RELAXATION MODEL OF SOLAR CORONAL HEATING IN MERGING MAGNETIC FLUX TUBES

A thesis submitted to the University of Manchester for the degree of Master of Science in the Faculty of Engineering and Physical Sciences

2014

By
Francesca Arese Lucini
School of Physics and Astronomy
4.3.6 Finding the relaxed state ........................................ 55
4.3.7 Energy .................................................................... 56

5 Merging flux tubes on the solar corona .......................... 59
5.1 Helicity calculation for finite length loop ...................... 59
5.2 Circular cross section ................................................ 62
5.3 Reversed Helicity ....................................................... 64
5.4 Multiple merging flux tubes ........................................ 65
5.5 Study of the magnetic energy released $\Delta W$ ................. 69

6 Conclusions ................................................................. 71
6.1 Summary and Conclusions ........................................... 71
6.2 Future Work ............................................................ 74

Bibliography ................................................................. 76

Word count: 35371
List of Tables

2.1 Typical values for a coronal loop. ........................................... 24

6.1 Typical values for a coronal loop (2). ................................. 72

6.2 Typical values for the energy released in the case of 4 merging flux ropes. $\Delta W^*$ are the dimensionless values while $\Delta W$ are dimensional and are measured in Joules. To find the dimensional energy we used the same formula used in the case of two merging tubes ($\Delta W = \frac{LSB^2}{\mu_0} \Delta W^*)$. $\alpha$ is in all cases set to be equal to 1.4. .......................... 72

6.3 Obtained values of the energy released in the case of multiple flux tubes. As shown in fig.(5.1), the energy released gets larger when the number of flux tubes increases. Even in this table $\alpha = 1.4$. ................................. 73

6.4 Comparison between the dimensionless energy release $\Delta W$ obtained computationally and $\Delta E$ calculated with the interpolation function (fig.(5.12)). .................................................. 73
List of Figures

1.1 Picture of Sun’s activity by SDO [http://sdo.gsfc.nasa.gov] . . . . . . . . 15

1.2 Picture of solar flares and flux tubes by SDO [http://sdo.gsfc.nasa.gov] 16

2.1 Model of a twisting flux tube; a cylinder of length L with circular or
rectangular cross section. Instabilities of the photospheric layer cause
the footpoints to twist. This twisting is characterized by the value of
\( \alpha \) in eq. (2.14) which also affects the amount of energy released (31). . 23

3.1 Magnetic reconnection in the proximity of a current sheet [http://
en.wikipedia.org/wiki/Magnetic_reconnection] . . . . . . . . . . 26

3.2 Petschek’s magnetic reconnection in the proximity of a current sheet
[https://www.teaching.physics.manchester.ac.uk/COURSES/PC3511/
Chapter_4_part_2_2012.pdf] . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27

3.3 Field lines link the toroidal flux [5]. In this case the linking number
is 5; the field lines are wined 5 times around the toroidal flux. . . . . 28

3.4 Toroidal pinch configuration [35] . . . . . . . . . . . . . . . . . . . . . . . 31

3.5 Evolution of a flux tube at four different times. The coronal loop un-
dergoes the kink instability which causes a progressive displacement
of the tube (31). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 32

3.6 Evolution of flux tubes in the magnetic carpet [30] . . . . . . . . . . 35

3.7 2D model of a coronal loop arising from the network elements in the
granule cell. These sources are placed at \( z = -L, L \). The separatrix
joins the loops and is represented by dashed lines (30) . . . . . . . . . . 35
4.1 Picture of MAST inside tokamak

4.2 Section of merging flux tubes in MAST

4.3 Snapshots of merging flux tubes with rectangular cross section

4.4 Model of the flux tube with squared cross section $a = b = 1 \overrightarrow{B} \cdot \vec{n} = 0$

4.5 Poloidal field lines for $a = 1$, $b = 1$ and $\alpha = 1.5$. When getting closer to the center, the magnitude of the magnetic flux increases.

4.6 Flux as a function of (left) $y$ when $x = a^2$ and $\alpha = 1.5$; (right) $x$ when $y = b^2$ and $\alpha = 1.5$ for $a = b = 1$.

4.7 Flux as a function of $x$ when $y = b^2$ when varying the value of $\alpha$. $\alpha = 0.7, 1.5, 2 \text{ and } 3$. The peak of the flux function increases when increasing the value of $\alpha$.

4.8 Magnetic flux for a cylinder with square cross section $a = b = 1$ and constant value of $\psi_b = 1$.

4.9 $B_z$ vs $\alpha$ for a cylinder with square cross section away from the center, for example for $x = a/3$ and $y = b/3$.

4.10 $B_z$ vs $\alpha$ for a cylinder with square cross section $a = b = 1$ and constant value of $\psi_b = 1$.

4.11 a) $B_y$ as a function of $x$ for fixed value of $y = a/2$ b) $B_x$ as a function of $y$ for fixed value of $x = a/2$ and for both cases $\alpha = 1.5$.

4.12 $\psi_b$ vs $\alpha$ for $a = b = 1$. $\psi_b$ is still constant with respect to $x$ and $y$ but it does depend on $\alpha$. $\psi_b \to 0$ when $\alpha \to \alpha_{11}$.

4.13 a) $K$ vs. $\alpha$ with $\psi_b = 1$ and b) Normalized $K$ vs $\alpha$ for $a = b = 1$ and $\psi_b$ given by eq.(4.27).

4.14 Cross section of two merging tubes. The red line is the current sheet where reconnection occurs. Also refer to fig.(3.1).

4.15 Normalized current vs $\alpha$ for $a = b = 1$ and $\psi_b$ given by eq.(4.27).
4.16 Contour of the poloidal field lines for the initial and final state according to our model for relaxation. $\alpha_i = 1.25$ and $\alpha_f = 0.782$. The initial tubes have squared cross section ($a = b = 1$), they undergo magnetic reconnection and relax to a final state with rectangular cross section ($a = 1, b = 2$) [9].

4.17 Normalized energy vs $\alpha$ for $a = b = 1$ and $\psi_b$ given by eq. (4.27).

4.18 Profile of the $\alpha_f(\alpha_i)$ and $\Delta W(\alpha_i)$ in the case of $\psi_b$ given by eq. (4.27) and values of $a = b = 1$ for the initial state, and $a = 1, b = 2$ for the final state. This is the same figure illustrated in [9], which confirms the validity of our MATLAB programs.

5.1 Contour plot of the poloidal field lines for a) $\alpha = 1.5$, b) $\alpha = 4$, c) $\alpha = 7$, d) $\alpha = 10$ and in all cases $a = 1$ and $b = 1$. Darker the color, higher is the value of the magnetic flux. $B_z$ (of the poloidal field lines hence $\psi$), is pointing outwards in the center of the contour and in d) is pointing inwards in the four areas at the corners of the squared cross section. Configuration c) and d) could be produced when we have multiple streams of toroidal current that produce poloidal fields. In this case the poloidal field is much less tidy and has more than one maxima and minima in strength for different values of $x$ and $y$.

Thanks to the flux we took (eq. (4.13)) and the relation between the flux and the field (eq. (4.2)), even the magnetic field has an oscillating configuration.

5.2 a) $B_z$ vs $\alpha$ with $x = y = a/2$ b) $\psi_b$ vs $\alpha$ when $\phi_t = 1$ c) normalized K vs $\alpha$ d) normalized $W$ vs $\alpha$. In all cases $a = b = 1$. We can notice how all quantities tend to infinity in correspondence to the eigenvalues.
5.3 Comparison of the normalized helicity between the rectangular case and the circular cross section case with fixed value of $\alpha = 1.5$. The big rectangle has $a = b = 1$ while the small rectangle has $a = b = \frac{1}{\sqrt{2}}$ and the circle has $R = 1/2$. $K$ for the circular case has values that lie in the middle of the values generated by the two rectangular cases.

5.4 Comparison between two merging tubes with a) constant $\alpha$ and b) with opposite in sign $\alpha$. We can see that in case a) the field lines at $y = b/2 = a$ go in opposite directions therefore they tend to break. In case b) field lines go in the same direction therefore they avoid spontaneous reconnection.

5.5 Energy released for a reversed helicity configuration compared with the case of constant $\alpha$. In both cases each tube has a rectangular cross section with $a = b = 1$ therefore the final state will have $a = 1, b = 2$.

5.6 Energy release $\Delta W$ and $\alpha_f$ as a function of $\alpha_i$ in the case of two and four merging flux tubes. Each tube has been modeled as having a square cross section ($a = b = 1$), therefore the final state has $a = 1, b = 2$ in the case of two loops merging and $a = b = 2$ in the case of four merging loops.

5.7 Initial state has two flux tube each having a square cross section ($a = b = 1$), the partial state has $a = 1$ and $b = 2$ and the final state has $a = b = 2$ as in the case of four merging loops.

5.8 Energy release $\Delta W$ (dashed line) and $\alpha_f$ (continuous line) as a function of $\alpha_i$ in the case of two and four merging flux tubes and the partial state which is a possible intermediate step that is not relaxed yet because it is not a state of minimum energy since we are still in the presence of current sheets. Each tube has been modeled as having a square cross section ($a = b = 1$). The red curve represents 2 merging tubes, the green is for the partial state and black is for 4 merging flux tubes.
5.9 Case of 4 merging flux tubes with reversed helicity. a) $-\alpha, \alpha, \alpha, \alpha$

b) $-\alpha, -\alpha, \alpha, \alpha$
c) $-\alpha, -\alpha, -\alpha, \alpha$

5.10 Energy release $\Delta W$ as a function of $\alpha_i$ in the case of four merging flux tubes considering the possibility of reversed helicity. a) one $-\alpha$, b) two $-\alpha$ compared with the case of constant-$\alpha$. Each tube has been modeled as having a square cross section with $a = b = 1$, therefore the cross section for the three different initial configurations is the same.

5.11 Energy release $\Delta W$ as a function of the number of merging flux tubes with fixed area $a \times b$ ($a = b = 1$) illustrated in two different ways. What changes when considering a different number of merging tubes is the length of the square for the single tube, not the total area. This is different from what we have done in fig.(5.6); in the latter case, the area of the initial state, was different from the area of the final state. We are now comparing energy releases in the same volume.

5.12 Energy release $\log_e(\Delta W)$ as a function of the logarithm of the number of merging flux tubes with fixed $a = b = 1$. We can see that the equation for the interpolation is $y = 2.1393x - 3.5254$. Therefore

$$\Delta W = 0.0294 \text{num}^{2.1393}$$

6.1 Cross section of two flux tubes with different areas. Notice that $b_1 + b_2 = 2a$.

6.2 Difference of the magnetic pressure between two flux tubes with different cross sections. $\alpha_1 = 1.5, a = 1$ and we vary the position of the boundary(change values of $b_1$). ($\alpha_1, \alpha_2 > 0$)
Abstract

In this year of work my final goal was to give a possible explanation to the coronal heating problem; starting off by studying and understanding the interaction of flux tubes in the Mega Ampere Spherical Tokamak - MAST- we have applied the same theories used for MAST to the solar corona, environment populated by plenty of flux tubes that merge together and in doing so, release energy. We suggest that the relaxation theory developed by Taylor in 1974, can be used to explain the release of magnetic energy when several flux tubes merge together on the solar corona. Starting by analyzing the interaction of two flux tubes (in MAST and on the solar corona), we shall explore properties of numerous solar loops merging together.

Throughout the whole dissertation, the flux tubes are modeled as cylindrical loops with rectangular (mainly squared) cross section, therefore we shall use cartesian coordinates to describe them. It is important to understand the origin of the flux tube in these two different environments; in MAST these are generated by the magnetic fields that should confine hot plasma inside the tokamak device, while in the solar corona, they are generated by the twisting of the photospheric footpoints; different twists of the footpoints (difference in velocity, strength etc. ) are represented by the difference in value of $\alpha$ where $\alpha = \frac{\vec{j} \cdot \vec{B}}{B^2}$ and according to the value of $\alpha$, the energy released will vary in magnitude.
Declaration

I declare that no portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Francesca Arese Lucini

University of Manchester

Jodrell Bank Center of Astrophysics

Oxford Road

Manchester
Copyright

i. The author of this thesis (including any appendices and/or schedules to this thesis) owns certain copyright or related rights in it (the Copyright) and s/he has given The University of Manchester certain rights to use such Copyright, including for administrative purposes.

ii. Copies of this thesis, either in full or in extracts and whether in hard or electronic copy, may be made only in accordance with the Copyright, Designs and Patents Act 1988 (as amended) and regulations issued under it or, where appropriate, in accordance with licensing agreements which the University has from time to time. This page must form part of any such copies made.

iii. The ownership of certain Copyright, patents, designs, trade marks and other intellectual property (the Intellectual Property) and any reproductions of copyright works in the thesis, for example graphs and tables (Reproductions), which may be described in this thesis, may not be owned by the author and may be owned by third parties. Such Intellectual Property and Reproductions cannot and must not be made available for use without the prior written permission of the owner(s) of the relevant Intellectual Property and/or Reproductions.

iv. Further information on the conditions under which disclosure, publication and commercialisation of this thesis, the Copyright and any Intellectual Property and/or Reproductions described in it may take place is available in the University IP Policy (download pdf file from http://documents.manchester.ac.uk/DocInfo.aspx?DocID=487), in any relevant Thesis restriction declarations deposited in the University Library, The University Library’s regulations (see http://www.manchester.ac.uk/library/aboutus/regulations) and in The University’s policy on presentation of Theses.
Francesca Arese Lucini
Jodrell Bank Centre for Astrophysics,
Alan Turing Building,
University of Manchester,
Oxford Road,
Manchester,
M13 9PL.

Supervisor : Philippa Browning
Chapter 1

The solar corona and the solar coronal heating problem

The solar corona is the high temperature portion of the Sun’s atmosphere that lies above the photosphere and chromosphere. The photosphere is the star’s outer shell and it lies at the top of convection cells called granules; these are cells with rising hot plasma in the middle surrounded by falling cooler plasma. On the other hand, the corona is composed by extremely rarefied gas (with an average density of a few micrograms per cubic meter) and can reach temperatures of $10^6$ K or more. Most studies on the solar corona concentrate on the very important physical fact that the temperatures reached by the solar corona are much bigger than the temperatures in the photosphere (6000 K) (14).

The best way to see the solar corona is during a total solar eclipse; during clear days at ground level the sky brightness exceeds that of the corona by 3 to 5 orders of magnitude so the corona is totally invisible to a naked eye. During a solar eclipse, the disc of the photosphere is covered but the portions of the sky around the Sun are visible. It is usual procedure to divide the solar corona into three distinct area; the active regions, the quiet Sun regions and the coronal holes. Space observations of the solar corona are done by satellites, such as TRACE or spacecrafts (e.g Voyager) in highly ionized X emission lines with emit EUV radiation at temperatures of $10^6$
K.

*Active regions* As the name suggests most of the activity of the Sun happen in these regions that are only a small part of the total surface of the star. These regions are located in areas of strong magnetic field concentrations and can be distinguished by the coronal loops, that are hotter and denser than the background corona. Typical phenomena in active regions are plasma heating, solar flares and coronal mass ejections due to magnetic reconnection processes. It is commonly accepted that Coronal mass ejections are the most important manifestation of solar activity and cause the rise of the corona’s temperature; they also play a central role in the solar cycle.

*Quiet Sun* Opposed to the active regions, the remaining area outside the active regions is called Quiet Sun. However, many dynamic processes occur in this area such as nano flares and explosions.

*Coronal Holes* are areas in the Sun that are colder and less dense than the average of the rest of the corona. These areas are dominated by open magnetic field lines, that act as efficient conduits used to transport heated plasma into the solar wind. Because these holes are mostly empty, they appear darker than the quiet Sun region.
There are two different magnetic zones in the solar corona and these have different properties; the first dominated by closed field lines, the second has open field lines. The open field line regions exist in the polar region of the Sun and in coronal holes on the disk; they are the source of the solar wind. In the closed field regions coronal loops are formed and are produced by the heated plasma that remains trapped by the closed field lines. The magnitude of the solar magnetic field varies from values of 2000 G in sunspots to 0.5 G in the Quiet Sun region in the photosphere.

Very important in the study of the Sun is the *Dynamo Theory* that studies the conditions under which the plasma motion generates electric and magnetic fields in the solar interior. The motions of the plasma, which is a conducting fluid, induces electric currents that will themselves produce a magnetic field. In the Sun the source of the motion of the conductor is the outflow of energy from the solar core that makes temperature gradients responsible for convective motions of the fund near the surface (14). Dynamo theory could be used to explain magnetic fields on other stars and planets which makes it a constant field of study for many scientists. When applied to the Sun, this theory can be compared with the observations provided by various satellites and therefore it is taken as a test case for future *stellar dynamo theory* (26).
As previously mentioned the corona reaches extremely high temperatures, even though the layer before it, the photosphere, is much cooler. This problem was first considered by Grotrian (1939) and Edlen (1942) when they observed that the emission lines seen during a total eclipse required extremely high temperatures to reach the observed stages of ionization. This appears to not be in agreement with thermodynamic laws which would require the temperature to drop as one moves further away from the Sun’s core. This is known as the coronal heating problem and the only way to solve it is to accept that the solar corona is being heated. Numerous possible theories have been proposed, but it has been difficult to prove which of these theories are correct. The difficulty lies in the identification of source of energy and how this energy release is converted into heat. The energy that is required from the heating source $E_H$ has to balance the two major energy losses, $E_R$ (radiative loss) and $E_C$ (thermal conduction). All of these energies depend on the spatial location. The corona is very inhomogeneous so the heating requirement varies a lot depending on the region under analysis. Assuming that the magnetic field is well structured and organized, neighboring structures are isolated and therefore can be treated separately. This allows us to express the energies in terms of the space coordinate $s$ so that

$$E_H(s) - E_R(s) - E_c(s) = 0 \quad (1.1)$$

It must be noted that the difference between the initial energy and the energy at minimum state depends on the amount of distortion of the field lines during magnetic reconnection.

It is recognized that the energy source capable of sustaining the basic energy losses are the motions at the photospheric level which display the footpoints on the corona. These motions increase the corona’s free energy at a rate which is given by the Poynting flux through the following equation:

$$\vec{F} = -\frac{1}{4\pi} \vec{B}_v \cdot \vec{B}_h \cdot \vec{v}_h \quad (1.2)$$
where $\vec{B}_v$ and $\vec{B}_h$ are the vertical and horizontal components of the field and $v_h$ is the horizontal velocity of the footpoint \(^{(20)}\).

Studies have recently been made to understand if nano flares, compared to flares, could be more responsible for the coronal heating problem. The first to consider the problem was Parker \(^{(28)}\). The corona can be maintained at high temperatures by the combination of small, weak and rapid releases of magnetic energy through magnetic reconnection which are initialized by turbulence caused by the movement of the photosphere. In the nanoflare scenario, heating is a sporadic rather than a continuous process. Hudson \(^{(17)}\) demonstrated that even if nano flares release a smaller amount of energy, they could maintain the coronal temperature if they occur with high frequency. It has not been possible to determine if the nanoflares occur frequently enough to allow the heating of the corona but different models are being made to try and predict a priori the frequency of energy releasing events of different sizes \(^{(8), (2)}\).

The overall goal of our work is to explain how the solar corona is heated using a model based on relaxation theory. The basic theoretical framework is summarized first. In Chapter 2, we describe the MHD equations, and in Chapter 3 we shall explain the relaxation theory used to study the release of energy of the solar corona. In chapter 4, we shall illustrate the work done in Browning et al article for the Mega Ampere Spherical Tokamak and write down the general equations that will be used to describe MAST flux tubes and coronal loops. In Chapter 5 we can finally approach the relaxation problem in the solar corona and concentrate on studying what happens when two or more flux tubes merge together. Finally, conclusions and possible future works are presented in Chapter 6.
Chapter 2

Magnetohydrodynamics

Plasma is a many particle system dominated by collective behaviour. It can be approached in two different ways; the first is the *kinetic theory* that models the distribution function for each specie present in the plasma and therefore each specie is treated as a separate fluid. The second approach is *Magnetohydrodynamics* that treats the plasma as a single electrically conductive fluid which interacts with the magnetic fields. To obtain a complete set of equations we use equations of fluid dynamics and Maxwell’s equation with some approximations; the net charge density is identical to zero and the displacement current is assumed to be small compared to the conduction current. The first assumption is made considering that in first approximation the plasma is electrically neutral. The second approximation holds because the fluid motion tends to be slow compared to the characteristic time scales of the a plasma. MHD theory was originally considered valid for collisional plasma, where the collisional mean free path is short compared with the characteristic length scale of the system, for low frequency phenomena and subrelativistic velocities. It has later been applied to collisionless plasma such as the one present in Tokamaks and the Earth’s magnetosphere and it works well for these configurations. Each plasma state is identified by the mass density \( \rho \), the temperature \( T \), the velocity \( \mathbf{v} \) and the magnetic field \( \mathbf{B} \) at each point in space \( \mathbf{r} \) ant time \( t \). The main MHD equations are the induction equation, the momentum equation, the mass continuity
equation and the energy equation (32).

The induction equation is given by

\[ \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B} \]  

(2.1)

\( \eta \) denotes the electric resistivity assumed to be a scalar but not necessarily constant in space. This equation tells us how the magnetic field \( \vec{B} \) evolves with time when we can neglect the displacement current because the velocities are sub relativistic \( v \ll c \).

Ideal MHD treats the plasma as a perfect conductor and neglects the ohmic resistivity and therefore it neglects the displacement current in Ampere’s law, which is therefore reduced to

\[ \mu_0 I = \oint \vec{B} \cdot d\vec{l} \]  

(2.2)

From the induction equation one can identify the Magnetic Reynolds number that shows the relative importance of plasma motion and Ohmic resistivity and has the following expression

\[ Re = \frac{Lv}{\eta} = \mu_0 \sigma Lv \]  

(2.3)

where \( L \) and \( v \) are typical length scale and velocity. In the limit of \( Re \gg 1 \) the numerator dominates, therefore we are treating the plasma as highly conductive. Ideal MHD is to be considered the limit of infinite magnetic Reynolds number (defined in eq. (2.3)). It can be demonstrated that in a perfectly conducting plasma the magnetic field lines are frozen to the plasma and the magnetic flux through any fluid surface is conserved; if a volume of plasma is surrounding a field line it will keep lying on the same field line at any future time. This relation between the fluid and \( \vec{B} \) fixes the topology of the magnetic field; if the magnetic field lines have a certain geometry at \( t = 0 \), they will keep having the same geometry at any consecutive time until the system has \( \eta = 0 \).
The momentum equation has the following expression

\[ \rho \frac{d\vec{v}}{dt} = -\nabla p + \vec{j} \times \vec{B} + \rho \vec{g} \] (2.4)

that gives the time rate of change of the fluid momentum. The terms on the right hand side of the equation are the total force per unit volume, the pressure gradients, Lorentz force and gravity. The left hand side is the mass per unit volume multiplied by the acceleration seen by a moving fluid element and it is given by a convective derivative so that

\[ \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \] (2.5)

Equation (2.4) shows that the rate of change of the momentum of a fluid element is equal to the sum of the electric field, magnetic field and pressure forces acting on that fluid element.

The third equation is the mass continuity equation which is given by

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \] (2.6)

and it is identical to the mass continuity equation of fluid dynamics. It states that the rate of change of the mass in a small volume is equals to the mass flux into the same volume. It must be noted that the momentum equation do not define a closed system of equations, therefore there must be and equation of state in order to close the system of equations. The equation of state specifies the plasma pressure as a function of the temperature and density and it is given by the perfect gas law

\[ p = \frac{k_B \rho T}{m} \] (2.7)

where \( m \) is the mean particle mass. Finally an energy equation is needed. No energy leakages such as conduction radiation and Ohmic heating will be considered.
In this simple case, the energy equation can be written as

\[
\frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) = 0 \quad (2.8)
\]

where \( \gamma = \frac{C_P}{C_V} \) and its called the adiabatic index.

The combination of these equations give 9 non linear coupled equations to solve the 9 unknown variables that are the 3 components of \( \vec{B} \) and \( \vec{v} \), pressure \( p \), density \( \rho \) and temperature \( T \).

Given these equations many parameters and properties of the plasma can be determined and these will help in analysing the plasma. The Lorentz force can be written as

\[
\vec{j} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = -\nabla \left( \frac{B^2}{2\mu_0} + (\vec{B} \cdot \nabla)\vec{B} \right) \quad (2.9)
\]

and this expression tells us that the total pressure force is given by the sum of the magnetic pressure force and the thermal pressure force. The relative importance of the thermal force and the magnetic force is determined by the plasma beta value that is defined as the ratio of the thermal pressure and the magnetic pressure

\[
\beta = \frac{p}{\frac{B^2}{2\mu_0}} = \frac{2\mu_0 p}{B^2}. \quad (2.10)
\]

Magnetic field lines have tension, so they can carry waves with a certain velocity, called *Alfven speed*, given by

\[
v_A = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{B^2}{\mu_0 \rho}} \quad (2.11)
\]

where \( \rho \) is the total mass density and \( T = \frac{B^2}{\mu_0} \) is the magnetic tension. This is the speed at which the information on the changes in the magnetic field propagate. This is similar to the sound speed, in the sense that the speed of light represents the speed at which information about pressure changes propagate. Therefore, there are two natural timescales for a plasma of typical length \( L \). It is always interesting
to study the equilibrium in which forces balance. This happens when plasma flows are weak compared to the Alfvén speed and sound speed and the fields are slowly evolving when compared to the Alfvén time \( t = \frac{L}{v_a} \) and sound time \( t = \frac{L}{c} \). In this case the momentum equation is given by

\[
\vec{j} \times \vec{B} - \nabla p + \rho \vec{g} = 0
\]  

(2.12)

### 2.1 Models of astrophysical flux tubes

In the solar corona, MHD equations are used to study and analyze the plasma. \( Re \gg 1 \) and therefore the coronal plasma is highly conductive. Moreover the timescales are slow compared to Alfvén time and the flows are slow compared to Alfvén speed; this allows us to study the equilibrium equation; we can neglect gravity and the pressure gradients because the Lorentz force is dominant. Since \( \beta \ll 1 \) the magnetic forces are dominant over the thermal forces. In this particular case we have force free fields that are ruled by the following expression

\[
\vec{j} \times \vec{B} = 0
\]  

(2.13)

Figure 2.1: Model of a twisting flux tube; a cylinder of length L with circular or rectangular cross section. Instabilities of the photospheric layer cause the footpoints to twist. This twisting is characterized by the value of \( \alpha \) in eq. (2.14) which also affects the amount of energy released [31].
This means that the current density and the force free magnetic fields are parallel and therefore satisfy the following equation

$$\nabla \times \vec{B} = \alpha(r)\vec{B}$$  \hspace{1cm} (2.14)

where $\alpha(r)$ is

$$\alpha(r) = \mu_0 \frac{\vec{B}(r) \cdot \vec{J}(r)}{B^2}.$$  \hspace{1cm} (2.15)

and it depends on the spatial coordinates. Applying the solenoidal condition $\vec{\nabla} \cdot \vec{B} = 0$ to eq.(2.14) gives

$$\vec{B} \cdot \nabla \alpha(r) = 0$$  \hspace{1cm} (2.16)

which shows that $\alpha$ has constant value along each field line \(^{[21]}\).

<table>
<thead>
<tr>
<th>Typical length scale(m)</th>
<th>10^6</th>
</tr>
</thead>
<tbody>
<tr>
<td>ion species</td>
<td>Hydrogen</td>
</tr>
<tr>
<td>magnitude $\vec{B}$ (T)</td>
<td>0.01</td>
</tr>
<tr>
<td>temperature (eV)</td>
<td>100</td>
</tr>
<tr>
<td>density ($\frac{1}{m^3}$)</td>
<td>10^{15}</td>
</tr>
<tr>
<td>plasma $\beta$</td>
<td>10^{-4}-10^{-2}</td>
</tr>
<tr>
<td>Lundquist number</td>
<td>10^{13}</td>
</tr>
<tr>
<td>Sweet-Parker width (m)</td>
<td>10^{13}</td>
</tr>
</tbody>
</table>

Table 2.1: Typical values for a coronal loop.

Often, the curvature of a flux tube, its bending back into the photosphere, is not considered and therefore flux tubes can be modeled as cylindrical loops of twisted magnetic field, embedded in a potential field layer and an outer conducting wall at large radius. With the present observations, magnetic field can be computed only at the photospheric level \(^{[11]}\). The boundary conditions applied are that the boundary along the $x$ and the $y$ direction $B_n = 0$. This condition is not valid in the $z$ direction, where for $z = 0$, $L$, where $L$ is the length of the cylinder, magnetic field lines cross the surface of the loop \(^{[23]}\).
Chapter 3

Magnetic Reconnection, Helicity
and Relaxation Theory

3.1 Magnetic Reconnection

Magnetic reconnection is a process occurring frequently both in laboratory and in astrophysical plasma. It can be defined as a topological change of the magnetic field in a localized region caused by a change in the connectivity of its field lines. This change allows the release of stored magnetic energy. Therefore, magnetic reconnection has been proposed as a mechanism for the heating of the solar corona. It must be noted that reconnection is relevant only in highly conductive plasmas. In MHD theory, processes that convert magnetic energy into other forms of energy can be distinguished as ideal or non-ideal. The first, converts magnetic energy into kinetic energy without dissipation while the second dissipates magnetic energy into kinetic energy and heat. Magnetic reconnection must therefore, be treated as a non-ideal process in MHD. What stores the magnetic energy is the current sheet which increases the energy density of the magnetic field.

To understand the physical role that magnetic reconnection plays in the solar activity it is essential to also investigate what happens before reconnection begins. The prephase starts with a slow evolution due to external forces. During this phase,
energy is supplied to the system in the form of magnetic energy. During reconnection
the system shows a fast evolution, which may involve a change of the magnetic
topology, associated with a conversion of magnetic to kinetic energy.

(6) The term reconnection was introduced by Dungey in 1953. Using the MHD
equations, Dungey argued that the current produced by the motion of particles
would take the form of a thin sheet in which diffusion would cause the field lines
passing through the current sheet to change their connectivity. This process was
describes as reconnection. In 1957 Sweet and Parker developed their quantitative
model for reconnection focusing on the problem of two dimensional steady state
reconnection (the electric field $E$ in the invariant direction is uniform in space) in an
incompressible plasma. They assumed that reconnection occurs in a current sheet
whose length is set by a global scale of the field. Fig. (3.1) shows that in a thin
current sheet non-ideal effects allow the field lines to break and reconnect.

![Figure 3.1: Magnetic reconnection in the proximity of a current sheet](http://en.wikipedia.org/wiki/Magnetic_reconnection)

Under these conditions Sweet and Parker determined that the speed of the
plasma flowing into the current sheet is given by

$$v_i = v_A S^{-\frac{1}{2}}$$

(3.1)

where $S$ is the *Lundquist number* which represents the ratio between the time
scales of resistive diffusion and typical Alfven waves. The outflow speed of the plasma
from the current sheet is $v_o = v_A$ and it does not depend on $S$. (6) The reconnection
rate in two dimensions is measured by the electric field at the reconnection site.
The electric field prescribes the rate at which magnetic flux is transported from one
3.1. MAGNETIC RECONNECTION

topological domain to another.

In astrophysical plasmas $S$ is very large, so Sweet Parker reconnection is too slow to account for phenomena such as solar flares and flux tubes. In 1964, Petschek proposed a model with an increased rate of reconnection.

![Petschek's magnetic reconnection in the proximity of a current sheet](https://www.teaching.physics.manchester.ac.uk/COURSES/PC3511/Chapter_4_part_2_2012.pdf)

In Petschek’s model, the length of the current sheet is reduced when compared to the Sweet-Parker model. Furthermore his theory encloses the current sheet in an exterior field with a global scale length of $L_e$ as shown in fig. (3.2).

It seems as if reconnection in two dimensions has been understood clearly, whilst reconnection in three dimensions is still being studied. It looks as if in 3D reconnection could happen in places where there is non vanishing field strength \((6), (10)\).

Applying reconnection theory to the solar corona, we can see that coronal loops can be heated by reconnection in many ways. The classical model for such heating was proposed by Parker \((27)\) and developed by many others. For example, Priest \((30)\) proposed that the heating would occur in terms of the formation and dissipation of current sheets by the twisting of footpoints of an essential uniform magnetic field. According to Priest, the corona can be described by a tectonics model that attempts to take in to consideration the effect of the magnetic carpet on coronal surface. It suggests that many current sheets are continually forming and dissipating at the separatrices as discussed further below.
3.2 Helicity and relaxation theory

During early 19th century Gauss discovered a formula that counts the linking of two curves called the Gauss linking number and its a double integral over the two curves. The magnetic helicity sums the Gauss linking number over every pair of field lines within a volume. The total magnetic helicity is defined as

\[ K = \int \vec{A} \cdot \vec{B} dV \]  (3.2)

where \( \vec{A} \) is the magnetic vector potential satisfying \( \nabla \times \vec{A} = \vec{B} \).

In the case of two interlinked flux tubes, the helicity describes the number of times that each closed field line of the first tube is linked with the field lines of the second tube.

The helicity integral measures topological properties of field lines. It is a measure of how much a magnetic field is "knotted". It has been demonstrated to be a invariant of plasma motion in ideal MHD; if we have an infinite conductive plasma, \( dK/dt = 0 \) (3.2). This means that if one field line is linked to another n times, then the two field lines must remain linked n times during all of plasma’s subsequent motion. This property is important in ideal MHD because K is conserved for every flux tube volume (4). Of course in any physical plasma, the conductivity is finite; for a small value of resistivity, topological properties of lines of force are no longer preserved and can undergo reconnection. For small plasma resistivity (3), while the
local helicity may no longer be conserved, the global helicity can still be considered a constant on magnetic reconnection time scales. Also, $K$ is invariant to gauge transformation when $S$ is a magnetic surface ($\vec{B} \cdot \vec{n} = 0$). Taylor has demonstrated that after an initial turbulent phase, such as a kink instability for astrophysical configurations, the plasma settles into a more inactive state after seeking for a preferred configuration, the relaxed state which is a particular minimum energy state. Other experiments have been made to demonstrate the conservation of magnetic helicity during plasma relaxation such as in the Madison Symmetric Torus reversed-field pinch.$^{(19)}$

To find the particular relaxed state it is necessary to minimize the magnetic energy given by the following expression

$$W = \frac{1}{2} \int (\nabla \times \vec{A})^2 dV$$

(3.3)

with the constraint that the total magnetic helicity is conserved where the helicity is given by eq (3.2). It must be noted that a recent article$^{(37)}$, which presents resistive MHD simulations which suggest that the conservation of $K$ is not a strong enough constraint to undergo relaxation. It could be possible that some other topological constraints should be considered. However we shall follow the main stream of plenty of other scientists and see which plausible results the application of Taylor’s theory produces.

The solution of the minimization is the force free field equation

$$\nabla \times \vec{B} = \alpha \vec{B}$$

(3.4)

$$\vec{B} \cdot \nabla \alpha = 0.$$  

(3.5)

In eq. (2.14) $\alpha$ was constant along a field line, in the case of a force free field $\alpha$ is a spatial constant (note that in laboratory plasma literature $\alpha$ is usually denoted with $\mu$).$^{(29)}$ For a constant-$\alpha$ or force free field the plasma energy decays during
relaxation, and \( K \) remains constant. If we were to consider a resistive plasma then

Faraday’s law would be

\[
\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} + \eta \nabla^2 \vec{B} \tag{3.6}
\]

\[
\frac{\partial \vec{A}}{\partial t} = \vec{v} \times (\nabla \times \vec{A} - \nabla \phi + \eta \vec{j}) \tag{3.7}
\]

therefore the decay rate of the magnetic energy and helicity is proportional to the electric resistivity \( \eta \) \(^{(35)}\)

\[
\dot{K} \approx -2\eta \int \vec{j} \cdot \vec{B} dV \tag{3.8}
\]

\[
\dot{W} \approx -\eta \int j^2 dV. \tag{3.9}
\]

From these equations, one can derive how energy and helicity vary with time; however, the decay of the helicity is much slower than the energy decay if the currents are concentrated into current sheets.

The final relaxed state depends only on \( \alpha \) and it has been shown that it is totally determined by \( K \) and the magnetic flux \( \Psi \). This process that allows the plasma to access a particular minimum energy state is called plasma relaxation. Since the states are of minimum energy, all relaxed states are stable against perturbations that leave helicity invariant.

3.3 Relaxation models for solar coronal heating

The first to introduce the relaxation theory was J. B. Taylor in 1974 \(^{(35)}\) regarding laboratory plasma in a toroidal pinch as shown in fig (3.4). In his paper Taylor treats plasma as a highly conductive fluid with small resistivity. Taylor demonstrated that in this model turbulence allows magnetized plasma to spontaneously reach a
3.3. RELAXATION MODELS FOR SOLAR CORONAL HEATING

unique configuration of lowest magnetic energy subject only to the constraint of conservation of the total helicity. It must be noted that the relaxed states are stable for perturbations that conserve helicity since these are states of minimum energy. Jensen and Chu (18) demonstrated that Taylor’s configuration is well defined when the shape of a conducting boundary, the boundary conditions and the total helicity of the system are given. The boundary condition assumed during the relaxation model is $B_n = 0$ which consequently tells that the magnetic flux is invariant. It must be noted that the relaxation occurs on short time scales when compared to resistive diffusion time and on these short timescales helicity is conserved. For higher timescales both the energy and helicity decay with $\eta$.

During the years scientists, for example (15), (21), (2), (7), (25), (24), (22) applied this theory to propose a coronal heating model driven by the fact that coronal plasma has the same characteristics as the laboratory plasma described above and the turbulence Taylor was discussing in his article is similar to the kink instabilities that happen in the corona in the sense that they both minimize the magnetic energy. These recent studies have brought to light that merging flux tubes on the solar coronal could be responsible for the coronal heating. They also discuss both the storage and the release of energy during the process; the storage of energy in the field occurs when the initial state of the magnetic field is sufficiently stressed beyond the relaxed state. Energy is stored over timescales longer than the Alfvén
travel time, which is expected to be the dominant situation in the solar active regions. Under the presence of an instability which allows magnetic reconnection and form a current at the flux tube’s footpoints, the energy is then released.

The initial system that is considered is a cylindrical loop and the magnetic field is untwisted, which means that the only component of the magnetic field is in the axial direction. Fig. (3.5) shows the evolution of a twisting flux tube when it is modeled.

![Figure 3.5: Evolution of a flux tube at four different times. The coronal loop undergoes the kink instability which causes a progressive displacement of the tube](image)

If the twisting of the field is localised to a small region then the introduction of an azimuthal field component will give a zero total axial current in the final state. When we refer to a zero net current we mean that

\[
I = \int_S \vec{j} \cdot d\vec{S} = 0 \quad \text{(3.10)}
\]

\[
\mu_0 I = \oint B_\theta dl = \oint B_\phi dl \quad \text{(3.11)}
\]

where \(S\) is the cross section of the loop and the second equation is Ampere’s law that expresses the relation between \(B_\theta\) and the current \(I\). Hence, if the net current is zero, the azimuthal field vanishes everywhere outside the loops.
When a current sheet is formed, there is localised heating inside the sheet and in the reconnection outflow. Most of the studies done on the dynamics of the system do not take in consideration the radiative losses and thermal conduction, the value of the obtained temperatures need to be interpreted as the maximum values of the relaxation state. The plasma is constrained to follow the field lines that exit the photosphere and bend back in it. What supplies the energy is the movement of the footpoints which produces current which are then dissipated under the form of heat. The way in which the system dissipates heat is given by magnetic reconnection in localized current sheets. The release of energy causes the system to relax in a minimum state energy where the field is a linear or constant $\alpha$ force-free state along all field lines (35). As the field evolves in the relaxed state, the field lines change in connectivity and here the reconnection begins. The convective motions are also cause of the twisting of the footpoints and therefore cause the injection of helicity which is then conserved throughout the process. It also allows the reconfiguration of the coronal fields rendering the field lines to undergo instability (16); for more flux tubes the kinetic energy of the convective zone is transported through the photosphere and stored in the flux tube. The time scale in which the twisting occurs is long compared to the Alfvén time and so the coronal loop transitions through a series of force-free field equilibrium expressed $\nabla \times \vec{B} = \alpha \vec{B}$ where $\alpha = \frac{\vec{j} \cdot \vec{B}}{B^2}$. In order to trigger the energy release the flux tube must undergo a certain turbulence; for the coronal problem, a typical trigger for turbulence is a kink instability which deforms the fields. MHD models show that the energy is released from the magnetic fields during the nonlinear phase of an ideal kink instability (16).

As previously discussed, Priest wrote an article illustrating a model for the coronal heating. (30) The tectonic model for the coronal heating consists in the study of the motions of the photosphere that drive the formation and dissipation of current sheets along the separatrices contributing to the dissipation of energy. The aim is to understand how the presence of unstable sources (the current sheets) affect the release of energy on the solar corona and how these change the magnetic field that
would otherwise be uniform throughout the entire surface. Results from *SOHO* have shown that the surface of the Sun is covered with a magnetic carpet, in which magnetic fragments are continually emerging, merging and canceling. A picture of the typical behavior of the photospheric magnetic flux is illustrated in fig (5.10). (30)

First of all a bipolar region is formed in a granule cell of the photosphere but the two poles of this region separate quickly. During the second phase, each polarity moves towards the boundary of the conductive cell and fragments into a high number of network elements. In the last phase, the network elements move along the boundary and eventually they can fragment into smaller elements or cancel. These network elements are the footpoints of the coronal loops and consequently photospheric flux is related to a coronal loop. The motion of the network elements will force coronal magnetic fields to interact and reconnect with the overlying field and form and dissipate current sheets on separatrix surfaces. Priest not only describes the tectonics of the magnetic carpet, he also gives explanation on how the photospheric motion could release heat. According to Priest heat will be issued the birth of bipolar regions will conduct to magnetic reconnection with the already existing magnetic field. Reconnection will be driven when the footpoints twist (the network points around the granule region), at the null points and at the separatrix surfaces, where the current sheets will both be formed and be canceled because of the movement of independent flux elements.

In the next chapter, we shall discuss how the relaxation theory is applied to laboratory plasmas following the guidelines of Browning et all article (9). We shall also show how the energy released by two merging flux tubes depends on the value of $\alpha$. 
3.3. RELAXATION MODELS FOR SOLAR CORONAL HEATING

Figure 3.6: Evolution of flux tubes in the magnetic carpet \( (30) \).

Figure 3.7: 2D model of a coronal loop arising from the network elements in the granule cell. These sources are placed at \( z = -L, L \). The separatrix joins the loops and is represented by dashed lines \( (30) \).
Chapter 4

Relaxation model of merging flux tubes in MAST

Figure 4.1: Picture of MAST inside http://en.wikipedia.org/wiki/Spherical_tokamak

4.1 Overview

In 2013 Browning et all [9] proposed an article in which relaxation theory is used to model merging of flux tubes in the Mega Ampere Spherical Tokamak operating at Culham, UK since 1999. In MAST, the plasma resembles astrophysical plasmas; the Lundquist number is high and the $\beta$ value is small.
A spherical tokamak is a fusion device that uses magnetic fields to confine the plasma. It is different from a tokamak device because instead of having a 'donut' shape with a large hole in the middle, the spherical tokamak reduces the hole to small dimensions forcing the confined plasma inside the device to have an almost spherical shape (fig.(4.3)).

In this configuration (fig.(4.2), fig.(4.2) and fig.(4.3)) two toroidal flux ropes with parallel toroidal currents are produced around poloidal field coils. Changing the current causes the flux ropes to move away from the coils and attract at the midplane due to their parallel currents. As in astrophysical plasmas, the relaxed state is a linear force free field that satisfies eq (2.14).

Assuming that initially the flux ropes have rectangular cross section and that they are governed by force free fields, Browning et al (9) suggest they undergo magnetic reconnection subject only to the constraint of helicity conservation and
reach a state of minimum energy.

There is another article published recently that concentrates on the study of this particular fusion device \cite{34}; according to Stanier et all, the strong magnetic field \((B \approx 0.5 \, \text{T})\), low temperatures and densities \((T \approx 10 \, \text{eV}, n \approx 5 \times 10^{18} \, \text{m}^{-3})\) make it the perfect device for magnetic reconnection experiments and for obtaining results that could be relevant to the astrophysical case. In this article though, there is no reference to relaxation theory; results are presented by simulating two-fluid models in both MHD and resistive MHD for the two merging tubes in MAST using cartesian coordinates.

![Model of the flux tube](image)

Figure 4.4: Model of the flux tube with squared cross section \(a = b = 1\) \(\vec{B} \cdot \vec{n} = 0\) at \(x = 0, a\) and \(y = 0, b\) and \(\vec{B} \cdot \vec{n} \neq 0\) at \(z = 0, L\).

### 4.2 General equations

Lets start off by pointing out some common physics for both merging laboratory and solar flux tubes.

As we have previously explained, the relaxed state is given by the conservation of helicity and it is a state of minimum energy. The plasma of a flux tube, which is in an initial state, undergoes instability and through the process of magnetic reconnection it reaches a final state of lower energy with the constraint that the total helicity is
conserved. We shall make the following approximations: the plasma flows are weak, the fields are slowly evolving, it is legitimate to neglect gravity (even in the case of a solar flux tube) and the plasma is highly conductive (low resistivity effects); in this case resistive effects such as dissipation are due by reconnection within thin current layers. As previously discussed, the relaxed state is a force free field and it is given by the following formula

\[ \nabla \times \vec{B} = \alpha \vec{B} \] \hspace{1cm} (4.1)

Here we set up the equations for modeling the equilibrium of a single magnetic flux tube within a rectangular cross section boundary. Later, this model will be used to model the merging of two (or more) initial flux ropes through relaxation theory. In this chapter, we follow the analysis of (9) developed for merging plasmas in MAST, but note it should also apply to solar coronal flux tubes.

In Browning et all. it has been proposed that the plasma flux rope which merge together could be modeled according to Taylor’s relaxation theory. We will assume that the flux ropes have rectangular cross section \((a \times b)\) therefore we will conveniently describe them using cartesian coordinates. A magnetic field, which is assumed to be independent on \(z\) coordinate, can be expressed by the following components

\[ \vec{B} = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, \alpha \psi \right) \] \hspace{1cm} (4.2)

where \(\psi \equiv \psi(x, y)\) is the poloidal magnetic flux which does not depend on \(z\).

Given eq.(4.1), and using the expression of the magnetic field of eq.(4.2), we can demonstrate that the magnetic flux will satisfy the following equation

\[ \nabla^2 \psi + \alpha^2 \psi = 0. \] \hspace{1cm} (4.3)
To do this let us solve eq. (4.1) for the magnetic field we are dealing with, therefore

\[ \nabla \times \vec{B} = \left( \alpha \frac{\partial \psi}{\partial y}, -\alpha \frac{\partial \psi}{\partial x}, -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right) = \left( \alpha \frac{\partial \psi}{\partial y}, -\alpha \frac{\partial \psi}{\partial x}, \alpha^2 \psi \right) \] (4.4)

calling \(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\nabla^2 \psi\) the z component of \(\vec{B}\) is eq. (4.3).

We could also check that if eq. (4.1) and eq. (4.2) hold, and if all quantities do not depend on \(z\) then \(B_z = \alpha \psi\):

\[ \nabla \times \vec{B} = \left( \frac{\partial B_z}{\partial y}, -\frac{\partial B_z}{\partial x}, -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right) = \left( \alpha \frac{\partial \psi}{\partial y}, -\alpha \frac{\partial \psi}{\partial x}, B_z \right) \] (4.5)

Analyzing each component, we find that along the \(x\) direction \(B_z = \alpha \psi(y) + f(x)\) and along the \(y\) direction \(B_z = \alpha \psi(x) + f(y)\) therefore \(B_z = \alpha \psi(x,y) + \text{constant}\) as required.

It is particularly important to discuss the boundary conditions at the wall for a fusion device, and the boundary conditions necessary to model the coronal flux tubes. We shall first of all state that \(\psi\) is constant along the magnetic field lines. This is because \((\vec{B} \cdot \nabla)\psi\) is zero when substituting the components of the magnetic field given in eq. (4.2). Let us show this here below:

\[ (\vec{B} \cdot \nabla)\psi = \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \alpha \psi \frac{\partial \psi}{\partial z} = 0 \] (4.6)

because no quantity depends on \(z\). Therefore since no field line can cross the boundary of a flux tube on the wall

\[ \psi = \psi_b = \text{constant} \] (4.7)

This statement is like assuming that the wall is a perfect conductor. This condition is a restriction on the resistivity of the plasma: if the wall is an isolator, a perfectly conducting plasma layer in front of the wall serves the same purpose as the
conducting wall. Our condition is therefore equivalent as stating that, at the wall

$$\vec{n} \cdot \vec{B} = 0$$  \hspace{1cm} (4.8)

This equation is valid if we assume that the plasma, which is flowing along the field lines, will not hit the wall and therefore the plasma will be magnetically confined. This same conditions is an appropriate assumption in the case of coronal flux ropes \(\text{(7)}\). Now that we know that the magnetic flux for \(x = 0, a\) and \(y = 0, b\) must be constant, we will assume that, in order to find non trivial solutions for all values of \(\alpha, \psi_b \neq 0\) at the boundary and the differential equation for the magnetic flux (eq.(4.3)), which is similar to the equation for the harmonic oscillator, has the following form of solution

$$\psi = C \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$  \hspace{1cm} (4.9)

We can immediately notice that for this solution, the flux is null at the boundary (for \(x = 0, a\) and \(y = 0, b\)). When we substitute this last equation into eq.(4.3), we get

$$- \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + \alpha^2 = 0$$  \hspace{1cm} (4.10)

so

$$\alpha_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$  \hspace{1cm} (4.11)

This equation gives us the expression for the eigenvalues of our solution. A lot of information on the eigenvalues, and their respective periodic or axisymmetric eigenfunctions, and their relative importance is given in Gimblett’s article \(\text{(13)}\); these are values of \(\alpha\) for which the toroidal flux is zero and this result consequently sets a maximum value for the current \(I\). We will not enter in too much detail and we shall only say that the eigenvalues are values of \(\alpha\) for which the magnetic flux tends to
CHAPTER 4. RELAXATION MODEL OF MERGING FLUX TUBES IN MAST

infinity and so do many other quantities (fig. (5.2)). The first eigenvalue, $\alpha_{mn} = \alpha_{11}$, is

$$\alpha_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2. \quad (4.12)$$

4.3 Analysis of the main quantities

After what was presented above in the brief introduction, I will not consider particular solutions with $\psi_b = 0$ because

- According to Taylor’s article $\alpha < \alpha_{11}$ in the state of lowest energy in the case of fusion devices.

- For $\psi_b = 0$, $B = 0$ which means that there is no magnetic field in the system.

  We already know that in tokamaks for example $\vec{B}_{tor} \gg \vec{B}_{pol}$ so we cannot make this assumption.

Following (18), we can write the magnetic flux in the following form

$$\psi = \psi_b \left[ 1 + \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right] \quad (4.13)$$

It is immediate to see that this expression for the flux function satisfies the boundary conditions (for $x = 0, a$ and $y = 0, b$, $\psi \neq 0$ but it is still constant). For this solution of eq. (4.2), the values of $\alpha$ are not limited to the eigenvalues.

We want to substitute eq. (4.13) into eq. (4.3). Showing the whole method we first calculate $\nabla^2 \psi$

$$\nabla^2 \psi = -\psi_b \left[ \left( \frac{m\pi x}{a} \right)^2 + \left( \frac{n\pi y}{b} \right)^2 \right] \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \quad (4.14)$$

substituting into eq. (4.13)
4.3. ANALYSIS OF THE MAIN QUANTITIES

\[ -\psi_b \left[ \left( \frac{m\pi x}{a} \right)^2 + \left( \frac{n\pi y}{b} \right)^2 \right] \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) + \alpha^2 \psi_b + \alpha^2 \psi_b \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) = 0 \] (4.15)

\[ + \alpha^2 \psi_b + \alpha^2 \psi_b \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) = 0 \] (4.16)

and find the equation

\[ \left[ - \left( \frac{m\pi x}{a} \right)^2 - \left( \frac{n\pi y}{b} \right)^2 + \alpha^2 \right] \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) = -\alpha^2. \] (4.17)

We now want to find the 2D Fourier coefficients \( a_{nm} \). To do this we must integrate using the following formula

\[ a_{nm} = \frac{\int_{0}^{b} \int_{0}^{a} f(x, y) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) dx \, dy}{\int_{0}^{b} \int_{0}^{a} \sin^2 \left( \frac{m\pi x}{a} \right) \sin^2 \left( \frac{n\pi y}{b} \right) dx \, dy} \] (4.18)

where in our case \( f(x, y) = -P \) is a constant defined in eq.(4.19). Renaming the variables in the following way

\[ P = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 - \alpha^2 \] (4.19)

\[ z_1 = \frac{m\pi}{a} \] (4.20)

\[ z_2 = \frac{n\pi}{b} \] (4.21)

we can calculate the integrals and find \( a_{nm} \). Let’s start by calculating the numerator by solving the integrals which gives

\[ \int_{0}^{b} \int_{0}^{a} P \sin(z_1 x) \sin(z_2 y) \, dx \, dy = \frac{4Pab}{mn\pi^2} \] (4.22)

if \( n \) and \( m \) are odd, \( a_{nm} = 0 \) if \( n \) or \( m \) are even.
For the denominator we have

\[
\int_0^a \int_0^b \sin^2(z_1x)\sin^2(z_2y) \, dx \, dy = \frac{ab}{4}
\]

(4.23)

therefore putting these two results together we find that

\[
a_{nm} = \frac{16\alpha^2}{mn\pi^2 \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 - \alpha^2 \right]}
\]

(4.24)

as long as \(m\) and \(n\) are both odd. If \(n\) and/or \(m\) is even, \(a_{nm} = 0\).

### 4.3.1 Magnetic flux and poloidal field lines

Let us now investigate thoroughly the quantities of interest, starting with the magnetic flux \(\psi(x,y)\). In order to compute \(\psi\) and the poloidal field, we need to fix the values of \(a\), \(b\), \(\psi_b\) and \(\alpha\). The only restriction that we must take into account is that \(\alpha < \alpha_{11}\) for the laboratory case, where \(\alpha_{11}\) is defined in eq.(4.24). For example, for \(a = 1\) and \(b = 1\), \(\alpha_{11} = 4.442\). Larger the value of \(a\) and/or \(b\), the smaller the value of \(\alpha_{11}\). A series of programs have been written in Matlab to evaluate \(\psi(x,y)\) and other quantities derived from this.

- For fixed values of \(\psi_b\) and \(\alpha\) : increasing the values of \(a\) and \(b\), the value of the flux function increases.
- For fixed values of \(a\), \(b\) and \(\psi_b\) : increasing the values of \(\alpha\), the flux increases.
- Swapping the values of \(a\) and \(b\), the values of the magnetic flux do not change (apart from coordinate swap).
- For \(\alpha \to 0\), \(a_{nm} = 0\) therefore \(\psi \to \psi_b\). In my program for small values of \(\alpha\), each term of the sum present in eq.(4.13) exceeds the tolerance immediately and therefore the whole sum is zero very fast.

Fig.(4.5) is a contour map of the flux function, which is constant along field lines, and therefore this contour map represents the poloidal field lines of the magnetic
4.3. **ANALYSIS OF THE MAIN QUANTITIES**

Field. From the figure it is clear that we have taken a rectangular section of the flux tube and satisfied the boundary condition $\vec{B} \cdot \vec{n} = 0$ on the side boundaries $x = 0, a$ and $y = 0, b$.

In fig. (4.6) is illustrated the flux for fixed values of $x$ or $y$. Note the symmetry between $x$ and $y$, as expected. We can also show how the magnetic flux varies in magnitude when changing values of $\alpha$ as shown in fig. (4.7).

![Figure 4.5: Poloidal field lines for $a = 1$, $b = 1$ and $\alpha = 1.5$. When getting closer to the center, the magnitude of the magnetic flux increases.](image)

Figure 4.5: Poloidal field lines for $a = 1$, $b = 1$ and $\alpha = 1.5$. When getting closer to the center, the magnitude of the magnetic flux increases.

![Figure 4.6: Flux as a function of (left) $y$ when $x = \frac{a}{2}$ and $\alpha = 1.5$; (right) $x$ when $y = \frac{b}{2}$ and $\alpha = 1.5$ for $a = b = 1$.](image)

Figure 4.6: Flux as a function of (left) $y$ when $x = \frac{a}{2}$ and $\alpha = 1.5$; (right) $x$ when $y = \frac{b}{2}$ and $\alpha = 1.5$ for $a = b = 1$.

Looking at both fig. (4.6) and fig. (4.7) we can see that the flux has its maximum values at the center so for $x = a/2$ and $y = b/2$. When we get closer to the boundary, the intensity of the flux drops to the value of $\psi_b(\psi_b = 1)$. This is because, closer to the boundary, the sum of eq. (4.13) is zero therefore $\psi = \psi_b$. (We have constructed
46 CHAPTER 4. RELAXATION MODEL OF MERGING FLUX TUBES IN MAST

Figure 4.7: Flux as a function of $x$ when $y = \frac{b}{2}$ when varying the value of $\alpha$. $\alpha = 0.7, 1.5, 2$ and 3. The peak of the flux function increases when increasing the value of $\alpha$.

We also notice that, as said before, when $a$ and $b$ have smaller values, the maximum of the flux function is smaller.

It is interesting to show the dependence of the magnetic flux $\psi$ (fig. 4.8) on $\alpha$. We can see from the fig. 4.8 that $\psi$ increases when $\alpha$ increases, which means that the magnetic flux increases with the magnitude of the current becomes bigger.

Figure 4.8: Magnetic flux for a cylinder with square cross section $a = b = 1$ and constant value of $\psi_b = 1$. 
4.3. **ANALYSIS OF THE MAIN QUANTITIES**

4.3.2 **Components of the magnetic field $\vec{B}$**

Now that we have explored the magnetic flux, we can derive the various components of the magnetic field $\vec{B}$ accordingly with eq.(4.2) and show how this depends on position. In a generic position, away from the boundary $(x = 0, a \text{ or/and} y = 0, b)$ and away from the center $(x = a/2, y = b/2)$, for example, for $x = a/3$ and $y = b/3$ the plot for $B_x$ is shown in fig.(4.9). (We shall not show the plots for the component of the magnetic field along the $y$ direction, because both $B_x$ and $B_y$ have the same dependence on $\alpha$, therefore the plots would look the same but have opposite direction).

![Figure 4.9: $B_x$ vs $\alpha$ for a cylinder with square cross section away from the center, for example for $x = a/3$ and $y = b/3$.](image)

As for $B_z$, given that $B_z = \alpha \psi$, the plots for the magnetic field along $z$ will look very similar to fig.(4.8). Even if so, the plot for $B_z$ is shown in fig.(4.10).

Another thing that we can study is the way the component of the magnetic field $B_y$ changes direction along the $x$ axis for fixed value of $\alpha = 1.5$. The plot of $B_x$ as a function of $y$ for fixed value of $x$, shows the same result. This change of magnitude is shown in fig.(4.11). From the figure we can see that $B_y$ changes direction for $x = a/2$ and the same happens for $B_x$. 
CHAPTER 4. RELAXATION MODEL OF MERGING FLUX TUBES IN MAST

Figure 4.10: $B_z$ vs $\alpha$ for a cylinder with square cross section $a = b = 1$ and constant value of $\psi_b = 1$.

Figure 4.11: a) $B_y$ as a function of $x$ for fixed value of $y = a/2$ b) $B_x$ as a function of $y$ for fixed value of $x = a/2$ and for both cases $\alpha = 1.5$

4.3.3 Conserved quantities

The study of the main quantities of our particular system, allows us to compute the relaxation theory following Taylor’s guidance. To do this, lets us first point out that the flux for the initial and the final state is given in both cases by eq.(4.13) with the $a_{nm}$ coefficients expressed by eq.(4.24). For every single flux tube, that will consequently merge with one or more other flux tubes, the toroidal flux is to be considered a conserved quantity; for both the initial and final regime, $\psi_b$ is constructed so to normalize the total toroidal flux. The toroidal flux is given by
4.3. ANALYSIS OF THE MAIN QUANTITIES

\[ \phi_t = \int B_z dx dy = \int \alpha \psi dx dy = \int \alpha \psi_b \left[ 1 + \sum_{nm} a_{nm} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \right] dx dy \]  \hspace{1cm} (4.25)

Solving the integral we find an expression for \( \phi_t \)

\[ \phi_t = \alpha ab \psi_b \left[ 1 + \frac{4}{\pi^2} \sum_{nm} \frac{a_{nm}}{nm} \right] \] \hspace{1cm} (4.26)

Applying the constraint that the toroidal flux must be conserved, we can normalize this quantity setting \( \phi_t = 1 \). Hence, we find the expression of \( \psi_b \) as a function of \( \alpha \) as shown in eq. (4.27)

\[ \psi_b = \frac{1}{\alpha ab \left[ 1 + \frac{4}{\pi^2} \sum_{nm} \frac{a_{nm}}{nm} \right]} \] \hspace{1cm} (4.27)

Fig. (4.12) plots eq. (4.27).

Figure 4.12: \( \psi_b \) vs \( \alpha \) for \( a = b = 1 \). \( \psi_b \) is still constant with respect to \( x \) and \( y \) but it does depend on \( \alpha \). \( \psi_b \to 0 \) when \( \alpha \to \alpha_{11} \).

Extremely important to understand is the total magnetic helicity \( K \) which is assumed to be conserved during the relaxation process, following the same procedure developed by Taylor (35). Let us therefore calculate \( K \). To do so, we shall start by
analyzing the vector potential. $\vec{A}$ is defined in eq. (4.28)

$$\vec{B} \equiv \nabla \times \vec{A} \quad (4.28)$$

In must be noted that $\vec{A}$ is not unique because by definition we can add a curl free component to the magnetic potential without changing the magnetic field. If $\vec{B}$ satisfies eq. (4.28) then,

$$\vec{B} = \nabla \times (\vec{A} + \nabla g) \quad (4.29)$$

since

$$\nabla \times \nabla g = 0 \quad (4.30)$$

where $\nabla g$ is curl free. Therefore $\vec{A}$ has a degree of freedom: this condition is called gauge invariance. Given that $\vec{A}$ is not unique, even $K$ is not unique (unless $\vec{B} \cdot \vec{n} = 0$ on all boundaries) and is given by

$$K = \int \vec{A} \cdot \vec{B} dV + \int \nabla g \cdot \vec{B} dV = \int \vec{A} \cdot \vec{B} dV + \int g\vec{B} \cdot d\vec{S} \quad (4.31)$$

using Gauss’s Theorem. Consistent with the work illustrated in Berger and Field article [4], we define the relative helicity to be

$$K_{rel} = \int \vec{A} \cdot \vec{B} dV - \int \vec{A}_d d\vec{l} \int \vec{A}_s d\vec{l} \quad (4.32)$$

where the last integral is done along the short ($A_s$) and along the long ($A_l$) diameter of the torus (fig. (3.4)). Since

$$\int \vec{A}_l d\vec{l} = (A_z)_{boundary} \cdot l \quad (4.33)$$

if we apply the condition that $A_z = 0$ at the boundary, then the correction term of the gauge invariance is zero which brings us back to the formula for the helicity
4.3. ANALYSIS OF THE MAIN QUANTITIES

\((K = \int \vec{A} \cdot \vec{B} dV)\). We are choosing a particular gauge to make things simpler; since the form of the vector potential is not unique we can always do this. Since we have now solved the problem of not uniqueness of \(\vec{A}\) we shall display an adequate form of the vector potential so that we can then analytically calculate an an expression for \(K\), needful for our further computations. Lets notice that starting from eq.(4.1), if

\[
\vec{A} = \frac{1}{\alpha} \vec{B} \tag{4.34}
\]

then

\[
\vec{\nabla} \times \vec{A} = \frac{1}{\alpha} \vec{\nabla} \times \vec{B} = \frac{1}{\alpha} \alpha \vec{B} = \vec{B} \tag{4.35}
\]

So the particular form \(\vec{A} = \frac{1}{\alpha} \vec{B}\) satisfies both eq.(4.1) and the definition of vector potential, but we haven’t yet applied our gauge condition. If we request \(A_z\) to be 0 for \(x = a\) and \(y = b\),

\[
A_z = \frac{1}{\alpha} B_z + \nabla g = 0 \tag{4.36}
\]

therefore the expression for the vector potential is

\[
\vec{A} = \frac{1}{\alpha} \vec{B} - \frac{1}{\alpha} B_z(wall) \tag{4.37}
\]

Note that we are studying a cylindrical flux tube with squared cross section merging together to make the model simpler, but when considering the boundary conditions we impose for each flux rope that the two ends of the cylinder \((z = 0\) and \(z = L)\) bend together to form a toroidal configuration (the geometry of the plasma in MAST). In solar flux tubes we need to see which gauge will be satisfied because the geometry is different (flux tubes on the solar corona are not toroidal). For now we are considering just the laboratory case, we will deal with solar physics later on in the next chapter.
4.3.4 Total helicity

Since the volume we are considering is bounded by flux surfaces $\vec{B} \cdot \vec{n} = 0$ (we built the cylinder to have this particular property), we can calculate the total helicity being sure that answer will be unique and it will not depend on the gauge of the vector potential (4), (12), (36). Thus we can proceed in calculating an explicit expression for the helicity. As previously mentioned, the total helicity is defined as

$$K \equiv \int \vec{A} \cdot \vec{B} \, dV$$  \hspace{1cm} (4.38)

Thanks to eq.(4.37), we can solve the integral for $K$ as follows:

$$K = \frac{1}{\alpha} \int \left[ B_x^2 + B_y^2 + B_z^2 - B_z(\text{wall})B_z \right] \, dxdydz \hspace{1cm} (4.39)$$

where $B_z(\text{wall})$ is the component of the magnetic field along the $z$ axis for $x = 0, a$ and $y = 0, b$.

Using eq.(4.2), we can solve each term of the integral separately

$$\frac{1}{\alpha} \int B_x^2 \, dxdydz = \frac{1}{\alpha} \int \left( \frac{\partial \psi}{\partial y} \right)^2 \, dxdydz = \frac{\psi_b^2}{\alpha} \int \sum_{nm} a_{nm}^2 \left( \frac{n\pi}{b} \right)^2 \cos^2 \left( \frac{m\pi x}{a} \right) \, dxdydz = ... = \frac{\psi_b^2}{\alpha} \sum_{nm} a_{nm}^2 \left( \frac{n\pi}{b} \right)^2$$  \hspace{1cm} (4.40)

$$\frac{1}{\alpha} \int B_y^2 \, dxdydz = \frac{\psi_b^2}{\alpha} \sum_{nm} a_{nm}^2 \left( \frac{m\pi}{a} \right)^2$$  \hspace{1cm} (4.41)

where $L$ is a typical length scale of the device along the $z$ direction and $\psi_b$ depends on $\alpha$ as a consequence of the normalization of $\phi = 1$ (eq.(4.27)). We can solve the $\int B_y^2 dxdydz$ with the same procedure and obtain

$$\frac{1}{\alpha} \int B_y^2 \, dxdydz = \frac{\psi_b^2}{\alpha} \sum_{nm} a_{nm}^2 \left( \frac{m\pi}{a} \right)^2$$  \hspace{1cm} (4.42)

Regarding the third term,
4.3. ANALYSIS OF THE MAIN QUANTITIES

\[ \frac{1}{\alpha} \int B_z^2 dx dy dz = L \int (\alpha \psi)^2 dx dy = \]

\[ = \alpha L \psi_b^2 \int \left[ 1 + \sum_{nm} a_{nm}^2 \sin^2 \left( \frac{m\pi x}{a} \right) \sin^2 \left( \frac{n\pi y}{b} \right) + 2 \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right] dx dy \]

\[ = ... = \alpha L \psi_b^2 ab \left[ 1 + \frac{1}{4} \sum_{nm} a_{nm}^2 + 8 \sum_{nm} \frac{a_{nm}}{mn\pi^2} \right] \]

Finally, for the fourth and last term

\[ \frac{1}{\alpha} \int B_z B_z (\text{wall}) dx dy dz = \frac{1}{\alpha} \int \alpha \psi B_z (\text{wall}) dx dy dz = \]

\[ = \alpha L \psi_b^2 \int \left[ 1 + \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right] dx dy \]

\[ = \alpha L \psi_b^2 \left[ ab + \sum_{nm} a_{nm} \left( -\frac{a}{m\pi} \cos \left( \frac{m\pi x}{a} \right) \right)_{x=a,y=b} \left( -\frac{b}{n\pi} \cos \left( \frac{n\pi y}{b} \right) \right)_{x=a,y=b} \right] \]

\[ = \alpha L \psi_b^2 ab \left[ 1 + 4 \sum_{nm} \frac{a_{nm}}{mn\pi^2} \right] \]

which shows that \( B_z (\text{wall}) = \alpha \psi_b \). Reordering and straightening, the expression for the \( K \) is given in eq.(4.51)

\[ K = \frac{\psi_b^2 abL}{4\alpha} \sum_{nm} a_{nm}^2 \left( \frac{(n\pi/b)^2}{a^2} + \frac{(m\pi/a)^2}{a^2} + a^2 \right) + 16\alpha^2 \sum_{nm} \frac{a_{nm}}{mn\pi^2} \]  

In fig.(4.13) is shown the dependence of \( K \) on \( \alpha \) when normalized and therefore, considering the toroidal flux as a conserved quantity and using eq.(4.27).

4.3.5 Current

We can calculate the magnitude of the current generated by the highly conductive plasma in the area where the magnetic field reverses and non-ideal effects allow the breaking and reconnecting of the field lines. The infinitesimal area where this
process occurs is called a current sheet. The current density $J$ is infinity but the current $I$, that is the integral of the current density over a surface ($I = \int \vec{J} \, dx \, dy$), is different from infinity. This tells us that the current sheet will be a delta function.

We can use Ampere’s law (eq. (4.52)) to calculate $I$

$$\mu_0 I = \oint \vec{B} \cdot d\vec{l} \quad (4.52)$$

where the choice of the closed curve is a rectangle that circles around the current sheet and $\mu_0$ is the magnetic permeability constant. In the $y$ direction the components of the magnetic field tend to zero because we are considering an infinitesimal layer while in the $x$ direction, the current travels in opposite directions for $y = 0$ and $y = b$ but do not cancel out, actually they sum up so we are left with

$$\mu_0 I = \int B_x(x, 0) \, dx - \int B_x(x, b) \, dx = 2 \int B_x(x, 0) \, dx \quad (4.53)$$

because $\int B_x(x, 0) \, dx = - \int B_x(x, b) \, dx$.

Solving the integral

$$\mu_0 I = 2 \int B_x(x, 0) \, dx = 2\psi_b \sum_{nm} a_{nm} \frac{n\pi}{b} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right)_{y=0} \, dx \quad (4.54)$$
4.3. ANALYSIS OF THE MAIN QUANTITIES

Going forward with the calculations, the expression for the current is

\[ I = \frac{4\psi_b}{\mu_0} \sum_{nm} \frac{an}{bm} a_{nm} \]  

(4.55)

Figure 4.14: Cross section of two merging tubes. The red line is the current sheet where reconnection occurs. Also refer to fig. (3.1).

Figure 4.15: Normalized current vs \( \alpha \) for \( a = b = 1 \) and \( \psi_b \) given by eq. (4.27).

4.3.6 Finding the relaxed state

We have finished analyzing and exploring the properties of a single flux tube in MAST. Following Browning et al. article (11), in MAST there are two flux ropes
that merge together and undergo magnetic reconnection. The authors of the article decided to use two different methods to investigate the release of energy and the final state; MHD computation and relaxation theory. We shall describe the latter, reach the same conclusions and then apply the same method to the solar corona.

Figure 4.16: Contour of the poloidal field lines for the initial and final state according to our model for relaxation. \( \alpha_i = 1.25 \) and \( \alpha_f = 0.782 \). The initial tubes have squared cross section \((a = b = 1)\), they undergo magnetic reconnection and relax to a final state with rectangular cross section \((a = 1, b = 2)\) (9).

What we are ready to do right now is to find the final state. To do so, we need to find values of \( \alpha_f \) using a root finding function in MATLAB. The equation that must be solved, assuming that the total helicity \( K(\alpha,x,y) \) is conserved during the process of reconnection of the two merging flux tubes in MAST, is

\[
4K_N(\alpha_f, a, 2a) - 2K_N(\alpha_i, a, a) = 0 \tag{4.56}
\]

The factors of 4 and 2 are needed because for the final state, \( K \propto \phi^2 \) and \( \phi = 2 \) (since we have two tubes), while for the initial state each flux tube is normalized in having \( \phi = 1 \). Once solved the equation, we find a value of \( \alpha \) for the final state which will be smaller that \( \alpha_i \), as anticipated on the Taylor’s paper (35).

4.3.7 Energy

The magnetic energy has been found to be
4.3. ANALYSIS OF THE MAIN QUANTITIES

\[ W = \frac{1}{2\mu_0} \int |\vec{B}|^2 dxdy = \frac{1}{2\mu_0} \int (B_x^2 + B_y^2 + B_z^2) dxdy = \quad (4.57) \]

\[ = \frac{1}{2\mu_0} \int \left[ \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2 + \alpha^2 \psi^2 \right] dxdy \quad (4.58) \]

Following the same calculation done for the helicity K and solving the integral \( \alpha^2 \int B_z^2 dxdy \) we find that the expression for the magnetic energy is eq.\( (4.59) \)

\[ W(\alpha, a, b) = \frac{\psi^2 ab}{2\mu_0} \left[ \alpha^2 + \sum_{nm} \left[ \frac{a_{nm}^2}{4} \left( \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \alpha^2 \right) + \frac{8a_{nm}\alpha^2}{\pi^2nm} \right] \right] \quad (4.59) \]

Figure 4.17: Normalized energy vs \( \alpha \) for \( a = b = 1 \) and \( \psi_b \) given by eq.\( (4.27) \).

According to Browning et all \( (9) \), the relation between the energy and the helicity is given by \( K = 2\mu_0W - \psi_b\phi_t \).

Thanks to the root finding function, the knowledge of \( \alpha_f \) allows us to calculate the amount of energy released during the relaxation phase and the fusion process (fig.\( (4.18) \)), which is the main purpose of this analysis according to the following equation

\[ \Delta W = 2W_i(\alpha_i, a, 2a) - 4W_f(\alpha_f, a, 2a) \quad (4.60) \]

where \( W(\alpha,x,y) \) is the normalized energy shown in fig.\( (4.17) \).
Figure 4.18: Profile of the $\alpha_f(\alpha_i)$ and $\Delta W(\alpha_i)$ in the case of $\psi_b$ given by eq.(4.27) and values of $a = b = 1$ for the initial state, and $a = 1, b = 2$ for the final state. This is the same figure illustrated in (9), which confirms the validity of our MATLAB programs.
Chapter 5

Merging flux tubes on the solar corona

Now that we have understood and analyzed the case of the Mega Ampere Spherical Tokamak, we are ready to talk about the most important part of the dissertation; the analysis of the amount of energy released from the solar corona starting from different initial configurations.

As we have previously pointed out, a difference between fusion and solar plasmas, is the fact that in the latter case, the constant $\alpha$ can assume any value (we could have $\alpha \gg \alpha_{11}$). All the properties that we pointed out in the previous chapter still hold and we will show the behavior of these quantities for $\alpha \gg \alpha_{11}$ in fig.(5.1) and fig.(5.2).

5.1 Helicity calculation for finite length loop

Another difference between fusion flux tubes and solar flux tubes is their geometry. The geometry with which we model the each tube is essential in the computation of the relaxed state because it affects the calculation of the total helicity. Regarding MAST, we modeled the tube to by a cylinder with squared cross section and having the same value of $K$ at the ends of the cylinder ($z = 0$ and $z = L$) because in the actual experiment, each rope has a toroidal configuration therefore, $K$ must be the
CHAPTER 5. MERGING FLUX TUBES ON THE SOLAR CORONA

Figure 5.1: Contour plot of the poloidal field lines for a) $\alpha = 1.5$, b) $\alpha = 4$, c) $\alpha = 7$, d) $\alpha = 10$ and in all cases $a = 1$ and $b = 1$. Darker the color, higher is the value of the magnetic flux. $B_z$ (of the poloidal field lines hence $\psi$), is pointing outwards in the center of the contour and in d) is pointing inwards in the four areas at the corners of the squared cross section. Configuration c) and d) could be produced when we have multiple streams of toroidal current that produce poloidal fields. In this case the poloidal field is much less tidy and has more than one maxima and minima in strength for different values of $x$ and $y$. Thanks to the flux we took (eq.(4.13)) and the relation between the flux and the field (eq.(4.2)), even the magnetic field has an oscillating configuration.

same in every section of the tube. On the solar corona, the configuration of the tube is different; each end of the flux rope is reconnected with the underlying photosphere layer and therefore it does not bend together generating a donut-geometry. To keep things simpler, even on the corona, each flux rope is modeled by a cylinder with rectangular cross section. This difference in geometry may or may not affect K. Starting from Berger and Field’s (4), Finn and Antonsen (12) and Valori’s (36) articles the eq.(4.38) is not gauge invariant in the solar case, because there are magnetic field lines that cross the flux surface at the photosphere ends. Hence we must use the expression of the relative helicity given in eq.(5.1)

$$K = \int \left( \vec{A} + \vec{A}' \right) \cdot \left( \vec{B} - \vec{B}' \right) dV \quad (5.1)$$
where $\vec{B}'$ and $\vec{A}'$ are respectively a vacuum magnetic field and its vacuum vector potential whose normal component at the boundary is the same as the normal component of $\vec{B}$ at the boundary ($\vec{B} \cdot \vec{n} = \vec{B}' \cdot \vec{n}$). Following articles (12), (7) we can write the field and the corresponding vector potential in the following form

$$\vec{B} = \nabla \psi \times \vec{z} + I \vec{z}$$  \hspace{1cm} (5.2)$$

$$\vec{A} = \psi \vec{z} + \vec{A}_\perp$$  \hspace{1cm} (5.3)$$

where $\vec{A}_\perp \equiv (A_x, A_y)$ and $\vec{A}$ satisfies the condition eq.(4.28). The expressions for $\vec{B}'$ and $\vec{A}'$ are similar except for the fact that the vacuum field has no azimuthal
current \((I'_z = 0)\). Substituting in eq.(5.1) and applying the boundary conditions,

\[ K = \int \psi IdV. \quad (5.4) \]

For our particular system,

\[ I = 2 \int B_x(x, 0)dx \quad (5.5) \]

\[ \psi = \psi_b \left[ 1 + \sum_{nm} a_{nm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \right] \quad (5.6) \]

Solving this integral for K, we obtain the same numerical values of the total helicity used in fusion device (eq.(4.51)).

### 5.2 Circular cross section

Most of the analysis that scientists have done when modeling solar flux ropes is to consider a circular cross section and express the quantities in cylindrical coordinates. Let us compare the rectangular cross section case that we have explored for MAST, and use for solar plasma as well, with the circular cross section case which is used to model merging tubes in tokamaks and in other solar relaxation models. Following Taylor’s article, the magnetic field components are as displayed in eq.(5.7)

\[ B_r = 0 \quad (5.7) \]

\[ B_\theta = B_0 J_1(\alpha r) \quad (5.8) \]

\[ B_z = B_0 J_0(\alpha r) \quad (5.9) \]

where \(B_0\) is the normalization constant, \(\alpha \equiv \alpha\) in astrophysics configuration, and \(J_1\) and \(J_0\) are the Bessel functions of the first kind. The normalized helicity is
5.2. **CIRCULAR CROSS SECTION**

given by Taylor’s formula

\[
\frac{K}{\phi^2} = \frac{L}{2\pi R} \left[ \frac{\alpha R [J_0^2(\alpha R) + J_1^2(\alpha R)] - 2J_0(\alpha R)J_1(\alpha R)}{J_1^2(\alpha R)} \right]
\]  

(5.10)

where \(L\) is the length of the cylinder and \(R\) is the radius. According to (4), (35) in this case the gauge of the vector potential is found by applying the condition \(A_z(R) = 0\) at the boundary as we have shown in the previous section. The calculation of the helicity for a finite loop that we have done previously, can be applied to a cylinder with circular cross section. This process is elaborated in (7).

It is interesting to see the range of values that we have for the normalized helicity when using different relaxation models; fig.(5.3) illustrates the comparison between a cylinder with circular cross section with the two different squared cross sections with which a flux rope could be modeled.

![Figure 5.3: Comparison of the normalized helicity between the rectangular case and the circular cross section case with fixed value of \(\alpha = 1.5\). The big rectangle has \(a = b = 1\) while the small rectangle has \(a = b = \frac{1}{\sqrt{2}}\) and the circle has \(R = 1/2\). \(K\) for the circular case has values that lie in the middle of the values generated by the two rectangular cases.](image)

From the figure we can see that there is a difference in the value of the helicity depending on which method we use; this is because the representation of the fields is different and the helicity is dependent on \(a\) and \(b\) (\(K\) grows when \(a\) and/or \(b\) increase).
5.3 Reversed Helicity

Until now we assumed that the value of $\alpha$ is the same for every single merging flux tube when applying the relaxation theory. In astrophysics, the dynamics of the flux ropes could be more complicated; for example we could be dealing with reversed helicity, which means that merging ropes have different, or more precisely opposite in sign, values of $\alpha$ ($\alpha \equiv \frac{\vec{r} \cdot \vec{j}}{\vec{B}^2}$). It is therefore interesting to analyze a simple case of reversed helicity (fig. 5.4); let's model two merging flux tubes, one characterized by $\alpha$ and the other by $-\alpha$. We already know that since $\alpha$ is not constant we are not in a state of minimum energy. In this particular configuration the poloidal fields in the two tubes have opposite directions thanks to the difference in sign of $\alpha$ and it must be noted that, for the same reason, $B_z$ has the same direction in both tubes. The different direction of the field lines, do not generate a current sheet at $y = b$ and the process of breaking and reconnecting of the field lines is not spontaneous; under the influence of an external force there could still be the possibility for the two tubes to merge, reach a relaxed state and consequently release energy. Let's see the relation between the computed quantities when using positive or negative $\alpha$'s

\begin{align*}
a_{nm}(-\alpha) &= -a_{nm}(\alpha), \quad \psi_b(-\alpha) = -\psi_b(\alpha), \quad \psi(-\alpha) = -\psi(\alpha) \quad (5.11) \\
B_z(\alpha) &= B_z(-\alpha), \quad B_{x/y}(-\alpha) = -B_{x/y}(\alpha) \quad (5.12) \\
W_i(-\alpha) &= W_i(\alpha) \quad (5.13)
\end{align*}

where $W_i$ is the initial energy.

Since $B_z$ is the same then the toroidal flux $\phi_t$ is the same in both the tubes but $K(-\alpha) = -K(\alpha)$ therefore computing the relaxation equation eq.(4.56) (this equation still stands because the toroidal flux tube is the same in both ropes), we notice that the normalized helicity is zero for this reversed helicity configuration; this means that $\alpha = 0$ in the final state (this can be seen by looking at the graphs for the helicity, for example fig. 4.3 or fig. 5.2) which tell us that $K = 0$ when
5.4. MULTIPLE MERGING FLUX TUBES

Figure 5.4: Comparison between two merging tubes with a) constant $\alpha$ and b) with opposite in sign $\alpha$. We can see that in case a) the field lines at $y = b/2 = a$ go in opposite directions therefore they tend to break. In case b) field lines go in the same direction therefore they avoid spontaneous reconnection.

$\alpha = 0$). Fig.(5.5) compares the release of energy between two merging flux tubes with constant $\alpha$ and the case of two merging tube with reversed helicity.

Figure 5.5: Energy released for a reversed helicity configuration compared with the case of constant $\alpha$. In both cases each tube has a rectangular cross section with $a = b = 1$ therefore the final state will have $a = 1, b = 2$.

As we can see from the figure, $\Delta W$ grows faster when studying the reversed helicity case with respect to the constant $\alpha$ case; this could be explained by the fact that in the case of no current sheet, the system doesn’t undergo relaxation naturally therefore this event has to be triggered by external forces that contribute when calculating the magnetic energy and consequently increase the amount of energy released.

5.4 Multiple merging flux tubes

For MAST we were concerned with just two merging flux tubes because this is what practically happens inside the tokamak. On the solar corona there could be
more than two flux ropes merging, so it is appropriate to study multiple merging finite length tubes. We shall model each one as having a square cross section \((a = b = 1)\). Let’s start by analyzing 4 merging loops with constant-\(\alpha\). For this configuration the equation that must be solved in order to find the final value of \(\alpha\), \(\alpha_f\), is given by eq. (5.14)

\[
16K_n(\alpha_f, 2a, 2b) - 4K_n(\alpha_i, a, b) = 0 \quad (5.14)
\]

\[
\Delta W = 4W(\alpha_i, a, b) - 16W(\alpha_f, 2a, 2b) \quad (5.15)
\]

where the coefficients are necessary because, as previously explained in the case of two constant-\(\alpha\) flux tube, the final state is proportional to \(\phi_t^2\) (the area of the final tube is now 4 times bigger), while the initial state is expressed by the sum of four tubes each with \(\phi_t = 1\). Let’s compare the energy released for 4 merging flux tubes with the case of 2 merging flux tubes in fig. (5.6).

\[
\text{Figure 5.6: Energy release } \Delta W \text{ and } \alpha_f \text{ as a function of } \alpha_i \text{ in the case of two and four merging flux tubes. Each tube has been modeled as having a square cross section } (a = b = 1), \text{ therefore the final state has } a = 1, b = 2 \text{ in the case of two loops merging and } a = b = 2 \text{ in the case of four merging loops.}
\]

Even though the area of the final state is different in the two studied cases, we can compare them and see that the amount of energy released for 4 merging loops is bigger than for 2 and that \(\alpha_f(4) < \alpha_f(2)\); this seems reasonable and can be explained by the fact that in the case of 4 flux tubes interacting, there are 4
5.4. MULTIPLE MERGING FLUX TUBES

distinct current sheets (two along $x$ and two along $y$) therefore the system undergoes magnetic reconnection more frequently. The current of the current sheet is the same in value, both in the $x$ and in the $y$ direction, as in the case of two merging flux tubes discussed in chapter 2.

We can also consider 4 merging tubes as the result of a pair of 2 merging tubes and analyze a possible intermediate state (fig.(5.7)). When confronting the case of 2 and 4 initial tubes with the partial state we can see how the energy release increases as a function of $\alpha_i$ in fig.(5.8). The intermediate state has been built in order to have double the energy of 2 merging tubes as shown in the figure.

![Figure 5.7: Initial state has two flux tube each having a square cross section ($a = b = 1$), the partial state has $a = 1$ and $b = 2$ and the final state has $a = b = 2$ as in the case of four merging loops.](image1)

![Figure 5.8: Energy release $\Delta W$ (dashed line) and $\alpha_f$ (continuous line) as a function of $\alpha_i$ in the case of two and four merging flux tubes and the partial state which is a possible intermediate step that is not relaxed yet because it is not a state of minimum energy since we are still in the presence of current sheets. Each tube has been modeled as having a square cross section ($a = b = 1$). The red curve represents 2 merging tubes, the green is for the partial state and black is for 4 merging flux tubes.](image2)
We can now consider 4 merging flux tubes when dealing with reversed helicity as shown in fig. (5.9). There are 3 distinct initial configurations which are possible; a) $-\alpha, \alpha, \alpha, \alpha$ (one $-\alpha$ and three $\alpha$), b) $-\alpha, -\alpha, \alpha, \alpha$ (same amount of positive and negative $\alpha$’s) and c) $-\alpha, -\alpha, -\alpha, \alpha$ (three $-\alpha$ and one $\alpha$) as illustrated in fig. (5.9). The position in which the tubes are organized does not affect the relaxation process nor the amount of energy released.

Following the analysis done for the reversed helicity in the case of 2 merging loops, the relaxation equations for case a) and b) are eq. (5.16) and eq. (5.18)

\begin{align*}
a) & 2K_n(\alpha_i, a, b) - 16K_n(\alpha_f, 2a, 2b) = 0 \\
& \Delta W = 4W_n(\alpha_i, a, b) - 16W_n(\alpha_f, 2a, 2b)
\end{align*}

\begin{align*}
b) & K = 0 \Rightarrow \alpha = 0 \\
& \Delta W = 4W_n(\alpha_i, a, b) - 16W_n(0, 2a, 2b)
\end{align*}

Case a) and c) release the same amount of energy and are characterized by the same equation (eq. (5.16)) for two main reasons; first of all, the magnetic energy does not depend on the sign of $\alpha$ ($W_i(\alpha) = W_i(-\alpha)$) and second of all, even though $K(-\alpha) = -K(\alpha)$ (the helicity does depend on the sign of $\alpha$), eq. (5.16) produces the same value independent of the sign of the equation.

The comparison between the energy released in case a) (and/or c)) and in case
5.5. **STUDY OF THE MAGNETIC ENERGY RELEASED $\Delta W$**

b) with the energy released by constant $\alpha$ merging tubes is represented in fig.(5.10).

Figure 5.10: Energy release $\Delta W$ as a function of $\alpha_i$ in the case of four merging flux tubes considering the possibility of reversed helicity. a) one -$\alpha$, b) two -$\alpha$ compared with the case of constant-$\alpha$. Each tube has been modeled as having a square cross section with $a = b = 1$, therefore the cross section for the three different initial configurations is the same.

The figure shows that b) releases the biggest amount of energy, and constant $\alpha$ the smallest as we were expecting. As a matter of fact the process of two tubes with opposite $\alpha$’s merging is not spontaneous; in order to undergo magnetic reconnection, the system needs to be fed by an external force. This force will have an effect on the amount of energy released.

5.5 **Study of the magnetic energy released $\Delta W$**

Let’s focus now on the energy released when considering multiple flux tubes interacting. Following the procedure done so fare, let’s start by studying the trend of the energy released in the case of constant $\alpha$ with respect to the number of merging flux tubes. This dependence is shown in fig.(5.11).

We can see from the fig.(5.11a)) that the energy release has a power dependence on the number of merging tubes. On the $x$-axis we haven’t exactly plotted the number of tubes merging but its square root. For example, for $num = 3$ where ‘num’ is the label of the axis, we have 9 flux tubes merging each with squared cross section and length equal to $a/3$.

The general formula used to calculate the energy released is, following what we
CHAPTER 5. MERGING FLUX TUBES ON THE SOLAR CORONA

Figure 5.11: Energy release $\Delta W$ as a function of the number of merging flux tubes with fixed area $a \times b$ ($a = b = 1$) illustrated in two different ways. What changes when considering a different number of merging tubes is the length of the square for the single tube, not the total area. This is different from what we have done in fig. (5.6); in the latter case, the area of the initial state, was different from the area of the final state. We are now comparing energy releases in the same volume.

have done in previous sections

$$\Delta W = num^2 W_N(a/num, a/num, \alpha_i) - num^4 W_n(a, a, \alpha_f)$$  \hspace{1cm} (5.20)

or in terms of the number of flux tubes considering ($num = \sqrt{N}$)

$$\Delta W = NW_N(a/\sqrt{N}, a/\sqrt{N}, \alpha_i) - N^2 W_n(a, a, \alpha_f)$$  \hspace{1cm} (5.21)

If we fit the logarithm of $\Delta W$ on excel we obtain fig. (5.12):

Figure 5.12: Energy release $\log_e(\Delta W)$ as a function of the logarithm of the number of merging flux tubes with fixed $a = b = 1$. We can see that the equation for the interpolation is $y = 2.1393x - 3.5254$. Therefore $\Delta W = 0.0294num^{2.1393}$
Chapter 6

Conclusions

6.1 Summary and Conclusions

This thesis presented the application of the relaxation theory with the aim to present a possible explanation to the solar corona heating problem. Starting by following Browning’s article (9), we computed the properties of MAST and under the guidance of Taylor’s theory (35), we have presented a model for the merging of flux tubes in a spherical tokamak. One of the main motivations in studying magnetic reconnection in MAST is to be able to develop the same model and apply it to the solar corona assuming that a turbulence is triggered along a current sheet by the twisting of photospheric footpoints that correspond to a typical value of $\alpha$. Of course, an important constraint took into consideration is the conservation of the total helicity.

Thanks to the analysis we have performed, it is clear that both in fusion and in solar plasmas (fig. 4.18, fig. 5.5) the energy release $\Delta W$ for two merging tubes is positive and it increases in value when the initial value of $\alpha$ increases.

We are now ready to make estimates of numeral values for the energy using typical observational values of $\alpha$ and other quantities such as the length of the cylindrical loop, the magnetic field etc... Considering a typical value of $\alpha = 1.4$, the dimensionless energy release is $\Delta W = 2.8 \times 10^{-2}$. 
Using the table above [6.1] which shows typical values for solar flares we are able to include the following numerical values in the case of two merging tubes with constant $\alpha$ on the solar corona (7):

- $\phi_t = \int_S B_z dxdy = 10^{10}$ Wb
- $\mu_0 I = \int \vec{B} \cdot d\vec{l} = 3.2 \times 10^{10}$ A
- $\Delta W = \frac{L S B^2}{\mu_0} \Delta W^* \approx 8 \times 10^{20} J \times 2.8 \times 10^{-2} \approx 2.2 \times 10^{19} J$

where S is the surface of the cross section and $\Delta W^*$ is the dimensionless value of the energy released.

To show the results of multiple flux tubes, when considering both constant and reversed $\alpha$, we shall start by referring to fig.(5.10) and dimensionalise the obtained energies

<table>
<thead>
<tr>
<th>constant $\alpha$</th>
<th>$\Delta W^*$</th>
<th>$\Delta W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>one $\alpha$</td>
<td>$1.05 \times 10^{-1}$</td>
<td>$8.4 \times 10^{19}$ J</td>
</tr>
<tr>
<td>two $\alpha$</td>
<td>$1.3 \times 10^{-1}$</td>
<td>$1 \times 10^{20}$ J</td>
</tr>
</tbody>
</table>

Table 6.2: Typical values for the energy released in the case of 4 merging flux ropes. $\Delta W^*$ are the dimensionless values while $\Delta W$ are dimensional and are measured in Joules. To find the dimensional energy we used the same formula used in the case of two merging tubes ($\Delta W = \frac{L S B^2}{\mu_0} \Delta W^*$). $\alpha$ is in all cases set to be equal to 1.4.

As shown from fig.(5.10) and the table (6.1), the amount of energy released is larger in the case of two reversed $\alpha$’s.

It is appropriate to give values for the amount of energy released in the case of multiple merging ropes, therefore looking at fig.(5.11), lets evaluate $\Delta W$ and show results in table (6.1).
As shown in fig. (5.11), the energy released gets larger when the number of flux tubes increases. Even in this table $\alpha = 1.4$.

We have also shown how the energy is described by a power equation that depends on the values of $\sqrt{N}$ where $N$ is the number of merging flux tubes. Let’s compare the values obtained with the power function, to the dimensionless values of energy we have obtained in MATLAB.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\Delta W^*$</th>
<th>$\Delta W$</th>
<th>$\Delta E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$1.05 \times 10^{-1}$</td>
<td>$8.4 \times 10^{19}$</td>
<td>$1.2 \times 10^{-1}$</td>
</tr>
<tr>
<td>9</td>
<td>$2.8 \times 10^{-1}$</td>
<td>$2.3 \times 10^{20}$</td>
<td>$3 \times 10^{-1}$</td>
</tr>
<tr>
<td>16</td>
<td>$5.2 \times 10^{-1}$</td>
<td>$4.1 \times 10^{20}$</td>
<td>$5.7 \times 10^{-1}$</td>
</tr>
<tr>
<td>25</td>
<td>$8.3 \times 10^{-1}$</td>
<td>$6.6 \times 10^{20}$</td>
<td>$9.1 \times 10^{-1}$</td>
</tr>
<tr>
<td>36</td>
<td>$1.2 \times 10^{-1}$</td>
<td>$9.6 \times 10^{20}$</td>
<td>$1.4$</td>
</tr>
</tbody>
</table>

Table 6.3: Obtained values of the energy released in the case of multiple flux tubes.

From table (6.1), we can see that the values differ; when evaluating the energy in joules, this difference will not have an affect on the magnitude of the amount of energy released.

To give an idea of how much energy is released, we can equal the dissipated magnetic energy to the thermal energy $W = \frac{2}{3} n K T V$ where $n$ is the number of particles per unit volume of both ions and electrons, $T$ is the temperature, $K$ is the Boltzmann constant and $V$ is the volume of the region therefore

$$T = \frac{2W}{3nkV} \approx 5 \times 10^7 K$$

(6.1)

when using the value of $\Delta W \approx 2.2 \times 10^{19} J$. This value is higher than the observational values of about $10^6 K$. This is probably due to the fact that the system we are considering is extremely simplified and does not include resistive effects.
6.2 Future Work

Future work built upon the result of this thesis are numerous. First of all the model used could be made more realistic, therefore in the case of MAST it could be extended to a toroidal geometry and in the case of solar loops could consider the variations along the axis and in the \( \theta \) direction. It would be interesting to model resistive MHD effects as well and see how much this affects on the rise of temperature of the solar corona.

What we have started to work on, is the development of an approximate model of relaxation when dealing with the merging of flux tubes with unequal values of \( \alpha \) (we would still be approximating the solar loop to a cylinder with rectangular cross section). What happens in this case is that the boundary between the two tubes will balance in a position that is different than \( y = a \) (the middle). The area of the cross section of the two tubes will be different and therefore the magnetic pressure of the two loops will also differ. An appropriate approximation is to consider the boundary between the two tubes to be a straight line, independent from the position along \( x \); what will really happen is that the boundary for different \( x \) will have different values of \( y \). Using this approximation though, enables us to calculate the movement of the boundary according to the values of the \( \alpha \)'s and obtain balance between the average magnetic pressure (fig.6.1), which will not depend on \( x \).

![Figure 6.1: Cross section of two flux tubes with different areas. Notice that \( b_1 + b_2 = 2a \).](image-url)
6.2. FUTURE WORK

We shall therefore start by calculating the magnetic pressure defined by

\[ P = \frac{|B|^2}{2\mu_0}. \]  

(6.2)

The average magnetic pressure is given by

\[ P_m = \frac{1}{2\mu_0} \int_0^L \int_0^a (B_x^2 + B_y^2 + B_z^2)dx\,dz. \]  

(6.3)

For \( y = 0, b_1, B_y = 0 \) hence, along the boundary the average magnetic pressure is

\[ P_m = \frac{1}{2\mu_0} \int_0^L \int_0^a (B_x^2(x,0) + B_z^2(x,0))dx\,dz. \]  

(6.4)

![Figure 6.2: Difference of the magnetic pressure between two flux tubes with different cross sections. \( \alpha_1 = 1.5, a = 1 \) and we vary the position of the boundary(change values of \( b_1 \). \( \alpha_1, \alpha_2 > 0 \))](image)

Calculating the integrals, following the same procedure applied for the calculation of the total helicity or the magnetic energy, we obtain an expression for \( P_m \) that is

\[ P_m = \frac{L\psi_b^2}{2\mu_0} \left[ \alpha^2 a + \sum_{nm} a_{nm}^2 \frac{a}{2} \left( \frac{n\pi}{b} \right)^2 \right]. \]  

(6.5)

At this point we can compute the difference between the two average pressures
that correspond to the two ropes using the following formula

\[ \Delta P = P_m(a, b_1, \alpha_1) - P_m(a, 2a - b_1, \alpha_2) \]  \hspace{1cm} (6.6)

and plot this difference for different values of \( b_1 \) (considering different positions of the barrier) in fig. (6.2).

As expected, when \( b_1 = a = 1 \), the pressure is the same when \( \alpha_2 = \alpha_1 \). What is surprising is that according to fig. (6.2), \( \Delta P = 0 \) even for \( \alpha \approx 3.7 \). Another thing is that for \( b_1 \neq a \), it seems as if the magnetic pressure is never equal to zero; if this were true, two flux tubes merging with different values of \( \alpha \) would never balance out and therefore would never be in a state of equilibrium.

This study needs more investigation and it would certainly lead to an interesting new path of research.
Bibliography


[33] Schrijver and Siscoe, Heliophysics Plasma physics of the Local Cosmos *Cambridge Univ Press* (2011)

