APPLICATIONS OF MEROMORPHIC LÉVY PROCESSES ON A STOCHASTIC GRID

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Doctor of Philosophy

Applications of meromorphic Lévy processes on a stochastic grid

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This thesis investigates several problems in financial and actuarial mathematics when the underlying driving process is a Lévy process $X$. Due to the structure and nature of these processes there arise various difficulties in applying them to ‘real world’ problems. Usually analytical expressions for the solutions are not known or too complicated to be implemented easily and quickly. The main contribution of this thesis is to provide a straightforward, ‘light weight’ and flexible method to tackle such questions.

A main assumption is the usage of a stochastic time grid. We introduce a Poisson process with intensity $n$ independent of $X$ whose jump times are spinning the grid points of this new grid. It is well known that the distances $\xi(n)$ between these points are exponentially distributed with mean $\frac{1}{n}$ and form an i.i.d. sequence of random variables. Therefore we might approximate $X_1$ by $X_{g(n,n)}$ for a large $n$ where $g(n,n)$ is the sum of $n$ $\xi(n)$’s, as $E[g(n,n)] = 1$. This concept interacts well together with the class of ‘Meromorphic Lévy processes’, introduced in [39]. This class contains all Lévy processes whose Lévy measure is an infinite mixture of exponentials which can generate both finite and infinite jump activity. In particular, they have the nice property of possessing a probability density function in a (semi)-explicit form at time $\xi(n)$, not only for the position but also for the supremum and the joint case of the process.

Hence we are able to derive explicit formulas for the American put problem (Chapter 6), for the ruin probability in finite time (Chapter 4) as well as developing the theory of meromorphic Lévy processes further (Chapter 3 and 5). We provide error bounds and support our results by numerical computations.
Declaration

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Chapter 1

Introduction

Lévy processes are named after the French mathematician Paul Lévy (1886 – 1971) who made numerous contributions to the theory of stochastic processes and probability amongst others. In particular, he dealt with the properties and the understanding of stochastic processes with independent and stationary increments. Due to his efforts the notation ‘Lévy processes’ for processes with these properties became standard by the 1980s. Applebaum [3] gives a useful short overview about the importance and the impact of Lévy processes. In short, they are analogues of random walks in continuous time. Moreover, they combine at first glance quite different stochastic processes like Brownian motion, Poisson processes, stable and meromorphic Lévy processes. There exist many books about Lévy processes. We will mainly rely on the ones of Applebaum [3], Bertoin [13], Kyprianou [42] and Sato [57].

Lévy processes are applied in various academic disciplines. We could mention for example continuous-state branching processes which are time-changed Lévy processes with no positive jumps (see [44] and Chapter 10 in [42]). They are popular in mathematical biology because they can be seen as a generalisation of Galton-Watson processes. These processes have their origin in the investigation why certain family names became extinct in the Victorian age. Another example is queing theory (see e.g. Chapter 4 in [42] for an overview). A very large field of applications of Lévy processes and one which plays a big role in this thesis is mathematical finance. In 1900 Bachelier [9] was the first to propose that a stock price could be modelled by a Lévy process (in this case a Brownian motion). Research in this area received an enormous boost when Black, Scholes and Merton proposed a mathematical model of the financial market in 1973.
[15, 49]. They were awarded the Nobel prize for economics in 1997 (Fisher Black died in 1995 and could not be an official laureate). Their model assumes that the stock price evolves as a geometric Brownian motion, which yields amongst others a closed form formula for the price of a European option. A financial option means that buyers of this contract are given the right but not the obligation to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date \( T \) units into the future (compare with [63]). In the case of European options the specified date can be only \( T \), in the case of American style options at any time between 0 and \( T \).

However, there exists strong evidence that Brownian motion does not represent real data very well. On the one hand it is a stochastic process, which is very easy to handle but on the other hand this process does not possess jumps for instance. Therefore we cannot expect to model for example shocks and extreme behaviour on the market very well. Also, it is well known that the tails of the normal distribution do not represent market observations in a realistic manner as the normal distribution might underrepresent extreme real events. Also because of these reasons we wanted to overcome these limitations to the Black-Scholes model. There exist many models and attempts how to model the financial market by Lévy processes with jumps. For compactness reasons we mention only some, e.g. the CGMY model [25], Variance-Gamma processes [47], hyperbolic processes including the Kou model [35] and Normal Inverse Gaussian processes [10, 11]. We refer to the books [20, 60, 64] for an overview of Lévy processes in mathematical finance.

As mentioned before, a key point of interest in the research of mathematical finance is the pricing of options. This can be understood as solving optimal stopping problems. They have been considered for a long time in probability theory and have many applications in stochastic analysis, mathematical statistics, mathematical finance and engineering amongst others. For a general overview of the theory of optimal stopping problems the reader is referred to the book of Peskir and Shiryaev [55]. The review by Pedersen [54] is also helpful for getting an insight.

In general, we consider a measurable function \( G : \mathbb{R}^d \to \mathbb{R} \) for some \( d \geq 1 \) called gain function, which also satisfies some regularity conditions. Let \( X \) is a Markov process taking values in \( \mathbb{R}^d \) on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \). Let \( \mathcal{M} \) denote the set of stopping times with respect to \( (\mathcal{F}_t)_{t \geq 0} \) taking values between 0 and \( T \) for \( T > 0 \).
CHAPTER 1. INTRODUCTION

Then one version of an optimal stopping problem is given by

\[ v(x) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_x \left[ e^{-r\tau} G(X_\tau) \right], \quad (1.0.1) \]

where \( x \in \mathbb{R}^d, v : \mathbb{R}^d \to \mathbb{R} \) is the value function, \( T \in [0, \infty] \) the expiry date, \( r \geq 0 \) is the interest rate and \( \tau \) is a stopping time with respect to the natural filtration generated by \( X \). Hereby, we denote the law of a Lévy process started at \( x \in \mathbb{R} \) with \( \mathbb{P}_x \). If \( x = 0 \), we write \( \mathbb{P}_0 = \mathbb{P} \) for simplicity. Accordingly, let \( \mathbb{E}_x \) and \( \mathbb{E} \) be the notation for the expectation operators.

It is well known that the problem (1.0.1) is equivalent to finding the smallest superharmonic function \( \hat{v} \) which dominates the gain function \( G \) on \( \mathbb{R}^d \). If we define the continuation set \( C \) as \( C = \{x \in \mathbb{R}^d : \hat{v}(x) > G(x)\} \), the stopping set \( D \) as \( D = \{x \in \mathbb{R}^d : \hat{v}(x) = G(x)\} \) and the first entry time \( \tau_D \) as \( \tau_D = \inf \{t \geq 0 : X_t \in D\} \), then, with some restrictions on \( \hat{v} \), it turns out that \( \hat{v} = v \) and \( \tau_D \) is optimal for (1.0.1).

Therefore, one has to solve two tasks: Compute the value function \( v \) and find the optimal stopping time \( \tau_D \) where the supremum is attained. Therefore the expectation of a functional of the random process has to be optimised.

Strongly connected with the search of the arbitrage-free price of an American put option which can be seen as the classic example for options is the McKean optimal stopping problem. Let \( \mathcal{N}_t \) for \( t \geq 0 \) denote the set of stopping times with respect to \( (\mathcal{F}_t)_{t \geq 0} \) taking values between 0 and \( T - t \) for \( T > 0 \).

\[ v(t, x) = \sup_{\tau \in \mathcal{N}_t} \mathbb{E}_{t,x} \left[ e^{-r\tau} \max \{0, K - X_{\tau+t}\} \right]. \quad (1.0.2) \]

If \( T = \infty \), the problem (1.0.2) becomes one-dimensional and it is possible to solve it for one-sided Lévy processes (see e.g. [42]). For a finite horizon \( T \) the problem appears to be very complicated.

If the process is a Brownian motion or a diffusion, from Peskir and Shiryaev [55] et al. it is known that the (unknown) value function \( v \) and the unknown boundary function \( b \) solve a free-boundary problem. This means the following.

Let \( L_X \) denote the infinitesimal generator of \( X \), \( C \) the continuation region

\[ \{(t, x) \in [0, T] \times (0, \infty) : v(t, x) > (K - x)\}, \]

\( D \) the stopping region

\[ \{(t, x) \in [0, T] \times (0, \infty) : v(t, x) = (K - x)\}, \]
and the stopping time $\tau_D$ the first entry time in $D$

$$\tau_D = \inf \{0 \leq s \leq T - t : X_{t+s} \in D\}.$$ 

Then we derive (after some calculations, compare with the reference mentioned before) the following free-boundary problem:

$$\mathbb{L}_X v + \frac{dv}{dt} = r \cdot v$$

$$v(t, x) = (K - x)^+ \quad \text{for } x = b(t)$$

$$\frac{dv}{dx}(t, x) = -1 \quad \text{for } x = b(t) \ ('\text{smooth fit}')$$

$$v(t, x) > (K - x)^+ \quad \text{for } x \in C$$

$$v(t, x) = (K - x)^+ \quad \text{for } x \in D.$$ 

Peskir and Shiryaev present a lot of properties of the value and the boundary function. The most important ones are:

- $v$ is continuous on $[0, T] \times \mathbb{R}_+$. 
- $v$ is $C^{1,2}$ on $C$ and $D$. 
- $x \mapsto v(t, x)$ is decreasing. 
- $x \mapsto v(t, x)$ is decreasing with $v(T, x) = (K - x)^+$. 
- $t \mapsto b(t)$ is increasing and continuous with $b(T-) = K$. 

It turns out that the optimal stopping boundary can be characterised as the unique solution of a nonlinear Volterra integral equation of the second kind, which in general requires a numerical solution method. 

If the process is a more general Lévy process there exist some solution methods to the problem (1.0.2) in the case of infinite horizon, see for example Alili and Kyprianou [2] or [42]. In the introduction of Chapter 6 we give some more references. 

If $T < \infty$, in general, no analytical expressions are known and therefore, numerical methods come into play. 

It is important to state again that even in the infinite time horizon case expressions for the optimal stopping time or boundary are quite complicated. In Chapter 6 we present a different approach to solve (1.0.2) if the underlying stochastic process $X$ is a Lévy
process with the following property: For each \( q > 0 \) the process \( X \) possesses a density function of a specific form at time \( \xi^{(q)} \) where \( \xi^{(q)} \) is an exponentially distributed random variable independent of \( X \) and with mean \( \frac{1}{q} \). These processes were introduced in [39] and are called 'Meromorphic Lévy processes'. We will give an overview in Section 2.2.

The popular stochastic processes in mathematical finance can be obtained by taking the limit of some parameters in the class of meromorphic Lévy processes (see [37]). By using a stochastic time grid (see Section 2.3) and applying a dynamic programming argument we are able to derive a simple, easy to implement formula for the price of an American option driven by meromorphic Lévy processes.

Not only in mathematical finance but also in actuarial mathematics more and more research is based on the usage of Lévy processes ([34], [42],[31]). An important question here is to find expressions for the ruin probability. This means we model the capital of an insurance company as a stochastic process \((R_t)_{t \geq 0}\) called risk process. Roughly speaking, the ruin probability is the probability that at some point in the future the capital of the company becomes negative. In the literature, see for instance [7], the standard model is the Cramér-Lundberg model in which \( R \) is a compound Poisson process with drift. The ultimate probability of ruin \( \psi(u) \) is defined as

\[
P(\inf_{t \geq 0} R_t < 0 | R_0 = u)
\]

and the probability of ruin in finite time

\[
\psi(u, T) := P(\inf_{0 \leq t \leq T} R_t < 0 | R_0 = u) \text{ for } T < \infty.
\]

However only in the case that \( X \) is a Brownian motion (by using the reflection principle [16]) or in the case of Cramér-Lundberg model with exponentially distributed jumps [4] explicit formulas for the ruin probability in finite time are known. In general we have to use different methods. We are giving an overview in the introduction of Chapter 4.

Not only applications of Lévy processes in the real world turn out to be quite difficult to use and implement. If we consider e.g. the law of a Lévy process \( X \) then the starting point is its characteristic exponent \( \Psi \) (see Section 2.1). For some Lévy processes there exist formulae for the law at time \( t > 0 \), such as Brownian motion,
Poison processes and some more. However, usually these are not available. It might be possible to apply Fourier inversion to the characteristic exponent to say something about $\mathbb{P}(X_t \in dx)$ but this can be again difficult and inefficient (For more information we refer to the introduction of Chapter 3). In Chapters 3 and 5 we derive simple and easy to implement formulas for the probability density function of $X_{g(n,n)}$ where $X$ is a meromorphic Lévy process and $g(n,n)$ an Erlang distributed random variable with mean 1 for $n \geq 1$.

1.1 Aim and Contribution of this thesis

We have briefly described the difficulties of applying Lévy processes to real world problems. Either a solution to a specific probabilistic problem is not known or if it is the case, the solution might be very complicated, hard to implement and therefore not be very helpful in practice.

This thesis presents some 'light weight' methods which are probabilistic in nature and do not need many involved proofs or tools from other fields. Furthermore, the computer implementation of them are rather straightforward as mainly only elementary operations are required. In detail the following work is done in each chapter.

Chapter 2 contains a brief overview of the background and the mathematical tools used in the following chapters. After the presentation of some properties of Lévy processes the focus lies on the class of 'Meromorphic Lévy processes' which are crucial for the work in this thesis. As mentioned before, they have the nice property of possessing a probability density function consisting of a (possibly infinite) mixture of exponentials at a random exponential time. This includes Brownian motion plus (hyper)exponential jumps, but also the so called $\beta$-class for example, cf. [37] and [39].

In general, a meromorphic Lévy process is any Lévy process whose Lévy measure is an infinite mixture of exponentials. This can generate both finite and infinite jump activity. Lévy processes well known in mathematical finance can in a straightforward way be obtained as a limit of meromorphic Lévy processes, cf. again [37]. Finally, we explain the concept of a stochastic grid.

In Chapter 3 we develop further the theory of 'Meromorphic Lévy processes'. These
are a large family of Lévy processes with the specific property of possessing a probability density function at an exponential time $\xi^{(q)}$ for $q > 0$ in a semi-explicit form. Note that $E[\xi^{(q)}] = \frac{1}{q}$. By $q$-times convoluting the density functions (see Section 2.2) of a meromorphic process $X$ at this random time we derive recursive formulas for the probability density function at an Erlang distributed time $g(q, q)$ whose expectation is 1 and with a vanishing variance as $q$ increases. By doing so, we find an approximation of the density function of $X$ at time 1 by calculating the density function at time $g(q, q)$. Based on this, we are able to derive an explicit formula for a special case of meromorphic Lévy processes which includes Brownian motion. Additionally we provide error bounds and show convergence for our method in the case of Brownian motion. We conclude with some figures to investigate some examples.

Chapter’s 4 methodology is similar to Chapter’s 3 one. By replacing the time 1 by an Erlang distributed time $g(q, q)$ we present a recursive formula for the ruin probability in finite time for the family of Meromorphic Lévy processes. This is nothing else than a formula for law of the infimum. Again we work out an explicit formula for this special class of the Lévy processes and provide error bounds. After remarking how to prove convergence in the Brownian motion case we illustrate our results again by some numerics.

Chapter 5 follows the idea of Chapter 3. Working on a stochastic grid (see Section 2.3), i.e. replacing the deterministic time 1 by an Erlang distributed time $g(q, q)$ for $q > 0$ we now provide a formula for the density function of the position of a discretised meromorphic Lévy process $\tilde{X}$. This means we introduce a grid in space such that $\tilde{X}$ only takes values on a lattice. The advantage of this method lies in its combination of speed and accuracy. Additionally, we derive formulas for both the law of the supremum $\tilde{S}$ of $\tilde{X}$ and the joint law $(\tilde{X}, \tilde{S})$ at an Erlang distributed time $g(q, q)$. Finally, we analyse the space error and present some figures comparing our method with the exact cases of Brownian motion and the Kou model [35].

In Chapter 6 we introduce an algorithm for the pricing of finite expiry American options driven by meromorphic Lévy processes. The idea is to tweak Carr’s ‘Canadaisation’ method, cf. Carr [24] (see also Bouchard et al [18]), in such a way that the adjusted algorithm is viable for any meromorphic Lévy process. We work out the algorithm in detail for the classic example of the American put, and we illustrate the
results with some numerics.

Chapter 7 contains a short conclusion of the work done in this thesis. Also, we present for each chapter some ideas how research might be continued and explain some open questions.

1.2 Publication details

Chapter 6 has been accepted for publication in the journal 'Stochastic Processes and their Applications'. It can be found on Arxiv:


http://authors.elsevier.com/sd/article/S0304414915000824.
Chapter 2

Preliminaries and Background

2.1 Lévy processes

Definition 2.1 (Definition 1.1 in [42]). A stochastic process \( X = \{X_t : t \geq 0\} \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is said to be a Lévy process if the following conditions are satisfied:

(i) \( \mathbb{P}(X_0 = 0) = 1 \).

(ii) The paths of \( X \) are right continuous with left limits \( \mathbb{P} \)-almost surely.

(iii) For \( 0 \leq s \leq t \), \( X_t - X_s \) is independent of \( \{X_u : 0 \leq u \leq s\} \) (‘independent increments’).

(iv) For \( 0 \leq s \leq t \), \( X_t - X_s \) is equal in distribution to \( X_{t-s} \) (‘stationary increments’).

Lévy processes can be defined for a more general space but the restriction to the real numbers is sufficient for this thesis.

Examples of Lévy processes are amongst others Brownian motion (this is the only continuous one), Compound Poisson processes and crucial for this thesis, the class of Meromorphic Lévy processes which will be explained in more detail in Section 2.2.

In this section we denote the law of a Lévy process started at \( x \in \mathbb{R} \) with \( \mathbb{P}_x \). If \( x = 0 \), we write \( \mathbb{P}_0 = \mathbb{P} \) for simplicity. Accordingly, let \( \mathbb{E}_x \) and \( \mathbb{E} \) be the notation for the expectation operators.

We define the supremum and the infimum of the process \( X \) as \( \overline{X}_t := \sup_{0 \leq s \leq t} X_s \) taking values in \([0, \infty)\) and \( \underline{X}_t := \inf_{0 \leq s \leq t} X_s \) taking values in \((-\infty, 0]\).
It is well known that for any \( t > 0 \), \( X_t \) possesses an infinitely divisible distribution, see e.g. Definition 1.2 in [42]. Therefore (see also the calculations before Definition 1.5 in [42]) we can deduce that the characteristic exponent \( \Psi_t \) of \( X_t \) which is
\[
\Psi_t(\theta) := -\log(\mathbb{E}[e^{i\theta X_t}]），
\]
can be written as follows
\[
\Psi_t(\theta) = t\Psi_1(\theta).
\]

Theorem 1.3 in [42] links infinitely divisible distributions with a specific form of the characteristic exponent.

\textbf{Theorem 2.1.1} (Lévy-Khinchine formula). A probability law \( \mu \) of a real-valued random variable is infinitely divisible with characteristic exponent \( \Psi \) if and only if there exists a triplet \((a, \sigma, \Pi)\), where \( a \in \mathbb{R} \), \( \sigma \geq 0 \) and \( \Pi \) is a measure concentrated on \( \mathbb{R}\setminus\{0\} \) satisfying \( \int (1 \wedge x^2)\Pi(dx) < \infty \), such that
\[
\Psi(\theta) = i\theta a + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{(|x|<1)})\Pi(dx) \quad (2.1.1)
\]
for every \( \theta \in \mathbb{R} \).

Hence we can deduce that for every Lévy process there exists a triplet \((a, \sigma, \Pi)\) such that the characteristic exponent has the form of (2.1.1). We call
\[
\Psi_1(\theta) := \Psi(\theta)
\]
the characteristic exponent of the Lévy process.

On the other hand we can see that for every triplet \((a, \sigma, \Pi)\) there exists a Lévy process with a characteristic exponent having the expression of (2.1.1), see for instance Theorem 1 of [13].

\textbf{Theorem 2.1.2} (Lévy-Khinchine formula for Lévy processes). Suppose that \( a \in \mathbb{R} \), \( \sigma \geq 0 \) and \( \Pi \) is a measure concentrated on \( \mathbb{R}\setminus\{0\} \) such that \( \int (1 \wedge x^2)\Pi(dx) < \infty \). From this triple define for each \( \theta \in \mathbb{R} \)
\[
\Psi(\theta) = i\theta a + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{(|x|<1)})\Pi(dx). \quad (2.1.2)
\]
Then there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which a Lévy process is defined having characteristic exponent \( \Psi \).
a is called the drift, \(\sigma\) the Gaussian coefficient and the measure \(\Pi\) the Lévy measure.

In the case of bounded variation the expression (2.1.2) can be simplified:

**Lemma 2.1.3** (Lemma 2.12 in [42]). A Lévy process with Lévy-Khinchine exponent corresponding to the triple \((a, \sigma, \Pi)\) has paths of bounded variation if and only if

\[
\sigma = 0 \quad \text{and} \quad \int_{\mathbb{R}} (1 \wedge |x|) \Pi(dx) < \infty.
\]

The characteristic exponent can then be written as

\[
\Psi(\theta) = -i d \theta + \int_{\mathbb{R}} (1 - \exp(i \theta x)) \Pi(dx),
\]

where \(d \in \mathbb{R}\) and is related to \(a\) and \(\Pi\) as follows:

\[
d = - \left( a + \int_{|x|<1} x \Pi(dx) \right). \tag{2.1.3}
\]

For practical reasons we sometimes use the Laplace exponent \(\psi\)

\[
\psi(z) = -\Psi(-iz) \tag{2.1.4}
\]

defined for all \(z\) for which it exists.

Next, we briefly want to present an important concept for Lévy processes which will be important for the subclass of processes presented in Section 2.2. It can be found in [42]. We will only present the statements which will be necessary for this thesis.

**Theorem 2.1.4.** Let \(X\) be a Lévy process which is not a Compound Poisson process.

Let \(\xi^{(p)}\) be an exponentially distributed random variable with mean \(\frac{1}{p}\) for \(p > 0\) independent of \(X\). Define \(\overline{G}_t := \sup \{s < t : X_s = X_s\}\) respectively \(G_t := \sup \{s < t : X_s = X_s\}\) for \(t > 0\). Then the pairs \((\overline{G}_{\xi^{(p)}}, X_{\xi^{(p)}})\) and \((\xi^{(p)} - \overline{G}_{\xi^{(p)}}, X_{\xi^{(p)}} - X_{\xi^{(p)}})\) are independent, yielding the factorisation

\[
\frac{p}{p - iv + \Psi(\theta)} = \Psi_p^+(v, \theta) \Psi_p^-(v, \theta) \tag{2.1.5}
\]

where \(v, \theta \in \mathbb{R}\) and

\[
\Psi_p^+(v, \theta) = \mathbb{E} \left[ e^{iv \overline{G}_{\xi^{(p)}} + i \theta X_{\xi^{(p)}}} \right]
\]

\[
\Psi_p^-(v, \theta) = \mathbb{E} \left[ e^{iv \overline{G}_{\xi^{(p)}} + i \theta X_{\xi^{(p)}}} \right].
\]

The pair \(\Psi_p^+(v, \theta)\) and \(\Psi_p^-(v, \theta)\) are called **Wiener-Hopf factors**.
Remark 2.1.5. • A consequence of equation (2.1.5) is that

\[ \Psi^X_p(\theta) = E \left[ e^{i\theta X(p)} \right] = \frac{p}{p + \Psi(\theta)} = \Psi_p^+(\theta)\Psi_p^-(\theta) = E \left[ e^{i\theta \overline{X}(p)} \right] E \left[ e^{i\theta \underline{X}(p)} \right] \]

where \( \Psi^X_p, \Psi_p^+ \) and \( \Psi_p^- \) are the characteristic functions of \( X(p), \overline{X}(p) \) and \( \underline{X}(p) \).

Therefore we have

\[ X(p) \overset{d}{=} \overline{X}(p) + \underline{X}(p). \tag{2.1.6} \]

• There do not exist many explicit expressions for the Wiener-Hopf factors. For instance, for standard Brownian motion they have the form \( \Psi_p^+(\theta) = \frac{\sqrt{2p}}{\sqrt{2p - i\theta}} \) and \( \Psi_p^-(\theta) = \frac{\sqrt{2p}}{\sqrt{2p + i\theta}} \). In the next Section 2.2 we will present the class of Meromorphic Lévy processes which possess Wiener-Hopf factors in the (semi-)explicit form of a possibly infinite product.

2.2 Meromorphic Lévy processes

Let us discuss meromorphic Lévy processes in more detail and base our short review on [39] but also [37] and [38]. We start recalling some basic definitions.

Definition 2.2.1. A function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) is called completely monotone if \( f \in C^\infty \) and \((-1)^n f^{(n)}(x) \geq 0\) for all \( n \geq 0 \) and \( x > 0 \).

The class of completely monotone functions is denoted as \( \text{CM} \).

By the Bernstein theorem (see Theorem 1.4 in [59]) the following result can be obtained.

Theorem 2.2.2. A function \( f \) is completely monotone if and only if it can be represented as the Laplace transform of a positive measure on \([0, \infty)\), i.e.

\[ f(x) = \int_0^\infty \exp(-zx)\mu(dz). \tag{2.2.1} \]

An important subclass of \( \text{CM} \) is the class of discrete completely monotone functions, denoted by \( \text{DCM} \) and defined as follows:

Definition 2.2.3. A function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) with a representation (2.2.1) is called discrete completely monotone if the measure \( \mu \) in (2.2.1) is finite with an infinite support and no finite accumulation points.
In this case we can write \( \mu \) as the infinite mixture of atoms of size \( a_m \) at points \( b_m \), i.e.

\[
\mu(dz) = \sum_{m \geq 1} a_m \delta_{b_m}(dz) \quad (2.2.2)
\]

where \( a_m > 0, b_m \geq 0 \) for all \( m \geq 1 \), \( b_m \to \infty \) as \( m \to \infty \) and \( \delta \) denotes the Dirac measure. (2.2.1) and (2.2.2) lead to the following corollary:

**Corollary 2.2.4.** A discrete completely monotone function \( f : \mathbb{R}_{>0} \to \mathbb{R} \) can be represented as

\[
f(x) = \sum_{m \geq 1} a_m e^{-b_mx} \quad \text{for all } x > 0. \quad (2.2.3)
\]

Meromorphic Lévy processes are now defined as follows.

**Definition 2.2.5.** A Lévy process \( X \) is meromorphic if the functions \( x \mapsto \Pi^+(x) := \Pi((x, \infty)) \) and \( x \mapsto \Pi^-(x) := \Pi((-\infty, -x)) \) are both discrete completely monotone for \( x > 0 \).

Then, with (2.2.3) it is clear that \( X \) is meromorphic if its Lévy measure \( \Pi(dx) \) has a density with respect to the Lebesgue measure which is given by

\[
\pi(x) = 1_{\{x>0\}} \sum_{j \geq 0} a_j \tau_j \exp(-\tau_j x) + 1_{\{x<0\}} \sum_{j \geq 0} \tilde{a}_j \tilde{\tau}_j \exp(\tilde{\tau}_j x), \quad (2.2.4)
\]

for some positive coefficients \( a_j, \tilde{a}_j, \tau_j \) and \( \tilde{\tau}_j \). Hereby, the sequences \((\tau_j)_{j \geq 1}\) and \((\tilde{\tau}_j)_{j \geq 1}\) are strictly increasing and \( \tau_j \to \infty \) and \( \tilde{\tau}_j \to \infty \) as \( j \to \infty \).

Theorem 1 of [39] presents equivalent statements of Definition 2.2.5.

**Theorem 2.2.6.** The following statements are equivalent:

- \( X \) is meromorphic.
- \( \Pi^+(x), \Pi^-(x) \in \mathcal{CM} \) and the Laplace exponent \( \psi(z) \) is meromorphic.
- For every \( q > 0 \) and \( x > 0 \) the functions \( \mathbb{P}(\overline{X_{\xi(q)}} > x), \mathbb{P}(-\overline{X_{\xi(q)}} > x), \mathbb{P}(X_{\xi(q)} > x) \) and \( \mathbb{P}(-X_{\xi(q)} > x) \) belong the the class \( \mathcal{DCM} \) where \( \xi(q) \) is an exponentially distributed random variable with mean \( \frac{1}{q} \) independent of \( X \).
- For every \( q > 0 \) and \( z \in \mathbb{C} \) we have the factorisation

\[
q - \psi(z) = q \prod_{n \geq 1} \frac{1 - \frac{z}{\xi_n}}{1 - \frac{z}{\rho_n}} \prod_{n \geq 1} \frac{1 + \frac{z}{\xi_n}}{1 - \frac{z}{\rho_n}}
\]
where all roots \( \{ \zeta_n, -\hat{\zeta}_n \} \) of \( \psi(z) - q \) are real and interlace with the poles \( \{ \rho_n, -\hat{\rho}_n \} \)

\[
\ldots < -\hat{\rho}_2 < -\hat{\zeta}_n < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \ldots
\]

- The Laplace exponent \( \psi(z) \) is a real meromorphic function which satisfies

\[
\text{Im} \left( \frac{\psi(z)}{z} \right) > 0
\]

for all \( z \) in the half-plane \( \text{Im}(z) > 0 \).

We do not want to dive deeper into the theory of meromorphic Lévy processes but we want to mention two key properties.

The first one is that the Wiener-Hopf factors of such processes are (semi-)explicitly known and can be written down as a possibly infinite product, compare with [39], Theorem 2 (i):

The Wiener-Hopf factors are given by:

\[
\phi_q^+(iz) := \mathbb{E} \left[ \exp(-zX_{\xi(q)}) \right] = \prod_{j \geq 1} \frac{1 + \frac{z}{\rho^+_q(j)}}{1 + \frac{z}{\zeta^+_q(j)}}
\]

\[
\phi_q^-(iz) := \mathbb{E} \left[ \exp(zX_{\xi(q)}) \right] = \prod_{j \geq 1} \frac{1 + \frac{z}{\rho^-_q(j)}}{1 + \frac{z}{\zeta^-_q(j)}}
\]

where

- \( q > 0 \),

- \( \xi^{(q)} \) is an exponentially distributed random variable with mean \( \frac{1}{q} \) independent of \( X \),

- \( (-\rho^{(q)}_-(j))_{j \geq 0} \) (resp. \( (\rho^{(q)}_+(j))_{j \geq 0} \)) is the sequence of all real negative (resp. real positive) poles of the function \( z \mapsto q + \Psi(-iz) \) (or, equivalently, of \( z \mapsto \psi(z) - q \)) and

- \( (-\zeta^{(q)}_-(j))_{j \geq 0} \) (resp. \( (\zeta^{(q)}_+(j))_{j \geq 0} \)) is the sequence of all real negative (resp. real positive) zeros of the function \( z \mapsto q + \Psi(-iz) \) (or, equivalently, of \( z \mapsto \psi(z) - q \)).

As mentioned in the introduction, the key property of a meromorphic Lévy process \( X \) that makes them applicable and which yield explicit expressions in the coming chapters is the fact that the laws of \( X_{\xi^{(q)}} \), \( \nabla_{\xi^{(q)}} \) and \( \nabla_{\xi^{(q)}} \) are mixtures of exponentials.

In the thesis the following results for the supremum and infimum will be used.
Proposition 2.2.7 ([39], Theorem 1 and 2). Let $X$ be a meromorphic Lévy process. For $q > 0$ we have:

$$g_s(x) := \mathbb{P}(X_{\xi(q)} \in dx) = \sum_{j=0}^{N^+} c_+^{(q)}(j) \zeta_+^{(q)}(j) \exp(-\zeta_+^{(q)}(j)x)dx$$ (2.2.5)

for $x > 0$,

$$g_I(x) := \mathbb{P}(X_{\xi(q)} \in dx) = \sum_{j=0}^{N^-} c_-^{(q)}(j) \zeta_-^{(q)}(j) \exp(\zeta_-^{(q)}(j)x)dx$$ (2.2.6)

for $x < 0$ and

$$w_+ := \mathbb{P}(X_{\xi(q)} = 0)$$

$$w_- := \mathbb{P}(X_{\xi(q)} = 0).$$

Here we have $N_-, N_+ \in \mathbb{N} \cup \{\infty\}$ and sequences $(c_-^{(q)}(j))$, $(c_+^{(q)}(j))$, where for all (relevant) $j$: $c_-^{(q)}(j) > 0$ and $c_+^{(q)}(j) > 0$; and

$$\sum_{j=0}^{N_+} c_+^{(q)}(j) = 1 = \sum_{j=0}^{N_-} c_-^{(q)}(j),$$

if $X$ is of unbounded variation and

$$\sum_{j=0}^{N_+} c_+^{(q)}(j) = 1 - w_+$$

$$\sum_{j=0}^{N_-} c_-^{(q)}(j) = 1 - w_-$$

if $X$ is of bounded variation. Also

$$w_+ = \lim_{w \to \infty} \prod_{k=0}^{w} \frac{c_+^{(q)}(k)}{s_+^{(q)}(k)}$$

and

$$c_+^{(q)}(j) = \left(1 - \frac{\zeta_+^{(q)}(j)}{\rho_+^{(q)}(j)}\right) \prod_{k \geq 0, k \neq j} \frac{1 - \frac{\zeta_+^{(q)}(j)}{\rho_+^{(q)}(k)}}{1 - \frac{\zeta_+^{(q)}(j)}{\zeta_+^{(q)}(k)}}.$$

Expressions for $c_+^{(q)}(j)$ have a similar form.

Kuznetsov [39] provides a formula for the density of the process:
Theorem 2.2.8 ([39], Theorem 1 and 2). Let \( X \) be a meromorphic Lévy process which is not compound Poisson. Then

\[
\mathbb{P} \left( X_{\xi(q)} \in \mathrm{d}x \right) = \left( 1_{\{x<0\}} \sum_{j=0}^{N_-} d_-^q(j)e^{-\zeta_-^q(j)x} + 1_{\{x>0\}} \sum_{j=0}^{N_+} d_+^q(j)e^{-\zeta_+^q(j)x} \right) \mathrm{d}x \quad (2.2.7)
\]

for \( N_-, N_+ \in \mathbb{N} \cup \{\infty\} \) and sequences \((d_-^q(j)), (d_+^q(j))\), where for all (relevant) \( j \):

\[
d_-^q(j) > 0, \quad d_+^q(j) > 0; \quad \text{and} \quad \sum_{j \geq 0} \left( \frac{d_+^q(j)}{\zeta_+(j)} + \frac{d_-^q(j)}{\zeta_-(j)} \right) = 1.
\]

Expressions for \( d_+^q(j) \) and \( d_-^q(j) \) are given in Proposition 2.2.9.

Note that \( N_- = N_+ = 0 \) corresponds to Brownian motion (with drift), while \( N_-, N_+ \in \mathbb{N} \) corresponds to a Brownian motion with drift plus (one- or two-sided) so-called hyper-exponential jumps (a generalisation of exponential jumps). See e.g. [6], [35], [51] and [52]. For \( N_- = N_+ = \infty \) this expression for the density holds if and only if \( X \) is a meromorphic Lévy process (the proof relies heavily on the results in [38]).

Proposition 2.2.9. Let \( X \) be a Lévy process which is not compound Poisson. Then \( X \) is a meromorphic Lévy process if and only if \( X \) satisfies (2.2.7) for some (and then all) \( q > 0 \). Furthermore, if \( X \) indeed satisfies (2.2.7), then \((-\zeta_-^q(j))_{j \geq 0}\) (resp. \((\zeta_+^q(j))_{j \geq 0}\)) is the sequence of all real negative (resp. real positive) zeros of the function \( z \mapsto q + \Psi(-iz) \) (or, equivalently, of \( z \mapsto \psi(z) - q \)), and for all \( j \geq 0 \)

\[
d_-^q(j) = \frac{q}{\psi'(\zeta_-^q(j))}, \quad d_+^q(j) = -\frac{q}{\psi'(-\zeta_+^q(j))}.
\]

Here \( \Psi \) is the characteristic exponent of \( X \) and \( \psi \) the Laplace exponent.

Proof. The ‘only if’ part can be found in Kuznetsov et al. [39], Theorem 2, in this thesis also stated as Theorem 2.2.8. Let us now assume that \( X \) satisfies (2.2.7) for some (and then all) \( q > 0 \). By using Theorem 1 in Kuznetsov et al. [39], it is sufficient to show that the functions \( x \mapsto \mathbb{P}(X_{\xi(q)} > x) \) and \( x \mapsto \mathbb{P}(X_{\xi(q)} < -x) \) are discrete completely monotone functions on \( \mathbb{R}_{>0} \) (recall (2.2.3)). Now

\[
\mathbb{P}(X_{\xi(q)} > x) = \int_{x}^{\infty} \sum_{j=0}^{N_+} d_+^q(j) \exp(-\zeta_+^q(j)x) \mathrm{d}x = \sum_{j=0}^{N_+} \frac{d_+^q(j)}{\zeta_+^q(j)} \exp(-\zeta_+^q(j)x)
\]
If we adopt the notation from [39] for the definition of discrete completely monotone functions (page 24) and choose \( a_j = \frac{\Phi(q)}{\zeta_+(j)} > 0 \) and \( b_j = \zeta_+(j) > 0 \) which is in fact increasing we see that indeed \( x \mapsto P(X_{\xi(q)} > x) \) is a discrete completely monotone function. This can be proven for \( x \mapsto P(X_{\xi(q)} < -x) \) in the same way.

In the case of no positive jumps we can use the following proposition:

**Proposition 2.2.10 ([40]).** Assume that \( Y \) is a meromorphic spectrally negative Lévy process. For \( q > 0 \), let \( \phi(q) \) be the unique positive root and let \( \zeta_-(\cdot) \) be the negative roots of the equation \( \psi(z) - q = 0 \), where \( \psi(z) = -\Psi(-iz) \) denotes the Laplace exponent of \( Y \).

Then we have \( \overline{Y}_{\xi(q)} \overset{d}{=} \exp(\phi(q)) \) and the law of the infimum is a mixture of exponential distributions, namely

\[
P(Y_{\xi(q)} \in dx) = \left[ \sum_{j=0}^{N_-} c_-^{(q)}(j) \zeta_-^{(q)}(j) e^{\zeta_-^{(q)}(j)x} \right] dx
\]

for \( x < 0 \), where

\[
c_-^{(q)}(j) := -\left( \frac{1}{\zeta_-^{(q)}(j)} + \frac{1}{\phi(q)} \right) \frac{q}{\Psi(-\zeta_-^{(q)}(j))}.
\]

If the process has bounded variation, then the atom at zero is

\[
w_- = P(Y_{\xi(q)} = 0) = \frac{q}{\phi(q) a},
\]

where \( a \) is the drift in (2.1.2).

The following observation which was already remarked in [37] and [41] will be needed in this thesis:

**Lemma 2.2.11.** Let \( X \) be a Lévy process and let \( g(q,q) \) be the sum of \( q \) i.i.d. exponentially distributed random variables \( \xi^{(q)} \) with common mean \( \frac{1}{q} \). Then we have

\[
X_{g(q,q)} \rightarrow X_1 \text{ almost surely as } q \rightarrow \infty,
\]

and

\[
X_{g(q,q)} \overset{d}{=} \sum_{j=1}^{q} (S_j^{(q)} + I_j^{(q)}),
\]

where \( (S_j^{(q)})_{j \geq 1} \) (respectively \( (I_j^{(q)})_{j \geq 1} \)) are independent copies of \( X_{\xi(q)} \) (respectively \( X_{\xi(q)} \)).
Proof. Note that \( g(q,q) \to 1 \) almost surely for \( q \to \infty \). A Lévy process is right-continuous and does not jump at any fixed time, hence also not at time 1. Therefore (2.2.11) follows. Equation (2.2.12) can be recovered from e.g. [41], Theorem 1.

Let us briefly outline a few prominent examples of meromorphic Lévy processes. The \( \beta \)-class was introduced in [37]. It has a characteristic exponent given by

\[
\Psi(z) = \frac{\sigma^2}{2} z^2 + i a z + \frac{c_1}{\beta_1} (B(\alpha_1, 1 - \lambda_1) - B(\alpha_1 - i z/\beta_1, 1 - \lambda_1)) \\
+ \frac{c_2}{\beta_2} (B(\alpha_2, 1 - \lambda_2) - B(\alpha_2 + i z/\beta_2, 1 - \lambda_2))
\]

where \( B(x,y) = \Gamma(x) \Gamma(y)/\Gamma(x+y) \) is the Beta function and the parameter ranges are \( a \in \mathbb{R}, \sigma, c_i, \alpha_i, \beta_i > 0 \) and \( \lambda_i \in (0,3) \setminus \{1,2\} \). The corresponding Lévy measure has density

\[
1_{\{x<0\}} c_2 e^{\alpha_2 \beta_2 x} \Theta_k(-x \beta_2) + 1_{\{x>0\}} c_1 e^{-\alpha_1 \beta_1 x} \Theta_k(x \beta_1)
\]

Note that setting \( \sigma = 0 \) and \( \lambda_i \in (0,2) \) yields a process with paths of bounded variation, all other choices paths of unbounded variation. Furthermore infinite (resp. finite) activity in the jump component can be obtained by setting \( \lambda_i \in (1,3) \) (resp. \( \lambda_i \not\in (1,3) \)).

A second example is the \( \theta \)-class introduced in [38]. The Lévy measure of such processes has a density of the form

\[
1_{\{x<0\}} c_2 \beta_2 e^{\alpha_2 \beta_2 x} \Theta_k(-x \beta_2) + 1_{\{x>0\}} c_1 \beta_1 e^{-\alpha_1 \beta_1 x} \Theta_k(x \beta_1)
\]

where \( \Theta_k \) is the \( k \)-th order (fractional) derivative of the theta function \( x \mapsto \theta_3(0,e^{-x}) \) (The definition and the properties of theta functions can be found e.g. in [53]):

\[
\Theta_k(x) = \frac{d^k}{dx^k} \theta_3(0,e^{-x}) = 1_{\{k=0\}} + 2 \sum_{n \geq 1} n^k e^{-n^2 x}, \quad x > 0.
\]

The parameter \( \chi = k+1/2 \) corresponds to the exponent of the singularity of the above density at \( x = 0 \), and when \( \chi \in \{1/2, 3/2, 5/2\} \) (resp. \( \chi \in \{1/2, 3/2, 5/2\} \)) the characteristic exponent can be expressed in terms of trigonometric (resp. digamma) functions.

As a final example we mention general hypergeometric Lévy processes, these were introduced in [41] as an example of how to use Vigon’s theory of philanthropy (cf.
for constructing new Lévy processes. They have a characteristic exponent given by
\[ \Psi(z) = \frac{\sigma^2}{2} z^2 + i\alpha z + \psi_1(-iz)\psi_2(iz), \]
where \( \sigma > 0, \alpha \in \mathbb{R} \) and \( \psi_i \) is the Laplace exponent of a (possibly killed) subordinator from the above mentioned \( \beta \)-class, which hence takes the form
\[ \psi(z) = \kappa + \delta z + \frac{c}{\beta} (B(1 - \alpha + \gamma, -\gamma) - B(1 - \alpha + \gamma + z/\beta, -\gamma)) \]
for a killing rate \( \kappa \geq 0 \), drift \( \delta \in \mathbb{R} \) and \( \gamma \in (-\infty, 0) \cup (0, 1) \).

Note that \( N_+ = N_- = 0 \) corresponds to Brownian motion (with drift) while \( N_+, N_- \in \mathbb{N} \) corresponds to the class of hyperexponential Lévy processes (a generalisation of exponential jumps) which also include the Kou model [35].

These examples illustrate how rich the class of meromorphic Lévy processes is. Furthermore it is useful to notice that many Lévy processes which play a prominent role in mathematical finance can in a straightforward way be obtained by considering limiting cases of meromorphic Lévy processes, see [37] and [39]. Examples include generalised tempered stable processes (cf. [26]), Kobol processes (cf. [19] and [21]), tempered stable processes (cf. [12]) and CGMY processes (cf. [25]).

**Remark 2.2.12.** As indicated before the class of meromorphic Lévy processes is large. Meromorphic Lévy processes can possess both paths of bounded variation and of unbounded variation. Furthermore infinite (resp. finite) activity in the jump component are possible.

However, they are not dense in the class of all Lévy processes in the sense of weak convergence of probability measures. The reason lies in the property of possessing a completely-monotone Lévy measure.

In (2.2.7) all coefficients are required to be positive. If they were allowed to be negative and \( N_+, N_- \) to be positive integers, then we would obtain so-called mixed-exponential jump diffusions (see Definition 2.1 in [50] or page 2069 in [23]). These processes are dense in the class of all Lévy processes in the sense of weak convergence of probability measures. This is true as the mixed-exponential distribution is dense in the class of all distributions in the sense of weak convergence which has been shown in [17].
2.3 Stochastic Grid

In Section 2.2 we presented the class of meromorphic Lévy processes. As mentioned before, its law at an independent, exponentially distributed time consists of a (possibly infinite) mixture of exponentials. This suggests that it might be useful to consider making exponentially distributed steps in time, i.e. introducing a stochastic time grid. Before explaining this in more detail, let us remark that this idea is not new. Carr [24] used this method in relation to problems in Mathematical finance (more details can be found in Chapter 6) and called it \textit{Canadisation}. In the field of Actuarial Science this approach is known as \textit{Erlangisation}, see for instance [7] or [58].

Let us introduce the idea in detail. For any $n \in \mathbb{N}$, enlarge the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $X$ lives to contain a Poisson process $N^{(n)}$ with intensity $n$, independent of $X$. Denote the $k$-th jump time of $N^{(n)}$ by $T^{(n)}_k$, i.e.

$$T^{(n)}_k := \inf\{t \geq 0 \mid N^{(n)}_t \geq k\} \quad (2.3.1)$$

with $T^{(n)}_0 := 0$. Note that for any $T \geq 0$, if $(k(n))_{n \geq 0}$ is a sequence of natural numbers such that $k(n)/n \to T$ as $n \to \infty$ then $T^{(n)}_{k(n)} \to T$ a.s. by the law of large numbers. Our stochastic grid is now the sequence

$$0 = T^{(n)}_0 < T^{(n)}_1 < \ldots.$$ 

It follows that the distances between consecutive grid points $T^{(n)}_k$ form a sequence of independent random variables with an identical exponential distribution with parameter $\frac{n}{T}$. If we increase the Poisson parameter $n$ the grid gets finer as the common mean of the exponential distributions vanishes.

The sum of the distances between the grid points up to $n$, i.e.

$$\sum_{k=0}^{n} T^{(n)}_k =: g(n, \frac{n}{T})$$

is nothing else than an Erlang distributed random variable with parameters $n$ and $\frac{n}{T}$. As mentioned before its expectation is

$$\frac{n}{\frac{n}{T}} = T.$$ 

The variance equals

$$\frac{n}{\frac{n}{T}^2} = \frac{T^2}{n}.$$
which vanishes as \( n \to \infty \) for a fixed \( T \).

**For simplicity we will often assume \( T \) to be equal to 1 in the sequel.**

The following figure illustrates the basic properties of this stochastic grid:

Figure 2.1: Path of a Poisson process with intensity \( \lambda = n = 5 \) and the induced stochastic grid.

In the next figures it can be seen how fine the grid becomes with an increasing \( \lambda = n \).

Figure 2.2: Path of a Poisson process with intensity 1 and the induced stochastic grid.
Figure 2.3: Path of a Poisson process with intensity 10 and the induced stochastic grid.

Figure 2.4: Path of a Poisson process with intensity 300 and the induced stochastic grid.

With the help of this stochastic grid, we can now address various problems in the field of probability theory (Chapter 3 and 5), Actuarial Science (Chapter 4) and Mathematical Finance (Chapter 6) when the driving process is a meromorphic Lévy process.
Chapter 3

Approximating the law of the position and joint law of a meromorphic Lévy process

3.1 Introduction

Let $X$ be a Lévy process which we have described in more detail in Section 2.1. The key of interest in this chapter is the investigation of the law of the position of a Lévy process at time $T > 0$. It is well known that the probability distribution of these stochastic processes at any time $t > 0$ possesses the property of infinite divisibility. Sato ([57], Chapter 5) lists sufficient conditions for the existence of a (smooth) probability density function. Briefly speaking, the Lévy measure $\Pi$ has to fulfill a boundness condition. However, even if there exists a density function there are not many explicit formulas available for them. Examples include Brownian motion, Normal inverse Gaussian processes [10, 11], Variance Gamma processes [47] or symmetric stable processes [67, 57]. Therefore usually Fourier inversion has to be used to obtain $\mathbb{P}(X_t \in dx)$ for $t > 0$ if the characteristic exponent (2.1.2) is known analytically. There is an efficient method which is called the 'Fast Fourier inversion' (see e.g. the book [66]).

Our approach is based on the usage of a stochastic grid, see Section 2.3. We want to investigate the law of the position of a Lévy process at a random time near $T > 0$. For simplicity we set $T = 1$. Again, this means that we replace the deterministic time $T$
by a sum of \( n \) independent exponentially distributed random variables \((\xi_i(n))_{i \geq 1}\) with common mean \( \frac{1}{n} \), i.e. \( n \) denotes the intensity of the underlying Poisson process for this time grid. Then instead of considering the distribution of \( X_1 \) we investigate the one of
\[
X_{\sum_{i=1}^{n} \xi_i(n)}.
\]
Our method works for all Lévy processes where its density is known at the random time \( \xi(n) \) in an (semi)-explicit form and that is equivalent to the one of meromorphic Lévy processes (see Section 2.2).

Obviously, if we choose \( n = 1 \) then we try to approximate the time 1 by \( \xi(1) \) and the law of \( X_1 \) by \( X_{\xi(1)} \). The accuracy of the time cannot be expected to be very high as \( \mathbb{E}[\xi(1)] = 1 \) but the variance equals 1 as well. Therefore, the law of \( X_{\xi(1)} \) is normally not very close to the law of \( X_1 \). Indeed, this conclusion is supported by some numerical examples (see Section 3.4).

If we choose a larger \( n \) and consider the sequence of i.i.d. exponentially distributed random variables \((\xi_i(n))_{i \geq 1}\) with expectation \( \frac{1}{n} \) then \( g(n, n) := \sum_{i=1}^{n} \xi_i(n) \) is an Erlang distributed random variable with parameters \( n \) and \( n \). Its expectation is 1 but its variance equals \( \frac{1}{n} \) which vanishes with a growing \( n \). Hence, we may expect that the approximation \( X_{g(n,n)} \) of the law for \( X_1 \) improves as we increase \( n \). Indeed, we saw in Theorem 2.2.11 we saw that \( X_{g(n,n)} \rightarrow X_1 \) a.s. as \( n \rightarrow \infty \).

The rest of the chapter is organised as follows. In Section 3.2, after briefly recalling some basic assumptions and properties we present recursive formulas for the probability density functions of \( X_{g(n,n)} \), which also depend on whether the process is of bounded or unbounded variation. The key tool to derive these formulas is a \( k \)-fold convolution. In the case of Brownian motion we are able to write down more explicit formulas. Next, we present a formula for the joint distribution, i.e. for \( \mathbb{P}_x \left( X_{g(k+1,n)} \in [u, \underline{x}], X_{g(k+1,n)} \in [\underline{u}, 0) \right) \) and suitable values of \( u, \underline{u} \) and \( \underline{x} \). In Section 3.3 we analyse the error for the formulas we derived in Section 3.2. Furthermore, we show that the Taylor series of our formula for the density of the position of \( X_{g(n,n)} \) in the case of Brownian motion converges to the Gaussian function. In Section 3.4 we support our results by some numerics.
3.2 Density of the position and the distribution of the joint law

We will continue to rely on a stochastic time grid which we already explained in Section 2.3. Firstly, we want to consider $X_{g(n,n)}$ and calculate a formula for the density. For this we use two key properties.

The first one, (2.1.6), is a consequence of the Wiener-Hopf factorization (cf. Theorem 2.1.4) for Lévy processes which says

$$X_{\xi(n)} = d\overline{X}_{\xi(n)} + \overline{X}_{\xi(n)}.$$ 

Making use of stationary and independent increments this extends to (2.2.12), i.e.

$$X_{g(n,n)} = d\sum_{j=1}^{n}(S_{j}^{(n)} + I_{j}^{(n)}),$$

where $(S_{j}^{(n)})_{j\geq 1}$ and $(I_{j}^{(n)})_{j\geq 1}$ are sequences of i.i.d. random variables, and $S_{j}^{(n)}$ (respectively $I_{j}^{(n)}$) are copies of $\overline{X}_{\xi(n)}$ (respectively $\overline{X}_{\xi(n)}$) for every $j \in \mathbb{N}$.

Remark 3.2.1. The approach in paper [41] where a novel Monte Carlo simulation method for Lévy processes was developed, also relies on a stochastic grid. Basically, the method there firstly consists of sampling a sufficiently large number of i.i.d. copies $S_{i}^{n}$ respectively $I_{i}^{n}$ of $\overline{X}_{\xi(n)}$ respectively $\overline{X}_{\xi(n)}$ for every $i \in \mathbb{N}$. Then the authors are able to present a result for the joint law $\mathbb{P}(X_{g(n,n)} \in dx, \overline{X}_{g(n,n)} \in dy)$ which converges to $\mathbb{P}(X_{1} \in dx, \overline{X}_{1} \in dy)$ for $n \to \infty$. Furthermore they suggest to approximate $\mathbb{E}[g(X_{1}, \overline{X}_{1})]$ for some function $g$, i.e. by

$$\mathbb{E}[g(X_{1}, \overline{X}_{1})] \approx \frac{1}{m} \sum_{i=1}^{m} g(V^{(i)}(n,n), J^{(i)}(n,n))$$

where $(V^{(i)}(n,n))_{i\geq 1}$ and $(J^{(i)}(n,n))_{i\geq 1}$ are independent copies of $X_{g(n,n)}$ respectively $\overline{X}_{g(n,n)}$ and $m$ is the number of created copies. The name of this whole idea is 'Wiener-Hopf Monte Carlo method'.

Our calculations below consist of three steps. First (by relying on (2.2.12)) we find a formula for the density of

$$S_{k,n} := \sum_{i=0}^{k} S_{i}^{n},$$
then one for

\[ I_{k,n} := \sum_{i=0}^{k} I_{i}^{n} \]

and finally one for

\[ X_{g(n,n)} \overset{d}{=} S_{n,n} + I_{n,n}. \]

As we deal with independent random variables \( S_{i}^{n} \) and \( I_{i}^{n} \) and we have a formula for their densities (see (2.2.5) and (2.2.6)) we use a \( n \)-fold convolution. After lengthy but straightforward calculations we arrive at the following recursive formulas.

For \( S_{k,n} \) and \( I_{k,n} \) we obtain:

**Theorem 3.2.2.** Let \( X \) be a meromorphic Lévy process. Let \( (-\zeta_{-}^{(n)}(j))_{j \geq 0} \) (resp. \( (\zeta_{+}^{(n)}(j))_{j \geq 0} \)) be the sequence of all real negative (resp. real positive) zeros of the function \( z \mapsto n + \Psi(-iz) \) (or, equivalently, if \( z \mapsto \psi(z) - n \)).

In the case of unbounded variation we have:

\[
\begin{align*}
\mathbf{P}(S_{k,n} \in dx) &= \sum_{j=0}^{N_{+}} \exp(-\zeta_{+}^{(n)}(j)x) \sum_{i=0}^{k-1} M_{+}^{(n)}(i,j,k)x^{i} dx \quad \text{for } x > 0, \\
\mathbf{P}(I_{k,n} \in dx) &= \sum_{j=0}^{N_{-}} \exp(\zeta_{-}^{(n)}(j)x) \sum_{i=0}^{k-1} M_{-}^{(n)}(i,j,k)x^{i} dx \quad \text{for } x < 0.
\end{align*}
\]

(3.2.1) (3.2.2)

If the process has bounded variation, define the atom at zero by \( w_{+}^{(n)} := \mathbf{P}(X_{\xi(n)} = 0) \) and \( w_{-}^{(n)} := \mathbf{P}(X_{\xi(n)} = 0) \). In the case we have:

\[
\begin{align*}
\mathbf{P}(S_{k,n} \in dx) &= \sum_{j=0}^{N_{+}} \exp(-\zeta_{+}^{(n)}(j)x) \sum_{i=0}^{k-1} R_{+}^{(n)}(i,j)x^{i} dx \quad \text{for } x > 0, \\
\mathbf{P}(I_{k,n} \in dx) &= \sum_{j=0}^{N_{-}} \exp(\zeta_{-}^{(n)}(j)x) \sum_{i=0}^{k-1} R_{-}^{(n)}(i,j)x^{i} dx \quad \text{for } x < 0
\end{align*}
\]

(3.2.3) (3.2.4)

(3.2.5) (3.2.6)

\[ \mathbf{P}(S_{k,n} = 0) = (w_{+}^{(n)})^{k}, \quad \mathbf{P}(I_{k,n} = 0) = (w_{-}^{(n)})^{k}. \]

Proof. Expressions for the coefficients \( M_{+}^{(n)}(.,j,.) \), \( M_{-}^{(n)}(.,j,.) \), \( R_{+}^{(n)}(.,j) \) and \( R_{-}^{(n)}(.,j) \) can be found in Appendix A for \( j = 0, \ldots, N_{+} \) respectively \( j = 0, \ldots, N_{-} \).

We start our proof with the unbounded case and use the principle of induction. For \( k = 1 \) it is obvious that (3.2.1) and (3.2.2) coincide with the formulas in Proposition
2.2.7. Let us assume that (3.2.1) and (3.2.2) are true for a \( k \geq 2 \). We start with (3.2.2) and note that we have to show:

\[
f^{(k+1,n)}_I(x) = f^{(1,n)}_I * f^{(k,n)}_I(x).
\]

for \( x \leq 0 \). However, this is a matter of a multiple applications of integration by parts, i.e. using (D.1) and sorting the terms in an obvious way. The case \( x \geq 0 \) can be obtained similarly.

If we now have a process of bounded variation, we have to show for \( x \leq 0 \)

\[
f^{(k+1,n)}_{I,b}(x) = f^{(1,n)}_{I,b} * f^{(k,n)}_{I,b}(x) \tag{3.2.7}
\]
as well. Here we can use the result from the unbounded case. The main difficulty lies in the fact that the expression (3.2.7) needs to be divided into 4 parts because of the atom in 0. This means if we write (3.2.7) as a convolution integral we have one part similar to the unbounded case, two as mixtures involving \( w \) and (3.2.2) and one only with \( w \). However, again after lengthy but straightforward computations, using again (D.1) and putting the terms together we obtain (3.2.3) and (3.2.4). Again, we can deal with the case \( x \geq 0 \) in a similar way.

Now we can use relation (2.2.12). Convoluting (3.2.1) and (3.2.2) respectively the formulas for the supremum and infimum, (3.2.3) and (3.2.5) in the bounded case, leads to the following form for the density of \( X_{g(k,n)} \).

**Theorem 3.2.3.** Let \( X \) be a meromorphic Lévy process. Let \((-\zeta^{(n)}_-(j))_{j \geq 0}\) (resp. \((\zeta^{(n)}_+(j))_{j \geq 0}\)) be the sequence of all real negative (resp. real positive) zeros of the function \( z \mapsto n + \Psi(-iz) \) (or, equivalently, of \( z \mapsto \psi(z) - n \)).

For the case of unbounded variation we have:

\[
f^{(k,n)}(x) := \mathbb{P}(X_{g(k,n)} \in dx) = \sum_{j=0}^{N_+} \exp(-\zeta^{(n)}_+(j)x) \sum_{i=0}^{k-1} W^{(n)}_+(i,j)x^i 1_{\{x \geq 0\}} dx + \sum_{j=0}^{N_-} \exp(\zeta^{(n)}_-(j)x) \sum_{i=0}^{k-1} W^{(n)}_-(i,j)x^i 1_{\{x \leq 0\}} dx \tag{3.2.8}
\]

If the process has bounded variation, define the atom at zero by \( w^{(n)}_+ := \mathbb{P}(X_{\xi(n)} = 0) \).
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and \( w_{-}^{(n)} := \mathbb{P}(X_{g(n)} = 0) \),

\[
f_{b}^{(k,n)}(x) := \mathbb{P}(X_{g(k,n)} \in dx) = \sum_{j=0}^{N_{+}} \exp(-\zeta_{+}^{(n)}(j)x) \sum_{i=0}^{k-1} R_{+}^{(n)}(i, j)x^{i}1_{\{x>0\}}dx + \sum_{j=0}^{N_{-}} \exp(\zeta_{-}^{(n)}(j)x) \sum_{i=0}^{k-1} R_{-}^{(n)}(i, j)x^{i}1_{\{x<0\}}dx,
\]

and

\[
\mathbb{P}(X_{g(k,n)} = 0) = (w_{+}^{(n)} w_{-}^{(n)})^{k}.
\]

**Proof.** Expressions for the coefficients \( W_{+}^{(n)}(., .), W_{-}^{(n)}(., .), R_{+}^{(n)}(., .) \) and \( R_{-}^{(n)}(., .) \)
can be found in Appendix A for \( j = 0, \ldots, N_{+} \) respectively \( j = 0, \ldots, N_{-} \).

To prove (3.2.8) and (3.2.9), once again this is a matter of multiple integration by parts, using (D.1) and sorting sums in the right way. \( \square \)

In the case that the process is of unbounded variation and \( N_{+} = N_{-} = 0 \), i.e. Brownian motion or a process of the \( \beta \)-class (see Section 2.2) with only one positive and negative root \( \zeta_{+/-}^{(n)}(0) \), the formulas (3.2.1), (3.2.2) and (3.2.8) can be simplified as follows:

**Lemma 3.2.4.** Let \( X \) be a meromorphic Lévy process of unbounded variation with \( N_{+} = N_{-} = 0 \). By using the results in Theorem 3.2.2 we obtain the following formulas:

\[
f_{S}^{(k,n)}(x) = \mathbb{P}(S_{g(k,n)} \in dx) = \exp(-\zeta_{+}^{(n)}(0)x) \frac{\zeta_{+}^{(n)}(0)^{k}}{(k-1)!} x^{k-1}dx \quad \text{for } x > 0, \quad (3.2.11)
\]

\[
f_{I}^{(k,n)}(x) = \mathbb{P}(I_{g(k,n)} \in dx) = \exp(\zeta_{-}^{(n)}(0)x) \frac{\zeta_{-}^{(n)}(0)^{k}(-1)^{k-1}}{(k-1)!} x^{k-1}dx \quad \text{for } x < 0, \quad (3.2.12)
\]

\[
f^{(k,n)}(x) = \mathbb{P}(X_{g(k,n)} \in dx) = \frac{\zeta_{+}^{(n)}(0)^{k} \zeta_{-}^{(n)}(0)^{k}}{(k-1)!} \exp(-\zeta_{+}^{(n)}(0)x) \quad (3.2.13)
\]

\[
\cdot \sum_{i=0}^{k-1} \frac{1}{(\zeta_{+}^{(n)}(0) + \zeta_{-}^{(n)}(0))^{2k-i-1} i! (k-1-i)!} x^{i}1_{\{x \geq 0\}}dx + \frac{\zeta_{+}^{(n)}(0)^{k} \zeta_{-}^{(n)}(0)^{k}}{(k-1)!} \exp(\zeta_{-}^{(n)}(0)x) \quad (3.2.14)
\]

\[
\cdot \sum_{i=0}^{k-1} \frac{(-1)^{i}}{(\zeta_{+}^{(n)}(0) + \zeta_{-}^{(n)}(0))^{2k-i-1} i! (k-1-i)!} x^{i}1_{\{x \leq 0\}}dx.
\]
Proof. We will prove the results by induction.

Formulas (3.2.11) and (3.2.12) are easily obtained by convoluting \( k \)-times (2.2.5) respectively (2.2.6).

Let us now have a look at (3.2.13) and let us assume \( x > 0 \) throughout the proof. (The case \( x < 0 \) can be obtained in a similar way). For \( k = 1 \), we have:

\[
\begin{align*}
 f_I^{(1,n)} \ast f_S^{(1,n)}(x) &= \int_{\mathbb{R}} f_I^{(1,n)}(x-y) f_S^{(1,n)}(y) 1_{\{x > 0, y > 0, x-y < 0\}} dy \\
 &= \int_{\mathbb{R}} f_I^{(1,n)}(x-y) f_S^{(1,n)}(y) dy \\
 &= \int_{\mathbb{R}} \exp(\zeta^{-}(0)(x-y)) \zeta^{+}(0) \zeta^{-}(0) \exp(-\zeta^{+}(0)y) dy \\
 &= \frac{\zeta^{+}(0) \zeta^{-}(0)}{\zeta^{+}(0) + \zeta^{-}(0)} \exp(-\zeta^{+}(0)x) \\
 &= f^{(1,n)}(x).
\end{align*}
\]

Define \( a := \zeta^{-}(0) \zeta^{+}(0) \) and \( b := \zeta^{+}(0) + \zeta^{-}(0) \). (3.2.11) and (3.2.12) are very useful for guessing the number sequence \( \frac{(2k-2-i)!}{i!(k-1-i)!} \) in (3.2.13). This sequence is known from the Padé approximation for the exponential function, see [8]. In order to prove the result we will use the relationship \( f^{(g(k+1,n))}(x) = f^{(g(1,n))}(x) \ast f^{(g(k,n))}(x) \):

\[
\begin{align*}
 f^{(k+1,n)}(x) &= f^{(1,n)}(x) \ast f^{(k,n)}(x) \\
 &= \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0\}} dy \\
 &= \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y > 0\}} dy + \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y < 0\}} dy \\
 &= \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y > 0, y > x\}} dy + \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y > 0, y < x\}} dy \\
 &\quad + \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y < 0, y > x\}} dy + \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y < 0, y < x\}} dy \\
 &= \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0\}} dy + \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y > 0\}} dy \\
 &\quad + \int_{\mathbb{R}} f^{(1,n)}(x-y) f^{(k,n)}(y) 1_{\{x > 0, y < 0\}} dy
\end{align*}
\]
\[
\begin{align*}
&= \int_0^{\infty} \frac{a}{b} \exp(\zeta^{(n)}_-(x-y)) \left( \frac{\zeta^{(n)}_-(\zeta^{(n)}_+)^k}{(k-1)!} \right) \exp(-\zeta^{(n)}_+ y) \sum_{i=0}^{k-1} \frac{1}{b^{2k-i-1}} \frac{(2k-2-i)!}{i!(k-1-i)!} y^i x^i \\
&\quad + \int_0^{\infty} \frac{a}{b} \exp(-\zeta^{(n)}_+(x-y)) \left( \frac{\zeta^{(n)}_+(\zeta^{(n)}_-)^k}{(k-1)!} \right) \exp(-\zeta^{(n)}_- y) \sum_{i=0}^{k-1} \frac{1}{b^{2k-i-1}} \frac{(2k-2-i)!}{i!(k-1-i)!} y^i x^i \\
&\quad + \int_0^{-\infty} \frac{a}{b} \exp(-\zeta^{(n)}_+(x-y)) \left( \frac{\zeta^{(n)}_+(\zeta^{(n)}_-)^k}{(k-1)!} \right) \exp(-\zeta^{(n)}_- y) \sum_{i=0}^{k-1} \frac{(-1)^i}{i!(k-1-i)!b^{2k-i}} y^i x^i \\
&= \sum_{i=0}^{k-1} \frac{a^{k+1}(2k-2-i)!}{i!(k-1-i)!(k-1)!b^{2k-i}} \\
&= \left( \exp(\zeta^{(n)}_- x) \right) \int_0^{\infty} \frac{a}{b} \exp(-by) y^i dy + \exp(-\zeta^{(n)}_+ x) \int_0^{\infty} y^i dy + \exp(-\zeta^{(n)}_- x) \int_{-\infty}^{0} \exp(by) y^i (-1)^i \\
&= \sum_{i=0}^{k-1} C(i) \exp(\zeta^{(n)}_- x) \left[ \sum_{l=0}^{i} (-1)^l \frac{1}{(b)^{l+1}} \frac{i!}{(i-l)!} y^{i-l} \exp(-by) \right]_x^{\infty} \\
&\quad + \sum_{i=0}^{k-1} C(i) \exp(-\zeta^{(n)}_+ x) \left[ \frac{y^{i+1}}{i+1} \right]_0^x \\
&\quad + \sum_{i=0}^{k-1} C(i) \exp(-\zeta^{(n)}_+ x)(-1)^i \left[ \sum_{l=0}^{i} (-1)^l \frac{1}{b^{l+1}} \frac{i!}{(i-l)!} y^{i-l} \exp(by) \right]_{-\infty}^0 \\
&= \sum_{i=0}^{k-1} C(i) \exp(\zeta^{(n)}_+ x) \left[ \sum_{l=0}^{i} \frac{1}{b^{l+1}} \frac{i!}{(i-l)!} x^{i-l} \right] \\
&\quad + \sum_{i=0}^{k-1} C(i) \exp(-\zeta^{(n)}_+ x) \frac{x^{i+1}}{i+1} \\
&\quad + \sum_{i=0}^{k-1} C(i) \exp(-\zeta^{(n)}_+ x) \frac{1}{b^{i+1}} i! \\
&= \sum_{i=0}^{k-1} C(i) \exp(-\zeta^{(n)}_+ x) \left[ \sum_{l=0}^{i} \frac{1}{b^{l+1}} \frac{i!}{(i-l)!} x^{i-l} \right] \\
&\quad + \sum_{i=0}^{k-1} C(i) \exp(-\zeta^{(n)}_+ x) \frac{x^{i+1}}{i+1} \\
&\quad + \sum_{i=0}^{k-1} C(i) \exp(-\zeta^{(n)}_+ x) \frac{1}{b^{i+1}} i! \\
&= \sum_{i=0}^{k-1} \sum_{l=0}^{i} \exp(-\zeta^{(n)}_+ x) x^l \frac{a^{k+1}(2k-2-i)!}{(k-1)!(k-1-i)!b^{2k-i}} \frac{1}{b^{l+1} i!} \\
&\quad + \sum_{i=0}^{k-1} a^{k+1} x^{i+1} \exp(-\zeta^{(n)}_+ x)(2k-2-i)! \\
&\quad + \sum_{i=0}^{k-1} (i+1)!b^{2k-i}(k-1)!(k-1-i)! \\
&\end{align*}
\]
\[ + \sum_{i=0}^{k-1} \frac{a^{k+1} \exp(-\zeta_+^{(n)} x)(2k - 2 - i)!}{i!b^{2k+1}(k-1)!(k-1 - i)!} =: R. \]

We separate \( R \) into a \( x^0 \)-, a \( x^i \)- and a \( x^k \)- part for \( i = 1 \ldots k - 1 \):

\[
R_1 := 2 \exp(-\zeta_+^{(n)} x) \sum_{i=0}^{k-1} \frac{a^{k+1}(2k - 2 - i)!}{b^{2k+1}(k-1)!(k-1 - i)!},
\]

\[
R_2 := \exp(-\zeta_+^{(n)} x) \sum_{i=0}^{k-1} \frac{a^{k+1} x^i(2k - 2 - i)!}{b^{2k-i+1}!(k-1 - i)!(k-1)!} x^i,
\]

\[
+ \sum_{i=1}^{k-1} \exp(-\zeta_+^{(n)} x) \frac{a^{k+1} x^i(2k - 2 - i)!}{b^{2k-i+1}!(k-1 - i)!(k-1)!} x^i,
\]

\[
R_3 := \exp(-\zeta_+^{(n)} x) \frac{a^{k+1}(k - 1)!}{(k-1)!k!b^{2k+1}x^k}.
\]

We have to show that

\[
R_1 = \exp(-\zeta_+^{(n)} x) \frac{a^{k+1}(2k)!}{k!k!b^{2k+1}},
\]

\[
R_2 = \exp(-\zeta_+^{(n)} x) \frac{a^{k+1}}{k!} \sum_{i=1}^{k-1} \frac{x^i(2k - 2 - i)!}{b^{2k-i+1}!(k-1 - i)!i!},
\]

\[
R_3 = \exp(-\zeta_+^{(n)} x) a^{k+1} \frac{x^k}{b^{k+1}k!}.
\]

This is obvious for \( R_3 \).

For \( R_1 \) we have:

\[
2 \exp(-\zeta_+^{(n)} x) \sum_{i=0}^{k-1} \frac{a^{k+1}(2k - 2 - i)!}{b^{2k+1}(k-1)!(k-1 - i)!}
\]

\[
= 2 \exp(-\zeta_+^{(n)} x) \sum_{i=0}^{k-1} \frac{a^{k+1}}{b^{2k+1}} \left( \frac{2k - 2 - i}{k - 1} \right)
\]

\[
= 2 \exp(-\zeta_+^{(n)} x) \frac{a^{k+1}}{k!b^{2k+1}} \left( \frac{2k - 2}{k - 1} \right)(2k - 1)
\]

\[
= 2 \exp(-\zeta_+^{(n)} x) \frac{a^{k+1}(2k - 1)!}{k!(k-1)!(k+1)!}b^{2k+1}
\]

\[
= \exp(-\zeta_+^{(n)} x) \frac{a^{k+1}(2k - 1)!}{k!(k-1)!(k+1)!}b^{2k+1}
\]

\[
= \exp(-\zeta_+^{(n)} x) \frac{a^{k+1}(2k)!}{k!k!b^{2k+1}}.
\]
Hereby we used in the third line a sum formula for binomial coefficients, i.e. the so-called ‘NW-Diagonal-Sum’ property (see [32], chapter 4):

\[
\sum_{i=0}^{d} \binom{2d-i}{d} = \binom{2d}{d} \frac{2d+1}{d+1}.
\]  

(3.2.14)

Let us now investigate the first part of \( R_2 \):

\[
\exp(-\zeta^{(n)}_+ x) \sum_{i=0}^{k-1} \sum_{l=1}^{k-1} \frac{a^{k+1}x^i(2k-2-i)!}{b^{2k-i+1}l!(k-1-i)!(k-1)!} \]

\[
= \exp(-\zeta^{(n)}_+ x) \sum_{i=1}^{k-1} \sum_{l=1}^{k-1} \frac{a^{k+1}x^i(2k-2-l)!}{b^{2k-i+1}l!(k-1-l)!(k-1)!} \]

\[
= \exp(-\zeta^{(n)}_+ x) \sum_{i=1}^{k-1} \frac{a^{k+1}x^i}{i!} \sum_{l=1}^{k-1} \frac{1}{b^{2k-i+1}k!(k-i-1)!} \]

\[
= \exp(-\zeta^{(n)}_+ x) \sum_{i=1}^{k-1} \frac{a^{k+1}x^i}{i!} \frac{(2k-i-1)!}{b^{2k-i+1}k!(k-i-1)!}.
\]

Again we used (3.2.14).

Together with the second part of \( R_2 \) we obtain:

\[
\exp(-\zeta^{(n)}_+ x) \sum_{i=1}^{k-1} \frac{a^{k+1}x^i}{b^{2k-i+1}} \left( \frac{(2k-i-1)!}{k!(k-i-1)!l!} + \frac{(2k-i-1)!}{(k-1)!(k-i)!l!} \right) \]

\[
= \exp(-\zeta^{(n)}_+ x) \sum_{i=1}^{k-1} \frac{a^{k+1}x^i}{b^{2k-i+1}} \left( \frac{(2k-i-1)!(2k-i)}{k!(k-i)!l!} \right) \]

\[
= \exp(-\zeta^{(n)}_+ x) \sum_{i=1}^{k-1} \frac{a^{k+1}x^i}{b^{2k-i+1}} \left( \frac{(2k-i)!}{k!(k-i)!l!} \right) \]

\[
= \exp(-\zeta^{(n)}_+ x) \sum_{i=1}^{k-1} \frac{a^{k+1}x^i}{b^{2k-i+1}} \left( \frac{(2k-i)!}{k!(k-i)!l!} \right).
\]

To conclude, one advantage of the explicit formula (3.2.4) is the reduction of the computer calculation time in comparison to the recursive one (3.2.8).

Next, we derive a formula for the joint law \( \mathbb{P}_x (X_{g(n,n)} \in [x, \bar{x}], X_{g(n,n)} \in [u, 0]) \).
Theorem 3.2.5. [Joint law] Let $X$ be a meromorphic Lévy process of unbounded variation. Let $u, \underline{x}, \bar{x} \in \mathbb{R}$ with $u < 0 \leq \underline{x} \leq \bar{x} < \infty$. Let
\[
c^{(0,n)}(x) := \mathbb{P}_x \left( X_{g(0,n)} \in [\underline{x}, \bar{x}], X_{g(0,n)} \in [u, 0) \right)
\]
\[
= \begin{cases} 
1 & \text{if } \underline{x} < x < \bar{x} \\
0 & \text{else.}
\end{cases}
\]
For every $x \in [u, \infty)$ we have:
\[
c^{(1,n)}(x) := \mathbb{P}_x \left( X_{g(1,n)} \in [\underline{x}, \bar{x}], X_{g(1,n)} \in [u, 0) \right)
\]
\[
= \sum_{j=0}^{N_+} \exp(-\zeta_+^{(n)}(j)x) C^{(n)}_-(j, +, 0, 1) \mathbf{1}_{\{x \geq 0\}} \\
+ \sum_{j=0}^{N_-} \exp(\zeta_-^{(n)}(j)x) C^{(n)}_+(j, -, 0, 1) \mathbf{1}_{\{u \leq x \leq 0\}} \\
+ \sum_{j=0}^{N_+} \exp(-\zeta_+^{(n)}(j)x) C^{(n)}_-(j, -, 0, 1) \mathbf{1}_{\{u \leq x \leq 0\}}.
\]
Furthermore, for every $k \geq 2$ we have:
\[
c^{(k,n)}(x) := \mathbb{P}_x \left( X_{g(k,n)} \in [\underline{x}, \bar{x}], X_{g(k,n)} \in [u, 0) \right)
\]
\[
= \sum_{j=0}^{N_+} \exp(-\zeta_+^{(n)}(j)x) C^{(n)}_-(j, +, 0, k) \mathbf{1}_{\{x \geq 0\}} \\
+ \sum_{j=0}^{N_-} \exp(\zeta_-^{(n)}(j)x) \sum_{i=0}^{k-1} C^{(n)}_+(j, -, 0, k) x^i \mathbf{1}_{\{u \leq x \leq 0\}} \\
+ \sum_{j=0}^{N_+} \exp(-\zeta_+^{(n)}(j)x) \sum_{i=0}^{k-1} C^{(n)}_-(j, -, 0, k) x^i \mathbf{1}_{\{u \leq x \leq 0\}}. \tag{3.2.15}
\]
Expressions for $C^{(n)}_-(\cdot, +, 0, \cdot), C^{(n)}_+(\cdot, -, 0, \cdot)$ and $C^{(n)}_-(\cdot, -, 0, \cdot)$ can be found in Appendix A.

Proof. Let us start to calculate $c^{(1,n)}(x) = \mathbb{P}_x \left( X_{g(1,n)} \in [u, \bar{x}], X_{g(1,n)} \in [u, 0) \right)$. This
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\[ \mathbb{E}_x \left[ 1_{\{u \leq X_{\xi(n)}(n) \leq 0\}} 1_{\{z \leq X_{\xi(n)}(n) \leq \bar{x}\}} \right] \]

\[ = \mathbb{E} \left[ 1_{\{u - z \leq X_{\xi(n)}(n) \leq -z\}} 1_{\{z - x \leq X_{\xi(n)}(n) \leq \bar{x} - x\}} \right] \]

\[ = \mathbb{E} \left[ 1_{\{u - z \leq X_{\xi(n)}(n) \leq -z\}} 1_{\{z - x \leq X_{\xi(n)}(n) + \bar{X}_{\xi(n)}(n) \leq \bar{x} - x - \bar{X}_{\xi(n)}(n)\}} \right] \]

\[ \overset{(*)}{=} \int_{u - z \leq -x - y} \int_{-x}^{\bar{x} - x - y} g_I(y) g_S(z) \, dz \, dy. \] (3.2.16)

For (*) we used relation (2.1.6) and for (**) expressions for the densities of $X_{\xi(n)}$ and $\bar{X}_{\xi(n)}$, (2.2.5) and (2.2.6). After an easy calculation we obtain $c^{(1,n)}(x)$ for $x > u$.

Next, let us assume that (3.2.15) is true for a fixed $k > 1$. We would like to prove the relation for $k + 1$.

However, it is obvious that $c^{(k+1,n)}(x)$ fulfils the following backwards recursion:

\[ c^{(k+1,n)}(x) = \mathbb{P}_x \left( X_{\xi(k+1,n)}(n) \in [x, \bar{x}) \right) \]

\[ = \mathbb{P}_x \left( c^{(k,n)}(X_{\xi(n)}) 1_{\{X_{\xi(n)} \geq 0\}} \right) \]

\[ = \mathbb{P} \left( c^{(k,n)}(x + \bar{X}_{\xi(n)} + \bar{X}_{\xi(n)}) 1_{\{X_{\xi(n)} \geq -x\}} \right). \] (3.2.17)

Plugging in the argument of $c^{(k+1,n)}(.)$ into (3.2.15) we have to consider 3 cases, dependent on the coefficients $C_{\xi(n)}(\cdot, +, 0, \cdot)$, $C_{\xi(n)}(\cdot, -, 0, \cdot)$ and $C_{\xi(n)}(\cdot, -, 0, \cdot)$. For each of these cases (3.2.17) we receive expressions consisting of double integrals. Hereby, we integrate over the densities $g_S$ respectively $g_I$ of $\bar{X}_{\xi(n)}$ respectively $\bar{X}_{\xi(n)}$. However, the derivation involves several summations using Proposition D.0.1. We omit details but very long but straightforward calculations and a clever rearranging of the sum terms yield the result.

Remark 3.2.6. In the paper \cite{50} the authors obtain similar to our results recursions for the densities of the position (Lemma 2.4) and the distribution of the joint law of the Lévy process (Proposition 2.5). Hereby the underlying Lévy process is a member of the class of the so-called mixed-exponential jump diffusions. These can be obtained by allowing the coefficients in (2.2.7) to be negative and $N_+, N_-$ to be positive integers,
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see Definition 2.1 in [50] or page 2069 in [23]. It is therefore important to emphasise that mixed-exponential jump diffusions do not belong to the class of meromorphic Lévy processes. Indeed, as they do not possess a completely-monotone Lévy measure it can be shown that they are dense in the class of all Lévy processes in the sense of weak convergence (see also Remark 2.2.12).

3.3 Error estimate

We want to estimate how good our approximation for the law of $X_1$ from Theorem 3.2.3 is. For this, we use a result from Ferreiro-Castilla et al. [30] and use their notation: We will write $a \lesssim b$ for two positive quantities $a$ and $b$, if $\frac{a}{b}$ is uniformly bounded independent of any parameters and $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

**Proposition 3.3.1** ([30]). Let $t > 0$ and let $X$ be a Lévy process which fulfils

$$\int_{|x|>1} x^2 \Pi(dx) < \infty. \quad (3.3.1)$$

Then, for any $n \in \mathbb{N}$, we have:

(i) $E \left[ (X_{g(n,n)} - X_1)^2 \right] \simeq n^{-\frac{1}{2}}$ and $E \left[ |X_{g(n,n)} - X_1| \right] \lesssim n^{-\frac{1}{4}}$ \quad (3.3.2)

(ii) $E \left[ (\overline{X}_{g(n,n)} - \overline{X}_1)^2 \right] \simeq n^{-\frac{1}{2}}$ and $E \left[ |\overline{X}_{g(n,n)} - \overline{X}_1| \right] \lesssim n^{-\frac{1}{4}}$. \quad (3.3.3)

If, in addition, $X$ has paths of bounded variation then we have the sharper bound:

$$E \left[ |X_{g(n,n)} - X_1| \right] \lesssim n^{-\frac{1}{2}} \quad \text{and} \quad E \left[ |\overline{X}_{g(n,n)} - \overline{X}_1| \right] \lesssim n^{-\frac{1}{2}}. \quad (3.3.4)$$

**Proposition 3.3.2.** Let $X$ be a Lévy process of unbounded variation. Let $\overline{G}$ be a Lipschitz-continuous function with the following form

$$\overline{G}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x < \alpha_n \\
\overline{H}(x) & \text{if } \alpha_n \leq x \leq 0
\end{cases}$$

for some Lipschitz-continuous function $\overline{H}$ with Lipschitz constant $-\frac{1}{\alpha_n}$ for $\alpha_n < 0$ and every $n \geq 1$, $H(\alpha_n) = 0$ and $H(0) = 1$. Then for $x \in \mathbb{R}$

$$|\mathbb{P}(X_{g(n,n)} \geq 0) - \mathbb{P}(X_1 \geq 0)|$$

$$\leq |\mathbb{E}[\overline{G}(X_{g(n,n)}) - \overline{G}(X_1)]| + \mathbb{E}[\varepsilon(X_{g(n,n)}, \alpha_n)] + \mathbb{E}[\varepsilon(X_1, \alpha_n)]$$
where \( \varepsilon(x, \alpha_n) := |\overline{H}(x)\mathbf{1}_{\{\alpha_n \leq x \leq 0\}}| \) for every \( n \geq 1 \).

Proof.

\[
|\mathbb{P}(X_{g(n,n)} \geq 0) - \mathbb{P}(X_1 \geq 0)| \\
= |\mathbb{E}[\mathbf{1}_{\{X_{g(n,n)} \geq 0\}}] - \mathbb{E}[\mathbf{1}_{\{X_1 \geq 0\}}]| \\
= |\mathbb{E}\left[\mathbf{1}_{\{X_{g(n,n)} \geq 0\}} - \mathbf{1}_{\{X_1 \geq 0\}}\right]| \\
\leq |\mathbb{E}(\overline{G}(X_{g(n,n)}) - \overline{G}(X_1))| + \mathbb{E}[\varepsilon(X_{g(n,n)}, \alpha_n)] + \mathbb{E}[\varepsilon(X_1, \alpha_n)]. \quad (3.3.5)
\]

The last inequality (3.3.5) can be seen as follows:

\[
\left|\mathbb{E}[\mathbf{1}_{\{X_{g(n,n)} \geq 0\}}] - \mathbb{E}[\mathbf{1}_{\{X_1 \geq 0\}}]\right| - \mathbb{E}[\overline{G}(X_{g(n,n)})] - \mathbb{E}[\overline{G}(X_1)] \\
\leq \left|\mathbb{E}[\mathbf{1}_{\{X_{g(n,n)} \geq 0\}}] - \mathbb{E}[\mathbf{1}_{\{X_1 \geq 0\}}]\right| - \mathbb{E}[\overline{G}(X_{g(n,n)})] + \mathbb{E}[\overline{G}(X_1)] \\
as |x| - |y| \leq |x - y| \\
= \mathbb{E}\left[\mathbf{1}_{\{X_{g(n,n)} \geq 0\}} - \overline{G}(X_{g(n,n)}) - \mathbf{1}_{\{X_1 \geq 0\}} + \overline{G}(X_1)\right] \\
\leq \left|\mathbb{E}[\mathbf{1}_{\{X_{g(n,n)} \geq 0\}}] - \overline{G}(X_{g(n,n)})\right| \leq \mathbb{E}[\varepsilon(X_{g(n,n)}, \alpha_n)] + \mathbb{E}[\varepsilon(X_1, \alpha_n)] \\
= \mathbb{E}[\varepsilon(X_{g(n,n)}, \alpha_n)] + \mathbb{E}[\varepsilon(X_1, \alpha_n)].
\]

* can be obtained by using the triangle inequality and ** by the Jensen inequality.

Now we want to estimate \( \mathbb{E}[\varepsilon(X_{g(n,n)}, \alpha_n)] \) and \( \mathbb{E}[\varepsilon(X_1, \alpha_n)] \). Hereby we have to assume that \( X \) fulfils the following condition for a fixed \( C > 0 \):

\[
\mathbb{P}(\alpha_n \leq X_1 \leq 0) < C|\alpha_n|. \quad (3.3.6)
\]

Proposition 3.3.3. Let \( X \) be a meromorphic Lévy process of unbounded variation which fulfils (3.3.6). Then

\[
\mathbb{E}[\varepsilon(X_{g(n,n)}, \alpha_n)] \\
\leq \sum_{j=0}^{N-1} \sum_{i=0}^{n-1} W_-(n)(i, j) \frac{i!(-1)^i}{\zeta_-(n)^{j+1}} \\
- \sum_{j=0}^{N-1} \sum_{i=0}^{n-1} \sum_{l=0}^i W_-(n)(i, j) \frac{i!}{(i-l)!}(-1)^l \frac{1}{\zeta_-(n)^{j+1}} \exp(\zeta_-(n)^{j}\alpha_n)\alpha_n^{j-l} =: D(\alpha_n)
\]
and

$$\mathbb{E}[\epsilon(X_1, \alpha_n)] < C|\alpha_n|$$

where $\mathbb{E}[\epsilon(X_{g(n,n)}, \alpha_n)]$ and $\mathbb{E}[\epsilon(X_1, \alpha_n)]$ are defined in Proposition 3.3.2.

**Proof.** The term $\mathbb{E}[\epsilon(X_{g(n,n)}, \alpha_n)]$ can be calculated explicitly with the results of Theorem 3.2.3.

$$\mathbb{E}[\epsilon(X_{g(n,n)}, \alpha_n)] = \mathbb{E}[[\mathcal{H}(X_{g(n,n)})1_{\{\alpha_n \leq X_{g(n,n)} \leq 0\}}]]$$

$$\leq \mathbb{E}[1_{\{\alpha_n \leq X_{g(n,n)} \leq 0\}}]$$

$$= \int_{\mathbb{R}_-} f^{(n,n)}(x) dx$$

$$= \sum_{j=0}^{N_n-1} \sum_{i=0}^{n-1} W^{(n)}_-(i,j) \frac{i!(-1)^i}{\zeta^{(n)}_-(j)^{i+1}}$$

$$- \sum_{j=0}^{N_n-1} \sum_{i=0}^{n-1} \sum_{l=0}^{i} W^{(n)}_-(i,j) \frac{i!}{(i-l)!} (-1)^l \frac{1}{\zeta^{(n)}_-(j)^{l+1}} \exp(\zeta^{(n)}_-(j)\alpha_n) \alpha_n^{i-l}.$$ 

The term $\mathbb{E}[\epsilon(X_1, \alpha_n)]$ can be estimated by assumption (3.3.6).

$$\mathbb{E}[\epsilon(X_1, \alpha_n)] = \mathbb{E}[[\mathcal{H}(X_1)1_{\{\alpha_n \leq X_1 \leq 0\}}]]$$

$$\leq \mathbb{E}[1_{\{\alpha_n \leq X_1 \leq 0\}}]$$

$$= \mathbb{P}(\alpha_n \leq X_1 \leq 0)$$

$$< C|\alpha_n|.$$ 

□

**Theorem 3.3.4.** Let $X$ be a meromorphic Lévy process of unbounded variation which fulfills (3.3.6). Then

$$|\mathbb{P}(X_{g(n,n)} \geq 0) - \mathbb{P}(X_1 \geq 0)| \lesssim -\frac{1}{\alpha_n}n^{-\frac{1}{4}} + D(\alpha_n) - C\alpha_n,$$

where $\alpha_n < 0$ for every $n \geq 1.$
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Proof.

\[
|\mathbb{P}(X_{g(n,n)} \geq 0) - \mathbb{P}(X_1 \geq 0)| \\
= |\mathbb{E}[1_{\{X_{g(n,n)} \geq 0\}}] - \mathbb{E}[1_{\{X_1 \geq 0\}}]| \\
= |\mathbb{E} \left[1_{\{X_{g(n,n)} \geq 0\}} - 1_{\{X_1 \geq 0\}} \right]| \\
= |\mathbb{E} \left[\overline{H}(X_{g(n,n)}) - \overline{H}(X_1) \right]| \\
+ \mathbb{E}[\epsilon(X_{g(n,n)}, \alpha_n)] + \mathbb{E}[\epsilon(X_1, \alpha_n)] \\
\leq |\mathbb{E} \left[\overline{H}(X_{g(n,n)}) - \overline{H}(X_1) \right]| + D(\alpha_n) - C\alpha_n \\
\leq E \left[|\overline{H}(X_{g(n,n)}) - \overline{H}(X_1)| \right] + D(\alpha_n) - C\alpha_n \\
\leq E \left[-\frac{1}{\alpha_n} |X_{g(n,n)} - X_1| \right] + D(\alpha_n) - C\alpha_n \\
\lesssim -\frac{1}{\alpha_n} n^{-\frac{1}{4}} + D(\alpha_n) - C\alpha_n.
\]

Hereby we used relation Proposition 3.3.3 for the bounds of \(\mathbb{E} \left[\epsilon(X_{g(n,n)}, \alpha_n)\right]\) and \(\mathbb{E} \left[\epsilon(X_1, \alpha_n)\right]\). In the second-last inequality we used the fact that \(\overline{H}\) is a Lipschitz-continuous function with constant \(-\frac{1}{\alpha_n}\). The last inequality is obtained from Proposition 3.3.1. \(\square\)

Corollary 3.3.5. Let \(X\) be a Lévy process of unbounded variation which fulfills (3.3.6). Then

\[
|\mathbb{P}(X_{g(n,n)} \geq 0) - \mathbb{P}(X_1 \geq 0)| \lesssim D(-n^{-\frac{1}{8}}) + (C + 1)n^{-\frac{1}{8}}. \quad (3.3.7)
\]

Proof. Let us choose \(\alpha_n = -n^{-\frac{1}{8}}\) in Theorem 3.3.4 for \(n \geq 1\).

It is clear that \(-n^{-\frac{1}{8}} < -n^{-\frac{1}{2}}\) for all \(n \geq 1\). Then we have from Theorem 3.3.4

\[
-\frac{1}{\alpha_n} n^{-\frac{1}{4}} + D(-n^{-\frac{1}{8}}) - C\alpha_n \\
= -\frac{1}{-n^{-\frac{1}{8}}} n^{-\frac{1}{4}} + D(-n^{-\frac{1}{8}}) - C(-n^{-\frac{1}{8}}) \\
= n^{\frac{1}{2}} n^{-\frac{1}{4}} + D(-n^{-\frac{1}{8}}) + Cn^{-\frac{1}{8}} \\
= D(-n^{-\frac{1}{8}}) + (C + 1)n^{-\frac{1}{8}}
\]

which is decreasing for \(n \geq 1\) and eventually becomes smaller than any chosen \(\delta > 0\). \(\square\)

Remark 3.3.6. The bound coefficients \(C\) and later \(U\) in assumptions (3.3.6) and (4.2.14) do not depend on \(n\). Therefore they do not play a role when investigating the
errors for growing \( n \) in the case that we exclude Lévy processes with an atom at 0, i.e. Compound Poisson processes.

To motivate assumptions (3.3.6) and (4.2.14) note that in this thesis we consider meromorphic Lévy processes which possess a probability density function (2.2.7) in the form of infinite sums. In case of practical applications and numerical computations these need to be truncated. As a result, we obtain a finite sum which is bounded by, say, \( W \). Then

\[
P(\alpha_n \leq X_{\xi(n)} \leq 0) = \int_{\alpha_n}^{0} f_{X_{\xi(n)}}(t)dt \leq W|\alpha_n|.
\]

However, a Lévy process does not necessarily have to possess a density function at time 1 (see [57], Proposition 28.3 for the conditions on the existence) and moreover, even it has one which is bounded at time \( \xi(n) \) this does not have to be true at time 1. Nevertheless, assumptions (3.3.6) and (4.2.14) are fulfilled by many processes used in this thesis, for example Brownian motion or jump diffusions with exponentially distributed jumps, both with a sufficiently high diffusion coefficient and the latter with suitable jump parameters.

In the case of Brownian motion we can show that the Taylor series of the density (3.2.13) converges to the Taylor series of the Gaussian function.

We start with calculating the Taylor series of (3.2.13) which we will need for Theorem 3.3.8.

**Lemma 3.3.7.** The Taylor series of the density (3.2.13) for a Brownian motion for \( x > 0 \) up to \( 2n \) is given by

\[
h^n(x) = \sum_{i=0}^{n-1} U_i x^i + \sum_{i=n}^{2n} W_i x^i \tag{3.3.8}
\]

where

\[
U_0 = \frac{\sqrt{2}\sqrt{n(2n-2)!}}{2^{2n-1}(n-1)!^2} \tag{3.3.9}
\]

\[
U_1 = 0 \tag{3.3.10}
\]

\[
U_{i+2} = -U_i \frac{n}{(2n-3-i)\frac{i+2}{2}} \text{ for } i = 0, \ldots, n-1 \tag{3.3.11}
\]
and $W_i$ are real coefficients for $i > n - 1$.

Due to symmetry reasons we skip a formula for $x < 0$.

**Proof.** First we note that if $X$ is a Brownian motion we have:

$$
\zeta_+(0) = \zeta_-(0) = \sqrt{2}\sqrt{n}.
$$

(3.3.12)

We also want to remark that we do not specify the coefficients $W_i$ as we do not need them for the convergence result (3.3.8) below.

The Taylor series coefficients can be calculated as follows:

\[
\begin{align*}
(\zeta_+^{(n)}(0))^n(\zeta_-^{(n)}(0))^n \exp(-\zeta_+(0)x) \\
\cdot \sum_{i=0}^{n-1} \sum_{l=0}^{n-1} \frac{1}{(\zeta_+^{(n)}(0) + \zeta_-^{(n)}(0))^{2n-i-1}} \frac{(2n - 2 - i)!}{i!(n - 1 - i)!(n - 1)!} x^i \\
= (\zeta_+^{(n)}(0))^n(\zeta_-^{(n)}(0))^n \\
\cdot \sum_{i=0}^{n-1} \sum_{l=0}^{n-1} \frac{1}{(\zeta_+^{(n)}(0) + \zeta_-^{(n)}(0))^{2n-i-1}} \frac{(2n - 2 - i)!\zeta_+^{(n)}(0)^l}{i!(n - 1 - i)!(n - 1)!} x^{i+l} \\
= (\zeta_+^{(n)}(0))^n(\zeta_-^{(n)}(0))^n \\
\cdot \sum_{i=0}^{n-1} \sum_{l=0}^{n-1} \frac{1}{(\zeta_+^{(n)}(0) + \zeta_-^{(n)}(0))^{2n-i-1}} \frac{(2n - 2 - i)!\zeta_+^{(n)}(0)^l}{i!(n - 1 - i)!(n - 1)!} x^{i+l} \\
= \sum_{k=0}^{n-1} \sum_{i=0}^{k} \frac{2^{\frac{3}{4}+\frac{1}{2}+\frac{1}{2}i}n^{\frac{3}{4}+\frac{1}{2}+\frac{1}{2}i}(-1)^{-i}}{4^n} \frac{(2n - 2 - i)!}{i!(k - i)!(n - 1 - i)!(n - 1)!} x^k + \sum_{k=n}^{n-2} W_k x^k.
\end{align*}
\]

We start to prove formula (3.3.9), i.e. $k = 0$:

\[
\sum_{i=0}^{0} \frac{2^{\frac{3}{4}+\frac{1}{2}+\frac{1}{2}0+0}n^{\frac{3}{4}+\frac{1}{2}+\frac{1}{2}0}(-1)^{0-0}}{4^n} \frac{(2n - 2 - 0)!}{0!(0 - 0)!(n - 1 - 0)!(n - 1)!} x^0 \\
= \frac{2^{\frac{3}{4}+\frac{1}{2}0}(2n - 2)!}{4^n(n - 1)!(n - 1)!} x^0 \\
= \frac{2^{\frac{3}{2}+\frac{1}{2}0}(2n - 2)!}{2^{2n}(n - 1)!(n - 1)!} x^0 \\
= \sqrt{2}\sqrt{n} \frac{(2n - 2)!}{2^{2n-1}(n - 1)!} x^0.
\]
Relation (3.3.10) can be derived as follows:

\[
\sum_{i=0}^{1} \frac{2^{\frac{3}{2} + \frac{1}{2} + in^{\frac{1}{2} + \frac{1}{2}}(-1)^{1-i} (2n - 2 - i)!}{4^n i!(k - i)!{(n - 1 - i)!}(n - 1)!} x^1
\]

\[
= - \frac{2^{\frac{3}{2} + \frac{1}{2} + n^{\frac{1}{2} + \frac{1}{2}}(2n - 2)!}{4^n (n - 1)!{(n - 1 - i)!}(n - 1)!} x^1 + \frac{2^{\frac{3}{2} + \frac{1}{2} + 1 + n^{\frac{1}{2} + \frac{1}{2} + 1}(2n - 3)!}{4^n (n - 2)!{(n - 1)!}(n - 1)!} x^1
\]

\[
= - \frac{4n (2n - 2)!}{4^n (n - 1)!{(n - 1 - i)!}(n - 1)!} x^1 + \frac{8n (2n - 3)!}{4^n (n - 2)!{(n - 1)!}(n - 1)!} x^1
\]

\[
= - \frac{8n (2n - 3)!{(n - 1)!}}{4^n (n - 1)!{(n - 1)!}} x^1 + \frac{8n (2n - 3)!{(n - 1)!}}{4^n (n - 1)!{(n - 1)!}} x^1
\]

\[
= 0.
\]

For a general \(k\) we now have to prove:

\[
- \sum_{i=0}^{k} \frac{2^{\frac{3}{2} + \frac{1}{2} + k + in^{\frac{1}{2} + \frac{1}{2}}(-1)^{k-i} (2n - 2 - i)!}{4^n i!(k - i)!{(n - 1 - i)!}(n - 1)!} (2n - 3 - k)^{\frac{k+2}{2}}
\]

\[
= \sum_{i=0}^{k+2} \frac{2^{\frac{3}{2} + \frac{1}{2} + (k+2) + i{n^{\frac{1}{2} + \frac{1}{2}}}(2n - 2 - i)!}{4^n i!{(k + 2 - i)!}(n - 1 - i)!} (2n - 2 - i)!}
\]

This is equivalent to show:

\[
- \frac{1}{(2n - 3 - k)(k + 2)} \sum_{i=0}^{k} \frac{(2n - 2 - i)!(-1)^{i2^i}}{i!(k - i)!{(n - 1 - i)!} (2n - 3 - k)^{\frac{k+2}{2}}}
\]

\[
= \sum_{i=0}^{k+2} \frac{(2n - 2 - i)!(-1)^{i2^i}}{i!{(k + 2 - i)!}(n - 1 - i)!}.
\]

However this can be done with the help of Theorem D.0.4. Let us assume that \(k\) is even (The case that \(k\) is odd is trivial).

From Theorem D.0.4 we know that

\[
\sum_{i=0}^{k} \frac{(-1)^{i2^i}(2n - 2 - i)!}{i!{(n - 1 - i)!}!(k - i)!} = \begin{cases} 
(-1)^{\frac{k}{2}} \frac{(2n - 2 - k)!}{(n - 1 - \frac{k}{2})!}, & \text{if } k \text{ is even}, \\
0, & \text{if } k \text{ is odd}.
\end{cases}
\]
Therefore

\[
\begin{align*}
\sum_{i=0}^{k+2} \frac{(2n-2-i)(-1)^i}{i!(k+2-i)!(n-1-i)!} \\
\sum_{i=0}^{k} \frac{(2n-2-i)(-1)^i}{i!(k-i)!(n-1-i)!} \\
\frac{(-1)^{\frac{k+2}{2}} (2n-4-k)!}{(n-1-\frac{k+2}{2})!(\frac{k+2}{2})!} \\
\frac{(-1)^{\frac{k}{2}} (2n-2-k)!}{(n-1-\frac{k}{2})!(\frac{k}{2})!} \\
\frac{(2n-4-k)!\frac{k}{2}!(n-1-\frac{k}{2})!}{(2n-2-k)!(\frac{k+2}{2})!(n-1-\frac{k+2}{2})!} \\
\frac{(n-1-\frac{k}{2})}{(2n-3-k)(2n-2-k)\frac{k+2}{2}} \\
\frac{1}{(2n-3-k)2\frac{1}{2}(k+2)} \\
\frac{-1}{(2n-3-k)(k+2)}. \\
\end{align*}
\]

\[
\square
\]

For the case of Brownian motion, we are now able to show convergence.

**Theorem 3.3.8. [Convergence for Brownian motion]** Let \( b^n \) be the Taylor series up to the \( n \)-th term of the density of Brownian motion at time 1, i.e.

\[
b^n(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{i=0}^{n} \frac{x^{2i}(-1)^i}{2i+1}.
\]

(3.3.13)

Let \( h^n(x) \) be the function defined in (3.3.8). Then

\[
|h^n(x) - b^n(x)| \to 0 \quad \text{as} \quad n \to \infty
\]

for every \( x > 0 \).
Proof. First we show that $U_0 \to \frac{\sqrt{2}}{2\sqrt{\pi}}$:

$$U_0 = \frac{n^n(2n-2)!2^n}{(n-1)!(2\sqrt{2}\sqrt{n})^{2n-1}} = \frac{\sqrt{2}\sqrt{n}(2n-2)!}{(n-1)!2^{2n-1}} = \frac{\sqrt{2}}{2^n-1}\Gamma(2n) = \frac{\sqrt{2}}{2^n-1}\Gamma(n)(2n-1) = \frac{\sqrt{2}}{2^n-1}\Gamma(n)(n+1) = \frac{\sqrt{2}}{2^n-1}(n-1)! = \frac{\sqrt{2}}{2^n-1}.$$ 

Hereby we used that

$$\Gamma(n - \frac{1}{2}) = \frac{\Gamma(n + \frac{1}{2})}{n - \frac{1}{2}}$$

and

$$\lim_{n \to \infty} \frac{\Gamma(n + b)n^{-b}}{\Gamma(n)} \text{ for } b \in \mathbb{R}. \quad (3.3.14)$$

The latter equation can be found in [1], page 257. Next, we note again that the coefficients for the $i + 1$th term of (3.3.13) can be calculated by the product of the coefficients of $i$th term and $-\frac{1}{2(i+1)}$. It is an easy exercise to see that formula (3.3.11) converges to this relation, i.e. for $i = 0, \ldots, n - 1$

$$\frac{2n}{(2n-i-3)(i+2)} \to -\frac{1}{(i+2)} \text{ as } n \to \infty.$$

\[\square\]

### 3.4 Numerics

In this section we discuss some numerics for the results obtained by a computer implementation of the result in Theorems 3.2.3 and Lemma 3.3.7.

Next, we briefly describe the two main tasks for creating the graphs of different value functions presented further below. The first step is the calculation of the solutions $\zeta_\psi^{(n)}(i)$ and $\zeta_\psi^{(n)}(j)$ of $z \mapsto \psi(z) - n$ and of the coefficients $c_\psi^{(n)}(i)$ and $c_\psi^{(n)}(j)$ by using
the relations in Chapter 2 for \(i \in \{0, \ldots, N_+\}\) and \(j \in \{0, \ldots, N_-\}\). In case a meromorphic Lévy process is used for which \(N_+ = \infty\) or \(N_- = \infty\) the infinite sums need to be truncated. For this we used the ad-hoc approach of ensuring that the truncated sums represent at least 99\% of the mass of \(X_{\xi(n)}\), i.e. (compare with (2.2.7)) we determined \(M^+, M^-\) such that
\[
\sum_{j=0}^{M^+} c_+^{(n)}(j) \zeta_+(j) - \sum_{j=0}^{M^-} c_-^{(n)}(j) \zeta_-(j) \geq 0.99
\]
and then normalised the coefficients.

The second main step is the recursive computation of the formulas for the densities using the formulas (3.2.8) and (3.2.9). A program for this purpose was written in the language C++. We found that in certain cases the algorithm may need more than the default ‘double precision’ as working precision in order to have enough significant digits left in the end result. For this we made use of the ‘GMP/MPFR/MPFRC++’ packages. Furthermore we made use of ‘OpenMP’ to parallelise the operations. The Lévy processes used in this chapter are Brownian motion with drift, a jump diffusion with exponentially distributed jumps, a compound Poisson process with exponentially distributed random variables \(U_i\) (see (4.1.1)) and a process from the \(\beta\)-class, see also [40] and Section 2.2.

All figures show the following observations.

Firstly, it shows the quick convergence of our method. The difference between the density for a large \(n\) (e.g. \(n = 3000\)) which we expect to be very near to the true value and a small \(n\) (e.g. \(n = 24\) or \(n = 120\)) is not very large. Secondly, they support the convergence result Theorem 3.3.8.

For all graphs, we consider an approximation of the time 1, i.e. \(k = n\) for the functions \(f^{(k,n)}(x)\) from formula (3.2.13).
Figure 3.1: Plots of the relative difference between the exact values and the approximating density functions $f^{(n,n)}(x)$ from formula (3.2.13) where the driving process $X$ is a standard Brownian motion. Furthermore $n = 24, 120, 600, 3000$ and $N^- = N^+ = 0.$
Figure 3.2: Plots of the relative difference between the approximating density functions $f^{(3000,3000)}(x)$ and the approximating density functions $f^{(n,n)}(x)$ from formula (3.2.8) where the driving process $X$ is a jump diffusion with exponentially distributed jumps, e.g. $X_t = \mu t + \sigma^2 B_t + \sum_{i=0}^{N_t} Y_i$. The parameters are $\mu = 1.01, \sigma = 1, \lambda = 1$ and $\alpha = 1$ where $Y \sim Exp(\alpha)$ and $N \sim Poi(\lambda)$. Furthermore $n = 1, 24, 600$ and $N^- = N^+ = 1$. 
Figure 3.3: Plots of the relative difference between the between the exact values and the approximating density functions \( f^{(n,n)}_b(x) \) from formula (3.2.9) where the driving process \( X \) is a compound Poisson process with drift, e.g. \( X_t = \mu t + \sum_{i=0}^{N_t} U_i \). The parameters are \( \mu = 1.01, \lambda = 1 \) and \( \alpha = 1 \) where \( Y \sim Exp(\alpha) \) and \( N \sim Poi(\lambda) \). Furthermore \( n = 24, 120, 600 \) and \( N^- = N^+ = 0 \).
Figure 3.4: Plots of the different approximating density functions $f^{(300,300)}(x)$ from formula (3.2.8) where the driving process $X$ is a $\beta$-class with infinite variation and infinite activity. Parameter values are $\mu = 1, \sigma = 0.5, 1, 3, \alpha_1 = \alpha_2 = 20, \beta_1 = \beta_2 = 40, c_1 = c_2 = 120$ and $\lambda_1 = \lambda_2 = 2.1$. Furthermore $n = 300$ and $N^- = N^+ = 6$. 
Chapter 4

An approximation of the ruin probability in finite time driven by meromorphic Lévy processes

4.1 Introduction

The study of ruin probabilities can be tracked back many years. The idea is to model the capital of an insurance company as a stochastic process \((R_t)_{t \geq 0}\) called risk process. Roughly speaking, the ruin probability is the probability that at some point in the future the capital of the company falls below 0. In the literature, see for instance [7], the standard model is the following

\[
R_t = u + c t - \sum_{i=1}^{N_t} U_i
\]  

(4.1.1)

where:

- \(u\) denotes the initial capital of the company.
- \(ct\) is the premium flow up to time \(t\).
- \(U_i\) is a sequence of i.i.d. random variables and can be interpreted as the size of claim \(i\).
- \(N_t\) is a random variable and counts the number of claims in the interval \([0, t]\).
CHAPTER 4. RUIN PROBABILITY IN FINITE TIME

Here, the most popular version is the Cramér-Lundberg model where $N_t$ is a Poisson process with intensity $\lambda > 0$ and it is independent from the $U_i$'s.

The ultimate probability of ruin $\psi(u)$ is defined as

$$\mathbb{P}(\inf_{t \geq 0} R_t < 0 | R_0 = u).$$

The probability of ruin in finite time

$$\psi(u, T) := \mathbb{P}(\inf_{0 \leq t \leq T} R_t < 0 | R_0 = u)$$

for $T < \infty$. In our studies, for reasons of convenience we are interested in studying the survival probability

$$\varrho(u, T) = 1 - \psi(u, T).$$

In recent years the Cramér-Lundberg model has been extended by replacing the risk process $(R_t)_{t \geq 0}$ in (4.1.1) by a spectrally one-sided Lévy process, see e.g. [34], [42] or [31]. That means, a Lévy process with either only positive or negative jumps. In our case this process should possess a positive drift term (representing the income of the insurance company via its premiums), negative jumps (representing the claims the company has to pay) and possibly a Brownian part $\sigma$ which can be interpreted as the influence from external factors on the development of the capital. The advantages of this idea are clear. Firstly, the Cramér-Lundberg model can be obtained as special case of the Lévy model as the compound Poisson process is such a process. Secondly, now infinite many small jumps (claims) in every finite interval are allowed.

Before we present our approach, let us briefly review some literature concerning the study of ruin probabilities.

In the infinite case, e.g. $T = \infty$, the probability of ruin can be calculated via the well known 'Pollaczek-Khintchine formula' (see [42, 7]) in the setting of the Cramér-Lundberg model. In order to overcome the limitations of that model several attempts were made to replace the process $R_t$ by a Lévy process, see e.g. [34].

In the finite case, e.g. $T < \infty$, only a few examples of an explicit formula for the ruin probability are known, e.g. when the driving process $R_t$ in (4.1.1) is replaced by a Brownian motion or in the case when $U_i$ follows an exponential distribution (see [4]). Usually, other methods have to come into play. One possibility is to calculate the finite-time ruin probability recursively. Here one way is assume that the claim
sizes have an integer-valued distribution and that general probability distributions can approximated by such distributions (see e.g. [27, 28]). Another approach is the so called ‘Seal’s formula’ which goes back to [56] and was developed further by others, see [7]. The ruin probability can be expressed recursively as an integral also involving the density of the claim sizes.

Another way is to use (double) Laplace inversion to obtain the ruin probability (see [7]).

Other approaches include the approximations of the process \( R_t \) by e.g. diffusions, special assumptions on \( R_t \), e.g. to be heavy-tailed, or comparisons to the ultimate probability of ruin (see [7]).

Another idea is to randomize the horizon \( T \) by an Erlang or phase-type distributed random variable (see [5]) but here also several restrictions on the distribution of the claim sizes come into play.

Recent ideas for Lévy processes is to focus on the Wiener-Hopf factors (see Theorem 2.1.4 and Remark 2.1.5). But firstly, as mentioned before there are only few examples where these factors are known explicitly. Secondly even if a expression for them is given Fourier inversion has to be applied which involves double integration, first in \( \theta \) and then in \( p \). To the authors’ best knowledge there are only few more publications working on ruin probabilities in finite time driven by Lévy processes. One example is [31] who obtain some asymptotic results and estimates for the ruin probability for a specific class of Lévy processes called ‘Tempered stable processes’.

We use the same setup described in Section 2.1 and can write

\[
\varrho(x, T_k^{(n)}) = \mathbb{P}_x(\text{No ruin until } T_k^{(n)}) = \mathbb{E}_x \left[ 1\{X_{T_k^{(n)}} \geq 0\} \right] =: V_k^{(n)}(x).
\] (4.1.2)

The usefulness of this setup relies on the following two facts. For any expiry date \( T \geq 0 \), choosing a sequence \((k(n))_{n \geq 1}\) such that \( k(n)/n \to T \) as \( n \to \infty \) we have for any \( x \in \mathbb{R} \) that \( V_k^{(n)}(x) \to \varrho(x, T) \) as \( n \to \infty \). Furthermore, for any \( n \in \mathbb{N} \) the sequence of functions \((V_k^{(n)})_{k \geq 0}\) satisfies the following recursion:

\[
V_0^{(n)}(x) = 1\{x \geq 0\}, \\
V_k^{(n)}(x) = \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{T_k^{(n)}}) 1\{X_{T_k^{(n)}} \geq 0\} \right] \text{ for } k \geq 1.
\] (4.1.3)
where $\xi^{(n)}$ is an exponentially distributed random variable with mean $1/n$, independent of $X$. This is easily seen by a dynamic programming argument (see Theorem 4.2.1).

To work out the recursion (4.1.3) and therefore a formula for the ruin probability explicitly, we need expressions for the law of $X$ evaluated at an independent exponentially distributed random variable $\xi^{(n)}$ with mean $1/n$. This is the case for the class of meromorphic Lévy processes introduced in [39] and we refer to Section 2.2 for further details. This property is key in our algorithm, as becomes clear in Section 4.2.

As mentioned before we may assume that the meromorphic Lévy process has only negative jumps, e.g. $\Pi(0,\infty) = 0$. All considerations remain valid for general meromorphic Lévy processes though.

**Remark 4.1.1.** A natural question is whether it would not be better to use a classic deterministic grid rather than a stochastic grid. That is to say, setting $t_k = kT/n$ for $k = 0, 1, \ldots, n$. Denoting the resulting value function with expiry date $t_k$ by $U_k^{(n)}$ the sequence of functions $(U_k^{(n)})_{k=0,1,\ldots,n}$ is determined by the recursion

$$U_0^{(n)}(x) = 1_{\{x \geq 0\}}, \quad U_k^{(n)}(x) = \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_1/n) 1_{\{X_1/n \geq 0\}} \right] \quad \text{for } k \geq 1, \quad (4.1.4)$$

The problem is that this recursion involves the joint law of $(X_1/n, X_1)$. Even if this is known (which generally is not the case except for a few notable exceptions) it is typically not of a friendly enough nature to work out the recursion (4.1.4) explicitly. Indeed consider for instance a Brownian motion.

## 4.2 Main result & proof

Let us fix $T = 1$. Then we can choose $(k(n))_{n \geq 1} = n$ and as already stated in Section 2.3 it follows that $k(n)/n \to 1$ and $\varrho(x, T_n^{(n)})$ defined in (4.1.2) converges to $\mathbb{P}_x(\text{No ruin until } 1) = \varrho(x, 1)$.

First we want to prove that the survival probability defined in (4.1.2) fulfils the backward recursion (4.1.3).
Theorem 4.2.1. Let $X$ a Lévy process. For each $n \geq 1$ and $k \geq 0$ let the function $V_k^{(n)}$ be given by

$$V_k^{(n)}(x) := \mathbb{P}_x(\text{No ruin until } T_k^{(n)}) = \mathbb{E}_x \left[ 1 \{ X_{T_k^{(n)}} \geq 0 \} \right]. \quad (4.2.1)$$

We have the following. For any $n \geq 1$, the sequence of functions $(V_k^{(n)})_{k \geq 0}$ satisfies the following recursion:

$$V_0^{(n)}(x) = 1_{\{ x \geq 0 \}}, \quad V_k^{(n)}(x) = \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{\xi^{(n)}}) 1_{\{ X_{\xi^{(n)}} \geq 0 \}} \right] \quad \text{for } k \geq 1, \quad (4.2.2)$$

where $\xi^{(n)}$ is independent of $X$ and follows an exponential distribution with mean $1/n$. Moreover, if $(k(n))_{n \geq 1}$ is a sequence such that $k(n)/n \to 1$ as $n \to \infty$ then:

$$T_{k(n)}^{(n)} \to 1 \text{ a.s. and } V_{k(n)}^{(n)}(x) \to \varrho(x, 1) \text{ for all } x \in \mathbb{R}.$$  

Proof. This is just the dynamic programming principle applied to (4.2.1). Indeed for any $k \geq 1$ we see that

\begin{align*}
V_k^{(n)}(x) &= \mathbb{E}_x \left[ 1 \{ X_{T_k^{(n)}} \geq 0 \} \right] \\
&= \mathbb{E}_x \left[ \mathbb{E}_x \left[ 1 \{ X_{T_k^{(n)}} \geq 0 \} 1_{\{ \inf_{s \in (T_1^{(n)}, T_k^{(n)})} X_s \geq 0 \}} \right] \right] \\
&= \mathbb{E}_x \left[ 1 \{ X_{T_1^{(n)}} \geq 0 \} \mathbb{E}_x \left[ 1_{\{ \inf_{s \in (T_1^{(n)}, T_k^{(n)})} X_s \geq 0 \}} | T_1^{(n)} \right] \right] \\
&= \mathbb{E}_x \left[ 1 \{ X_{T_1^{(n)}} \geq 0 \} \mathbb{E}_x \left[ 1_{\{ \inf_{s \in (T_1^{(n)}, T_k^{(n)})} X_s \geq 0 \}} | T_1^{(n)} \right] \right] \\
&= \mathbb{E}_x \left[ 1 \{ X_{T_1^{(n)}} \geq 0 \} \mathbb{E}_{x_{T_1^{(n)}}} \left[ 1_{\{ \inf_{s \in (T_1^{(n)}, T_k^{(n)})} X_s \geq 0 \}} | T_1^{(n)} \right] \right] \\
&= \mathbb{E}_x \left[ 1 \{ X_{T_1^{(n)}} \geq 0 \} V_{k-1}^{(n)}(X_{T_1^{(n)}}) \right] \\
&= \mathbb{E}_x \left[ 1 \{ X_{\xi^{(n)}} \geq 0 \} V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right].
\end{align*}
where \((*)\) uses that if we condition on \(\mathcal{F}_{T_{1}^{(n)}}\), \(X_s - X_{T_{1}^{(n)}}\) has the same law as \(X_{T_{1}^{(n)}}\).

The convergence is a consequence from the considerations in Sections 2.2 and 2.3. Basically, as a Lévy process is right-continuous and does not jump at any fixed time, hence also not at time 1 the result follows as \(T_{k(n)}^{(n)} \longrightarrow 1\) a.s.

By using the recursion in Theorem 4.2.1 and the properties for meromorphic Lévy processes (2.2.6) and (2.2.7) we can derive a formula for the ruin respectively survival probability on a stochastic time grid.

**Proposition 4.2.2.** Let \(X\) a meromorphic Lévy process except a Compound Poisson process. Fix some \(n \geq 1\). For any \(k \geq 0\) the function \(V_{k}^{(n)}\) can be expressed as follows:

\[
V_{k}^{(n)}(x) = 1 - \sum_{j=0}^{N_{-}} e^{-\zeta^{(n)}(j)x} \sum_{i=0}^{k-1} U_{-}^{(n)}(i, j, k)x^{i},
\]

(4.2.3)

if \(x \geq 0\) and

\[
V_{k}^{(n)}(x) = 0
\]

if \(x < 0\). Expressions for the coefficients involved can be found in Appendix B.

**Proof.** We can prove the result by induction. The case \(k = 0\) is trivial. Let us assume that (4.2.3) is true for a fixed \(k\). Hence we have to consider

\[
\mathbb{E}_{x}\left[V_{k}^{(n)}(X_{\xi^{(n)}})1_{\{X_{\xi^{(n)}} \geq 0\}}\right].
\]

We simplify this as follows.

\[
\mathbb{E}_{x}\left[V_{k}^{(n)}(X_{\xi^{(n)}})1_{\{X_{\xi^{(n)}} \geq 0\}}\right] = \mathbb{E}_{x}\left[\left(1 - \sum_{j=0}^{N_{-}} e^{-\zeta^{(n)}(j)x} \sum_{i=0}^{k-1} U_{-}^{(n)}(i, j, k)x^{i} \right)1_{\{X_{\xi^{(n)}} \geq 0\}}\right] \tag{*}
\]

\[
= \mathbb{P}(x + X_{\xi^{(n)}} \geq 0)
\]

\[
- \mathbb{E}\left[\sum_{j=0}^{N_{-}} e^{-\zeta^{(n)}(j)(x + X_{\xi^{(n)}}) + X_{\xi^{(n)}}} \sum_{i=0}^{k-1} U_{-}^{(n)}(i, j, k)(x + X_{\xi^{(n)})}^{i}1_{\{x + X_{\xi^{(n)} \geq 0\}}\right]
\]

\[
= \mathbb{P}(x + X_{\xi^{(n)}} \geq 0)
\]

\[
- \mathbb{E}\left[\sum_{j=0}^{N_{-}} e^{-\zeta^{(n)}(j)(x + X_{\xi^{(n)}}) + X_{\xi^{(n)}}} \sum_{i=0}^{k-1} U_{-}^{(n)}(i, j, k) \sum_{i_{1}+i_{2}+i_{3}=i} X_{\xi^{(n)}}^{i_{1}} X_{\xi^{(n)}}^{i_{2}} X_{\xi^{(n)}}^{i_{3}}1_{\{x + X_{\xi^{(n)} \geq 0\}}\right]
\]

\[
= \mathbb{P}(x + X_{\xi^{(n)}} \geq 0)
\]
\[
\sum_{j=0}^{N_-} e^{-\zeta^{(n)}(j)x} \sum_{i=0}^{k-1} U^{(n)}_-(i, j, k) \sum_{i_1 + i_2 + i_3 = i} \binom{i}{i_1, i_2, i_3} x^{i_1} \cdot \mathbb{E} \left[ e^{-\zeta^{(n)}(j)X_{\xi^{(n)}}} X^{i_2}_{\xi^{(n)}} \right] \mathbb{E} \left[ e^{-\zeta^{(n)}(j)X_{\xi^{(n)}}} X^{i_3}_{\xi^{(n)}} 1_{\{x + X_{\xi^{(n)}} \geq 0\}} \right].
\]

(*) uses relation (2.1.6) and (**) the independence of \(X_{\xi^{(n)}}\) and \(X_{\xi^{(n)}}\). In the latter calculation \(\binom{i}{i_1, i_2, i_3} := \frac{i!}{i_1! i_2! i_3!}\) denotes the multinomial coefficient.

From this point, it is a matter of straightforward but lengthy calculations by using the probability densities for the position (2.2.7) and infimum (2.2.6) of a meromorphic Lévy process to verify the statement for \(k + 1\). Hereby, we use also Proposition D.0.1.

If we consider a Lévy process with bounded variation and finite activity we derive the following result:

**Proposition 4.2.3.** Let \(X\) a meromorphic Lévy process with bounded variation and finite activity. Fix some \(n \geq 1\). For any \(k \geq 0\) the function \(V^{(n,b)}_k\) can be expressed as follows:

\[
V^{(n,b)}_k(x) = 1 - \sum_{j=0}^{N_-} e^{-\zeta^{(n)}(j)x} \sum_{i=0}^{k-1} U^{(n)}_-(i, j, k) x^i,
\]

(4.2.4) if \(x \geq 0\) and

\[
V^{(n,b)}_k(x) = 0
\]

if \(x < 0\). Expressions for the coefficients involved can be found in Appendix B.

**Proof.** The idea of the proof follows the one of Proposition 4.2.2. The difference in the computations is the appearance of the atom in 0, i.e. the probabilities \(\mathbb{P}(X_{\xi^{(n)}} = 0)\) and \(\mathbb{P}(X_{\xi^{(n)}} = 0)\) are positive. This leads to the coefficients \(U^{(n)}_-(i, j, k)\) which are very similar to \(U^{(n)}_-(i, j, k)\).

**Remark 4.2.4.** If we consider a Lévy process with bounded variation and finite activity it is a fact that it is irregular downwards, see e.g. [42], Chapter 6. It means that

\[
\mathbb{P}(\tau_0^- > 0) = 1,
\]

where

\[
\tau_0^- := \inf \{t > 0 | X_t < 0\}.
\]
In this case it is obvious that the ruin probability \( V_k^{(n,b)} \) in Proposition 4.2.3 is discontinuous in \( x = 0 \).

Although we cannot deduce immediately from formula (4.2.4) we support this observation with some numerics. If we consider a compound Poisson process (4.1.1) with a drift \( \mu = 1 \), claim sizes \( U_i \) which are exponentially distributed with parameter 1 and a Poisson process counting the number of jumps with intensity 1 we are able to compare our results with the exact values due to the existence of an explicit formula, see [7]. The exact value in \( x = 0 \) is 0.47622 and our result is 0.47582 with \( k = n = 300 \) in formula (4.2.4). Obviously, both functions are as expected discontinuous as for a negative value for \( x \) near 0 we have a ruin probability of 1 in both cases.

Moreover, we see that the error for our method is quite small.

**Remark 4.2.5.** A simplification could be to assume that ruin can only occur at the end of each exponential period. Then we define for each \( n \geq 1 \) and \( k \geq 0 \) the function \( \bar{V}_k^{(n)} \) by

\[
\bar{V}_k^{(n)}(x) = \mathbb{P}_x(\text{No ruin until } T_k^{(n)}) = \mathbb{E}_x \left[ 1 \left\{ \min\{x, X_{\tau_1(n)}, X_{\tau_2(n)}, \ldots, X_{\tau_k(n)} \} \geq 0 \right\} \right]. \tag{4.2.5}
\]

In this case, we can work out a result similar to the one in Theorem 4.2.1. First, we can derive the following recursion

\[
\bar{V}_0^{(n)}(x) = 1_{\{x \geq 0\}}, \quad \bar{V}_k^{(n)}(x) = \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{\xi(n)}) \right] \quad \text{for } k \geq 1. \tag{4.2.6}
\]

Then we can also find a formula equal in appearance to (4.2.3), only with different coefficients \( U_k^{(n)}(i,j,k) \).

Although it is clear that this approach is less accurate (as the formula (4.2.3) takes also the infimum of the process into account) we support with Figure 4.4 that an approximation with formula (4.2.5) yields a relatively small error for a sufficiently high number of exponential periods.

**Lemma 4.2.6.** In the case that the process is of unbounded variation and \( N_+ = N_- = 0 \), i.e. \( X \) is a Brownian motion or \( \beta \)-class process (see Section 2.2) with only one positive and negative zero, formula (4.2.3) can be simplified as follows for \( x \geq 0 \):

\[
V_k^{(n)}(x) = 1 - \exp(-\zeta_-^{(n)}(0)x) 1_{\{k \geq 1\}} - \exp(-\zeta_-^{(n)}(0)x) \sum_{i=1}^{k-1} x^i \sum_{g=1}^{k-1} \frac{a_{i+1+g} T(i+g-2,g-1)}{b^{i+2+2g} i!}, \tag{4.2.7}
\]
where \( a := \zeta_+^{(n)}(0)\zeta_-^{(n)}(0) \), \( b := \zeta_+^{(n)}(0) + \zeta_-^{(n)}(0) \) and

\[
T(w, u) = \frac{(w + u)!(w - u + 1)}{u!(w + 1)!}
\]

(4.2.8)

with \( T(w, u) = 0 \) for \( u > w \) and for negative values of \( w \) and \( u \).

**Proof.** First we state the following relation. Hereby \( g_S \) and \( g_I \) denote the probability densities (see also (2.2.5) and (2.2.6)) of \( S \) respectively \( I \) which are copies of \( X_{\xi^{(n)}} \) respectively \( X_{\xi^{(n)}} \).

For \( m \geq 1 \),

\[
\int_{-t}^{\infty} \int_0^\infty (t + r + w)^m \exp(-\zeta_-^{(n)}(0)(r + w))g_S(dr)g_I(dw) = \sum_{h=1}^{m+1} t^h \frac{m!}{h!} \frac{a}{b^m - 2h} =: p(m).
\]

(4.2.9)

This can be proved by using twice the binomial theorem and \( m \)-times integration by parts. We now have to verify:

\[
\mathbb{E} \left[ 1 - \exp(-\zeta_-^{(n)}(0)(t + S + I)) \right] =: A
\]

\[
+ \mathbb{E} \left[ \exp(-\zeta_-^{(n)}(0)(t + S + I)) \sum_{i=1}^{k-1} (t + S + I)^i \sum_{g=1}^{k-i} \frac{a^{i-1+g}}{b^{g-2+2g}} T(i + g - 2, g - 1) \right] =: B
\]

\[
= 1 - \exp(-\zeta_-^{(n)}(0)x)
\]

\[
- \exp(-\zeta_-^{(n)}(0)x) \sum_{i=1}^k x^i \sum_{g=1}^{k-i+1} \frac{a^{i-1+g}}{b^{g-2+2g}} T(i + g - 2, g - 1).
\]

(4.2.11)

Define

\[
D(m) = \sum_{g=1}^{k-m} \frac{a^{m+g}}{b^{m-2+2g}} T(m + g - 2, g - 1).
\]

Then we have for

\[
A = 1 - \exp(-\zeta_-^{(n)}(0)t) - \frac{a}{b} \exp(-\zeta_-^{(n)}(0)t)t
\]
By comparing (4.2.9):

\[
B = \exp(-\zeta_n^2(0)) \sum_{i=1}^{k-1} \sum_{g=1}^{k-i} \frac{a^{i-1+g} T(i + g - 2, g - 1)}{i!} p(i)
\]

\[
= \exp(-\zeta_n^2(0)) \sum_{i=1}^{k-1} \sum_{h=1}^{k-2} \frac{a^{h+i} b^{i+2-h} h!}{b^h} D(i)
\]

\[
= \exp(-\zeta_n^2(0)) \sum_{i=1}^{k-1} \sum_{h=1}^{k-2} \frac{a^{h+i} b^{i+2-h} h!}{b^h} D(i + h - 2)
\]

\[
= \exp(-\zeta_n^2(0)) \sum_{i=1}^{k-1} \sum_{h=1}^{k-2} \sum_{r=1}^{i-2+1} \frac{a^{h+i} b^{i+2-h} h!}{b^h} \frac{T(h + i - 2)}{b^{h+i} b^{i+2-h} h!} D(i + h - 2)
\]

\[
= \exp(-\zeta_n^2(0)) \sum_{i=1}^{k-1} \sum_{h=1}^{k-2} \frac{a^{h+i} b^{i+2-h} h!}{b^h} T(h + i - 2, i - 1).
\]

By comparing (4.2.11) and (4.2.12) we have to show for \( h = 1 \ldots k \):

\[
\sum_{i=1}^{k-1} a^i b^{2i} T(h + i - 2, i - 1) = \sum_{i=1}^{k-1} \sum_{r=1}^{i-2+1} \frac{a^{i+r-1}}{b^{2i+2r-2}} T(h + i + r - 4, r - 1).
\]

However we can write the right-hand side of this equation by setting \( z := i + r \) as

\[
\sum_{i=1}^{k-1} \sum_{r=1}^{i-2+1} \frac{a^{i+r-1}}{b^{2i+2r-2}} T(h + i + r - 4, r - 1)
\]

\[
= \sum_{z=1}^{k-1} \sum_{r=0}^{z-1} \frac{a^z}{b^{2z}} T(h + z - 3, r).
\]

The sequence \( T(w, v)_{w \geq 0, v = 0, \ldots, w} \) contains the entries of the 'Catalan triangle', see e.g. [62]. In particular, it has the property that the \( v \)-th entry in the \( w \)-th row is the sum of all entries up to index \( v \) in the \( w - 1 \)-th row. This means:

\[
T(w, v) = \sum_{g=0}^{v} T(w - 1, g),
\]

and (4.2.13) follows. \( \square \)

As in Chapter 3 we provide an error estimate. With both the help of Proposition 3.3.1 and our approach to approximate the law of the process by a Lipschitz continuous function, see Section 3.3, we are able to obtain results for the infimum as well. Hereby we have to assume that \( X \) fulfils the following condition:

\[
P(a_n \leq X_1 \leq U | a_n) < U | a_n|
\]

(4.2.14)

for a fixed \( U > 0 \).
Proposition 4.2.7. Let $X$ be a Lévy process of unbounded variation which fulfils (3.3.6) and (4.2.14). Then for $x, y \geq 0$

$$|\mathbb{P}(X_{g(n,n)} \geq 0) - \mathbb{P}(X_1 \geq 0)| \lesssim K(-n^{-\frac{1}{8}}) + (U + 1)n^{-\frac{1}{8}}. \quad (4.2.15)$$

where $K(-n^{-\frac{1}{8}})$ is defined as follows

$$\mathbb{P}(-n^{-\frac{1}{8}} \leq X_{g(n,n)} \leq 0) = 1 - \sum_{j=0}^{N} e^{-c_{-}^{(n)}(j)n^{-\frac{1}{8}}} \sum_{i=0}^{n-1} U_{i}^{(n)}(i, j, k)n^{-\frac{1}{8}}.$$

Proof. Equation (4.2.15) can be derived in the same manner as (3.3.7). First, by using a Lipschitz-continuous function similar to $G$ we derive a result like in Proposition 3.3.2. Then we can estimate $\mathbb{E}[\epsilon(X_{g(n,n)}, \alpha_n)]$ and $\mathbb{E}[\epsilon(X_1, \alpha_n)]$ similar to Proposition 3.3.3 by using (4.2.3) and assumption (4.2.14). Finally, we are able to calculate a result like in Theorem 3.3.4 and Corollary 3.3.5 which gives us (4.2.15).

\[\square\]

4.3 Reinsurance

A natural idea might be the question if it is possible to minimise the ruin probability. In the real world one way to achieve this is the usage of reinsurance. This means roughly speaking the following: As already explained in the Introduction, the insurer earns money by receiving premiums from its customers. He can now decide to insure his risk or claims by buying reinsurance from a reinsurer. That means it gives away some of its premium income to the reinsurer but therefore he will not have to pay the whole amount of its claims. The question now what is the optimal reinsurance choice. That means how much of his premium should the insurer take in order to pay the reinsurer for protection?

There exist plenty of contract forms, we will concentrate on 'proportional reinsurance' and explain it below. We omit a detailed introduction to reinsurance and refer the reader to the books of Asmussen [7] and Schmidli [61].

Unfortunately, this section does not provide a positive result. On the contrary, it explains why introducing dynamic reinsurance in our framework is a difficult task and why we could not find a working solution.

We begin with some notation and short explanations.
4.3.1 Setup

Let us consider the capital process of the insurer. This means we look for time \( t > 0 \) and some initial capital \( x \) at

\[
X_{i}^{\text{in}} = x + c_{\text{in}}(t) + Y_{i}^{\text{in}}.
\]

Hereby, \( c_{\text{in}}(t) \) denotes the premium of the insurer obtained by its customers and \( Y_{i}^{\text{in}} \) accumulates the claims, both up to time \( t \).

Let \( \tilde{\mathcal{F}}^{(n)} \) denote the naturally enlarged filtration generated by the pair \((X, N^{(n)})\) (cf. Definition 1.3.38 in [14]). At each random time point \( T_{k}^{(n)} \) (see Section 2.3) the insurer can choose a reinsurance treaty \( b_{k} \in (0, 1) \). This means that if the insurer has chosen \( b_{k} \) he will have to pay \( h(Z_{k}, b) \) in the \( k \)-th exponential time period.

We assume a proportional reinsurance treaty:

\[
h(x, b) = x \cdot b.
\]

where \( h : (0, \infty) \times \mathcal{B} \to (0, \infty) \). Hereby, \( \mathcal{B} = (0, 1)^{n} \) should contain functions \( b \) with the following assumptions in accordance with [61]:

- The insurer has to pay a positive part of the claims, but at most the full percentage. If \( b \) is chosen to be 0 it has the meaning of a full reinsurance. If \( b = 1 \) the insurer is not taking any reinsurance. Hence

\[
0 \leq h(x, b) \leq x.
\]

Let \((Z_{i}^{(n)})_{i \geq 1}\) be the sequence of i.i.d. random variables describing the claims in each exponential time period \( \xi^{(n)} \). Let \( \mu := \mathbb{E}\left[Z_{i}^{(n)}\right] \) denote its common expectation. As before, let us fix as before on \( T = 1 \), i.e. we consider the capital process until \( T = 1 \).

We approximate 1 again by the sum of \( n \) exponentially distributed i.i.d. random variables \((\xi_{i}^{(n)})_{i \geq 1}\) with mean \( 1/n \). This yields that the total claims up to time

\[
g(n, n) := \sum_{i=1}^{n} \xi_{i}^{(n)}
\]

can be written in distribution as

\[
Y_{g(n, n)}^{\text{in}} \overset{d}{=} \sum_{i=1}^{n} Z_{i}^{(n)}.
\]
$c_{in}(\cdot)$ consists each of two parts, the net premium and a safety loading. The former is needed to fulfill the expected claims, i.e. the net premium equals $\mu$. Then the safety loading is added for other costs and in order to make some gains. We denote by $\theta$ and $\Omega$ the safety loadings of insurer and reinsurer. There are different ways to include the safety loadings into the premium calculation, we are using the simplest one, the \textit{Expected premium principle}. For $t > 0$ we have:

$$c_{in}(t) = (1 + \theta)\mu t.$$  

The reinsurer has to pay a part of the claims, but at most the full percentage. As mentioned before, we are using the expected premium principle. Then the reinsurer sets his premium as follows for an exponential time period $\xi(n)$.

$$c_{re}(\xi(n)) = (1 + \Omega)\xi(n) E[h(Z^{(n)}, b)]$$

$$= (1 + \Omega)\mu \xi(n) - (1 + \Omega)E[h(Z^{(n)}, b)] \xi(n)$$

$$= (1 + \Omega)\mu \xi(n) - (1 + \Omega)E[bZ^{(n)}] \xi(n)$$

$$= (1 + \Omega)\mu \xi(n) - (1 + \Omega)b\mu \xi(n). \quad (4.3.1)$$

Now we have to modify the insurer’s premium in order to take account of reinsurance. Hence it follows that the ’new’ premium $c_{in}$ now consists of the original premium minus the premium the insurer has to pay for the reinsurer which we stated in (4.3.1). This means for each exponential time period $\xi(n)$.

$$c_{in}(\xi(n)) = c_{in}(\xi(n)) - c_{re}(\xi(n))$$

$$= (1 + \theta)\mu \xi(n) - ((1 + \Omega)\mu \xi(n) - (1 + \Omega)b\mu \xi(n))$$

$$= (1 + \Omega)b\mu \xi(n) - (\Omega - \theta)\mu \xi(n). \quad (4.3.2)$$

This modified premium should be the ’drift’ part of the modified capital process. A natural assumption to prevent almost sure ruin is to assume the \textit{Net premium principle}: $c_{in}(\xi(n))$ has to be bigger than $E[h(Z^{(n)}, B)]$. In our case this is equivalent to assume that $b > 1 - \frac{\theta}{\mu}$. Therefore we investigate the following model in each exponential time period $\xi(n)$.

$$X_{\xi(n)}^{in,b} = x + c_{in}\xi(n) + h(Z^{(n)}, b)$$

$$= x + \xi(n)((1 + \Omega)b\mu - (\Omega - \theta)\mu) + bZ^{(n)}. \quad (4.3.3)$$
Note that if we choose \( b = 1 \), then we come back to the original setup \( X_{\xi(n)} \).

### 4.3.2 Difficulties

If we want to allow for reinsurance in formula (4.2.3) we have to modify the coefficients \( U_i^{(n)}(j,k) \) for every \( i < k, 0 < k < n \) and \( j = 0, \ldots, N_- \). Let us denote the coefficients with reinsurance by \( U_i^{(n)}(j,k,b) \). Note that \( U_i^{(n)}(j,k) \) are dependent on the solutions \( \zeta_-(n)(j) \) and \( \zeta_+(n)(j) \) to the equation \( \psi_X^{(n)}(z) - n = 0 \) where \( \psi \) is the Laplace exponent of \( X^{in} \) and hence also dependent on \( c_-(n)(j) \) and \( c_+(n)(j) \) (see also Section 2.2).

Therefore \( U_i^{(n)}(j,k,b) \) is dependent on \( \zeta_-(n)(j,b) \) and \( \zeta_+(n)(j,b) \) which are solutions to the equation \( \psi_X^{in,b}(z) - n = 0 \) where \( \psi_X^{in,b} \) is the Laplace exponent of \( X^{in,b} \) and \( c_-(n)(j,b) \) and \( c_+(n)(j,b) \).

The first difficulty lies in the calculation of \( \zeta_-(n)(j,b) \) and \( \zeta_+(n)(j,b) \). Usually no analytic expressions for the solutions are known and one has to settle for numerical methods.

Let us consider (4.3.3) and assume that \( X^{in,b}_{\xi(n)} \) is a process of unbounded variation. Let \( \psi^Z \) be the Laplace exponent of \( Z^{(n)} \). Note that \( \psi^Z \) equals \( \psi^{X^{in,1}} \) except the drift part.

We start with the following property of characteristic functions:

\[
\psi^{bZ}(z) = \psi^Z(bz) \quad \text{(4.3.4)}
\]

This means: Let \( s_{+,-Z,n}^{(n)}(j) \) or \( s_{+,-Z,n}^{(n)}(j) \) be the solutions to the equation \( \psi^Z(z) - n = 0 \).

Then the solutions \( s_{+,-Z,n}^{bZ,n}(j) \) to the equation \( \psi^{bZ}(z) - n = 0 \) can be expressed as

\[
s_{+,-Z,n}^{bZ,n}(j) = \frac{s_{+,-Z,n}^{(n)}(j)}{b}.
\]

We need to calculate the Laplace exponent of \( X^{in,b} \) which is the one of \( Z \) plus drift.

Hence

\[
\psi_{bX}(z) = \psi_X(bz) = (1 + \theta)\mu bz \quad \text{This is the wrong drift'}.
\]

Therefore, we have to adjust:

\[
\psi_{bX}(z) = \psi_X(bz) - (1 + \theta)\mu bz + (1 + \Omega)\mu bz - ((\Omega - \theta)\mu)z = \psi_X(bz) - (1 - b)\mu(\Omega - \theta)z. \quad \text{(4.3.5)}
\]

Therefore, in order to obtain solutions for \( \zeta_-(n)(j,b) \) and \( \zeta_+(n)(j,b) \) we have to solve

\[
\psi_X(bz) - (1 - b)\mu(\Omega - \theta)z - n = 0.
\]
As mentioned before, solving this equation is not easy and there only exist some explicit expressions (Brownian motion or Compound Poisson process with exponential claims, cf. [40]). The idea then is to introduce a grid with grid size $\gamma$ from $0, \gamma, 2\gamma, \ldots, 1$ such that the reinsurance $b$ takes a value on a grid point.

Note that if the process is of bounded variation we have to consider (2.1.3) and adjust the drift part with parts of the Lévy measure as well.

Assume now that we have calculated $\zeta_-(n)(j,b)$ and $\zeta_+(n)(j,b)$ for different $b$ and we also know the coefficients $c_-(n)(j,b)$ and $c_+(n)(j,b)$.

We want to apply a dynamic reinsurance, e.g. choosing at the beginning of each exponential period $\xi_k$ a reinsurance level $b_k$. This means that by looking at the coefficients $U_{i,j,k,b}^-(n)$ in formula (4.2.3) we have to find a $b_k$ such that we maximise our survival probability for every $x > 0$. However this is a very complicated task as two sums (the first one over the number of roots $\zeta_-(n)(j,b)$ and $\zeta_+(n)(j,b)$ and the second one over the number of exponential time periods away from $T_0^{(n)}$.) and there is no analytic way how to find an optimal solution.

Again one possibility would be to find the optimal reinsurance by implementing it on a computer. This requires a very involved numerical scheme.

Also in the case that there is only one positive and negative root $\zeta_+(n)(0,b)$ and $\zeta_-(n)(0,b)$, we can not seem to find a useful expression for the optimal reinsurance level $b_k$.

If we used instead a static reinsurance treaty, which means that we choose a reinsurance level $b$ at time 0 staying the same over all exponential periods $\xi(n)$, we could investigate formula (4.2.7). However, we would still have to deal with a sum and therefore it is only possible to find a numerical optimal solution to $b$ and not an analytical one.

### 4.4 Some numerics

In this section we discuss some numerics for the ruin probability in finite time, obtained by a computer implementation of the result in Proposition 4.2.2. In particular we present some graphs of approximating functions $V_k^{(n)}$.

Next, we briefly describe the two main tasks for creating the graphs of different value functions presented further below. The first step is the calculation of the solutions $\zeta_+(n)(i)$ and $\zeta_-(n)(j)$ of $z \mapsto \psi(z) - n$ and of the coefficients $c_-(n)(j)$ by using Proposition
2.2.10 for \(i \in \{0, \ldots, N_+\}\) and \(j \in \{0, \ldots, N_-\}\). In case a meromorphic Lévy process is used for which \(N_+ = \infty\) or \(N_- = \infty\) the infinite sums need to be truncated. For this we used the ad-hoc approach of ensuring that the truncated sums represent at least 99\% of the mass of \(X_{\xi(n)}\), i.e. (compare with (2.2.7)) we determined \(M^+, M^-\) such that

\[
\sum_{j=0}^{M^+} \frac{c_+(n)(j)}{\zeta_+(j)} - \sum_{j=0}^{M^-} \frac{c_-(n)(j)}{\zeta_-(j)} \geq 0.99
\]

and then normalised the coefficients. It is worth noting that \(M\) is larger in the case of meromorphic Lévy process with finite variation than in the cases with infinite variation. Particularly, if \(\sigma > 0\) the largest part of the probability mass is concentrated on \(\zeta_+(0)\) and \(\zeta_-(0)\) whereas the other zeroes contribute only a small part.

The second main step is the recursive computation of the ruin probability using formula (4.2.3) and (4.2.4). A program for this purpose was written in the language C++. We found that in certain cases the algorithm may need more than the default ‘double precision’ as working precision in order to have enough significant digits left in the end result. For this we made use of the ‘GMP/MPFR/MPFRC++’ packages. Furthermore we made use of ‘OpenMP’ to parallelise the operations. The Lévy processes used in this section are Brownian motion with drift, a compound Poisson process with exponential distributed random variables \(U_i\) and the \(\beta\)-class, see Section 2.2.

Figures 4.1 and 4.2 show the quality of the approximating functions from (4.2.1) in comparison to exact values. Moreover, our method is working fine not only for quite simple stochastic processes but also for more advanced like from the \(\beta\)-class and this is illustrated in Figure 4.3. Finally, we end with the observation that the error might not increase significantly if we do not use the infimum in our recursion Theorem 4.2.1, see also Remark 4.2.5. With other words, we consider in Figure 4.4 the functions \(V_k(n)(x)\) from (4.2.1) respectively \(\tilde{V}_k(n)(x)\) from (4.2.5).
Figure 4.1: Plots of the relative difference between three approximating value functions from (4.2.1), where \( n = k \) for \( n = 75, 150 \) and \( n = 300 \), and the exact ruin probability. The plots are ‘zoomed in’ to the part of the \( x \)-axis where the difference is largest. The underlying compound Poisson process (4.1.1) with finite variation and finite activity has the following parameters: \( \sigma = 0 \), drift \( \mu = 1 \), the claim sizes \( U_i \) are exponentially distributed with parameter 1 and the Poisson process counting the number of jumps has intensity 1. Furthermore \( M^+ = M^- = 0 \).
Figure 4.2: Plots of the relative difference between three approximating value functions from (4.2.1), where \( n = k \) for \( n = 32 \), \( n = 128 \) and \( n = 512 \), and the exact ruin probability. The plots are ‘zoomed in’ to the part of the \( x \)-axis where the difference is largest. The underlying process is a standard Brownian motion. Therefore \( M^+ = M^- = 0 \).
Figure 4.3: Plots of the relative difference between the approximating function $V_k^{(n)}(x)$ from (4.2.1) for $k = n = 50, 200, 500$, and the approximating function $V_{800}^{(800)}(x)$ (taken as 'true value') from formula (4.2.1) where the driving process $X$ is from the $\beta$-class with parameter values $\sigma = 1, \mu = 0.6, \alpha_1 = \alpha_2 = 2, \beta_1 = \beta_2 = 80, c_1 = 0, c_2 = 5$ and $\lambda_1 = \lambda_2 = 2.5$. Furthermore we choose $M^+ = 1, M^- = 3$. 
Figure 4.4: Plots of the relative difference between the approximating functions $V_{50}^{(50)}(x)$ from (4.2.1) respectively $\bar{V}_{50}^{(50)}(x)$ from (4.2.5) and $V_{200}^{(200)}(x)$ from (4.2.1) respectively $\bar{V}_{200}^{(200)}(x)$ from (4.2.5) where the driving process $X$ is from the $\beta$-class with parameter values $\sigma = 1, \alpha_1 = \alpha_2 = 20, \beta_1 = \beta_2 = 15, c_1 = 5, c_2 = 8$ and $\lambda_1 = \lambda_2 = 2.4$. The drift $\mu$ is chosen as the solution of $\Psi(-I) = -r$. Furthermore we choose $M^+ = 3, M^- = 9$. 
Chapter 5

A tree method for the law of Lévy processes

5.1 Introduction

Let $X$ be a Lévy process which we have described in Section 2.1. In Chapter 3 we derived formulas for the density of the position of a meromorphic Lévy process after $n$ exponential periods. These results constitute an advantage in comparison to the original density of meromorphic Lévy process (see Section 2.2). We saw by both Theorem 3.3.4 and numerical evidence in Section 3.3 that the densities in Theorem 3.2.3 and Lemma 3.3.7 are more accurate than the one in original case (Theorem 2.2.8). Moreover, they are useful and valuable approximations to Lévy processes whose law is in general very complicated to calculate at a deterministic time.

However, computing the coefficients given in Appendix A can take a while. If we consider for example a $\beta$-class process and choose both the number of solutions $N_+, N_-$ and the number of exponential periods $n$ to be quite large then the time to run the computer program is quite long.

Therefore, in this chapter, we still work on a stochastic time grid (see Section 2.3) but we discretise the law of our meromorphic Lévy process. With other words, we consider the density of the process $X$ at time $\xi^{(n)}$ where again $\xi^{(n)}$ is an exponentially distributed random variable with mean $\frac{1}{n}$. Therefore we approximate $X_{\xi^{(n)}}$ by a discrete random variable $Y^{(n)}$ taking values on a space grid $\{-N\delta, -N\delta + \delta, -N\delta + 2\delta, \ldots, 0, \delta, 2\delta, \ldots, N\delta\}$. Then $X_{g(n,u)} \approx \tilde{X}_n^{(n)}$ where $(\tilde{X}_k^{(n)})_{k=0,1,...}$ is the process defined as $\tilde{X}_0^{(n)} = 0$ and $\tilde{X}_k^{(n)} = ...$
\( \tilde{X}_{k-1}^{(n)} + Y_k^{(n)} \). Hereby \( Y_k^{(n)} \) is an independent copy of \( Y^{(n)} \). Note that \( (\tilde{X}_k^{(n)})_{k=0,1,...} \) is nothing else than a random walk.

By basically using convolution and hereby using the fact that

\[
\sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{n} x_i^{(n)}
\]

we derive an explicit formula for the law of the position of the 'discretised meromorphic Lévy process' \( \tilde{X}_n^{(n)} \) after \( n \) exponential time periods. We observe numerically that this is a good approximation to \( X_{g(n,n)} \) and that the calculation costs can be much less in comparison than for the method in Chapter 3. Moreover, we analyse the error we make by discretising \( X \). We finish with recursive results for the probability mass functions of the supremum \( \tilde{S}_n^{(n)} \) and joint case \( (\tilde{X}_n^{(n)}, \tilde{S}_n^{(n)}) \) after \( n \) exponential time periods. Hereby, \( \tilde{S}_n^{(n)} \) denotes the discretised version of \( X_{g(n,n)} \) which is the supremum of \( X_{g(n,n)} \) after \( n \) exponential time periods.

The straightforward idea is now to approximate \( X_{\xi(n)} \) by a discrete random variable \( \tilde{S}^{(n)} \) taking values on some grid \( \{0, \delta, 2\delta, \ldots, N\delta\} \) and similarly approximate \( X_{\xi(n)} \) by a discrete random variable \( \tilde{I}^{(n)} \) taking values on \( \{-N\delta, -N\delta + \delta, \ldots, 0\} \). The value of \( N \) could be different if appropriate. As an example in the case of a spectrally negative Lévy process with a jump activity we would need the grid range for \( \tilde{I}^{(n)} \) to be larger than the one for \( \tilde{S}^{(n)} \).

Then we should have for large enough \( n, N \) and small enough \( \delta \):

\[
X_1 \approx X_{g(n,n)} \approx \sum_{j=1}^{n} (\tilde{S}^{(n)}(j) + \tilde{I}^{(n)}(j)) = \tilde{X}^{(n)}_n,
\]

where \( (\tilde{S}^{(n)}(j))_{j\geq 1} \) is a sequence of i.i.d random variables with common law equal to the law of \( \tilde{S}^{(n)} \) and \( (\tilde{I}^{(n)}(j))_{j\geq 1} \) is a sequence of i.i.d. random variables with common law equal to the law of \( \tilde{I}^{(n)} \).

**Theorem 5.1.1** (Position). Let \( \delta \in \mathbb{Q}, \delta > 0 \) determine the step size. Let \( (Z^{(i)})_{i\geq 1} \) be an i.i.d. sequence of random variables taking values in \( \{-\delta N, -\delta N + \delta, \ldots, \delta N - \delta, \delta N\} \) with common probability mass function \( p_Z \). Set \( Y_k := \sum_{i=1}^{k} Z^{(i)} \) with probability mass function \( p_{Y_k} \). Then \( Y_k \) takes values on \( \{-k\delta N, -k\delta N + \delta, \ldots, k\delta N - \delta, k\delta N\} \),

\[
p_{Y_k}(-k\delta N) = p_Z(-\delta N)^k
\]

(5.1.1)
and
\[ p_{Y_k}(\delta(l - kN)) = \frac{k a_{l-1}}{l} \quad \text{for all } l = 1, \ldots, 2kN \] (5.1.2)

where \( a_{-1} = 0 \) and
\[ a_l = \frac{1}{p_Z(-\delta N)} \sum_{j=-1}^{l-1} p_Z(\delta(l-j-N)) ((l-j)p_{Y_k}(\delta(j+1 - kN)) - a_j) \quad \text{for all } l = 1, \ldots, 2kN. \] (5.1.3)

**Proof.** For simplicity we set \( \delta = 1 \). Denote \( \hat{Z}^{(i)} := Z^{(i)} + N \) and \( \hat{Y}_k := Y_k + kN \) so that these new defined random variables take only values on \( \mathbb{Z}_+ \cup \{0\} \). Let us denote by \( G_Z \) and \( G_{\hat{Y}_k} \) the probability generating functions of \( \hat{Z}^{(i)} \) and \( \hat{Y}_k \). Denote by \( p_Z \) and \( p_{\hat{Y}_k} \) the probability mass functions of \( \hat{Z}^{(i)} \) and \( \hat{Y}_k \). Then
\[ G_{\hat{Y}_k}(r) = (G_Z(r))^k. \]

Taking the logarithm and then differentiating both sides yields
\[ G'_{\hat{Y}_k}(r) = k G_{\hat{Y}_k}(r) \frac{G'_Z(r)}{G_Z(r)}. \] (5.1.4)

To obtain an expansion of the right hand side denote \( H(r) := G_{\hat{Y}_k}(r) \frac{G'_Z(r)}{G_Z(r)} \) so that
\[ G_Z(r)H(r) = G_{\hat{Y}_k}(r) G'_Z(r). \]

With the general definition of probability generating functions and with \( H(r) = \sum_{j=0}^{2kN-1} a_j r^j \) this becomes
\[ \left( \sum_{j=0}^{2N} p_Z(j) r^j \right) \left( \sum_{j=0}^{2kN-1} a_j r^j \right) = \left( \sum_{j=0}^{2kN} p_{\hat{Y}_k}(j) r^j \right) \left( \sum_{j=0}^{2N-1} (j+1)p_Z(j+1) r^j \right) \]
and hence, after working out the products and equating powers of we have for all \( m \)
\[ \sum_{j=0}^{m} a_j p_Z(m-j) = \sum_{j=0}^{m} (m-j+1) p_Z(m-j+1) p_{\hat{Y}_k}(j). \]

Working out the recursive formula for the \( a's \) that follows from this yields, with \( a_{-1} = 0 \)
\[ a_m = \frac{1}{p_Z(0)} \sum_{j=-1}^{m-1} p_Z(m-j) ((m-j)p_{\hat{Y}_k}(j+1) - a_j) \quad \text{for all } m = 1, \ldots, 2kN. \]

Now reconsider (5.1.4) which reads
\[ G'_{\hat{Y}_k}(r) = k H(r) \Rightarrow \sum_{j=0}^{2N-1} (j+1)p_{\hat{Y}_k}(j+1) r^j = k \sum_{j=0}^{2kN-1} a_j r^j. \]

Equating the powers of \( r \) and translating back to \( p_Z \) and \( p_{\hat{Y}_k} \) yields the result. \( \square \)
Remark 5.1.2. It would seem as this is such a generic formula it is unlikely it has not been quoted in the literature before. Yet no reference could be found.

Remark 5.1.3. In Theorem 5.1.1 \((Z^{(i)})_{i \geq 1}\) is taking values on a grid covering the interval \([-\delta N, \delta N]\). Let us define \(R := \delta N\). Therefore if we change the step size \(\delta\) and still want to cover the interval \([-R, R]\), we have to adjust \(N\) and vice versa.

Hence, for the convergence results below we assume that \(N := \frac{R}{\delta}\) ensuring that the grid does not collapse when taking the limit of the step size \(\delta\) to be 0.

Note that by multiplying the probability generating functions in the proof of Theorem 5.1.1 this is equivalent to use convolution in a clever way. If we choose in Theorem 5.1.1 \(Z := \tilde{X}^{(n)}\) then we have the following convergence result:

Theorem 5.1.4. Let \(X\) be a Lévy process and let \(Y^{(n)}\) be its discretised version on a grid \([-\delta N, -\delta N + \delta, \ldots, \delta N - \delta, \delta N]\). Let \(F_X\) be the CDF of \(X^{(n)}\). Define a majorant step function of \(F_X\) by \(\tilde{F}_X(x) := F_X(\delta \lceil \frac{x}{\delta} \rceil)\) and respectively a minorant function \(F_{\tilde{X}}\) of \(F_X\) by \(\tilde{F}_X(x) := F_X(\delta \lfloor \frac{x}{\delta} \rfloor)\) for \(N > 0\) and \(\delta > 0\). Then for every \(x \in \mathbb{R}\)

\[
\tilde{F}_X(x) \to F_X(x) \quad \tilde{F}_X(x) \to F_X(x)
\]

as \(\delta \to 0\).

Moreover, \(\tilde{F}_X\) and \(F_X\) are the cdfs of two discrete random variables \(Y^{(n)}\) and \(W^{(n)} := X^{(n)}\) bounding \(X^{(n)}\). It is

\[
\tilde{X}_n^{(n)} \overset{d}{\to} X_{g(n,n)} \\
X_n^{(n)} \overset{d}{\to} X_{g(n,n)}
\]

as \(\delta \to 0\).

Proof. Note that the cdf \(F_X\) is a continuous function. This can be easily seen by investigating the pdf, see (2.2.7).

Let \(x > 0\). It is obvious that

\[
\delta \lceil \frac{x}{\delta} \rceil \downarrow x \text{ as } \delta \to 0.
\]

This implies immediately \(\tilde{F}_X(x) \to F_X(x)\) by using the continuity of \(F_X\). Similarly we can prove the relation for the relation for \(F_X(x)\) and \(x < 0\).
By Lévy’s continuity theorem for characteristic functions we deduce that $\tilde{\phi}(x) \to \phi(x)$ and $\hat{\phi}(x) \to \phi(x)$ as $\delta \to 0$ for all $x \in \mathbb{R}$ where $\phi$, $\tilde{\phi}$ and $\hat{\phi}$ denote the characteristic functions of $X$, $Y^{(n)}$ and $W^{(n)}$. As the convolution of distribution functions of independent random variables is the power of their characteristic functions it is easy to see that $\tilde{\phi}^n(x) \to \phi^n(x)$ for $x \in \mathbb{R}$. This means nothing else than $\tilde{F}_X^{(sn)}(x) \to F_X^{(sn)}(x)$ and $F_X^{(sn)}(x) \to F_X^{(sn)}(x)$ as $\delta \to 0$ and for all $x \in \mathbb{R}$.

5.2 Error analysis

Let us now have a closer look at the errors. For every $x \in \mathbb{R}$ we fix $k \geq 2$, $\delta > 0$ and $N$ and we make the following observation:

$$F_X^{(s(k))}(x) \approx F_X^{(s(k))}(\lfloor \frac{x}{\delta} \rfloor \delta)$$

$$= \int_{-\infty}^{\infty} F_X^{(s(k)-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - y) f(y) dy$$

by the definition of convolution,

$$\approx \int_{-kN\delta}^{kN\delta} F_X^{(s(k-1))}(\lfloor \frac{x}{\delta} \rfloor \delta - y) f(y) dy$$

if $\delta N$ is large enough,

$$\approx \sum_{i=-kN}^{kN} F_X^{(s(k-1))}(\lfloor \frac{x}{\delta} \rfloor \delta - \delta i) \delta f(i\delta)$$

by a standard approximation for the Riemann integral,

$$\approx \sum_{i=-kN}^{kN} F_X^{(s(k-1))}(\lfloor \frac{x}{\delta} \rfloor \delta - \delta i) \mathbb{P}(W^{(n)} = \delta i)$$

by a rough approximation for the Riemann integral,

$$= F_X^{(s(k))}(x).$$

Similarly, we obtain a observation for $\tilde{F}_X^{(s(k))}(x)$. 

\hfill \square
Let us have a closer look at the 4 approximations above. For every $x \in \mathbb{R}$ we fix a $k \geq 2$, $N$ and $\delta$ and investigate:

(i) $F_X^{(k)}(x) - F_X^{(k)}(\lfloor \frac{x}{\delta} \rfloor \delta)$,

(ii) $\int_{-\infty}^{\infty} F_X^{(k-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - y) f(y) dy - \int_{-kN\delta}^{kN\delta} F_X^{(k-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - y) f(y) dy$,

(iii) $\int_{-kN\delta}^{kN\delta} F_X^{(k-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - \delta i) f(i\delta) dy - \sum_{i=-kN}^{kN} F_X^{(k-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - \delta i) f(i\delta)$,

(iv) $\sum_{i=-kN}^{kN} F_X^{(k)}(\lfloor \frac{x}{\delta} \rfloor \delta - \delta i) f(i\delta) - \sum_{i=-kN}^{kN} F_X^{(k-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - \delta i) \mathbb{P}(W^{(n)} = \delta i)$.

Lemma 5.2.1. Let $X$ be a meromorphic Lévy process of unbounded variation. Let $F$ be the CDF of $X^{(n)}$ where $\xi^{(n)}$ is an exponentially distributed random variable with mean $\frac{1}{n}$ for $n > 0$. Let $-(\zeta^{(n)}_+(j))_{j \geq 0}$ (resp. $(\zeta^{(n)}_+)_j$) be the sequence of all real negative (resp. real positive) zeros of the function $z \mapsto n + \Psi(-iz)$ (or, equivalently, of $z \mapsto \psi(z) - n$) where $\Psi$ and $\psi$ denote the characteristic respectively Laplace exponent of $X$.

(i) can be calculated as follows:

\[ \varepsilon_1(k, \delta) := F_X^{(k)}(x) - F_X^{(k)}(\lfloor \frac{x}{\delta} \rfloor \delta) \]

\[ = \sum_{j=0}^{N_+} \sum_{i=0}^{k-1} \sum_{r=0}^{i} W^{(n)}_+(i, j) \frac{i!}{(i-r)!} \left( \frac{1}{\zeta^{(n)}_+(j)} \right)^{r+1} \left( \lfloor \frac{x}{\delta} \rfloor \delta \right)^{i-r} \exp(-\zeta^{(n)}_+(j)(\lfloor \frac{x}{\delta} \rfloor \delta)) - x^{i-r} \exp(-\zeta^{(n)}_+(j)x) \]

where expressions for $W^{(n)}_+(\cdot, \cdot)$ can be found in Appendix A.

(ii) has an upper bound:

\[ \int_{-\infty}^{\infty} F_X^{(k-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - y) f(y) dy - \int_{-kN\delta}^{kN\delta} F_X^{(k-1)}(\lfloor \frac{x}{\delta} \rfloor \delta - y) f(y) dy \]

\[ \leq \sum_{j=0}^{N_-} \sum_{i=0}^{k-1} \sum_{r=0}^{i} W^{(n)}_-(i, j) \frac{i!}{(i-r)!} \left( \frac{1}{\zeta^{(n)}_-(j)} \right)^{r+1} (-kN\delta)^{i-r} \exp(-\zeta^{(n)}_-(j)kN\delta) \]

\[ + \sum_{j=0}^{N_+} \sum_{i=0}^{k-1} \sum_{r=0}^{i} W^{(n)}_+(i, j) \frac{i!}{(i-r)!} \left( \frac{1}{\zeta^{(n)}_+(j)} \right)^{r+1} (kN\delta)^{i-r} \exp(-\zeta^{(n)}_+(j)kN\delta) =: \varepsilon_2(k, N, \delta) \]

where expressions for $W^{(n)}_+(\cdot, \cdot)$ and $W^{(n)}_-(\cdot, \cdot)$ can be found in Appendix A.
(iii) can be bounded by $\varepsilon_3(k, N, \delta)$, i.e.

$$
\left| \int_{-kN\delta}^{kN\delta} \frac{x}{\delta} \left( \left\lfloor \frac{x}{\delta} \right\rfloor \delta - y \right) f(y) dy - \sum_{i=-kN}^{kN} F_X^{(k-1)} \left( \left\lfloor \frac{x}{\delta} \right\rfloor \delta - \delta i \right) \delta f(i\delta) \right| \quad (5.2.3)
$$

$$
< \varepsilon_3(k, N, \delta)
$$

where $\varepsilon_3(k, N, \delta)$ can be calculated with the help of Theorem 3.2.3. Due to compactness reasons we refer to the proof for a detailed expression.

Finally, (iv) can be simplified as follows:

$$
\left| \sum_{i=-kN}^{kN} F_X^{(k-1)} \left( \left\lfloor \frac{x}{\delta} \right\rfloor \delta - \delta i \right) \delta f(i\delta) - \sum_{i=-kN}^{kN} F_X^{(k-1)} \left( \left\lfloor \frac{x}{\delta} \right\rfloor \delta - \delta i \right) \right| \quad (5.2.4)
$$

$$
< \varepsilon_4(k, N, \delta)
$$

where $\varepsilon_4(k, N, \delta)$ can be calculated with the help of Theorem 3.2.3. Due to compactness reasons we refer to the proof for a detailed expression.

The total maximal error for approximations (i) – (iv) is then

$$
\varepsilon_1(k, \delta) + \varepsilon_2(k, N, \delta) + \varepsilon_3(k, N, \delta) + \varepsilon_4(k, N, \delta).
$$

Proof. Ad (5.2.1): This can be calculated with the help of Theorem 3.2.3. Without loss of generality assume $x \geq 0$:

$$
\varepsilon_1(k, \delta) = F_X^{(k)}(x) - F_X^{(k)} \left( \left\lfloor \frac{x}{\delta} \right\rfloor \delta \right)
$$

$$
= \int_{\left\lfloor \frac{x}{\delta} \right\rfloor \delta}^{x} f_X^{(k)}(y) dy
$$

$$
= \sum_{j=0}^{x} \sum_{i=0}^{N_+} W^{(n)}_+(i, j) \frac{\zeta^{(n)}_+(j) y^i}{i!} \frac{1}{(i-r)!} \frac{1}{\zeta^{(n)}_+(j)^{r+1}}
$$

$$
\cdot \left[ \left( \left\lfloor \frac{x}{\delta} \right\rfloor \delta \right)^{-r} \exp(-\zeta^{(n)}_+(j) \left( \left\lfloor \frac{x}{\delta} \right\rfloor \delta \right)) - x^{-r} \exp(-\zeta^{(n)}_+(j) x) \right].
$$
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Ad (5.2.2): This can be calculated again with the help of Theorem 3.2.3.

\[ \int_{-\infty}^{\infty} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - y) f(y) dy - \int_{-kN\delta}^{kN\delta} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - y) f(y) dy \]

\[ = \int_{-\infty}^{-kN\delta} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - y) f(y) dy + \int_{kN\delta}^{\infty} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - y) f(y) dy \]

\[ \leq \int_{-\infty}^{\infty} f(y) dy + \int_{kN\delta}^{\infty} f(y) dy \]

\[ = \int_{j=0}^{kN\delta} \sum_{i=0}^{k-1} e^{x(i)}(\zeta^{(n)}(j)y) \sum_{r=0}^{N_{\delta}} \sum_{i=0}^{k-1} W_{r}^{(n)}(i, j)y^{i} + \int_{kN\delta}^{\infty} \sum_{j=0}^{N_{\delta}} e^{x(-\zeta^{(n)}(j)y)} \sum_{i=0}^{k-1} W_{r}^{+}(i, j)y^{i} \]

\[ = \sum_{j=0}^{k-1} \sum_{i=0}^{k-1} \sum_{r=0}^{N_{\delta}} W_{r}^{(n)}(i, j) \frac{1}{(i - r)!} \zeta^{(n)}(j)^{i} \frac{1}{\zeta^{(n)}(j)^{i} + 1} (kN\delta)^{i} e^{x(-\zeta^{(n)}(j)kN\delta)} \]

\[ + \sum_{j=0}^{k-1} \sum_{i=0}^{k-1} \sum_{r=0}^{N_{\delta}} W_{r}^{+(n)}(i, j) \frac{1}{(i - r)!} \zeta^{(n)}(j)^{i} \frac{1}{\zeta^{(n)}(j)^{i} + 1} (kN\delta)^{i} e^{x(-\zeta^{(n)}(j)kN\delta)} = \varepsilon_{2}(k, N, \delta). \]

Ad (5.2.3): Note that \( F \) is a continuous function. Hence we can use a simple estimate for Riemann integrals:

\[ | \int_{-kN\delta}^{kN\delta} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - y) f(y) dy - \sum_{i=-kN}^{kN} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - \delta i) f(i\delta) | \]

\[ \leq kN\delta \max_{z \in [-kN, kN]} (F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - z) f(z))' \]

\[ = \varepsilon_{3}(k, N, \delta). \]

We can calculate \( \varepsilon_{3}(k, N, \delta) \) by using Theorem 3.2.3 but omit it due to compactness reasons.

Ad (5.2.4):

\[ | \sum_{i=-kN}^{kN} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - \delta i) f(i\delta) - \sum_{i=-kN}^{kN} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - \delta i)f(W^{(n)} = \delta i) | \]

\[ \Leftrightarrow \sum_{i=-kN}^{kN} F_{X}^{(k-1)}(\frac{x}{\delta} \mid \delta - \delta i) | f(i\delta) - \mathbb{P}(W^{(n)} = \delta i) |. \]

By construction of \( F_{X} := F(\delta \frac{x}{\delta}) \) we obtain the following:

\[ | \delta f(i\delta) - \mathbb{P}(W^{(n)} = \delta i) | = | \delta f(i\delta) - \int_{(i-1)\delta}^{i\delta} f(y) dy |. \]
As $f \in \mathbf{C}^1$ on the interval $[(i-1)\delta, i\delta]$ we can apply the mean value theorem and get for some $r_x \in [x, i\delta]$ where $x > (i-1)\delta$:

$$(i\delta - x)f'(r_x) = f(i\delta) - f(x).$$

Therefore we have as $f$ has a bounded derivative on the interval $[(i-1)\delta, i\delta]$:

$$\left| \int_{(i-1)\delta}^{i\delta} f(y) dy - \delta f(i\delta) \right| = \left| \int_{(i-1)\delta}^{i\delta} (i\delta - y) f'(r_x) dy \right| \leq (i\delta - (i-1)\delta) \sup_{(i-1)\delta \leq x \leq i\delta} |f'(x)| = \delta \sup_{(i-1)\delta \leq x \leq i\delta} |f'(x)|.$$

Without loss of generality let us assume $(i-1)\delta > 0$, therefore $x > 0$ as well. Then

$$\delta \sup_{(i-1)\delta \leq x \leq i\delta} \left| \sum_{j=0}^{N_x} c_{n+}^{(n)}(i) \zeta_+^{(n)}(j)^2 \exp(-\zeta_+^{(n)}(j)x) \right| \leq \delta \sup_{(i-1)\delta \leq x \leq i\delta} \left| \sum_{j=0}^{N_x} c_{n+}^{(n)}(i) \zeta_+^{(n)}(j)^2 \exp(-\zeta_+^{(n)}(j)(i-1)\delta) \right|.$$

Finally,

$$\sum_{i=-kN}^{kN} F_X^{(k-1)}(\frac{x}{\delta} \delta - \delta i) |\delta f(i\delta) - \mathbb{P}(W^{(n)} = \delta i)| < \sum_{i=-kN}^{kN} F_X^{(k-1)}(\frac{x}{\delta} \delta - \delta i) (\delta \sup_{(i-1)\delta \leq x \leq i\delta} \left| \sum_{j=0}^{N_x} c_{n+}^{(n)}(i) \zeta_+^{(n)}(j)^2 \exp(-\zeta_+^{(n)}(j)x) \right|$$

$$< \sum_{i=-kN}^{kN} F_X^{(k-1)}(\frac{x}{\delta} \delta - \delta i) (\delta \left| \sum_{j=0}^{N_x} c_{n+}^{(n)}(i) \zeta_+^{(n)}(j)^2 \exp(-\zeta_+^{(n)}(j)(i-1)\delta) \right|$$

$$= \varepsilon_4(k, N, \delta)$$

We can calculate $\varepsilon_4(k, N, \delta)$ by using Theorem 3.2.3 but omit it due to compactness reasons.

**Remark 5.2.2.** In this chapter we only investigate the error in space. As mentioned before we calculate the law of $X$ at a random time $g(n, n)$. Estimates for the error in time can be found in Section 3.3.
5.3 Law of the supremum and the joint law

Let us assume for notational convenience only that the step grid size $\delta = 1$.

In this section we derive recursive formulas for the law of the supremum and the joint law of the discretised process $\tilde{X}_n^{(n)}$. Hereby we will use the following definition of the running supremum. With the notation in this chapter this is

$$\tilde{S}_i^{(n)} = \max(\tilde{S}_{i-1}^{(n)}, \tilde{X}_{i-1}^{(n)} + \tilde{S}(n)(i)).$$

(5.3.1)

Note that formula (5.3.1) already appears in [41] with $X$ instead of $\tilde{X}$ being a Lévy process.

We would like to remark that the definition (5.3.1) constitutes an advantage to the usual one of the maximum of a random walk,

$$\tilde{S}_i^{(n)} = \max_{0 \leq j \leq i} \tilde{X}_j^{(n)}.$$

(5.3.2)

By comparing (5.3.1) and (5.3.2) the former one takes into account the excursions between positions of the random walk $\tilde{X}_k^{(n)}$ and $\tilde{X}_{k+1}^{(n)}$ for some $k \geq 1$ while the latter does not.

**Theorem 5.3.1 (Supremum).** Let $X$ be a Lévy process and let $g(l, n)$ be the sum of $l$ i.i.d. exponentially distributed random variables $\xi^{(n)}$ with common mean $\frac{1}{n}$. Let $\tilde{S}^{(n)}$ respectively $\tilde{I}^{(n)}$ denote the discretised version of $X_\xi^{(n)}$ respectively $X_\xi^{(n)}$. Let $(\tilde{S}^{(n)}(j))_{j \geq 1}$ be a sequence of i.i.d random variables with common law equal to the law of $\tilde{S}^{(n)}$ and $(\tilde{I}^{(n)}(j))_{j \geq 1}$ is a sequence of i.i.d. random variables with common law equal to the law of $\tilde{I}^{(n)}$. Again, let $\tilde{X}_i^{(n)}$ denote the discretised version of $X_{g(l, n)}$ and $Y^{(n)}$ of $X_\xi^{(n)}$. Define

$$z[r, 1] := \begin{cases} \mathbb{P}(\tilde{S}^{(n)} = r) & \text{if } r \geq 0 \\ 0 & \text{if } r < 0. \end{cases}$$

$$z[r, s] := \sum_{t=-N}^{N} \mathbb{P}(Y^{(n)} = t)z[r - t, s - 1]$$

for $s > 1$ and $0 \leq r \leq sN$.

Then

$$\mathbb{P}(\tilde{S}_i^{(n)} = j) = \sum_{t=0}^{j} \mathbb{P}(\tilde{S}^{(n)} = t)z[j - t, i - 1].$$

(5.3.3)
CHAPTER 5. TREE METHOD FOR LÉVY PROCESSES

Proof.

\[ \mathbb{P}(\tilde{S}_i^{(n)} = j) = \mathbb{P}(\max(\tilde{S}_{i-1}^{(n)}, \tilde{X}_{i-1}^{(n)} + \tilde{S}^{(n)}(i)) = j) \]

\[ = \mathbb{P}(\max(\tilde{S}_{i-1}^{(n)}, \sum_{g=1}^{i-1} (\tilde{S}^{(n)}(g) + \tilde{I}^{(n)}(g)) + \tilde{S}^{(n)}(i)) = j) \]

\[ = \mathbb{P}(\max(\max(\tilde{S}_{i-2}^{(n)}, \sum_{g=1}^{i-2} (\tilde{S}^{(n)}(g) + \tilde{I}^{(n)}(g))) + \tilde{S}^{(n)}(i - 1)), \]

\[ \sum_{g=1}^{i-1} (\tilde{S}^{(n)}(g) + \tilde{I}^{(n)}(g)) + \tilde{S}^{(n)}(i) = j) \]

\[ = \ldots \]

\[ = \mathbb{P}(\max(\max(\ldots \max(0, \tilde{S}^{(n)}(1)), \sum_{g=1}^{1} (\tilde{S}^{(n)}(g) + \tilde{I}^{(n)}(g)), \ldots), \ldots), \]

\[ \sum_{g=1}^{i-2} (\tilde{S}^{(n)}(g) + \tilde{I}^{(n)}(g)) + \tilde{S}^{(n)}(i - 1)), \sum_{g=1}^{i-1} (\tilde{S}^{(g)}(g) + \tilde{I}^{(g)}(g)) + \tilde{S}^{(n)}(i) = j) \]

\[ = \mathbb{P}(\tilde{S}^{(n)}(1) + \max(\tilde{S}^{(n)}(2) + \tilde{I}^{(n)}(1) + \max(\tilde{S}^{(n)}(3) + \tilde{I}^{(n)}(2) \]

\[ + \max(\ldots + \max(0, \tilde{S}^{(n)}(i)) \ldots, 0), 0), 0) = j) \]

\[ \overset{(d)}{=} \mathbb{P}(\tilde{S}^{(n)}(1) + \max(Y^{(n)} + \max(Y^{(n)} + \max(\ldots + \max(Y^{(n)} + \tilde{S}^{(n)}(i), 0), \ldots, 0), 0), 0)) = j). \]

\[ \square \]

Remark 5.3.2. Unfortunately we were not able to obtain a simpler formula for the discretized supremum after \( n \) exponential periods than (5.3.3). Consider the case \( N = \delta = 1 \).

Let \( s_0 := \mathbb{P}(\tilde{S}^{(n)} = 0) \) and \( i_0 := \mathbb{P}(\tilde{I}^{(n)} = 0) \). As \( N = 1 \) we have \( s_1 := \mathbb{P}(\tilde{S}^{(n)} = 1) = 1 - s_0 \) and \( i_1 := \mathbb{P}(\tilde{I}^{(n)} = 1) = 1 - i_0 \). We made the following observations and results by analysing numbers gained from an implementation of (5.3.3) on the computer by using Maple and C for \( n = 1, \ldots, 100 \).

As mentioned before, the discretized supremum after \( n \) exponential periods can take values on the grid \( \{0, 1, 2, \ldots, nN\} \).

We derived the following formulas which gave identical values in comparison to (5.3.3)
for at least \( n = 1, \ldots, 100 \):

\[
\mathbb{P}(\tilde{S}_n^{(n)} = n) = i_0^{-1} \sum_{l=0}^{n} s_0^l (-1)^l \binom{n}{l}
\]

\[
\mathbb{P}(\tilde{S}_n^{(n)} = n - 1) = i_0^{-2} \sum_{l=0}^{n} s_0^{n-l} (-1)^{n-l} W_{n,l,n-1}
\]

\[
+ i_0^{-1} \sum_{l=0}^{n} s_0^{n-l} (-1)^{n-l-1} \left\{ W_{n,l,n-1} + n \binom{n-1}{l} \right\}
\]

\[
\mathbb{P}(\tilde{S}_n^{(n)} = n - 2) = i_0^{-3} \sum_{l=0}^{n} s_0^{n-l} (-1)^{n-l} W_{n,l,n-2}
\]

\[
+ i_0^{-2} \sum_{l=0}^{n} s_0^{n-l} (-1)^{n-l-1}
\]

\[
\left\{ W_{n,l,n-2} + (n-1) \binom{n-2}{l-1} + \binom{n-1}{2} \binom{n}{l} + (n-3 + (n-2)^2) \binom{n-1}{l} \right\}
\]

\[
+ i_0^{-1} \sum_{l=0}^{n} s_0^{n-l} (-1)^{n-l} \left\{ W_{n,l,n-2} + \frac{(n-3)n}{2} \binom{n-2}{l} + (n-2 + (n-1)^2) \binom{n-1}{l} \right\},
\]

where \( W_{n,l,n-k} \) := \( \sum_{u=0}^{k} \binom{n-u}{l-u} \binom{n-2-j}{k-j} \). It is obvious that the general structure of the formula should look like the following:

\[
a[i, n] = \sum_{l=i-1}^{n} i_0^l \sum_{r=0}^{n} s_0^r B(r, l, i, n).
\]

We tried to obtain the coefficients in two different ways. Note again that these are numerical observations rather than derived in a formal mathematical way. From our computer implementation we suspect the following.

The first way can be described as follows: For each \( n, i, l \) we can write the coefficients as sum of binomial coefficients. That means

\[
a[i, n] = \sum_{l=i-1}^{n} i_0^l \sum_{r=0}^{n} s_0^r B(r, l, i, n),
\]

where we found

\[
B(r, l, i, n) = (-1)^{n-i-l-1} \binom{l}{i-1} \binom{i}{r} + \sum_{t=0}^{n-i-1} \binom{n-t}{r} D(t, r, l, i, n).
\]

Hereby the coefficients \( D(., ., ., .) \) are polynomials dependent on \( t, i, l, r, n \) and the their degree can be divided into three cases for all relevant \( n, i, l \):

(i) If \( l \leq i \) then the degree in \( n \) of \( D(., ., ., .) \) is \( l + r \).
(ii) If \( l > i \) and \( t > l - i \) then the degree in \( n \) of \( D(.,. ,.,.,.) \) is \( i + t \).

(iii) If \( l > i \) and \( t \leq l - i \) then the degree in \( n \) of \( D(.,. ,.,.,.) \) is \( l + t \).

In the case (i) \( l \leq i \) means \( l = i - 1 \) or \( l = i \). If \( l = i - 1 \) we get

\[
D(t, r, l, i, n) = \prod_{g=1}^{l+r} (n-g) \frac{1}{(l+r)!} \frac{1}{(i-1)!} (-1)^{l+r-i-1} \tag{5.3.4}
\]

and if \( l = i \)

\[
D(t, r, l, i, n) = \prod_{g=1}^{i+r-1} (n-g) \frac{1}{(l+r)!} \frac{1}{(i-1)!} (n-4(r-1)-(l+1)^2)(-1)^r \quad \text{for } r > 0. \tag{5.3.5}
\]

In the case (ii) we obtain

\[
D(t, r, l, i, n) = \frac{1}{t+i} \frac{(-1)^{t+l}}{2(l+1)(l-1)!} \prod_{g=1}^{t+i-1} (n-g)((l-i+1)n-(l+1)^2-(t-l+i-1)(l+1)). \tag{5.3.6}
\]

Unfortunately it was not possible to derive such formulas for case (iii). We just can see that for \( t = 0 \)

\[
D(t, r, l, i, n) = \prod_{g=1}^{l} (n-g) \frac{1}{2(i-1)!(l-i+1)} \tag{5.3.7}
\]

and for \( t > 0 \)

\[
D(t, r, l, i, n) = \prod_{g=1}^{l} (n-g) \frac{1}{2(i-1)!} \frac{1}{\prod_{s=1}^{i} (i+s)} C(t, i, l, n) \tag{5.3.8}
\]

where the degree of \( C(t, i, l, n) \) in \( n \) is \( t \).

A second approach to tackle the problem was the following: For each \( i \in \{1, \ldots, n-2\} \)
and each \( l \), choose a small \( n_0 \). We consider the sequence \( (B(r, l, i, n))_{n \geq n_0} \) which we divide into two classes, \( (B(n-r, l, i, n))_{n} =: (B_u(n-r, l, i, n))_{n} \) and \( (B(r, l, i, n))_{n} =: (B_d(r, l, i, n))_{n} \) for \( r \leq \left\lfloor \frac{n}{2} \right\rfloor \). Then the sequences \( (B_u(n-r, l, i, n))_{n} \) and \( (B_d(r, l, i, n))_{n} \) for \( r \leq \left\lfloor \frac{n}{2} \right\rfloor \) are polynomials in \( n \). Their degree is \( 2l - i + r - 1 \) respectively \( l - 1 + r \) if \( i \geq 3 \) or \( i = 2, l \geq 2 \). In the case \( i = 2, l = 1 \) their degree is \( r \).

If \( n_0 \) was odd choose \( n_1 = n_0 + 1 \). Then the sequences \( (B_u(n-\frac{n_1}{2}, l, i, n))_{n} \) and \( (B_d(\frac{n_1}{2}, l, i, n))_{n} \) are again polynomials in \( n \) with degree \( 2l - i + n_1 - 1 \) respectively.
Proof. We have:

\[ P(\hat{X}_n^{(n)} = i, \hat{S}_n^{(n)} = j) \]

\[ = P(\tilde{I}^{(n)} = i - j) \sum_{l=0}^{N} \sum_{w=0}^{j-1} P(\tilde{S}(n) = l) P(\hat{X}_{n-1}^{(n)} = j - l, \hat{S}_{n-1}^{(n)} = w) \]  

(5.3.10)

\[ + \sum_{l=-nN}^{j} P(\tilde{I}^{(n)} = i - l) \sum_{m=0}^{N} P(\tilde{S}(n) = m) P(\hat{X}_{n-1}^{(n)} = l - m, \hat{S}_{n-1}^{(n)} = j). \]  

(5.3.11)

**Theorem 5.3.3** (Joint law). Let \( X \) be a Lévy process and use the same notation as in Theorem 5.3.1. Then

\[ P(\hat{X}_n^{(n)} = i, \hat{S}_n^{(n)} = j) = \sum_{w=0}^{N} P(\hat{X}_n^{(n)} = i, \hat{S}_n^{(n)} = j, \hat{S}_{n-1}^{(n)} = w) \]

\[ = \sum_{w=0}^{j} P(\hat{X}_n^{(n)} = i, \hat{S}_n^{(n)} = j, \hat{S}_{n-1}^{(n)} = w) \]

\[ + \sum_{w=j+1}^{N} P(\hat{X}_n^{(n)} = i, \hat{S}_n^{(n)} = j, \hat{S}_{n-1}^{(n)} = w) \]

\[ = \sum_{w=0}^{j} P(\hat{X}_n^{(n)} = i, \hat{S}_n^{(n)} = j, \hat{S}_{n-1}^{(n)} = w) \]

\[ + \sum_{w=0}^{j-1} P(\hat{X}_n^{(n)} = i, \hat{S}_n^{(n)} = j, \hat{S}_{n-1}^{(n)} = w) \]
\[ + \mathbb{P}(\tilde{X}_n^{(n)} = i, \tilde{S}_n^{(n)} = j = \tilde{S}_{n-1}^{(n)}) \]
\[ = \sum_{w=0}^{j-1} \mathbb{P}(\tilde{X}_n^{(n)} + \tilde{S}_n^{(n)} + \tilde{I}^{(n)} = i, \tilde{S}_{n-1}^{(n)} + \tilde{X}_{n-1}^{(n)} + \tilde{S}^{(n)} = j = \tilde{S}_{n-1}^{(n)} = w) \]
\[ + \mathbb{P}(\tilde{X}_{n-1}^{(n)} + \tilde{S}_n^{(n)} + \tilde{I}^{(n)} = i, \tilde{S}_{n-1}^{(n)} = j, \tilde{X}_{n-1}^{(n)} + \tilde{S}^{(n)} = j < j + 1) \]
\[ = \mathbb{P}(\tilde{I}^{(n)} = i - j) \sum_{w=0}^{j-1} \mathbb{P}(\tilde{X}_{n-1}^{(n)} + \tilde{S}^{(n)} = j, \tilde{S}_{n-1}^{(n)} = w) \]
\[ + \sum_{l=-nN}^{j} \mathbb{P}(\tilde{X}_{n-1}^{(n)} + \tilde{S}^{(n)} + \tilde{I}^{(n)} = i, \tilde{S}_{n-1}^{(n)} = j, \tilde{X}_{n-1}^{(n)} + \tilde{S}^{(n)} = l) \]
\[ = \mathbb{P}(\tilde{I}^{(n)} = i - j) \sum_{l=0}^{N} \sum_{w=0}^{j-1} \mathbb{P}(\tilde{X}_{n-1}^{(n)} = j - l, \tilde{S}^{(n)} = l, \tilde{S}_{n-1}^{(n)} = w) \]
\[ + \sum_{l=-nN}^{j} \mathbb{P}(\tilde{I}^{(n)} = i - l) \mathbb{P}(\tilde{X}_{n-1}^{(n)} + \tilde{S}^{(n)} = l, \tilde{S}_{n-1}^{(n)} = j) \]
\[ = \mathbb{P}(\tilde{I}^{(n)} = i - j) \sum_{l=0}^{N} \sum_{w=0}^{j-1} \mathbb{P}(\tilde{S}^{(n)} = l) \mathbb{P}(\tilde{X}_{n-1}^{(n)} = j - l, \tilde{S}_{n-1}^{(n)} = w) \]
\[ + \sum_{l=-nN}^{j} \mathbb{P}(\tilde{I}^{(n)} = i - l) \sum_{m=0}^{N} \mathbb{P}(\tilde{X}_{n-1}^{(n)} = l - m, \tilde{S}_{n-1}^{(n)} = j, \tilde{S}^{(n)} = m) \]
\[ = \mathbb{P}(\tilde{I}^{(n)} = i - j) \sum_{l=0}^{N} \sum_{w=0}^{j-1} \mathbb{P}(\tilde{S}^{(n)} = l) \mathbb{P}(\tilde{X}_{n-1}^{(n)} = j - l, \tilde{S}_{n-1}^{(n)} = w) \]
\[ + \sum_{l=-nN}^{j} \mathbb{P}(\tilde{I}^{(n)} = i - l) \sum_{m=0}^{N} \mathbb{P}(\tilde{S}^{(n)} = m) \mathbb{P}(\tilde{X}_{n-1}^{(n)} = l - m, \tilde{S}_{n-1}^{(n)} = j). \]

This concludes the proof.

Note that in the last line of the previous proof $\mathbb{P}(\tilde{I}^{(n)} = i - j) = 0$ for $0 \geq i - j \geq -N$ and $\mathbb{P}(\tilde{I}^{(n)} = i - l) = 0$ for $0 \geq i - l \geq -N$.  

\[ \square \]
5.4 Numerics

In this section we discuss some numerics for the law of a discretised meromorphic Lévy process, obtained by a computer implementation of the result in Theorem 5.1.1. Remember that we consider the interval $[-R, R]$ for a space grid where $R := \delta N$. $N$ hereby denotes the number of points in the grid with mesh $\lambda$.

Next, we briefly describe the three main tasks for creating the graphs of different value functions presented further below. The first step is the calculation of the solutions $\zeta_+^{(n)}(i)$ and $\zeta_-^{(n)}(j)$ of $z \mapsto \psi(z) - n$ and of the coefficients $c_-^{(n)}(j)$ by using Proposition 2.2.10 for $i \in \{0, \ldots, N_+\}$ and $j \in \{0, \ldots, N_-\}$. In case a meromorphic Lévy process $X$ is used for which $N_+ = \infty$ or $N_- = \infty$ the infinite sums need to be truncated. For this we used the ad-hoc approach of ensuring that the truncated sums represent at least 99\% of the mass of $X_{\xi(n)}$, i.e. (compare with (2.2.7)) we determined $M^+, M^-$ such that

$$
\sum_{j=0}^{M^+} c_+^{(n)}(j) \frac{\zeta_+^{(n)}(j)}{\zeta_+(j)} - \sum_{j=0}^{M^-} c_-^{(n)}(j) \frac{\zeta_-^{(n)}(j)}{\zeta_-(j)} \geq 0.99
$$

and then normalised the coefficients. It is worth noting that $M$ is larger in the case of meromorphic Lévy process with finite variation than in the cases with infinite variation. Particularly, if $\sigma > 0$ the largest part of the probability mass is concentrated on $\zeta_+(0)$ and $\zeta_-(0)$ whereas the other zeroes contribute only a small part.

The second main step is the discretization of $(S_{\xi(n)}^{(j)})_{j \geq 1}$ and $(I_{\xi(n)}^{(j)})_{j \geq 1}$ which are sequences of i.i.d. random variables with common law equal to $X_{\xi(n)}$ respectively $X_{\xi(n)}$. Let us assume that $X$ is of unbounded variation. We choose the following one.

As in Theorem 5.1.1 let us denote $\tilde{S}^{(n)}$ and $\tilde{I}^{(n)}$ the discretised approximations to $X_{\xi(n)}$ and $X_{\xi(n)}$ taking values on $\{0, \delta, 2\delta, \ldots, N\delta\}$ and $\{-N\delta, \ldots, -\delta, 0\}$. Because the probability density functions $g_{S}$ and $g_{I}$ of $X_{\xi(n)}$ and $X_{\xi(n)}$ at an exponential random
time are known (see Section 2.2) we can set

\[ P(\tilde{S}^{(n)} = 0) := \int_0^{\frac{4}{\sqrt{\delta}}} g_S(x) \, dx, \]  
\[ (5.4.1) \]

\[ P(\tilde{S}^{(n)} = i\delta) := \int_{(i-\frac{1}{2})\delta}^{(i+\frac{1}{2})\delta} g_S(x) \, dx \text{ for } i = 1, \ldots, \lceil N \rceil, \]  
\[ (5.4.2) \]

\[ P(\tilde{I}^{(n)} = 0) := \int_{-\frac{2}{\sqrt{\delta}}}^{-\frac{1}{\sqrt{\delta}}} g_I(x) \, dx, \]  
\[ (5.4.3) \]

\[ P(\tilde{I}^{(n)} = i\delta) := \int_{-(i+\frac{1}{2})\delta}^{-(i-\frac{1}{2})\delta} g_I(x) \, dx \text{ for } i = 1, \ldots, \lceil N \rceil. \]  
\[ (5.4.4) \]

By convoluting the probability mass functions of \( \tilde{S}^{\xi(n)} \) and \( \tilde{S}^{\eta(n)} \) given by (5.4.1)–(5.4.4) and using the relation (2.1.6) we obtain the probabilities \( p_Z \) in Theorem 5.1.1 for \( \tilde{X}^{\xi(n)} \) \( \overset{d}{=} \tilde{S}^{(n)} + \tilde{I}^{(n)} \).

The third main step is the recursive computation of this tree method using formulas (5.1.1), (5.1.2) and (5.1.3). A program for this purpose was written in the language C++. We found that in certain cases the algorithm may need more than the default ‘double precision’ as working precision in order to have enough significant digits left in the end result. For this we made use of the ‘GMP/MPFR/MPFR++’ packages. Furthermore we made use of ‘OpenMP’ to parallelise the operations.

The Lévy processes used in this section are Brownian motion with drift and the Kou model which is a special case of Hyperexponential processes, cf Section 2.2.

As stated before one advantage of the tree method presented is the amount of parameters to choose. Namely, next to the parameters for the chosen stochastic process we can set

- the range of the grid \( R \),
- the grid step size \( \delta \) in space and
- the size of each time step \( \xi^{(n)} \) which is controlled by the intensity \( n \) of the Poisson process generating the stochastic grid (see 2.3).

We would like to find the best mixture between highest accuracy, with other words a minimal error and fast running time of our program. Obviously there is a tradeoff
between these both. If we choose large values for $N$ and $n$ and a small one for $\delta$ then the computational cost for our program will be large. However we made the following observations based on our figures and computations presented below:

- The grid range $R$ can be chosen in dependence of $n$. If $n$ is large, i.e. there are many exponentially distributed time intervals for our stochastic grid and each of them has an expected length of $\frac{1}{n}$ then the variance of the increments of the Lévy process will be small. With other words the larger $n$ is the smaller $N$ needs to be and vice versa.

We are solving the following equation in $z$ for some $t$ close to 1;

$$\mathbb{P}(\|X_{\xi(n)}\| \leq z) > t. \quad (5.4.5)$$

Then we obtain solutions $z_1 < z_2 < \ldots$ and set $R = \max(\max_{i \geq 1} z_i, \delta)$.

From numerical experience a good choice for $R$ is $t = 0.9999$. For example in the case of Brownian motion we derive $R > \max(\delta, -\frac{1}{2} \log(1 - R)^{\frac{n}{\sqrt{n}}})$.

- From our experience a sufficient choice for the number of exponential time periods $n$ is $n \approx 100$. Although Theorem 3.3.4 is indicating that $n$ needs to be much bigger to ensure a decent approximation both the numerical results in Chapter 3 and in this one confirm our guess. However it can be noted that if we consider a process of infinite activity where the Brownian part $\sigma$ plays a minor role $n$ needs to be chosen larger. However, contrary to this observation if the underlying process is a Brownian motion and or process where the jumps do not have much influence, $n$ can be significantly smaller, cf. Figure 5.1.

- From our point of view the grid step size $\delta$ can be chosen approximately as $\delta = 0.01$. This ensures a fast running time (with $n = 100, r = 1$ in the case of Brownian motion around 3 minutes) and sufficient small error in space.

Also note if $\delta$ is chosen very small, then $\mathbb{P}(\tilde{S}^{(n)} = i\delta)$ and $\mathbb{P}(\tilde{I}^{(n)} = -i\delta)$ defined above for $i = 1, \ldots, [\frac{N}{\delta}]$ become small. Moreover if we apply our method, we basically multiply these small values very often which results in very small values.

To overcome this difficulty one has to increase the computation accuracy in the ‘GMP/MPFR/MPFRC++’ package which slows down the computation speed.
In Figure 5.1 we see that the error between our 'tree' method for the position (Theorem 5.1.1) and the exact values, in this case Brownian motion, is quite small for a small number $n$ of exponential time periods already.

Figure 5.1: Plots of the relative difference between the discretised law from Theorem 5.1.1 for $k = n = 30, 100, 200$ and the exact values where the driving process $X$ is a standard Brownian motion. We choose the step grid size $\delta = 0.01$ and the grid range $R = 1.5$.

In the next Figure 5.2 we compare the error for different values of $\delta$. The running time for this case was approximately 30 minutes, whereas it were 2 minutes for the case 0.01 and more than 2 hours for the case $\delta = 0.005$. 
Figure 5.2: Plots of the relative difference between the discretised law from Theorem 5.1.1 for $\delta = 0.1, 0.01$ and 0.005 and the exact values where the driving process $X$ is a standard Brownian motion. We choose the number of exponential time periods $n = 100$ and the grid range $R = 1.0$. 
Figure 5.3: Plots of the relative difference between the discretised law from Theorem 5.1.1 for $k = n = 10, 40, 160$ and the exact values where the driving process $X$ is a double exponential jump process (Kou model, see Remark D.0.5) with the parameters $\mu = 0, \sigma = 1, \eta_1 = 3, \eta_2 = 5, p = 0.7, q = 0.3$ and $\lambda = 4$. Furthermore we choose $M^+ = 1, M^- = 1$, the step grid size $\delta = 0.01$ and the grid range $R = 4.0$. 
Figure 5.4: Plots of the relative difference between the discretised law from Theorem 5.1.1 for $\delta = 1.0, 0.1$ and $0.01$ and the exact values where the driving process $X$ is a double exponential jump process (Kou model, see Remark D.0.5) with the parameters $\mu = 0, \sigma = 1, \eta_1 = 2.5, \eta_2 = 7, p = 0.52, q = 0.48$ and $\lambda = 3$. Furthermore we choose $M^+ = 1, M^- = 1$, the number of exponential time periods $n = 160$ and the grid range $R = 4.0$.

In the next two figures, we demonstrate that the 'tree method' also in the case of the supremum works well (Theorem 5.3.1). Firstly, we compare different values for the number $n$ of exponential time periods.
Figure 5.5: Plots of the relative difference between the discretised law from Theorem 5.3.1 for \( k = n = 5, 30, 100 \) and the exact values where the driving process \( X \) is a standard Brownian motion. We choose the step grid size \( \delta = 0.01 \) and the grid range \( R = 2.0 \).

In the next Figure 5.6 we compare the error for different values of \( \delta \).
Figure 5.6: Plots of the relative difference between the discretised law from Theorem 5.3.1 for $\delta = 0.1, 0.01$ and 0.001 and the exact values where the driving process $X$ is a standard Brownian motion. We choose the number of exponential time periods $n = 50$ and the grid range $R = 2.0$. 
Chapter 6

A variation of the Canadisation algorithm for the pricing of American options driven by Lévy processes

6.1 Introduction

Let $X$ be a general Lévy process (see Section 2.1) defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by $X$ which is naturally enlarged (cf. Definition 1.3.38 in [14]). We denote by $\mathcal{T}$ the set of $\mathbb{F}$-stopping times taking values in $[0, \infty)$.

Consider the following classic optimal stopping problem with expiry date $T \in [0, \infty)$:

$$v(T, x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[ e^{-r(\tau \wedge T)} f(X_{\tau \wedge T}) \right],$$

(6.1.1)

where $f$ is a payoff function and $r \geq 0$ the discount rate. Such problems have been studied thoroughly and have applications in many fields, where option pricing theory in modern mathematical finance is maybe one of the most appealing ones. See e.g. the textbook [55] for a recent overview of the general theory concerning optimal stopping problems, and e.g. [33] for an introduction to mathematical finance and explanation of the related terms used here. For example, if in a typical mathematical finance setup
the price of the risky asset $S$ is assumed to evolve as $S_t = \exp(X_t)$ for all $t \geq 0$ while the price of the riskless asset $S^0$ is given by $S^0_t = e^{rt}$ for all $t \geq 0$, then $v$ can be interpreted* as the fair price of an American option on the risky asset which expires after $T$ time units and $x = X_0 = \log S_0$. One of the classic examples is the American put option, which is characterised by the payoff function $f(x) = (K - e^x)^+$ for some $K > 0$. Here $a^+ := \max\{a, 0\}$. Such Lévy process driven market models have received considerable attention in recent years, in an attempt to overcome some of the limitations of the classic Black & Scholes model where $X$ is a Brownian motion. See e.g. [26] and the references therein.

It is well known that in general no closed form formula exists for (6.1.1). Results do exist for certain combinations of payoff functions $f$ and Lévy processes $X$ when $T = \infty$, but when $T < \infty$ — as we assume throughout this chapter — one typically has to settle for approximating $v$ rather.

In his celebrated paper [24] (see also [18]), Carr proposes a method for this he refers to as ‘Canadisation’. The idea is to introduce a ‘stochastic time grid’ (see Section 2.3). The value of an American option with expiry date $T$ is then approximated by the value of the same American option but with the expiry date replaced by a grid point close to $T$. This approximating value function can be determined by performing a backwards induction over the grid points. Carr works out this algorithm for the American put option driven by a Brownian motion. See also [22] and the references therein for more general applications of Carr’s algorithm.

As shown in [22] Carr’s Canadisation is also viable for Brownian motion with exponential jumps. However, it does not seem easy to apply this algorithm to more general Lévy processes due to the fact that the backwards induction over the grid points is involved. This chapter introduces a variation of the Canadisation algorithm, where some changes to the original setup are made so that the backwards induction over the grid points becomes more straightforward. Indeed, the resulting adjusted algorithm allows to approximate $v$ by elementary functions not just for Brownian motion plus exponential jumps but for any meromorphic Lévy process (see Section 2.2).

*In addition it is required that $S/S^0$ is a martingale. If $X$ is not a Brownian motion there are typically multiple measures $Q$ equivalent to each other under which $S/S^0$ is a martingale. For this chapter it is not relevant which measure is used, as long as $X$ is a Lévy process under the chosen measure and $P$ is understood to be the chosen measure.
CHAPTER 6. PRICING OF AMERICAN OPTIONS

Recall they have the property that the law of the process evaluated at an independent, exponentially distributed time is an infinite mixture of exponentials (cf. Proposition 2.2.9). This property is key in our algorithm, as becomes clear in Section 6.4. Indeed, if this is the case every step in the algorithm can be worked out explicitly in terms of elementary functions provided the payoff function \( f \) in (6.1.1) can (piecewise) be expressed as a linear combination of functions of the form \( Ax^i e^{Bx} + C \) for \( A, B, C \in \mathbb{R} \) and \( i \in \mathbb{N} \). This is for instance the case for the classic example of the American put option, where \( f(x) = (K - e^x)^+ \) for some \( K > 0 \). For any payoff function not of this type it is a straightforward exercise to approximate it by a function of this type and estimate the error due to this approximation, cf. Remark 6.4.3.

Pricing American options driven by Lévy processes has received considerable attention in recent years. Indeed several other possible techniques were developed, including (but not limited to) tree-based methods (see e.g. [46]) and variational methods (see e.g. [48]). See also the overview in [45]. A detailed comparison of all available methods with the one proposed in this chapter would be a very substantial project and we do not embark on such a task here. The main appeal of the method presented in this chapter lies in the fact that it is a 'light weight' method which is probabilistic in nature and does not need any involved proofs or tools from other fields. Furthermore the computer implementation is rather straightforward as only elementary operations are required (together with the straightforward operation of numerical root finding). Finally the method is flexible and can easily be adjusted to deal with situations where for instance the option has a Bermudan type structure, i.e. where exercising is only allowed in certain subsets of \([0, T]\), or there is some path dependency in the payoff (for instance when the payoff also depends on the path supremum). Cf. Remark 6.4.3.

The rest of this chapter is organised as follows. In Section 6.2 we introduce the algorithm in detail and spend some time discussing the difference with the original Canadisation approach. In Section 6.3 we make the algorithm rigorous. In Section 6.4 we work out the algorithm in detail for the prominent example of the American put, where the driving process is a meromorphic Lévy process (or a Brownian motion plus (hyper)exponential jumps). In that section we also provide some more details concerning meromorphic Lévy processes. We find that the functions approximating \( v \) generated by our algorithm have easy to implement, explicit formulae. We conclude
6.2 The algorithm and comparison with Canadianization

Let us introduce the algorithm in detail. For the basic ideas of a stochastic grid we refer to Section 2.3. Again, for any \( n \in \mathbb{N} \), enlarge the above probability space on which \( X \) lives to contain a Poisson process \( \mathcal{N}^{(n)} \) with intensity \( n \), independent of \( X \). Denote the \( k \)-th jump time of \( \mathcal{N}^{(n)} \) by \( T^{(n)}_k \), i.e.

\[
T^{(n)}_k := \inf\{ t \geq 0 \mid \mathcal{N}^{(n)}_t \geq k \} \quad (6.2.1)
\]

with \( T^{(n)}_0 := 0 \). Let \( \tilde{\mathcal{F}}^{(n)} \) denote the naturally enlarged filtration generated by the pair \((X, \mathcal{N}^{(n)})\).

Furthermore, denote by \( \tilde{T}^{(n)} \) the set of \( \tilde{\mathcal{F}}^{(n)} \)-stopping times on this enlarged probability space that only take values on the grid points \( \{0 = T^{(n)}_0, T^{(n)}_1, \ldots\} \), that is

\[
\tilde{T}^{(n)} := \{ \tau \mid \tau \text{ is an } \tilde{\mathcal{F}}^{(n)} \text{-stopping time } \& \tau \in \{0 = T^{(n)}_0, T^{(n)}_1, \ldots\} \}.
\]

Now consider modifying the original optimal stopping problem (6.1.1) in three steps. First, replace the deterministic expiry date \( T \) by the random variable \( T^{(n)}_k \) (for suitably chosen \( k \)). Second, replace the set of stopping times \( T \) over which is optimised by \( \tilde{T}^{(n)} \). Finally, replace the discount factor \( e^{-r \tau} \) by \( D^{(n)}(\tau) \) defined as

\[
D^{(n)}(\tau) := \sum_{i=0}^{\infty} 1_{\{\tau = T^{(n)}_i\}} e^{-r_i/n}. \quad (6.2.2)
\]

Together this amounts to defining for each \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \):

\[
V^{(n)}_k(x) := \sup_{\tau \in \tilde{T}^{(n)}} \mathbb{E}_x \left[ D^{(n)}(\tau \wedge T^{(n)}_k) f(X_{\tau \wedge T^{(n)}_k}) \right]. \quad (6.2.3)
\]

The usefulness of this setup relies on the following two facts. For any expiry date \( T \geq 0 \), choosing a sequence \((k(n))_{n \geq 1}\) such that \( k(n)/n \to T \) as \( n \to \infty \) we have for any \( x \in \mathbb{R} \) that \( V^{(n)}_{k(n)}(x) \to v(T, x) \) as \( n \to \infty \) (cf. Theorem 6.3.1 (ii)). Furthermore, for any \( n \in \mathbb{N} \) the sequence of functions \((V^{(n)}_k)_{k \geq 0}\) satisfies the following recursion:
$V_0^{(n)}(x) = f(x), \quad V_k^{(n)}(x) = \max \left\{ f(x), e^{-r/n} \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{\xi(n)}) \right] \right\}$ for $k \geq 1, \quad (6.2.4)$

where $\xi^{(n)}$ is an exponentially distributed random variable with mean $1/n$, independent of $X$. This is easily seen by a dynamic programming argument. Indeed, consider (6.2.3) for some $x \in \mathbb{R}$. The available stopping times allow to stop either immediately or at $T_1^{(n)}$ or later. Stopping immediately yields a payoff of $f(x)$ while the (current) value of waiting until $T_1^{(n)}$ or later equals the discounted expected value of the option with expiry date $T_{k-1}^{(n)}$ for $x = X_{T_1^{(n)}}$. Hence $V_k^{(n)}(x)$ equals the maximum of the two.

The main point of this setup is that the recursion (6.2.4) is very straightforward. Indeed, as already alluded to in Section 2.2, for any meromorphic Lévy process the law of $X_{\xi^{(n)}}$ is a mixture of exponentials. As a consequence (6.2.4) can be worked out explicitly for example if $f(x) = (K - e^x)^+$, where first the expectation is computed explicitly and next the maximum is easily determined, cf. Section 6.4.

Now let us for comparison recall the original Canadisation algorithm introduced by Carr. With $T_k^{(n)}$ as defined in (2.3.1), define for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\widehat{W}_k^{(n)}(x) = \sup_{\tau} \mathbb{E}_x \left[ e^{-r(\tau \wedge T_k^{(n)})} f(X_{\tau \wedge T_k^{(n)}}) \right],$$

where the supremum is taken over all $\tilde{F}^{(n)}$-stopping times. It can be shown (cf. also [18]) that this yields the following recursion:

$$\widehat{W}_k^{(n)}(x) = \sup_{\tau \in T} \mathbb{E}_x \left[ e^{-(r+n)\tau} f(X_{\tau}) + e^{-r\xi^{(n)}} \widehat{W}_{k-1}^{(n)}(X_{\xi^{(n)}}) \right]$$

To work out this recursion, general theory of optimal stopping (cf. e.g. [55]) dictates that the optimal stopping time in (6.2.5) is the first time that $X$ enters a subset $S$ of the state space $\mathbb{R}$, say $\tau_S$. Hence, in order to find an explicit expression for (6.2.5) one requirement is to compute the mean of $e^{-(r+n)\tau_S} f(X_{\tau_S})$. Carr worked with Brownian motion, in which case closed form expressions are available for the first exit time from an interval. These expressions then allow to compute the expectation in (6.2.5) for a candidate optimal stopping time, and standard arguments from the theory of optimal stopping allow then to verify whether the candidate is indeed optimal.
When we allow $X$ to have jumps this problem becomes much more involved. Even though for meromorphic Lévy processes expressions for the first exit time from an interval are in principle available (cf. Theorem 3 and Theorem 5 in [39]), these are quite complicated and implementing them would be very involved. Furthermore having such expression is not the whole story in this case. Indeed, suppose that it is guessed that the optimal stopping region $S$ consists of a union of intervals, say $S = [a, b] \cup [c, d]$ for some $a < b < c < d$. As a consequence of the presence of jumps then typically none of the three functions $\tilde{V}_k^{(n)}|_{(-\infty, a)}$, $\tilde{V}_k^{(n)}|(b, c)$ and $\tilde{V}_k^{(n)}|(d, \infty)$ can be determined in isolation. Indeed, if $X$ starts in $(-\infty, a)$ it might very well visit one or both of the intervals $(b, c)$ and $(d, \infty)$ before entering $S$ for the first time. Hence it is not possible to determine the optimal choice of $a$ in isolation: all four optimal points have to be determined simultaneously, leading to a typically very complicated optimisation problem.

Clearly the problem becomes yet more complicated if it is a priori not clear what a sensible guess for the shape of $S$ is.

In contrast, the recursion we propose in this chapter, namely (6.2.4), is of a much more straightforward nature. In the classic example of an American put option, i.e. $f(x) = (K - e^x)^+$, it is straightforward to show (cf. Proposition 6.4.1) that $S = (-\infty, x^*]$ for some $x^*$ (depending on $n$ and $k$) so that (6.2.4) yields

$$V_k^{(n)}(x) = \begin{cases} f(x) & \text{if } x \leq x^* \\ e^{-r/n}E_x[V_k^{(n)}(X_{\xi(n)})] & \text{if } x > x^*. \end{cases} \tag{6.2.6}$$

Working out this recursion (in $k$) and determining the $x^*$'s is easily possible (though it does require an amount of algebra) for any meromorphic Lévy process.

If the payoff function $f$ is such that it does not (easily) allow to simplify (6.2.4) to an analogue of (6.2.6) a possible way forward is as follows. First determine an explicit formula for

$$x \mapsto e^{-r/n}E_x[V_k^{(n)}(X_{\xi(n)})], \tag{6.2.7}$$

so that $V_k^{(n)}(x)$ is the maximum of (6.2.7) and $f(x)$. With formulae available for both (6.2.7) and $f$, determining the maximum of the two is rather straightforward to implement using a computer. See Remark 6.4.3 (ii) for more details, in particular also
concerning an explicit formula for (6.2.7).

**Remark 6.2.1.** A natural question is whether it would not be better to use a classic deterministic grid rather than a stochastic grid. That is to say, setting \( t_k = kT/n \) for \( k = 0, 1, \ldots, n \) one could consider approximating \( v \) by narrowing the set of stopping times over which is optimised to the set of stopping times taking values in \( \{t_0, t_1, \ldots, t_n\} \) only. Denoting the resulting value function with expiry date \( t_k \) by \( Z_k^{(n)} \), the sequence of functions \( (Z_k^{(n)})_{k=0,1,\ldots,n} \) is determined by the recursion

\[
Z_0^{(n)}(x) = f(x), \quad Z_k^{(n)}(x) = \max \left\{ f(x), e^{-r/n}E_x \left[ Z_{k-1}^{(n)}(X_{1/n}) \right] \right\} \quad \text{for } k = 1, \ldots, n.
\]

(6.2.8)

The problem is that this recursion involves the law of \( X_{1/n} \). Even if this is known (which generally is not the case except for a few notable exceptions) it is typically not of a friendly enough nature to work out the recursion (6.2.8) explicitly. Indeed consider for instance a Brownian motion.

### 6.3 Main result & proof

This section is dedicated to making the algorithm outlined in the above Section 6.2 rigorous.

**Theorem 6.3.1.** Let \( f \) be a bounded and continuous function. Let the value function \( v \) be given by (6.1.1). For each \( n \geq 1 \) and \( k \geq 0 \) let the function \( V_k^{(n)} \) be given by (6.2.3). We have the following.

(i) For any \( n \geq 1 \), the sequence of functions \( (V_k^{(n)})_{k \geq 0} \) satisfies the following recursion:

\[
V_0^{(n)}(x) = f(x), \quad V_k^{(n)}(x) = \max \left\{ f(x), e^{-r/n}E_x \left[ V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] \right\} \quad \text{for } k \geq 1,
\]

(6.3.1)

where \( \xi^{(n)} \) is independent of \( X \) and follows an exponential distribution with mean \( 1/n \).
(ii) For any $T \in [0, \infty)$, if $(k(n))_{n\geq 1}$ is a sequence such that $k(n)/n \to T$ as $n \to \infty$ then:

$$T_{k(n)}^{(n)} \to T \ a.s. \quad \text{and} \quad V_{k(n)}^{(n)}(x) \to v(T, x) \ \text{for all} \ x \in \mathbb{R}.$$ 

Proof. Ad (i). This is just the dynamic programming principle applied to (6.2.3). Indeed for any $k \geq 1$

$$V_{k(n)}^{(n)}(x) = \sup_{\tau \in \mathcal{T}(n)} \mathbb{E}_x \left[ \left( 1_{\{\tau = 0\}} + 1_{\{\tau \geq T_{k(n)}^{(n)}\}} \right) D^{(n)}(\tau \land T_{k(n)}^{(n)}) f(X_{\tau \land T_{k(n)}^{(n)}}) \right]$$

$$= \sup_{\tau \in \mathcal{T}(n)} \mathbb{E}_x \left[ 1_{\{\tau = 0\}} f(x) + 1_{\{\tau \geq T_1^{(n)}\}} \mathbb{E}_x \left[ D^{(n)}(\tau \land T_{k(n)}^{(n)}) f(X_{\tau \land T_{k(n)}^{(n)}}) \middle| \mathcal{F}_{T_{k(n)}^{(n)}} \right] \right]$$

$$= \sup_{\tau \in \mathcal{T}(n)} \mathbb{E}_x \left[ 1_{\{\tau = 0\}} f(x) + 1_{\{\tau \geq T_1^{(n)}\}} e^{-r/n} \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{\tau}) \right] \right]$$

$$= \max \left\{ f(x), e^{-r/n} \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{\tau}) \right] \right\}.$$ 

Ad (ii). $T_{k(n)}^{(n)} \to T \ a.s. \ as \ n \to \infty$ is just an application of the law of the large numbers. For proving that $V_{k(n)}^{(n)}(x) \to v(T, x)$ as $n \to \infty$ for any $x \in \mathbb{R}$ we have to do a bit more work. In this proof we will also need an auxiliary sequence of functions that is generated by our algorithm if we leave out the step of adjusting the discounting factor, that is, let for $n \geq 1, k \geq 0$ the function $\hat{V}_{k(n)}^{(n)}$ be given by (compare with (6.2.3)):

$$\hat{V}_{k(n)}^{(n)}(x) = \sup_{\tau \in \mathcal{T}(n)} \mathbb{E}_x \left[ e^{-r(\tau \land T_{k(n)}^{(n)})} f(X_{\tau \land T_{k(n)}^{(n)}}) \right]. \quad (6.3.2)$$

Take any $x \in \mathbb{R}$. The result follows from the following three steps.

Step 1:

$$\limsup_{n \to \infty} V_{k(n)}^{(n)}(x) \leq v(T, x). \quad (6.3.3)$$

Step 2:

$$\liminf_{n \to \infty} \hat{V}_{k(n)}^{(n)}(x) \geq v(T, x). \quad (6.3.4)$$

Step 3:

$$\lim_{n \to \infty} \left| \hat{V}_{k(n)}^{(n)}(x) - V_{k(n)}^{(n)}(x) \right| = 0. \quad (6.3.5)$$

Step 1. We appeal to Carr’s Canadisation as outlined in Section 6.2, however we follow the setup from [18]. Define for $n \geq 1$ the sequence of functions $(U_{k(n)}^{(n)})_{k \geq 0}$
as follows (recall that $T$ is the set of stopping times with respect to the completed filtration generated by $X$):

$$U_0^{(n)}(x) = f(x), \quad U_k^{(n)}(x) = \sup_{\tau \in T} \mathbb{E}_x \left[ e^{-(r+n)\tau} f(X_\tau) + n \int_0^\tau e^{-(r+n)u} U_{k-1}^{(n)}(X_u) \, du \right] \text{ for } k \geq 1.$$  

(6.3.6)

It is shown in [18] (see in particular Section 4) that

$$U_k^{(n)}(x) \to v(T, x) \quad \text{as } n \to \infty.$$  

(6.3.7)

By a dynamic programming argument the sequence $(\hat{V}_k^{(n)})_{k \geq 0}$ as defined in (6.3.2) satisfies the recursion

$$\hat{V}_0^{(n)}(x) = f(x), \quad \hat{V}_k^{(n)}(x) = \max \left\{ f(x), \mathbb{E}_x \left[ e^{-r\xi^{(n)}(n)} \hat{V}_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] \right\} \quad \text{for } k \geq 1,$$

where $\xi^{(n)}$ is an exponentially distributed random variable with mean $1/n$, independent of $X$. Comparing this recursion with (6.3.6), we see that $\hat{V}_k^{(n)} \leq U_k^{(n)}$, since in (6.3.6) the supremum is at least as large as taking either $\tau = 0$ or $\tau = \infty$.

Since $\hat{V}_0^{(n)} = U_0^{(n)} = f$ it follows that $\hat{V}_k^{(n)} \leq U_k^{(n)}$ for all $k, n$, which together with (6.3.7) indeed yields (6.3.3).

**Step 2.** Let $\varepsilon > 0$ be arbitrary and let us show that

$$\liminf_{n \to \infty} \hat{V}_k^{(n)}(x) \geq v(T, x) - \varepsilon.$$  

(6.3.8)

For this, first note that since $X$ is right-continuous there exists a $\tau'$ taking values in $0 = t_0 < t_1 < \ldots < t_N = T$ such that

$$\mathbb{E}_x \left[ e^{-r\tau'} f(X_{\tau'}) \right] \geq v(T, x) - \varepsilon.$$  

(6.3.9)

Define an approximating sequence of stopping times for $\tau'$ by setting for any $n \geq 1$

$$\tilde{T}^{(n)} \ni \sigma^{(n)} := \inf \left\{ T_k^{(n)} \mid T_k^{(n)} \geq \tau' \right\}.$$  

Now we claim that as $n \to \infty$:  

$$\mathbb{E}_x \left[ e^{-r\sigma^{(n)}} f(X_{\sigma^{(n)}}) \right] \to \mathbb{E}_x \left[ e^{-r\tau'} f(X_{\tau'}) \right]$$  

(6.3.10)
and

\[ \mathbb{E}_x \left[ e^{-r(x(n))} f(X_{\sigma(n)}) - e^{-r(\sigma(n),T_{k(n)})} f(X_{\sigma(n) \wedge T_{k(n)}}) \right] \to 0. \tag{6.3.11} \]

Using this we indeed arrive at (6.3.8):

\[ \lim_{n \to \infty} \hat{V}^{(n)}(x) \geq \lim_{n \to \infty} \mathbb{E}_x \left[ e^{-r(\sigma(n),\tau')} f(X_{\tau'}) \right] \]

\[ = \mathbb{E}_x \left[ e^{-r t_i} f(X_{t_i}) \right] \]

\[ \geq v(T, x) - \varepsilon, \tag{6.3.10} \]

so let us prove (6.3.10) & (6.3.11).

Ad (6.3.10): By construction of \( \tau' \) and \( \sigma^{(n)} \) we have

\[ \mathbb{E}_x \left[ e^{-r t_i} f(X_{t_i}) \right] - \mathbb{E}_x \left[ e^{-r \sigma^{(n)}} f(X_{\sigma^{(n)}}) \right] = \sum_{i=0}^{N} \mathbb{E}_x \left[ 1_{\{\tau' = t_i\}} \left( e^{-r t_i} f(X_{t_i}) - e^{-r \sigma^{(n)}_{t_i}} f(X_{\sigma^{(n)}_{t_i}}) \right) \right], \]

where

\[ \sigma^{(n)}_{t_i} := \inf \left\{ T_{k(n)} \mid T_{k(n)} \geq t_i \right\}, \tag{6.3.12} \]

so it is enough to show that for each \( i = 0, \ldots, N \)

\[ e^{-r \sigma^{(n)}_{t_i}} f(X_{\sigma^{(n)}_{t_i}}) \stackrel{L^1}{\longrightarrow} e^{-r t_i} f(X_{t_i}) \quad \text{as} \ n \to \infty. \tag{6.3.13} \]

Since \( f \) is continuous and bounded, say by \( M \), we have

\[ \left| e^{-r \sigma^{(n)}_{t_i}} f(X_{\sigma^{(n)}_{t_i}}) - e^{-r t_i} f(X_{t_i}) \right| \leq \left| e^{-r \sigma^{(n)}_{t_i}} f(X_{\sigma^{(n)}_{t_i}}) - e^{-r t_i} f(X_{\sigma^{(n)}_{t_i}}) \right| + \left| e^{-r t_i} f(X_{\sigma^{(n)}_{t_i}}) - e^{-r t_i} f(X_{t_i}) \right| \]

\[ \leq M \left( e^{-r \sigma^{(n)}_{t_i}} - e^{-r t_i} \right) + e^{-r t_i} \left| f(X_{\sigma^{(n)}_{t_i}}) - f(X_{t_i}) \right| \]

so that by uniform integrability and the continuous mapping theorem we have that (6.3.13) follows from

\[ \sigma^{(n)}_{t_i} \stackrel{P_x}{\to} t_i \quad \text{and} \quad X_{\sigma^{(n)}_{t_i}} \stackrel{P_x}{\to} X_{t_i} \quad \text{as} \ n \to \infty. \tag{6.3.14} \]
Now, recalling (6.3.12) and that $T_{k(n)}$ is a sum of i.i.d. exponentials with parameter $n$, due to the lack of memory property $\sigma_i(n) - t_i$ is equal in distribution to an exponentially distributed random variable with mean $1/n$, independent of $X$, say $\xi(n)$. So the first part of (6.3.14) is obvious and the second part is a direct consequence of stochastic continuity of $X$.

Ad (6.3.11): as above, it suffices to show

$$\sigma(n) - (\sigma(n) \wedge T_{k(n)}) \xrightarrow{P_x} 0 \quad \text{and} \quad X_{\sigma(n)} - X_{\sigma(n) \wedge T_{k(n)}} \xrightarrow{P_x} 0 \quad \text{as } n \to \infty. \quad (6.3.15)$$

For the first one, for any $\delta > 0$ we have

$$\begin{align*}
\mathbb{P}_x \left( \left| \sigma(n) - (\sigma(n) \wedge T_{k(n)}) \right| \geq \delta \right) &= \mathbb{P}_x \left( \sigma(n) \geq T_{k(n)} + \delta \right) \\
&= \mathbb{P}_x \left( \sigma(n) \geq T_{k(n)} + \delta \quad \& \quad T_{k(n)} \leq T - \delta/2 \right) \\
&\quad + \mathbb{P}_x \left( \sigma(n) \geq T_{k(n)} + \delta \quad \& \quad T_{k(n)} > T - \delta/2 \right) \\
&\leq \mathbb{P}_x \left( T_{k(n)} \leq T - \delta/2 \right) + \mathbb{P}_x \left( \sigma(n) \geq T + \delta/2 \right).
\end{align*}$$

The first probability on the last line vanishes as $n \to \infty$ on account of $T_{k(n)} \to T$ a.s. To see that the second probability vanishes as well, note that by construction $\tau' \leq T$ so that by definition of $\sigma(n)$ the difference $\sigma(n) - T$ is bounded above by $\inf\{T_k(n) \mid T_k(n) \geq T\} - T$, but again by the lack of memory property this random variable follows an exponential distribution with mean $1/n$.

Finally, the second statement in (6.3.15) follows as above from the first statement and stochastic continuity of $X$.

**Step 3.** Some standard manipulations, using Hölder’s inequality and the boundedness of $f$, show that the definitions (6.3.2) and (6.2.3) yield:

$$\begin{align*}
\left| \hat{V}^{(n)}_{k(n)}(x) - V^{(n)}_{k(n)}(x) \right| &\leq M \sup_{\tau \in \mathcal{T}(n)} \sum_{i=0}^{k(n)} \mathbb{E}_x \left[ 1_{\{\tau = T_i^{(n)}\}} \left| e^{-r_i/n} - e^{-rT_i^{(n)}} \right| \right] \\
&\leq M \sup_{\tau \in \mathcal{T}(n)} \sum_{i=0}^{k(n)} \mathbb{E}_x \left[ 1_{\{\tau = T_i^{(n)}\}} \right]^{1/2} \mathbb{E}_x \left[ \left( e^{-r_i/n} - e^{-rT_i^{(n)}} \right)^2 \right]^{1/2}. \quad (6.3.16)
\end{align*}$$

Recalling that $T_i^{(n)}$ is the sum of $i$ i.i.d. exponentials with mean $1/n$, it is straightforward to compute
\[ \mathbb{E}_x \left[ \left( e^{-ri/n} - e^{-rT_i^{(n)}} \right)^2 \right] = \frac{n^i}{(n + 2r)^i} - 2e^{-ri/n} \frac{n^i}{(n + r)^i} + e^{-2ri/n}. \]

As this expression is increasing in \( i \) the supremum in (6.3.16) above is attained by \( \tau = T_{k(n)}^{(n)} \). Plugging this in it is clear from the vanishing variance of \( T_{k(n)}^{(n)} \) (or directly from the above display) that (6.3.16) indeed vanishes as \( n \to \infty \).

\[ \square \]

**Remark 6.3.2.** The assumption in the above Theorem 6.3.1 that \( f \) is bounded and continuous keeps the proof compact. However the boundedness can be weakened to a more liberal integrability condition, and continuity can be weakened to (at least) a discrete set of discontinuities — provided compound Poisson processes are excluded. Details are left to the reader.

### 6.4 Example: the American put driven by a meromorphic Lévy process

In this section we work out in detail our algorithm for an American put option driven by a meromorphic Lévy process. This is probably the most classic example of an option in a Black & Scholes type financial market. If the option has strike price \( K > 0 \) and the price \( S \) of the risky asset on which the option is written is assumed to evolve as \( S_t = \exp(X_t) \) for \( t \geq 0 \) then the payoff function is given by

\[ f(x) = (K - e^x)^+. \] (6.4.1)

Let us now return to working out our algorithm in the case of the American put. We start by noting that for the payoff function (6.4.1) it is straightforward to derive the following simplification of the recursion for the functions \( (V_k^{(n)})_{k \geq 0} \) stated in Theorem 6.3.1 (i).

**Proposition 6.4.1.** Suppose that \( \mathbb{P}(X_1 < 0) > 0 \), i.e. \( X \) is not a subordinator. Let \( n \geq 1 \). There exists a decreasing sequence of points \( (\bar{x}_k^{(n)})_{k \geq 0} \), with \( \bar{x}_0^{(n)} := \log K \), such that for each \( k \geq 1 \) we have:

(i) the point \( \bar{x}_k^{(n)} \) is the unique solution on \( (-\infty, \bar{x}_{k-1}^{(n)}) \) of the equation in \( z \)
\[ e^{-r/n} \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] - f(x) = 0 \]

(ii) \( V_k^{(n)} \) is given by

\[
V_k^{(n)}(x) = \begin{cases} 
    f(x) & \text{if } x \leq \bar{x}_k^{(n)} \\
    e^{-r/n} \mathbb{E}_x \left[ V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] & \text{if } x > \bar{x}_k^{(n)}
\end{cases}
\]

(iii) \( V_k^{(n)} > V_{k-1}^{(n)} \) on \( (\bar{x}_k^{(n)}, \infty) \).

\textbf{Proof.} The proof is by induction. First we consider \( k = 1 \). Denote for \( s > 0 \)
\[ F_1(s) = e^{-r/n} \mathbb{E} \left[ \left( K - se^X_{\xi^{(n)}} \right)^+ \right] - (K - s)^+ \]
(note that this is just the left hand side of the equation in (i) after the substitution \( s = e^x \)). Then \( F_1(0+) = e^{-r/n} K - K < 0 \) and \( F_1(\exp(\bar{x}_0^{(n)})) = F_1(K) > 0 \) as \( X \) not being a subordinator implies that \( \mathbb{P}(X_{\xi^{(n)}} < 0) > 0 \). Furthermore, as \( s \mapsto (K - s\alpha)^+ \) is convex for any \( \alpha \) the first term in the formula of \( F_1 \) is also convex and thus \( F_1 \) has a unique zero on \( (0, K] \). Combining these observations with Theorem 6.3.1 (i) yields (i) and (ii) above. This also shows that \( V_1^{(n)}(x) > (K - e^x)^+ \) as \( X \) not being a subordinator implies \( \mathbb{P}_x(X_{\xi^{(n)}} < \log K) > 0 \), so (iii) is also established.

Now suppose that (i)-(iii) above hold for \( k-1 \) (IH (i), IH (ii) and IH (iii)). Analogue to above, denote for \( s > 0 \) the left hand side of the equation in (i) after the substitution \( s = e^x \):

\[
F_k(s) = e^{-r/n} \mathbb{E} \left[ V_{k-1}^{(n)}(\log s + X_{\xi^{(n)}}) \right] - f(\log s).
\]

By IH (ii) we again have \( F_k(0+) = e^{-r/n} K - K < 0 \). Also

\[
f(\bar{x}_{k-1}^{(n)}) = e^{-r/n} \mathbb{E}_{\bar{x}_{k-1}^{(n)}} \left[ V_{k-2}^{(n)}(X_{\xi^{(n)}}) \right] < e^{-r/n} \mathbb{E}_{\bar{x}_{k-1}^{(n)}} \left[ V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right],
\]

the equality by IH (i) and the inequality by IH (iii) together with \( X \) not being a subordinator. This translates to \( F_k(\exp(\bar{x}_{k-1}^{(n)})) > 0 \). As above, the first term in the formula of \( F_k \) is convex since \( s \mapsto V_{k-1}^{(n)}(\log s + \alpha) \) is convex for any \( \alpha \), as is clear
from the definition (6.2.3). Consequently $F_k$ has a unique zero on $(0, K]$, located in $(-\infty, \bar{x}^{(n)}_{k-1})$, and in combination with Theorem 6.3.1 (i), statements (i) and (ii) follow.

It remains to show (iii). It follows from the above and IH (ii) that $V_k^{(n)}(x) > V_{k-1}^{(n)}(x) = (K - e^x)$ on $(\bar{x}^{(n)}_k, \bar{x}^{(n)}_{k-1}]$. For $x > \bar{x}^{(n)}_{k-1}$ we have by (ii), IH (ii), IH (iii) and $X$ not being a subordinator that

$$V_k^{(n)}(x) = e^{-r/n}E_x \left[ V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] > e^{-r/n}E_x \left[ V_{k-2}^{(n)}(X_{\xi^{(n)}}) \right] = V_{k-1}^{(n)}(x)$$

and we are done. \(\square\)

Note that the above result shows that in the approximating value functions it is optimal to stop as soon as $X$ falls below a certain level. Or, equivalently, as soon as the stock price $S = \exp(X)$ falls below a certain level. This is structurally consistent with the exact optimal exercise strategy in the original problem, which is to stop as soon as $S$ falls below a certain time dependent boundary (cf. e.g. [33]).

If $X$ satisfies (2.2.7), i.e. in particular if $X$ is a meromorphic Lévy process, then it turns out the expectations in the above Proposition 6.4.1 can be worked out explicitly. The proof uses induction over $k$ and consists of tedious yet straightforward computations, it is therefore omitted.

**Proposition 6.4.2.** Suppose that $X$ satisfies (2.2.7) and is not compound Poisson. Fix some $n \geq 1$. For any $k \geq 0$ the function $V_k^{(n)}$ can be expressed in piecewise form as follows:

$$V_k^{(n)}(x) = \begin{cases} 1_{\{x \geq x_0^{(n)}\}} \sum_{j=0}^{N_+} e^{-\zeta^{(n)}(j)x} \sum_{i=0}^{k-1} A_+^{(n)}(i, j, 0, k) x^i \\
\quad + \sum_{m=1}^{k} 1_{\{x \in [\bar{x}_m^{(n)}, \bar{x}_{m-1}^{(n)}]\}} \left( \sum_{j=0}^{N_+} e^{\zeta^{(n)}(j)x} \sum_{i=0}^{k-m} A_+^{(n)}(i, j, m, k) x^i \\
\quad + \sum_{j=0}^{N_-} e^{-\zeta^{(n)}(j)x} \sum_{i=0}^{k-m} A_-^{(n)}(i, j, m, k) x^i - B^{(n)}(m, k) e^x + C^{(n)}(m, k) \right) \\
\quad + 1_{\{x < \bar{x}_k^{(n)}\}}(K - e^x), \end{cases}$$

where $x_0^{(n)} = \log K$. Expressions for $\bar{x}_k^{(n)}$ and all the coefficients involved can be found in Appendix C.
Remark 6.4.3. (i) Note that in the Brownian motion case, the structure of the formulae for the \( V^{(n)}_k \)'s from Proposition 6.4.2 matches the corresponding formulae found by Carr with his Canadisation method (cf. Section 6.2). That is, the formulae coincide except for the values of the coefficients \( A, B \) and \( C \), and for the values of the boundary points \( \bar{x} \).

(ii) Any payoff function \( f \) that can (piecewise) be expressed as a linear combination of functions of the form \( Ax^i e^{Bx} + C \) for \( A, B, C \in \mathbb{R} \) and \( i \in \mathbb{N} \) can be dealt with in similar fashion and yields a similar structure as (6.4.2). If a result as in Proposition 6.4.1 is not available, i.e. if it is a priori not clear how the maximum in the recursive relation (6.3.1) works out, a way forward is to implement a dynamic decision rule where first a formula for

\[
F_k(x) = e^{-r/n} \mathbb{E}_x \left[ V^{(n)}_{k-1}(X_{\xi(n)}) \right]
\]

is computed (which is still of the same structure as in (6.4.2)) and then \( V_k = \max\{F_k, f\} \) can be determined in a straightforward way.

Finally, a payoff function that is not of this convenient form mentioned above can easily be approximated by a function which is of this convenient form, and it is straightforward to estimate the error due to this approximation.

(iii) It would be possible to further extend this approach to deal with more complicated payoffs involving path dependency, for instance to cases where the payoff at time \( t \) depends also on the running supremum \( \overline{X}_t = \sup_{s \leq t} X_s \) as is the case for ‘lookback options’. A step in the algorithm would then not only require \( X_{\xi(n)} \) but the pair \( (X_{\xi(n)}, \overline{X}_{\xi(n)}) \) rather. Explicit expressions for the law of this pair is available for meromorphic Lévy processes.

6.5 Some numerics for the American put

In this section we discuss some numerics for the American put, obtained by a computer implementation of the result in Proposition 6.4.2. In particular we present some graphs of approximating value functions \( V^{(n)}_k \) and some graphs of the approximative optimal
exercise boundary. For the latter, as is well known (cf. e.g. [33]) the optimal stopping time $\tau^*$ in (6.1.1) with $f(x) = (K - e^x)^+$ is of the form

$$
\tau^* = \inf\{u \geq 0 \mid v(T - u, X_u) = f(X_u)\} = \inf\{u \geq 0 \mid X_u \leq b(T - u)\}
$$

(6.5.1)

for some non-increasing function $b$ on $\mathbb{R}_{>0}$ referred to as the optimal exercise boundary, which may hence be defined as

$$
b(t) := \sup\{y \in \mathbb{R} \mid v(t, y) = f(y)\} \leq \log(K) \quad \text{for } t > 0.
$$

Hence, since for large $n$ we have $V_k^{(n)}(\cdot) \approx v(k/n, \cdot)$ for all $k \geq 1$ and recalling the result of Proposition 6.4.1 it should be expected that the set of points $\{(k/n, x_k^{(n)})\}_{k \geq 1}$, in the captions of the plots referred to as ‘boundary points’, forms an approximation of the set $\{(k/n, b(k/n))\}_{k \geq 1}$.

Next, we briefly describe the two main tasks for creating the graphs of different value functions and boundary points presented further below.

The first step is the calculation of the solutions $\zeta^{(n)}_+(i)$ and $\zeta^{(n)}_-(j)$ of $z \mapsto n+\Psi(-iz)$ and of the coefficients $d^{(n)}_+(i)$ and $d^{(n)}_-(j)$ by using Proposition 2.2.9 for $i \in \{0, \ldots, N_+\}$ and $j \in \{0, \ldots, N_-\}$. In case a meromorphic Lévy process is used for which $N_+ = \infty$ or $N_- = \infty$ the infinite sums need to be truncated. For this we used the ad-hoc approach of ensuring that the truncated sums represent at least 99% of the mass of $X_{\xi^{(n)}}$, i.e. (compare with (2.2.7)) we determined $M^+, M^-$ such that

$$
\sum_{j=0}^{M^+} d^{(n)}_+(j) \frac{\zeta_+(j)}{\zeta_-(j)} - \sum_{j=0}^{M^-} d^{(n)}_-(j) \geq 0.99
$$

and then normalised the coefficients. It is worth noting that $M$ is larger in the case of meromorphic Lévy process with finite variation than in the cases with infinite variation. Particularly, if $\sigma > 0$ the largest part of the probability mass is concentrated on $\zeta_+(0)$ and $\zeta_-(0)$ whereas the other zeroes contribute only a small part.

The second main step is the recursive computation of the value function and the boundary points using formula (6.4.2). A program for this purpose was written in the

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†We do not provide a rigorous proof of this, which seems slightly tricky due to the fact that the principle of smooth pasting dictates that at least when the driving Lévy process has a Gaussian part $v$ ‘pastes smoothly’ onto $f$, i.e. their first derivatives in the space component coincide. Consequently it is not a straightforward application of the Implicit function theorem.
language C++. We found that in certain cases the algorithm may need more than
the default ‘double precision’ as working precision in order to have enough significant
digits left in the end result. For this we made use of the ‘GMP/MPFR/MPFRC++’
packages. Furthermore we made use of ‘OpenMP’ to parallelise the operations.

The Lévy processes used in this section are Brownian motion with drift, Brownian
motion with drift plus exponentially distributed jumps (Kou model) and the β-class,
see Section 6.4. The Figures 6.1 and 6.2 contain different value functions and its
boundary points for processes of the β-class. We compare different values of $\lambda_1 = \lambda_2$
which are responsible for the small jump behaviour. One process is similar to a CGMY
process ($\lambda_1 = 1.8$, finite variation and infinite activity, see [25]). The other processes
have finite variation and finite activity ($\lambda_1 = 0.95$) respectively infinite variation and
infinite activity ($\lambda_1 = 2.1$).
Figure 6.1: Plots of the approximating value function $V^{(n)}_k$ where the driving process $X$ is of the $\beta$-class with parameter values $\sigma = 0, \alpha_1 = 10, \alpha_2 = 100, \beta_1 = \beta_2 = 0.5, c_1 = c_2 = 100$ and $\lambda_1 = \lambda_2 = 0.95, 1.8, 2.1$. The drift $\mu$ is chosen as the solution of $\Psi(-I) = -r$. Furthermore $n = 100, k = 20, r = 0.05, K = 10000$ and $M^+ = 31, M^- = 61$. 
In Figures 6.3 and 6.4 the value functions and boundary points for a process of the \( \beta \)-class with infinite variation and infinite activity are plotted for different values of the Gaussian coefficient \( \sigma \). If we compare over certain intervals \((\bar{x}_i^{(q)}, \bar{x}_j^{(q)})\), \(i > j\) the curves of the value functions and boundary points for different \(\sigma_1\) and \(\sigma_2\), \(\sigma_1, \sigma_2 \in \{0, 1, 2, 6\}\) we observe the following. If the curve of the value function for \(\sigma_1\) is steeper on \((\bar{x}_i^{(q)}, \bar{x}_j^{(q)})\) than the one for a different \(\sigma_2\), then the curve of the boundary points for \(\sigma_1\) appears to be more flat on that interval and vice versa. This is in accordance to general optimal stopping theory. Note that for \(\sigma = 0\) there seems to be a jump in the boundary at 0, that is Figure 6.4 indicates that \(b(0^+) < \log K\) if \(\sigma = 0\) while \(b(0^+) = \log K\) if \(\sigma > 0\). This is consistent with the observation in [43].
Figure 6.3: Plots of the approximating value function $V_k^{(n)}$ where the driving process $X$ is of the $\beta$-class with infinite variation and infinite activity. Parameter values are $\sigma = 0, 1, 2, 6, \alpha_1 = 56, \alpha_2 = 56.4, \beta_1 = \beta_2 = 2, c_1 = c_2 = 1.5$ and $\lambda_1 = \lambda_2 = 2.8$. Furthermore $n = 500, k = 100, r = 0.05, K = 10000$ and $M^- = M^+ = 1$. 
Figure 6.4: Four sets of boundary points obtained using the same parameter values and settings as in Figure 6.3.

Figure 6.5 illustrates the convergence. Hereby we compare the relative difference between the value functions for several $n$ and $k$ with a constant ratio $k/n = 0.2$. 
The plots are ‘zoomed in’ to the part of the x-axis where the difference is largest. The case $n = 800$ and $k = 160$ is taken as the base. The underlying $\beta$-class process with infinite variation and infinite activity has the following parameters: $\sigma = 1, \alpha_1 = 1, \alpha_2 = 1, \beta_1 = \beta_2 = 80, c_1 = c_2 = 1$ and $\lambda_1 = \lambda_2 = 2.3$. The drift $\mu$ is chosen as the solution of $\Psi(-I) = -r$. Furthermore $r = 0.05, K = 10000$ and $M^+ = M^- = 1$.

As is well known (see e.g. [2] and the references therein), optimal stopping problems driven by Lévy processes can exhibit both smooth fit and continuous fit. The former means that the value function $v$ connects in a ‘smooth’ way with the payoff function $f$ at the exercise boundary, i.e. that it holds for all $T > 0$ that

$$v(T, b(T)+) = f(b(T)) \quad \text{and} \quad \frac{\partial v}{\partial x}(T, b(T)+) = f'(b(T))$$
while the former means that at most a continuous connection is guaranteed, i.e. that it holds for all \( T > 0 \) that

\[
v(T, b(T)^+) = f(b(T)), \text{ but not necessarily } \frac{\partial v}{\partial x}(T, b(T)^+) = f'(b(T)).
\]

Figure 6.6 illustrates that smooth fit appears for an underlying \( \beta \)-process with \( \sigma = 0 \) but with infinite variation and infinite activity.

Figure 6.6: Plots of the approximating value function \( V_{k}^{(n)} \) and the payoff function \( f(x) = (K - \exp(x))^+ \) where the driving process \( X \) is of the \( \beta \)-class with infinite variation and infinite activity. Parameter values are \( \sigma = 0, \alpha_1 = 56, \alpha_2 = 56.4, \beta_1 = \beta_2 = 2, c_1 = c_2 = 1.5 \) and \( \lambda_1 = \lambda_2 = 2.8 \). Furthermore \( n = 500, k = 30, r = 0.05, K = 10000 \) and \( M^- = M^+ = 1 \).

Finally, in contrast to Figure 6.6, Figure 6.7 illustrates that (only) continuous fit appears for a Lévy process with finite variation and finite activity.
Figure 6.7: Plots of the approximating value function $V_k^{(n)}$ and the payoff function $f(x) = (K - \exp(x))^+$ where the driving process $X$ is of the $\beta$-class with finite variation and finite activity. Parameter values are $\sigma = 0, \alpha_1 = 1, \alpha_2 = 1, \beta_1 = \beta_2 = 5, c_1 = 5000, c_2 = 20$ and $\lambda_1 = \lambda_2 = 0.1$. Furthermore $n = 100, k = 20, r = 0.2, K = 5000$ and $M^+ = 1, M^- = 6$. 
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Chapter 7

Conclusion

This thesis combines the concept of working on a stochastic grid and using meromorphic Lévy processes to develop a powerful method for the application of these stochastic processes to both ‘real world problems’ and questions in probability theory. It provides a straightforward, ‘light weight’ and flexible way to tackle questions from different mathematical disciplines. Therefore this dissertation does not only add a valuable gain to the theory of Lévy processes. Moreover, it might help people in the industry to use these results as well. It might be worth to keep in mind that usually companies or businesses usually do not want to derive the best ultimate solution to a specific problem but preferably more one which gives both a good approximation or intuition to the question and which can be applied in a simple and quick way.

Numerical evidence and Corollary 3.3.5 show that our method of using a stochastic grid and approximating the law of a meromorphic Lévy process at a deterministic time by an Erlang distributed time $g(n,n)$ works well and is a clear improvement to the case of the original meromorphic Lévy process. However, it would be nice to find a stronger result for the error analysis. As presented in Section 3.3 the difference between the cumulative distribution functions of the deterministic and stochastic time case will eventually converge to 0. Unfortunately, the convergence speed is sublinear which does not support the numerical examples. It is not clear if the result in Corollary 3.3.5 can be improved with today’s tools for analysing Lévy processes but it might be possible in the future.

Also, with the provided formulas in Theorems 3.2.2 and 3.2.5 it should be possible to tackle various practical problems in several scientific disciplines.
Chapter 4 includes a powerful tool for the analysis of the capital process of insurance companies, a formula for the ruin probability in finite time. As stated before we did not succeed to solve this minimising problem with reinsurance. However, this question belongs to the broad class of control problems in stochastics. In the Schmidli’s book [61] a good summary can be found. Therefore it might be possible to apply our method, e.g. the application of meromorphic Lévy processes on a stochastic grid, to dividend problems for instance.

In Chapter 5 we derived a recursive formula for the discretised version of the supremum of a meromorphic Lévy process. A further project would be to continue working with the contents of Remark 5.3.2 and find an explicit formula. Moreover, this might be also possible for the joint law (Theorem 5.3.3).

Also, as already mentioned in Remark 6.4.3 it would be interesting to derive formulas for other financial options.

Also as many popular Lévy processes in mathematical finance can be obtained as the limit of meromorphic Lévy processes one could compare our model (Proposition 6.4.2) with models at deterministic time horizon, e.g. the Kou model [35] or the CGMY model [25]. Finally, analysing the error in Proposition 6.4.2 caused by using the stochastic time grid would be an interesting challenge.
Appendix A

Formulae for the coefficients appearing in Chapter 3

This Appendix contains expressions for the coefficients that appear in chapter 3.2, recursively in $k$ for some $n \geq 1$ fixed. Throughout the sequel, fix some $k \geq 2$. As usual, we understand an empty sum as equal to zero. Define

\[
\Lambda_1^{(n)}(i, j) = \sum_{q=0, q \neq j}^{N_+} \frac{c_+^{(n)}(q) \zeta_+^{(n)}(q)}{(\zeta_+^{(n)}(j) - \zeta_+^{(n)}(q))^{(i+1)}},
\]

\[
\Lambda_2^{(n)}(i, j, l) = \sum_{q=0, q \neq j}^{N_+} \frac{M_+^{(n)}(i, q, l - 1)}{(\zeta_+^{(n)}(j) - \zeta_+^{(n)}(q))^{(i+1)}} (-1)^i i!,
\]

\[
\Lambda_3^{(n)}(i, j) = \sum_{q=0, q \neq j}^{N_-} \frac{c_-^{(n)}(q) \zeta_-^{(n)}(q)}{(\zeta_-^{(n)}(j) - \zeta_-^{(n)}(q))^{(i+1)}},
\]

\[
\Lambda_4^{(n)}(i, j, l) = \sum_{q=0, q \neq j}^{N_-} \frac{M_-^{(n)}(i, q, l - 1)}{(\zeta_-^{(n)}(j) - \zeta_-^{(n)}(q))^{(i+1)}} (-1)^i i!.
\]

(A.1)

The coefficients appearing in the formulas (3.2.1) and (3.2.2) can now be expressed as follows – note that the expressions for $M_+^{(n)}(\cdot, j, \cdot)$ (resp. $M_-^{(n)}(\cdot, j, \cdot)$) below are
valid for all $j \in \{0, \ldots, N_+\}$ (resp. $j \in \{0, \ldots, N_-\}$):

$$M_+^{(n)}(0, j, 1) = c_+^{(n)}(j) \zeta_+^{(n)}(j)$$

$$M_+^{(n)}(0, j, k) = \sum_{l=0}^{k-2} (-1)^l l! \Lambda_1^{(n)}(l, j) M_+^{(n)}(l, j, k - 1)$$

$$- \left( \sum_{l=0}^{k-2} \Lambda_2^{(n)}(l, j, k) \right) \zeta_+^{(n)}(j) c_+^{(n)}(j)$$

$$M_+^{(n)}(l, j, k) = \sum_{h=0}^{k-2-l} (-1)^h \frac{(h + l)!}{l!} \Lambda_1^{(n)}(l, j) M_+^{(n)}(l + h, j, k - 1)$$

$$+ \frac{\zeta_+^{(n)}(j) c_+^{(n)}(j)}{l} M_+^{(n)}(l - 1, j, k - 1) \text{ for all } l \in \{1, \ldots, k - 2\}$$

$$M_+^{(n)}(k - 1, j, k) = \frac{\zeta_+^{(n)}(j) c_+^{(n)}(j)}{k - 1} M_+^{(n)}(k - 2, j, k - 1)$$

$$M_-^{(n)}(0, j, 1) = c_-^{(n)}(j) \zeta_-^{(n)}(j)$$

$$M_-^{(n)}(0, j, k) = -\sum_{l=0}^{k-2} (-1)^l l! \Lambda_3^{(n)}(l, j) M_-^{(n)}(l, j, k - 1)$$

$$+ \left( \sum_{l=0}^{k-2} \Lambda_4^{(n)}(l, j, k) \right) \zeta_-^{(n)}(j) c_-^{(n)}(j)$$

$$M_-^{(n)}(i, j, k) = -\sum_{h=0}^{k-2} (-1)^{h-l} \frac{h!}{l!} \Lambda_5^{(n)}(h - l, j) M_-^{(n)}(h, j, k - 1)$$

$$- \frac{\zeta_-^{(n)}(j) c_-^{(n)}(j)}{l} M_-^{(n)}(l - 1, j, k - 1) \text{ for all } l \in \{1, \ldots, k - 2\}$$

$$M_-^{(n)}(k - 1, j, k) = -\frac{\zeta_-^{(n)}(j) c_-^{(n)}(j)}{k - 1} M_-^{(n)}(k - 2, j, k - 1)$$

With the help of the coefficients $M_+^{(n)}(\cdot, j, \cdot)$ (resp. $M_-^{(n)}(\cdot, j, \cdot)$) we can express the coefficients in the formula (3.2.8) as follows – note that we as usual understand an empty sum as equal to zero, and that the expressions for $D_+^{(n)}(\cdot, j, \cdot)$ and $B_+^{(n)}(\cdot, j)$ (resp. $D_-^{(n)}(\cdot, j, \cdot)$ and $B_-^{(n)}(\cdot, j)$) below are valid for all $j \in \{0, \ldots, N_+\}$ (resp. $j \in \{0, \ldots, N_-\}$): Define

$$D_+^{(n)}(i, j, l) = \sum_{q=0}^{N_+} \sum_{r=0}^{l-1} M_-^{(n)}(r, q, n) \frac{(r + l - i)!(-1)^r}{(\zeta_-^{(n)}(j) + \zeta_-^{(n)}(q))^{r+l+1-i}}$$

$$D_-^{(n)}(i, j, l) = \sum_{q=0}^{N_-} \sum_{r=0}^{l-1} M_+^{(n)}(r, q, n) \frac{(r + l - i)!}{(\zeta_+^{(n)}(j) + \zeta_+^{(n)}(q))^{r+l+1-i}}$$
If the process has bounded variation, define:

\[
W_{+}^{(n)}(i, j) = \sum_{l=i}^{n-1} M_{+}^{(n)}(i, j, n) D_{+}^{(n)}(i, j, l) \begin{pmatrix} l \\ i \end{pmatrix} (\begin{pmatrix} n \\ h \end{pmatrix})
\]

\[
W_{-}^{(n)}(i, j) = \sum_{l=i}^{n-1} M_{-}^{(n)}(i, j, n) D_{-}^{(n)}(i, j, l) \begin{pmatrix} l \\ i \end{pmatrix} (-1)^{i-l}
\]

If the process has bounded variation, define:

\[
T_{+}^{(n)}(i, j) = \sum_{h=0}^{n-1} M_{+}^{(n)}(i, j, n-h)(w_{+}^{(n)})_h \begin{pmatrix} n \\ h \end{pmatrix}
\]

\[
T_{-}^{(n)}(i, j) = \sum_{h=0}^{n-1} M_{-}^{(n)}(i, j, n-h)(w_{-}^{(n)})_h \begin{pmatrix} n \\ h \end{pmatrix}
\]

\[
P_{+}^{(n)}(i, j, l) = \sum_{q=0}^{N_-} \sum_{r=0}^{l-1} T_{-}^{(n)}(r, q) \frac{(r + l - i)!(-1)^r}{(\zeta_{-}^{(n)}(j) + \zeta_{+}^{(n)}(q))^{r+l+1-i}}
\]

\[
P_{-}^{(n)}(i, j, l) = \sum_{q=0}^{N_+} \sum_{r=0}^{l-1} T_{+}^{(n)}(r, q) \frac{(r + l - i)!}{(\zeta_{-}^{(n)}(j) + \zeta_{+}^{(n)}(q))^{r+l+1-i}}
\]

\[
G_{+}^{(n)}(i, j) = \sum_{l=i}^{n-1} T_{+}^{(n)}(i, j) P_{+}^{(n)}(i, j, l) \begin{pmatrix} l \\ i \end{pmatrix}
\]

\[
G_{-}^{(n)}(i, j) = \sum_{l=i}^{n-1} T_{-}^{(n)}(i, j) P_{-}^{(n)}(i, j, l) \begin{pmatrix} l \\ i \end{pmatrix} (-1)^{l-i}
\]

With the help of the coefficients \( G_{+}^{(n)}(., ., j) \) (resp. \( G_{-}^{(n)}(., ., j) \)) we can express the coefficients in the formula (3.2.9) as follows – note that we as usual understand an empty sum as equal to zero, and that the expressions for \( R_{+}^{(n)}(., ., j) \) (resp. \( R_{-}^{(n)}(., ., j) \)) below are valid for all \( j \in \{0, \ldots, N_+\} \) (resp. \( j \in \{0, \ldots, N_-\} \)):

\[
R_{+}^{(n)}(i, j) = G_{+}^{(n)}(i, j) + (w_{+}^{(n)})^n T_{-}^{(n)}(i, j) + (w_{-}^{(n)})^n T_{+}^{(n)}(i, j)
\]

\[
R_{-}^{(n)}(i, j) = G_{-}^{(n)}(i, j)
\]

Define

\[
\mu_1(j, q) := \zeta_{+}^{(n)}(j) - \zeta_{-}^{(n)}(q)
\]

for \( j = 0, \ldots, N_+ \) and \( q = 0, \ldots, j - 1, j + 1, \ldots N_+ \)

\[
\mu_2(j, q) := \zeta_{+}^{(n)}(j) - \zeta_{-}^{(n)}(q)
\]

for \( j = 0, \ldots, N_- \) and \( q = 0, \ldots, j - 1, j + 1, \ldots N_- \)

\[
\mu_3(j, q, w) := \zeta_{-}^{(n)}(j) + \zeta_{+}^{(n)}(q)
\]

for \( j = 0, \ldots, N_+, q = 0, \ldots, j - 1, j + 1, \ldots N_+ \)

and for \( w = 0, \ldots, N_- \)

\[
\mu_5(j, w) := \zeta_{-}^{(n)}(w) + \zeta_{+}^{(n)}(j)
\]

for \( j = 0, \ldots, N_+ \) and for \( w = 0, \ldots, N_- \)
APPENDIX A. FORMULAE FOR SECTION 3.2

\[ C_{-}^{(n)}(j,+,0,1) = \sum_{q=0}^{N^{+}} c_{-}^{(n)}(j)c_{+}^{(n)}(q) \frac{\zeta^{(n)}(j)}{\zeta^{(n)}(j) + \zeta^{(n)}(q)} \]

\[(\exp(-\zeta^{(n)}_{+}(q)x) - \exp(\zeta^{(n)}_{+}(q)x))(1 - \exp((\zeta^{(n)}_{+}(q) + \zeta^{(n)}_{+}(j))u)) \]

\[ C_{-}^{(n)}(j,+,0,k+1) = \sum_{q=0}^{N^{+}} \sum_{w=0}^{N^{+}} C_{-}^{(n)}(w,+,0,k) \frac{1}{\mu_{5}(q,j)\mu_{5}(q,w)}(1 - \exp((\mu_{5}(q,j))u)) \]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0}^{N^{+}} \sum_{h=0}^{k-1} c_{-}^{(n)}(j)c_{-}^{(n)}(j)\zeta^{(n)}_{+}(q)c_{+}^{(n)}(q)(-1)^h \frac{\mu_{5}(j,i)}{\mu_{1}(j,q)\mu_{5}(j,q)^{h+1}} \mu_{3}(j,q,w) \]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0}^{N^{+}} \sum_{h=0}^{N^{+}-k-1} h!(1-\zeta^{(n)}_{+}(j,-,h,k)) \exp(u(\mu_{5}(j,q))) \frac{\mu_{5}(j,q)\mu_{1}(j,q)^{h+1}}{\mu_{5}(j,q)\mu_{1}(j,q)^{h+1}} \]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0}^{N^{+}} \mu_{5}(j,w)^{k-l-1} \mu_{1}(w,q)^{k-h-1} \]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0}^{N^{+}} \mu_{5}(j,w)^{k-l} \mu_{1}(j,q)^{k-h} + u^{k-1} \zeta^{(n)}_{+}(j,-,k-1,1) \]

\[ + \sum_{q=0}^{N^{+}} \sum_{r=0}^{k-1} \zeta^{(n)}_{+}(j,-,r,k) \mu_{5}(q,j)^{r+2} \]

\[ + \sum_{q=0}^{N^{+}} \sum_{r=0}^{k-1} \sum_{l=0}^{r+1} \frac{r!}{(r-l+1)!} \zeta^{(n)}_{+}(j,-,r,k) \mu_{5}(q,j)^{l+1} \mu_{5}(q,j)^{l+1} - u^{r+1-l} \exp(\mu_{5}(q,j)u) \]

\[ + \sum_{q=0}^{N^{+}} \sum_{w=0}^{N^{+}} \sum_{l=0}^{k-1} C_{-}^{(n)}(w,-,l,k)\mu_{2}(j,w)^{l+1} \mu_{5}(q,w)^{l+1} \]

\[ + \sum_{q=0}^{N^{+}} \sum_{k=0}^{N^{+}-k-2} \sum_{h=0}^{k-1} \sum_{r=0}^{k-1} \mu_{5}(q,w)^{r+1} \mu_{5}(q,w)^{r+1} \sum_{g=0}^{l} \frac{1}{\mu_{2}(j,w)^{g+1} \mu_{5}(q,w)^{l+1-g}} \]

\[ + u^{k-1} \zeta^{(n)}_{+}(j,-,k-1,1) \mu_{5}(q,w)\mu_{2}(j,w)^{k-1-g} \]

\[ - \sum_{q=0}^{N^{+}} \sum_{l=0}^{k-1} \frac{u^{l}}{\mu_{5}(q,j)^{k}} \sum_{g=0}^{l} C_{-}^{(n)}(j,-,g+l-1,k)\mu_{5}(q,j)^{k-1-g} \frac{(l+g-1)}{l!} \]

\[ - \frac{1}{k\mu_{5}(q,j)} \zeta^{(n)}_{+}(j,-,k-1,1) \]

\[ C_{+}^{(n)}(j,-,0,1) \]
\[ 
\sum_{q=0}^{N^{-}} C_+^{(n)}(q) + \sum_{q=0}^{N^{-}} \frac{\zeta_+(n)(q)}{\zeta_-(n)(q) + \zeta_+(n)(j)} \left( \exp(-\zeta_+(n)(j)x) - \exp(\zeta_+(n)(j)x) \right) 
\]

\[ 
C_+^{(n)}(j, -, 0, k + 1) = \sum_{q=0}^{N^{-}} \sum_{w=0}^{N^{-}} C_+^{(n)}(q, +, 0, k) \frac{1}{\mu_5(j, q) \mu_5(j, q)} 
\]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0, q \neq w}^{N^{+}} \sum_{l=0}^{k-1} C_+^{(n)}(w, -, l, k)(-1)^{l+l!} \frac{1}{\mu_1(w, q)^{l+1} \mu_3(w, q, j)} 
\]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0, q \neq w}^{N^{+}} \frac{1}{\mu_1(w, q)^k \mu_5(w, j)^k} 
\]

\[ \sum_{l=0}^{k-1} l \sum_{l=0}^{k-1} C_+^{(n)}(w, -, l, k)(-1)^{l+l!} \mu_1(w, q)^{l+1} \mu_5(w, j)^{k-g-1} 
\]

\[ + \sum_{w=0}^{N^{-}} \sum_{q=0, q \neq w}^{N^{-}} \sum_{l=0}^{k-1} \frac{l! C_+^{(n)}(w, -, l, k)}{\mu_5(j, q)^{l+1} \mu_5(j, q)} 
\]

\[ C_+^{(n)}(j, -, i, k + 1) = \sum_{w=0}^{N^{+}} \sum_{q=0, q \neq w}^{N^{+}} \sum_{l=1}^{k-1} C_+^{(n)}(w, -, l, k)(-1)^{l-i} \sum_{g=0}^{i} \left( \begin{array}{c} i \\ g \end{array} \right) \frac{(-1)^{l+1}}{i!} \frac{1}{\mu_1(w, q)^{l-i+1} \mu_3(w, q, j)^{2g+1}} 
\]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0, q \neq w}^{N^{+}} \frac{1}{\mu_1(w, q)^k \mu_5(w, j)^k} 
\]

\[ \sum_{l=0}^{k-1} l \sum_{l=0}^{k-1} C_+^{(n)}(w, -, l, k)(-1)^{l+l!} \mu_1(w, q)^{l+1} \mu_5(w, j)^{k-g-1} 
\]

\[ + \sum_{w=0}^{N^{-}} \sum_{q=0, q \neq w}^{N^{-}} \sum_{l=1}^{k} C_+^{(n)}(w, -, l, k)(-1)^{l-l+1} \frac{1}{l! \mu_5(j, w)^{l+1}} 
\]

for \( i \in \{0, \ldots, k-2\} \)

\[ C_+^{(n)}(j, -, k - 1, k + 1) = \sum_{w=0}^{N^{+}} \sum_{q=0, q \neq w}^{N^{+}} C_+^{(n)}(w, -, k - 1, k) \sum_{g=0}^{k-1} \left( \begin{array}{c} k - 1 \\ g \end{array} \right) \frac{(-1)^{g}}{\mu_1(w, q)^{k-g} \mu_3(w, q, j)^{2g+1}} 
\]

\[ + \sum_{w=0}^{N^{+}} \sum_{q=0, q \neq w}^{N^{+}} \frac{1}{\mu_1(w, q)^k \mu_5(j, q)^k} 
\]

\[ \sum_{l=0}^{k-1} l \sum_{l=0}^{k-1} C_+^{(n)}(w, -, l, k)(-1)^{k-l+1} \frac{1}{(k-1)! \mu_5(j, w)^{l+1}} 
\]
\[ C_{+}^{(n)}(j, -, k, k + 1) = \sum_{w=0}^{N^+} \sum_{l=0}^{k} C_{+}^{(n)}(w, -, l, k)(-1)^{k-l} \frac{l!}{k!} \frac{1}{\mu_5(j, w)^{l+1}} \]

\[ C_{-}^{(n)}(j, -, 0, 1) = \sum_{q=0}^{N^+} c_{-}^{(n)}(q)C_{-}^{(n)}(j) \frac{\zeta_{-}^{(n)}(j)}{\zeta_{-}^{(n)}(j) + \zeta_{-}^{(n)}(q)} \left( \exp(-\zeta_{-}^{(n)}(j)x) - \exp(\zeta_{+}^{(n)}(j)x) \right) \left( \zeta_{+}^{(n)}(q) + \zeta_{-}^{(n)}(j) \right)u \]

\[ C_{-}^{(n)}(j, -, 0, k + 1) = -\sum_{q=0}^{N^+} \sum_{w=0}^{N^-} C_{-}^{(n)}(w, +, 0, k) \frac{1}{\mu_5(q, j)} \frac{1}{\mu_5(q, w)} \left( \exp((\mu_5(q, j))u) \right) \]

\[ + \sum_{w=0}^{N^+} \sum_{q=0}^{N^-} \sum_{h=0}^{N^-} n!(-1)^h C_{+}^{(n)}(j, -, h, k) \frac{\exp(u(\mu_5(j, q)))}{\mu_5(j, q)\mu_1(j, q)^{h+1}} \]

\[ + \sum_{w=0}^{N^+} \sum_{q=0}^{N^-} \sum_{h=0}^{N^-} \mu_5(w, j)^{w} \mu_1(w, q)^{w} \left( \frac{1}{\mu_5(w, j)^{w} \mu_1(w, q)^{w}} \right) \sum_{r=0}^{k-2} \sum_{r-h}^{k-2} \sum_{h=0}^{r-h} \sum_{g=0}^{r-h} r! \frac{1}{h!} \frac{1}{l!} \mu_5(j, w)^{k-g-1} \mu_1(w, j)^{k-(r-h)+g} \]

\[ + u^{k-1} C_{+}^{(n)}(j, -, -k - 1, k) \frac{1}{\mu_5(w, j)\mu_1(w, q)} \]

\[ + \sum_{w=0}^{N^+} \sum_{q=0}^{N^-} \sum_{h=0}^{N^-} \sum_{l=0}^{r} \frac{r!}{(r-l+1)!} (-1)^{l+1} \frac{1}{\mu_5(q, j)^{l+1}} u^{r-l-1} \exp(\mu_5(q, j)u) \]

\[ + \sum_{w=0}^{N^+} \sum_{q=0}^{N^-} \sum_{l=0}^{N^-} l! C_{-}^{(n)}(w, -, l, k) \frac{1}{\mu_5(q, w)^{l+1} \mu_5(q, j)} \]

\[ + \sum_{q=0}^{N^+} \sum_{w=0}^{N^-} u^{k-1} C_{-}^{(n)}(w, -, -k - 1, k) \frac{1}{\mu_2(j, w)\mu_5(q, w)} \]

\[ + \sum_{q=0}^{N^+} \sum_{w=0}^{N^-} \sum_{l=0}^{k-1} \sum_{g=1}^{k} u^l \sum_{g=1}^{k-2} C_{-}^{(n)}(w, -, g, k) g! \frac{1}{\mu_2(j, w)^{k-1-d} \mu_5(q, w)} \]

\[ + \sum_{q=0}^{N^+} \sum_{w=0}^{N^-} \sum_{l=0}^{k-1} \sum_{l=1}^{k} C_{-}^{(n)}(j, -, l, k) \frac{l!}{l!} \sum_{g=0}^{l-i} \frac{1}{\mu_2(j, w)^{l-i-g} \mu_5(q, j)^{g+1}} \]

\[ + \sum_{q=0}^{N^+} \sum_{w=0}^{N^-} \sum_{l=1}^{k} \sum_{l=1}^{k-1} C_{-}^{(n)}(j, -, l, k) \frac{l!}{l!} \frac{1}{\mu_5(q, j)^{l+i+2}} \]

\[ C_{-}^{(n)}(j, -, i, k + 1) = \sum_{q=0}^{N^+} \sum_{w=0}^{N^-} \sum_{l=0}^{k} C_{-}^{(n)}(j, -, l, k) \frac{l!}{l!} \sum_{g=0}^{l-i} \frac{1}{\mu_2(j, w)^{l-i-g} \mu_5(q, j)^{g+1}} \]

\[ - \sum_{q=0}^{N^+} \sum_{l=1}^{k} C_{-}^{(n)}(j, -, l, k) \frac{l!}{l!} \frac{1}{\mu_5(q, j)^{l+i+2}} \] for \( i \in \{0, \ldots, k-1\} \)

\[ C_{-}^{(n)}(j, -, k, k + 1) = -\frac{1}{k\mu_5(q, j)}. \]
Appendix B

Formulae for the coefficients appearing in Chapter 4

This Appendix contains expressions for the coefficients that appear in Proposition 4.2.2 and Proposition 4.2.3, recursively in \(k\) for some \(n \geq 1\) fixed. Recall that the starting point for the recursion is (4.2.3) for \(k = 0\), which contains only one coefficient, namely \(U_-(n, 0, 0) = 0\). Throughout the sequel, fix some \(k \geq 1\). As usual, we understand an empty sum as equal to zero. Define

\[
\rho_1^{(n)}(h, j) = (-1)^h \sum_{q=0}^{N_+} \zeta_+(q) \zeta_+(q) \left( -\zeta_+(q) - \zeta_-(j) \right)^{-(h+1)},
\]

\[
\rho_2^{(n)}(h, j) = (-1)^h \sum_{q=0, q \neq j}^{N_-} \zeta_-(q) \zeta_-(-q) \left( \zeta_-(q) - \zeta_-(j) \right)^{-(h+1)},
\]

\[
\chi_1^{(n)}(i, l, j, k) = \sum_{q=0, q \neq j}^{N_-} U_-(n)(i, q, k)(-1)^i i! \rho_1(n)(i - l, q) \left( \zeta_-(q) - \zeta_-(j) \right)^{-(l+1)},
\]

\[
\chi_2^{(n)}(i, j, k) = \sum_{l=0}^{i} \chi_1^{(n)}(i, l, j, k),
\]

\[
\chi_3^{(n)}(j, k) = \sum_{i=0}^{k-2} \chi_2^{(n)}(i, j, k),
\]

where \(h \in \mathbb{N}\), \(j \in \{0, \ldots, N_-\}\) for \(\rho_1^{(n)}, \rho_2^{(n)}, \chi_1^{(n)}, \chi_2^{(n)}, \chi_3^{(n)}, i \in \{0, \ldots, k-1\}\) and \(l \in \{0, \ldots, i\}\).

The coefficients appearing in formula (4.2.3) for \(V_+(n)\) can now be expressed as follows.

The expressions for \(U_-(n, ., j, .)\) below are valid for all \(j \in \{0, \ldots, N_-\}\). We present
first the coefficients for the case that the underlying process is unbounded and then give a brief remark on the bounded case:

\[
\begin{align*}
U^{(n)}_-(0, j, k) &= c^{(n)}_-(j) + c^{(n)}_-(j)\zeta^{(n)}_-(j)\chi^{(n)}_3(j, k) \\
&\quad - \sum_{h=0}^{k-2} U^{(n)}_-(h, j, k - 1) \sum_{l=0}^{h} \rho^{(n)}_1(h - l, j) \rho^{(n)}_2(l, j) h!
\end{align*}
\]

\[
U^{(n)}_-(i, j, k) = - \sum_{h=i-1}^{k-2} U^{(n)}_-(h, j, k - 1) \rho^{(n)}_1(h - i + 1, j) \frac{h!}{l!} c^{(n)}_-(j) \zeta^{(n)}_-(j)
\]

\[
U^{(n)}_-(h, j, k - 1) \sum_{l=0}^{h-i} \rho^{(n)}_1(h - i - l, j) \rho^{(n)}_2(l, j) \frac{h!}{l!}, \\
\forall i \in \{1, \ldots, k - 2\}
\]

\[
U^{(n)}_-(k - 1, j, k) = -U^{(n)}_-(k - 2, j, k - 1) \rho^{(n)}_1(0, j) \frac{1}{k-1} c^{(n)}_-(j) \zeta^{(n)}_-(j)
\]

In the case of bounded variation, we can show that:

\[
U^{(n)}_\text{bounded}(0, j, k) = U^{(n)}_-(0, j, k) - c^{(n)}_-(j)(-1) \sum_{h=0}^{k-2} U^{(n)}_-(h, j, k - 1) h! \rho^{(n)}_1(h, j)
\]

\[
U^{(n)}_\text{bounded}(i, j, k) = U^{(n)}_-(i, j, k) - c^{(n)}_-(j)(-1) \sum_{h=i}^{k-2} U^{(n)}_-(h, j, k - 1) \frac{h!}{l!}
\]

\[
\frac{h!}{l!} \rho^{(n)}_1(h - i, j), \quad \forall i \in \{1, \ldots, k - 2\}
\]

\[
U^{(n)}_\text{bounded}(k - 1, j, k) = U^{(n)}_-(k - 1, j, k)
\]
Appendix C

Formulae for the coefficients appearing in Chapter 6

This Appendix contains expressions for the coefficients that appear in Proposition 6.4.2, recursively in \( k \) for some \( n \geq 1 \) fixed. Recall that the starting point for the recursion is (6.4.2) for \( k = 0 \), which contains only one coefficient, namely \( \bar{x}_0^{(n)} := \log K \).

Throughout the sequel, fix some \( k \geq 1 \).

Let the intervals \( I^k_l = [\bar{x}_l^{(n)}, \bar{x}_{l-1}^{(n)}] \) for \( l = 1, \ldots, k-1 \), \( I_0 = [\bar{x}_0^{(n)}, \infty) \) and \( I_k = (-\infty, \bar{x}_{k-1}^{(n)}) \). Furthermore let \( F_-(x;l,j,k) \) be an antiderivative of \( x \mapsto V_{k-1}^{(n)}(x)e^{\zeta^{(n)}(j)x} \) on the interval \( I^k_l \) and \( F_+(x;l,j,k) \) an antiderivative of \( x \mapsto V_{k-1}^{(n)}(x)e^{-\zeta^{(n)}(j)x} \) on the same interval \( I^k_l \). Recalling the formula of \( V_{k-1}^{(n)} \) from (6.4.2), note that \( F_+(\cdot;l,j,k) \) and \( F_-(\cdot;l,j,k) \) consist of sums of antiderivatives of functions of the form \( x \mapsto x^ie^{bx} \) for \( i \in \mathbb{N} \), it is hence straightforward to apply partial integration and express them in terms of elementary functions. For compactness we do not spell out these formulae in all detail here.

Furthermore, define
\[ \Upsilon_0 = \sum_{q=0}^{N_-} \frac{d^{(n)}_-(q)}{\zeta^{(n)}_-(q)} + 1 - \sum_{q=0}^{N_+} \frac{d^{(n)}_+(q)}{\zeta^{(n)}_+(q)} + 1, \]

\[ \Upsilon_1^{(n)}(h, j) = (-1)^h \left( \sum_{q=0, q \neq j}^{N_-} d^{(n)}_-(q) \left( \zeta^{(n)}_-(q) - \zeta^{(n)}_-(j) \right)^{(h+1)} \right) - \sum_{q=0}^{N_+} d^{(n)}_+(q) \left( -\zeta^{(n)}_+(q) - \zeta^{(n)}_+(j) \right)^{(h+1)}, \]

\[ \Upsilon_2^{(n)}(h, j) = (-1)^h \left( \sum_{q=0}^{N_-} d^{(n)}_-(q) \left( \zeta^{(n)}_-(q) + \zeta^{(n)}_+(j) \right)^{(h+1)} \right) - \sum_{q=0, q \neq j}^{N_+} d^{(n)}_+(q) \left( -\zeta^{(n)}_+(q) + \zeta^{(n)}_+(j) \right)^{(h+1)}. \]

where \( h \in \mathbb{N}, j \in \{0, \ldots, N_-\} \) for \( \Upsilon_1^{(n)} \) and \( j \in \{0, \ldots, N_+\} \) for \( \Upsilon_2^{(n)} \).

The coefficients appearing in the formula (6.4.2) for \( V_k^{(n)} \) can now be expressed as follows – note that we as usual understand an empty sum as equal to zero, and that the expressions for \( A^{(n)}_+ (\ldots, j, \ldots) \) (resp. \( A^{(n)}_- (\ldots, j, \ldots) \)) below are valid for all \( j \in \{0, \ldots, N_+\} \) (resp. \( j \in \{0, \ldots, N_-\} \)):

i) On the interval \([\bar{x}^{(n)}_k, \bar{x}^{(n)}_{k-1}]\):

\[ A^{(n)}_-(0, j, k, k) = 0; \]

\[ A^{(n)}_+(0, j, k, k) = e^{-r/n} d^{(n)}_+(j) \left( F_+(\bar{x}^{(n)}_{k-1}; k, j, k) - F_+(\bar{x}^{(n)}_0; 0, j, k) \right) + \sum_{l=1}^{k-1} \left( F_+(\bar{x}^{(n)}_l; l, j, k) - F_+(\bar{x}^{(n)}_l; l, j, k) \right); \]

\[ B^{(n)}(k, k) = e^{-r/n} \Upsilon_0; \]

\[ C^{(n)}(k, k) = e^{-r/n} K, \]

where \( \bar{x}^{(n)}_k \) is the unique solution on \((-\infty, \bar{x}^{(n)}_{k-1})\) of the equation in \( z \):

\[ \sum_{q=0}^{N_+} e^{\zeta^{(n)}_+(q)} A^{(n)}_+(0, q, k, k) - B^{(n)}(k, k) e^z + C^{(n)}(k, k) = K - e^z. \]

ii) On the interval \([\bar{x}^{(n)}_m, \bar{x}^{(n)}_{m-1}]\) for \( m = 1, \ldots, k - 1 \):
\[ A_{-}^{(n)}(0, j, m, k) = e^{-r/n} d_{-}^{(n)}(j) \left( F_{-}(\overline{x}_{k-1}^{(n)}; k, j, k) - F_{-}(\overline{x}_{0}^{(n)}; 0, j, k) \right. \\
+ \sum_{l=m+1}^{k-1} \left( F_{-}(\overline{x}_{l-1}^{(n)}; l, j, k) - F_{-}(\overline{x}_{l}^{(n)}; l, j, k) \right) \\
+ e^{-r/n} \sum_{h=0}^{k-m-1} A_{-}^{(n)}(h, j, m, k-1) \Upsilon_{1}^{(n)}(h, j) h!; \]
\[ A_{-}^{(n)}(i, j, m, k) = e^{-r/n} \sum_{h=1}^{k-m-i} A_{-}^{(n)}(i + h - 1, j, m, k-1) \Upsilon_{1}^{(n)}(h-1, j) \frac{(h+i-1)!}{i!} \\
+ e^{-r/n} d_{-}^{(n)}(j) \left( A_{-}^{(n)}(i-1, j, m, k-1), \ \forall i \in \{1, \ldots, k-m\}; \right) \]
\[ A_{+}^{(n)}(0, j, m, k) = e^{-r/n} d_{+}^{(n)}(j) \left( F_{+}(\overline{x}_{m-1}^{(n)}; m, j, k) - F_{+}(\overline{x}_{0}^{(n)}; 0, j, k) \right. \\
+ \sum_{l=1}^{m-1} \left( F_{+}(\overline{x}_{l-1}^{(n)}; l, j, k) - F_{+}(\overline{x}_{l}^{(n)}; l, j, k) \right) \\
+ e^{-r/n} \sum_{h=0}^{k-m-1} A_{+}^{(n)}(h, j, m, k-1) \Upsilon_{2}^{(n)}(h, j) h!; \]
\[ A_{+}^{(n)}(i, j, m, k) = e^{-r/n} \sum_{h=1}^{k-m-i} A_{+}^{(n)}(i + h - 1, j, m, k-1) \Upsilon_{2}^{(n)}(h-1, j) \frac{(h+i-1)!}{i!} \\
- e^{-r/n} d_{+}^{(n)}(j) \left( A_{+}^{(n)}(i-1, j, m, k-1), \ \forall i \in \{1, \ldots, k-m\}; \right) \]
\[ B^{(n)}(m, k) = e^{-r/n} \Upsilon_{0} B^{(n)}(m, k-1); \]
\[ C^{(n)}(m, k) = e^{-r/n} C^{(n)}(m, k-1). \]

iii) On the interval \([\overline{x}_{0}^{(n)}, \infty]:\)
\[ A_{-}^{(n)}(0, 0, k) = e^{-r/n} d_{-}^{(n)}(j) \left( F_{-}(\overline{x}_{k-1}^{(n)}; k, j, k) - F_{-}(\overline{x}_{0}^{(n)}; 0, j, k) \right. \\
+ \sum_{l=1}^{k-1} \left( F_{-}(\overline{x}_{l-1}^{(n)}; l, j, k) - F_{-}(\overline{x}_{l}^{(n)}; l, j, k) \right) \\
+ e^{-r/n} \sum_{h=1}^{k-1-i} A_{-}^{(n)}(h-1, 0, k-1) \Upsilon_{1}^{(n)}(h-1, j)(h-1)!; \]
\[ A_{-}^{(n)}(i, 0, k) = e^{-r/n} \sum_{h=1}^{k-1-i} A_{-}^{(n)}(i + h - 1, 0, k-1) \Upsilon_{1}^{(n)}(h-1, j) \frac{(h+i-1)!}{i!} \\
+ e^{-r/n} d_{-}^{(n)}(j) \left( A_{-}^{(n)}(i-1, 0, k-1), \ \forall i \in \{1, \ldots, k-1\}. \right) \]
Appendix D

Useful formulas

**Proposition D.0.1.** Let \( i \) be a natural number or 0, \( b \) real. Then we have:

\[
\int \exp(bx)x^i = \sum_{g=0}^{i} \frac{i!}{(i-g)!} \frac{(-1)^g}{bg+1} x^{i-g} \exp(bx)
\]  

(D.1)

*Proof.* We will prove this result by induction. For \( i = 0 \) we have:

\[
\int \exp(bx) = \frac{1}{b} \exp(bx) = \sum_{g=0}^{0} \frac{1}{b} \exp(bx)
\]

Let us now assume that (D.1) is true for \( i - 1 \). Then

\[
\int \exp(bx)x^i = \frac{1}{b} \exp(bx)x^i - \int \exp(bx)x^{i-1} \frac{i}{b}
\]

\[
= \frac{1}{b} \exp(bx)x^i - \sum_{g=0}^{i-1} \frac{(i-1)!}{(i-1-g)!} \frac{(-1)^g}{bg+1} x^{i-1-g} \exp(bx) \frac{i}{b}
\]

\[
= \frac{1}{b} \exp(bx)x^i - \sum_{g=0}^{i-1} \frac{i!}{(i-1-g)!} \frac{(-1)^g}{bg+1} x^{i-1-g} \exp(bx)
\]

\[
= \frac{1}{b} \exp(bx)x^i - \sum_{g=1}^{i} \frac{i!}{(i-g)!} \frac{(-1)^g-1}{bg+1} x^{i-g} \exp(bx)
\]

\[
= \frac{1}{b} \exp(bx)x^i + \sum_{g=1}^{i} \frac{i!}{(i-g)!} \frac{(-1)^g}{bg+1} x^{i-g} \exp(bx)
\]

\[
= \sum_{g=0}^{i} \frac{i!}{(i-g)!} \frac{(-1)^g}{bg+1} x^{i-g} \exp(bx)
\]

\(\square\)
**Lemma D.0.2.** For any $n \in \mathbb{N}$ and $k \leq n - 1$ we have:

$$
\sum_{j=0}^{k} (-1)^j \binom{n-1}{j} \binom{n-1}{k-j} = \begin{cases} 
\binom{n-1}{\frac{k}{2}} (-1)^{\frac{k}{2}} & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd.}
\end{cases} \tag{D.2}
$$

**Proof.** Recall the binomial theorem for $m \in \mathbb{N}$ and $z, y \in \mathbb{R}$:

$$(z + y)^m = \sum_{l=0}^{m} \binom{m}{l} z^{m-l} y^l. \tag{D.3}$$

Consider $g(x) = (1 - x)^{n-1}(1 + x)^{n-1}$ for $x \in \mathbb{R}$ which we can write with (D.3) as

$$g(x) = \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j x^j \sum_{j=0}^{n-1} \binom{n-1}{j} x^j. \tag{D.4}$$

Combining the same powers of $x$ gives us:

$$g(x) = \sum_{k=0}^{2n-2} x^k \sum_{j=0}^{k} (-1)^j \binom{n-1}{j} \binom{n-1}{k-j}. \tag{D.5}$$

This means that we have for each $k \leq 2n - 2$ the following coefficient for $x^k$:

$$\sum_{j=0}^{k} (-1)^j \binom{n-1}{j} \binom{n-1}{k-j}. \tag{D.6}$$

On the other hand we can write $g$ as:

$$g(x) = (1 - x)^{n-1}(1 + x)^{n-1} = (1 - x^2)^{n-1}$$

$$= \sum_{r=0}^{n-1} x^{2r} (-1)^r \binom{n-1}{r}. \tag{D.7}$$

where the last line follows again from (D.3).

If we now investigate the coefficients (D.7) of $x^k$ for $k \leq 2n - 2$ we see that that they are equal to 0 if $k$ is odd. If $k$ is even we see that they are equal to $(-1)^{\frac{k}{2}} \binom{n-1}{\frac{k}{2}}$. This observation together with the combination of (D.6) and (D.7) yields (D.2). \qed

**Lemma D.0.3** ([29]). The Bernstein polynomial $B_{i,n}$ of $n$-th degree is defined for $n \in \mathbb{N}$, $0 \leq i \leq n$ and $x \in \mathbb{R}$ as

$$B_{i,n}(x) = \binom{n}{i} x^i (1 - x)^{n-i}.$$

Then the $p$-th derivative of $B_{i,n}(x)$ is given by:

$$D^{(p)} B_{i,n}(x) = \frac{n!}{(n-p)!} \sum_{j=\max(0,i+p-n)}^{\min(i,p)} (-1)^j \binom{p}{j} B_{i-j,n-p}(x). \tag{D.8}$$
Theorem D.0.4. For any $n \in \mathbb{N}$ and $k \leq n - 1$ we have:

$$
\sum_{i=0}^{k} \frac{(-1)^i 2^i (2n - 2 - i)!}{i! (n - 1 - i)! (k - i)!} = \begin{cases} 
(-1)^{\frac{k}{2}} \frac{(2n-2-k)!}{(n-1-\frac{k}{2})!} & \text{if } k \text{ is even,} \\
0 & \text{if } k \text{ is odd.}
\end{cases} \quad (D.9)
$$

Proof. Let us define the function

$$
g_k(x) = \frac{1}{k!} x^{2n-2-k} (1 + x)^k. \quad (D.10)
$$

We can then derive with the help of the binomial theorem (D.3):

$$
g_k(x) = \frac{1}{k!} x^{2n-2-k} \sum_{i=0}^{k} \binom{k}{i} x^{k-i} \\
= \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} x^{2n-2-i}
$$

It is an easy exercise to calculate the $n-1$-th derivative of $g_k(x)$:

$$
g_k^{(n-1)}(x) = \sum_{i=0}^{k} \frac{(2n - 2 - i)! x^{n-1-i}}{i! (k - i)! (n - 1 - i)!}. \quad (D.11)
$$

We note that

$$
g_k^{(n-1)}\left(-\frac{1}{2}\right) \frac{(-1)^{n-1}}{(-\frac{1}{2})^{n-1}} = \sum_{i=0}^{k} \frac{(-1)^i 2^i (2n - 2 - i)!}{i! (n - 1 - i)! (k - i)!}. \quad (D.12)
$$

By looking at Lemma D.0.3 we also see that

$$
g_k(x) = (-1)^k \frac{(2n - 2 - k)!}{(2n - 2)!} B_{2n-2-k,2n-2}(-x). \quad (D.13)
$$

Therefore we are keen to evaluate $D^{(n-1)} B_{2n-2-k,2n-2}(-\frac{1}{2})$. However this can be done
with Lemma D.0.3:

\[
D^{(n-1)}B_{2n-2-k,2n-2}(\frac{1}{2}) = \frac{(2n-2)!}{(n-1)!} \sum_{j=\max(0,2n-2-k,n-1-2(n-2))}^{\min(2n-2-k,n-1)} (-1)^{j+n-1} \binom{n-1}{j} B_{2n-2-k-j,2n-2-(n-1)}(\frac{1}{2})
\]

\[
= \frac{(2n-2)!}{(n-1)!} \sum_{j=n-1-k}^{n-1} (-1)^{j+n-1} \binom{n-1}{j} B_{2n-2-k-j,n-1}(\frac{1}{2})
\]

\[
= \frac{(2n-2)!}{(n-1)!} \sum_{j=n-1-k}^{n-1} (-1)^{j+n-1} \binom{n-1}{j} \binom{n-1}{2n-2-k-j}(\frac{1}{2})^{2n-2-k-j}(\frac{1}{2})^{j+n-1}
\]

\[
= \frac{(2n-2)!}{(n-1)!} \frac{1}{2}^{n-1} \sum_{j=0}^{k} (-1)^{k-j} \binom{n-1}{k} \binom{n-1}{n-1-j}
\]

\[
= \frac{(2n-2)!}{(n-1)!} \frac{1}{2}^{n-1} (-1)^{k} \sum_{j=0}^{k} (-1)^{j} \binom{n-1}{k+j} \binom{n-1}{2n-2-k -(n-1-k+j)}
\]

By using the result in Lemma D.0.2 we conclude that for \(k\) even

\[
D^{(n-1)}B_{2n-2-k,2n-2}(\frac{1}{2}) = \frac{(2n-2)!}{(n-1)!} \frac{1}{2}^{n-1} (-1)^{k} \binom{n-1}{\frac{k}{2}} (-1)^{\frac{k}{2}}
\]

and for \(k\) odd

\[
D^{(n-1)}B_{2n-2-k,2n-2}(\frac{1}{2}) = 0.
\]

Therefore by relation (D.13) we obtain for \(k\) even

\[
g_{k}^{(n-1)}(-\frac{1}{2}) = (-1)^{k} \frac{(2n-2-k)!}{(2n-2)!} B_{2n-2-k,2n-2}(-x)
\]

\[
= (-1)^{k} \frac{(2n-2-k)!}{(2n-2)!} \frac{(2n-2)!}{(n-1)!} \frac{1}{2}^{n-1} (-1)^{k} \binom{n-1}{\frac{k}{2}} (-1)^{\frac{k}{2}}
\]

\[
= \frac{(2n-2-k)!}{(n-1-\frac{k}{2})!} \frac{1}{2}^{n-1} (-1)^{\frac{k}{2}}
\]

and for \(k\) odd

\[
g_{k}^{(n-1)}(-\frac{1}{2}) = 0.
\]

Applying (D.12) yields the result. □
Remark D.0.5 (Kou model). Kou [35, 36] suggested that the price of an asset $S$ at time $t > 0$ as

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t + d \sum_{i=1}^{N_t} \exp(Y_i)
\]  
(D.14)

where $\mu \in \mathbb{R}$, $\sigma > 0$, $B_t$ is a standard Brownian motion, $N_t$ is a Poisson process with intensity $\lambda$ and $(Y_i)_{i \geq 1}$ is a sequence of i.i.d. random variables with a double exponential probability distribution. That means that $Y := Y_i$ for all $i > 0$ has the following density:

\[
f_Y(y) = p \eta_1 \exp(-\eta_1 y)1_{\{y > 0\}} + (1-p) \eta_2 \exp(\eta_2 y)1_{\{y < 0\}}
\]  
(D.15)

with $0 < p < 1$ respectively $(1-p)$ denoting the probability of jumps upwards respectively downwards and $\eta_1 > 1, \eta_2 > 0$.

Without going too much into details Kou derives a formula for $\mathbb{P}(S_1 > a)$ for $a \in \mathbb{R}$ (Theorem B.1 in [35]).

Also important for this thesis, the Laplace exponent of $S_1$ is given by:

\[
G(x) = \tilde{\mu} x + \frac{1}{2} \sigma^2 x^2 + \lambda \frac{p \eta_1}{(\eta_1 - x)} + \lambda \frac{(1-p) \eta_2}{(\eta_2 + x)} - \lambda
\]  
(D.16)

which gives us for the equation $G(x) = q$ for $q > 0$ the solutions $-\infty < -\zeta^{(q)}_-(1) < -\zeta^{(q)}_-(0) < \zeta^{(q)}_+(0) < \zeta^{(q)}_+(1) < \infty$. Note that $\tilde{\mu} := r - \lambda \left( \frac{p \eta_1}{(\eta_1 - 1)} + \frac{(1-p) \eta_2}{(\eta_2 + 1)} - 1 \right) - 0.5$ with $r > 0$ chosen such that $(\exp(-rt)S_t)_{t \geq 0}$ is a martingale.
Bibliography


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