\eta}_f + \psi_i(x_f) - \psi_i(\hat{x}_f) \quad (4.73)
\]

where \( \lambda_i \) is the eigenvalues of \( J_f \) for \( i = 1, \ldots, M \). Let the Lyapunov candidate be

\[
V_i = \tilde{\eta}_f^T P \tilde{\eta}_f^T, \quad i = 1, \ldots, M \quad (4.74)
\]
and
\[ \psi_i(x_f, \hat{x}_f) = \psi_i(x_f) - \psi_i(\hat{x}_f) \]  \hspace{1cm} (4.75)

For \( k \leq p \) we have
\[
\dot{\tilde{\eta}}_{f_i} = (A - \lambda_i LC)\tilde{\eta}_{f_i} - LC\tilde{\eta}_{f_{i+1}} + \psi_i(x_f, \hat{x}_f)
\]  \hspace{1cm} (4.76)

with \( i = N_{k-1} + 1, \ldots, N_k - 1 \) and
\[
\dot{\tilde{\eta}}_{f_i} = (A - \lambda_i LC)\tilde{\eta}_{f_i} + \psi_i(x_f, \hat{x}_f)
\]

for \( i = N_k \).

For \( k > p \), we have
\[
\dot{\tilde{\eta}}_{f_{i_1}} = (A - \alpha_k LC)\tilde{\eta}_{f_{i_1}} - \beta_k LC\tilde{\eta}_{f_{i_2}} - LC\tilde{\eta}_{f_{i_1+2}} + \psi_i(x_f, \hat{x}_f)
\]
\[
\dot{\tilde{\eta}}_{f_{i_2}} = (A - \alpha_k LC)\tilde{\eta}_{f_{i_2}} + \beta_k LC\tilde{\eta}_{f_{i_1}} - LC\tilde{\eta}_{f_{i_2+2}} + \psi_{f_{i_2}}(x_f, \hat{x}_f)
\]

with \( j = 1, \ldots, n_{k/2} - 1 \) and
\[
\dot{\tilde{\eta}}_{f_{i_1}} = (A - \alpha_k LC)\tilde{\eta}_{f_{i_1}} - \beta_k LC\tilde{\eta}_{f_{i_2}} + \psi_{f_{i_1}}(x_f, \hat{x}_f)
\]
\[
\dot{\tilde{\eta}}_{f_{i_2}} = (A - \alpha_k LC)\tilde{\eta}_{f_{i_2}} + \beta_k LC\tilde{\eta}_{f_{i_1}} + \psi_{f_{i_2}}(x_f, \hat{x}_f)
\]

for \( j = n_{k/2} \).

Let \( W_i = \tilde{\eta}_{f_i}^T P \tilde{\eta}_{f_i} \). Choose \( V_k = \sum_{j=1}^{n_k} \sigma^{2(j-1)} W_{j+N_{k-1}} \) for \( k = 1, \ldots, p \) and \( V_k = \sum_{j=1}^{n_{k/2}} \sigma^{2(j-1)} (W_{i_{1(j)}} + W_{i_{2(j)}}) \) for \( k = p+1, \ldots, q \), where \( \sigma > 0 \). Then we consider the Lyapunov function \( V = \sum_{i=1}^{q} V_k \); with \( \sigma = 1 \) and \( L = P^{-1}C^T \), we have the following results.

**Case 1:** For distinct eigenvalues, we obtain
\[
\dot{V} \leq M \sum_{i=1}^{M} \tilde{\eta}_{f_i}^T \left( A^T P + PA - 2\alpha C^T C + \kappa PP + \frac{\rho_0}{\kappa} I_m \right) \tilde{\eta}_{f_i}
\]

**Case 2:** For multiple eigenvalues, we obtain
\[
\dot{V} \leq M \sum_{i=1}^{M} \tilde{\eta}_{f_i}^T \left[ A^T P + PA - 2(\alpha - 1)C^T C + \kappa PP + \frac{\rho_0}{\kappa} I_m \right] \tilde{\eta}_{f_i}
\]

Therefore, conditions (3.50) and (3.51) guarantee that \( \dot{V} < 0 \) for both cases. Hence we conclude that \( \tilde{\eta}_{f_i}(t) \to 0, \forall i = 1, \ldots, N \) as \( t \to \infty \). This completes the proof.
If there is a solution of \( P = P^T > 0 \), observer (4.48) is stable and able to track the original state signal as shown in Theorem 4.4.1. Without the nonlinear terms \( \Phi(x) - \Phi(\hat{x}) \), the observed follower agents’ states will be able to track the leader states \( (L^{-1} L \otimes I_m)x_l \) as \( t \to \infty \), as shown in [67]. With the inclusion of the nonlinear terms \( \Phi(x) - \Phi(\hat{x}) \), Lyapunov stability analysis was used to establish a stable condition for the containment observer for each subsystem. With a stable containment observer, the observed signal of each subsystem is guaranteed to follow the original state signal. The signal observed will now be used in the containment controller later in the chapter.

4.4.2 Simulation and Discussion

For the following simulation example, the state space (4.34) used is the same as that in the previous section. It can be seen that the pair of matrices \((A, C)\) are now observable; the connection topology for the subsystems are the same as in Figure 4.1, which results in the same Laplacian matrix \( L \) in (4.37), and the partition of \( L_1 \) in (4.38). Note that the eigenvalues of the Laplacian matrix \( L_1 \) are distinct, with \( \{0.8213, 1, 2, 2.3329 \pm 0.6708j, 3.5129\} \) with \( \alpha = 0.8213 \).

As previously mentioned, it is difficult to obtain the solution of \( P = P^T > 0 \) from the standard LMI routine. Thus, a solution of \( P = P^T > 0 \) from ARE is obtained as

\[
P = \begin{bmatrix} 0.5382 & -0.0509 \\ -0.0509 & 0.2948 \end{bmatrix}
\]

(4.77)

and observer gain \( L = P^{-1}C^T \) is chosen and obtained as

\[
L = \begin{bmatrix} 1.8889 \\ 0.3259 \end{bmatrix}
\]

(4.78)

where \( \kappa = 0.0025 \) is used, and from (4.1), \( \gamma = 0.05 \). The initial values of the leader and followers’ subsystems also need to be changed. The new values for the followers
are set as
\[
\begin{align*}
x_1 &= [0.1; -0.1] \\
x_2 &= [0.2; -0.2] \\
x_3 &= [0.3; -0.3] \\
x_4 &= [0.4; -0.4] \\
x_5 &= [0.5; -0.5] \\
x_6 &= [0.6; -0.6]
\end{align*}
\] (4.79)

The new leader values are set as
\[
\begin{align*}
x_7 &= [0.7; -0.7] \\
x_8 &= [0.8; -0.8] \\
x_9 &= [0.9; -0.9]
\end{align*}
\] (4.80)

Without the nonlinear term \( \phi(x_i) - \phi(\hat{x}_i) \), we obtained
\[
eigenvalue(A - \alpha LC) = \begin{bmatrix} -1.6968 \\ -0.1921 \end{bmatrix}
\]

which is clearly on the left side of the plane, and thus Hurwitz-stable. This means that the observer is stable and able to estimate the states in the subsystem without the nonlinear terms \( \phi(x_i) - \phi(\hat{x}_i) \). With the nonlinear term \( \phi(x_i) - \phi(\hat{x}_i) \), the Lyapunov method is executed to provide a solution of \( P = P^T > 0 \). If a solution of \( P = P^T > 0 \) exists, a stable observer is also possible which estimates the states of the system. Figure 4.9 shows that the estimated signal \( \hat{x}_i \) tracking the state \( x_i \) when \( \gamma = 0.05 \) for subsystem 1. When \( \gamma \) is increased to 0.3, the tracking becomes slightly slower, and the trajectory of the signal changes by a small amount, as shown in Figure 4.10. When \( \gamma \) is increased to 1, a larger change in trajectory is observed, with a longer tracking action of the estimated state \( \hat{x}_i \) onto state \( x_i \), as shown in Figure 4.11; this happens to all subsystems within the group.

### 4.4.3 Conclusion

In this section, a containment observer was proposed, based on Luenberger observer, for multi-agent systems with Lipschitz nonlinearity, in networks subject to a directed
Figure 4.9: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.05$

Figure 4.10: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.3$
spanning forest topology. Based on the structure of the Laplacian matrix, the observer gain $L$ is selected by getting a solution of $P = P^T > 0$ from Lyapunov stability analysis. The solution of $P = P^T > 0$ for the observer design was provided from Algebraic Riccati Equation (ARE) standard routine. It was observed that the estimated substates were able to track the substates under certain amount of nonlinearity influence. Hence, this observer is going to be used in the controller that is going to be designed in the next section.

### 4.5 Observer-based Containment Controller

The containment controller, with an output-feedback-based or containment-observer-based controller, utilizes the value obtained from the observer that has been designed in the previous section. As previously mentioned, if the exact state value is difficult to obtain, an observer is a useful tool to estimate the exact value of the states.
Hence, with the system (4.1), the output feedback consensus controller takes the following structure:

\[ u_i = -K(\hat{x}_i - x_i) \]  

(4.81)

where \( \hat{x}_i \in \mathbb{R}^n \) for \( i = 1, \ldots, N \) are the estimated states, \( x_i \in \mathbb{R}^n \) is the states of the leader, and the consensus controller gain \( K \in \mathbb{R}^{p \times n} \) is to be determined later. With system (4.1) and (4.81), system (4.1) can be rewritten as

\[ \dot{x}_i = (A - BK)x_i + BKx_l + BK\hat{x}_i + \phi(x_i) \]  

(4.82)

With (4.82), and by having \( \zeta = x_i - x_l \) the augmented closed-loop network dynamics can be stacked into a compact form as

\[
\begin{bmatrix}
\dot{\zeta} \\
\dot{\tilde{\xi}}
\end{bmatrix} = 
\begin{bmatrix}
I_N \otimes (A - BK) & I_N \otimes BK \\
0 & I_N \otimes A - \mathcal{L} \otimes LC
\end{bmatrix}
\begin{bmatrix}
\zeta \\
\tilde{\xi}
\end{bmatrix} 
+ 
\begin{bmatrix}
\Phi(x) - \Phi(x_l) \\
\Phi(x) - \Phi(\hat{x})
\end{bmatrix}
\]

(4.83)

Note that this approach is similar that used in the consensus with output feedback controller design in Section 3.5 (pg.79). Two transformations are now introduced \( \xi = (\mathcal{L} \otimes I_n)\zeta \), and \( \tilde{\xi} = (\mathcal{L} \otimes I_n)\tilde{x} \), and the overall followers network dynamics can be written as

\[
\begin{bmatrix}
\dot{\xi}_f \\
\dot{\tilde{\xi}}_f
\end{bmatrix} = 
\begin{bmatrix}
I_M \otimes (A - BK) & I_M \otimes BK \\
0 & I_M \otimes A - \mathcal{L}_1 \otimes LC
\end{bmatrix}
\begin{bmatrix}
\xi_f \\
\tilde{\xi}_f
\end{bmatrix} 
+ 
\begin{bmatrix}
\Phi(x_f) - \Phi(x_l) \\
\Phi(x_f) - \Phi(\hat{x}_f)
\end{bmatrix}
\]

(4.84)

By using the same transformation (4.59) for \( \tilde{\eta}_f \), the transformed network dynamics are as follows:

\[
\begin{bmatrix}
\dot{\tilde{\xi}}_f \\
\dot{\tilde{\eta}}_f
\end{bmatrix} = 
\begin{bmatrix}
I_M \otimes (A - BK) & I_M \otimes BK \\
0 & I_M \otimes A - J_f \otimes LC
\end{bmatrix}
\begin{bmatrix}
\xi_f \\
\tilde{\eta}_f
\end{bmatrix} 
+ 
\begin{bmatrix}
\Psi(x_f) - \Psi(x_l) \\
\Psi(x_f, \hat{x}_f)
\end{bmatrix}
\]

(4.85)

Similar to the previous section, the estimation error dynamics in (4.85) are now turned into a diagonally-dominant matrix that will assist the stability analysis. The stability
analysis from the observer depends on the analysis of the stability of the controller of the follower, as described in the same equation.

### 4.5.1 Stability analysis

The stability analysis comprises two parts - the controller and the observer. In linear system, the separation principle is observed in the implementation of observer-based control. In nonlinear systems, the separation principle does not hold; here the Lyapunov method is used to analyse the stability of the nonlinear system. Hence, a summary of the results is stated in the following theorem.

**Theorem 4.5.1** For a network-connected dynamic system (4.1) with the associated Laplacian matrix that satisfies Assumptions 4.2.1 to Assumption 4.2.3 and Assumption 4.4.1, together with Lemma 4.2.1, the observer-based containment controller (4.81) with \( K = B^TP_1 \), and \( L = P_2^{-1}C^T \), solve the containment problem if there exist solutions \( P_1 = P_1^T > 0 \) and \( P_2 = P_2^T > 0 \) specified in one of the following cases:

1. If the eigenvalues of the matrix \( L_1 \) are distinct, i.e., \( n_k = 1 \) for \( k = 1, \ldots, q \), the matrices \( P_1 \) and \( P_2 \) satisfy

\[
A^T P_1 + P_1 A - P_1 BB^T P_1 + \kappa_1 P_1 P_1 + \frac{\gamma^2}{\kappa_1} I_n < 0
\]

and

\[
A^T P_2 + P_2 A - 2\alpha C^T C + \kappa_2 P_2 P_2 + \frac{\gamma^2}{\kappa_2} I_n + P_1 BB^T P_1 < 0
\]

with \( \kappa_1 \) and \( \kappa_2 \) being any positive real numbers, \( \alpha = \min\{\lambda_1, \ldots, \lambda_p, \alpha_{p+1}, \ldots, \alpha_q\} \), \( \gamma \) defined in (2.31, pg.44) and \( \gamma_0 \) defined in (4.70, pg.118).

2. If multiple eigenvalues exist for the matrix \( L_1 \), i.e., \( n_k > 1 \) for any \( k \in \{1, \ldots, q\} \), the matrices \( P_1 \) and \( P_2 \) satisfy

\[
A^T P_1 + P_1 A - P_1 BB^T P_1 + \kappa_1 P_1 P_1 + \frac{\gamma^2}{\kappa_1} I_n < 0
\]

and

\[
A^T P_2 + P_2 A - 2(\alpha - 1) C^T C + \kappa_2 P_2 P_2 + \frac{\gamma_0^2}{\kappa_2} I_n + P_1 BB^T P_1 < 0
\]

with \( \kappa_1 \) and \( \kappa_2 \) being any positive real numbers.
\textbf{Proof 4.5.1} Let $V_1 = \sum_{i=1}^{M} \xi_i^T P_1 \xi_i$ and taking the derivative of $V_1$ along (4.85) yields

$$\dot{V}_1 = \sum_{i=1}^{M} [(A - BK)\xi_i + BK\tilde{\eta}_i + \phi(x_i) - \phi(x_i)]^T P_1 \xi_i$$

$$+ \sum_{i=1}^{M} \xi_i^T P_1 [(A - BK)\xi_i + BK\tilde{\eta}_i + \phi(x_i) - \phi(x_i)]$$

$$= \sum_{i=1}^{M} \xi_i^T (A^T P_1 + P_1 A - 2P_1 BK)\xi_i + 2\xi_i^T P_1 BK \tilde{\eta}_i$$

$$+ \sum_{i=1}^{M} 2\xi_i^T P_1 (\phi(x_i) - \phi(x_i))$$

$$\leq \sum_{i=1}^{M} \xi_i^T (A^T P_1 + P_1 A - P_1 BB^T P_1 + \kappa_1 P_1)\xi_i + \sum_{i=1}^{M} \tilde{\eta}_i^T P_1 BB^T P_1 \tilde{\eta}_i$$

$$+ \sum_{i=1}^{M} \frac{1}{\kappa_1} \|\phi(x_i) - \phi(x_i)\|^2$$

$$\leq \sum_{i=1}^{M} \xi_i^T (A^T P_1 + P_1 A - P_1 BB^T P_1 + \kappa_1 P_1 + \frac{\gamma^2}{\kappa_1} C_m)\xi_i$$

$$+ \sum_{i=1}^{M} \tilde{\eta}_i^T P_1 BB^T P_1 \tilde{\eta}_i$$

where $K = B^T P_1$, and we have used the inequality $2a^T b \leq \kappa a^T a + b^T b/\kappa$ for vectors $a$ and $b$ with the same dimension.

Let the Lyapunov function be $V = V_1 + V_2$ with $V_2 = \nabla V$; using Theorem 4.5.1, two cases are considered for stability analysis, respectively.

\textbf{Case 1:} If the eigenvalues of the matrix $L_1$ are distinct, we have

$$\dot{V} \leq \sum_{i=1}^{M} \xi_i^T (A^T P_1 + P_1 A - P_1 BB^T P_1 + \kappa_1 P_1 + \frac{\gamma^2}{\kappa_1} C_m)\xi_i$$

$$+ \sum_{i=1}^{M} \tilde{\eta}_i^T \left( A^T P + PA - 2\alpha C^T C + \kappa PP + \frac{\phi_0}{\kappa} I_m \right) \tilde{\eta}_i$$

$$+ \sum_{i=1}^{M} \tilde{\eta}_i^T P_1 BB^T P_1 \tilde{\eta}_i$$

$$= \sum_{i=1}^{M} \xi_i^T M_1 \xi_i + \sum_{i=1}^{M} \tilde{\eta}_i^T M_2 \tilde{\eta}_i$$

where

$$M_1 = A^T P_1 + P_1 A - P_1 BB^T P_1 + \kappa_1 P_1 + \frac{\gamma^2}{\kappa_1} C_m$$

$$M_2 = A^T P_2 + P_2 A - 2\alpha C^T C + \kappa_2 P_2 + \frac{\phi_0}{\kappa} I_n + P_1 BB^T P_1$$
Conditions 4.86 and 4.87 guarantee \( \dot{V} < 0 \) for \( \kappa_1 > 0, \kappa_2 > 0, \gamma \) defined in 2.31 (pg.44) and \( \gamma_0 \) defined in (3.47), which implies \( \xi_{f_i}(t) \to 0 \) and \( \tilde{\eta}_{f_i}(t) \to 0 \) as \( t \to \infty \) for all \( i \).

Case 2: If the matrix \( \mathcal{L}_1 \) has multiple eigenvalues, we similarly have

\[
\dot{V} \leq \sum_{i=1}^{M} \xi_{f_i}^T M_1 \xi_{f_i} + \sum_{i=1}^{M} \tilde{\eta}_{f_i}^T \overline{M}_2 \tilde{\eta}_{f_i}
\]

(4.92)

where

\[
\overline{M}_2 = A^T P_2 + P_2 A - 2(\alpha - 1) C^T C + \kappa_2 P_2 P_2 + \frac{\gamma_0^2}{\kappa_2} I_n + P_1 B B^T P_1
\]

Conditions (4.88) and (4.89) guarantee \( \dot{V} < 0 \), which implies \( \xi_{f_i}(t) \to 0 \) and \( \tilde{\eta}_{f_i}(t) \to 0 \) as \( t \to \infty \) for all \( i \). Hence, leader-follower containment is achieved. This completes the proof.

4.5.2 Simulation and Discussion

In this section, an example is shown which gives details of the containment observer-based control design. For this example, the state-feedback containment controller is now replaced with an observer-based containment controller. The system under consideration is the same as the system (4.35).

The solution of \( P_1 = P_1^T > 0 \) is obtained from ARE as

\[
P_1 = \begin{bmatrix} 2.4080 & -0.4652 \\ -0.4652 & 2.7814 \end{bmatrix}
\]

(4.93)

where \( K = B^T P_1 \) is selected and obtained as

\[
K = \begin{bmatrix} -0.4652 & 2.7814 \end{bmatrix}
\]

(4.94)

Similarly, \( P_2 = P_2^T > 0 \) is obtained using ARE, since the eigenvalues of the Laplacian matrix \( \mathcal{L} \) are distinct, as follows:

\[
P_2 = \begin{bmatrix} 1.9269 & -0.4641 \\ -0.4641 & 1.6687 \end{bmatrix}
\]

(4.95)

where \( L = P_2^{-1} C^T \) is chosen as

\[
L = \begin{bmatrix} 0.5562 \\ 0.1547 \end{bmatrix}
\]

(4.96)
Figure 4.13 and Figure 4.14 show the simulation plots for substate 1 and 2 of the containment of the followers with multiple leaders. The controller (4.4, pg. 99) enabled the followers to access the leader’s information. It can be seen that conditions (4.86) to (4.87) are sufficient for the controller to achieve the containment when there are nonlinearities in the system. As shown in Figure 4.15 and Figure 4.16 the containment of all followers was realized when $\gamma = 0.2$. As $\gamma$ is increased to 0.7 the follower required more time to be contained by the leaders and the trajectory changes slightly compared to the previous figures. Unlike the state-feedback containment controller in the previous section, where the controller struggled to contain substates 2 from each subsystem, observer-based containment controller managed to contain both substates 1 and 2 when $\gamma$ was increased.

\[ x_{n+1}, x_{n+2}, x_{n+3} \]

\[ \dot{x}_1, \dot{x}_2, \dot{x}_3, \ldots, \dot{x}_N \]

\[ e_1, e_2, e_3, \ldots, e_N \]

\[ \mathcal{L} \]

Figure 4.12: Observer-based containment controller design
Figure 4.13: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.05$

Figure 4.14: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.05$
Figure 4.15: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.2$

Figure 4.16: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.2$
Figure 4.17: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.7$.

Figure 4.18: Plot of actual and estimated states for subsystem 1 with $\gamma = 0.7$.
4.5.3 Conclusion

In this section, an observer-based containment controller was proposed for multi-agent systems with Lipschitz nonlinearity, in networks subject to directed spanning forest topology. The stability of the observer is achieved by carefully selecting the controller gain $K$ and observer gain $L$, together with the knowledge of graph theory. The solution of the stability analysis was obtained via ARE since solution with LMI could not guarantee containment was reached by the leaders under the influence of the nonlinearity element. The controller were tested with a few different values of nonlinearity represented by the change of $\gamma$. In all cases, containment of the follower agents by the leader agents was reached with a certain amount of nonlinearity influence.

4.6 Conclusion of the Chapter

In this chapter, a state-feedback containment controller, a containment observer, and an observer-based containment controller were proposed. Since there were no existing publications relating to containment of multi-agent systems based on Lipschitz nonlinearity, this chapter provided one of the main contributions of the thesis which is the design of containment control for multi-agent systems with Lipschitz nonlinearity.

Several theories were provided that explained the conditions and restrictions of each controller and observer required for consensus to be reached based on Lyapunov stability analysis. The solution for the stability analysis could be obtained via LMI, but could not guarantee the containment of the follower agents. Hence, ARE was used to get the solution of $P = P^T > 0$ instead.

In the simulation examples, all controller were tested with certain values of Lipschitz nonlinearity by changing the value of Lipschitz constant $\gamma$. It is observed that consensus were reached in each of the test. These findings validate the theoretical results obtained. However, the increase of the nonlinearity constant affected the trajectory and output of each agent or subsystem. Therefore, the design objective of these controllers and observers has been met - to contain a number of subsystems in a nonlinear system with Lipschitz nonlinearity.
Chapter 5

Case Studies

5.1 Introduction

We have seen an increasing amount of work related to cooperative control strategies amongst the control system field. They can be applied both as centralized or decentralized. Decentralized control strategy or also known as distributed control strategy forms part of cooperative control. A branch of distributed control is the consensus control which is the topic of interest in this chapter.

Chapter 3 displayed the theoretical aspects of consensus and containment control. There has been a number of practical applications of distributed control system that applied consensus control in recent publications; specifically in formation control that is used in mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs), satellites, aircraft, spacecraft, and automated highway systems [9]. Most are simplified to general linear systems when consensus control were applied to solve the consensus problem. Previously in Chapter 3, a simple second-order example was employed for readability to demonstrate the theoretical ideas. Consensus problem was solved for the second-order system.

For the practical application, two real physical systems that are closely resembled the Lipschitz nonlinear system such as Chua’s circuit [110] and the flexible robot [111]. Several control strategies have been employed for both systems since both systems were published.
In consensus control, certain publications such as [72, 73] used flexible mobile robot to demonstrate their ideas in consensus control. Based on the same paper and Spong [111], this chapter will concentrate on two examples of flexible robot formations to demonstrate our ideas to solve consensus problem.

## 5.2 Flexible Robot Arm Position

A model used is the elastic joint robot, as shown in Figure 5.1. It is a single-link manipulator with revolute joints actuated by a DC motor. The elasticity of the joint can be modelled as a linear tensional spring [111], and the elastic coupling of the motor shaft to the link introduces an additional degree of freedom. The states of the system are motor position, the link position and velocity.

![Elastic joint robot model](image)

Figure 5.1: Elastic joint robot model

The corresponding state-space model is

\[
\begin{align*}
\dot{\theta}_m &= \omega_m \\
\dot{\omega}_m &= \frac{k}{J_m} (\theta_1 - \theta_m) - \frac{B}{J_m} \omega_m + \frac{K_r}{J_m} u \\
\dot{\theta}_1 &= \omega_1 \\
\dot{\omega}_1 &= -\frac{k}{J_1} (\theta - \theta_m) - \frac{mgh}{J_1} \sin \theta_1
\end{align*}
\] (5.1)
with $J_m$ being the inertia of the motor, $J_1$ the inertia of the link, $\theta_m$ the angular rotation of the motor, $\theta_1$ the angular position of the link, $\omega_m$ the angular velocity of the motor, and $\omega_1$ the angular velocity of the link.

This described the system as nonlinear, and the system dynamics proposed by Spong [111] are

$$\dot{x} = Ax + Bu + \phi(x)$$

$$y = Cx$$

with $x = \begin{bmatrix} \theta_m & \omega_m & \theta_1 & \omega_1 \end{bmatrix}^T$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix},$$

$$\phi(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \gamma \sin x_3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Spong [111] presented a nonlinear I/O linearising control law for this system. The control law guarantees closed-loop stability and tracking ability for any desired trajectory of the robotic link. However, Spong’s work was in single system. It is interesting to see if a number of flexible mobile robot’s parameters controlled with consensus controller in a distributed manner. Therefore, the next two examples will demonstrate our ideas of distributed control on flexible mobile robot with the application of state-feedback consensus controller and observer-based consensus controller designed in Chapter 3.
5.2.1 Example 1

The nonlinear system dynamics (5.2) can be transferred to a multi-agent system dynamics in the form of

\[ \dot{x}_i = Ax_i + Bu_i + \phi(x_i) \]
\[ y_i = Cx_i \]

with the same matrices \( A, B, C \) and \( \phi(x_i) \) for all \( i = 1, \ldots, N \) where \( N \) represents the number of agents. Let \( x_{i1}(t) \) be the angular rotation of the motor, \( x_{i2}(t) \) be the angular velocity of the motor, \( x_{i3}(t) \) be the angular rotation of the link of the \( i^{th} \) manipulator, and \( x_{i4} \) be the angular velocity of the link of the \( i^{th} \) manipulator. \( \gamma \) is determined to be 0.333, with \( \kappa_1 = 1 \) and \( \kappa_2 = 3 \). It can be verified that \( (A, B, C) \) is controllable and observable. The connections of the system is shown in Figure 5.2.

\[ \] 

Figure 5.2: Flexible Mobile Robots Directed Spanning Tree Connection.

In this example, state-feedback consensus controller is applied. It depends on solution of matrix \( P = P^T > 0 \) from the Lyapunov stability analysis. Due to the dynamics of the system, it is difficult to find the solution of \( P = P^T > 0 \) with the standard Linear Matrix Inequality (LMI) procedure. Hence, a new way of finding a solution for \( P \) is needed, which is where the Algebraic Riccati Equation (ARE) procedure becomes useful.
Based on Lemma 3.3.3, the Hamiltonian matrix with respect to the nonlinear system dynamics in this example needs to have no eigenvalues on the imaginary axis. Hence, we can obtain eigenvalues of the Hamiltonian matrix $H_1$ to be $9.8574, -9.8574, 2.8793 + 5.4484i, 2.8793 - 5.4484i, -2.8793 + 5.4484i, -2.8793 - 5.4484i, 1.0130$, and $-1.0130$. We then obtained controller gain matrix $K$ to be

$$K = \begin{bmatrix} 0.7044 & 0.1533 & -0.4935 & 0.0886 \end{bmatrix}$$

(5.5)

where

$$P = \begin{bmatrix} 4.3414 & 0.0326 & -3.1209 & 0.5353 \\ 0.0326 & 0.0071 & -0.0228 & 0.0041 \\ -3.1209 & -0.0228 & 3.8188 & -0.2690 \\ 0.5353 & 0.0041 & -0.2690 & 0.2898 \end{bmatrix}$$

(5.6)

Figure 5.3 shows the state-feedback controller design model for flexible mobile robots. The plots for $\gamma = 0.333$ are shown in Figure 5.4 and Figure 5.5. When $\gamma$ was increased to 0.8, it is observed that the subsystem still reached consensus after significant oscillations as shown in Figure 5.6 and Figure 5.7. The signal was smooth when $\gamma = 0.333$.

5.2.2 Example 2

In this example, the state-feedback controller is now replaced by an observer-based consensus controller in nonlinear system (5.4). Based on Lemma 3.3.3, the Hamiltonian matrix with respect to the nonlinear system dynamics of this example need to have no eigenvalues on the imaginary axis. Hence, we can obtain eigenvalues of the Hamiltonian matrix $H_1$ to be $-11.9415, 11.9415, -0.7624 + 5.0943i, -0.7624 - 5.0943i, 0.7624 + 5.0943i, 0.7624 - 5.0943i, -0.7827, 0.7827$ which means that no eigenvalues exist on the imaginary axis. We can then obtain the controller gain matrix

$$K = \begin{bmatrix} 0.0100 & 0.0183 & 0.0104 & -0.0067 \end{bmatrix}$$

(5.7)

where

$$P = \begin{bmatrix} 7.3256 & 0.0005 & 7.3256 & 0.0005 \\ 0.0005 & 0.0008 & 0.0005 & -0.0003 \\ 7.3256 & 0.0005 & 7.3257 & 0.0005 \\ 0.0005 & -0.0003 & 0.0005 & 0.0002 \end{bmatrix}$$

(5.8)
The system is found to be controllable and observable. Similarly, there is no suitable $P_1 > 0$, and $P_2 > 0$ can be obtained with LMI standard procedure. Therefore, the
Figure 5.4: Plot of Leader substate 1 and follower substates 1 with consensus state feedback controller, under the influence of Lipschitz nonlinearity at $\gamma = 0.333$.

Figure 5.5: Plot of Leader substate 1 and follower substates 1 with consensus state feedback controller, under the influence of Lipschitz nonlinearity at $\gamma = 0.333$. 
Figure 5.6: Plot of Leader substate 1 and follower substates 1 with consensus state feedback controller, under the influence of Lipschitz nonlinearity at $\gamma = 0.8$.

Figure 5.7: Plot of Leader substate 1 and follower substates 1 with consensus state feedback controller, under the influence of Lipschitz nonlinearity at $\gamma = 0.8$. 
ARE procedure is used, and the plots are shown in Figure 5.8 and Figure 5.9 for \( \gamma = 0.333 \). When \( \gamma \) is increased to 0.8, it can be seen that the subsystem still reaches consensus after severe oscillation, compared with the smooth signal when \( \gamma = 0.333 \). For both cases, the observer enables the estimated substates of the subsystems to track the substates of the subsystem.

It can also be seen that the observer enables the estimated substates to track the substates, as shown in Figure 5.12 for actual and estimated substate 1, and Figure 5.13 for actual and estimated substate 2, from follower subsystem 1; this also happens to all other substates for other subsystems.

![Figure 5.8: Plot of leader substate 1 and follower substates 1 consensus controller with state estimation, under the influence of Lipschitz nonlinearity at \( \gamma = 0.333 \)](image)

The simulation in Figure 5.4-5.7 is then compared to Figure 5.8-5.11. It is clear that, with the application of the observer, the reaction of the controller (3.61) is slightly slow, compared to that of the state feedback controller (3.4). This is due to the controller (3.4) having a faster response, and also because of the application of an observer (3.40) in the slower-responding controller (3.61). It should also be noted
Figure 5.9: Plot of Leader substate 2 and follower substates 2 consensus controller, with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 0.333$.

Figure 5.10: Plot of Leader substate 1 and follower substates 1 consensus controller, with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 0.8$. 
Figure 5.11: Plot of Leader substate 1 and follower substates 1 consensus controller, with state estimation under the influence of Lipschitz nonlinearity at $\gamma = 0.8$.

Figure 5.12: Plot of substate 1 and estimated substates 1 of subsystem 1, under the influence of Lipschitz nonlinearity at $\gamma = 0.333$. 
that, for both cases, it is difficult to obtain the solution of $P$ from the LMI procedure.

### 5.3 Conclusion

In this chapter, the flexible robot model has been introduced. Based on a few publications relating to a single system with Lipschitz nonlinearity for this model, consensus control has been introduced. Two types of consensus controller were applied based on the theoretical results from the previous chapter; the state feedback controller and the observer-based controller. The difference between these two controllers was that the state feedback controller depended on the relative information of the agents, while the latter only required the relative output information from each agent. Both controllers were able to achieve consensus.

The solution for $P = P^T > 0$ from the stability analysis was obtained from ARE routine that guaranteed consensus was reached. If LMI was used to get the solution, consensus could be not be guaranteed. This is due to the nonlinearity element of the
system. The outcomes of each examples were compared and showed identical plots for both controllers which further validates the theoretical results from the previous chapter.
Chapter 6

Conclusions

In conclusion, the overall objectives of this thesis have been met. In Chapter 2, a thorough introduction to consensus control has been presented. This involves the explanation of the basic of linear matrix knowledge and graph theory. From the relative information of the agents in a multi-agent system, eigenvalues may be obtained from Graph Theory, and can be utilized for both the consensus control and containment control strategies. This enables the system to be analysed for its stability and convergence, so that both consensus and containment can be reached. The multi-agent system was introduced, with a class of nonlinearity in the form of Lipschitz. In relation to stability analysis, several publications were identified that help with the process of reaching a stable condition for the system. The Lyapunov stability analysis concept was introduced, together with the controller and observer design. Initially, the nonlinear systems were identified in single systems, then consensus control was applied, and the stability of the system was assessed using Lyapunov stability analysis.

Chapter 3 represented the analysis of the proposed state feedback consensus controller, consensus observer and observer-based controller, with the inclusion of Lipschitz nonlinearity in the system. The design of the state feedback controller is primarily discussed in Chapter 2. The proposed observer design was based on the Luenberger observer. Another type controller - the observer-based controller - was proposed that utilizes the relative output measurement values. The design procedure of all controllers and observers were described in detail, and were based on the recent results in the available literature. Stability analysis was performed, using conditions based on
the Lyapunov stability method and careful study of the Laplacian structure. These solutions were obtained with the Linear Matrix Inequality (LMI) and Algebraic Riccati Equation (ARE) methods. The performances of these controllers and observers were evaluated using simulation studies, with the results presented in the same chapter. Both controllers successfully solved the consensus problem for a general linear system with Lipschitz nonlinearity. Simulations were presented to show the validation of the theoretical design and the outcome of the state feedback controller was compared with the proposed observer-based controller, and discussed in detail.

Chapter 4 presented the results of a multi-agent system with Lipschitz nonlinearity, with the application of two containment controllers; namely state-feedback containment controller and observer-based containment controller. The design of containment control was discussed in detail, together with the differences between containment control and consensus control, with respect to the structure of the Laplacian matrices and the the type of information utilized. Two types of containment controller were proposed: the state feedback controller and the observer-based controller under directed spanning forest connection topology. The stability of the system was analysed using the Lyapunov stability analysis method; this method depends on the solution of matrix $P$, obtained from either the standard Linear Matrix Inequality (LMI) or Algebraic Riccati Equation (ARE). Both controllers successfully obtained containment with the element of Lipschitz nonlinearity. Simulations were presented demonstrate the effectiveness of the controllers and observer, and the results are discussed in detail.

Chapter 5 gave two examples of the applications of consensus control, based on the simulation of flexible mobile robots, modelled as state-space systems with Lipschitz nonlinearity. The state-feedback controller, observer and an observer-based controller were applied for the consensus control of the system. Conditions for stability were given based on Lyapunov stability analysis. Both controllers were able to achieve consensus with the observer proven to be stable. Simulation examples were given to verify the theoretical findings and discussed in detail.
6.1 Direction and Future Research

With proposed consensus control design performed in chapter 3, 4 and 5, consensus and containment problem have been solved for multi-agent system with Lipschitz nonlinearity. Observer-based consensus controller was proposed when not all relative states measurements were obtainable. This type of control design is new and became the main contribution of this thesis. There are two other contributions such as the state-feedback containment controller and observer-based containment controller. With the application of these controllers in the field of consensus and containment control, several interesting fields of research can be considered in the future. Specifically for consensus control of multi-agent systems with Lipschitz nonlinearity, several areas can be considered; namely reducing the affects of time-delay, switching, and noise in directed connection topology. In particular, more research areas can be explored in containment control for nonlinear systems under directed and undirected connection topology. We might suggest these research areas to be the direction of future research.
Bibliography


