Time and Risk Preferences: Theoretical Models for Individual Decision Making

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Abstract

Thesis submitted by Jinrui Pan for the Degree of Doctor of Philosophy in the University of Manchester, and entitled, "Time and Risk Preferences: Theoretical Models and Applications." Date of submission 2014.

This thesis makes contributions to two important areas of behavioural economics, namely individual decision making over time and under risk.

Following the Introduction, Chapter 2 presents a new discounting function for analysing intertemporal choice. Liminal discounting, the model developed here, generalises exponential discounting in a parsimonious way. It allows for well-known departures, whilst maintaining its elegance and tractability. It also can be seen as an extension of quasi-hyperbolic discounting to continuous time. A liminal discounter has a constant rate of time preference before and after some threshold time; the liminal point. A preference foundation is provided, showing that the liminal point is derived endogenously from behaviour.

Chapter 3 proposes an axiomatic model featuring a differential treatment of attitudes towards risk and time. Such distinction has been strongly suggested by experimental research when studying intertemporal choice, since the future is inherently risky. In the proposed model, non-linear probability distortions are incorporated into a dynamic model with discounted utility. Time is captured by a general discounting function independent of probabilities and outcomes. Utility of outcomes is captured by standard vNM utility independent of time. A two-parameter probability weighting function captures intertemporal probabilistic risk attitudes, with one parameter being constant over time, the other being time-dependent. An index of optimism is derived that depends on both parameters, which allows to model the observed high risk tolerance for delayed lotteries. Further, a preference foundation is provided. Interestingly, the model allows behaviour to be consistent with discounted expected utility, when risk is sufficiently distant from the present.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

Behavioural economics is a fascinating and rapidly growing field that has drawn attention predominantly as a discipline that catalogues anomalies and explores alternative ways to model real life choices. It is recognised as an umbrella of approaches that seek to extend the standard economics framework to account for relevant features that are absent in the standard economics framework (Diamond and Vartiainen, 2007).

To understand and model deviations from the standard economic model, the fundamental insight that needs to be addressed is individual choice. Researchers explore additional psychological and sociological factors that shape individual decision making, examine decision processes and focuses directly on the question of how decisions are made. Alternative behavioural models of individual choice that are built upon information extracted from the process, can help us to understand the functioning of economic institutions, hence to design better institutions.

Risk and time are two essential elements to many individual decisions. They are present when future consequences depend on today's decisions and are not entirely certain. Such decision making requires evaluating consequences that are distributed over time. For instance, consumers decide how much money to put aside for saving; graduates choose to find a job or pursue a higher qualification; governments take actions to reduce the future effects of global warming; people adopt healthy lifestyles to improve their future health. In order to make the best decision at present a precise analysis of what may happen in the future, or at least what the individual thinks it may happen in the future, is needed.

This thesis contributes to developments of theoretical modelling on individual decision making over time, from two distinctive perspectives. The classical approach to evaluate one's time preferences is Samuelson's (1937) *discounted utility theory*. It views decision makers as maximising a weighted sum of utility with the weights representing exponentially declining discount weights. Such a model implies constant rate of time preferences.

Numerous empirical studies (see Frederick, Loewenstein and O'Donoghue, 2002 for a comprehensive review of experimental literature) have challenged the validity of discounted utility theory as descriptive models of intertemporal choice. In particular, models allowing decreasing discount rates instead of constant ones become more appealing, which has been known as *hyperbolic discounting* (Ainslie, 1975).

Chapter 2 follows the above approaches concerning discount weights, and presents a model that generalises exponential discounting in a parsimonious way. It also can be seen as an extension of Phelps and Pollak's (1968) quasi-hyperbolic discounting model, which was developed in discrete time, to continuous time. The discrete-time quasi-hyperbolic discount function is used to model that discount rates are much greater in the short-run than in the long-run (Ainslie, 1992; Loewenstein and Thaler, 1989), which has been applied extensively in economic theory (Luttmer and Mariotti, 2003; O'Donoghue and Rabin, 2001; Barro, 1999; Asheim, 1997; Laibson, 1997). Extending quasi-hyperbolic discounting to continuous time is desired for economic application.

Another strand of the literature investigates people's intertemporal choices from the perspective of risk preference. When time preferences are modelled from discount weights as described above, it is commonly assumed that the promised outcomes will be delivered without risk. This assumption is flawed as the future is naturally bonded with uncertainty and risk. Consequently, one's risk preference must have an influence on the rate of time preference. Evaluating future prospects is directly related to the evaluation of the likelihoods of their outcomes at the point when the prospect is realised.

For decision under risk the classical approach is *expected utility theory* (Von Neumann and Morgenstern, 1944). Similarly with discounted utility theory, decision makers are to maximise a weighted sum of utilities with the weights represent probabilities. Violations of expected utility theory have also been observed from experiments. Evidence shows that individuals tend to underweight and/or overweight probabilities. In other word, the probabilities or the likelihoods are normally perceived non-linearly. *Rank-dependent utility* (Quiggin, 1981) was developed to incorporate a probability transformation function associates decision weights to such outcome, therefore, to capture the subjective perception of objective probabilities.

In economics, risk preferences are conventionally assumed to be unaffected by the passage of time and time preferences to be unaffected by the presence of risk, even though some authors have pointed out parallels between the two domains (Quiggin and Horowitz, 1995; Prelec and Loewenstein, 1991) In contrast to this view, there is mounting evidence of complex interactions between behaviour under risk and behaviour over time that challenges the standard models of risk taking and time discounting (Anderson and Stafford, 2009; Ahlbrecht and Weber, 1997; Keren and Roelofsma, 1995).

Chapter 3 adopts rank-dependent utility to model individuals' perception on the risk future prospects may bear. It focuses on interaction between time and risk, and develops a model that captures the effect of time on decision weights associated with future prospects.

Both chapters are written in article form and therefore self-containing, thus some notation and definitions are repeated. Proofs are placed in the respective appendix of the chapter.

Chapter 2

Liminal Discounting Model

2.1 Introduction

The theory of *discounted utility* is the most widely used framework for analysing *intertemporal choice*. Descriptive discounting models capture the property that most economic agents place less weight on the future than on the present, i.e., they act as though they *discount* future payoffs. The underlying psychological reason of such behaviour is that people are *impatient*. Samuelson (1937) proposed the Exponential Discounting model, in which the notion of impatience is quantified by a single discount rate. The exponential discounting model implies *dynamically consistent* behaviour due to a *constant* level of impatience, i.e. time preferences held at one point in time do not change with the passage of time.

The purpose of this chapter is to integrate the concept of liminality[1] into

[1] The concept "liminality" is adopted from the psychology literature, where it means a subjective state that is a threshold between psychologically distinctive domains. It is also used in social anthropology, where it indicates a transition point between historical periods.

intertemporal choice. A model of liminal discounting[2] is developed. As with exponential and many other non-exponential discounting models (Abdellaoui, Attema and Bleichrodt, 2010), liminal discounting retains a stationary instantaneous utility for outcomes. This utility is discounted by a constant rate of time preference up to a threshold time; the *liminal point*. After this point, the discount rate may change, but then remains constant afterwards. Violations of constant discounting occur only when comparing the near and distant future. Jamison and Jamison (2011) first presented the parametric discount function considered here, calling it *split rate quasi-hyperbolic discounting*. A preference foundation for liminal discounting over timed outcomes is also provided in this chapter.

Despite its many appealing properties, the exponential discounting model fails to match several empirical regularities (see Manzini and Mariotti, 2008; Frederick, Loewenstein and O'Donoghue, 2002, for a review). One of these well-known experimental findings is that *discounting is not constant* (Read and Read, 2004; Bleichrodt and Johannesson, 2001; Van Der Pol and Cairns, 2000; Laibson, 1997; Kirby and Maraković, 1995; Loewenstein and Prelec, 1992; Benzion, Rapoport and Yagil, 1989; Mazur, 1987; Thaler, 1981). If an early reward and another, later and larger reward are perceived as being equivalent, then delaying both rewards equally will result in a strict preference for the later and larger rewards, revealing *decreasing impatience*.

As a direct consequence of decreasing impatience, individuals' preferences can be *dynamically inconsistent*. In other words, the passage of time may

[2] The model developed here is called "Two-Stage Exponential Discounting" in Pan, Webb and Zank (2015).

change one's time preferences. Consider the classic example of Thaler (1981), in which a person who prefers to receive one apple today rather than two apples tomorrow, will often prefer to receive two apples in one year plus one day rather than one apple in one year. If such person is dynamically consistent, i.e. her preferences between 'today' and 'tomorrow' remain the same for one year, and she resets the clock at zero[3] whenever she makes a decision, then in one year from now she will prefer to receive one apple on that day rather than two apples one day later. Thus, between the point when she is making plans for one year later, and the point when one year has passed, her preferences must have changed. She is dynamically inconsistent with herself one year ago.

Since the initial discussion of dynamically inconsistent preferences in Strotz (1955), many economic models are adapted to incorporate decreasing impatience, or *hyperbolic discounting* (Asheim, 1997; Kirby and Maraković, 1995; Benzion, Rapoport and Yagil, 1989; Thaler, 1981; Phelps and Pollak, 1968; Pollak, 1968). Yet, more recent evidence finds support for both exponential discounting and hyperbolic discounting model (Andersen, Harrison, Lau and Rutstroem, 2011; Abdellaoui, Attema and Bleichrodt, 2010). It suggests that modest, tractable deviation from exponential discounting is desired. One such model, *quasi-hyperbolic discounting* (Olea and Strzalecki, 2014; Attema, Bleichrodt, Rohde and Wakker, 2010; Hayashi, 2003; Laibson, 1997; Phelps and Pollak, 1968) has been applied extensively in economic theory (Luttmer and Mariotti, 2003; O'Donoghue and Rabin, 2001; Barro, 1999; Asheim, 1997; Laibson, 1997).

Under quasi-hyperbolic discounting, decision makers exhibit decreasing im-

[3] Such "resetting the clock" behaviour is called time invariance in Halevy (2011).

patience only at time point 0, i.e., at present, and constant impatience thereafter. Quasi-hyperbolic discounting was developed in discrete time. Liminal discounting extends quasi-hyperbolic discounting to continuous time. Such extension is absent from the literature and is important for economic applications.

The outline of this chapter is as follows: Section 2.2 contains the notation and definitions. Section 2.3 reviews the most relevant known discount functions. Section 2.4 presents liminal discounting model and Section 2.5 provides preference foundation for such model. Section 2.6 concludes and discusses the possible applications.

2.2 Preliminaries

Let $[0, X]$, with $X > 0$, represent the set of *outcomes*, and $[0, T]$, with $T > 0$ be the set of *time points* at which an outcome can occur. A *timed outcome* $(t : x)$ is interpreted as a promise to receive of an outcome $x \in [0, X]$ at time point $t \in [0, T]$, with no risk attached. Such timed outcomes are the objects of choice.

A preference relation $\succcurlyeq$ is defined over the set of timed outcomes $[0, T] \times [0, X]$. As usual, the symbol $\succ$ denotes strict preference while $\sim$ denotes indifference ($\preccurlyeq$ and $\prec$ denote reversed weak and strict preferences, respectively).

A preference relation $\succcurlyeq$ is *complete* if for all $(t : x), (t' : x') \in [0, T] \times [0, X]$, either $(t : x) \succcurlyeq (t' : x')$ or $(t' : x') \succcurlyeq (t : x)$ holds. It is *transitive* if for all $(t : x), (t' : x'), (t'' : x'') \in [0, T] \times [0, X]$, $(t : x) \succcurlyeq (t' : x')$ and $(t' : x') \succcurlyeq (t'' : x'')$ jointly imply $(t : x) \succcurlyeq (t'' : x'')$. It is a *weak order* if it is complete and transitive. It is *monotonic* if, for all $(t : x), (t : x') \in [0, T] \times [0, X]$,

$(t : x) \succcurlyeq (t : x')$ if and only if $x \geqslant x'$. It is *impatient* if, for all $(t : x), (t' : x) \in [0, T] \times [0, X]$, $(t : x) \succcurlyeq (t' : x)$ if and only if $t' \geqslant t$. We will always assume that $(t : 0) \sim (t' : 0)$, for all $t, t' \in [0, T]$, and include this condition in the definition of impatience. A preference relation $\succcurlyeq$ is *continuous* if, for all $(t : x) \in [0, T] \times [0, X]$, the sets $\{(t' : x') : (t : x) \succcurlyeq (t' : x')\}$ and $\{(t' : x') : (t : x) \preccurlyeq (t' : x')\}$ are closed subsets of $[0, T] \times [0, X]$.

Weak order is a rationality property deeply rooted in the economic theory of choice. Monotonicity and impatience are also universally assumed in economic models, which are populated by agents for whom more of a good thing is better, and especially for whom a good thing is better if it comes sooner.

A preference relation $\succcurlyeq$ is represented by a function V if V assigns to each timed outcome a real value, such that for all $(t : x), (t' : x') \in [0, T] \times [0, X]$, the following holds:

$$(t : x) \succcurlyeq (t' : x') \quad \Leftrightarrow \quad V(t : x) \geqslant V(t' : x').$$

A necessary condition for $\succcurlyeq$ to admit such a representation is that $\succcurlyeq$ is a weak order. It has been showed that weak ordering and continuity of $\succcurlyeq$ are sufficient for the existence of a *continuous* utility representation (Debreu, 1964, Proposition 4). Monotonicity and impatience ensure that such a representation is non-decreasing in x and non-increasing in t.

In order to obtain representations for $\succcurlyeq$ which separate the effect of time preference from outcomes preference, the following assumption[4] must hold.

[4] Such condition has been used previously by Debreu (1960) and others for additive measurement representations.

Axiom 2.2.1 (Thomsen Separability)**.** *For all* $x, x', x'' \in [0, X]$ *and all* $t, t', t'' \in [0, T]$*, if*

$$(t : x) \sim (t' : x') \qquad \& \qquad (t'' : x') \sim (t : x'')$$
$$\Rightarrow$$
$$(t'' : x) \sim (t' : x'')$$

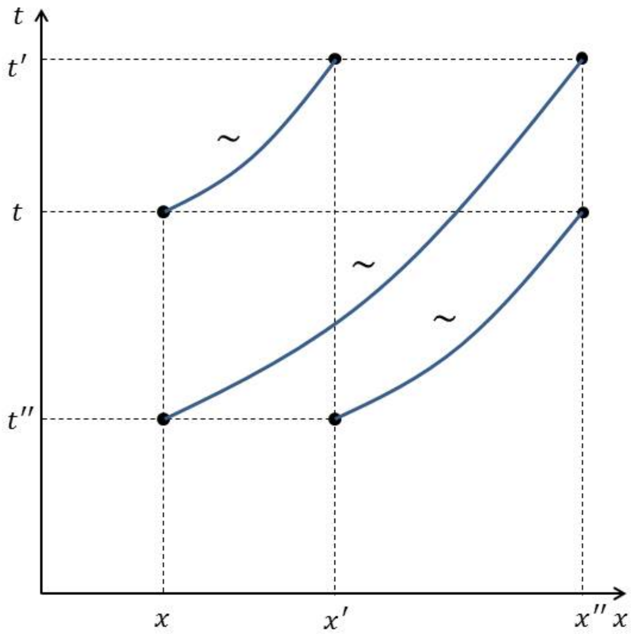


Figure 2.1: Thomsen Separability

Figure 2.1 illustrates Axiom 2.2.1 for positive outcomes. It can also be exemplified by a simple example as follows. Suppose that a decision maker is indifferent between receiving £20 today and receiving £35 tomorrow, and he is also indifferent between getting £10 today and getting £35 the day after tomorrow, Axiom 2.2.1 then states that he should be indifferent between receiving £10 tomorrow and receiving £20 the day after tomorrow. Thomsen Separability isolates the effect of time preference from outcome preference. This means that there exists some kind of independence between the attributes of time and outcomes, and such independence makes the elicitation of utility simpler and more transparent.

Structural Theorem (Fishburn and Rubinstein, 1982)**.** *The following statements are equivalent:*

(i). The preference relation $\succcurlyeq$ *over* $[0,T] \times [0,X]$ *is represented by a function such that,*

$$V(t:x) = D(t)u(x), \tag{2.1}$$

where there are continuous real-valued functions u *on* $[0,X]$ *and* D *on* $[0,T]$*. In addition,* $u(0) = 0$ *and* u *is strictly increasing while* D *is strictly decreasing and positive.*

D *and* u *are unique up to separate positive factors and a joint positive power, i.e.,* D *and* u *are jointly cardinal.*

(ii). The preference relation $\succcurlyeq$ *over* $[0,T] \times [0,X]$ *is a continuous, monotonic, impatient weak order that satisfies thomsen separability.*

In this representation, V is the total utility of a future prospect $(t : x)$ evaluated as time $t = 0$; $u(x)$ is the utility function on outcomes; $D(t)$ is the discount function. Since people put less weight to the future, i.e., they are impatient, the discount function declines as the delay t increases. Given the standard normalization $D(0) = 1$ and assuming impatience, the following is implied,

$$1 = D(0) \geqslant D(t) \geqslant D(t') \geqslant 0,$$

where $0 \leqslant t \leqslant t'$.

Equation 2.1 represents an axiomatisation of generalised discounting utility models. These models combine an instantaneous utility function that reflects attitudes towards outcomes with a discount function that captures the effect of the passage of time. In the following section, several models that differ in the assumptions they impose on the discount functions, but not in the underlying utility structure, are discussed.

2.3 Discounting

This section reviews the essential discount functions, as applied to choice over timed outcomes.

2.3.1 Constant Discounting

Shortly after Fisher's (1930) graphical indifference curve analysis, which was difficult to extend to more than two time periods, Samuelson (1937) introduced a generalised model of intertemporal choice that is able to be extended to multiple time periods - exponential discounting model, in which it features the exponential discount function:

$$D(t) = \delta^t$$

with δ being a constant discount factor and $0 < \delta < 1$.

In Samuelson's simplified model, all the psychological reasons for discounting the future were compressed into a single parameter, the discount rate.

The key property of exponential discounting, that distinguishes it from other models, is *stationarity*:

Definition 2.3.1 (Stationarity)**.** *A preference relation* $\succcurlyeq$ *satisfies stationarity if for all* $(t : x), (s : y), (t + \tau : x), (s + \tau : y) \in [0, T] \times [0, X]$ *the following holds:*

$$(t : x) \sim (s : y) \quad \Leftrightarrow \quad (t + \tau : x) \sim (s + \tau : y).$$

This formulation of stationarity is due to Fishburn and Rubinstein (1982). Koopmans (1960) and Bleichrodt, Rohde and Wakker (2008) formulate such condition for sequences of outcomes.

Stationarity asserts that a decision maker's preferences between two timed outcomes depends only on the absolute time interval between delivery of the outcomes. In other words, if the two time points are accelerated or delayed by the same amount, then preferences will be preserved. The assumption of stationarity permits an individual's time preference to be summarised as a single discount rate, hence characterises exponential discounting model as constant discounting model.

It appears that, stationarity does not have a very strong justification, either from the normative or from the positive perspective. Indeed Fishburn and Rubinstein themselves explicitly state that 'we know of no persuasive argument for stationarity as a psychologically viable assumption' (Fishburn and Rubinstein, 1982, p. 681). Both intuition, and experimental evidence (Thaler, 1981; Benzion, Rapoport and Yagil, 1989) indicate the impact of a constant time difference between two outcomes becomes less significant as both outcomes are made more remote, namely, the *common difference effect*.

Definition 2.3.2 (Common Difference Effect)**.** *A preference relation* $\succcurlyeq$ *exhibits* *common difference effect if for all* $(t : x), (s : y), (t + \tau : x), (s + \tau : y) \in [0, T] \times [0, X]$*, and* $y > x, \tau > 0$ *the following holds:*

$$(t : x) \sim (s : y) \quad \Rightarrow \quad (t + \tau : x) \preccurlyeq (s + \tau : y).$$

The above formally defines the famous example in Thaler (1981). An individual who is indifferent, say, between one apple today and two apples tomorrow will most likely prefer two apples in one year plus one day to one apple in one year. Empirical studies on time preference have confirmed such observations, implying that people are more patient to wait for the larger reward when the receipt of rewards are equally delayed, hence behaviour that exhibits decreasing impatience (Frederick, Loewenstein and O'Donoghue, 2002; Read, 2004). More evidence see for example Thaler (1981), Benzion, Rapoport and Yagil (1989), Shelley (1993) and Kirby and Maraković (1995) for money rewards, and Chapman (1996), Lazaro, Barberan and Rubio (2001), van der Pol and Cairns (2002) for health outcomes.

Recent empirical studies also observed increasing instead of decreasing impatience (Chesson and Viscusi, 2003; Gigliotti and Sopher, 2003; Read, Airoldi and Loewenstein, 2005; Sayman and Öncüler, 2008). Nonetheless, these studies observed the discount rates are not constant, which have led to the development of alternative discounting models, discussed in the following subsection.

2.3.2 Non-constant Discounting

Hyperbolic Discounting

Decreasing Impatience has led psychologists (Ainslie, 1992; Loewenstein and Prelec, 1992) to adopt discount functions in the family of generalized hyperbolas:

$$D(t) = (1 + \beta^t)^{-\alpha/\beta} \tag{2.2}$$

with $\alpha > 0, \beta > 0$.

The parameter α determines how much the function departs from constant discounting. As α tends to zero, the limiting case yields the exponential discount function, $D(t) = e^{-\beta t}$.

Hyperbolic discount functions imply that discount rates decrease over time. So people are assumed to be more impatient for trade-offs between money and delay near the present than for the same trade-offs pushed further away in time. It can account for common difference effect.

Particular attention has been paid to the case in which $\alpha = \beta$, implying that

$$D(t) = (1 + \beta^t)^{-1}$$

which is proposed by Harvey (1986) and Mazur (1987), often called proportional discounting.

Quasi-hyperbolic Discounting

There is some controversy in the literature as to whether decreasing impatience holds in general or whether violations of constant discounting occur only in the first time interval. The latter hypothesis is referred to as the immediacy effect.

Definition 2.3.3. *The preference relation* $\succcurlyeq$ *exhibits immediacy bias if there are* $x, y \in [0, X]$*,* $t \in [0, T]$ *and* $\tau > 0$ *such that*

$$\begin{aligned}(0 : x) &\succ (\tau : y)\\(t : x) &\prec (t + \tau : y)\end{aligned}$$

Immediacy effect is the special case of the common difference effect. It means that, the delay τ, for outcome y over x, becomes acceptable when the timed outcomes are translated t units into the future. Such preferences are incompatible with exponential discounting.

For the underlying reason, Phelps and Pollak (1968) introduced quasi-hyperbolic discounting. Laibson (1997) demonstrated the usefulness of quasi-hyperbolic discounting for economics applications. The quasi-hyperbolic discount function is given by

$$D(t) = \begin{cases} 1 & \text{if } t = 0 \\ \alpha\beta^t & \text{if } t > 0 \end{cases}$$

for some $\alpha \in (0, 1]$, and some $\beta > 0$.

Quasi-hyperbolic discounting model deviates from exponential discounting in a simple, yet profound way. A penalty, beyond the discount factor, is applied to any outcome that is not received immediately. When comparing two non-immediate timed outcomes, the penalties cancel out and the decision maker acts in accordance with exponential discounting. When one of the timed outcomes is paid immediately, only the delayed outcome is penalised, biasing the decision maker towards immediate payments.

Quasi-hyperbolic discounting is equivalent to constant discounting for all the future except the present. Some studies found support for the immediacy effect (Bleichrodt and Johannesson, 2001; Frederick, Loewenstein and O'Donoghue, 2002); others also found violations of constant discounting for later time intervals (Kirby and Herrnstein, 1995; Kirby, 1997; Lazaro, Barberan and Rubio, 2001).

Split Function Quasi-hyperbolic Discounting

As an extension of quasi-hyperbolic discounting, split function has the form:

$$D(t) = \begin{cases} \beta^t & \text{if } t \leqslant \lambda \\ \gamma\beta^t & \text{if } t > \lambda \end{cases}$$

with $\lambda \in [0, T]$, $\gamma \in (0, 1]$.

Under such form, outcomes occurring in the future, after λ, are penalised by a fixed factor γ. If $\lambda = 0$, then the model coincides with quasi-hyperbolic discounting, therefore allows for immediacy bias. Split Function quasi-hyperbolic discounting allows the "present", the time interval up to λ, to be subjective.

Thaler's (1981) example, however, forces either $\lambda \in$ [now, tomorrow], or $\lambda \in$ [1 year, 1 year and 1 day] to hold. These are restrictive requirements. As such, split function quasi-hyperbolic discounting cannot simultaneously explain minor adaptation of immediacy bias. If $\lambda \in$ [now, tomorrow], then immediacy bias with a front-end delay of anything larger than one day cannot be explained. If $\lambda \in$ [1 year, 1 year and 1 day], then immediacy bias with translation t less than 364 days, or greater than 366 days cannot be explained (assuming it is not a leap year).

2.4 Liminal Discounting

In this section we present the liminal discount function. The term 'liminal' is derived from the Latin *limen*, meaning a 'threshold'. The term is used in various fields (anthropology, sociology, medicine to name a few) to refer to being in states of transition. We use the term to mean the following: the decision maker's discount rate will change at some known point. Their discount rate will change, they know when, but it remains the same for the present.

The liminal discount function has the following form:

$$D(t) = \begin{cases} \alpha^t & \text{if } t \leqslant \lambda \\ (\alpha/\beta)^\lambda \beta^t & \text{if } t > \lambda \end{cases}$$

with $\lambda \in T$, $\alpha, \beta \in (0, 1)$.

Here λ is called the liminal point. It is a threshold, that separates periods before and after a change in time preference. When evaluating timed-outcomes that occur before λ, the decision maker makes the evaluation by using an exponential discount function with discount factor α. For timed-outcomes occurring after λ, the decision maker still makes the evaluation by using an exponential discount function, but with discount factor β. The weight $(\alpha/\beta)^\lambda$ ensures that the discount function is continuous everywhere.

Liminal discount function, with $\lambda \in [0, T]$ exhibits decreasing impatience if $\alpha \leqslant \beta$, and increasing impatience if $\alpha \geqslant \beta$. Notice that, since the utility function does not change after λ, we can make meaningful comparisons of the liminal discounter's discount factors.

It should be remarked that the psychological and philosophical interpretations of λ could be interesting. If we expand the theoretical environment to the whole society instead of just among a group of individuals, one may find that the value of λ can switch among various aspects of society dramatically. For instance, corporations, who are effectively immortal, could have λ of 200 years; while for individuals, who will die long before 200 years, it seems less plausible.

2.5 Preference Foundation

This section provides a preference foundation for the liminal discounting model. The result is presented in the framework of choice over timed-outcomes. The approach allows the liminal point to be detected from observed behaviour. First, two concepts are introduced: *stationarity-after-t* and *stationarity-before-t*.

Definition 2.5.1 (Stationarity-after-t)**.** *A preference relation $\succcurlyeq$ satisfies stationarity-after-t if for all $(t:x), (t+\tau:y), (s:x), (s+\tau:y) \in [0,T] \times [0,X]$ with $\tau > 0$ and $s > t$, the following holds:*

$$(t:x) \succcurlyeq (t+\tau:y) \quad \Rightarrow \quad (s:x) \succcurlyeq (s+\tau:y).$$

Stationarity-after-t demands that, when comparing two timed-outcomes with the *soonest* outcome occurring at time t, preferences are preserved when delaying each outcome by the same amount. Note that it is a one way implication; the preference regarding the earlier timed-outcome implying the preference regarding the later timed-outcome. One can verify, by assuming structural theorem, liminal discounting preferences satisfy stationary-after-t when $t > \lambda$.

Suppose that a violation of stationarity-after-t is observed. For liminal discounters, this can only happen due to their liminal point λ is later than t. Consequently, such an observation tells us, when the experiments are conducted to identify subjects' liminal points λ, there is no need to look before t if violations of stationarity-after-t are observed. Similarly, a violation of the following condition, stationarity-before-t, will rule out the possibilities of subjects' liminal points being any time after that particular t.

Definition 2.5.2 (Stationarity-before-t). *A preference relation $\succcurlyeq$ satisfies stationarity-before-t if for all $(t : x), (t - \tau : y), (s : x), (s - \tau : y) \in [0, T] \times [0, X]$ with $0 < \tau < s < t$, the following holds:*

$$(t : x) \succcurlyeq (t - \tau : y) \quad \Rightarrow \quad (s : x) \succcurlyeq (s - \tau : y).$$

Stationarity-before-t states that, when comparing two timed-outcomes with the *latest* outcome occurring at time t, preferences are preserved when bringing each outcome forward in time by the same amount. Again, this is a one-way implication. One may also verify, by substitution of the preference functional, that liminal discounting preferences satisfy stationarity-before-t when $t \leqslant \lambda$.

The essential axiom, *liminal stationarity*, is presented as following:

Axiom 2.5.1 (Liminal Stationarity). *For all time points $t \in [0, T]$, preferences $\succcurlyeq$ are stationary-before-t, or stationary-after-t, or both.*

To explain the necessity of Axiom 2.5.1, one can consider an arbitrary time $t \in [0, T]$, and a observed violation of stationarity-before-t. Then, since preferences are assumed to be liminal discounting preferences, it must be that the liminal point λ being in the interval $[0, t)$ is the cause of such violation. Then consider the interval after t, since λ cannot belong to this interval, stationarity-after-t must hold for this t under consideration. Hence, for all $t \in [0, T]$, the contradiction of one condition (stationarity-before/after-t), combined with the assumption that liminal discounting holds, always implies the other condition.

Liminal stationarity does not exclude violations of stationarity that occur when comparing the near and distant future. For example, take some t and suppose preferences are stationary-before-t. Then, it may well be the case that preference reversals occur when timed-outcomes before t are delayed to after t. Stationarity if not implied by the simultaneous satisfaction of stationarity-before-t and stationarity-after-t for one $t \in [0, T]$. Full stationarity requires the simultaneous satisfaction of stationarity-before-t and stationarity-after-t for all $t \in [0, T]$.

The following theorem provides the preference foundation for the liminal discounting model:

Theorem 2.5.1. *The following statements are equivalent:*

(i). The preference relation $\succcurlyeq$ over $[0, T] \times [0, X]$ is represented by a function

such that,

$$V(t:x) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant \lambda \\ (\alpha/\beta)^\lambda \beta^t u(x) & \text{if } t > \lambda \end{cases}$$

for some $\alpha, \beta \in (0,1)$, $\lambda \in [0,T]$ and a continuous, strictly increasing $u : [0,X] \to \mathbb{R}$.

(ii). The preference relation $\succcurlyeq$ over $[0,T] \times [0,X]$ is a continuous, monotonic, impatient and thomsen separable weak order that satisfies liminal stationarity .

The following Proposition outlines the uniqueness results pertaining to Theorem 2.5.1. A liminal point λ is *meaningful* if it is in $(0,T)$ and $\alpha \neq \beta$.

Proposition 2.5.1 (Uniqueness Results)**.** *Let the representation obtained in Theorem 2.5.1 hold for some $\alpha, \beta \in (0,1)$, $\lambda \in [0,T]$ and $u : [0,X] \to \mathbb{R}$. If a liminal point is not meaningful, then $\lambda \in \{0,T\}$ or $\alpha = \beta$, so liminal discounting collapses to exponential discounting and the uniqueness results expressed in Fishburn and Rubinstein (1982) hold. Now consider a meaningful liminal point. In this case, the liminal point is uniquely determined. Then, α, β and u are unique up to a joint power $\sigma > 0$ and factor $\tau > 0$ for u.*

The proof of Theorem 2.5.1 and Proposition 2.5.1 are provided in Appendix.

2.6 Discussion and Concluding Remarks

The aim of this chapter is to present liminal discounting, with an axiomatic foundation provided. The characterisation provides simple, testable condition (liminal stationarity) that merit empirical study. In Pan, Webb and Zank (2015), liminal discounting was also extended to dynamic choice by developing time consistent and time invariant (Halevy, 2011) version of the model. It turns out that whether the model captures time consistent or time invariant behaviour depends only on the interpretation of one parameter, the liminal point. If the liminal point is expressed in calendar time, say 24th December, then the model is time consistent. If it is expressed in waiting time, say one year from now, then the model is time invariant.

Furthermore, Pan, Webb and Zank (2015) gave these models a common, game theoretic application to the infinite-horizon, alternating-offers bargaining model of Rubinstein (1982). Rubinstein's model[5] provides clear predictions under exponential discounting, showing that there is a unique subgame perfect equilibrium that prescribes an immediate agreement. Pan, Webb and Zank (2015) investigated the cases where one of the players has liminal discounting preferences. Under such assumption, the subgame perfect equilibrium is different as there are incentives to delay agreement. The change in discount rate that occurs after the player's liminal point affects the subgame perfect equilibrium payoffs in a predictable way. The fact that the player's preferences will change at some point must be integrated into the determination of the equilibrium payoffs, whenever the agreement may be reached.

[5] The basic framework in Rubinstein's bargaining game is as follows. There are two players, 1 and 2, and a normalised surplus of size 1. The players have exponential discounting preferences with linear utility. The players alternate in proposing and considering offers regarding how the surplus should be divided. Player 1 proposes first at $t = 0$ and player 2 may accept or reject the proposal. If player 2 accepts the proposal, the game ends at that point and the payoffs are those specified in player 1's offer. If player 2 rejects the proposal, the game continues, and the players' previous roles are exchanged. Indefinite disagreement yields zero payoffs for both players.

2.7 Appendix

2.7.1 Proof of Theorem 2.5.1

First suppose that the preference relation over timed-outcomes is represented as in statement (i) of the theorem. That this implies statement (ii) is straightforward. Weak ordering and continuity follow immediately. Monotonicity follows as u is strictly increasing and $\alpha, \beta > 0$, and impatience follows as $\alpha, \beta \in (0, 1)$. Thomsen Separability follows from substitution of the representing functional stated in Structural Theorem Equation 2.1. The first two indifferences in Thomsen Separability imply

$$D(t)u(x) = D(t')u(x')$$
$$D(t'')u(x') = D(t)u(x'').$$

These two equalities jointly imply,

$$\frac{D(t')}{D(t'')} = \frac{u(x)}{u(x'')}.$$

The equality $D(t'')u(x) = D(t')u(x'')$ follows immediately, as does the equivalent required by the final indifference $(t'' : x) \sim (t' : x'')$.

The necessity of liminal stationarity, given statement (i) is explained next. Recall $\lambda \in [0, T]$, the following cases need to be considered:

Case I: If $\lambda = 0$, then both conditions of liminal stationarity hold.

Case II: If $\lambda = T$, then both conditions of liminal stationarity hold.

Case III: Suppose $\lambda \in (0, T)$. Taking any time $t \in [0, T]$, stationarity-before-t holds whenever $t \leqslant \lambda$.

Proof. The first preference relation in Definition 2.5.2, $(t : x) \succcurlyeq (t - \tau : y)$, represented by the liminal discount functional form when $t \leqslant \lambda$ in statement (i), yields

$$\alpha^t u(x) \geqslant \alpha^{t-\tau} u(y).$$

Multiplying $\left(\dfrac{\alpha^s}{\alpha^t}\right)$ on both sides of the above inequity gives

$$\alpha^s u(x) \geqslant \alpha^{s-\tau} u(y)$$

as $\dfrac{\alpha^s}{\alpha^t} > 0$. This directly implies the second preference relation in Definition 2.5.2, $(s : x) \succcurlyeq (s - \tau : y)$. $\square$

Case IV: Suppose $\lambda \in (0, T)$. Taking any time $t \in [0, T]$, stationarity-after-t holds whenever $t > \lambda$.

Proof. The first preference relation in Definition 2.5.1, $(t : x) \succcurlyeq (t + \tau : y)$, represented by the liminal discount functional form when $t > \lambda$ in statement (i), yields

$$\left(\frac{\alpha}{\beta}\right)^{\lambda} \beta^t u(x) \geqslant \left(\frac{\alpha}{\beta}\right)^{\lambda} \beta^{t+\tau} u(y).$$

Cancelling the common term, and multiplying $\left(\dfrac{\beta^s}{\beta^t}\right)$ on both sides of the above inequity gives

$$\beta^s u(x) \geqslant \beta^{s+\tau} u(y)$$

as $\dfrac{\beta^s}{\beta^t} > 0$. This directly implies the second preference relation in Definition 2.5.1, $(s : x) \succcurlyeq (s + \tau : y)$. $\square$

These cover all cases, so liminal stationarity is established.

Next, the sufficiency of statement (ii) for deriving statement (i) is proved, i.e., assuming the preference conditions, the representation in statement (i) is derived. The process is constructed through the following lemmas:

Lemma 2.7.1. *For all $t, t' \in [0, T]$, with $t' < t$, if stationarity-before-t and stationarity-after-t' hold simultaneously, then stationarity holds everywhere.*

Proof. Under weak ordering, and using Definition 2.5.1 and Definition 2.5.2 behind liminal stationarity, it is straightforward to establish that for any $t < t'$, if $\succcurlyeq$ satisfies stationarity-after-t then it satisfies stationarity-after-t'. Similarly, for any $t' < t$, if $\succcurlyeq$ satisfies stationarity-before-t then it satisfies stationarity-before-t'.

Suppose that the conditions of the claim, stationarity-before-t and stationarity-after-t', are true. The restriction of preferences to $[0, t] \times [0, X]$ satisfies all the conditions to admit an exponential discounting representation (Fishburn and Rubinstein, 1982). The same holds for preferences restricted to $[t', T] \times [0, X]$. By the uniqueness results attached to Fishburn and Rubinstein's theorem, one can choose the same discount factor δ for each case. Then, there will be a u such that $(t : x)$ mapped to $\delta^t u(x)$ represents preferences on $[0, t] \times [0, X]$, and a $\tilde{u}$ such that $(t : x)$ mapped to $\delta^t \tilde{u}(x)$ represents preferences on $[t', T] \times [0, X]$. By assumption, there is a set $[t', t] \times [0, X]$ where both functions must represent preferences, so they can be chosen to be equal. Then preferences over the whole set of timed-outcomes admit one exponential discounting representation, so stationarity must necessarily hold everywhere. $\square$

Lemma 2.7.2. *Suppose the conditions in statement (ii) of Theorem 2.5.1 hold,* *there exists an unique liminal point λ.*

Proof. Firstly, if the preference relation $\succcurlyeq$ is stationary, then one may choose either $\lambda = 0$ or $\lambda = T$.

Now, suppose that the conditions of statement (ii) hold, but stationarity does not hold. The following terms are defined:

$$t_* = \sup\{t \in [0, T] : \succcurlyeq \text{ satisfies stationarity-before-}t\}$$
$$t^* = \inf\{t \in [0, T] : \succcurlyeq \text{ satisfies stationarity-after-}t\}.$$

Hence, t_* represents the largest t satisfying stationarity-before-t, and t^* represents the smallest t satisfying stationarity-after-t. Liminal stationarity demands that $[0, T] = [0, t_*] \cup [t^*, T]$. By connectedness, if the union of $[0, t_*]$ and $[t^*, T]$ cover $[0, T]$, they must have a non-empty intersection. The situation where $t_* > t^*$ has been ruled out, as otherwise stationarity would hold everywhere as Lemma 2.7.1 argued. Therefore, there is a unique point in this intersection, $t_* = t^* := \lambda$, as required. □

Now the following lemmas determines the structure of $D : [0, T] \to \mathbb{R}$, considering it's behaviour on $[0, \lambda]$ and $[\lambda, T]$ separately.

Lemma 2.7.3. *Suppose the conditions in statement (ii) of Theorem 2.5.1 hold, $D(t)$ satisfies $D(t + s) = D(t)D(s)$ for all $t, s, t + s \leqslant \lambda$.*

Proof. For $t, s, t + s \leqslant \lambda$, and $x, x' \in [0, X]$, preferences satisfy stationarity-before-λ, then the following equivalence holds:

$$(0 : x) \sim (t : x') \;\;\Rightarrow\;\; (s : x) \sim (t + s : x').$$

The existence of suitable x and x' is straightforward. Substituting the separable representation in Equation 2.1,

$$D(0)u(x) = D(t)u(x')$$
$$\Rightarrow$$
$$D(s)u(x) = D(t+s)u(x').$$

Together with $D(0) = 1$, D satisfies the following local functional equation:

$$D(t+s) = D(t)D(s)$$

for all $t, s, t+s \in [0, \lambda]$. $\square$

The above lemma is the second of Cauchy's functional equations, restricted to a connected subset of the reals. The classic approach to solve this applies to the case where the equation holds on all of $\mathbb{R}$. Aczél and Luce (2007) has shown that there is an extension of D that preserves the functional equation. Their results apply here as D is strictly positive. The general, continuous solution gives $D(t) = \pi\alpha^t$ for all $t \leqslant \lambda$ for non-zero α and π. The initial condition, $D(0) = 1$, gives $\pi = 1$.

The following lemma is presented to derive a local functional equation on $[\lambda, T]$.

Lemma 2.7.4. *Suppose the conditions in statement (ii) of Theorem 2.5.1 hold, $D(t)$ satisfies $D(t+s) = D(t)D(s)$ for all $t, s, t+s \geqslant \lambda$.*

Proof. Stationarity-after-λ guarantees that, for $t, s, t+s \geqslant \lambda$, and $x, x' \in [0, X]$,

$$(\lambda : x) \sim (t : x') \quad \Rightarrow \quad (\lambda + s : x) \sim (t + s : x').$$

Substituting the separable representation in Equation 2.1,

$$D(\lambda)u(x) = D(t)u(x')$$
$$\Rightarrow$$
$$D(\lambda + s)u(x) = D(t + s)u(x'),$$

which together give,

$$\frac{D(\lambda)}{D(\lambda + s)} = \frac{D(t)}{D(t + s)}.$$

There is no $t \in [\lambda, T]$ with $D(t) = 1$. Define $\tilde{D}(t) = \dfrac{D(t)}{D(\lambda)}$ for all $t \in [\lambda, T]$. Notice that $\tilde{D}(\cdot)u(\cdot)$ still represents preferences and that $\tilde{D}(\lambda) = 1$. Substituting the above rescaled representation gives,

$$\tilde{D}(t + s) = \tilde{D}(t)\tilde{D}(s)$$

for all $t, s, t + s \in [\lambda, T]$.

The general, continuous solution is of the form $\tilde{D}(t) = \tilde{\pi}\beta^t$ for all $t \in [\lambda, T]$, for non-zero $\tilde{\pi}$ and β. The initial condition gives, $\tilde{D}(t) = \tilde{\pi}\beta^t$. Recall that $D = D(\lambda)\tilde{D}$ when $t \in [\lambda, T]$. $\square$

Summing up all the above lemmas, it has been shown that:

$$V(t:x) = D(t)u(x) = \begin{cases} \alpha^t u(x) & \text{if } t \leqslant \lambda \\ (\alpha/\beta)^\lambda \beta^t u(x) & \text{if } t > \lambda \end{cases}$$

as required.

2.7.2 Proof of Proposition 2.5.1

Assume preferences admit a liminal discounting representation $V : [0,T] \times [0,X] \to \mathbb{R}$ for some parameters $\alpha, \beta \in (0,1)$, $\lambda \in [0,T]$ and utility function $u : [0,X] \to \mathbb{R}$. The uniqueness of λ, when $\lambda \notin \{0,T\}$, has been explained in the proof of Theorem 2.5.1 in last subsection. Either λ is unique, or else stationarity must hold everywhere.

Since V represents preferences, it can be replaced by $f \circ V$ whenever f is strictly increasing. In general, such transformations need not retain the separable form. Suppose α is replaced with any $\tilde{\alpha} \in (0,1)$ and utility u replaced with $\tilde{u} = u^k$ with $k = \ln(\tilde{\alpha})/\ln(\alpha)$. One can verify that $\ln(\alpha^t u(x)) = (1/k)\ln(\tilde{\alpha}^t \tilde{u}(x))$ for all $(x,t) \in [0,\lambda] \times [0,X]$, and because ln is strictly increasing and $k > 0$, it must be that preferences over $[0,\lambda] \times [0,X]$ are represented by $\tilde{\alpha}^t \tilde{u}(x)$.

Similarly, one may verify that $\ln(\beta^t u(x)) = (1/k)\ln(\beta^{kt}\tilde{u}(x))$ for all $(x,t) \in [\lambda,T] \times [0,X]$, hence preferences over $[\lambda,T] \times [0,X]$ are represented by $\tilde{\beta}^t \tilde{u}(x)$ with $\tilde{\beta} = \beta^k$. By the same reasoning, one may choose any $\tilde{\beta} \in (0,1)$, and proceed as above replacing u and α appropriately.

Once α and β are chosen, utility must be a ratio scale. This follows from well-known results on separable representations, given that the location of utility is fixed. To see this, recall that we included the condition $(t : 0) \sim (t' : 0)$, for any $t, t' \in [0, T]$, in the definition of impatience. Then $u(0) = 0$ holds, or else the representation would not exhibit impatience.

Having chosen either of α or β, however, the other is uniquely determined. To see this, one may take any $x < y$ and find a unique t such that $(0 : x) \sim (t : y)$. Choose x and y such that $t > \lambda$. Substituting the representation and rearranging gives:

$$\beta = \left[\frac{u(x)}{\alpha^{\lambda} u(y)} \right]^{\frac{1}{t-\lambda}} .$$

Given that u is a ratio scale, the right hand side of the above equation is dimensionless. Therefore β, for given α (or vice versa), is uniquely determined.

Chapter 3

Separating Risk and Time Preferences: Discounted Utility with Increasing Optimism

3.1 Introduction

Most decisions we make today inevitably involve risky options that are to be resolved at some point in the future. Fisher (1930) identified *uncertainty* as an essential aspect in intertemporal decision-making. The other aspect that influences how future outcomes are perceived today is *time* (Samuelson, 1937). Traditionally, the interaction between time and uncertainty in intertemporal choice has been modelled by Discounted Expected Utility (DEU). In this model, Expected Utility (EU) has been used to account for uncertainty via subjective probabilities, and exponential discounting has been used to account for time via constant discount factor.

The EU model has been challenged by the works of Allais (1953) and Ellsberg (1961). Numerous alternatives to EU have been developed in static settings, which incorporate a non-linear treatment of probabilities (Quiggin, 1981, 1982; Yaari, 1987; Luce, 1991; Tversky and Kahneman, 1992). In dynamic settings, this raises the questions of how such distortions of probabilities develop over time.

The model proposed in this paper incorporates non-linear probability distortions into a dynamic model with discounted utility. The objects of choices are timed lotteries (simple probability distributions over outcomes). Time is captured by a general discounting function independent of probabilities and outcomes. Utility of outcomes is captured by standard von Neumann-Morgenstern utility, which doesn't change over time.

The novel aspect of the model is that probability treatment is modelled by a probability weighting function as in rank-dependent utility (Quiggin, 1981, 1982) and prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) that depends on time. Specifically, the weighting functions are parametric with one constant parameter across time while the second is time-dependent. As the latter parameter measures optimism, which in our model may increase over time, we call the model Discounted Utility with increasing Optimism (DUO). Since in our model increased optimism is equivalent to decreased pessimism, DUO reflects these two probabilistic risk attitudes jointly.

Empirical evidence shows that probability attitudes and time are interdependent (Keren and Roelofsma, 1995; Ahlbrecht and Weber, 1997; Weber and Chapman, 2005; Noussair and Wu, 2006; Anderson and Stafford, 2009; Baucells and Heukamp, 2010; Coble and Lusk, 2010; Abdellaoui, Diecidue and Öncüler, 2011). These results confirmed that subjects' probabilistic risk attitudes are significantly influenced by time. In particular, experiments have revealed that people tend to be more optimistic if the risk is delayed into the future. Baucells and Heukamp (2010) presented an experiment in which they add a common delay in a choice between two risky lotteries. Their results showed that the common delay implies a reduced preference for sure outcomes. In other words, subjects become more risk tolerant for delayed lotteries. Abdellaoui, Diecidue and Öncüler (2011) also found evidence of increasing risk tolerance for lotteries with delayed resolution. They showed that the impact of time is completely absorbed by the probability weighting function, and that such delay does not affect the utility. Higher risk tolerance can be explained by increased optimism. For this reason, DUO can capture increased optimism over time through a parametric weighting function.

Two of the most common empirical findings on decision-making under risk are that of optimism/pessimism about obtaining outcomes (overweighting small, respectively, underweighting large probabilities) and the diminishing effect of optimism/pessimism for intermediate probabilities. This has been modelled through probability weighting functions that have an inverse-S shape (Tversky and Kahneman, 1992; Prelec, 1998; Wakker, 2010). Gonzalez and Wu (1999) suggested that curvature and elevation are two independent psychological components underlying this inverse-S shape. They interpret curvature as reflecting the ability of an individual to discriminate between probabilities. Elevation reflects the propensity of an individual to accept a lottery, with higher elevation indicating a higher propensity to accept. Gonzalez and Wu provided empirical evidence in support of two-parameter probability weighting functions

,where one parameter measures elevation and the other measures curvature.

Abdellaoui, L'Haridon and Zank (2010) provided theoretical foundations for such two-parameter weighting functions accomplishing a complete separation of curvature and elevation. In their model, the elevation parameter controls the proportion of increased optimism to decreased pessimism, while the curvature parameter reflects diminishing effect of optimism and pessimism for intermediate probabilities. Also, their empirical evidence indicate that the elevation parameter is approximately 1/3. Several other studies suggest that the inverse-S shaped weighting function has a fixed point[1] at 0.32 (Tversky and Kahneman, 1992; Prelec, 1998; Bleichrodt and Pinto, 2000; Bleichrodt, Pinto and Wakker, 2001). In DUO, a dynamic version of Abdellaoui, L'Haridon and Zank's model with time-independent elevation parameter and time-dependent curvature parameter is presented. We define a measure of optimism that depends on both the curvature and the elevation parameters, which is time dependent. Further, preference foundations are proposed.

The chapter is organised as follows. First, basic notation of decision under risk and rank-dependent utility theory are set out in Section 3.2. Section 3.3 reviews the features of probability weighting functions. Then the proposed static index of relative optimism is developed in Section 3.4. Section 3.5 extends this index for the intertemporal context. Section 3.6 provides a preference foundation for the proposed representation. Concluding remarks are discussed in the end, followed by the Appendix with proofs.

[1] Note that such observations for the value of the elevation parameter are obtained under static settings, there's yet any evidence to suggest whether the elevation parameter would stay constant over time.

3.2 Preliminaries

The *outcome set* is $\mathbb{R}$, with real numbers designating money. A *prospect* is a finitely supported probability distribution[2] over $\mathbb{R}$. The generic notation for a prospect is $P = (p_1 : x_1, \ldots, p_m : x_m)$, yielding outcomes x_j with probability p_j for each j. Here m is a natural number that can be different for different prospects. It is implicitly understood that probabilities p_j are non-negative and sum to one, i.e., $p_j \geqslant 0$ and $\sum_{j=1}^{m} p_j = 1$. As a notational convention, outcomes are always assumed rank-ordered. The rank is assumed to agree with the natural ordering on the real-valued outcomes, i.e., for the notation of a prospect P, it is implicitly understood that $x_1 \geqslant \cdots \geqslant x_m$. The set of all prospects is denoted by L.

In its general form, rank-dependent utility theory requires a utility function to evaluate outcomes, and a probability weighting function that evaluates the probabilities associated with ranked outcomes, respectively. *Rank-dependent Utility* (RDU) holds if any prospect P is evaluated by

$$RDU(P) = \sum_{i=1}^{m} \left[w\left(\sum_{j=1}^{i} p_j\right) - w\left(\sum_{j=1}^{i-1} p_j\right)\right] u(x_i), \tag{3.1}$$

the RDU of the prospect[3]. Formally, the utility function assigns a real value to each outcome, and is strictly increasing and continuous. It is also unique up to scale and location, that is, u can be replaced by any $v = au + b$ for $a > 0$ and real value b, i.e., u is cardinal. The probability weighting function

[2] Extensions to prospects that are continuous distributions with infinite support, are possible for future research, using techniques developed in Kothiyal, Spinu and Wakker (2011).

[3] Note that $p_1 + \cdots + p_{i-1} = 0$ for $i = 1$, irrespective of the numbers $p_1, p_2, \ldots$. This also applies for any i that satisfies $i - 1 < 1$.

is an strictly increasing continuous mapping $w : [0,1] \to [0,1]$, with $w(0) = 0$ and $w(1) = 1$. In Equation 3.1, the weighting functions apply to cumulative probabilities. The differences in these distorted probabilities (i.e., the term $w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)$ in the notation above) are referred to as *decision weights* of the corresponding outcomes x_i, $i = 1, \ldots, m$. Under RDU, the weighting functions are uniquely determined. Note that RDU reduces to EU is the weighting functions are identity functions.

We assume that the prospects can be resolved and paid out within a finite and discrete period of time T such that $T = \{0, \ldots, n\}$ for $n \geqslant 2$. A prospect P that is resolved and paid out at time period $t \in T$ is called a *temporal prospect*, denoted by $P_t = (p_{1,t} : x_{1,t}, \ldots, p_{m_t,t} : x_{m_t,t})$. Since m indicates the number of outcomes in a temporal prospect, it may vary over time. Let $\mathcal{L} = L^{n+1}$ denote the set of all temporal prospects.

A *preference relation* $\succcurlyeq$ is assumed over $\mathcal{L}$, and its restriction to subsets of $\mathcal{L}$ (e.g., all degenerate temporal lotteries, L) is also denoted by $\succcurlyeq$. As usual, the symbol $\succ$ denotes strict preference while $\sim$ denotes indifference ($\preccurlyeq$ and $\prec$ denote reversed weak and strict preferences, respectively).

Under *Discounted Expected Utility* (DEU), the representing functional for $\succcurlyeq$ evaluates any $\mathcal{P} \in \mathcal{L}$ by

$$DEU(\mathcal{P}) = \sum_{t=0}^{n} D(t) EU(P_t). \tag{3.2}$$

Here $D(t) = \sigma^t$ is normally assumed to be exponential discounting function, with σ being constant "pure" time preference. A temporal prospect P_t is evaluated by expected utility

$$EU(P_t) = \sum_{i=1}^{m_t} p_{i,t} u(x_{i,t}), \tag{3.3}$$

with utility function $u : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and cardinal.

The focus of this chapter is on a more general representation. Under *Discounted Rank-dependent Utility* (DRU), the representing functional for $\succcurlyeq$ evaluates any $\mathcal{P} \in \mathcal{L}$ by

$$DRU(\mathcal{P}) = \sum_{t=0}^{n} D(t) RDU_t(P_t). \tag{3.4}$$

That is, a temporal prospect P_t is represented by rank-dependent utility

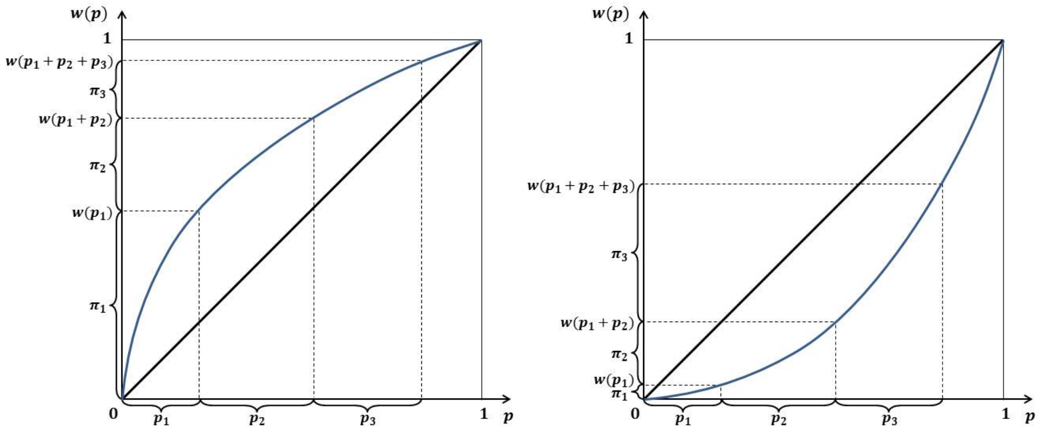
$$RDU_t(P_t) = \sum_{i=1}^{m_t} \left[w_t(\sum_{j=1}^{i} p_{j,t}) - w_t(\sum_{j=1}^{i-1} p_{j,t}) \right] u(x_{i,t}). \tag{3.5}$$

Like under EU, the utility function $u : \mathbb{R} \to \mathbb{R}$ for RDU is continuous, strictly increasing and cardinal, and is independent of time. The probability weighting function, which transforms the objective probabilities into subjective measurements, is time-dependent. It captures *intertemporal probabilistic risk attitude* (Wakker, 1994, 2010).

3.3 Probability Weighting Functions

Probability weighting functions, as argued before, play an important role in the proposed model. It is also key to understand RDU theory as a deviation from EU theory. This section reviews some essential properties of probability weighting functions under static settings. More specially, it discusses and highlights the role of probability weighting functions in capturing the effect of probability weighting on risk behaviour, which is normally referred as *probabilistic risk attitude*.

The inputs of probability weighting functions w are cumulative probabilities, hence, depending on ranking of outcomes. Recall that the outcomes are ordered from best to worst. Naturally, as shown in Figure 3.1, the part of domain of w that is close to 0, is for small cumulative probabilities p and is relevant for the best outcomes. The part of domain of w that is close to 1, is for large cumulative probabilities p and is relevant for the worst outcomes.



(a) Concave w, optimism (b) Convex w, pessimism

Figure 3.1: Probability weighting functions and Probabilistic risk attitude

The decision weight π of an outcome x_i, is formally defined as $\pi_i = w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)$, $j = 1, \ldots, m$. From the definition and illustration in Figure 3.1, one can see that the decision weight of an outcome will be affected by the steepness of w around cumulative probability associated with the outcome rather than by the absolute level of w. In the case of EU, the decision weights are the probabilities of obtaining the respective outcome, i.e., $\pi_j = p_j$, $j = 1, \ldots, m$.

Consider a simple prospect $P = (p_1 : x_1,\, p_2 : x_2,\, p_3 : x_3)$. A weighting function, w, is *concave* if for all probabilities p_1, p_2, p_3 such that $p_1 + p_2 + p_3 \leqslant 1$, $w(p_1+p_2) - w(p_1) \geqslant w(p_1+p_2+p_3) - w(p_1+p_2)$, i.e., $\pi_2 \geqslant \pi_3$ is satisfied. From Figure 3.1a, one can also observe this property, as w is steep towards small cumulative probabilities, with large differences of w and hence, large decision weights are obtained for higher ranked, more favourable outcomes. Meanwhile, w is shallow/flat towards large probabilities, yielding small decision weights for lower ranked, less favourable outcomes. Such weighting functions exhibit strong sensitivity towards changed in probabilities away from 0 but relatively little sensitivity towards changes in probabilities away from 1. Moreover, a concave weighting function characterizes *probabilistic risk proneness*, also known as *optimism*. It is demonstrated in the Figure 3.1a as w overweights cumulative probabilities, i.e., $w(p) > p$ for all $p \in (0, 1)$.

Similarly, A weighting function, w, is *convex* if for all probabilities p_1, p_2, p_3 such that $p_1 + p_2 + p_3 \leqslant 1$, $w(p_1 + p_2) - w(p_1) \leqslant w(p_1 + p_2 + p_3) - w(p_1 + p_2)$, i.e., $\pi_2 \leqslant \pi_3$ is satisfied. It implies large decision weights are obtained for lower ranked, less favourable outcomes, and small decision weights for higher ranked more favourable outcomes (Figure 3.1b). Such weighting functions exhibit little sensitivity towards changes in probabilities away from 0 but extreme sensitivity towards changes in probability away from 1. *Probabilistic risk aversion*, or *pessimism* is captured by w underweighting cumulative probabilities, i.e., $w(p) < p$ for all $p \in (0, 1)$. Naturally, a linear weighting function characterizes *probabilistic risk neutrality*, which is the case of EU ($w(p) = p$ for all $p \in [0, 1]$).

Nonetheless, optimistic and pessimistic attitudes of paying special attention towards favourable and unfavourable outcomes respectively, a commonly found derivation from EU, is frequently reported in the studies (van Osch and Stiggelbout, 2008; Sherrick, Sonka, Lamb and Mazzocco, 2000; Showers, 1992). Considerably more than 50 percent of all car drivers assume that they belong to the best 50 percent of car drivers (Guppy, 1993). Such phenomenon is often called unrealistic optimism or overconfidence (Hoelzl and Rustichini, 2005; van den Steen, 2004; Wenglert and Rosén, 2000). It can be accommodated by probability weighting and rank dependence whereas it was not possible under EU. Recall that the outcomes are ranked from best to worst. The above observations indicate a weighting function that overweights small probabilities and underweights large probabilities.

Another common empirical finding is that of large decision weights for unlikely extreme outcomes (best and worst) (MacCrimmon and Larsson, 1979; Kahneman and Tversky, 1979; Allais, 1953). Such decision weights indicate that individuals are extremely sensitive to changes in cumulative probabilities close to 0 and 1, i.e., they exhibit diminishing sensitivity for probabilities.

As a result, the family of inverse-S shaped weighting functions seem to be more plausible to capture above findings, which is also confirmed by many empirically estimated weighting functions (Baucells and Heukamp, 2010; Abdellaoui, Vossmann and Weber, 2005; Abdellaoui, 2000; Gonzalez and Wu, 1999; Wu and Gonzalez, 1996; Tversky and Kahneman, 1992; Camerer and Weber, 1992; Kahneman and Tversky, 1979; Karmarkar, 1979, 1978).

Relative to EU, the weighting function is shallower in the middle region, and steeper near both end points (Figure 3.2). Formally, that is, for some σ, $w(\sigma) = \sigma$; for $p \in (0, \sigma), w(p) > p$ and is concave; for $p \in (\sigma, 1), w(p) < p$

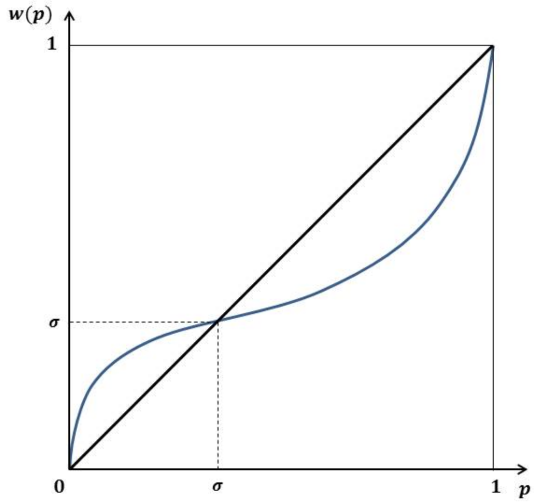


Figure 3.2: Inverse-S shaped probability weighting function

and is convex. Empirical studies, also suggest that the point that intersects the diagonal σ approximately equals to 1/3 (Prelec, 1998; Abdellaoui, 2000; Bleichrodt and Pinto, 2000; Bleichrodt, Pinto and Wakker, 2001; Abdellaoui, Barrios and Wakker, 2007).

While RDU being a more descriptive model than EU, it has a disadvantage of having more parameters than EU. Consequently, it is more difficult to estimate those parameters from data. The following section will discuss more on the psychological meaning of the parameters, and propose an index of measurement for individual's probabilistic risk attitudes.

3.4 An Index of Relative Optimism

Under EU, risk attitudes are modelled through utility, more specifically, curvature of utility. Classical economics identifies risk aversion with concave utility, and the Arrow-Pratt utility index is used as a measure of risk aversion (Arrow, 1965; Pratt, 1964).

Utility has been a controversial concept throughout the history of economics, with various interpretation over time. The nature of utility has been debated extensively in the literature (see Abdellaoui, Barrios and Wakker, 2007 for a comprehensive review of the historic development of the concept). The classical decision-theoretic studies invariably assumed EU for analysing risky decisions. Under such assumption, a difference between marginal utility and risk attitude necessarily implies that there must be a non-linear relation between risky and riskless utility. Under RDU, however, aspects of risk attitude not captured by marginal utility can be explained by probability weighting, so that the main reason for classical decision-theoretic studies to distinguish between risky and riskless utility disappears.

In the RDU models, the utility function $u(x)$, captures sensitivity towards outcomes, while the probability weighting function $w(p)$, capturing sensitivity towards probabilities. RDU generalises EU as it reduces to EU when w is the identity, i.e., $w(p) = p$. A probability weighting function permits objective probabilities to be weighted non-linearly. As mentioned before, the term *probabilistic risk attitude* is used to refer to the effect of probability weighting on risk attitudes, i.e. optimism and pessimism (Wakker, 1994, 2001; Abdellaoui, 2002; Wakker, 2010). A concave weighting function features overweighting, and enhances optimism and risk seeking. A convex weighting function features underweighting and enhances pessimism and risk aversion. Under RDU, risk averse behaviour depends on both utility and probability weighting. For instance, a risk averse person can have strictly convex utility if probability weighting generates sufficient risk aversion (or pessimism) (Chateauneuf and

Cohen, 1994, Corollary 2).

Gonzalez and Wu (1999) provided psychological arguments for the curvature and elevation of weighting functions. They interpreted the curvature of the weighting function as reflecting the ability of an individual to discriminate between probabilities. For instance, an individual who put more decision weight on a 1% probability if added to a 99% probability of a good outcomes than if added to a 10% probability of the same outcome, shows less ability to discriminate (diminishing sensitivity). The interpretation given to elevation is that it reflects how confident/prone an individual is to the domain of prospects (optimism and pessimism).

They further argued that curvature and elevation are logically independent aspects and that this should be reflected in two separate parameters within the weighting function, like the ones proposed by Goldstein and Einhorn (1987); Lattimore, Baker and Witte (1992); Prelec (1998). Such weighting functions with two parameters, which influence curvature and elevation, respectively, provide a plausible account for discriminablity of probabilities and attraction to prospects, and hence, for observed probabilistic risk attitude in general.

Abdellaoui, L'Haridon and Zank (2010) proposed a class of parametric weighting functions (see also Diecidue, Schmidt and Zank, 2009) that achieve a natural separation between a parameter controlling for curvature and a parameter controlling for elevation. The constant relative sensitivity (CRS) weighting functions have the form:

$$w(p) = \begin{cases} \sigma^{1-\gamma} p^{\gamma} & \text{if } 0 \leqslant p \leqslant \sigma \\ 1 - (1-\sigma)^{1-\gamma}(1-p)^{\gamma} & \text{if } \sigma < p \leqslant 1 \end{cases}$$

with $0 \leqslant \sigma \leqslant 1$ and $\gamma > 0$.

It is easy to observe that the CRS weighting functions are power functions (concave) on the interval $[0, \sigma]$, which enhances optimism and risk seeking, and dual power functions (convex) on the interval $[\sigma, 1]$, which enhances pessimism and risk aversion. Evidence shows the intersection point between the weighting functions and the diagonal σ is approximately 0.33 (Prelec, 1998; Abdellaoui, 2000; Bleichrodt and Pinto, 2000; Bleichrodt, Pinto and Wakker, 2001; Abdellaoui, Barrios and Wakker, 2007).

One can easily validate that for such weighting function, the best-ranked outcomes receive large decision weights and the worst-ranked outcomes receive even larger decision weights. Through such decision weights, the effect of optimism and pessimism can be modelled.

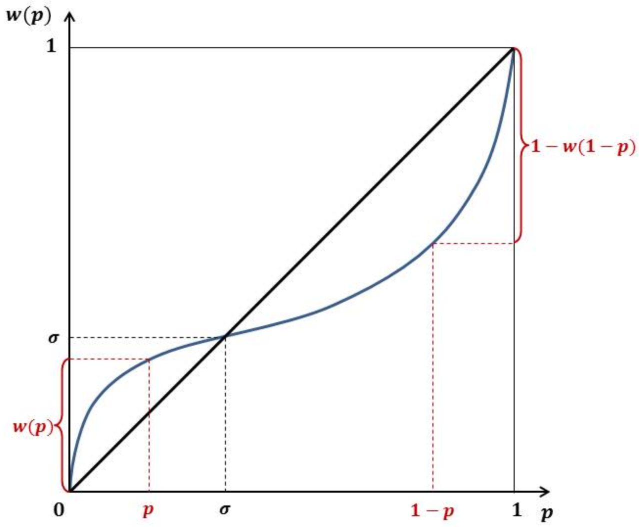


Figure 3.3: Illustration of Relative Optimism Index

For simplicity, a binary prospect is denoted by $(p : x, 1 - p : y)$ and $x > y$. Here p is the probability of obtaining better outcome x, $1-p$ is the probability of obtaining worse outcome y. We define the proportion of the decision weights generated by changes away from σ influenced by optimism (i.e., changed for probabilities in $[0, \sigma]$) and the decision weights generated by corresponding dual changes away from σ by pessimism as the index of *relative optimism* θ, see Figure 3.3 for simple illustration.

Formally, the index of relative optimism is defined as,

$$\theta = \begin{cases} \dfrac{w(p)}{1 - w(1-p)} & \text{if } p \leqslant \sigma \\ \dfrac{w(1-p)}{1 - w(p)} & \text{if } p > \sigma. \end{cases}$$

By substituting the CRS parametric form, the relative optimism index under both above circumstances[4] is given by $\theta = \left(\dfrac{\sigma}{1-\sigma}\right)^{1-\gamma}$, which depends only on parameters that represent curvature and elevation of the probability weighting function, respectively.

The CRS weighting functions allow for a comparative analysis based on the index of relative optimism, θ. With extensive empirical evidence indicating σ is around $1/3$ as mentioned before, it is plausible to keep σ constant for such a comparative analysis. The other parameter γ reflects degree of curvature of the weighting function (Abdellaoui, L'Haridon and Zank, 2010). As a result,

[4] Recall CRS functional form, hence, when $p \leqslant \sigma$,

$$\theta = \frac{w(p)}{1 - w(1-p)} = \frac{\sigma^{1-\gamma} p^{\gamma}}{1 - [1 - (1-\sigma)^{1-\gamma} p^{\gamma}]} = \left(\frac{\sigma}{1-\sigma}\right)^{1-\gamma};$$

when $p > \sigma$

$$\theta = \frac{w(1-p)}{1 - w(p)} = \frac{\sigma^{1-\gamma}(1-p)^{\gamma}}{1 - [1 - (1-\sigma)^{1-\gamma}(1-p)^{\gamma}]} = \left(\frac{\sigma}{1-\sigma}\right)^{1-\gamma}.$$

the graph of the weighting function is rotated around (σ, σ). An individual with more rotated around (σ, σ) weighting functions is more optimistic and also more pessimistic, and more sensitive towards probabilities of extreme outcomes. More precisely, such individual's behaviour suggests more deviation from EU.

Mathematically, since $\sigma < \dfrac{1}{2}$ (hence $\dfrac{\sigma}{1-\sigma} < 1$) and $0 < \gamma < 1$, the closer γ is to 1, the higher is the value of θ, keeping σ constant. Hence the weighting function capturing more (relative) optimistic risk attitude is closer to linearity.

3.5 Intertemporal Relative Optimism

This section proceeds further with a discussion of the relative optimism index in the intertemporal context. The time-dependent CRS weighting functions have the form

$$w_t(p_{i,t}) = \begin{cases} \sigma^{1-\gamma_t} p_{i,t}^{\gamma_t} & \text{if } 0 \leqslant p_{i,t} \leqslant \sigma \\ 1 - (1-\sigma)^{1-\gamma_t}(1-p_{i,t})^{\gamma_t} & \text{if } \sigma < p_{i,t} \leqslant 1 \end{cases}$$

with $0 \leqslant \sigma \leqslant 1$ and $\gamma_t > 0$.

A temporal binary prospect, denoted by $(p : x, 1-p : y)_t$ for $x > y$, is used for simplicity of exposition. The probability of getting the better outcome x in period t is denoted by p, and the probability of getting the worse outcome y in period t is denoted by $1-p$. A decision maker with CRS probability weighting function has relative optimism at time period t as $\theta_t = \left(\dfrac{\sigma}{1-\sigma}\right)^{1-\gamma_t}$.

Baucells and Heukamp (2010) tested the following preference reversal:

$$\begin{aligned}(1:300)_{\text{now}} &\succ (0.8:400, 0.2:0)_{\text{now}}\\ (1:300)_{6\text{ months}} &\prec (0.8:400, 0.2:0)_{6\text{ months}}\end{aligned}$$

It shows that the more risky prospect with larger reward would be preferred after the prospects are delayed by 6 months, while now the safer prospect is chosen.

Recall that an individual's preference $\succcurlyeq$ is defined over the set of temporal lotteries $\mathcal{L}$. Assume such preferences at period t and s, can each be modelled by RDU_t, RDU_s, underlying basic utility function u, time-dependent probability weighting functions w_t, w_s, respectively.

Definition 3.5.1. *Increasing relative optimism holds if for all p, q, x, y, t, s, with $x < y$, $x, y \in \mathbb{R}$, $t < s$, $t, s \in T$ and $\sigma < q < p \leqslant 1$:*

$$\begin{aligned}(p:x, 1-p:0)_t &\succ (q:y, 1-q:0)_t\\ &\Rightarrow\\ (p:x, 1-p:0)_s &\prec (q:y, 1-q:0)_s\end{aligned}$$

Substituting Equation 3.4 and Equation 3.5, the two resulting inequalities together give:

$$\frac{w_t(p)}{w_t(q)} > \frac{w_{t+1}(p)}{w_{t+1}(q)}.$$

Note $p, q \in (\sigma, 1]$, it is more convenient to use the notation of the dual weighting function, i.e.,

$$\frac{1-w_t(1-p)}{1-w_t(1-q)} < \frac{1-w_s(1-p)}{1-w_s(1-q)}.$$

Such weighting function is dual to the (regular) weighting function that corresponds to cumulative distribution of the prospects, which assign to the outcome the probability of that outcome or better (see Zank, 2010; Abdellaoui, L'Haridon and Zank, 2010; Abdellaoui, 2002, for formal definitions of both notations). It corresponds to the decumulative distribution of the prospects, and graphically is represented by the (regular) weighting function flipped horizontally.

Adopting the parametric form of time-dependent CRS probability weighting function , the above inequality results in

$$\left(\frac{p}{q}\right)^{\gamma_t} < \left(\frac{p}{q}\right)^{\gamma_s},$$

which implies $\gamma_t < \gamma_s$ since $p > q$.

Therefore, by the definition of the decision maker's relative optimism index presented in previous section, the following proposition holds.

Proposition 3.5.1. *The DUO decision maker exhibits increasing relative optimism if: $\theta_t < \theta_s$ for every $t < s$, $t, s \in T$.*

From the discussion above, one can clearly see that the parameter controlling for curvature of the weighting functions decreases[5] over time. It indicates that the deviation from EU diminishes over time (see Figure 3.4).

In other words, people tend to be more optimistic when risk is delayed. This suggests a possible hypothesis that, if the prospect is sufficiently far away

[5] In CRS weighting functions, $1 - \gamma$ reflects the curvature of the weighting functions (Abdellaoui, L'Haridon and Zank, 2010). Also we have $1 - \gamma_t > 1 - \gamma_s$ as $\gamma_t < \gamma_s$ for all $t < s$.

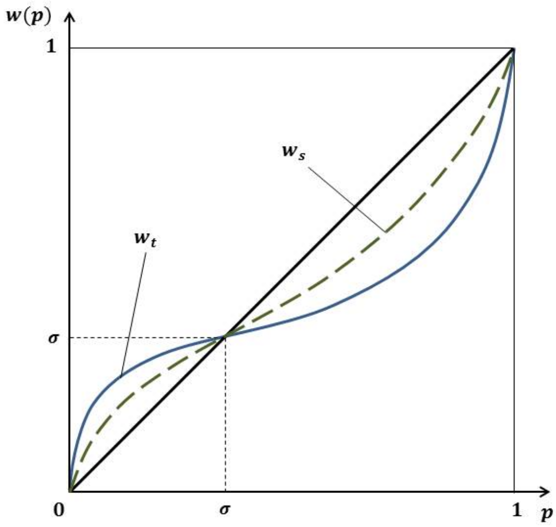


Figure 3.4: Dynamic Inverse-S Probability Weighting Function with $t < s$, $t, s \in T$

from the present, the decision maker exhibits less probability distortion, i.e., they behave as expected utility maximisers.

3.6 A Preference Foundation

Preference foundations give necessary and sufficient conditions for a decision model, stated directly in terms of the empirical primitive: the preference relation. Since preferences are directly observable, preference foundations identify the empirical meaning of a model.

This section presents such an axiomatic preference foundation for the DUO model with a two-parameter weighting function, i.e., CRS, that incorporate increasing (relative) optimism over time. The result is presented in the framework of choices over temporal prospects.

We are interested in conditions for a preference relation $\succcurlyeq$ on the set of all temporal prospects $\mathcal{L}$, in order to represent the preference relation by a function V. That is, the *representation* or *representing function* $V : \mathcal{L} \to \mathbb{R}$, assigns to each temporal prospect a real number, such that for all $\mathcal{P}, \mathcal{Q} \in \mathcal{L}$,

$$\mathcal{P} \succcurlyeq \mathcal{Q} \;\Leftrightarrow\; V(\mathcal{P}) \geqslant V(\mathcal{Q}).$$

If such a representing function exists then $\succcurlyeq$ must be a *weak order*, i.e. $\succcurlyeq$ is *complete* ($\mathcal{P} \succcurlyeq \mathcal{Q}$ or $\mathcal{P} \preccurlyeq \mathcal{Q}$ for all $\mathcal{P}, \mathcal{Q} \in \mathcal{L}$) and *transitive* ($\mathcal{P} \succcurlyeq \mathcal{Q}$ and $\mathcal{Q} \succcurlyeq \mathcal{R}$ implies $\mathcal{P} \succcurlyeq \mathcal{R}$ for all $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{L}$). Weak ordering entails a ranking of prospects with ties permitted.

In general being a weak order alone is not sufficient for $\succcurlyeq$ to admit such representation. Further conditions are required to guarantee the existence of a representing function. In the following content, by $\mathbb{R}^{m_t}_{\downarrow}$ we denote the set of rank-ordered m_t-tuples from $\mathbb{R}$ for $t \in T$, i.e.,

$$\mathbb{R}^{m_t}_{\downarrow} = \{(x_{1,t}, \dots, x_{m_t,t}) \in \mathbb{R}^{m_t} : x_{1,t} \geqslant \cdots \geqslant x_{m_t,t}\}.$$

The preference relation $\succcurlyeq$ satisfies *continuous* if for all $t \in T$, and $p_{j,t} \in [0,1]$ for $j = 1, \dots, m_t$, the sets

$$\{(x_{1,t}, \dots, x_{m_t,t}) : (p_{1,t} : x_{1,t}, \dots, p_{m_t,t} : x_{m_t,t}) \succcurlyeq (p_{1,t} : y_{1,t}, \dots, p_{m_t,t} : y_{m_t,t})\}$$

and

$$\{(x_{1,t}, \dots, x_{m_t,t}) : (p_{1,t} : x_{1,t}, \dots, p_{m_t,t} : x_{m_t,t}) \preccurlyeq (p_{1,t} : y_{1,t}, \dots, p_{m_t,t} : y_{m_t,t})\}$$

are closed sets in the Euclidean space $\mathbb{R}_{\downarrow}^{m_0} \times \cdots \times \mathbb{R}_{\downarrow}^{m_n}$. Closed refers to the "product topology", i.e., it generates the natural continuity in $\prod_{t=0}^{n} \mathbb{R}_{\downarrow}^{m_t}$.

Monotonicity is defined with respect to the preference order over outcomes at all time periods. That is, $\succcurlyeq$ satisfies *monotonicity* if for all $(x_{1,t}, \ldots, x_{m_t,t})$, $(y_{1,t}, \ldots, y_{m_t,t}) \in \prod_{t=0}^{n} \mathbb{R}_{\downarrow}^{m_t}$, $x_{j,t} \geqslant y_{j,t}$ for all j implies $(p_{1,t} : x_{1,t}, \ldots, p_{m_t,t} : x_{m_t,t}) \succcurlyeq (p_{1,t} : y_{1,t}, \ldots, p_{m_t,t} : y_{m_t,t})$, with a strict preference if $x_{j,t} > y_{j,t}$ for at least one j with $p_{j,t} > 0$. It states that, assuming temporal prospects are resolved and paid out at the same time $t \in T$, monotonicity holds if the preference relation over temporal prospects agrees with the natural ordering of the outcomes. In other words, the more money the better.

With weak ordering, continuity and monotonicity defined above, for each temporal prospect that is resolved and paid at time period t, a unique certainty equivalent CE_t exists (Wakker, 1989).

Now, we focus on the additive separability across and within time dimensions for the representing function V. In order to formally define the condition, some useful notation needs to be introduced.

Definition 3.6.1. *For a subset $S \subseteq T$, and $\mathcal{P}, \mathcal{R} \in \mathcal{L}$, define*

$$\mathcal{R}_S\mathcal{P} = \begin{cases} R_t & \text{if } t \in S \\ P_t & \text{if } t \notin S. \end{cases}$$

Note that $\mathcal{R}_S\mathcal{P}$ is the set of temporal lotteries that is identical to $\mathcal{P}$ except that, all lotteries that are resolved and paid within the period of time S have been replaced by $\mathcal{R}$.

The following independence condition for time is required.

Axiom 3.6.1 (Time Independence)**.** *The preference relation $\succcurlyeq$ satisfies time independence on $\mathcal{L}$ if for any $\mathcal{R}_S\mathcal{P}, \mathcal{R}_S\mathcal{Q}, \mathcal{R}'_S\mathcal{P}, \mathcal{R}'_S\mathcal{Q} \in \mathcal{L}$, it holds that*

$$\mathcal{R}_S\mathcal{P} \succcurlyeq \mathcal{R}_S\mathcal{Q}$$
$$\Leftrightarrow$$
$$\mathcal{R}'_S\mathcal{P} \succcurlyeq \mathcal{R}'_S\mathcal{Q}$$

for $S \subseteq T$.

The time independence requires that preferences between temporal prospects be independent of the time periods, in which common prospects are resolved and paid. It ensures the representative functional across time periods is additive separable, and further enforces that within the time periods, the functional is additive separable, i.e. Lemma 1 in Abdellaoui, L'Haridon and Zank (2010) is true.

Assumption 3.6.1. *For each time period of a temporal lottery $\mathcal{P}$, the axioms in Theorem 2 of Abdellaoui, L'Haridon and Zank (2010) are satisfied.*

Theorem 2 of Abdellaoui, L'Haridon and Zank (2010) characterises the class of RDU preferences with weighting functions which are CRS weighting functions. Together with time independence, the preference relation $\succcurlyeq$ on $\mathcal{L}$ is represented by

$$V(\mathcal{P}) = \sum_{t=0}^{n} \Phi_t[RDU_t(P_t)], \tag{3.6}$$

where $\Phi_t : \mathbb{R} \to \mathbb{R}$ are continuous and jointly cardinal, with uniquely defined weighting function

$$w_t(p_{i,t}) = \begin{cases} \sigma_t^{1-\gamma_t} p_{i,t}^{\gamma_t} & \text{if } 0 \leqslant p_{i,t} \leqslant \sigma_t \\ 1-(1-\sigma_t)^{1-\gamma_t}(1-p_{i,t})^{\gamma_t} & \text{if } \sigma_t < p_{i,t} \leqslant 1. \end{cases}$$

Now we introduce an additional condition that forces, for each period, the preference relation can be represented by a linear transformation of RDU representation. It is similar with the idea of derived trade-offs for outcomes used in Wakker (1994) and for probabilities used in Köbberling and Wakker (2003) and Abdellaoui (2002).

Definition 3.6.2. *For a temporal lottery $\mathcal{P}_s\mathcal{R} \in \mathcal{L}$, $s \in T$, define $(p : x, 1-p : z)_s\mathcal{R}$ as a written for $\mathcal{P}_s\mathcal{R}$ with P_s replaced by a binary lottery that getting x with probability p, and z otherwise, $x > z$.*

Axiom 3.6.2 (Intertemporal Trade-off Consistency)**.** *The preference relation $\succcurlyeq$ satisfies intertemporal trade-off consistency if for some probability p:*

$$\begin{array}{rcl} (p : x, 1-p : z)_s\mathcal{P} & \sim & (p : y, 1-p : z')_s\mathcal{Q} \\ (p : x', 1-p : z)_s\mathcal{P} & \sim & (p : y', 1-p : z')_s\mathcal{Q} \\ (p : x, 1-p : \tilde{z})_{s'}\mathcal{R} & \sim & (p : y, 1-p : \tilde{z}')_{s'}\mathcal{S} \\ & \Leftrightarrow & \\ (p : x', 1-p : \tilde{z})_{s'}\mathcal{R} & \sim & (p : y', 1-p : \tilde{z}')_{s'}\mathcal{S} \end{array}$$

Lemma 3.6.1. *Assuming Equation 3.6 holds, $\Phi_t : \mathbb{R} \to \mathbb{R}$ are linear and positive if the preference relation $\succcurlyeq$ is a continuous monotonic weak order that satisfies intertemporal trade-off consistency.*

With Lemma 3.6.1 true, we separate the discount factor from RDU_t for each time period t, $t \in T$. An additional condition is needed to have the same probabilistic risk attitude across time, but different relative sensitivity over time.

Axiom 3.6.3 (Constant Elevation)**.** *The preference relation $\succcurlyeq$ satisfies constant elevation if*

$$\begin{array}{ccc} (\sigma_s : x, 1-\sigma_s : z)_s\mathcal{P} & \sim & (\sigma_s : y, 1-\sigma_s : z')_s\mathcal{P} \\ & \Leftrightarrow & \\ (\sigma_{s'} : x, 1-\sigma_{s'} : z)_{s'}\mathcal{Q} & \sim & (\sigma_{s'} : y, 1-\sigma_{s'} : z')_{s'}\mathcal{Q} \end{array}$$

Lemma 3.6.2. *The RDU representation with time-dependent CRS weighting function has one parameter σ that is independent of time, iff the preference relation $\succcurlyeq$ satisfies constant elevation.*

The following theorem provides the preference foundation for DRU with time-dependent CRS weighting functions. In addition, the elevation parameter σ is constant overtime while the curvature parameter γ is time-dependent.

Theorem 3.6.1. *The following two statements are equivalent for a preference relation $\succcurlyeq$ on $\mathcal{L}$:*

(i). The preference relation $\succcurlyeq$ on $\mathcal{L}$ is represented by DRU, i.e., Equation 3.4 and Equation 3.5, with time-dependent CRS weighting functions and time-independent σ in each time period, i.e.,

$$w_t(p_{i,t}) = \begin{cases} \sigma^{1-\gamma_t} p_{i,t}^{\gamma_t} & \text{if } 0 \leqslant p_{i,t} \leqslant \sigma \\ 1-(1-\sigma)^{1-\gamma_t}(1-p_{i,t})^{\gamma_t} & \text{if } \sigma < p_{i,t} \leqslant 1 \end{cases}$$

with $0 \leqslant \sigma \leqslant 1$ *and* $\gamma_t > 0$.

(ii). The preference relation $\succcurlyeq$ *satisfies weak order, monotonicity, continuity, time independence and further, within each time period Assumption 3.6.1 holds and across time the preference relation* $\succcurlyeq$ *satisfies intertemporal trade-off consistency and constant elevation.*

The parameters γ_t *and* σ *are uniquely determined, the utility function* u *is cardinal and the discounting function* D *is linear and positive.*

The proof of Theorem 3.6.1 naturally follows by proofing Lemma 3.6.1 and Lemma 3.6.2 jointly, which are provided in the Appendix.

3.7 Conclusion

The main objective in this chapter was to separate the risk and time preferences in intertemporal risky decision-making.

With additive separability, lotteries that are realised in different time periods can be modelled by the summary of linear transformations of RDU representations. Within each distinct time period, the highlight of this chapter is to provide theoretical and axiomatic analysis for a parametric weighting functions suggested by Abdellaoui, L'Haridon and Zank (2010), called constant relative sensitivity function in dynamic framework. This function contains two parameters indicating probabilistic risk attitudes which provides a better explanatory power.

One of the parameters represents the relative strength of optimism vs. pessimism, which is constant over time in the dynamic framework, while the other parameter measures the diminishing effect of optimism and pessimism when moving away from extreme probabilities 0 and 1. which changes over time. Risk and time preferences are disentangled in the way the effect of time on risk preferences is captured by the time-dependent probability weighting function, generating probabilistic optimism resulting in a higher risk tolerance for delayed lotteries.

3.8 Appendix

3.8.1 Proof of Lemma 3.6.1

The following cases need to be considered:

Case I: When $p = 0$ or $p = 1$, one gets x, y, x', y' or $z, z', \tilde{z}, \tilde{z}'$for certain. Assuming $p = 1$, recall Axiom 3.6.2, the first two conditions jointly give the following:

$$\Phi_s[u_s(x)] - \Phi_s[u_s(x')] = \Phi_s[u_s(y)] - \Phi_s[u_s(y')].$$

Also the last two conditions jointly give the following:

$$\Phi_{s'}[u_{s'}(x)] - \Phi_{s'}[u_{s'}(x')] = \Phi_{s'}[u_{s'}(y)] - \Phi_{s'}[u_{s'}(y')].$$

The above two equations indicate that, $\Phi_s \circ u_s$ is proportional of $\Phi_{s'} \circ u_{s'}$, hence $\Phi_s \circ u_s$ is a linear increasing transformation of $\Phi_{s'} \circ u_{s'}$.

Case II: When $p \in (0, 1)$, the first two conditions in Axiom 3.6.2 jointly implies,

$$\Phi_s[w_s(p)u_s(x)+[1-w_s(p)]u_s(z)]-\Phi_s[w_s(p)u_s(x')+[1-w_s(p)]u_s(z)]$$
$$=\Phi_s[w_s(p)u_s(y)+[1-w_s(p)]u_s(z')]-\Phi_s[w_s(p)u_s(y')+[1-w_s(p)]u_s(z')]$$

Therefore,

$$[w_s(p)u_s(x)+[1-w_s(p)]u_s(z)]-[w_s(p)u_s(x')+[1-w_s(p)]u_s(z)]$$
$$=[w_s(p)u_s(y)+[1-w_s(p)]u_s(z')]-[w_s(p)u_s(y')+[1-w_s(p)]u_s(z')],$$

i.e.,

$$u_s(x)-u_s(x')=u_s(y)-u_s(y').$$

Similarly, by using the last two conditions in Axiom 3.6.2 jointly, it implies

$$u_{s'}(x)-u_{s'}(x')=u_{s'}(y)-u_{s'}(y')$$

Therefore, u_s is proportional of $u_{s'}$. Together with the previous case, we have Φ_s is proportional of $\Phi_{s'}$, i.e., Φ_s is linear. This implies that there exists constant D_t such that preferences are represented by DRU (i.e.,Equation 3.4).

3.8.2 Proof of Lemma 3.6.2

From the two preference relations in Axiom 3.6.3, which are represented by DRU, we have:

$$w_s(\sigma_s)u_s(x) + [1 - w_s(\sigma_s)]u_s(z) = w_s(\sigma_s)u_s(y) + [1 - w_s(\sigma_s)]u_s(z') \quad (3.7)$$

$$w_{s'}(\sigma_{s'})u_{s'}(x) + [1 - w_{s'}(\sigma_{s'})]u_{s'}(z) = w_{s'}(\sigma_{s'})u_{s'}(y) + [1 - w_{s'}(\sigma_{s'})]u_{s'}(z'), \quad (3.8)$$

respectively.

From Equation 3.7:

$$\begin{aligned} w_s(\sigma_s)[u_s(x) - u_s(y)] &= [1 - w_s(\sigma_s)][u_s(z') - u_s(z)] \\ \frac{w_s(\sigma_s)}{1 - w_s(\sigma_s)} &= \frac{u_s(z') - u_s(z)}{u_s(x) - u_s(y)}. \end{aligned} \quad (3.9)$$

From Equation 3.8:

$$\begin{aligned} w_{s'}(\sigma_{s'})[u_{s'}(x) - u_{s'}(y)] &= [1 - w_{s'}(\sigma_{s'})][u_{s'}(z') - u_{s'}(z)] \\ \frac{w_{s'}(\sigma_{s'})}{1 - w_{s'}(\sigma_{s'})} &= \frac{u_{s'}(z') - u_{s'}(z)}{u_{s'}(x) - u_{s'}(y)}. \end{aligned} \quad (3.10)$$

Combine Equation 3.9 and Equation 3.10,

$$\frac{w_s(\sigma_s)}{1 - w_s(\sigma_s)} = \frac{w_{s'}(\sigma_{s'})}{1 - w_{s'}(\sigma_{s'})}$$

Therefore,

$$w_s(\sigma_s) = w_{s'}(\sigma_{s'}) \quad (3.11)$$

Since w_t is monotonic increasing, we have $\sigma_s = \sigma_{s'} = \sigma$.

Chapter 4

Discussion and Conclusion

All in all, my thesis consists of chapters that present theoretical models of individual choice. Such models can help us to understand the functioning of economic institutions, and design better institutions. One cannot evaluate the soundness of a theoretical models without seeing how useful the model is in structuring out thinking of general economics. Applicability requires that the models and stylized facts compound to an integrated theory that is flexible, adequately parsimonious, and permits us to construct testable hypotheses. This suggests enhancing communication between applications and the underlying theory. To develop the theory further it helps to have feedback from areas where the theory could be applied.

Both main chapters in this thesis have potential opportunities for applications of behavioural economics. Chapter 2 presented an alternative discounting function for measuring intertemporal choice, liminal discounting, which generalises exponential discounting, as well as quasi-hyperbolic discounting. Pan, Webb and Zank (2015) gives interesting application of liminal discounting to bargaining games of Rubinstein (1982) under the assumption that the players have liminal discounting preferences, rather than exponential discounting.

Furthermore, liminal discounting model is also more flexible when applying to typical dynamic behaviour such as self-control problems. Lack of self-control refers to the tendency of economic agents to make decisions that are in conflict with their long-term interest, which lead to addictive behaviour, undersaving, or procrastination. For instance, when it comes to the matter of exercise taxes, one of the most criticism is their regressivity, with lower income groups spending a much larger share of their income on goods such as cigarettes than do higher income groups (Gruber and Kőszegi, 2004). Yet alternative conclusion could be made if we review the model with two distinctive liminal points capturing time-inconsistent behaviour for lower and higher income groups, respectively. By applying liminal discounting, it may shred new light on models of consumption of 'sin' goods.

In Chapter 3 we added the element of risk into the process of analysing intertemporal choice, due to the very nature of future prospects. We adopted one of the most profound non-expected utility theories, rank-dependent utility, to model individuals' risk attitudes towards future prospects, capturing interactions between time and risk preferences. These interactions need to be accounted for both in theoretical and empirical models, particularly because many important policy questions involve tradeoffs over time and in the presence of risk. More specifically, we presented a time-dependent two-parameter probability weighting function to capture the effect of time on probabilistic risk attitudes. In Chapter 3, the elevation parameter is constant across time while the curvature parameter is time-dependent. Alternative models can be derived if we keep curvature parameter constant over time instead. Such models can account for important empirical findings on different fields. For instance, Groneck, Ludwig and Zimper (2013) looked into household saving behaviour by treating probabilities as ambiguous survival beliefs (unknown probabilities). They also indicate that the curvature parameter can be explained as the level of ambiguity. DUO model can be extended to investigate behaviour under uncertainty as one of my future research agenda.

Preference foundations also play an essential role in this thesis. A list of conditions are given, such as liminal stationarity in Chapter 2 and Trade-off consistency in Chapter 3, in terms of observable preferences, that hold if and only of the decision model holds. These preference foundations show how to verify or falsify decision models descriptively, and provide the terms to justify or criticize models normatively. These observable behavioural conditions will contribute on further test of the intrinsic soundness of the decision models presented in the thesis through future experiments.

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