ROLL WAVES AND
EROSION-DEPOSITION WAVES IN
GRANULAR FLOWS

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Debris flows can be highly destructive and pose a significant threat to both life and property in those areas in which they naturally occur. Such flows can be especially hazardous when large amplitude surges form, which cause more damage than continuous flows of the same mass flux. It is therefore important to understand how these surges form and subsequently behave.

The most likely explanation for their formation is the spontaneous development of roll waves - small shock-like disturbances typically observed in thin liquid films - which merge and coarsen as they travel downslope, in turn growing in amplitude and wavespeed.

There have also been observations of naturally occurring debris flows which develop surges with regions of completely stationary material between them. The terminology of ‘erosion-deposition’ waves is introduced to describe these waves, according to the process by which they propagate steadily through a flow by eroding at the static layer ahead of the wave front and depositing a stationary layer behind it. This behaviour is particularly novel and the pulses can be even more destructive than their roll wave counterparts.

A combination of experimental observations, travelling-wave solutions and numerical simulations are used here to study the behaviour of both roll waves and erosion-deposition waves in granular flows.
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Acknowledgements

I would like to thank my friends, family, colleagues in the mathematics and physics departments and my supervisor, Prof. Nico Gray, for their support during my studies.
Chapter 1

Introduction

Roll waves may be described as a periodic wave formation in a thin flow of fluid or granular material down an inclined channel, in which steep-fronted, steady travelling wave profiles propagate downslope through the flow with a constant wavespeed that is everywhere greater than the magnitude of the individual particle velocities. They have been observed in many various types of fluid and granular flows; including flows of water down an open inclined channel (Cornish, 1910, 1934; Dressler, 1949), thin viscous fluid flows (Kapitza & Kapitza, 1949), flows of dense granular material (Forterre & Pouliquen, 2003) or mud (Balmforth & Liu, 2004) down inclined channels and in natural debris flows (Zanuttigh & Lamberti, 2007).

The characteristic behaviour of erosion-deposition waves, that of regions of stationary material forming between periodic waves which otherwise resemble roll waves, have also been observed in various natural debris flows (Davies et al., 1992; Zanuttigh & Lamberti, 2007).

1.1 Format

This alternative format thesis consists of three papers, all of which are related to roll waves and/or erosion-deposition waves in granular free-surface flows. The rationale for writing this thesis in alternative format is that virtually all of the research that I have carried out during the course of my PhD is contained within three papers which have all been accepted or are under review for publication in peer-reviewed journals.

Firstly, in a depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows
(chapter 2), an additional viscous-like term is derived in the governing depth-averaged granular flow equations. The inclusion of such a term is necessary to smooth out the naturally occurring discontinuities in inviscid roll wave solutions. This newly derived diffusive term correctly predicts the growth rate and cut-off frequency of spatial instabilities to a steady uniform flow without any adjustable parameters, which were previously required to achieve quantitative agreement with existing experiments. The particular form of the new term also allows for physical solutions to a propagating front problem - with grain free regions ahead of a finite gradient front - which is another improvement of this model over previous work.

Secondly, in *erosion-deposition waves in shallow granular free-surface flows* (chapter 3), small-scale laboratory experiments are performed in which a granular flow of carborundum particles down a rough inclined chute develops discrete surge-like waves with stationary material between them. These are modelled by using a complicated basal friction law, which encompasses the dynamic, intermediate and static friction regimes and allows mobile and static regions of flow to develop. It is also shown that the inclusion of the previously derived viscous-like term in the governing equations is vital in order to produce steady travelling-wave solutions with stationary layers. Numerical results are shown to be in good quantitative agreement with the experimental observations.

Finally, in *arrested coarsening of granular roll waves* (chapter 4), the coarsening process of roll waves is studied by performing numerical simulations on periodic domains. Small random disturbances to a uniform flow lead to the spontaneous growth of various sized roll waves, with the larger, faster waves catching up to and merging with smaller, slower waves. It is observed that for sufficiently long domains, the coarsening process is arrested before the system reaches a one-wave state, due to an inherent limiting wave size. An upper bound on the wave peak height is found and a coarsening law is derived, which predicts the final number and average separation distance of waves in the arrested state. Finally, the simulations are scaled up to show that the coarsening of roll waves can explain the development of large amplitude surges in natural debris flows.
1.2 Contributions

My main contributions to \textit{a depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows} (chapter 2) are sections 6 to 8, which is entirely my own work except for some stylistic changes to the writing by J.M.N.T Gray. My other contributions to this paper include the roll wave experiment and the corresponding photograph (figure 1), as well as finding the alternative expressions for the various depth-averaged viscosity coefficients $\dot{\nu}$ (4.9), $\nu$ (4.16) and $\nu_F$ (5.6), where the full angle dependence is revealed whilst the particle diameter and solid volume fraction dependence drops out.

The vast majority of the second paper \textit{erosion-deposition waves in shallow granular free-surface flows} (chapter 3) is my own work. However, there was input from J.M.N.T Gray in parts of the writing; particularly in the abstract, introduction, and conclusions sections, as well as in discussion of the diffusive term in the governing equations. He also made some general improvements to the writing throughout the paper.

Finally, \textit{arrested coarsening of granular roll waves} (chapter 4) is very much a joint collaboration between myself, D. Razis and K. Van Der Weele, with all of the work being completed during the course of three visits to the authors’ respective institutions. In particular I was a main contributor to the mathematics of sections 2 to 4, including the important results of the relationship between the wave speed and the peak height (subsection 4.1) and the upper bound on the peak height (subsection 4.2), which were formulated along with D. Razis. The numerical simulations were all carried out by myself, based on the numerical scheme used in the previous paper (chapter 3). Also, the experimental results and photographs are all from my experimental setup. Furthermore, I was solely responsible for the creation of all the figures in this paper. Much of the writing, including some of sections 2 to 4, as well as the mathematics of section 5, was carried out mostly by K. Van Der Weele and D. Razis in consultation with myself. There was also input from J.M.N.T. Gray in the form of ideas at the beginning of the project, and the addition of a detailed discussion around the diffusive term in the governing equations as well some minor general improvements, at completion.
Chapter 2

A depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows
A depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows

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The $\mu(I)$-rheology is a nonlinear viscous law, with a strain-rate invariant and pressure dependent viscosity, that has proved to be effective at modeling dry granular flows in the intermediate range of the inertial number, $I$. This paper shows how to incorporate the rheology into depth-averaged granular avalanche models. To leading order, the rheology generates an effective basal friction, which is equivalent to a rough bed friction law. A depth-averaged viscous-like term can be derived by integrating the in-plane deviatoric stress through the avalanche depth, using pressure and velocity profiles from a steady-uniform solution to the full $\mu(I)$-rheology. The resulting viscosity is proportional to the thickness to the three halves power, with a coefficient of proportionality that is angle dependent. When substituted into the depth-averaged momentum balance this term generates second order derivatives of the depth-averaged velocity, which are multiplied by a small parameter. Its inclusion therefore represents a singular perturbation to the equations. It is shown that a granular front propagating down a rough inclined plane is completely unaffected by the rheology, but, discontinuities, which naturally develop in inviscid roll-wave solutions, are smoothed out. By comparison with existing experimental data, it is shown that the depth-averaged $\mu(I)$-rheology accurately predicts the growth rate of spatial instabilities to steady-uniform flow, as well as the dependence of the cut-off frequency on the Froude number and inclination angle. This provides strong evidence that, in the steady-uniform flow regime, the predicted de-
crease in the viscosity with increasing slope is correct. Outside the range of angles where steady-uniform flows develop, the viscosity becomes negative, which implies that the equations are ill-posed. This is a signature of the ill-posedness of the full $\mu(I)$-rheology at both high and low inertial numbers. The depth-averaged rheology can be regularized by specifying non-negative maximum and minimum bounds for the angle dependent coefficient in the viscosity.

1. Introduction

Avalanche models have proved to be very successful at predicting many types of phenomena in shallow granular free-surface flows. The earliest published application dates back to Grigorian, Eglit & Iakimov (1967) who used a modified shallow water type theory to compute snow avalanche paths in the Ural Mountains. A formal derivation of the equations was first presented by Savage & Hutter (1989). They depth-averaged the mass and momentum equations, assuming a Mohr-Coulomb internal rheology and a constant Coulomb basal friction law, which produced a system of conservation laws that looked like the shallow water equations with source terms. An additional complexity was that the depth-averaged pressure gradient was multiplied by an earth pressure coefficient, which changed its value dependent on whether the downslope flow was diverging or converging. Savage & Hutter (1991) and Greve & Hutter (1993) showed that these equations were able to predict the motion of finite mass of granular material from initiation to run-out on a slope with variable topography. Subsequent models have been generalized to two-dimensional flows over complex topography (e.g. Gray, Wieland & Hutter 1999; Wieland, Gray & Hutter 1999; Pudasaini & Hutter 2003; Mangeney-Castelnau et al. 2003; Bouchut & Westdickenberg 2004; Luca et al. 2009) and have been extensively used in the snow avalanche community for hazard zone mapping in alpine regions (e.g. Sampl & Zwinger 2004; Christen, Kowalski & Bartelt 2010; Fischer, Kowalski & Pudasaini 2012). Closely analogous theories have also been
A depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows
developed for other hazardous geophysical mass flows, including for debris flows
(e.g. Iverson 1997; Iverson & Denlinger 2001; Denlinger & Iverson 2001), pyroclastic
flows (e.g. Pitman et al. 2003; Doyle, Hogg & Mader 2011) and landslides (e.g.
Kuo et al. 2009; Mangeney et al. 2010).

Gray, Tai & Noelle (2003) simplified the equations by assuming that the in-
plane deviatoric stresses were sufficiently small that they could be neglected. This
reduced the Savage & Hutter (1989) type models to a shallow water structure with
source terms. By using shock capturing numerical methods (Tai, Noelle, Gray &
Hutter 2002) it was possible to use these hyperbolic theories to accurately calcu-
late high speed granular flows past obstacles, such as pyramids (Gray et al. 2003),
wedges (Hákonardóttir & Hogg 2005; Cui, Gray & Jóhannesson 2007), constrict-
tions (Vreman, Al-Tarazi, Kuipers, Sint & Bokhove 2007; Gray & Cui 2007) and
cylinders (Cui & Gray 2013), where shock waves developed. This has an impor-
tant practical application to the design of avalanche defences such as catching and
deflecting dams (Barbolini et al. 2009), which are used to protect people and in-
frastructure in mountainous regions. Understanding the flow around such obstacles
is also important for the interpretation of data collected at the instrumented pylon
at the European Avalanche avalanche test site in the Vallée de la Sionne (Sovilla,
Schaer, Kern & Bartelt 2008).

Pouliquen (1999a) made systematic measurements of the relationship between
the slope angle, the flow depth and the depth-averaged velocity for steady-uniform
flows on rough beds. He was able to collapse the data to determine a new rate-
dependent friction law, to replace the rate-independent Coulomb friction law used
by Savage & Hutter (1989). As a result he was able to calculate the shape of
a granular flow front as it propagated down a rough inclined plane, and showed
that it agreed remarkably well with experiments (Pouliquen 1999b). An improved
version of this basal friction law, which accounted for the friction at low Froude
numbers, was given by Pouliquen & Forterre (2002).
A depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows

Figure 1. An oblique head-on view of a series of granular roll-waves in an avalanche of 300-355$\mu$m carborundum particles flowing down a channel inclined at 35.1 degrees to the horizontal. All the grains are in motion, and two roll wave crests can be seen that have a sharp brightly illuminated front and a shallower darker lee side.
Forterre & Pouliquen (2003) investigated the spatial stability of steady-uniform flow, by imposing a controlled perturbation at the inlet and and measuring its development down the chute. In this way they were able to experimentally determine the threshold and the dispersion relation of the instability. The results were compared to a linear stability analysis of the problem, within the framework of depth-averaged Saint-Venant (shallow water) equations with a rough bed friction law. The theory was able to quantitatively predict the stability threshold and the phase velocity of the waves, but failed to predict the cut-off frequency. Instead, it predicted that the growth rate tended to a positive constant at high frequencies, which indicated that an important dissipative mechanism was missing in the theory. Such instabilities to steady-uniform flow lead to the formation of periodic granular roll-waves, such as those shown in figure 1. Roll-waves therefore provide a sensitive case against which to test new rheological laws and model formulations.

An even more sensitive problem, that is also deserving of study, is that of particle-size segregation induced fingering instabilities (Pouliquen, Delour & Savage 1997; Pouliquen & Vallance 1999), which cause the classical shallow water type models to become ill-posed (Woodhouse et al. 2012) when coupled to a simple depth-averaged theory for large particle transport (Gray & Kokelaar 2010b,a).

The Groupement de Recherche Milieux Divisés (GDR-MiDi 2004) collated experimental data and numerical results, from molecular dynamics and contact dynamics simulations, in six different configurations, with a view to determining the rheology of dense granular flows. Using dimensional analysis they identified two independent non-dimensional parameters in these systems; the effective-friction coefficient $\mu_{\text{eff}}$, which is just the ratio of the shear stress to the confining pressure, and the inertial number $I$, which is the ratio of a typical time-scale for particle rearrangement to a typical timescale for deformation. This suggested a simple local rheology, $\mu_{\text{eff}} = \mu(I)$, in which the effective friction was a function, $\mu$, of the inertial number, $I$. GDR-MiDi (2004) showed that this law was able to explain (a)
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Linear velocity profiles in plane shear flow and (b) Bagnold-like velocity profiles in chute flow. Jop, Forterre & Pouliquen (2005) showed how the function, $\mu(I)$, could be determined from the basal friction law measured by Pouliquen & Forterre (2002) and Jop, Forterre & Pouliquen (2006) converted the scalar rheology of GDR-MiDi (2004) into a full tensor constitutive law.

The $\mu(I)$-rheology has had a major impact on the field. Jop et al. (2006) used it to compute the steady downslope velocity down a pile constrained between rough side walls. The results agreed with experimental measurements of the free-surface velocity for a range of gap widths. Forterre (2006) performed a two-dimensional linear stability analysis of chute flow with the Bagnold velocity profile, and showed that the new rheology was able to predict the cut-off frequency, which matched the experiments of Forterre & Pouliquen (2003). More recently Lagrée, Staron & Popinet (2011) and Staron, Lagrée & Popinet (2012) have developed impressive two-dimensional time-dependent simulations for granular column collapses and silo flow, which lend considerable weight to the theory.

The success of the $\mu(I)$-rheology opens up the question as to whether it can be used to improve depth-averaged avalanche models, which are, for instance, unable to predict the cut-off frequency of the roll-wave instability. Forterre (2006) has already thought along these lines. He used a heuristic argument to include the depth-averaged in-plane stress gradients in the downslope momentum balance, to generate a non-linear viscous term. While this was able to match the experimental cutoff frequency for the instability, an additional scaling factor had to be used. In addition, the viscosity had a singularity as the thickness tended to zero, which also is problematical. This paper therefore seeks to re-examine the approximations necessary in the depth-integration of the $\mu(I)$-rheology and come up with an improved formulation, that is able to predict the cut-off frequency of the roll-wave instability (Forterre & Pouliquen 2003) and might provide a rationally based regularization.
2. Governing equations and the $\mu(I)$-rheology

Let $Oxz$ be a rectangular Cartesian coordinate system with the $x$-axis orientated down a slope at an angle $\zeta$ to the horizontal and the $z$-axis being the upward pointing normal. The velocity $\mathbf{u}$ has components $(u, w)$ in the $(x, z)$ directions, respectively, and the grains have constant intrinsic density, $\rho^*$. The solids volume fraction, $\Phi$, is assumed to be constant and uniform throughout the material (GDR-MiDi 2004), so the partial density, $\rho = \Phi \rho^*$, is constant and uniform, and mass balance implies that the granular material is incompressible

$$\nabla \cdot \mathbf{u} = 0. \quad (2.1)$$

The momentum balance equation is

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \right) = \nabla \cdot \mathbf{\sigma} + \rho \mathbf{g}, \quad (2.2)$$

where $t$ is time, $\nabla$ is the gradient operator, ‘$\cdot$’ is the dot product, $\otimes$ is the dyadic product, $\mathbf{\sigma}$ is the Cauchy stress tensor and $\mathbf{g}$ is the gravitational acceleration vector. The Cauchy stress is decomposed into an isotropic pressure $p$ and a deviatoric stress $\tau$

$$\mathbf{\sigma} = -p \mathbf{1} + \tau, \quad (2.3)$$

where $\mathbf{1}$ is the unit tensor. The constitutive model for the granular material is provided by the $\mu(I)$-rheology (Jop et al. 2005, 2006), which is a nonlinear viscous law with a pressure and strain-rate dependent viscosity of the form

$$\tau = \mu(I)p \frac{\mathbf{D}}{||\mathbf{D}||}, \quad (2.4)$$

where $\mu$ is the friction law and $I$ is the inertial number. The strain-rate $\mathbf{D}$ is defined in terms of the velocity gradient $\mathbf{L} = \nabla \mathbf{u}$ as

$$\mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T), \quad (2.5)$$
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>ζ₁</td>
<td>20.9°</td>
</tr>
<tr>
<td>ζ₂</td>
<td>32.76°</td>
</tr>
<tr>
<td>β</td>
<td>0.136</td>
</tr>
<tr>
<td>L</td>
<td>0.825 × 10⁻³m</td>
</tr>
</tbody>
</table>

Table 1. Rheological parameters measured by Forterre & Pouliquen (2003).

and $||D||$ is a second invariant of the strain-rate tensor

$$||D|| = \sqrt{\frac{1}{2}trD^2},$$

(2.6)

where, $tr$, is the trace. Note that (2.5) is the standard definition of the strain-rate tensor, which differs from the definition used by (Jop et al. 2006), so there is an extra factor of two in the definition of the non-dimensional inertial number

$$I = \frac{2||D||d}{\sqrt{p/\rho^*}},$$

(2.7)

where $d$ is the diameter of the grains. The inertial number is the ratio of the timescale for microscopic rearrangements of the particles at a given confining pressure, $d/\sqrt{p/\rho^*}$, to the time scale given by the bulk shear rate, $1/||D||$, and is equal to the square root of the Savage or Coulomb number (Savage 1984; Ancey, Coussot & Evesque 1999). Note that in the definition (2.7) the intrinsic solid density of the grains, $\rho^*$, is used rather than the partial density, $\rho$, which includes the interstitial pore space.

The rate dependence in the rheology (2.4) arises from the increase of the friction coefficient, $\mu$, with increasing inertial number, $I$. The dependence was determined from basal friction measurements that were made on an inclined plane (Pouliquen 1999a; Pouliquen & Forterre 2002). They observed that steady-uniform depth flows developed between two critical inclination angles $\zeta_1$ and $\zeta_2$. For slope angles below $\zeta_1$ there was no flow and for angles above $\zeta_2$ the flows accelerated. In the steady-uniform regime they determined the empirical basal friction law

$$\mu(\text{Fr}, h) = \mu_1 + \frac{\mu_2 - \mu_1}{\beta h} \frac{L}{\text{Fr}} + 1,$$

(2.8)

where the friction coefficients $\mu_1 = \tan \zeta_1$ and $\mu_2 = \tan \zeta_2$. The parameter $\beta$ is a
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dimensionless empirical constant (Pouliquen 1999a), whilst $L$ has the dimensions of a length and is characterized by the depth of flow over which a transition between the angles $\zeta_1$ and $\zeta_2$ occurs and, as such, is dependent on the material properties of the flowing particles and on the bed roughness conditions. The values of the rheological parameters used in this paper are those found by Forterre & Pouliquen (2003) and are given in table 1.

On an inclined plane the Froude number

$$\text{Fr} = \frac{\bar{u}}{\sqrt{gh \cos \zeta}}, \quad (2.9)$$

is defined as the ratio of the depth-averaged flow velocity, $\bar{u}$, to the gravity wave speed, $\sqrt{gh \cos \zeta}$, where $g$ is the constant of gravitational acceleration and $h$ is the flow thickness (see e.g Gray et al. 2003). In these steady-uniform flows the inertial number, $I$, is constant and there is a Bagnold velocity profile through their depth (GDR-MiDi 2004). Using the fact that the depth-averaged Bagnold velocity is equal to

$$\bar{u} = \frac{2I}{5d} \sqrt{\Phi g \cos \zeta} h^{3/2}, \quad (2.10)$$

Jop et al. (2005) substituted for the Froude number and the depth-averaged velocity in (2.8) to obtain a general expression for the friction as a function of the inertial number

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1}, \quad (2.11)$$

where the constant

$$I_0 = \frac{5\beta d}{2\sqrt{\Phi} L}. \quad (2.12)$$

The basal and internal friction laws are therefore intimately linked. The friction law (2.8) is only strictly valid for Froude numbers above $\beta$. For Froude numbers below this value Pouliquen & Forterre (2002) determined a transition law, which plays an important role in the development of static regions (see e.g. Mangeney, Bouchut, Thomas, Villette & Bristeau 2007; Johnson & Gray 2011). Note, that the
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use of the Froude number (2.9) corrects for an angle dependence in $I_0$ that was present in the original derivation of Jop et al. (2005) (in their appendix A).

The granular material is subject to kinematic conditions at its free-surface and its base. These are formulated in terms of functions, $F^s = z - s(x, t)$ and $F^b = b(x, t) - z$, which define outward pointing normals, $n^s = \nabla F^s / |\nabla F^s|$ and $n^b = \nabla F^b / |\nabla F^b|$, at the surface and base, respectively. The functions, $F^s$ and $F^b$, are identically zero at $z = s(x, t)$ and $z = b(x, t)$, which implies the material derivatives are zero

$$\frac{\partial F^s}{\partial t} + u^s \cdot \nabla F^s = 0, \quad z = s(x, t),$$

$$\frac{\partial F^b}{\partial t} + u^b \cdot \nabla F^b = 0, \quad z = b(x, t),$$

where the superscripts ‘$s$’ and ‘$b$’ on the velocity indicate evaluation at the surface and base, respectively. In addition, the free-surface of the flow is assumed to be traction free and there is a no slip at the base

$$\sigma^s n^s = 0, \quad z = s(x, t),$$

$$u^b = 0, \quad z = b(x, t).$$

The basal boundary condition (2.16) is different to that imposed by Savage & Hutter (1989) and Gray et al. (1999) who assumed tangential slip with a Mohr-Coulomb friction law that was independent of the internal properties of the material.

3. Non-dimensionalization and depth-integration

3.1. Scaling

The shallowness of the flow is now exploited in order to obtain simplified depth-averaged equations. The avalanche is assumed to be of a typical thickness, $H$, which is much smaller than the downslope length scale, $L$. Typical downstream flow speeds are assumed to be of the order of the gravity wave speed, $U = (gH)^{1/2}$, and mass balance implies that typical normal velocities in the $z$-direction are of magnitude $\epsilon U$, where the aspect ratio $\epsilon \ll 1$. Note that this velocity scaling differs.
A depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows from that of Savage & Hutter (1989), who used the faster scale $U = (gL)^{1/2}$. The pressure scaling, $pgH$, is based on a lithostatic balance in the normal momentum equation. Typical magnitudes for the strain-rate and hence the deviatoric stresses can then be determined from the constitutive relations (2.4)–(2.6). This suggests introducing non-dimensional variables, indicated by the hat, of the form

$$(x, z, s, b, h, d, \mathcal{L}) = L (\hat{x}, \hat{z}, \hat{s}, \hat{b}, \hat{h}, \hat{d}, \hat{\mathcal{L}}), \quad t = L/\sqrt{gH} \hat{t},$$

$$(u, w, |u|, \bar{u}) = \sqrt{gH} (\hat{u}, \hat{w}, \hat{|u|}, \hat{\bar{u}}),$$

$$(D_{xx}, D_{xz}, D_{zz}, ||D||) = \sqrt{g/H} (\hat{D}_{xx}, \hat{D}_{xz}, \hat{D}_{zz}, ||D||),$$

$$(p, \tau_{xx}, \tau_{xz}, \tau_{zz}, \sigma) = \rho gH (\hat{p}, \hat{\tau}_{xx}, \hat{\tau}_{xz}, \hat{\tau}_{zz}, \hat{\sigma}).$$

The non-dimensional mass balance equation and the downslope and normal components of the momentum balance are

$$\varepsilon \left( \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) = 0$$

$$\varepsilon \left( \frac{\partial \hat{u}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} (\hat{u}^2) + \frac{\partial}{\partial \hat{z}} (\hat{u}\hat{w}) \right) = -\varepsilon \frac{\partial \hat{p}}{\partial \hat{x}} + \varepsilon^2 \frac{\partial \hat{\tau}_{xx}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{xz}}{\partial \hat{z}} + \sin \zeta,$$

$$\varepsilon^2 \left( \frac{\partial \hat{w}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} (\hat{u}\hat{w}) + \frac{\partial}{\partial \hat{z}} (\hat{w}^2) \right) = -\frac{\partial \hat{p}}{\partial \hat{z}} + \varepsilon \frac{\partial \hat{\tau}_{xz}}{\partial \hat{x}} + \varepsilon \frac{\partial \hat{\tau}_{zz}}{\partial \hat{z}} - \cos \zeta,$$

which reveal the dominant balances in the equations. The surface and basal kinematic conditions (2.13) and (2.14) become

$$\frac{\partial \hat{s}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{s}}{\partial \hat{x}} - \hat{w} = 0, \quad \hat{z} = \hat{s}(\hat{x}, \hat{t}),$$

$$\frac{\partial \hat{b}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{b}}{\partial \hat{x}} - \hat{w} = 0, \quad \hat{z} = \hat{b}(\hat{x}, \hat{t}),$$

and the downslope and normal components of the surface traction (2.15) are

$$\varepsilon \hat{p} \frac{\partial \hat{s}}{\partial \hat{x}} - \varepsilon^2 \hat{\tau}_{xx} \frac{\partial \hat{s}}{\partial \hat{x}} + \hat{\tau}_{zx} = 0, \quad \hat{z} = \hat{s}(\hat{x}, \hat{t}),$$

$$-\varepsilon \hat{\tau}_{xz} \frac{\partial \hat{s}}{\partial \hat{x}} - \hat{p} + \varepsilon \hat{\tau}_{zz} = 0, \quad \hat{z} = \hat{s}(\hat{x}, \hat{t}).$$
To leading order in the small parameter, $\varepsilon$, the normal component of the momentum balance (3.4) and the surface traction condition (3.8) imply
\[
\frac{\partial \hat{p}}{\partial \hat{z}} = -\cos \zeta, \quad \hat{p}(\hat{s}) = 0,
\] (3.9)
which can be integrated to show that the pressure is lithostatic
\[
\hat{p} = (\hat{s} - \hat{z}) \cos \zeta.
\] (3.10)
The pressure at the base of the avalanche is
\[
\hat{p}_b = (\hat{s} - \hat{b}) \cos \zeta = \hat{h} \cos \zeta,
\] (3.11)
where $\hat{h} = \hat{s} - \hat{b}$ is the non-dimensional thickness, and the depth-averaged pressure
\[
\hat{p} = \frac{1}{\hat{h}} \int_{\hat{b}}^{\hat{s}} \hat{p} \, d\hat{z} = \frac{1}{2} \hat{h} \cos \zeta.
\] (3.12)
The leading order momentum balance and surface traction conditions in the downhill direction, (3.3) and (3.7), reduce to
\[
\frac{\partial \hat{\tau}_{xz}}{\partial \hat{z}} = -\sin \zeta, \quad \hat{\tau}_{xz}(\hat{s}) = 0,
\] (3.13)
which implies that
\[
\hat{\tau}_{xz} = (\hat{s} - \hat{z}) \sin \zeta.
\] (3.14)
Since the shear strain-rate and the second strain-rate invariant are equal to
\[
\hat{D}_{xx} = \frac{1}{2} \frac{\partial \hat{u}}{\partial \hat{z}} + O(\varepsilon^2), \quad ||\hat{D}|| = \frac{1}{2} \left| \frac{\partial \hat{u}}{\partial \hat{z}} \right| + O(\varepsilon),
\] (3.15)
the deviatoric shear stress
\[
\hat{\tau}_{xz} = \mu(I) \hat{p} \text{sgn} \left( \frac{\partial \hat{u}}{\partial \hat{z}} \right),
\] (3.16)
where ’sgn’ is the sign function. Using the lithostatic pressure relation (3.10) and assuming that the sign of $\partial \hat{u}/\partial \hat{z}$ is positive, equation (3.14) can be equated to (3.16) to show that
\[
\mu(I) = \tan \zeta.
\] (3.17)
From the definition of $\mu(I)$, in equation (2.11), it follows that the inertial number, $I$, is equal to the constant, $I_\zeta$, through the depth of the flow, where

$$I_\zeta = I_0 \left( \frac{\tan \zeta - \tan \zeta_1}{\tan \zeta_2 - \tan \zeta} \right),$$

(3.18)

is dependent on the inclination angle $\zeta$. As $\zeta \rightarrow \zeta_1$, the inertial number, $I_\zeta \rightarrow 0$. While as $\zeta \rightarrow \zeta_2$, the inertial number, $I_\zeta \rightarrow \infty$, so the whole range of $I$ is realized by simply changing the slope angle, $\zeta$, from $\zeta_1$ to $\zeta_2$. In non-dimensional variables the definition of the inertial number (2.7) becomes

$$I = \frac{2||\hat{D}||d}{\sqrt{\Phi_p}}.$$

(3.19)

Since $I$ is equal to the constant $I_\zeta$, using (3.10) and (3.15), equation (3.19) reduces to an ordinary differential equation for the velocity profile

$$\frac{\partial \hat{u}}{\partial \hat{z}} = \frac{I_\zeta \sqrt{\Phi \cos \zeta}}{d} (\hat{s} - \hat{z})^{1/2}.$$

(3.20)

This can be integrated, subject to the no slip condition (2.16) at the base, $\hat{u}(\hat{b}) = 0$, to give a Bagnold like velocity profile (e.g. GDR-MiDi 2004)

$$\hat{u} = \frac{2I_\zeta}{3d} \sqrt{\Phi \cos \zeta} \left( \hat{h}^{3/2} - (\hat{s} - \hat{z})^{3/2} \right).$$

(3.21)

The depth-averaged Bagnold velocity is

$$\overline{\hat{u}} = \frac{1}{\hat{h}} \int_\hat{b}^\hat{z} \hat{u} d\hat{z} = \frac{2I_\zeta}{5d} \sqrt{\Phi \cos \zeta} \hat{h}^{3/2},$$

(3.22)

and hence the average of the square of the downslope velocity is

$$\overline{\hat{u}^2} = \frac{1}{\hat{h}} \int_\hat{b}^\hat{z} \hat{u}^2 d\hat{z} = \frac{5}{4} \overline{\hat{u}^2}.$$

(3.23)

### 3.3. Depth Integration

The mass balance (3.2) and the downslope momentum balance (3.3) can now be integrated through the avalanche depth by using Leibniz’ rule (Abramowitz & Stegun 1970) to exchange the order of differentiation and integration (see e.g. Gray et al. 1999; Gray & Kokelaar 2010b). The resulting equations can then be simplified by using the surface and basal kinematic conditions, (3.5) and (3.6), and
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the downslope surface traction condition (3.7) to give

\[
\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} (\hat{h} \hat{u}) = 0, \tag{3.24}
\]

\[
\varepsilon \left( \frac{\partial}{\partial \hat{t}} (\hat{h}) + \frac{\partial}{\partial \hat{x}} (\hat{h} \hat{u}) \right) = \hat{h} \sin \zeta + |\nabla F^b| \hat{i} \cdot \sigma^b n^b + \varepsilon^2 \frac{\partial}{\partial \hat{x}} \left( \hat{h} \hat{\tau}_{xx} \right), \tag{3.25}
\]

where \( |\nabla F^b| = (1 + \varepsilon^2 (\partial \hat{b}/\partial \hat{x})^2)^{1/2} \) and \( \hat{i} \) is the unit normal in the downslope direction. An expression is still needed for the downslope component of the basal traction. Instead of being able to substitute a traction condition, as in Savage & Hutter (1989), the internal Cauchy stress (2.3) must be evaluated at the base. Using the \( \mu(I) \)-rheology (2.4) and the basal pressure (3.11) it follows that

\[
|\nabla F^b| \hat{i} \cdot \sigma^b n^b = -\mu(I \zeta) \hat{h} \cos \zeta \frac{\hat{D}^b_{xz}}{||\hat{D}^b||} - \varepsilon \hat{h} \cos \zeta \frac{\partial \hat{b}}{\partial \hat{x}} + O(\varepsilon^2). \tag{3.26}
\]

Since the no slip condition at the base, (2.16), implies that \( \hat{u}(\hat{b}) \) is identically zero, the \( xx \) component of the strain-rate at the base is zero, i.e. \( \hat{D}^b_{xx} = \partial \hat{u}/\partial \hat{x} = 0 \).

The incompressibility relation (3.2) then implies that \( \hat{D}^b_{zz} = 0 \), since \( \partial \hat{w}/\partial \hat{z} = -\partial \hat{u}/\partial \hat{x} = 0 \), and hence \( ||\hat{D}^b|| = ||\hat{D}^b_{xz}|| \) at the base. As a result the factor

\[
\frac{\hat{D}^b_{xz}}{||\hat{D}^b||} = \text{sgn}(\hat{D}^b_{xz}) = \text{sgn}(\hat{u}), \tag{3.27}
\]

because equations (3.20) and (3.22) imply that the sign of \( \hat{D}^b_{xz} \) is the same as the sign of the depth-averaged velocity. In addition, following Jop et al. (2006), the friction coefficient can be expressed in terms of the Froude number and the flow thickness

\[
\mu(I \zeta) = \mu(\text{Fr}, \hat{h}), \tag{3.28}
\]

as described in §2. The traction condition (3.26) can therefore be expressed entirely in terms of depth-averaged variables.

Although the gravitational force, \( \hat{h} \sin \zeta \), and the basal friction, \( \mu \hat{h} \cos \zeta \), are both order unity quantities, their difference, which arises in the source terms in equation (3.25), is typically much smaller. To formalize this, it is assumed that

\[
\hat{h} \sin \zeta - \mu \hat{h} \cos \zeta \text{sgn}(\hat{u}) = \varepsilon \hat{h} \cos \zeta \left( \tan \zeta - \mu(\text{Fr}, \hat{h}) \text{sgn}(\hat{u}) \right) + O(\varepsilon^2). \tag{3.29}
\]
To leading order the source terms are therefore of order $\varepsilon$, which allows the depth-averaged momentum balance (3.25) to be divided through by $\varepsilon$. When (3.26) and (3.29) are substituted into (3.25), together with the approximations for the depth-averaged pressure (3.10) and the depth-averaged downslope velocity squared (3.23), the leading order system of conservation laws becomes

$$\frac{\partial \hat{h}}{\partial t} + \frac{\partial}{\partial x}(\hat{h}\hat{u}) = 0,$$

(3.30)

$$\frac{\partial}{\partial t}(\hat{h}\hat{u}) + \frac{\partial}{\partial x}(\chi \hat{h}\hat{u}^2) + \frac{\partial}{\partial x}\left(\frac{1}{2}\hat{h}^2 \cos \zeta\right) = \hat{h}\hat{S},$$

(3.31)

where the shape factor $\chi$ is the ratio of the depth-averaged square of the velocity, $\bar{u}^2$, to the depth-averaged velocity squared, $\hat{u}^2$. Equation (3.23) implies that $\chi$ is formally equal to $5/4$ for the Bagnold velocity profile, but in virtually all granular flow models $\chi$ is assumed to be unity for simplicity. Finally the source term

$$\hat{S} = \cos \zeta \left(\tan \zeta - \mu(\text{Fr}, \hat{h}) \text{sgn} (\hat{u})\right) - \cos \zeta \frac{\partial \hat{b}}{\partial \hat{x}},$$

(3.32)

is the combination of gravity acceleration, effective basal friction and topography gradients. Equations (3.30)–(3.32) are the familiar shallow water type avalanche equations, which are commonly used in the literature and have proved their effectiveness over many years (e.g. Grigorian et al. 1967; Savage & Hutter 1989; Gray et al. 1999; Pouliquen 1999b; Pouliquen & Forterre 2002; Gray et al. 2003). What is interesting, is that the full shallow water structure emerges naturally out of the leading order balances, rather than having to include the depth-averaged pressure gradient and the basal topography gradients as order $\varepsilon$ terms, as in Savage & Hutter (1989) and Gray et al. (2003). It is also interesting how the combination of the no slip condition (2.16) and the internal rheology (2.4) naturally give rise to an effective basal friction in the source terms (3.32). This is the only effect of the rheology on the flow, as the depth-averaged in-plane deviatoric stress gradient, which was present in (3.25), does not contribute to the leading order momentum balance.
4. Inclusion of depth-averaged viscous-like terms

Despite the effectiveness of the shallow water type avalanche models, higher order terms are sometimes required to provide high wave-number cut-off of roll-wave instabilities (Forterre 2006) or to regularize models of segregation-induced fingering (Woodhouse et al. 2012). Trying to formally include all the order $\varepsilon$ effects is difficult and rapidly leads to theories that are too complex. A pragmatic approach is to simply include the in-plane deviatoric stress gradient that was present in equation (3.5). Let us therefore consider the modified downslope momentum balance

$$\frac{\partial}{\partial t}(\hat{h}\hat{u}) + \frac{\partial}{\partial x}(\chi \hat{h}\hat{u}^2) + \frac{\partial}{\partial x}\left(\frac{1}{2}\hat{h}^2 \cos \zeta \right) = \hat{h}\hat{S} + \varepsilon \frac{\partial}{\partial x}(\hat{h}\hat{\tau}_{xx}), \quad (4.1)$$

and seek an approximation for the depth-averaged in-plane deviatoric stress using the approximations for the pressure (3.11), inertial number (3.17) and velocity profile (3.21) that have already been obtained in §3.2. The $\mu(I)$-rheology (2.4) implies that

$$\hat{\tau}_{xx} = \mu(I)\rho \frac{\hat{D}_{xx}}{||\hat{D}||}. \quad (4.2)$$

Assuming $\hat{h} = \hat{h}(\hat{x}, \hat{t})$ and $\hat{s} = \hat{s}(\hat{x}, \hat{t})$, the $xx$-component of the strain-rate tensor, $\hat{D}$, can be calculated by differentiating the Bagnold velocity profile (3.21) with respect to $\hat{x}$

$$\hat{D}_{xx} = \frac{\partial \hat{u}}{\partial \hat{x}} = \frac{I_c \sqrt{\Phi \cos \zeta}}{d} \left(\hat{h}^{1/2} \frac{\partial \hat{h}}{\partial \hat{x}} - (\hat{s} - \hat{z})^{1/2} \frac{\partial \hat{s}}{\partial \hat{x}}\right). \quad (4.3)$$

Provided $\partial \hat{u}/\partial \hat{z}$ is positive, it follows from (3.15) and (3.20) that to leading order the second strain-rate invariant is

$$||\hat{D}|| = \frac{1}{2} \frac{\partial \hat{u}}{\partial \hat{z}} = \frac{I_c \sqrt{\Phi \cos \zeta}}{2d} (\hat{s} - \hat{z})^{1/2}. \quad (4.4)$$

Substituting (4.3) and (4.4) into (4.2), together with the expressions (3.11) and (3.17) for the lithostatic pressure and the inertial number, implies that

$$\hat{\tau}_{xx} = 2 \sin \zeta \left(\hat{h}^{1/2}(\hat{s} - \hat{z})^{1/2} \frac{\partial \hat{h}}{\partial \hat{x}} - (\hat{s} - \hat{z}) \frac{\partial \hat{s}}{\partial \hat{x}}\right). \quad (4.5)$$
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Integrating this through the depth of the avalanche and eliminating the free-surface height, using $\dot{s} = \dot{h} + \dot{b}$, implies that

$$h\dot{\tau}_{xx} = \int_{\dot{b}}^{\dot{s}} \dot{\tau}_{xx} \, d\dot{z} = \frac{1}{3} \dot{h}^2 \sin\zeta \frac{\partial \dot{h}}{\partial \dot{x}} - \dot{h}^2 \sin\zeta \frac{\partial \dot{b}}{\partial \dot{x}}. \quad (4.6)$$

This provides an approximation for the depth-averaged deviatoric in-plane stress arising in the momentum balance equation (4.1), but it is dependent on the gradient of the thickness, $\partial \dot{h}/\partial \dot{x}$, rather than having a dependence on the gradient of the depth-averaged velocity, $\partial \dot{u}/\partial \dot{x}$, as one might have expected. It can, however, be reformulated by using the depth-averaged Bagnold velocity (3.22) to substitute for the gradient

$$\frac{\partial \dot{h}}{\partial \dot{x}} = \frac{5 \dot{d}}{3 I_0 \sqrt{\Phi \cos \zeta}} h^{1/2} \frac{1}{\partial \dot{x}}, \quad (4.7)$$

in (4.6) to give a depth-averaged non-linear viscous law of the form

$$h\dot{\tau}_{xx} = \dot{\nu} \dot{h}^{3/2} \frac{\partial \dot{u}}{\partial \dot{x}} - \dot{h}^2 \sin\zeta \frac{\partial \dot{b}}{\partial \dot{x}}, \quad (4.8)$$

where $\dot{\nu} \dot{h}^{3/2}/2$ is the coefficient of depth-averaged viscosity. The dependence on the thickness to the three halves power is a direct result of the $\mu(I)$-rheology.

Using the definition of $I_0$, in equation (3.18), and a non-dimensionalized version of the definition of $I_0$, (2.12), the coefficient $\dot{\nu}$ can be expressed as

$$\dot{\nu} = \frac{5 \dot{d} \sin \zeta}{9 I_0 \sqrt{\Phi \cos \zeta}} \frac{1}{2 \beta} \sqrt{\frac{\cos \zeta}{\cos \zeta}} \left( \tan \zeta - \tan \zeta_1 \right). \quad (4.9)$$

This has a very interesting dependence on the slope inclination angle, $\zeta$, for $\zeta_1 \leq \zeta \leq \zeta_2$ as shown in figure 2. Avalanches on slopes with lower inclinations will experience a greater viscosity than if they are on higher gradient slopes. In particular, as the slope inclination tends to the lower limit, $\zeta \rightarrow \zeta_1$, the viscosity, $\dot{\nu} \rightarrow \infty$. This could potentially play an important role in flow arrest processes. Conversely, as the slope inclination approaches the case of accelerated flow, $\zeta \rightarrow \zeta_2$, the viscosity tends to zero. For angles $\zeta < \zeta_1$ and $\zeta > \zeta_2$ the viscosity is negative! This sounds a note of caution, as problems with negative viscosities will be ill-posed. This may in part be because the leading order steady-uniform flow assumptions in
Figure 2. The coefficient $\nu$ in the effective viscosity of the new depth-averaged $\mu(I)$-rheology is plotted as a function of the slope inclination angle $\zeta$ (solid line). There is a singularity as $\zeta \rightarrow \zeta_1$ and $\nu$ becomes negative in the shaded region for $\zeta < \zeta_1$. At $\zeta = \zeta_2$ the coefficient $\nu = 0$ and becomes negative in the shaded region $\zeta > \zeta_2$. The effective viscosity will therefore be negative in these regions, which will lead to ill-posedness. By setting thresholds $\nu_{\min} \geq 0$ and $\nu_{\max} \geq 0$, shown by the dot-dash lines, the depth-averaged rheology can be regularized.

§3.2 break down, but here it is also a signature of the underlying ill-posedness of the $\mu(I)$-rheology (Barker, Schaeffer, Bohórquez & Gray 2014) at both high and low inertial numbers. In the depth-averaged framework, the ill-posedness can easily be avoided by defining maximum and minimum values of $\hat{\nu}$, so the viscosity is always greater than or equal to zero, as shown by the dot-dash lines in figure 2.

The second term on the righthand side of (4.8) arises from the interaction of the in-plane stresses with the prescribed topography, $\hat{b}$. It has the same $\hat{h}^2$ thickness dependence as the depth-averaged pressure gradient term in (4.1), but is much smaller, being of order $\varepsilon$, and is therefore neglected. The first term in (4.8), however, introduces a singular perturbation into the new system of conservation laws

$$\frac{\partial \hat{h}}{\partial t} + \frac{\partial}{\partial x} (\hat{h} \hat{u}) = 0,$$

$$\frac{\partial}{\partial t} (\hat{h} \hat{u}) + \frac{\partial}{\partial x} (\chi \hat{h}^2 \hat{u}) + \frac{\partial}{\partial x} \left( \frac{1}{2} \hat{h}^2 \cos \zeta \right) = \hat{h} \hat{S} + \varepsilon \frac{\partial}{\partial x} \left( \hat{\nu} \hat{h}^{3/2} \frac{\partial \hat{u}}{\partial x} \right),$$

(4.11)
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since the highest order gradient in (4.11) is multiplied by the small parameter $\varepsilon$.

Equations (4.10)–(4.11) can be made dimensional again by applying the scalings

$$
\begin{align*}
x &= L \hat{x}, \quad (h, b, \mathcal{L}) = H(\hat{h}, \hat{b}, \hat{\mathcal{L}}), \quad t = L/\sqrt{gH \hat{t}}, \\
\tilde{u} &= \sqrt{gH} \hat{u}, & S &= \varepsilon \hat{S}, & \nu &= \sqrt{gH} \hat{\nu}, \\
\end{align*}
$$

(4.12)
to show that the depth-averaged mass and momentum balances are

$$
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h \tilde{u}) = 0, \tag{4.13}
$$

$$
\frac{\partial}{\partial t}(h \tilde{u}) + \frac{\partial}{\partial x}(\chi h \tilde{u}^2) + \frac{\partial}{\partial x} \left( \frac{1}{2} gh^2 \cos \zeta \right) = hgS + \frac{\partial}{\partial x} \left( \nu h^{3/2} \frac{\partial \tilde{u}}{\partial x} \right), \tag{4.14}
$$

where the dimensional source term is

$$
S = \cos \zeta (\tan \zeta - \mu(\text{Fr}, h) \text{sgn}(\tilde{u})) - \cos \zeta \frac{\partial b}{\partial x}, \tag{4.15}
$$

and the dimensional coefficient in the viscous law becomes

$$
\nu = \frac{2 \mathcal{L} \sqrt{g}}{9 - \beta} \frac{\sin \zeta}{\cos \zeta} \left( \frac{\tan \zeta_2 - \tan \zeta_1}{\tan \zeta - \tan \zeta_1} \right). \tag{4.16}
$$

For the large majority of situations, the new depth-averaged $\mu(I)$-rheology can be neglected, but when sharp gradients in $\tilde{u}$ develop, a boundary layer forms in which the viscous terms play a significant role. This system therefore has all the advantages of the classic shallow water type avalanche models (e.g. Grigorian et al. 1967; Savage & Hutter 1989; Gray et al. 1999; Pouliquen 1999b; Pouliquen & Forterre 2002; Gray et al. 2003), but has the extra physics necessary to obtain high wave-number cut-off (Forterre 2006) as well as the ability to regularize ill-posed models (Woodhouse et al. 2012).

5. Comparison with Forterre’s depth averaged $\mu(I)$-rheology

Forterre (2006) used a heuristic approach to incorporate depth-averaged viscous-stresses into the shallow water type avalanche equations. His argument was based on using (2.7) to rewrite the pressure, $p$, as a function of the inertial number as

$$
p = 4 \rho^* d^2 \frac{|D|}{I^2}, \tag{5.1}
$$
A depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows and substituting it into (2.4) to obtain an expression for the in-plane deviatoric stress

$$\tau_{xx} = 2\rho^* d^2 \mu(I) I^2 \dot{\gamma} \frac{\partial u}{\partial x},$$

(5.2)

where $\dot{\gamma} = 2||D||$. In equation (5.2), $\mu(I) = \tan \zeta$, and, $I = I_\zeta$, are constant at a given slope angle and are therefore independent of depth. In addition, Forterre (2006) assumed that $\dot{\gamma}$ was also independent of depth and was then able to explicitly integrate (5.2) through the avalanche thickness to obtain

$$h \tau_{xx} = \int_0^h \tau_{xx} dz = 2\rho^* d^2 \tan \zeta I^2 \dot{\gamma} \frac{\partial}{\partial x}(h\ddot{u}).$$

(5.3)

Forterre (2006) made the approximation that $\dot{\gamma} = \ddot{u}/(2h)$ in (5.3), but subsequently found that he needed to add an additional scaling factor, $a$, into the viscosity (in his momentum equation C2), which was set equal to 0.1, in order to get good agreement with the neutral stability curves for Forterre & Pouliquen’s (2003) roll-wave experiments.

Using (5.3), it follows that Forterre’s (2006) depth-averaged avalanche equations are

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\ddot{u}) = 0,$$

(5.4)

$$\frac{\partial}{\partial t}(h\ddot{u}) + \frac{\partial}{\partial x}(\chi h\ddot{u}^2) + \frac{\partial}{\partial x}\left(\frac{1}{2} h^2 g \cos \zeta\right) = hgS + \frac{\partial}{\partial x}\left(\nu_F \ddot{u} \frac{\partial}{\partial x}(h\ddot{u})\right),$$

(5.5)

where the constant in the effective viscosity can be expressed as

$$\nu_F = \frac{ad^2 \tan \zeta I^2 \Phi}{I^2_\zeta} = \frac{4aZ^2 \tan \zeta}{25\beta^2} \left(\frac{\tan \zeta_2 - \tan \zeta_1}{\tan \zeta - \tan \zeta_1}\right)^2,$$

(5.6)

by using (3.18) and (2.12). Note that just as in equation (4.16), $\nu_F$ also has a singularity as $\zeta \rightarrow \zeta_1$ and is equal to zero when $\zeta = \zeta_2$, but here it is does not go negative. Equations (5.3)–(5.4) are identical to equations (4.13)–(4.14) except for the structure of the non-linear viscous term, which is radically different. The new depth-averaged $\mu(I)$-rheology (4.14) has a viscous-like term with a gradient of the depth-averaged velocity, $\partial \ddot{u}/\partial x$, multiplied by a coefficient of viscosity, $\nu h^{3/2}/2$, which is degenerate as $h \rightarrow 0$, i.e. the diffusion coefficient is zero when $h = 0$. 


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Whereas Forterre’s (2006) viscous-like term in (5.5) is based on gradients of the depth-averaged momentum, $\partial(h\bar{u})/\partial x$, and is multiplied by an effective viscosity $\nu_F\bar{u}/(2h)$, which is degenerate in $\bar{u}$, but is singular in $h$, i.e. the viscosity tends to infinity as $h \rightarrow 0$.

6. Effect of viscosity on the shape of a granular flow front

The difference between the two depth-averaged viscous-like terms is immediately apparent when one considers the problem of a granular flow front propagating down a rough inclined plane. Pouliquen (1999b) showed experimentally that the flow front moves at constant speed, $u_F$, and develops a well defined shape. By changing to a front centred frame of coordinates defined by the change of variables

$$\xi = x - u_F t, \quad \tau = t,$$

the steady state mass balance equation (4.13) in the traveling frame becomes

$$\frac{d}{d\xi} (h(\bar{u} - u_F)) = 0.$$  (6.2)

Integrating this subject to the boundary condition that $h = 0$ at $\xi = 0$ implies that either

$$h = 0, \quad \text{or} \quad \bar{u} = u_F.$$  (6.3)

This allows a solution to be constructed with a front located at $\xi = 0$, ahead of which is a grain-free region for $\xi > 0$. Behind the front ($\xi \leq 0$) the depth-averaged velocity, $\bar{u}$, is equal to the front speed, $u_F$, and the frontal shape is given by, $h = h(\xi)$. Since, $\bar{u}$, is constant everywhere, there are no gradients in the depth-averaged velocity, and hence the new depth-averaged $\mu(I)$-rheology, derived in §4, does not induce any viscous stresses. Assuming that the shape factor, $\chi$, equals unity and the topography, $b$, is flat, the depth-averaged momentum balance equation (4.14) therefore simply reduces to the inviscid case, i.e.

$$\frac{dh}{d\xi} = \tan \zeta - \mu(\text{Fr}, h).$$  (6.4)
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Far upstream the solution is assumed to tend toward a steady-uniform flow, in which $h = h_0$ and $\bar{u} = \bar{u}_0$, and there is a balance between $\tan \zeta$ and $\mu(F_{r0}, h_0)$, where

$$F_{r0} = \frac{\bar{u}_0}{\sqrt{gh_0 \cos \zeta}},$$

(6.5)

is the steady-uniform Froude number. Using the definition of the friction law (2.8) it follows that the depth-averaged steady-uniform velocity

$$\bar{u}_0 = \frac{\beta \sqrt{g \cos \zeta}}{\mathcal{L} \gamma} h_0^{3/2},$$

(6.6)

where the parameter $\gamma$ is defined as

$$\gamma = \frac{\tan \zeta_2 - \tan \zeta_1}{\tan \zeta - \tan \zeta_1} = \frac{\beta h_0}{\mathcal{L} F_{r0}}.$$  

(6.7)

Since the depth-averaged velocity is constant everywhere, it is determined by (6.6), which using (2.12), (3.18) and (4.12) is the dimensional equivalent of the depth-averaged Bagnold velocity (3.22). It is useful to non-dimensionalize the problem using the steady-uniform flow depth, $h_0$, and depth-averaged velocity, $\bar{u}_0$, by

$$(h, x, \xi) = h_0(\tilde{h}, \tilde{x}, \tilde{\xi}), \quad \bar{u} = \bar{u}_0 \tilde{u}, \quad t = (h_0/\bar{u}_0) \tilde{t},$$

(6.8)

where the tilded variables are non-dimensional. These scalings imply that the differential equation (6.4) with the friction law (2.8) becomes

$$\frac{dh}{d\xi} = (\tan \zeta_2 - \tan \zeta_1) \left( \frac{1}{1 + \gamma} - \frac{1}{1 + \gamma h_0^{3/2}} \right),$$

(6.9)

where the tildes are now dropped for simplicity. Gray & Ancey (2009) showed that this could be integrated exactly to give $\xi = \xi(h)$. Pouliquen (1999b) numerically solved an equivalent version of this equation for the exponential form of the basal friction law (Pouliquen 1999a). The exact solution for the new depth-averaged rheology is shown by the dashed line in figure 3, for parameters $\zeta_1 = 20.9^\circ$, $\zeta = 29^\circ$, $\zeta_2 = 32.76^\circ$. The key point is that, for the front problem, the new depth-averaged $\mu(I)$-rheology gives the identical solution to the inviscid case. The results therefore not only automatically match Pouliquen’s (1999b) experimental data, but also naturally produce a completely grain-free region ahead of the front.
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Figure 3. Comparison between the non-dimensional front thickness $h$ for the new depth-averaged viscous model (dashed line) and Forterre’s (2006) depth-averaged viscous model (solid line) in the moving coordinate $\xi$ and for parameters $\zeta_1 = 20.9^\circ$, $\zeta = 29^\circ$ and $\zeta_2 = 32.76^\circ$. The new model has exactly the same solution as the inviscid case (Pouliquen 1999b; Gray & Ancey 2009) with a well defined front and a grain free region for $\xi > 0$. Conversely Forterre’s (2006) model does not allow grain free regions, as the inset diagram shows, and the shape of the front is sensitively dependent on the precise way in which $h \to 0$ as $\xi \to \infty$. In the case illustrated here the exponential solution $\epsilon \exp(r\xi)$, for $\xi > 0$ with $r = -1$ and $\epsilon = 10^{-3}$, is matched to a numerical solution for $\text{Fr}_0 = 1.02$ and $R_F = 113.18$, in the region $\xi < 0$, at $\xi = 0$.

Conversely, the front problem for Forterre’s (2006) depth-averaged rheology is affected by the viscous term. Even though mass balance still implies that if $h = 0$ at the front then $\bar{u} = u_F$ everywhere, the viscous term involves the gradient of $h\bar{u}$, which is not constant. Assuming that far upstream there is steady-uniform flow, then $\bar{u}$ is equal to the steady-uniform velocity, $u_0$, everywhere. It follows that for shape factor $\chi = 1$ and flat basal topography, the scalings (6.8) imply the non-dimensional depth-averaged momentum balance (5.5) in the moving frame is

$$h \frac{dh}{d\xi} = h(\tan \zeta_2 - \tan \zeta_1) \left( \frac{1}{1 + \gamma} - \frac{1}{1 + \gamma h^{3/2}} \right) + \frac{\text{Fr}_0^2}{R_F} \frac{dh}{h} \frac{d\bar{u}}{d\xi}, \quad (6.10)$$

where the equivalent of the Reynolds number for this rheology is

$$R_F = \frac{h_0^2}{\nu_F}. \quad (6.11)$$
The last term on the right-hand side of (6.10) is the viscous term, and has a singularity in the viscosity as $h \longrightarrow 0$, which is problematic at the flow front. To study the effect of this term, an expansion is made about $h = 0$ by the introduction of a re-scaled order unity thickness, $H$, that satisfies

$$h = \epsilon H, \quad \text{where} \quad \epsilon \ll 1.$$  \hfill (6.12)

Substituting this into the differential equation (6.10) implies that to leading order in $\epsilon$

$$\frac{d}{d\xi} \left( \frac{1}{H} \frac{dH}{d\xi} \right) = 0,$$  \hfill (6.13)

which can be integrated twice to show that

$$H = q e^{r\xi},$$  \hfill (6.14)

where $q$ and $r$ are constant of integration. Since the exponential function is strictly positive for finite $\xi$, the boundary condition $H = 0$ at $\xi = 0$ can only be satisfied if $q = 0$, and hence $H = 0$ everywhere, which is not desired.

Non-trivial solutions can be constructed to Forterre’s (2006) model for $r < 0$, but they only satisfy the boundary condition $h = 0$ in the limit as $\xi \longrightarrow \infty$. An example of such a solution is shown by the solid line in figure 3, for the same parameters as the inviscid case and with $Fr_0 = 1.02$ and $R_F = 113.18$. The asymptotic solution (6.14) is used in $\xi > 0$, with parameters $q = 1$, $r = -1$ and $\epsilon = 10^{-3}$. It is matched onto a numerical solution by applying the initial conditions

$$h = \epsilon, \quad \frac{dh}{d\xi} = -\epsilon, \quad \text{at} \quad \xi = 0,$$  \hfill (6.15)

and then integrating (6.10) for $h$ in $\xi < 0$. Overall the solution lies close to the new depth-averaged result, which is indicated by the dashed line. However, as the inset image shows, Forterre’s (2006) depth-averaged viscous term prevents the development of grain-free regions, and fills the whole domain with a thin layer of material. Since this very thin layer will be far below the actual grain size of the particles, one might argue that this is an irrelevance that could be ignored.
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In this case it is not, since it is possible to construct different solutions for other values of $r$ and $\epsilon$. The front shape in Forterre’s (2006) model is therefore sensitively dependent on the precise thickness of the precursor layer and the way in which it tends to zero as $\xi \rightarrow \infty$, which is not physically realistic. The new depth-averaged model (4.13)–(4.16), on the other hand, generates frontal shapes that agree with the experimental profiles measured by Pouliquen (1999b), and which are totally unaffected by the grain-free region.

7. Instability of steady-uniform flows

7.1. Non-dimensional steady-uniform flow equations

The new model (4.13)–(4.16) is non-dimensionalized about the steady-uniform flow thickness, $h_0$, and depth-averaged velocity, $\bar{u}_0$, using the scalings (6.8). Assuming the shape factor, $\chi$, equals unity and the basal topography, $b$, is flat, the non-dimensional depth-averaged mass and momentum balances become

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h\bar{u}) = 0,$$

$$F^2 h \left( \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} \right) + h \frac{\partial h}{\partial x} = h (\tan \zeta - \mu) + \frac{F^2 \partial}{R \partial x} \left( h^{3/2} \frac{\partial \bar{u}}{\partial x} \right),$$

where mass balance has been used to reduce the acceleration terms. The friction coefficient (2.8) can be written as

$$\mu(h, \bar{u}) = \tan \zeta_1 + \tan \zeta_2 - \tan \zeta_1 \frac{1 + \gamma h^{3/2}/\bar{u}}{1 + \gamma h^{3/2}/\bar{u}},$$

where $\gamma$ is defined in (6.7), and the Reynolds number, $R$, for the new rheology is defined in terms of the dimensional quantities as

$$R = \frac{\bar{u}_0 \sqrt{h_0}}{\nu}.$$  

For notational simplicity the steady-uniform Froude number is denoted by

$$F = Fr_0,$$
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\[ \zeta = 29.0^\circ \quad \gamma \approx 0.517 \quad \nu \approx 1.13 \times 10^{-3} \text{m}^3/\text{2s}^{-1} \quad F = 1.02 \quad R \approx 8.45 \]

Table 2. Parameters for the new depth-averaged rheology corresponding to the conditions in Forterre & Pouliquen’s (2003) experiments, shown in their figure 11.

which is defined by (6.5). It is convenient to use $F$ as a parameter in the theory, which then sets the dimensional thickness, $h_0$, and dimensional velocity, $\bar{u}_0$, as well as the Reynolds number, $R$, by solving (6.7), (6.6) and (7.4), respectively.

7.2. Linear stability analysis

To study the linear stability of the steady-uniform flow, small perturbations are made about the base state $h = 1$, $\bar{u} = 1$ with the introduction of new variables

\[ h = 1 + \epsilon \mathcal{H}, \quad \bar{u} = 1 + \epsilon \mathcal{U}, \quad \epsilon \ll 1. \quad (7.6) \]

Substitution of the variables (7.6) into the non-dimensional system of equations (7.1)–(7.2) and collecting terms in orders of $\epsilon$ shows that the $O(1)$ equations are trivially satisfied, whilst the $O(\epsilon)$ equations are

\[ \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial x} + \frac{\partial \mathcal{U}}{\partial x} = 0, \quad (7.7) \]

\[ F^2 \left( \frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{U}}{\partial x} \right) + \frac{\partial \mathcal{H}}{\partial x} = \Gamma \left( \frac{3}{2} \mathcal{H} - \mathcal{U} \right) + \frac{F^2}{R} \frac{\partial^2 \mathcal{U}}{\partial x^2}, \quad (7.8) \]

where the constant $\Gamma$ is defined as

\[ \Gamma = \frac{\gamma \left( \tan \zeta_2 - \tan \zeta_1 \right)}{(1 + \gamma)^2}. \quad (7.9) \]

Solutions to the linearized equations (7.7)–(7.8) are sought in the form

\[ (\mathcal{H}, \mathcal{U}) = (V_1, V_2) e^{i(kx - \omega t)} = (V_1, V_2) e^{-k_x x} e^{i(k_x x - \omega t)}, \quad (7.10) \]

for a complex wavenumber, $k = k_r + ik_x$, and real pulsation, $\omega$. This form is used because Forterre & Pouliquen (2003) initiated the perturbations to their flow by using sound waves with a given pulsation frequency at a fixed location near the
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top of the chute. The spatial growth rate can therefore be defined as

$$\sigma = -\Im(k) = -k_i,$$

and the phase velocity as

$$c = \frac{\omega}{\Re(k)} = \frac{\omega}{k_r}.$$  \hspace{1cm} (7.12)

Substituting for $\mathcal{H}$ and $\mathcal{U}$ using (7.10) reduces the linear stability equations to

$$\mathbf{A} \mathbf{V} = 0,$$  \hspace{1cm} (7.13)

where $\mathbf{V} = (V_1, V_2)^T$, $\mathbf{0} = (0, 0)^T$ and the matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} k - \omega & k \\ ik - \frac{\Gamma}{2} & \Gamma + iF^2(k - \omega) + \frac{F^2k^2}{R} \end{pmatrix}.$$  \hspace{1cm} (7.14)

For non-trivial solutions, the determinant, $\det \mathbf{A} = 0$, yields the dispersion relation

$$k^3 + \left( iR - \frac{iR}{F^2} - \omega \right) k^2 + R \left( \frac{5\Gamma}{2F^2} - 2i\omega \right) k + R \left( i\omega - \frac{\Gamma}{F^2} \right) \omega = 0.$$  \hspace{1cm} (7.15)

This has one distinct complex root and a pair of complex conjugate roots. Explicit solutions for these three roots may be obtained by Cardano’s solution to a cubic equation with complex coefficients. Of the three solution branches, only one has any region of positive spatial growth rate, $\sigma$, with a positive phase velocity (one solution always has negative growth with a positive phase velocity, the other always shows positive growth with a negative phase velocity, both implying spatial decay).

Note that this implicitly assumes $R$ is positive. If $R$ is negative one the roots grows without bound, which implies that the equations are ill-posed if the coefficient in the viscosity, $\nu$, is negative.

The spatial growth rate and phase velocity for this solution branch are shown in figure 4 (solid lines), which are in good quantitative agreement with the experimental measurements (markers) made by Forterre & Pouliquen (2003). The parameter values for Forterre & Pouliquen’s (2003) experiments and the implied values of $\nu$ and $R$ are summarized in table 2. In particular, the new-depth-averaged $\mu(I)$-rheology is able to predict a cut-off frequency, $\omega_c$, that is close to the experimental
value, and similar to the results of Forterre’s (2006) stability analysis using the full \( \mu(I) \)-rheology. This is a significant improvement on Forterre’s (2006) depth-averaged rheology, which was only able to match the results by including the scaling factor, \( a \), in the viscosity, \( \nu_F \). It should be noted that in the absence of rheology, the classical granular avalanche equations (3.30)–(3.32) do not predict a cut-off frequency (Forterre & Pouliquen 2003), and instead the growth rate tends to a positive constant as the frequency tends to infinity, so the presence of a viscous term is vital to match experiments.

7.3. Cut-off frequency for instability using the new depth-averaged rheology

The cut-off frequency, \( \omega_c \), is the frequency at which the growth rate is zero, that is \( \sigma = -k_i = 0 \), and the corresponding wavenumber has a critical value \( k_c = k_{rc} \in \mathbb{R} \). An expression for \( \omega_c \) is found by taking real and imaginary parts of the dispersion relation (7.15) with \( k = k_c \), to give

\[
\frac{F^2}{R} k_c^3 - \frac{F^2}{R} \omega_c k_c^2 + \frac{5\Gamma}{2} k_c - \Gamma \omega_c = 0, \tag{7.16}
\]
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Figure 5. Comparison between the model using the new depth-averaged $\mu(I)$-rheology (lines) and experimental data (symbols) of Forterre & Pouliquen (2003) on the cut-off frequency, $\omega_c$, as a function of the Froude number $F$ above the critical Froude number $F_c = 2/3$ for a range of slope angles $24^\circ \leq \zeta \leq 29^\circ$. All quantities are dimensionless, with the experimental data made so using the measured mean velocity and thickness at each point.

\[
(F^2 - 1)k_c^2 - 2F^2\omega_c k_c + F^2\omega_c^2 = 0,
\]
respectively. Solving the second of these gives a pair of quadratic roots for the critical wavenumber

\[
k_{c\pm} = \frac{F}{F \mp 1}\omega_c.
\]
Substitution of the roots (7.18) into (7.16) gives a quadratic for $\omega_c$,

\[
\pm \frac{F^4}{R(F \mp 1)^3}\omega_c^2 + \frac{5F}{2(F \mp 1)} - \Gamma = 0,
\]
(assuming $\omega_c \neq 0$), which implies

\[
\omega_c^2 = \mp \frac{(F \mp 1)^2}{F^4}R\Gamma \left( \frac{3F}{2} \pm 1 \right).
\]
This has four distinct, non-trivial solutions for $\omega_c$ given by

\[
\omega_{c1\pm} = \frac{(F \mp 1)}{F^2} \sqrt{\mp R\Gamma \left( \frac{3F}{2} \pm 1 \right)}, \quad \omega_{c2\pm} = -\frac{(F \mp 1)}{F^2} \sqrt{\mp R\Gamma \left( \frac{3F}{2} \pm 1 \right)}.
\]
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Two of the solutions $\omega_{c1,2}^+$ can immediately be discarded, since they are purely imaginary (with $\Gamma, R, F > 0$) and the frequency $\omega$ was assumed real. Next $\omega_{c2}^-$ is ignored since any non-zero real part is always negative. This leaves only one solution with positive real part for the cut-off frequency

$$\omega_c = \omega_{c1} = \frac{F + 1}{F^2} \sqrt{RF\left(\frac{3}{2}F - 1\right)},$$ (7.22)

which is purely real for $F$ greater than the critical Froude number for the instability

$$F > F_c = \frac{2}{3}.$$ (7.23)

The real part of $\omega_c$ is plotted as a function of $F - F_c$ in figure 5 to give the neutral stability curves for integer values of the slope inclination angle $\zeta$ between $24^\circ$ and $29^\circ$. New values of $R$ and $\Gamma$ (along with all of the constituent parameters) must be calculated from the definitions (7.4) and (7.9), respectively, for each inclination.

There is very good qualitative and quantitative agreement between the cut-off frequency predicted by the new depth-averaged $\mu(I)$-rheology (lines) and that of the experimental data measured by Forterre & Pouliquen (2003) (markers). Quite surprisingly the new model is in better agreement with the experimental data than the cut-off frequency predicted using the full two-dimensional $\mu(I)$-rheology (Forterre 2006). It is also an improvement on Forterre’s (2006) depth-averaged rheology, which needed an arbitrary fitting parameter. The near collapse of all of the neutral stability curves for different slope inclination angles is a direct result of the $\zeta$ dependence in the coefficient $\nu$ in the new depth-averaged $\mu(I)$-rheology, given by (4.16).

8. Granular roll-waves with the new depth-averaged $\mu(I)$-rheology

It is interesting to construct some solutions for the shape of granular roll-waves, to see the effect of the new depth-averaged $\mu(I)$-rheology. In order to do this, it is convenient to switch to a coordinate system that moves at the same speed, $u_w$, as
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the roll-wave, by making the coordinate transformation

$$\xi = x - u_w t, \quad \tau = t,$$

and then looking for steady-state solutions in the traveling frame to the non-dimensional system of conservation laws (7.1)–(7.2), which become

$$\frac{d}{d\xi} \left( h(\bar{u} - u_w) \right) = 0,$$  \hspace{1cm} (8.2)

$$F^2 h(\bar{u} - u_w) \frac{d\bar{u}}{d\xi} + h \frac{dh}{d\xi} = h(\tan \zeta - \mu) + \frac{F^2}{R} \frac{d}{d\xi} \left( h^{3/2} \frac{d\bar{u}}{d\xi} \right).$$  \hspace{1cm} (8.3)

The first of these can be integrated to show that, $h(\bar{u} - u_w)$, is constant. Since steady-uniform flow, in which $h = 1$ and $\bar{u} = 1$, must be a solution of these equations, it follows that the constant equals, $1 - u_w$, and hence that

$$\bar{u} = u_w + \frac{1 - u_w}{h}.$$  \hspace{1cm} (8.4)

Substituting (8.4) into the momentum balance (8.3) leads to the second order ordinary differential equation for $h(\xi)$, that governs roll-wave solutions,

$$\frac{d^2 h}{d\xi^2} = \frac{1}{2h} \left( \frac{dh}{d\xi} \right)^2 + \frac{Rh^{3/2}}{F^2(u_w - 1)} \left[ \left( 1 - \frac{F^2(u_w - 1)^2}{h^3} \right) \frac{dh}{d\xi} - \tan \zeta + \mu(h) \right],$$  \hspace{1cm} (8.5)

where the friction $\mu$ is expressed as a function of $h$ (only) as

$$\mu(h) = \tan \zeta_1 + \frac{(\tan \zeta_2 - \tan \zeta_1)(1 - u_w + u_wh)}{1 - u_w + u_wh + \gamma h^{5/2}}.$$  \hspace{1cm} (8.6)

It is convenient to write the second order ordinary differential equation (8.5) as a system of first order phase-plane equations in order to study the existence of limit cycles. Defining

$$\frac{dh}{d\xi} = n,$$  \hspace{1cm} (8.7)

and substituting this into (8.5) gives

$$\frac{dn}{d\xi} = \frac{n^2}{2h} + \frac{Rh^{3/2}}{F^2(u_w - 1)} \left[ \left( 1 - \frac{F^2(u_w - 1)^2}{h^3} \right) n - \tan \zeta + \mu(h) \right].$$  \hspace{1cm} (8.8)

The first order system of ordinary differential equations (8.7)–(8.8) has an equilibrium, or fixed point, at $(h, n) = (1, 0)$. For a given value of the steady-uniform Froude number, $F$, and the Reynolds number, $R$, (8.7)–(8.8) can be solved for $(h, n)$ with the speed of the wave, $u_w$, acting as a control parameter.
8.1. The existence of stable limit cycles

Before numerically solving (8.7)–(8.8) to find periodic roll-wave solutions, it is useful to constrain the values of \( u_w \) for which closed periodic orbits in phase space exist. The existence and classification of limit cycles in the phase-plane equations (8.7)–(8.8) is determined by a linearization in the neighbourhood of the fixed point \((1,0)\) (e.g. Needham & Merkin 1984; Jordan & Smith 1987; Strogatz 1994). Expanding about this point with

\[
h = 1 + \epsilon \mathcal{H}, \quad n = \epsilon \mathcal{N}, \quad \epsilon \ll 1,
\]

the \(O(1)\) equations are trivially satisfied, whilst the \(O(\epsilon)\) terms give the linear system

\[
\frac{d\mathcal{H}}{d\xi} = B\mathcal{H},
\]

where \(\mathcal{H} = (\mathcal{H}, \mathcal{N})^T\) and the matrix of coefficients is

\[
B = \begin{pmatrix}
0 & 1 \\
R\Gamma(2u_w - 5) & \frac{1}{2F^2(u_w - 1)} \\
2F^2(u_w - 1) & \frac{R[1 - F^2(u_w - 1)^2]}{F^2(u_w - 1)}
\end{pmatrix}.
\]

The eigenvalues, \(\lambda\), which classify the equilibrium point are found by solving

\[
\det(B - \lambda I) = 0,
\]

where \(I\) is the identity matrix. This yields the quadratic equation

\[
\lambda^2 - \frac{R[1 - F^2(u_w - 1)^2]}{F^2(u_w - 1)} \lambda - \frac{R(2u_w - 5)}{2F^2(u_w - 1)} = 0,
\]

which has roots

\[
\lambda_{1,2} = \frac{R[1 - F^2(u_w - 1)^2]}{2F^2(u_w - 1)} \left(1 \pm \sqrt{\Delta}\right),
\]

where the discriminant

\[
\Delta = 1 + \frac{2\Gamma F^2(u_w - 1)(2u_w - 5)}{R[1 - F^2(u_w - 1)^2]^2}.
\]

Considering the wavespeed \(u_w\) as the bifurcation parameter, the behaviour of the equilibrium point is considered for all values of \(u_w > 0\). Firstly, for \(u_w < 1\) and \(u_w \geq 5/2\) the discriminant \(\Delta \geq 1\), hence the eigenvalues \(\lambda_{1,2}\) are always real and
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of opposite sign, which implies that the equilibrium point is a saddle and there can be no limit cycles surrounding it. Next, in the special case $u_w = 1$, the linear system (8.10)–(8.11) is singular. Returning to depth-averaged velocity relation (8.4) with $u_w = 1$ implies that $\bar{u} = u_w = 1$ everywhere, and so the first-order ordinary differential equation (6.9) for the front problem is recovered. In this case it is already known that there can be no limit cycles surrounding the equilibrium. Finally, for $1 < u_w < 5/2$, the discriminant $\Delta < 0$ provided

$$ \frac{2\Gamma F^2(u_w - 1)(5 - 2u_w)}{R [1 - F^2(u_w - 1)^2]^2} > 1, $$

(8.16)

in which case the eigenvalues $\lambda_{1,2}$ are complex conjugates and there is a spiral which will be unstable or stable dependent on whether the real part

$$ \text{Re}\{\lambda_{1,2}\} = \frac{R [1 - F^2(u_w - 1)^2]}{2F^2(u_w - 1)}, $$

(8.17)

is positive or negative. The real part is zero at the bifurcation point

$$ u_w = u_c = 1 + \frac{1}{F}. $$

(8.18)

Since the above is true only for $1 < u_w < 5/2$, it follows the bifurcation point exists if and only if $F > 2/3$, which is exactly equivalent to the condition (7.23) required for linear instability of the uniform flow. A necessary condition for a limit cycle to exist is that the fixed point must be an unstable spiral, which requires that

$$ 1 < u_w < u_c = 1 + \frac{1}{F}. $$

(8.19)

Since a limit cycle will correspond to periodic roll-wave solution in physical space, the inequalities in (8.19) place a useful restriction on the range $u_w$.

8.2. Numerical integration of the phase-plane equations

In the unstable regime, in which the Froude number $F > 2/3$, the phase-plane equations (8.7)–(8.8) are integrated numerically using Matlab’s ode15s initial value problem solver using a prescribed value of $u_w \in [1, u_c]$. The integration is started from

$$ (h, n) = (1.001, 0), $$

(8.20)
which represents a small perturbation away from the fixed point (1,0), and the
solver steps forward from $\xi = 0$ (chosen arbitrarily). Provided a limit cycle exists,
the instability causes the perturbation to grow as $\xi$ increases until a stable roll-
wave is reached, as shown in figure 6(a) for the parameter values given in table 2
and a wave speed $u_w = 1.978613807$. Figure 6(b) shows the same solution in the
phase plane, with the trajectory spiralling away from the unstable fixed point until
the limit cycle is reached (the outer trajectory). When plotted in physical space the
limit cycle represents a periodic roll-wave solution to the system of equations (7.1)—
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Figure 7. Limit cycles in the phase-plane $(n, h)$, which correspond to roll-wave solutions using the new depth-averaged $\mu(I)$-rheology with parameters given in table 2. In each simulation the Froude number $F = 1.02$ is fixed and various wave speeds $u_w$ (labeled on the outer of the corresponding limit cycle) are prescribed.

(7.2). It travels with speed $u_w$, which is always greater than the non-dimensional depth-averaged speed, $\bar{u} = 1$. Figure 6(c) shows three complete periods of the limit cycle in physical space. The solution computed with the new depth-averaged $\mu(I)$-rheology therefore looks very similar to ones calculated using the inviscid theory (3.30)–(3.31), except that the discontinuous shocks are now replaced by a sharply varying smooth transitions at the wave-fronts.

The effect of varying $u_w$ on the solution is shown in figure 7, where only the final limit cycle is plotted in each case. It is found that convergent solutions only exist for values of $u_w$ close to the bifurcation point $u_c$, i.e. there exists a minimum wavespeed $u_w^{\text{min}}$ for which a limit cycle exists. For each $F$, it is possible to iterate between the known bounds $u_w \in [1, u_c]$ in order to find $u_w^{\text{min}}$. The amplitude of the roll-waves increases as the wavespeed $u_w$ is decreased away from $u_c$ until the largest
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Figure 8. Numerical relationship (solid lines) between the Froude number $F$ and (a) the roll-wave amplitude $A$ (measured peak to trough) and (b) the inverse of the wavelength $\Lambda$ at (c) the minimum wavespeed $u_{w}^{\min}$ for which there is a convergent solution (found iteratively). The difference $\Delta u_c$ between the critical wavespeed $u_c = 1 + 1/F$ and $u_{w}^{\min}$ is shown in panel (d). The critical Froude number $F_c = 2/3$ (dashed line) is the minimum Froude number for which roll-wave solutions exist.

amplitude wave is reached at $u_w = u_{w}^{\min}$. The overall shape of the limit cycle does not change greatly as $u_w$ is varied.

In order to examine how the size and shape of the roll-waves change with the steady-uniform Froude number, solutions are computed for various $F$ with the wavespeed set to $u_w = u_{w}^{\min}$. Figure 8(a) shows the roll-wave amplitude, $A$, defined here as the peak to trough distance, as a function of $F$. At the critical Froude number $F_c = 2/3$ the amplitude is zero. It then increases rapidly up to a maximum $A \approx 0.3$ for $F \approx 1.2$ before decreasing exponentially at larger values of $F$. Figure 8(b) shows a graph of the reciprocal of the wavelength $\Lambda$, defined here as the peak to peak distance in $\xi$. Since $1/\Lambda = 0$ at $F = F_c$ the wavelength is infinite at
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the critical Froude number. As $F$ increases, $1/\Lambda$ increases and the wavelength $\Lambda$ decreases monotonically. The corresponding value of $u_{\text{min}}^w$ is shown in figure 8(c). It behaves like the bifurcation point velocity, $u_c$, defined in (8.18), with approximately a $1 + 1/F$ dependence. The difference $\Delta u_c$ between the bifurcation point velocity, $u_c$, and the minimum velocity for convergent solutions, $u_{\text{min}}^w$, is shown in figure 8(d) and is of the order of $10^{-3}$. For a given Froude number, $F$, a general initial disturbance to the full system of conservation laws (7.1)–(7.2) will, under the right conditions, generate a range of roll-waves moving with closely similar velocities, $u_w$, in the range $[u_{\text{min}}^w, u_c]$. Faster waves will therefore catch up, and merge with, slower moving ones, generating complex coarsening dynamics.

9. Conclusions

In this paper it is shown how the $\mu(I)$-rheology (Jop et al. 2006) can be incorporated into depth-averaged models of granular avalanches. These are extensively used for the prediction of natural hazards (e.g. Grigorian et al. 1967; Savage & Hutter 1989; Iverson 1997; Gray et al. 1999; Iverson & Denlinger 2001; Mangeney et al. 2007), as well as for modeling small scale dry granular flows (e.g. Pouliquen & Forterre 2002; Gray et al. 2003; Johnson & Gray 2011; Cui & Gray 2013). Using the shallowness of the flow, a formal depth-integration of the mass and momentum balances yields the classical shallow water-like avalanche equations (3.30)–(3.31). To leading order, the only effect of the rheology is to generate an effective basal friction in the source terms (3.32) that is precisely the rough bed friction law (2.8) measured by Pouliquen & Forterre (2002). The gradient of the depth-averaged in-plane deviatoric stress $\tau_{xx}$ does not, however, contribute to the leading order depth-averaged momentum balance, so no viscous-like terms are generated. For most situations the classical system of avalanche equations (3.30)–(3.32) are therefore entirely appropriate.

There are some more subtle problems, such as the formation and growth of gran-
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ular roll-waves (Forterre & Pouliquen 2003), where a viscous-like term is required to get the correct physical behaviour. In §4 it is shown how the $\mu(I)$-rheology can be approximated, using the lithostatic pressure and Bagnold velocity profiles (GDR-MiDi 2004) during the integration of the in-plane deviatoric stress, $\tau_{xx}$, through the avalanche thickness, to generate a viscous-like term in the depth-integrated momentum balance. Formally this term is $O(\epsilon)$, so most of the time it does not play a role and the classical equations dominate. However, the viscous term involves second order derivatives of the depth-averaged velocity, $\bar{u}$, which is the highest gradient in the theory. Its inclusion in the conservation laws (4.13)–(4.14) therefore represents a singular perturbation, which plays an important role in smoothing out shocks that form with the inviscid theory, as shown for roll-wave fronts in figure 6.

The new depth-averaged $\mu(I)$-rheology is compared to a previous depth-averaged model of Forterre (2006), which is also based on the $\mu(I)$-rheology. It contains a viscous-like term that involves second order gradients of $h\bar{u}$ and has a singularity in the viscosity as $h \rightarrow 0$. By considering the formation of a steadily traveling granular front on a rough inclined plane, it is shown in §6 that Forterre’s (2006) theory generates a thin precursor layer ahead of the main front. Moreover, the precise shape of the front is dependent on the depth and rate of decay of the precursor layer. Since this layer is well below the grain size of particles, this behaviour is unphysical. The new depth-averaged $\mu(I)$-rheology, on the other hand, does not affect the problem at all, since it is only dependent on second order derivatives of $\bar{u}$, which is constant in this problem. The inviscid solution, with a clearly defined frontal shape and a completely grain-free region ahead of the front, is therefore reproduced exactly.

A linear stability analysis using the new depth-averaged $\mu(I)$-rheology predicts that steady-uniform flows become unstable above a critical Froude number $F_c = 2/3$. The spatial growth rate and phase velocity are in very good agreement with the experimental results of Forterre & Pouliquen (2003). In particular, the
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new model predicts a cut-off frequency, \( \omega_c \), that is in excellent agreement with experimental results over a wide range of slope inclination angles, \( \zeta \), and Froude numbers, \( F \), as shown in figure 5. The agreement is at least as good as the instability results of the full \( \mu(I) \)-rheology in two-dimensions (Forterre 2006). It is certainly an improvement on the inviscid model, which does not predict cut-off at all, as well as the depth-averaged model of Forterre (2006), which required an additional arbitrary fitting parameter. The near independence of the cut-off frequency, \( \omega_c \), on the slope inclination angle is a direct consequence of the dependence of the coefficient, \( \nu \), (defined in 4.16), on the slope inclination angle, \( \zeta \). The stability results therefore provide strong evidence that the increase in \( \nu \) as the inclination angle, \( \zeta \), decreases, is an important physical effect that may play a significant role in flow arrest processes. Indeed as \( \zeta \) tends to the minimum angle, \( \zeta_1 \), for steady-uniform flow the effective viscosity \( \nu h^{3/2}/2 \) tends to infinity. Conversely as \( \zeta \) tends to the maximum angle for steady-uniform flow, \( \zeta_2 \) the effective viscosity tends to zero. Outside this range, the viscosity is negative, which will lead to ill-posedness! Bizarrely this is also consistent with the full \( \mu(I) \)-rheology, which is ill-posed for very high and very low values of the inertial number (Barker et al. 2014). A simple regularization of the depth-averaged \( \mu(I) \)-rheology can be achieved by setting a maximum and minimum threshold for the coefficient, \( \nu \), which is positive for all inclination angles, as shown in figure 2.

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Chapter 3

Erosion-deposition waves in shallow granular free-surface flows
Erosion-deposition waves in shallow granular free-surface flows

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Debris flows can spontaneously develop regular large amplitude surge waves that are interspersed by periods in which the channel fill is completely stationary. These are important because each individual surge is much more destructive than a steady uniform flow with the same mass flux. In this paper small scale experiments are described, which exhibit similar behaviour. The flow consists of carborundum particles that flow down a rough inclined chute covered with a static erodible layer of the same grains. For inflow conditions close to the minimum depth required for steady-uniform flows to exist, small disturbances are unstable, creating waves that rapidly coarsen and grow in size. As the waves become sufficiently large the troughs between the wave crests drop below a critical thickness and come to rest. A series of steadily travelling waves develop that erode the static layer of particles in front of them and deposit grains behind them, to form a layer that is again stationary. This is, in turn, re-eroded and deposited by the next wave. We term these waves granular erosion-deposition waves. Although erosion and deposition problems are notoriously difficult, a simple model is developed that uses a depth-averaged version of the $\mu(I)$-rheology and Pouliquen and Forterre’s extended friction law. The viscous dissipation combines with dynamic, intermediate and static friction regimes to generate finite length waves with static and mobile regions. The existence of stationary layers fundamentally distinguishes erosion-deposition waves from granular roll waves, which form in slightly deeper flows and are always completely mobi-
lized. Numerical simulations show that the system of equations is able to model both erosion-deposition waves and granular roll waves. Moreover the computed wave amplitude, wave-speed and coarsening dynamics are in good quantitative agreement with experiments.

1. Introduction

On the 15th October 2000 an unintentional release of 150,000 m$^3$ of water from Lac Inférieur, over a period of 20 hours, caused a debris flow to form on the slope of Les Garette. This flowed downslope and arrived at the village of Fully, Switzerland, as a sequence of 1-1.5m thick surges, travelling at approximately 10 ms$^{-1}$, which were interspersed by periods in which the remaining channel fill was completely stationary (Zanuttigh & Lamberti 2007). Similar surges have been reported in the Jiang-Jia Ravine, China (Li, Jianmo, Bi & Luo 1983), Illgraben, Switzerland (McArdell, Zanuttigh, Lamberti & Rickenmann 2003) and in the Moscardo torrent, Italy (Marchi, Arattano & Deganutti 2002). Although no one was hurt in Fully it is important to understand how these surges form, because each pulse is much more destructive than a continuous flow of the same mass flux. Davies (1986) was perhaps the first to suggest that the most likely explanation for them was the spontaneous formation of roll waves (e.g. Cornish 1910; Dressler 1949; Needham & Merkin 1984; Kranenburg 1992; Balmforth & Mandre 2004; Balmforth & Liu 2004; Zanuttigh & Lamberti 2007), which merge and coarsen along the channel to form large amplitude wave surges.

In this paper experiments are described that exhibit similar discrete surge-like behaviour in dry granular flows, albeit at a much smaller scale. Figure 1 shows a photograph of the chute, which is fed by a hopper with a steady supply of mono-disperse carborundum grains. For slope inclinations just above the onset of flow, the free-surface is unstable and small waves develop that grow in amplitude, merge and coarsen as they flow downslope. In many respects they initially look like
Figure 1. A photograph showing an oblique view of two erosion-deposition waves travelling down a chute inclined at $\zeta = 35.1$ degrees to the horizontal. The wave fronts appear brighter due to the lighting and a long time exposure has been used so that moving grains are blurred and the static regions are sharp. The chute has a total length of 3.29m in the downslope $x$-direction, a width of 7.8cm in the cross-slope $y$-direction and the glass sidewalls have a height of 6cm in the $z$-direction normal to the plane. A movie showing the time-dependent evolution of the waves is available in the online supplementary material.

granular roll waves (e.g. Davies 1990; Vallance 1994; Forterre & Pouliquen 2003; Forterre 2006; Iverson, Logan, LaHusen & Berti 2010; Gray & Edwards 2014),
but sufficiently far downstream the troughs, between the wave peaks, drop below the minimum thickness for flow and deposit material, which forms a stationary erodible layer between the thicker mobile parts of the waves. In the oblique view of the chute in figure 1 a long shutter speed is used to reveal the moving grains, which appear blurred, and the stationary regions, which are in sharp focus. A further two overhead photographs (Fig. 2), taken 0.4 seconds apart, show a close-up of the static layer ahead of the wave being eroded by the propagating wave front. The major difference to conventional roll waves is therefore that completely stationary regions form spontaneously between the wave crests, which fundamentally changes the nature of the solution. To make the distinction clear from the outset we therefore term them “erosion-deposition” waves.
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The surges when viewed independently resemble a solitary granular avalanche propagating on an erodible bed (see e.g. Daerr & Douady 1999; Daerr 2001; Pouliquen & Forterre 2002; Börzsönyi, Halsey & Ecke 2005; Clement, Malloggi, Andreotti & Aranson 2007; Börzsönyi, Halsey & Ecke 2008; Takagi, McElwaine & Huppert 2011). We believe that each individual wave is directly related to the isolated triangular waves, first reported in the literature by Daerr (2001). Börzsönyi et al. (2005, 2008) also observed these waves, which have a static layer ahead and behind of them, and postulated a simple Burgers’-type equation for the evolving thickness. We have solved this equation numerically and found that an initial cap of material released on top of an erodible bed rapidly developed into a single triangular section of an N-wave (Whitham 1974). It is well known, however, that these decrease in height with increasing time, so this approach is not able to model steadily traveling solitary waves. Clement et al. (2007) used Aranson & Tsimring’s (2001; 2002) partial fluidization theory to construct a model consisting of a thin viscous film equation for the flow height and an order parameter for the fluidity, which was controlled by the Ginzburg-Landau equation. This model was able to generate solitary wave solutions, but really contains very little granular physics. Takagi et al. (2011) are the only ones to have reported multiple erosion-deposition waves, although, these were generated by regular pile collapses at the source and did not interact with one another. The erosion-deposition waves reported here are therefore somewhat different, as they form spontaneously in a steady-uniform flow and subsequently merge and coarsen very much like roll waves.

The theory developed in this paper builds on recent advances in modelling the constitutive behaviour of granular flows with the $\mu(I)$-rheology (Jop, Forterre & Pouliquen 2006). Numerical simulations using this rheology are now able to tackle complex flows, such as column collapses (Lagrée, Staron & Popinet 2011) and silos (Staron, Lagrée & Popinet 2012), but this is still challenging even in two-dimensions. Conversely, one-dimensional depth-averaged avalanche models (Grigo-
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rian, Eglit & Iakimov 1967; Savage & Hutter 1989; Gray, Wieland & Hutter 1999; Pouliquen 1999b; Pouliquen & Forterre 2002; Gray, Tai & Noelle 2003) have a much simpler description that has proven to be effective over many years. These are able to predict the critical Froude number for granular roll waves, $F_{rc} > 2/3$, consistent with the experiments of Forterre & Pouliquen (2003), but incorrectly predict growth at all frequencies. Forterre (2006) used linear stability analysis to show that the full $\mu(I)$-rheology could predict the cut-off frequency, and attempted to depth-average the deviatoric stresses to derive a shallow-water-like system of equations that could also predict cut-off. This was only partially successful as the frequency could only be matched by adding an ad hoc fitting parameter. Gray & Edwards (2014) recently developed a depth-averaged $\mu(I)$-rheology that was able to match the cut-off frequency without any fitting parameters over a wide range of angles, i.e. without including ad hoc viscosity. In particular, the angle dependence provides strong evidence for its accuracy and it is this approach that is adopted here. Indeed the viscous term plays a crucial role in modelling the erosion-deposition waves described in this paper.

2. Small scale experiments

Laboratory experiments using 315-355$\mu$m carborundum particles have been used to produce discrete surge waves similar to those observed in the October 2000 debris flow in Fully, Switzerland. The chute has glass sidewalls and the bed is roughened by attaching a mono-layer of spherical glass beads of diameter 400-600$\mu$m using double-sided sticky tape. It is inclined at $\zeta = 35.1 \pm 0.1$ degrees to the horizontal and is supplied from the top by a perspex hopper with a double gate that allows the start time and inflow thickness to be controlled. The width of the chute is 7.8cm and its length is 3.29m from the inflow gate to the outflow, where the material flows out freely. A coordinate system $Oxyz$ is defined with the $x$-axis pointing in the downslope direction, the $y$-axis across the chute and the $z$-axis pointing
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normal to the rough plane at \( z = 0 \). Experiments are initiated with a stationary layer of carborundum particles, deposited from a previous experiment, lining the bed. The flow is illuminated from the outflow end by a light source that is parallel to, and level with, the base. This creates bright areas in front of the wave peaks and shadows behind them, enhancing their appearance.

Data are acquired simultaneously using a combination of a camera, a scale and a laser profilometer. A high-speed camera (Teledyne DALSA Genie HM1400/HC1400) is used to capture still images of the flow normal to the chute from above, in a region between 2.21 and 3.29 metres downslope of the inflow gate. At the outflow of the chute the material is collected in a tray on a high-resolution bench scale (KERN FKB 8K0.05) which outputs the measured mass to an accuracy of 50mg at a rate of 5Hz to a computer. A profilometer (Micro-Epsilon scanCONTROL 2700-100 laser profile sensor) is positioned with its laser line covering the width of the channel and perpendicular to the downslope direction, at a distance of \( x_L = 3.21 \) m from the inflow gate. The profilometer measures the distance of the bulk flow particles away from the sensor, at a frequency of 100Hz and to an accuracy of \( \pm 0.2 \) mm for a maximum of 40 seconds, by the method of laser line triangulation. Measuring the distance of the bed from the sensor on the empty chute before or after an experiment allows the flow thickness profile \( h \) along the laser line to be calculated.

The results from these three sources of data acquisition are shown for one run of experiment in figure 3, for a time period of 0 to 24 seconds, where \( t = 0 \) s is an arbitrarily chosen start time before the bulk flow reaches the region of interest (2.21 to 3.29 metres from the inflow gate). The space-time plot in figure 3(a) is constructed by using the central column of pixels from each successive image. The wavefronts appear as bright lines, which are followed by darker regions behind the peak. The regions of horizontal lines indicate completely stationary grains between the surges. For an isolated erosion-deposition wave the wavefront and the boundary of the stationary region appear to be parallel straight lines, indicating that these
Figure 3. Experimental data for a period of 0 to 24 seconds. The space-time plot (a) is constructed by taking the middle column of each successive overhead still image and laying them out horizontally to give downslope position $x$ with time $t$. Horizontal lines indicate stationary grains and the wavefronts appear as diagonal white lines due to the illumination. Measurements from the laser profilometer (b) give the flow thickness $h$ on the centreline at $x = x_L = 3.21\text{m}$, which is near to the outflow. The balance data (c) gives the accumulated mass $M$ that has flowed out of the chute.
are travelling waves moving at constant speed. All the waves have approximately the same speed, but there are several merging events, where they coalesce to form another travelling wave, such as at \( t \approx 12s \) and \( x \approx 2.48m \) in figure 3(a). Merging events are more frequent closer to the inflow gate. For this experiment the mean wave-speed \( u_w \approx 0.25\text{ms}^{-1} \). The wave period, defined here as the time between successive wave fronts including the stationary regions, is found to have a mean value of \( T \approx 2.36s \) and a corresponding frequency of \( f \approx 0.42\text{Hz} \). The wavelength including the static regions has a mean value of \( \lambda \approx 0.59m \).

The flow thickness profile \( h \) on the centreline at \( x = x_L = 3.21m \) (near the end of the chute) is shown in figure 3(b). The thickness of the stationary regions between successive waves is defined as \( h_+ \approx 2.0\text{mm} \) and the maximum thickness at each roll wave peak is defined as \( h_w \approx 5.2\text{mm} \). This implies a mean wave amplitude of
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Figure 5. Space-time plots of the thickness profile of a flow of carborundum particles that is built from a sequence of images that view the flow through a glass sidewall and are aligned with the slope at a fixed $x$ position. The vertical axis gives the free-surface height and the horizontal axis gives the experiment time $t$ for a period of 4 seconds in (a) and 2 seconds in (b). The black line has been added to represent the position of the rough bed in relation to the bulk flow. Horizontal lines represent particles which are stationary and therefore show that particles come to rest between the wave surges. Note that the flow front appears more diffuse than it actually is, because sidewall friction implies that the flow front in the centre is slightly in advance of that at the sidewall. A movie showing the time-dependent evolution of the waves through the sidewall is available in the online supplementary material.

$A \approx 3.2$mm for this experiment, where the amplitude

$$A = h_w - h_+.$$  \hspace{1cm} (2.1)

Figure 3(c) shows the accumulated mass at the end of the chute, which is a smooth increasing function of time. Regions with positive gradient correspond to the mobile part of the erosion-deposition waves reaching the outflow, whilst flat regions indicate that no grains are exiting the chute and correspond to the stationary regions between the surges. The discrete nature of the pulses that develop spontaneously from a steady uniform flow are a novel feature of the experiment.

Measured variations in the shape of a single wave across the chute at $x = x_L$ are shown in the space-time plot in figure 4. This indicates that the thickness on
the centreline is slightly higher and arrives slightly earlier than the material at the
wall due to sidewall friction. This does not however appear to be a large effect. The
depth of the static region is approximately the same before and after the passage of
the wave and the variations in $h_+$ across the slope are of the order of the grain-size.

The high-speed camera has also been positioned to one side of the chute and
aligned with the slope angle $\zeta$, so that the flow thickness profile past a point can
be observed through the glass sidewalls. A space-time plot of the flow thickness
profile in time is shown in figure 5 at a point 2.5m downslope of the inflow gate.
It shows how the erosion-deposition waves travel downslope by eroding the static
layer of particles in front of them and depositing a static layer behind. When one
of these waves is travelling steadily with a constant wavespeed, then the erosion-
deposition process is in equilibrium. Interestingly these plots show that the lowest
point of particle motion is below the thickness of the static layers, but the erosion
does not penetrate all the way down to the rough bed.

3. Governing equations

There are essentially two ways of modelling erosion-deposition waves within a
depth-averaged framework, which are illustrated schematically in figure 6(a,b). In
figure 6(a) the free-surface lies at $z = s(x, t)$, the interface between mobile and im-
mobile regions lies at $z = b(x, t)$ and the difference, $\hat{h} = s - b$, defines the avalanche
depth. Modelling flows using this approach is notoriously difficult, because an em-
pirical/theoretical expression for the erosion/deposition rate, $d$, at the base of the
avalanche must be prescribed to close the system (e.g. Bouchaud, Cates, Prakash
& Edwards 1994; Douady, Andreotti & Daerr 1999; Gray 2001; Doyle et al. 2007;
Tai & Kuo 2008; Gray & Ancey 2009; Iverson 2012; Tai & Kuo 2012). However, in
formulating the basal friction law that will be adopted in this paper, Pouliquen &
Forterre (2002) implicitly assumed complete mobilization of particles throughout
the entire depth, i.e. the flows were either completely mobile or completely static.
Figure 6. A schematic diagram showing two possible representations of an eroding and deposing granular avalanche on a slope inclined at an angle, $\zeta$, to the horizontal. The stationary material is shaded grey. The flow thickness, in the normal $z$-direction, is considered to be either (a) the depth $\hat{h}(x, t)$ of particles between the free-surface $s(x, t)$ and base of the avalanche at $z = b(x, t)$, or (b) the entire depth $h(x, t)$ between the free-surface and the bed. Typical downslope velocity profiles $u(x, z, t)$ through the depth of the flow are illustrated. The travelling wave solution assumes the wave moves with constant speed $u_w$ and erodes a static layer of grains of thickness $h_w$ ahead of the wave and deposits the same thickness behind it. The maximum height of the wave at the peak is denoted $h_w$.

While this is a crude approximation, it will be shown here that it allows considerable progress to be made. The free-surface of the flow at $z = h(x, t)$ is therefore taken to be a measure of the flow thickness normal to the plane of both the mobile and static regions of particles, as shown in figure 6(b).

3.1. Depth-averaged equations with viscous dissipation

The avalanche is modeled using Gray & Edwards’s (2014) theory, which is derived by depth-averaging the $\mu(I)$–rheology (Jop et al. 2006) for dense granular flows. This model differs from standard shallow water type avalanche equations
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(Grigorian et al. 1967; Savage & Hutter 1989; Gray et al. 1999; Pouliquen 1999b; Pouliquen & Forterre 2002; Gray et al. 2003) by the inclusion of in-plane deviatoric normal stresses. For an avalanche of thickness $h$ and depth-averaged down slope velocity $\bar{u}$ the depth-averaged mass and momentum balance equations are

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) = 0,$$

(3.1)

$$\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(\chi h\bar{u}^2) + \frac{\partial}{\partial x}\left(\frac{1}{2}h^2g\cos\zeta\right) = hgS + \frac{\partial}{\partial x}\left(\nu h^{3/2}\frac{\partial \bar{u}}{\partial x}\right),$$

(3.2)

where $\chi = \bar{u}^2/\bar{u}_w^2$ is the shape factor and $g$ is the constant of gravitational acceleration. The $\mu(I)$-rheology implies that for steady uniform flow a Bagnold velocity profile develops (see e.g GDR-MiDi 2004; Gray & Edwards 2014) and the resulting shape factor $\chi = 5/4$. However, non-unity values of the shape factor change the characteristic structure of the inviscid equations, and cause problems when handling grain-free regions, so virtually all avalanche models (e.g. Grigorian et al. 1967; Savage & Hutter 1989; Gray et al. 1999; Pouliquen 1999b; Pouliquen & Forterre 2002; Gray et al. 2003) assume, as we do here, that $\chi = 1$. The source term

$$S = \sin\zeta - \mu(h, Fr)\left|\frac{\bar{u}}{\bar{u}_w}\right|^{\cos\zeta},$$

(3.3)

consists of the component of gravity in the down slope $x$-direction and the effective basal friction $\mu(h, Fr)$ between the avalanche and the rough plane, which oppose the direction of motion. The Froude number $Fr = \bar{u}/\sqrt{hg\cos\zeta}$ is defined as the ratio of the depth-averaged velocity to gravity wavespeed. The viscous term on the righthand side of (3.2) plays a critical role in the formation of erosion-deposition waves. In their derivation, Gray & Edwards (2014) showed that to leading order the $\mu(I)$-rheology implies that there is a lithostatic pressure and a Bagnold velocity profile through the avalanche depth. To first order the shallow water-like avalanche equations are recovered, with the only contributions from the $\mu(I)$-rheology coming from the shape factor $\chi$ and an effective basal friction, which is equivalent to the law of Pouliquen & Forterre (2002). Such equations have been used for many years (e.g. Grigorian et al. 1967; Pouliquen 1999b; Gray et al. 2003) and are sufficient for
many applications (e.g. Johnson & Gray 2011; Cui & Gray 2013). However, they are unable to predict the cut-off frequency of roll waves (Forterre & Pouliquen 2003).

Extension to second order leads to very complicated models that are often difficult to interpret. Gray & Edwards (2014) therefore used a pragmatic approach, in which they included the depth-averaged in-plane deviatoric stress in (3.2). This was evaluated by substituting the lithostatic pressure and Bagnold velocity into the $\mu(I)$-rheology (Jop et al. 2006) and integrating. They then used the leading order relationship between the depth-averaged Bagnold velocity and the thickness to reformulate the result into the viscous-like term, where other contributions with lower order gradients were neglected, since they do not contribute to the principal part. At this order, the leading order balance can be used to create many different formulations. The one given in (3.2) has been specifically chosen because for angles in the steady-uniform regime it does not introduce any singularities and it degenerates in grain-free regions. The coefficient

$$
\nu = \frac{2 \mathcal{L} \sqrt{g}}{9 \beta} \frac{\sin \zeta}{\sqrt{\cos \zeta}} \left( \tan \zeta_2 - \tan \zeta_1 \right),
$$

where $\mathcal{L}$ and $\beta$, and the angles $\zeta_1$ and $\zeta_2$ arise directly from Pouliquen & Forterre’s (2002) basal friction law, which will be described in greater detail below. Note that Gray & Edwards (2014)’s theory is only well-posed between the angles $\zeta_1 \leq \zeta \leq \zeta_2$. This is a signature of the ill-posedness of the $\mu(I)$-rheology for both high and low inertial numbers (Barker et al. 2014). The inclusion of the viscous term represents a singular perturbation to the problem. Most of the time it is negligibly small, but sometimes it plays a critical role, which proves to be the case for the erosion-deposition waves studied here. Strong evidence for this theory and the specific angle dependence in (3.4) is provided by the fact that Gray & Edwards (2014) were able to match the experimental cut-off frequency for roll waves (Forterre & Pouliquen 2003) over a wide range of angles without any fitting parameters. The
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The viscous term in (3.2) therefore has a strong physical basis rather than being just an ad hoc regularization with an unspecified coefficient of viscosity.

3.2. Pouliquen and Forterre’s basal friction law

The basal friction law also plays a pivotal part in allowing flows like those observed experimentally to be modeled by a framework which does not explicitly include the effects of erosion and deposition. Pouliquen (1999a) performed laboratory experiments on flows of spherical glass beads on a rough chute and found an empirical friction law that was valid for steady uniform flows at various slope angles. An extension of this by Pouliquen & Forterre (2002) found two critical slope inclination angles as functions of the flow thickness, namely $\zeta_{\text{stop}}(h)$ and $\zeta_{\text{start}}(h)$. The function $\zeta_{\text{stop}}(h)$ gives the slope angle at which a steady uniform flow leaves a deposit of thickness $h$, whilst $\zeta_{\text{start}}(h)$ is the angle at which a layer of thickness $h$ is mobilized. The two functions are illustrated in figure 7 for the parameters given in table 1,
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together with arrows to indicate a typical sequence of avalanches that they used to determine them. At a given thickness $h_1$, the inclination angle is increased until a layer of static grains is mobilized, which determines $\zeta_{\text{start}}(h_1)$, after which the avalanche thickness decreases until the layer stops, determining $\zeta_{\text{stop}}(h_2)$ at a new thickness $h_2$. This behaviour is also observed for flows of various particle types and bed roughness conditions (e.g. Pouliquen & Renaut 1996; Daerr & Douady 1999).

The thickness of a deposit left by a steady uniform flow at an inclination angle $\zeta$ is denoted by $h_{\text{stop}}(\zeta)$, which is the inverse function of $\zeta_{\text{stop}}(h)$. An empirical dependence was found by Pouliquen (1999a) between the ratio of the flow thickness $h$ to $h_{\text{stop}}(\zeta)$ and the Froude number $Fr$, which is given here as

$$Fr = \beta \frac{h}{h_{\text{stop}}(\zeta)}. \quad (3.5)$$

The constant $\beta = 0.136$ for spherical glass beads and 0.65 for sand on a rough bed of the same material (Forterre & Pouliquen 2003). Note that Forterre & Pouliquen (2003) found that there was also an offset at the origin for sand (in equation 3.5). Although this is very easy to program into the basal friction law, it is unclear how to include it into the $\mu(I)$-rheology, since it is no longer possible to use Jop, Forterre & Pouliquen’s (2005) method of inferring the rheology from the basal friction law. The offset is therefore neglected in this paper, although the value of $\beta = 0.65$ for sand is used to approximate the friction of our carborundum particles flowing on a rough bed of spherical glass beads. One consequence of using the simpler law (3.5) is that, in principle, it should be possible to observe erosion-deposition waves for all forms of particles, including ballotini, although the region of parameter space where this occurs may be small.

For steady uniform flows the depth-averaged momentum balance (3.2) is satisfied provided the source terms are identically zero, which implies that

$$\mu = \tan \zeta. \quad (3.6)$$
By defining the tangent of the critical stopping angle as

\[ \mu_{\text{stop}}(h) = \tan(\zeta_{\text{stop}}(h)), \]  

the friction coefficient for the static layer is found through the steady uniform flow relation (3.6) and the empirical law (3.5) to be

\[ \mu = \tan \zeta = \tan(\zeta_{\text{stop}}(h_{\text{stop}}(\zeta))) = \mu_{\text{stop}}(h_{\text{stop}}(\zeta)) = \mu_{\text{stop}} \left( \frac{h\beta}{Fr} \right). \]  

(3.8)

However the empirical law (3.5) and therefore the friction law (3.8) is only valid for flows in the steady regime, \( h \geq h_{\text{stop}}(\zeta) \), or equivalently for flows in which \( Fr \geq \beta \).

Defining the tangent of the critical starting angle as

\[ \mu_{\text{start}}(h) = \tan(\zeta_{\text{start}}(h)), \]  

(3.9)

Pouliquen & Forterre (2002) showed that a knowledge of the functions \( \mu_{\text{stop}}(h) \) and \( \mu_{\text{start}}(h) \) is sufficient to define an empirical friction law, \( \mu(h, Fr) \), for the whole range of possible flow thicknesses and Froude numbers. The static friction coefficient, which holds for \( Fr = 0 \), is defined as

\[ \mu(h, 0) = \tan(\zeta_{\text{start}}(h)) = \mu_{\text{start}}(h), \]  

(3.10)

since for stationary material the basal friction must balance the lithostatic pressure and gravitational forces exactly, so that the source terms balance and (3.6) still holds. In the intermediate friction regime when \( 0 < Fr < \beta \), the friction coefficient \( \mu \) is given by a power law extrapolation between the friction laws in the static and dynamic friction regimes as

\[ \mu(h, Fr) = \left( \frac{Fr}{\beta} \right)^\kappa \left( \mu_{\text{stop}}(h) - \mu_{\text{start}}(h) \right) + \mu_{\text{start}}(h), \]  

(3.11)

where \( \kappa = 10^{-3} \) is the power of the extrapolation chosen by Pouliquen & Forterre (2002). The forms of the functions \( \mu_{\text{stop}} \) and \( \mu_{\text{start}} \) are given by fits to experimental measurements as transitions between the relevant critical friction angles. They are written here as

\[ \mu_{\text{stop}}(H) = \tan \zeta_1 + \frac{\tan \zeta_2 - \tan \zeta_1}{1 + H/L}, \]  

(3.12)
Table 1. Material properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ζ</td>
<td>35.1°</td>
</tr>
<tr>
<td>ζ₁</td>
<td>32.9°</td>
</tr>
<tr>
<td>ζ₂</td>
<td>42.0°</td>
</tr>
<tr>
<td>ζ₃</td>
<td>33.9°</td>
</tr>
<tr>
<td>β</td>
<td>0.65</td>
</tr>
<tr>
<td>L</td>
<td>10⁻³m</td>
</tr>
<tr>
<td>ν</td>
<td>2.4 × 10⁻³m³/s⁻¹</td>
</tr>
</tbody>
</table>

where the dummy variable, \( H = h β / Fr \) or \( H = h \), is dependent on where \( μ_{\text{stop}} \) is to be evaluated (i.e. for \( Fr ≥ β \) or \( 0 < Fr < β \) respectively). There is no flow for inclination angles \( ζ < ζ_1 \), which is the asymptote of the curve \( ζ_{\text{stop}}(h) \) for large \( h \), and the flow is accelerated for \( ζ > ζ_2 = ζ_{\text{stop}}(0) \). The third critical angle, \( ζ_3 \), is the asymptote of the curve \( ζ_{\text{start}}(h) \) for large \( h \). The parameter \( L \) (having the dimensions of a length) is the characteristic depth of flow over which a transition between the angles \( ζ_1 \) and \( ζ_2 \) occurs and as such it is dependent on the properties of the grains and on the bed roughness. In summary, the friction coefficient can be written in each of the three flow regimes, which shall be referred to as dynamic, intermediate and static, respectively, as

\[
μ(h, Fr) = \begin{cases} 
μ_1 + \frac{θ - μ_1}{1 + h / L}, & Fr ≥ β, \\
\frac{(Fr)}{β}^κ (μ_1 - μ_3) + μ_3 + \frac{θ - μ_1}{1 + h / L}, & 0 < Fr < β, \\
μ_3 + \frac{θ - μ_1}{1 + h / L}, & Fr = 0,
\end{cases}
\]

where \( μ_1 = \tan ζ_1, μ_2 = \tan ζ_2 \) and \( μ_3 = \tan ζ_3 \) are the tangents of the critical angles, \( ζ_1, ζ_2 \) and \( ζ_3 \). Values of the friction angles, \( ζ_i \) \((i = 1..3)\), along with the parameters, \( β \) and \( L \), are estimated for the experimental setup of §2. They are given in table 1, together with the exact experimental slope angle, \( ζ \), and the resultant value of the effective viscosity, \( ν \). These values remain fixed throughout this paper.
4. Travelling erosion-deposition waves

Travelling-wave solutions to the depth-averaged mass and momentum conservation laws (3.1)-(3.2) are sought for an erosion-deposition wave moving downslope with speed $u_w$. A wavefront-centred coordinate system, $(\xi, \tau)$, is introduced by the transformation

$$\xi = x - u_w t, \quad \tau = t. \quad (4.1)$$

Assuming a steady solution in the moving frame by setting $\partial/\partial \tau = 0$, the system is reduced to a pair of ordinary differential equations (ODEs)

$$\frac{d}{d\xi}(h(\bar{u} - u_w)) = 0, \quad (4.2)$$

$$h(\bar{u} - u_w)\frac{d\bar{u}}{d\xi} + hg \cos \zeta \frac{dh}{d\xi} = hg \cos \zeta (\tan \zeta - \mu) + \frac{d}{d\xi}\left(\nu h^{3/2} \frac{d\bar{u}}{d\xi}\right), \quad (4.3)$$

where the acceleration terms have been simplified using (4.2). This form assumes that $\bar{u}/|\bar{u}| = 1$ everywhere and the friction coefficient $\mu = \mu(h, Fr)$ is given by (3.14)–(3.16). The mass balance equation can be integrated immediately, subject to the condition that $\bar{u} = 0$ in a stationary layer of thickness $h = h_+$, to show that

$$h(\bar{u} - u_w) = -h_+ u_w, \quad (4.4)$$

which implies that the depth-averaged velocity

$$\bar{u} = u_w \left(1 - \frac{h_+}{h}\right). \quad (4.5)$$

Substitution of (4.5) into the momentum balance (4.3) yields a second order ODE for $h$,

$$\frac{d^2h}{d\xi^2} = \frac{1}{2h} \left(\frac{dh}{d\xi}\right)^2 + \frac{h^{3/2}g \cos \zeta}{\nu h_+ u_w} \left[\left(1 - \frac{h_+^2 u_w^2}{h^{3}g \cos \zeta}\right) \frac{dh}{d\xi} - \tan \zeta + \mu\right], \quad (4.6)$$

which determines the thickness profile of a travelling-wave solution with erosion and deposition. This ODE looks similar to the one that governs roll waves (see e.g. Gray & Edwards 2014), but the fact that the velocity (4.5) equals zero, when $h = h_+$, fundamentally changes the structure of the solutions. It is this distinction
that has led to the introduction of the new terminology of an “erosion-deposition” wave.

In (4.6) the friction coefficient, \( \mu = \mu(h, Fr) \), transitions between three different expressions (3.14)–(3.16) dependent on whether \( Fr \geq \beta \), \( \beta > Fr > 0 \) or \( Fr = 0 \).

For steady uniform flows, Pouliquen’s (1999a) empirical law (3.5) implies that \( h = h_{\text{stop}}(\zeta) \) when \( Fr = \beta \). An explicit expression for \( h_{\text{stop}}(\zeta) \) is found by substituting (3.14) into (3.6) and setting \( Fr = \beta \) to give

\[
h_{\text{stop}}(\zeta) = L \gamma, \tag{4.7}
\]

where

\[
\gamma = \frac{\tan \zeta_2 - \tan \zeta_1}{\tan \zeta - \tan \zeta_1} - 1, \tag{4.8}
\]

is dependent on the inclination \( \zeta \). For the values of the parameters used here (table 1) the constant \( \gamma \approx 3.5358 \) and \( h_{\text{stop}} \approx 3.5358 \text{mm} \). It is not, however, true that \( Fr = \beta \) at \( h = h_{\text{stop}}(\zeta) \) for non-uniform flows, such as for the travelling-wave solutions that are being sought here. The actual flow thickness for which \( Fr = \beta \) is now defined as \( h = h_\star \), where \( h_\star \neq h_{\text{stop}} \) must be solved for as part of the problem.

It follows that the friction law varies in three regions in which \( h \geq h_\star \), \( h_\star > h > h_+ \) or \( h = h_+ \), corresponding to where \( Fr \geq \beta \), \( \beta > Fr > 0 \) or \( Fr = 0 \), respectively.

For the travelling wave the Froude number can be expressed solely in terms of the flow thickness \( h \), by substituting the depth-averaged velocity (4.5) into \( Fr = \bar{u}/\sqrt{gh \cos \zeta} \) to give

\[
Fr(h) = \frac{u_w(h - h_+)}{h^{3/2} \sqrt{g \cos \zeta}}. \tag{4.9}
\]

A Froude number equal to \( \beta \) now corresponds to a flow thickness of \( h_\star \), by definition, and so equating (4.9) to \( \beta \) when \( h = h_\star \) gives the propagation speed of the travelling wave,

\[
u_w = \frac{\beta h_\star^{3/2} \sqrt{g \cos \zeta}}{h_\star - h_+}. \tag{4.10}
\]

The Froude number may then be expressed independently of the wavespeed \( u_w \) by
substituting (4.10) into (4.9) to give
\[ \text{Fr}(h) = \frac{\beta h^3/2(h - h_+)}{h^3/2(h_+ - h_+)} \], (4.11)

which is dependent on the parameters \( h_*, h_+ \) and the known constant \( \beta \). Substitution of (4.11) into (3.14)-(3.16) allows the friction law to be expressed purely as a function of thickness, \( h \), i.e.
\[ \mu(h) = \begin{cases} 
\mu_1 + \frac{\mu_2 - \mu_1}{1 + h^{5/2}(h_+ - h_+)}, & h_* \leq h, \\
\left( \frac{h^3/2(h - h_+)}{h^{5/2}(h_+ - h_+)} \right)^\kappa (\mu_1 - \mu_3) + \mu_3 + \frac{\mu_2 - \mu_1}{1 + h/L}, & h_+ < h < h_*, \\
\mu_3 + \frac{\mu_2 - \mu_1}{1 + h/L}, & h = h_+. 
\end{cases} \] (4.12)
(4.13)
(4.14)

4.1. Initial value problem

The second order ODE (4.6) is solved numerically using MATLAB’s ode45 initial value problem solver by writing the equation as a pair of first order ODEs. The first of these is the definition of a new variable, \( n \), as
\[ \frac{dh}{d\xi} = n, \] (4.15)
and the latter is obtained by substituting the wavespeed (4.10) into (4.6) and writing it in terms of \( h \) and \( n \) to give
\[ \frac{dn}{d\xi} = \frac{n^2}{2h} + \frac{\sqrt{g \cos \zeta (h_+ - h_+)h^{3/2}}}{\beta \nu h_+^{3/2}h_+} \left[ \left(1 - \frac{\beta^2 h^3 h_+^2}{(h_+ - h_+)h^3} \right)n - \tan \zeta + \mu(h) \right]. \] (4.16)

The pair of coupled ODEs (4.15)-(4.16) are solved for \( h \) and \( n \) in the mobile section of the wave for a prescribed value of \( h_* \). The solution is started from the rear of the wave located at \( \xi = 0 \) (chosen arbitrarily) where the depth of the static region is assumed to be \( h_+ \) and the thickness gradient is zero, i.e
\[ h(\xi = 0) = h_+, \quad n(\xi = 0) = 0. \] (4.17)

For each value of the thickness \( h_* \) there is a unique value of \( h_+ \) that enables the static layer to be the same thickness at the front and rear of the wave. The value
Figure 8. Typical travelling-wave solutions (solid lines) to the coupled ODEs (4.15)-(4.16) showing how the flow thickness $h$ varies with down slope position $\xi$ (a,c,e) and $n = dh/d\xi$ (b,d,f). Each solution corresponds to a pair of input parameters $h_*$ (star marker) and $h_+$ (thickness of constant/stationary layer) which are given in equation (4.19) for panels (a,b), equation (4.20) for (c,d) and equation (4.21) for (e,f). The minimum value of the input parameter $h_*$ is $h_{\text{stop}}$ (dashed line). The wavespeed is determined from each $(h_+, h_*)$ pair by equation (4.10). The inset in (a) shows the smooth transition to zero gradient over a finite length.
of \( h_+ \in [0, h_{\text{stop}}] \) is found iteratively for each \( h_* \) by imposing the condition that

\[
h(\xi = \Lambda) = h_+, \quad n(\xi = \Lambda) = 0,
\]

(4.18)

at the front, \( \xi = \Lambda \). The wavelength \( \Lambda \) of the mobile section of the wave is a result of the integration, rather than being specified, and the static regions on either side can be of arbitrary length. Once \( h_* \) and \( h_+ \) are known the wavespeed \( u_w \) is recovered from (4.10) and the depth-averaged velocity is given by (4.5).

Three typical solutions in the physical domain, \((\xi, h)\) (for \(-0.05m \leq \xi \leq 0.55m\)), and phase-plane, \((n, h)\), are shown in figure 8 for the following pairs of the input parameters, \( h_* \) and the corresponding, iteratively determined \( h_+ \),

- figure 8(a,b): \( h_* = 3.56\text{mm}, \quad h_+ \approx 2.16\text{mm}, \)
  (4.19)
- figure 8(c,d): \( h_* = 4.00\text{mm}, \quad h_+ \approx 2.24\text{mm}, \)
  (4.20)
- figure 8(e,f): \( h_* = 6.00\text{mm}, \quad h_+ \approx 2.46\text{mm}. \)
  (4.21)

All of the waves appear to have a shock at the front and a finite gradient at the rear, but, as the inset in figure 8(a) shows, the transitions to the static layer are continuous. It is very surprising that despite the inclusion of depth-averaged viscous dissipation in the momentum balance (3.2) a finite length mobile wave arises naturally from the equations. The solutions generated here are therefore also able to explain the isolated triangular solitary waves observed on erodible beds (Daerr 2001; Börzsönyi et al. 2005; Clement et al. 2007; Börzsönyi et al. 2008; Takagi et al. 2011) as well as the multiple erosion-deposition waves that spontaneously form from a steady-uniform flow detailed in section §2. There are two types of solution (i) ones in which all the friction regimes are all activated (shown in panels a and c) and (ii) those where the mobile section of the wave is entirely in the intermediate friction regime (panel e).

A comparison between all three cases (4.19)–(4.21) and the first six wave crests shown in the space-time plot in figure 3(b) are shown in figure 9. In the experiments the mean wavespeed was found to be \( u_w \approx 0.25\text{ms}^{-1} \) and the wavelength was
λ ≈ 0.59m including stationary regions of mean thickness \( h^+ \approx 2.0 \text{mm} \). Case (4.19) has a wavespeed of \( u_w \approx 0.28 \text{ms}^{-1} \) and a wavelength excluding stationary regions of \( \Lambda \approx 0.49 \text{m} \). The solution has a peak thickness of \( h_w \approx 5.5 \text{mm} \), giving an amplitude \( A \approx 3.3 \text{mm} \), which is in good agreement with the experimental values of \( h_w \approx 5.2 \text{mm} \) and \( A \approx 3.2 \text{mm} \), respectively. In terms of these metrics case (4.19) provides the closest fit to the experimental data, which corresponds to the case in when the friction law changes from dynamic to intermediate at a thickness, \( h^* \), that is only slightly greater than the minimum value, \( h_{\text{stop}} \approx 3.5358 \text{mm} \). There is, however, a marked hump on the lee side of the wave that does not match the observed exponentially decaying tail. A better fit in terms of the shape is provided by case (4.20) although the front is a little too steep and its overall wavelength is too short. In this paper the friction law proposed by Pouliquen & Forterre (2002) has been used throughout. It should be noted, however, that in order to modify...
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Figure 10. Numerical relationship between the input parameters $h_\star$ and (a) $h_+$, for which a valid travelling-wave solution exists to the coupled ODEs (4.15)-(4.16) and the conditions (4.17)-(4.18). Unique solutions exist for $h_\star \leq h_\star^{\text{max}} = 3h_+$ (dashed-dotted line). The resulting solutions have (b) peak thickness $h_w$, (c) wavespeed $u_w$ and (d) wavelength $\Lambda$, exclusive of stationary regions, which varies with $h_\star$. Solutions with no transition in the friction law exist for $h_\star$ greater than the point at which $h_\star = h_w$ (solid markers).

It is interesting to contrast the steady-state erosion-deposition waves to the case of granular roll waves computed by Gray & Edwards (2014) at a fixed slope angle. At a given Froude number for steady-uniform flow, $\text{Fr}_0$, stable limit cycles can be found for roll waves at a range of wave speeds $u_w \in [1, u_c]$, where $u_c < 1 + 1/\text{Fr}_0$. The steady uniform Froude number and the wave speed therefore parameterize the different states. For granular erosion-deposition waves the thickness $h_\star$, at which the friction switches from intermediate to dynamic laws, is the only free parameter.
Figure 10(a) shows that there is a unique relationship between $h_*$ and the deposit depth $h_+$. It follows, that erosion-deposition waves are more constrained than roll waves. The graphs in figure 10 show that as $h_*$ increases, $h_+$ increases, while the peak thickness, wave velocity and wavelength decrease. Large amplitude erosion-deposition waves therefore travel faster than smaller ones. Solutions whose mobile region lies entirely within the intermediate friction regime lie to the right of the bullet point in each of the subplots. There is also a maximum value for $h_*$, which can be found by differentiating (4.11) with respect to $h$, equating it to zero and substituting $h = h_*$ to give

$$h_*^{\text{max}} = 3 h_+.$$  \hfill (4.22)

Admissible values of $h_*$ therefore lie in the range $[h_{\text{stop}}, h_*^{\text{max}}]$, which are shown by the dashed and dot-dashed lines in figure 10(a). For all the numerical simulations, presented in §6 and §7, the waves appear to develop into ones of maximum amplitude, with humps, as in figure 8(a) rather than ones with exponentially decaying tails. It is not entirely clear why this is so. It could be that these waves are representative of the most unstable mode, or it could be due to non-linear coarsening.

5. Non-existence of solutions without viscous dissipation

The depth-averaged $\mu(I)$-rheology (Gray & Edwards 2014) plays a crucial role in modelling erosion-deposition waves. To see this, consider the standard inviscid avalanche model, i.e. equations (3.1)-(3.2) with $\nu$ equal to zero,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h \bar{u}) = 0,$$  \hfill (5.1)

$$\frac{\partial}{\partial t} (h \bar{u}) + \frac{\partial}{\partial x} (h \bar{u}^2) + \frac{\partial}{\partial x} \left( \frac{1}{2} h^2 g \cos \zeta \right) = h g \cos \zeta (\tan \zeta - \mu),$$  \hfill (5.2)

where $\bar{u}$ has been assumed to be positive everywhere and the shape factor $\chi = 1$. Seeking travelling wave solutions, the depth-averaged mass balance can be integrated to give the same expression for the depth-averaged velocity as (4.5). Equa-
Figure 11. Travelling-wave solutions to the non-diffusive ODE (5.3), for values of the parameters in (4.19), with initial conditions (solid markers) at the rearward side of the wave and the wave peak. Trajectories are attracted towards a stable fixed point at \( h = h_{\text{crit}} \), where \( h_{\text{stop}} < h_{\text{crit}} \ll h_{\star} \).

In this example \( h_{\text{crit}} \) is only very slightly greater than \( h_{\text{stop}} \).

The fixed point of the ODE (5.3) occurs when

\[ M(h) = \tan \zeta - \mu(h) = 0. \]  

(5.4)

Assuming that the friction \( \mu(h) \) is given by the intermediate case (4.13) an expan-
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Figure 12. (a) The value of $M$ (solid line) as a function of $h$ for parameter values $h_\star = 3.56\text{mm}$ and $h_+ \approx 2.16\text{mm}$ assumed in the solution (4.19) shown in figure 8(a,b). There is a fixed point when $M(h_{\text{crit}}) = 0$ (dotted line) where the ODE (5.3) has zero gradient. An approximate value of $h_{\text{crit}}$ (solid marker) is found by linearizing the intermediate friction law (4.13) for small $\kappa$. (b) The approximate value of $h_{\text{crit}} - h_{\text{stop}}$ is given as a function of $h_\star$ to show that $h_{\text{stop}} \leq h_{\text{crit}} < h_\star$ for all valid travelling-wave solutions shown in figure 10(a).

The position is made about the point $h = h_{\text{stop}}$ by the introduction of a new variable $\tilde{h} > 0$, where

$$h = h_{\text{stop}} \left(1 + \kappa \tilde{h} + O(\kappa^2)\right), \quad (5.5)$$

and $\kappa = 10^{-3}$ is the power of extrapolation between the dynamic and static friction law regimes, (4.12) and (4.14), respectively. Substitution of (5.5) into (4.13) and...
linearizing about $\kappa = 0$ gives the $\tilde{h}$ expansion of (5.4) as

$$M(h) = \kappa \left[ (\mu_3 - \mu_1) \ln \left( \frac{h_{3/2}^3 (h_{\text{stop}} - h_{+})}{h_{3/2}^3 (h_{*} - h_{+})} \right) + \frac{\gamma (\mu_2 - \mu_1)}{(1 + \gamma)^2} \tilde{h} \right] + O(\kappa^2), \quad (5.6)$$

where the order unity terms equate to zero by equations (4.7) and (4.8). It follows that, to leading order, $M$ is equal to zero when

$$\tilde{h} = \frac{(1 + \gamma)^2 (\mu_3 - \mu_1)}{\gamma (\mu_2 - \mu_1)} \ln \left( \frac{h_{\text{stop}}^3 (h_{*} - h_{+})}{h_{*}^3 (h_{\text{stop}} - h_{+})} \right) = \tilde{h}_{\text{crit}}. \quad (5.7)$$

The quantity $\tilde{h}_{\text{crit}}$ is positive, since $\gamma$ is positive (for the range of angles considered here), $\mu_1 < \mu_3 < \mu_2$ and equation (4.11) implies that the argument in the logarithm is equal to $\beta/\text{Fr}(h_{\text{stop}})$, which is greater than unity. Figure 12(a) shows a plot of $M$ as a function of $h$ for the parameters $h_{*}$ and $h_{+}$ given in (4.19). The approximate solution $h_{\text{crit}} = h_{\text{stop}} (1 + \kappa \tilde{h}_{\text{crit}}) \approx 3.53585$ (solid marker) is less than $h_{*} = 3.56$. This implies that there is a fixed point in the range $[h_{+}, h_{*}]$, so the inviscid system (5.1)–(5.2) is unable to model the same case as the erosion-deposition wave illustrated in figure 8(a) (where $h_{w} > h_{*}$). Figure 12(b) shows that $h_{\text{stop}} \leq h_{\text{crit}} \ll h_{*}$ for the complete range of $h_{*}$. Inviscid solutions starting from the rearward side of the wave will therefore always reach a fixed point, where $dh/d\xi = 0$, before the friction law transitions from the intermediate case (4.13) to the dynamic case (4.12). The depth-averaged $\mu(I)$-rheology is therefore vital to produce the travelling-wave solutions (with $h_{w} > h_{*}$) that resemble the erosion-deposition waves observed experimentally in $\S$2.

6. Numerical method and periodic box simulations

The standard depth-averaged avalanche equations (e.g. Gray et al. 2003) represent a system of hyperbolic equations that require high-resolution shock capturing numerical methods (e.g. Nessyahu & Tadmor 1990) to solve them. Although our problems are still convection dominated the inclusion of the depth-averaged $\mu(I)$-rheology (Gray & Edwards 2014) changes the system into a set of convection–diffusion equations. This paper therefore uses the closely related semi-discrete high-
Figure 13. Flow thickness $h$ varying with downslope position $x$ (solid lines), obtained from numerical simulations on a periodic domain with the travelling-wave solution from figure 8(a) as the initial condition (dashed lines). The flow thickness profile is observed to remain unchanged, aside from small variations in the constant thickness $h_+$ of the stationary layer. These variations are greater with the 2nd order Runge-Kutta time-stepper (a) than with the 3rd order adaptive Runge-Kutta time-stepper (b). Two movies showing the time-dependent evolution of the waves are available in the online supplementary material.

resolution non-oscillatory central schemes of Kurganov & Tadmor (2000), which use high-order, large step-size ODE solvers for their time evolution. In order to solve the system, the depth-averaged equations (3.1)–(3.2) together with the friction law (3.14)–(3.16) are written in vector form as

$$
\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = S(w) + \frac{\partial}{\partial x} (Q(w, w_x)),
$$

where $w = (h, m)^T$ is the vector of conserved variables, $h$ and $m = h\bar{u}$. The resulting convection flux $f$ and source term $S$ are

$$
f = \begin{pmatrix} m \\ m^2/h + k^2/2g \cos \zeta \end{pmatrix}, \quad S = \begin{pmatrix} 0 \\ h g (\sin \zeta - \mu |m| \cos \zeta) \end{pmatrix},
$$

respectively, and the diffusive flux $Q$ is

$$
Q = \begin{pmatrix} 0 \\ \nu h^{1/2} \left( \frac{\partial m}{\partial x} - \frac{m}{h} \frac{\partial h}{\partial x} \right) \end{pmatrix}.
$$

Periodic box simulations are performed using the travelling-wave solution illus-
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trated in figure 8(a) as an initial condition. This is determined by the parameters summarized in table 1, together with \( h_+ \) and \( h_* \) given in equation (4.19) and depth-averaged velocity computed from (4.5) and (4.10). The domain length \( L \) is set to 0.6m and is discretized over 1200 grid points. The travelling-wave is allowed to evolve on the periodic domain and the resulting flow thickness is shown at \( t = 98.7s \) in figure 13 for (a) a second-order Runge-Kutta method with a step size of \( 10^{-2}s \) and (b) a third-order Runge-Kutta adaptive step method (Medovikov 1998). Both methods preserve the initial form of the wave (shown as a dashed line) over the mobile section, but the second-order Runge-Kutta method encounters some problems in properly handling the stationary region. The standard deviation of the static layer from its mean value is \( 1.73 \times 10^{-5} \) mm, which is significantly larger than \( 4.26 \times 10^{-9} \) mm for the third order Runge-Kutta scheme. However, simulation times with the 3rd order method are much longer than with the 2nd order Runge-Kutta time-stepper, particularly when stationary regions of flow develop, since the adaptive step size reduces drastically. As such it is impractical to adopt this time-stepper for all of the numerical results and the 2nd order Runge-Kutta time-stepper is used hereafter. Numerical simulations have also been performed for waves that don’t transition to the dynamic flow regime, such as those illustrated in figure 8(e). For simulations on a box of length 0.3m the solution is stable, but on the 0.6m box the solution transforms into one that resembles the solution in figure 8(c), where \( h_* < h_w \). This may be an indication that such waves, whose mobile section lies purely in the intermediate friction regime, are physically unstable.

6.1. Random perturbation about steady-uniform flow

For the same periodic domain and grid resolution the evolution of an erosion-deposition wave in time is computed for an initially steady-uniform flow of thickness \( h_0 \) and velocity

\[
\bar{u}_0 = \frac{\beta \sqrt{g \cos \zeta}}{L \gamma} h_0^{3/2}.
\]
Figure 14. The flow thickness $h$ obtained by a numerical simulation in a periodic domain with initial conditions $h(x,0) = h_{\text{stop}} + 10^{-4}R(x)$, where $R(x) \in [-1,1]$ is a zero-mean pseudo-random perturbation to the thickness $h_{\text{stop}}$ of a steady uniform flow. The perturbations grow in size and eventually coarsen to form a single erosion-deposition wave. The solution is plotted at various times, with the final state, first reached in (e), surviving forever (f). A movie showing the time-dependent evolution is available in the online supplementary material.
To trigger the instability a pseudo-random perturbation, \( R(x) \), which takes a different value in the interval \([-1, 1]\) at each grid point, is added to the initial condition, i.e.

\[
h(x, 0) = h_0 + \varepsilon R(x), \quad m(x, 0) = m_0 = h_0 \bar{u}_0,
\]

where \( \varepsilon = 10^{-4} \) is the magnitude of the zero-mean perturbation. Figure 14 shows a typical numerical simulation of the evolving flow thickness for the case

\[
h_0 = h_{\text{stop}} = L \gamma, \quad \bar{u}_0 = \bar{u}_{\text{stop}} = \beta \sqrt{\frac{L g \cos \zeta}{\text{Gr}}}. \quad (6.6)
\]

The surface becomes unstable and small waves form that interact and grow in size before reaching a final state in figure 14(e), which survives indefinitely as shown at a much later time (f). The wavespeed of the erosion-deposition wave is \( u_w = 0.28\text{ms}^{-1} \) and the peak height and stationary layer thickness are found to be \( h_w = 5.5\text{mm} \) and \( h_+ = 2.2\text{mm} \) respectively, resulting in an amplitude of \( A = 3.3\text{mm} \). These closely match those of the travelling-wave solution in figure 8(a), which provides further validation of the method.

7. Numerical simulations of erosion-deposition waves in a chute

To compare the model with the experimental results of §2, one-dimensional numerical simulations are performed for a chute geometry. Initially a steady uniform flow is assumed along the complete length of the chute

\[
h(x, 0) = h_0, \quad m(x, 0) = m_0 = h_0 \bar{u}_0, \quad \forall x \in [0, L]
\]

where \( h_0 \geq h_{\text{stop}} \). The boundary conditions at the start and end of the chute are still dominated by the those of the convective problem, since inclusion of the depth-averaged \( \mu(I) \)-rheology is a singular perturbation to the equations. For the problems of interest here the flows are subcritical (\( \text{Fr} < 1 \)) everywhere, which requires one upstream and one downstream condition to be specified for the hyperbolic system (e.g Weiyan 1992). The introduction of a diffusive term means that
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Figure 15. Results of a numerical simulation at time $t = 23.5 s$ showing (a) the flow thickness $h$ and (b) the depth-averaged velocity $\bar{u}$ with downslope position $x$ (solid lines). At the inflow boundary the flow thickness $h(0, t) = h_{\text{stop}} + 10^{-4} \sin (2\pi ft)$ is prescribed, where the sinusoidal perturbation has frequency $f = 0.47\text{Hz}$. Initially there is a steady uniform flow with $h_0 = h_{\text{stop}}$ and $\bar{u}_0 = \bar{u}_{\text{stop}}$ (dashed lines). The important experimental flow feature of stationary regions between waves, where $\bar{u} = 0$, is captured numerically. A movie showing the time-dependent evolution is available in the online supplementary material.

The system is now parabolic and an extra boundary condition is required, which is applied here at the outflow end ($x = L$), where free outflow is imposed via linear extrapolations of the values of $h$ and $m$ from the final two interior cells. At the upstream boundary, $x = 0$, the general inflow condition is

$$h(0, t) = h_0 + \varepsilon H(t), \quad (7.2)$$

where $\varepsilon = 10^{-4}$ is the magnitude of the perturbation, $H(t)$. For the simulations performed here, erosion-deposition waves develop in the range $h_0 \in [h_{\text{stop}}, 1.025h_{\text{stop}}]$. 
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Figure 16. Results of a numerical simulation at time $t = 23.5s$ showing (a) the flow thickness $h$ and (b) the depth-averaged velocity $\bar{u}$ with downslope position $x$ (solid lines). At the inflow boundary, a flow thickness $h(0, t) = h_{\text{stop}} + 10^{-4}R(t)$ is prescribed, where $R(t) \in [-1, 1]$ is a zero-mean, pseudo-random perturbation. A steady uniform flow of thickness $h_{\text{stop}}$ and corresponding depth-averaged velocity $\bar{u}_{\text{stop}}$ is imposed as an initial condition (dashed lines). A movie showing the time-dependent evolution is available in the online supplementary material.

The flow stops for $h_0 < h_{\text{stop}}$ and for $h_0 > 1.025h_{\text{stop}}$ granular roll waves form, which are fully mobilized and do not have stopping regions.

7.1. Sinusoidal inflow perturbation

A series of travelling erosion-deposition waves can be generated by imposing a small sinusoidal perturbation to a steady uniform flow of thickness $h_{\text{stop}}$, i.e.

$$h_0 = h_{\text{stop}}, \quad H(t) = \sin (2\pi ft). \quad (7.3)$$
Figure 17. Results of a numerical simulation for an inflow thickness \( h(0, t) = h_{\text{stop}} + 10^{-4}R(t) \), where \( R(t) \in [-1, 1] \) is a zero-mean, pseudo-random perturbation. Steady uniform flow of thickness \( h_{\text{stop}} \) and depth-averaged velocity \( \bar{u}_{\text{stop}} \) is imposed as an initial condition. Data is given between times \( \hat{t} \) of 0 to 24 seconds for (a) a space-time plot (with flow thickness given by the greyscale colourbar) of the downslope region \( 2.21m \leq x \leq 3.29m \) and (b) the flow thickness past the point \( x_L = 3.21m \), corresponding with the experimental results in figure 3. The important experimental flow features of stationary regions between waves of a constant wavespeed (straight diagonal lines) and merging events (where lines representing waves of a different speed meet), are all captured numerically.

where the oscillation frequency is set to \( f = 0.47Hz \). The computed flow thickness \( h \) and depth-averaged velocity \( \bar{u} \) are shown in figure 15 for a domain length \( L = 5m \) using 10,000 grid points. The perturbations rapidly grow into erosion-deposition waves in shallow granular free-surface flows.
waves that closely resemble the travelling-wave shown in figure 8(a). The distance between successive wavefronts $\lambda \approx 0.60m$ (including the stationary regions) and wave fronts pass a given point at regular intervals with a period $T \approx 2.13s$ implying a wavespeed $u_w \approx 0.28ms^{-1}$. This is consistent with the imposed perturbation. The resulting peak thickness $h_w \approx 5.5mm$, layer thickness $h_+ \approx 2.2mm$ and wave amplitude $A = 3.3mm$ are all in good quantitative agreement with the experimentally obtained values. However, since all the waves have the same height and wavespeed they do not produce wave merging events like those observed in figure 3(a).

7.2. Random inflow perturbation

In order to produce merging the form of the perturbation, $H(t)$, is now chosen to be a pseudo-random, zero-mean function $R(t)$, which takes a value in the interval $[-1, 1]$ at each time step, i.e. the inflow conditions are

$$h_0 = h_{\text{stop}}, \quad H(t) = R(t). \quad (7.4)$$

The spatial evolution of the computed flow thickness $h$ and the depth-averaged velocity $\bar{u}$ are shown in figure 16 at a fixed time. A corresponding video of the simulation is also available in the online supplementary material. The random inflow condition generates erosion-deposition waves with different characteristic frequencies and wavespeeds, which merge with one another. This delays the formation of stationary regions, which only start to develop between two and three metres downstream, as compared to the sinusoidal case, where they form just after the first metre. The merging events can be visualized by generating a space-time plot for the region between $x = 2.20$ and $3.29$ metres as shown in figure 17(a), which shows diagonal wavefront lines meeting in a similar way to the experiment shown in figure 3(a). Since the data acquisition was started at an arbitrary time the simulation time $\hat{t}$ is shifted, i.e. $\hat{t} = t - 7.5s$, in order to compare directly with the experimental data. The wavespeed of each roll wave is determined by the gradient of the corresponding diagonal line in the space-time plot (figure 17a) and is found here to have
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A value of $u_w \approx 0.27\text{ms}^{-1}$. The period between successive wavefronts, inclusive of stationary regions (given by the horizontal length of time between the wavefronts), is found to have a mean value of $T \approx 2.13\text{s}$ and a corresponding mean frequency of $f \approx 0.47\text{Hz}$. Combining these gives the mean wavelength of each roll wave, including a stationary region, as $\lambda \approx 0.58\text{m}$. Figure 17(b) shows the flow thickness past the point $x_L = 3.21\text{m}$ downslope as a function of time. It indicates that mean peak thickness $h_w \approx 5.4\text{mm}$ and the mean stationary layer depth $h_+ \approx 2.2\text{mm}$, implying a wave amplitude of approximately 3.2mm. These are all in good qualitative agreement with the experimentally obtained values. The Pouliquen-Forterre friction law (Pouliquen & Forterre 2002) together with the depth-averaged $\mu(I)$–rheology (Gray & Edwards 2014) are therefore able to quantitatively model the formation of erosion-deposition waves with stationary regions.

8. Contrast to numerical simulations of granular roll waves

Using exactly the same depth-averaged equations (3.1)–(3.2) together with the friction law (3.14)–(3.16) it is also possible to simulate the formation of granular roll waves, such as those observed in experiments (e.g. Davies 1990; Vallance 1994; Forterre & Pouliquen 2003; Zanuttigh & Lamberti 2007; Iverson et al. 2010; Gray & Edwards 2014). Roll waves develop in thicker flows ($h_0 > h_{\text{stop}}$) in which the Froude number, $F_r$, remains greater than $\beta$ throughout. This implies that the friction coefficient is always in the dynamic regime (3.14) although this is not enforced in the simulations. Initially the chute is therefore assumed to contain a steady uniform flow, as in (7.1), that is slightly deeper

$$h_0 = 1.2h_{\text{stop}}, \quad m_0 = h_0\bar{u}_0 = (1.2)^{5/2}h_{\text{stop}}\bar{u}_{\text{stop}},$$

where $\bar{u}_0 = (1.2)^{3/2}\bar{u}_{\text{stop}}$ follows from (6.4) and (6.6). Since roll waves take longer to form than erosion-deposition waves, the domain length is doubled to $L = 10\text{m}$
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Figure 18. Results of a numerical simulation at time $t = 40.0$ s showing (a) the flow thickness $h$ and (b) the depth-averaged velocity $\bar{u}$ with downslope position $x$ (solid lines). At the inflow boundary a flow thickness $h(0, t) = 1.2h_{\text{stop}} + 10^{-4}\sin(2\pi ft)$ with sinusoidal perturbation frequency $f = 0.47$ Hz is imposed. The initial steady uniform flow thickness $h_0 = 1.2h_{\text{stop}}$ and depth-averaged velocity $\bar{u}_0 = (1.2)^{3/2}\bar{u}_{\text{stop}}$ are shown by the dot-dashed lines. The minimum flow thickness and velocity, $h_{\text{stop}}$ and $\bar{u}_{\text{stop}}$, are shown with dashed lines. A movie showing the time-dependent evolution is available in the online supplementary material.

and is discretized over 20 000 grid points, to match the resolution of the previous simulations.

8.1. Sinusoidal inflow perturbation

Since the inflow condition is still subcritical ($Fr < 1$) only the upstream flow height and its sinusoidal perturbation need to be prescribed at $x = 0$, i.e.

$$h_0 = 1.2h_{\text{stop}}, \quad H(t) = \sin(2\pi ft), \quad (8.2)$$
Figure 19. Results of a numerical simulation at time $t = 40.0\,\text{s}$ showing (a) the flow thickness $h$ and (b) the depth-averaged velocity $\bar{u}$ with downslope position $x$ (solid lines). At the inflow boundary, a flow thickness $h(0, t) = 1.2h_{\text{stop}} + 10^{-4}R(t)$ is prescribed, where $R(t) \in [-1, 1]$ is a zero-mean, pseudo-random perturbation. The initial flow thickness $h_0 = 1.2h_{\text{stop}}$ and corresponding depth-averaged velocity $\bar{u}_0 = (1.2)^{3/2}\bar{u}_{\text{stop}}$ are shown with dot-dashed lines. The minimum flow thickness and velocity for which a steady uniform flow is possible, $h_{\text{stop}}$ and $\bar{u}_{\text{stop}}$, are also shown using dashed lines. A movie showing the time-dependent evolution is available in the online supplementary material.

where the perturbation frequency $f = 0.47\,\text{Hz}$ is the same as in (7.3). The computed flow thickness, $h$, and the depth-averaged velocity, $\bar{u}$, are shown in figure 18. Since the uniform flow is unstable to the small perturbations at the inflow, these grow into fully developed granular roll waves of the imposed frequency by approximately 8 metres downstream. This is significantly further than the sinusoidally induced erosion-deposition waves which were fully developed after about 1 metre. During
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the evolution none of the troughs between the wave crests lie below \( h_{\text{stop}} \) and so no stationary regions form. Examining the flow thickness at the outflow end of the domain, when a steady-state has been reached, it can be seen that although the each wave has a peak thickness of \( h_w \approx 5.5 \text{mm} \), similar to the erosion-deposition wave case, the troughs have a much greater thickness, \( h_{\text{trough}} \approx 3.8 \text{mm} \). This implies that continuous waves, despite developing from a greater inflow thickness than their erosion-deposition wave counterparts, have a much smaller wave amplitude of \( A \approx 1.7 \text{mm} \), for an inflow perturbation oscillating with the same amplitude and frequency.

8.2. Random inflow perturbation

The pseudo-random, zero-mean form of the inflow perturbation function, with \( R(t) \in [-1, 1] \) at each time step, is now applied, i.e.

\[
h_0 = 1.2 h_{\text{stop}}, \quad H(t) = R(t).
\]  

(8.3)

The flow thickness, \( h \), and depth-averaged velocity, \( \bar{u} \) are shown in figure 19 for one fixed time, \( t \), of a typical random simulation. A corresponding video of the simulation is also available in the online supplementary material in order to compare and contrast it to the formation of erosion-deposition waves. Continuous roll waves of a range of characteristic frequencies and wavespeeds develop downstream of the inflow. Examining the flow thickness at the outflow end of this domain, the wave peaks have a mean flow thickness of \( h_w \approx 4.8 \text{mm} \), whilst the troughs have a mean thickness of \( h_{\text{trough}} \approx 3.9 \text{mm} \). This implies a mean wave amplitude of only \( A \approx 0.9 \text{mm} \), despite many merging events and wave coarsening having taken place on a domain which is twice the length. This is consistent with the experimental observations of Forterre & Pouliquen (2003) who had to use a loudspeaker system to induce moderately large perturbations with a given frequency in order to see roll waves on their 2 metre long chute. The granular roll waves look qualitatively different in shape to erosion-deposition waves and their coarsening dynamics...
also appears to be very different. The most notable difference, however, is that erosion-deposition waves have stationary regions.

9. Conclusions

In this paper experiments are used to show that a granular avalanche close to the minimum depth for steady-uniform flow, $h_{\text{stop}}$, can spontaneously break down into a series of discrete erosion-deposition waves that are separated by regions of completely static grains. We believe that each individual wave is directly related to the isolated triangular waves, first reported in the literature by Daerr (2001). The waves are able to travel downslope at constant speed, steadily eroding and depositing a static erodible layer, ahead of and behind them, in exact balance, so that their shape is preserved. Remarkably, given the notorious difficulty in modelling erosion-deposition problems, a quantitative depth-averaged theory has been developed for these waves by combining the depth-averaged $\mu(I)$–rheology of Gray & Edwards (2014) with the dynamic, intermediate and static friction regimes in Pouliquen & Forterre’s (2002) extended friction law. Rather than progressively eroding and/or depositing the grains from the base of the avalanche, this approach treats the regions as either mobile, or static, throughout their depth. This necessarily imposes a limitation on the model’s applicability, since the depth of the erodible material has to be shallow, i.e. it is not possible to calculate the erosion and deposition on slopes with an arbitrary depth of static grains. However, it is encouraging that a model constructed in this fashion captures much of the observed behavior.

It is the combination of the viscous dissipation introduced through the $\mu(I)$–rheology (Gray & Edwards 2014) and Pouliquen & Forterre’s (2002) intermediate friction law (3.15) (for $0 < Fr < \beta$) that are able to bring the grains to rest. Indeed, we have shown in §5 that it is not possible to construct travelling erosion-deposition waves using a standard inviscid avalanche model (e.g. Savage & Hutter 1989; Gray et al. 1999; Pouliquen & Forterre 2002; Gray et al. 2003). This problem is therefore
one of those occasions where the rheology, which is normally negligibly small, plays a crucial role in the solution.

The depth-averaged $\mu(I)$-rheology changes the standard hyperbolic avalanche models into a system of convection-diffusion equations. A numerical method to solve the full depth-averaged equations (3.1)–(3.2) together with the friction law (3.14)–(3.16) has been developed, which is based on the semi-discrete high-resolution non-oscillatory central schemes of Kurganov & Tadmor (2000). This uses efficient high order, large step-size ODE solvers for their time evolution, which allow the diffusive terms to be included while preserving the advantages of explicit methods for convection dominated problems. The method is able to compute the formation of erosion-deposition waves from a steady-uniform flow of depth $h = h_{\text{stop}}$, which is on the boundary of being able to flow. Figure 16 shows results for a very small pseudo-random perturbation to the inflow thickness $h_{\text{stop}} = 3.53\text{mm}$ that is able to quantitatively reproduce typical peak flow thicknesses, $h_w = 5.4\text{mm}$, static deposit depths, $h_+ \approx 2.2\text{mm}$, wavespeeds, $u_w \approx 0.27\text{ms}^{-1}$ and typical wavelengths $\lambda = 0.58\text{m}$ consistent with those observed in the experiments described in §2. Moreover a space-time plot in figure 17 shows that variation in individual wave speeds leads to coarsening similar to that observed in experiment (as shown in fig. 3).

The theory presented here is also able to model the formation of granular roll waves for $h > 1.025h_{\text{stop}}$, such as those observed experimentally (e.g. Davies 1990; Vallance 1994; Forterre & Pouliquen 2003; Zanuttigh & Lamberti 2007; Iverson et al. 2010; Gray & Edwards 2014). Roll waves are in some ways simpler than erosion-deposition waves, because the friction stays entirely within the dynamic regime ($\text{Fr} > \beta$) given by (3.14). The introduction of viscous-dissipation allows continuous roll waves to form, with no stopping regions, i.e. $\bar{u} > 0$ everywhere. The amplitudes of granular roll waves are significantly smaller than their erosion-deposition wave counterparts, and they take a much greater distance to fully de-
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velop, which is consistent with experiments. As the inflow thickness is decreased, roll waves transition into erosion-deposition waves in a narrow regime just above \( h_{\text{stop}} \). Inflow thicknesses below \( h_{\text{stop}} \) rapidly come completely to rest. It is interesting, and perhaps very significant, that flows on the very cusp of stopping spontaneously self-organize to produce large amplitude waves that sustain the flow.

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Chapter 4

Arrested coarsening of granular roll waves
Arrested coarsening of granular roll waves

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We study a system in which granular matter, flowing down an inclined chute with periodic boundary conditions, organizes itself in a train of roll waves of varying size. Since large waves travel faster than small ones, the waves merge, and their number gradually diminishes. This \textit{coarsening process}, however, does not generally proceed to the ultimate one-wave state: Numerical simulations of the dynamical equations (being the granular analogue of the shallow water equations) reveal that the process is \textit{arrested} at some intermediate stage. This is confirmed by a theoretical analysis, in which we show that the roll waves cannot grow beyond a certain limiting size (which is fully determined by the system parameters), meaning that on long chutes the material necessarily remains distributed over more waves. We determine the average lifetime $\tau_N$ of the successive $N$-wave states, from the initial state with typically $N = 50$ waves (depending on the length of the periodic domain) down to the final state consisting of only a handful of waves ($N = N_{\text{arr}}$). At the latter value of $N$ the lifetime $\tau_N$ goes to infinity. At this point the roll waves all have become equal in size and are traveling with the same speed. Our theoretical predictions for the successive lifetimes $\tau_N$ and the value for $N_{\text{arr}}$ show good agreement with the numerical observations.
1. Introduction

When granular matter flows down an inclined channel as in Fig. 1, with a slope of about 35 degrees, one may witness the spontaneous formation of so-called roll waves. These are traveling waves with sharp fronts, propagating at a speed exceeding the velocity of the grains themselves, separated by stretches of almost uniform thickness. Dramatic examples of this, often quite destructive, may be observed in debris flows, mud flows and landslides around the world (de Blasio 2011). Indeed, granular roll waves are not only of intrinsic scientific interest but there is also an urgent practical need for understanding them better.

While roll waves in granular matter are a fairly new research theme (Schonfeld 1996), similar waves in water and other traditional fluids have been studied for decades. Pioneering work was done by Cornish who, in two books published in 1910 and 1934, described roll waves in steep water channels, conduits and spillways behind dams (Cornish 1910, 1934). The free-surface instability responsible for the formation of the roll waves was revealed by Kapitza & Kapitza (1949). In the same year, Dressler published a mathematical study on roll waves that is still essential reading for anyone interested in the subject (Dressler 1949; Whitham 1974; Debnath 2005). In the context of the present work, we also mention the paper by Chang, Demekhin, Kalaidin & Ye (1996), who realized that trains of roll waves in water undergo coarsening: since large waves move faster than small ones, they swallow the latter and thereby increase the typical distance between the waves. Finally, Balmforth & Mandre (2004) reported that this coarsening process – in a channel with periodic boundary conditions – does not necessarily continue all the way down to the 1-wave state, but may be interrupted at some intermediate stage with more than 1 wave. At this stage the surviving waves have all become identical (and hence have equal velocities) and chase each other indefinitely.

The purpose of this paper is to describe the coarsening process of roll waves in dry granular matter, or sand, in a chute with periodic boundary conditions. Also
Figure 1. Four photographs, 0.2 seconds apart, showing a section of about 50 cm of the experimental chute. In the first frame (a) we see four roll waves progressing down the chute. The leading wave is being overtaken by the second (b-c) and thereafter these two continue as one (d). The chute has a total length of 3.29 m, width 7.8 cm, and is inclined at an angle of $\zeta = 35.1$ degrees with the horizontal. The flowing grains are carborundum particles of 0.3-0.4 mm in diameter, with an average bed thickness of $h_0 = 4.2$ mm.
in this case we will see that the coarsening is generally *arrested* before it arrives at the 1-wave state. We will show how this can be explained from the dynamical equations that govern the system.

In Section II the governing equations of mass and momentum balance are introduced. They resemble the so-called shallow water equations for normal fluids, adapted where necessary to account for the special properties of granular matter, especially in the terms that represent friction and effective viscosity. Subsequently, in Section III we present the results of a series of numerical experiments, starting
out from a slightly perturbed sheet of flowing sand in a periodic chute of length \( L \), and following its evolution (by numerically integrating the dynamical equations) in the course of time. It is here that we see that, if \( L \) is sufficiently long, the coarsening never reaches the 1-wave state. In Section IV we describe the two opposing mechanisms that drive the coarsening process. The first mechanism is the *merging* of waves, caused by the fact that larger waves travel faster than small ones. The second mechanism is the *ripening* of waves: at each \( N \)-wave state, all the waves show a tendency to adapt their peak height to a given equilibrium value (and moreover, we show that there is an upper threshold this value can never exceed). Roll waves that are larger than this value spontaneously diminish their height (by shedding off mass) whereas smaller waves grow (by picking up some mass). The interplay between these two mechanisms controls the coarsening dynamics and determines at which stage it is arrested. In Section V we develop a theoretical model for the coarsening process, which is based on these two opposing mechanisms. The paper culminates in a formula for the lifetime \( \tau_N \) of the \( N \)-wave state. At the value \( N = N_{\text{arr}} \), depending on the length of the chute and on the properties of the sand, this lifetime diverges. This is equivalent to saying that the coarsening process is arrested at the stage with \( N_{\text{arr}} \) waves. Finally, Section VI contains concluding remarks.

Before we turn to the analysis, let us briefly consider the system we are dealing with. We study a periodically extended version of the chute of Fig. 1. This means that the sand that leaves the system at the bottom of the chute re-enters the system at the top, instantaneously and without change of its profile or velocity. This allows the coarsening process (which takes a considerable time, especially during the final stages) to run its full course until it freezes in the state with \( N_{\text{arr}} \) waves. But of course the periodicity also introduces a connectedness between the downstream and upstream sections that a linear chute does not posses; so one should be cautious when translating our results to linear chutes, especially if the
length $L$ of the periodic domain is short. This is one of the reasons why we will focus mainly on long chutes.

Let us also mention the importance of the angle of inclination $\zeta$. This should neither be too small (or the sand would simply sit still) nor too large (or the sand would be accelerated down the chute (Daerr & Douady 1999) until air drag and collisional stresses become important (Holyoake & McElwaine 2012)). Only for angles between $\zeta_1 = 32.9$ and $\zeta_2 = 42.0$ degrees there is a delicate, changing balance between the gravitational force that pushes the sand downwards and the friction from the floor of the chute that works in the opposite direction. In the setup of Fig. 1 the floor is roughened by means of spherical glass beads of 0.4 to 0.6 mm in diameter, which are glued to the chute, and the “sand” consists of carborundum particles of 0.3-0.4 mm in diameter. If we tilt the system at an angle of $\zeta = 35.1$ degrees, a sheet of thickness $h_0 = 4.2$ mm (12 particle diameters) shows excellent spontaneous formation of roll waves and subsequent coarsening dynamics. Throughout the paper we assume that our system has the above well-balanced properties. A typical experimental run is presented in Fig. 2, where we see a train of roll waves and several merging events; the height of the waves (upper figure) is measured by means of a laser profilometer, while the space-time plot in the lower figure has been reconstructed from successive still images taken by a high-speed camera positioned above the chute.

2. Governing equations

In order to study the behaviour of the moving granules we use a hydrodynamic-like description, as for normal fluids, with properly modified friction and viscosity to take into account the special, granular nature of the flow. In this description, the two fundamental quantities to be determined are the height of the granular sheet, $h(x, t)$, and the depth-averaged velocity $\bar{u}(x, t)$:
Figure 3. Schematic diagram showing two roll waves propagating down a chute inclined at an angle $\zeta$ to the horizontal. There is a downstream velocity profile $u(x, z, t)$ (independent of the transverse coordinate $y$) and the dashed line at $z = h_0$ represents the uniform thickness of a steady, unperturbed flow.

$$\tilde{u}(x, t) = \frac{1}{h(x, t)} \int_{0}^{h(x, t)} u(x, z, t) dz,$$

(2.1)

where $u(x, z, t)$ is the velocity field of the grains in the chute. We ignore any variations in the crosswise ($y$) direction. The dynamical equations from which we determine the above two quantities are the continuity equation (expressing the conservation of material along the chute):

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0,$$

(2.2)

and the depth-averaged momentum balance, which takes the following form:

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (\chi h^2 u)}{\partial x} = hg \sin \zeta - \mu h g \cos \zeta - \frac{\partial}{\partial x} \left( \frac{1}{2} h^2 g \cos \zeta \right) + \frac{\partial}{\partial x} \left( \nu h^{3/2} \frac{\partial \tilde{u}}{\partial x} \right),$$

(2.3)

which is reminiscent of the Savage-Hutter equation (Savage & Hutter 1989) but with an additional, novel viscous-like term (Gray & Edwards 2014) and a shape factor $\chi$ that is taken to be unity.

The continuity equation (2.2) simply states that a net influx of material into a
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fixed control volume, represented mathematically by $-\partial(h\bar{u})/\partial x$, gives rise to an increase in height $\partial h/\partial t$ in this control volume. Here we assume that the granular material has a constant density $\rho$, i.e., that it may be treated as an incompressible fluid. This has also been used in the momentum balance, which has been divided by $\rho$ throughout.

The momentum balance (2.3) expresses the fact that the change of momentum of the material contained in the control volume, given by the total (Stokesian) derivative on the left hand side, is equal to the resultant force on this volume. The contributing forces are [in the order in which they appear on the right hand side of Eq. (2.3)]:

(i) The component of the gravitational force that acts along the chute, $hg \sin \zeta$ (with $\zeta$ being the angle of the chute with the horizontal);

(ii) the friction the granular material experiences from the bottom of the chute, $-\mu hg \cos \zeta$, which has the usual form of a friction coefficient $\mu$ times the normal force of the bottom acting on the granular material;

(iii) the depth-averaged pressure gradient $-\partial(\frac{1}{2}h^2 g \cos \zeta)/\partial x$ due to height variations of the granular sheet, assuming a standard lithostatic pressure profile within the sheet;

(iv) the diffusive term $\partial/\partial x(\nu h^{3/2}\partial \bar{u}/\partial x)$, arising from depth-averaging the in-plane deviatoric stresses (Gray & Edwards 2014) assuming a $\mu(I)$-rheology (Jop, Forterre & Pouliquen 2006) in the granular sheet.

Of the above terms, the friction and diffusion require some further discussion, since it is here that the granular nature of the medium is incorporated. With respect to the friction, we note that the coefficient $\mu$ in a flowing sheet of sand is not simply a constant (as in standard Coulomb friction), but has the rather more intricate form measured by Pouliquen & Forterre (2002)

$$\mu = \mu(h, \bar{u}) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + \beta h^{3/2} \sqrt{g \cos \zeta}/(L \bar{u})}.$$ (2.4)
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This holds as long as the Froude number

\[ Fr = \frac{\bar{u}}{\sqrt{gh \cos \zeta}} \]  (2.5)

exceeds \( \beta \), an empirical constant which for our chute is estimated to be \( \beta = 0.65 \). For Froude numbers below \( \beta \), the friction is given by two alternative static and intermediate laws (Pouliquen & Forterre 2002). Here, however, the condition \( Fr > \beta \) is amply fulfilled, and the friction is always given by the dynamic friction law (2.4) although this is not imposed. The Froude number compares the depth-averaged velocity \( \bar{u} \) with the velocity of gravity waves in a tilted sheet of fluid. Its value is related to the direction in which information is transferred through the sheet (in the inviscid case): for \( Fr > 1 \) information can only propagate downstream, whereas for \( Fr < 1 \) the information can propagate both upstream and downstream. In Eq. (2.4) for \( \mu(h, \bar{u}) \), the quantities

\[ \mu_1 = \tan \zeta_1 \quad \text{and} \quad \mu_2 = \tan \zeta_2 \]  (2.6)

are the standard friction coefficients associated with the critical angles \( \zeta_1 \) and \( \zeta_2 \) discussed in the introduction. The inclination \( \zeta \) of our chute lies between these two values: \( \zeta_1 < \zeta < \zeta_2 \), since only in this regime do roll waves occur. The empirical constant \( \mathcal{L} = 1 \text{ mm} \) is a characteristic length scale associated with a transition from \( \zeta_1 \) to \( \zeta_2 \) in the basal friction law (Pouliquen 1999; Pouliquen & Forterre 2002; Forterre & Pouliquen 2008). In particular, the thickness of static material left on the chute, \( h_{\text{stop}} \), by a steady uniform flow when the inflow is closed, is

\[ h_{\text{stop}}(\zeta) = \gamma \mathcal{L}, \quad \text{where} \quad \gamma = \frac{\mu_2 - \mu_1}{\tan \zeta - \mu_1} - 1. \]  (2.7)

The definition of \( \gamma \) is a useful shorthand that will be used later (Gray & Aney 2009). For a steady uniform flow, with constant thickness and velocity, the only non-zero terms in Eq. (2.3) are those of gravity and friction (the first two terms on the right hand side), meaning that these must precisely balance each other, so \( \mu = \tan \zeta \). Denoting the uniform thickness by \( h_0 \), the corresponding depth-averaged
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velocity is then found to be, through (2.4):
\[ \bar{u}_0 = \frac{\beta \sqrt{g \cos \zeta}}{L \gamma} h_0^{3/2}, \]
where for notational convenience we have introduced \( B = \beta \sqrt{g \cos \zeta / (L \gamma)} \). For the parameters used in the present paper it has the value \( B = 522 \ (\text{m}^{1/2}\text{s})^{-1} \).

When the uniform flow thickness is reduced to \( h_{\text{stop}} \), the Froude number (2.5) falls to the threshold value \( Fr = \beta \). In our system, to ensure that the Froude number remains above \( \beta \) everywhere and the material keeps flowing without exhibiting any stopping regions (Edwards & Gray 2014), we take an initial thickness of \( h_0 = 1.2h_{\text{stop}} \).

To be specific, with \( \zeta_1 = 32.9^\circ \ (\mu_1 = 0.647) \), \( \zeta_2 = 42.0^\circ \ (\mu_2 = 0.900) \) and \( \zeta = 35.1^\circ \ (\tan \zeta = 0.703) \), we have \( \gamma = 3.536 \) and hence \( h_{\text{stop}} = L \gamma = 3.536 \ \text{mm} \). So our choice \( h_0 = 1.2h_{\text{stop}} \) corresponds to a uniform thickness of 4.2 mm. As we will see in the next Section (and as we have already seen in the experimental setup of the introduction), the uniform flow under these conditions happens to be unstable: small random perturbations in the flow thickness do not damp out but tend to grow into well-developed roll waves.

Figure 4 illustrates how \( \mu(h, \bar{u}) \) depends on the local flow thickness \( h(x, t) \) and the depth-averaged velocity \( \bar{u}(x, t) \). It always remains close to the traditional Coulomb value \( \tan \zeta = 0.703 \), but the small variations around this value are of crucial importance. If \( \mu(h, \bar{u}) \) were constant the material on the chute would simply accelerate indefinitely. By contrast, Fig. 4 shows that the friction force on the granular sheet is such that it slides more easily in regions where it is thick and flows slowly (large \( h \), small \( \bar{u} \)), enabling small height perturbations of the uniformly flowing sheet to develop into roll waves.

We close with a few words on the diffusion term in Eq. (2.3). This term prevents the steep wave fronts from developing into discontinuous jumps, keeping the slope of the wave front to a finite (albeit large) value. It is derived from the \( \mu(I) \)-rheology for granular flow (Jop et al. 2006), which is a nonlinear viscous law with a pressure and
Figure 4. Contour plot of the friction coefficient $\mu(h, \bar{u})$, given by Eq. (2.4), which is seen to decrease with growing flow thickness $h$ and to increase with growing velocity $\bar{u}$. The shaded region corresponds to combinations of $h$ and $\bar{u}$ when the granular layer is close to static (the corresponding Froude number is less than $\beta = 0.65$) and the given expression for $\mu$ does not apply. The dashed parabolic curve corresponds to the threshold $Fr = \beta$.

strain-rate dependent viscosity. To leading order this rheology implies that a steady uniform flow has a Bagnold velocity profile $u(z) \propto h^{3/2} - (h - z)^{3/2}$, and a lithostatic pressure $p(z) = \rho g (h - z) \cos \zeta$, while to first order, depth-integration of the mass and momentum equations yield the inviscid shallow water-like equations (2.2)-(2.3) without the viscous terms (Gray & Edwards 2014). This system of equations has proved to be highly effective at modeling granular avalanches (Savage & Hutter 1989; Gray & Cui 2007; Johnson & Gray 2011; Cui & Gray 2013) and can predict the critical Froude number $Fr_c = 2/3$ for roll waves (Forterre & Pouliquen 2003), but not the cut-off frequency. Formal extensions to higher order are complicated and generate a large number of extra terms that are difficult to interpret (Schied, Ruyer-Quil & Manneville 2006). Gray & Edwards (2014) therefore took a pragmatic approach and simply added the depth-averaged in-plane deviatoric stress to the
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momentum balance (2.3), i.e. the term
\[ \frac{1}{\rho} \int_0^h \frac{\partial \tau_{xx}}{\partial x} \, dz = \frac{\partial}{\partial x} \left( \nu \frac{h^{3/2}}{2} \frac{\partial \bar{u}}{\partial x} \right). \] (2.9)

Many different viscous terms are possible, because the leading order relationship between the depth-averaged Bagnold velocity and the thickness can be used to reformulate it at this order. The version selected in (2.9) has been specifically chosen to ensure that the effective viscosity \( \nu \frac{h^{3/2}}{2} \) is non-singular (for slope inclinations \( \zeta_1 < \zeta < \zeta_2 \)) and degenerates when \( h = 0 \). This approach is more powerful than one might expect, since Gray & Edwards (2014) showed that the system (2.2)-(2.3) was able to quantitatively match the experimental cut-off frequency for roll waves (Forterre & Pouliquen 2003) without any fitting parameters. In addition, Edwards & Gray (2014) have shown that at low flow rates this viscous term plays a crucial role in the formation of erosion-deposition waves, which have completely static regions ahead of and behind a solitary mobile wave (Daerr 2001; Börzsönyi, Halsey & Ecke 2005, 2008; Takagi, McElwaine & Huppert 2011).

The viscous-term is not simply an ad hoc regularization of the equations, since the coefficient \( \nu \) is determined directly from the \( \mu(I) \)-rheology. It takes the form Gray & Edwards (2014)
\[ \nu = \frac{2 \gamma L \sqrt{g} \sin \zeta}{9 \beta \sqrt{\cos \zeta}}, \] (2.10)
which for the system under consideration means that \( \nu = 2.4 \times 10^{-3} \text{m}^{3/2} \text{s}^{-1} \). From the definition of \( \gamma \) in equation (2.7) the coefficient \( \nu \) decreases with increasing slope angle provided it lies in the range where steady uniform flows develop, i.e. \( \zeta \in (\zeta_1, \zeta_2) \). Strong evidence for this is provided by the theory’s ability to match the cut-off frequency for roll waves in experimentsForterre & Pouliquen (2003) over a wide range of anglesGray & Edwards (2014). It is also important, however, to sound a note of caution, since outside this range of angles the coefficient \( \nu \) is negative and the theory is ill-posedGray & Edwards (2014). This is a signature of
the underlying ill-posedness of the $\mu(I)$-rheology at high and low inertial numbers Barker et al. (2014).

3. Numerical observations

The conservation equations (2.2)-(2.3) are numerically integrated using a solver based on the high-resolution non-oscillatory central scheme of Ref. 19, with the time evolution being carried out by a second-order Runge-Kutta method with a step size of $10^{-2}$ s.

The computational domain has periodic boundary conditions at the ends $x = 0$ and $x = L$, and the domain length $L$ is discretized over 40 grid points per metre (i.e., 2000 grid points for a typical domain with $L = 50$ m) unless otherwise stated. The results have been checked for consistency on higher resolution grids. The higher resolution helps to smooth the artificial spikes in $h(x,t)$ that can be observed in front of the wave peaks for our standard resolution, while in all other respects the results are the same. In view of the fact that simulations on the finer grid require much longer computing times, we choose the standard resolution mentioned above in order to be able to do as many runs as possible (because we need good statistics) on a sizeable domain.

The initial conditions are those of a randomly perturbed steady uniform flow,

$$h(x, t = 0) = h_0 + \epsilon H(x), \quad m(x, t = 0) = m_0 = h_0 \bar{u}_0,$$

where $m(x, t)$ represents the volumetric flux (per unit width of the chute) of the granular material. The uniform flow velocity $\bar{u}_0$ is related to $h_0$ via the relation (2.8) and $\epsilon = 10^{-4}$ m (= 0.1 mm) is the amplitude of the zero-mean perturbation $H(x)$, which picks a random value from the interval $[-1, 1]$ at each grid point. The uniform flow thickness is set to $h_0 = 1.2h_{\text{stop}} = 4.2$ mm in all simulations to ensure that the material keeps flowing continuously, with $Fr > \beta$ everywhere; the expression (2.4) for the friction coefficient $\mu(h, \bar{u})$ then holds throughout.

The system of governing equations (2.2)-(2.3) is written in the vector form of...
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Convection-diffusion equations required by the numerical scheme, as follows:

\[
\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{w})}{\partial x} = \mathbf{S}(\mathbf{w}) + \frac{\partial}{\partial x}(\mathbf{Q}(\mathbf{w}, \mathbf{w}_x)),
\]  

(3.2)

where \( \mathbf{w} = (h, m)^T \) is the vector of conserved variables \( h \) and \( m = h \bar{u} \). The vectors for the convection flux \( \mathbf{f} \), source term \( \mathbf{S} \) and diffusive flux \( \mathbf{Q} \) then take the form

\[
\mathbf{f} = \begin{pmatrix}
\frac{m^2}{h} + \frac{h^2}{2} g \cos \zeta \\
0
\end{pmatrix}, \quad
\mathbf{S} = \begin{pmatrix}
0 \\
h \bar{g} \left( \sin \zeta - \mu \frac{m}{|m|} \cos \zeta \right)
\end{pmatrix},
\]  

\[
\mathbf{Q} = \begin{pmatrix}
0 \\
\nu h^{1/2} \left( \frac{\partial m}{\partial x} - \frac{m}{h} \frac{\partial h}{\partial x} \right)
\end{pmatrix},
\]  

(3.3)

respectively, where the friction coefficient is given in terms of the conserved variables as

\[
\mu = \mu(h, m) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + \beta h^{5/2} \sqrt{g \cos \zeta} / (\mathcal{L} m)}.
\]  

(3.4)

In Fig. 5 we present the results of a sample simulation, carried out on a small domain of length \( L = 3.20 \) m, discretized over 3200 grid points, at several instants in time. Clearly, the upper plot (at \( t = 116.0 \) s) corresponds to an already advanced stage in the coarsening process. It shows two waves, A and B, which have originated from the coalescence of several smaller waves, which in turn emerged from the random initial conditions (3.1). The larger wave A travels with a greater speed than wave B, such that it catches up to B and merges with it, and from \( t = 155.3 \) s onward they are seen to form a single wave AB that is larger and faster than either of its composite parts. This is the basis of the coarsening behavior that is observed for larger wave trains on longer computational domains; the present example (Fig. 5) is only special insofar as the domain is sufficiently short to sustain a final 1-wave final state. That is, the coarsening in this case is not arrested in any intermediate multi-wave state but reaches its ultimate goal.

From the final two snapshots of Fig. 5 we also get a first glimpse of the second mechanism that will play an important role in our analysis of the coarsening process: The height of the wave AB does not maintain the exact value it has right after
Figure 5. Two roll waves A and B chasing each other in a chute of length 3.20 m with periodic boundary conditions. At $t = 155.3$ s the larger (and therefore faster) wave A is seen to catch up with B and the two waves merge. From this moment on, A and B continue as one roll wave (AB) forever.
the merging event (which happens to be 8.25 mm) but in the course of time relaxes to the somewhat lower value of 7.98 mm. This we call the equilibrium height of the roll wave, which depends on the various system parameters and the dimensions of the chute, as we shall see later.

The results of a typical simulation on a domain of length \( L = 50 \) m are shown in Fig. 6 at seven successive moments in time. Here the number of roll waves emerging from the perturbed uniform flow is approximately 80, which gradually decreases to a 6-wave state, at which point (for this particular run) the coarsening process is arrested; the six surviving waves are all seen to relax to the same equilibrium height 9.27 mm. The exact number of waves in the arrested state varies between the different simulations, owing to the randomness of the initial conditions. The coarsening process is most commonly arrested in a state with 6 or 7 waves. But arrested states with as many as 13 surviving states, or as few as 4, are also occasionally observed. The minimal value depends on the length of the chute: on the 50 m chute of Fig. 6 the coarsening process was never seen to get beyond the 4-wave states. On a similar chute with a periodic domain of \( L = 25 \) m (on which we also did extensive measurements) the corresponding minimal value was found to be 2 waves. The most commonly observed arrested state in this case consisted of 3 or 4 waves, and occasionally the coarsening process already came to a halt with as many as 10 waves.

In the context of coarsening, the key quantity of interest is the average lifetime \( \tau_N \) of the successive \( N \)-wave states. Obviously, this lifetime grows during the course of the coarsening process, i.e., it should be a decreasing function of \( N \). Figure 7 gives the measured values of \( \tau_N \) versus \( N \), collected from 100 simulation runs for periodic domains of length \( L = 50 \) m and \( L = 25 \) m. The \( \tau_N \) are averaged over the number of runs in which each \( N \)-wave state is observed. Especially toward the end of the coarsening process this number falls far below 100. Moreover, we exclude the arrested state of each run to avoid spurious divergences. (If one would include
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Figure 6. Snapshots of a long time numerical simulation for a chute of 50 m length, with periodic boundary conditions, illustrating the sequence of merging events up to the final stage where the coarsening process is arrested. (a) $t = 0$ s: The initial stage corresponding to a random perturbation of the uniform flow (the amplitude of the perturbation is too small to be visible on this scale), (b) $t = 74.4$ s: approximately 80 peaks, not necessarily fully developed roll waves yet, (c) $t = 122.8$ s: 40 roll waves, (d) $t = 176.6$ s: 20 roll waves, (e) $t = 373.2$ s: 10 roll waves, (f) $t = 510.6$ s: the start of the 6-wave state, and (g) $t = 40,000$ s: the same state at a much later time, when the 6 roll waves have become identical.
Figure 7. The numerically observed lifetimes $\tau_N$ (solid dots), averaged over the number of times each $N$-wave state is observed as a transient stage: (a) for a chute of length $L = 50$ m, where the coarsening is never observed to reach lower values than $N = 4$ (and hence there are no finite values of $\tau_N$ for $N \leq 4$), and (b) for a chute of length $L = 25$ m, where the minimal observed number of waves is $N = 2$ (and hence no finite lifetimes are found for $N \leq 2$). The error bars represent the standard error of each lifetime $\tau_N$. 
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these states, $\tau_N$ would diverge for all values of $N$ at which the coarsening is ever seen to be arrested; for the 50 m chute this would mean at all $N = 4, 5, \ldots, 13$.) The error bars in Fig. 7 represent the standard error of the measured data. Note that in both cases the average lifetime is seen to rise steeply towards the end of the coarsening process (for small values of $N$) and that there are no measurements at all for $N \leq 4$ and $N \leq 2$, respectively, in accord with our earlier observation that the coarsening is never seen to proceed beyond these values.

In the next Section we describe the mechanisms behind the coarsening process from which we subsequently derive a theoretical expression for $\tau_N$.

4. Two competing mechanisms

The $N$-wave state starts immediately after a merging event takes place in the $(N+1)$-wave state and ceases to exist after a time $\tau_N$ when a new merging occurs. At this point the $(N-1)$-wave state begins, and so on and so forth until finally the arrested state with $N = N_{\text{arr}}$ waves is reached.

There are two competing mechanisms which together set the lifetime $\tau_N$ of the $N$-wave state: On the one hand large roll waves travel faster than small ones, which leads to merging. On the other hand the waves (large and small) all tend to adjust their amplitude to a preferential height $h_{\text{eq}}$, which means that the velocity differences between them diminish in time, thereby decelerating the merging.

4.1. First mechanism: larger waves travel faster

Regarding the first mechanism, we will derive a quantitative relation between the wave speed $u_w$ and the height of the peaks by considering each roll wave as a discontinuous shock. That is, we consider the zero viscosity limit $\nu \to 0$, which is a sensible approximation, given the very steep slope at the shock front (see Fig. 8). The Rankine-Hugoniot conditions for the mass and momentum conservation across
the shock [cf. Eqs. (2.2) and (2.3)] are then given by (Gray & Cui 2007)

\[ [h(\bar{u} - u_w)]^+ = 0 \]  \hspace{1cm} (4.1)

and

\[ [h\bar{u}(\bar{u} - u_w) + \frac{1}{2}gh^2 \cos \zeta]^+_{-} = 0, \]  \hspace{1cm} (4.2)

respectively, where the jump bracket \([f]^+ - f_-\) denotes the difference in value of the enclosed quantity \(f\) on the forward ‘+’ and rearward ‘-’ sides of the jump, which travels downslope at the constant speed \(u_w\) of the traveling wave. The first jump condition (4.1) may be rearranged to give the rearward side depth-averaged velocity as

\[ \bar{u}_- = u_w + \frac{h_+}{h_-}(\bar{u}_+ - u_w). \]  \hspace{1cm} (4.3)

Replacing \(\bar{u}_-\) with (4.3) in the second jump condition (4.2) and canceling common factors \(h_+ - h_-\), which is permitted since \(h_+ \neq h_-\) (see Fig. 8), yields a quadratic equation for the wave speed \(u_w\):

\[ h_+u_w^2 - 2h_+\bar{u}_+u_w + h_+\bar{u}_+^2 - \frac{1}{2}g \cos \zeta (h_+ + h_-)h_- = 0, \]  \hspace{1cm} (4.4)

which has two roots:

\[ u_w(h_-) = \bar{u}_+ \pm \sqrt{\frac{1}{2}g \cos \zeta \left[ 1 + (h_-/h_+) \right]h_-}. \]  \hspace{1cm} (4.5)

Our numerical observations show that the flow thickness on the forward side of the jump is only slightly below the uniform flow thickness \(h_0\) (see Fig. 8), since the accumulated mass in the wave peak is taken evenly from across the wavelength. It is thus a reasonable approximation to set \(h_+ = h_0\) and \(\bar{u}_+ = \bar{u}_0\). Identifying the thickness on the rearward side of the shock with the peak height, \(h_- = h_w\), Eq. (4.5) then takes the form:

\[ u_w(h_w) = \bar{u}_0 \pm \sqrt{\frac{1}{2}g \cos \zeta \left[ 1 + (h_w/h_0) \right]h_w}, \]  \hspace{1cm} (4.6)

where we have kept only the root with the plus sign, since \(u_w > \bar{u}_0\) for a roll wave. This formula for the wave speed is depicted in Fig. 9 (solid line) together with numerical measurements on chutes of varying length. The agreement is seen
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Figure 8. The flow thickness $h_+$ on the forward side of the wave crest, here considered as an inviscid shock, is observed to be approximately equal to the uniform flow thickness $h_0$ (dashed line). The depicted wave, with a height on the rearward side of the jump of $h_- = 10.2$ mm, is one of two waves which remain in an arrested state after developing from a randomly perturbed uniform flow on a periodic chute of length $L = 25$ m; the snapshot is taken at a simulation time of $t = 465$ s. to be very good, justifying the various assumptions we made in the derivation of Eq. (4.6).

The analysis can be simplified further by linearizing the expression (4.6) about $h_w = h_0$. Up to first order in $(h_w - h_0)$ we then obtain:

$$u_w(h_w) = \bar{u}_0 + \sqrt{gh_0 \cos \zeta} + K(h_w - h_0) + ... = u_g + K(h_w - h_0) + ...,$$  (4.7)

where the dots indicate the neglected higher order terms in the small variable $(h_w - h_0)$. The velocity $u_g = \bar{u}_0 + \sqrt{gh_0 \cos \zeta}$ may be recognized as the speed of gravity waves in the uniform flow, and the constant $K$ is given by:

$$K = \frac{3}{4} \sqrt{\frac{g \cos \zeta}{h_0}}.$$  (4.8)

Figure 9 shows that the linear approximation (4.7) is in good agreement with both
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The speed of the roll wave $u_w$ as function of its peak height $h_w$. The solid dots are numerical data from a 50 m chute, the open dots correspond to a chute of 25 m, while the solid squares are collected from various short chutes with $0.3 < L < 3.2$ m. The solid curve is the theoretical prediction Eq. (4.6) from the inviscid shock wave approximation; the straight dashed line is a linearization of the latter about $h_w = h_0$, given by Eq. (4.7).

Figure 9. The speed of the roll wave $u_w$ as function of its peak height $h_w$. The solid dots are numerical data from a 50 m chute, the open dots correspond to a chute of 25 m, while the solid squares are collected from various short chutes with $0.3 < L < 3.2$ m. The solid curve is the theoretical prediction Eq. (4.6) from the inviscid shock wave approximation; the straight dashed line is a linearization of the latter about $h_w = h_0$, given by Eq. (4.7).

We have established the relation between the speed of a granular roll wave and its peak height.

4.2. Upper bound on the wave peak height

In the same inviscid limit ($\nu \to 0$) it can be shown that the peak height of the roll waves must always remain smaller than a certain upper bound. This is of crucial importance in the context of the present paper, since the arrested coarsening is a direct consequence of it: in a long chute the granular material cannot accumulate in a single roll wave (because the peak height of this wave would violate the upper bound) and therefore the mass must necessarily be distributed over two or more roll waves. Our derivation of the upper bound starts from the observation that the roll wave is a traveling wave solution of Eqs. (2.2)-(2.3), which we exploit (in the
spirit of Dressler 1949) to gather information on the characteristic shape of the wave.

A traveling wave solution to the equations of motion (2.2)-(2.3), propagating in the positive $x$ direction at speed $u_w$, is most conveniently described in the variables of the co-moving frame:

$$\xi = x - u_w t, \quad \tau = t. \quad (4.9)$$

Thus we follow the wave in the coordinate system $(\xi, \tau)$ that remains at all times centered at the wavefront; in this frame $h = h(\xi, \tau)$ and $\bar{u} = \bar{u}(\xi, \tau)$. As a matter of fact, for a traveling wave in the steady state we demand $\partial h / \partial \tau = 0$ and $\partial \bar{u} / \partial \tau = 0$.

The continuity and momentum equations (2.2)-(2.3) then reduce to

$$\frac{d}{d\xi}((\bar{u} - u_w)h) = 0, \quad (4.10)$$

$$\frac{d}{d\xi}(\bar{u} - u_w)h \frac{d\bar{u}}{d\xi} = hg \sin \zeta - \mu hg \cos \zeta - hg \cos \zeta \frac{dh}{d\xi}, \quad (4.11)$$

where in the latter equation we have taken the inviscid limit $\nu \to 0$, and divided throughout by $h$. Note that, as a consequence of the fact that we now focus on steady traveling wave solutions, all partial derivatives of the original equations have been replaced by the straight derivative $d/d\xi$ (since $\xi$ is the only remaining variable): the equations (4.10)-(4.11) are ordinary differential equations (ODEs).

The continuity equation (4.10) tells us that the discharge rate of granular material (or volumetric flux per unit width) with respect to the co-moving frame of reference is constant:

$$(\bar{u} - u_w)h = M. \quad (4.12)$$

The value of the constant $M$ can be determined by substituting the values of $h$ and $\bar{u}$ at any point along the wave. Taking the point where $h = h_+$ and $\bar{u} = \bar{u}_+$ (Fig. 8) we see that $M = (\bar{u}_+ - u_w)h_+$ is in fact negative, because $u_w$ is always larger than any other velocity by the definition of the roll wave.

For waves of a relatively long wavelength (of the order of 1 m and greater, as in Fig. 8), we observe that the peak of the roll wave is followed by a long tail of
practically uniform thickness. The flow thickness in this whole region is indistinguishable from \( h = h_+ \). The corresponding velocity \( \bar{u}_+ \) may thus be identified as the velocity of a flow of uniform thickness \( h_+ \), suggesting that we may use Eq. (2.8) to write:

\[
(\bar{u} - u_w)h = M = (Bh_+^{3/2} - u_w)h_+,
\]

where the constant \( B = \beta \sqrt{g \cos \zeta / (\mathcal{L} \gamma)} \) has been introduced in Section II. Rearranging Eq. (4.13) gives the depth-averaged velocity \( \bar{u} \) everywhere as a function of flow thickness \( h \):

\[
\bar{u} = u_w + \frac{h_+}{h} \left( Bh_+^{3/2} - u_w \right),
\]

which upon substitution into the momentum equation (4.11) gives a first order ODE for the flow thickness,

\[
\frac{dh}{d\xi} = \frac{h^3 g (\sin \zeta - \mu \cos \zeta)}{h^3 g \cos \zeta - h_+^2 \left( Bh_+^{3/2} - u_w \right)^2}.
\]

Somewhere along the roll wave profile, at the rear slope of the wave, the flow thickness attains the critical value \( h = h_{\text{crit}} \), where the denominator in the above expression is zero. At this point the Froude number measured in the co-moving frame (\( \text{Fr}_{\text{co}} = |\bar{u} - u_w| / \sqrt{gh \cos \zeta} \)) passes through 1, separating the wave in two fundamentally different sections: one section where the height exceeds \( h_{\text{crit}} \) and hence \( \text{Fr}_{\text{co}} < 1 \) (subcritical flow), and one section where \( h < h_{\text{crit}} \) and hence \( \text{Fr}_{\text{co}} > 1 \) (supercritical flow). Note that in the co-moving frame we are considering here, owing to the fact that the flux \( M \) is negative, the grains move to the left, from the shock front to the tail.

When the denominator in (4.15) becomes zero, it is necessary to also set the numerator to zero, in order to prevent the gradient \( dh/d\xi \) from becoming infinite (Dressler 1949). The numerator is zero for \( \tan \zeta = \mu \). Now, the friction coeffi-
cient $\mu$ defined by Eq. (2.4) can be expressed in terms of $h$ only by using Eq. (4.14):

$$\mu = \mu(h) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + \beta h^{5/2} \sqrt{g \cos \zeta}} \mathcal{L} \left[ u_w(h - h_+) + Bh_+^{5/2} \right].$$

(4.16)

and with this, the condition $\tan \zeta = \mu(h)$ at the critical point takes the following form:

$$Bh_+^{5/2} - u_w(h - h_+) - Bh_+^{5/2} = 0,$$

(4.17)

where we have made use of the definitions of the parameters $\gamma$ [see Eq. (2.7)] and $B$ [see below Eq. (4.13)]. Rearranging for the wave speed we obtain:

$$u_w = \frac{B \left( h_+^{5/2} - h_+^{5/2} \right)}{h - h_+}.$$

(4.18)

Similarly, setting the denominator of (4.15) equal to zero:

$$h^3 g \cos \zeta - h_+^2 \left( Bh_+^{3/2} - u_w \right)^2 = 0,$$

(4.19)

and solving for $u_w$ gives (we keep only the negative square root because only this corresponds to an admissible wave velocity $u_w > \bar{u}_+$):

$$u_w = Bh_+^{3/2} + \frac{h_+^{3/2} \sqrt{g \cos \zeta}}{h_+}.$$

(4.20)

Note that this wave speed holds for the entire roll wave (not only for the points featuring in the derivation of the above expression where $h = h_+$ or $h = h_{crit}$) since the ripened wave moves steadily forward, as a whole, without change of shape.

Combining (4.18) and (4.20), and excluding the solution $h = 0$ (since $h_{crit}$ must obviously be larger than zero), after some algebra yields:

$$(Bh_+ - \sqrt{g \cos \zeta})h_+^{3/2} + h_+ \sqrt{g \cos \zeta} h_+^{1/2} - Bh_+^{5/2} = 0,$$

(4.21)

which is satisfied at the critical thickness $h = h_{crit}$ where both the numerator and denominator of Eq. (4.15) are zero. It is also trivially satisfied for the uniform thickness $h = h_+$ in the tail of the roll wave, but this is not the solution we are looking for here.
The critical thickness $h_{\text{crit}}(h_+)$ which follows from Eq. (4.21) is an increasing function of $h_+$. Consequently, it attains its maximum value when $h_+$ is as close as possible to its upper bound $h_0$. The peak mass is then the accumulation of an extremely thin layer of grains taken away from the uniform flow thickness $h_0$; the corresponding domain length must obviously be quite long. The absolute, never attainable upper bound for the critical thickness (denoted by $h_{\text{crit}}^{\max}$) is obtained by setting $h_+ = h_0$ and in our system turns out to be equal to $h_{\text{crit}}^{\max} = 8.9$ mm.

The associated upper bound to the wave speed is found by substituting this critical thickness $h_{\text{crit}}^{\max}$ into either (4.18) or (4.20), yielding $u_{w}^{\max} = 0.7$ m/s. This in turn sets the upper bound for the peak height of the roll waves, which we will denote by $h_{w}^{\max}$. Substituting the wave speed $u_{w}^{\max}$ into our formula for $u_{w}(h_{w})$, Eq. (4.6), and solving for the peak height, we finally obtain

$$h_{w}^{\max} = \frac{h_0}{2} \left( -1 + \sqrt{1 + \frac{\left( \frac{u_{w}^{\max}}{2} - Bh_{0}^{3/2} \right)^2}{2gh_{0}\cos\zeta}} \right), \quad (4.22)$$

which for the parameter values in our system gives $h_{w}^{\max} = 16.2$ mm.

The above result is an absolute upper bound on the peak height of a traveling roll wave solution to the inviscid governing equations. It is larger than any of the roll waves observed in our numerical simulations (the largest one of which had a peak height $h_{w} = 10.4$ mm) for several reasons: (i) the effect of the viscous-like term in the full system of equations (2.2)-(2.3) will round off the peaks, (ii) a finite layer of granular matter must be taken away from the uniform flow thickness (to construct the wave peak) which means that the actual value of $h_{\text{crit}}$ will in practice always be smaller than $h_{\text{crit}}^{\max}$ and this in turn leads to a smaller peak height, and (iii) the finite resolution of our simulations slightly underestimates the actual peak heights. We have performed several sample simulations with a higher resolution and these yield roll waves that have a somewhat larger peak height than those with the standard resolution.

The existence of a maximum peak height is an important result: it means that
the waves cannot grow indefinitely. The fact that the coarsening process in long
chutes is arrested in a state with more than 1 wave is an immediate consequence
of this.

To close this section, we note that the above analysis involving the critical thick-
ness $h_{\text{crit}}$ is not restricted to the absolute maximum found by setting $h_+ = h_0$.
Indeed, the wave speeds and peak heights found by solving Eqs. (4.21)-(4.22) with
the numerically observed values of $h_+$ are all in very good agreement with the
previously derived relationship shown in Fig. 9.

4.3. Second mechanism: wave height relaxation

As for the second mechanism, the equilibrium wave height $h_{\text{eq}}$ gradually grows as
the coarsening process proceeds towards smaller values of $N$, just as one would
expect. As a matter of fact, our numerical simulations reveal a universal trend
indicating that $h_{\text{eq}}$ is an increasing function of the average separation distance
($s = L/N$) between successive waves, rather than of $N$ alone. When the equilibrium
heights $h_{\text{eq}}$ are plotted as a function of $s$ for chutes of different length $L$ (as in
Fig. 10) all data are seen to collapse onto a single curve:

$$h_{\text{eq}}(s) = h_0 + C s^\eta,$$

(4.23)

where $h_0$ is the thickness of the unperturbed bed, and $C$ and $\eta$ are fit parameters.
The best fit to our numerical data (given by the solid curve in Fig. 10) was obtained
with a prefactor $C = 0.0019$ and exponent $\eta = 0.45$.

The curve stops at $s \approx 15$ m. The measurement at $s = 15$ m in Fig. 10 cor-
responds to an arrested 1-wave state in a 15 m periodic chute, and this is the
largest wavelength we have ever observed. For longer chutes (with a periodic do-
main length $L > 15$ m) the number of waves in the arrested state is always greater
than 1. So when $15$ m $< L < 30$ m the coarsening process will be arrested in a state
with at least 2 waves. The 2-wave case (with a separation distance $L/2$ varying
between 7.5 m and 15 m) is the most ripened form for this domain length, and in
Figure 10. The equilibrium peak height $h_{eq}$ as a function of the average separation distance between the waves, $s = L/N$. The data from chutes with different domain lengths $L$ are seen to collapse onto a single universal curve; the solid line represents the fit to the data given by Eq. (4.23). It stops at $s = 15$ m, which is the maximal separation distance (for the parameters used in this paper). The solid diamonds correspond to a domain length $L = 15$ m, the open dots to $L = 25$ m, the solid dots to $L = 50$ m, and the open diamonds to $L = 100$ m. In the limit $s \to 0$ the peak height $h_{eq}$ reduces to the steady uniform flow thickness $h_0$ (dashed horizontal line): the amplitude of the roll wave ($N = 1$) vanishes, and hence the wave ceases to exist, as the chute length $L$ tends to zero.

In practice the coarsening will be often arrested in a state with several more waves depending on the initial, random perturbations of the uniform flow; note that this agrees precisely with our observations of the arrested states for the 25 m chute of Fig. 7b. Similarly, for $30 < L < 45$ m the coarsening will stop in a state with at least 3 waves (with $s = L/3$ varying between 10 m and 15 m) and usually several more. This periodic extension of the basic 15 m domain can be continued indefinitely. For instance, if $45 < L < 60$ m the arrested state will consist of minimally 4 roll waves (with $s = L/4$ varying between 11.25 m and 15 m), and
usually several more, in full agreement with the numerical data for our standard chute with $L = 50$ m.

The existence of a maximal separation distance (15 m for the parameters we are using) tells us that there is a natural length unit in the direction of the flow: increasing the length $L$ of the domain beyond this value is equivalent to periodically extending the unit system. The maximal separation distance, or wavelength, in a chute of any domain length $L$ can be inferred from the curve in Fig. 10. For increasing $L$, as the examples of the previous paragraph show, the maximal wavelength is selected from an ever-decreasing part of the upper end of this curve. In the limit $L \to \infty$, for the rarely observed arrested state with maximal wavelength (i.e., minimal number of waves), all roll waves will have a separation distance close to 15 m and a peak height of about 10.5 mm. Note that these numbers depend on the parameters in the governing equations and constitutive relations (such as $g$, $\gamma$ and $\mathcal{L}$) and in the concluding section we will briefly discuss how our results can be upscaled to be compatible with real geophysical flows, where granular roll waves are observed to reach amplitudes up to one meter or more.

Now let us turn our attention to a typical $N$-wave state during the coarsening process. During the lifetime of such a state, the roll waves tend to adjust their peak height to the equilibrium value $h_{eq}(s)$, by shedding off or picking up mass. This is illustrated in Fig. 11 for a 4-wave state in a periodic chute of 50 m; for clarity we here show the ripening of a final arrested state, which means that the convergence to $h_{eq}(s)$ is not interrupted by any merging event as it would for an intermediate $N$-wave state. The peak heights of the four waves are seen to oscillate, as they shed off and pick up material, and collectively converge to the value $h_{eq}(50/4) = 10.1$ mm.

A particularly insightful way to view the relaxation process is presented in Fig. 12a, which shows the ripening of the roll waves in $(h, dh/d\xi)$ phase space (where $\xi$ is the traveling wave coordinate introduced in the previous subsection). It is seen that the trajectories corresponding to the profiles of the developing roll
Figure 11. Relaxation of the peak heights of four roll waves toward the equilibrium value $h_{eq}(50/4) = 10.1$ mm in a chute of length $L = 50$ m. The time $t = 0$ s coincides with the merging event that commences the 4-wave state (which happens to be the state in which the coarsening process is arrested in this particular run). The oscillations correspond to the periodic shedding off and picking up of mass by the waves in their attempt to adjust their heights to $h_{eq}(50/4)$. The dashed envelope curves indicate the exponential decay (and growth) described by Eq. (4.25), with parameters $\lambda = 0.0021$ s$^{-1}$ and $\alpha = 0.07$.

Waves converge towards a stable limit cycle (Gray & Edwards 2014). Waves that are initially too large start in the outer part of the shaded region: they spiral inward to the limit cycle, cross it somewhere along the segment E-F, and then proceed in the inner part of the shaded region until they cross the segment E-F again, and so on. The amplitude of the excursions from the limit cycle decreases gradually until the trajectory becomes indistinguishable from the limit cycle itself. This explains the oscillatory convergence of the peak heights observed in Fig. 11; the peak height $h_{eq}(s)$ corresponds to the point indicated by the letter A in Fig. 12a, where $h(\xi)$ is maximal (10.1 mm) and $dh/d\xi = 0$.

Waves that are initially too small exhibit the same behavior, only now starting out from a position in the inner part of the shaded region. The start of an arrested
Figure 12. (a) In \((h, dh/d\xi)\) phase space, the ripening of roll waves to their equilibrium shape shows up as a convergence of the waves’ phase space trajectories to a stable limit cycle (solid closed curve). (b) The start of the final 2-wave state on a 25 m chute, just after the merger of the large leading wave with a third wave. These waves correspond to the boundaries I and II of the shaded area in the phase space plot (a), and in the course of time convergence towards the limit cycle, exploring the entire shaded region along the way. (c) Connection between the limit cycle and the profile of the same two roll waves in the long time limit, when they have reached the (arrested) equilibrium state. The points A-F along the roll waves correspond to the same six points on the limit cycle.
2-wave state is shown in Fig. 12b, where the waves I (initially too small) and II (initially too large) correspond to the boundaries I and II of the shaded area in Fig. 12a.

The oscillatory convergence to the equilibrium height is only observed when there are at least 2 waves in the system, and is a manifestation of the exchange of mass between the waves (by shedding off material, which is then being picked up by the next). In a situation with only 1 wave, such as may be encountered in a chute with a short domain length, the convergence is monotonous.

The letters B-F indicate five further characteristic points along the wave, and Fig. 12c shows the full correspondence between the limit cycle in \((h, \frac{dh}{d\xi})\) phase space and the structure of the ripened roll wave. At point B the derivative \(\frac{dh}{d\xi}\) goes through its minimum value, corresponding to the position on the steeply descending wavefront where the profile turns from being convex to concave. Point C is located near the foot of the wavefront, where \(h\) passes through the value \(h_0 = 4.2\) mm, soon thereafter to be followed by point D where the wave attains its lowest level \(h = 3.95\) mm. The part of the limit cycle between D and E, small as it may be, corresponds to the long (practically horizontal) tail of the roll wave. Finally, at point E the height \(h\) passes through the value \(h_0\) again and from here rises all the way to points F and A, after which the cycle is repeated.

Point F (where the derivative \(\frac{dh}{d\xi}\) goes through its maximum value) corresponds to the position on the upward flank of the wave where the profile changes from concave to convex. Interestingly, the segment of the limit cycle between E and F is seen to be practically straight: \(h = a + b\frac{dh}{d\xi}\), from which it can be inferred that the wave profile from E to F in Fig. 12c rises (in good approximation) exponentially, as \(h - a \propto \exp(\xi/b)\).

According to standard stability theory, the nearby phase space trajectories will approach the limit cycle in such a way that their distance to the limit cycle decreases exponentially in time, as \(\exp(-\lambda t)\), where \(-\lambda\) is the Lyapunov exponent.
In agreement with this, we find that the peak heights of the waves converge exponentially to the equilibrium value $h_{eq}$ exponentially (with $t_N$ denoting the moment when the $N$-wave state comes into existence):

$$h_{w,i}(t) - h_{eq} = [h_{w,i}(0) - h_{eq}] e^{-\lambda(t-t_N)},$$

(4.24)

for all waves $i = 1, 2, ..., N$. A similar relaxation phenomenon was observed by Chang, Demekhin & Kalaidin (2000) in a model for roll waves in water. The convergence rate (or negative Lyapunov exponent) $\lambda$ appears to be fairly independent of $N$. For sufficiently long chutes it also does not depend sensitively on the domain length $L$; from our simulations on the 50 m chute we find $\lambda = 0.0021 \text{ s}^{-1}$ and the same value fits the data from the 25 m chute reasonably well. For significantly shorter chutes, in which the waves are recycled more frequently and therefore interact more strongly, the adjustment to the equilibrium height is somewhat faster, corresponding to a higher value of $\lambda$.

For a “typical” wave, whose peak height (according to our simulations) at $t = t_N$ deviates only a few per cent from the target value $h_{eq}(s)$, the above relaxation to the equilibrium height can also be expressed as follows:

$$|h_{typ}(t) - h_{eq}| = \alpha h_{eq} e^{-\lambda(t-t_N)},$$

(4.25)

with $\alpha$ (the typical deviation) being of the order of 0.07, i.e., seven per cent. This is in fact the value which has been used in Fig. 11 to construct the dashed envelope curves. The parameter $\alpha$ is intrinsically noisy, conveying the memory of the random initial conditions, and in our various numerical runs is found to vary between 0.05 and 0.09 (without any systematic dependence on $N$ or $L$). Within these bounds we may therefore treat it as a free fit parameter.

Two different time scales can be assigned to the above two competing mechanisms: (i) the target time $t_{target,N}$, being the average distance between two successive waves divided by their typical velocity difference at the start of the $N$-wave state (which in the absence of the second mechanism would be equal to the catch-
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up time), and (ii) the relaxation time \( t_{\text{relax}} = 1/\lambda = 480 \text{ s} \) associated with the convergence to the preferential peak height. In the early stages of the coarsening process, when the average distance between the waves \( s = L/N \) is small, the time scale \( t_{\text{target},N} \) is much smaller than \( t_{\text{relax}} \) and we observe a number of merging events in rapid succession; the value of \( N \) thus quickly decreases. As time goes by, however, the system enters a different regime. When \( t_{\text{target},N} \) becomes of the same order of magnitude as \( t_{\text{relax}} \), and the influence of the relaxation is increasingly felt, the coarsening process slows down drastically. Some time later, when \( t_{\text{target},N} \) significantly exceeds \( t_{\text{relax}} \), the waves are given the opportunity to adjust their height to the equilibrium height before any merging event takes place. All the remaining waves in the system then travel with the same speed (because they all have the same height) and the coarsening process comes to a halt. The average distance between the waves in this arrested state, denoted \( s_{\text{arr}} \), may be viewed as the natural wavelength of the system, and the physical interpretation of the coarsening process from this perspective is that of an intricate mechanism for spontaneous wavelength selection.

The corresponding value of \( N \), which we call \( N_{\text{arr}} \), is derived in the next Section together with a theoretical expression for the lifetimes \( \tau_N \) that quantitatively reproduces the simulation results of Section III.

5. Derivation of the Coarsening law

5.1. Typical lifetime of the \( N \)-wave state

In the \( N \)-wave state, the average distance separating the waves is \( s = L/N \). This is the distance the larger wave must gain in order to catch up with the smaller one. The time it takes for this to happen (denoted by \( T_{\text{av}} \)) is determined by the velocity difference between the two waves, \( \delta u \), as follows:

\[
\int_0^{L/N} ds = \int_0^{T_{\text{av}}} \delta u \, dt. \tag{5.1}
\]
where the time variable $t$ has been put to zero at the start of the $N$-wave state. With $\delta u = K\delta h$ by (4.7) the above equation can also be written as:

$$\frac{L}{N} = K \int_0^{T_{av}} \delta h \, dt. \quad (5.2)$$

where $K$ is the constant given by Eq. (4.8); its value for the present setup is $K = 32.8 \, \text{s}^{-1}$. 

Now we use Eq. (4.25) to set $\delta h(t) = \alpha h_{eq}(s) \exp(-\lambda t)$ (the typical peak height difference between two neighboring waves in the system). With these substitutions the integral in Eq. (5.2) can be evaluated directly:

$$\frac{L}{N} = \alpha Kh_{eq}(s) \int_0^{T_{av}} e^{-\lambda t} \, dt = \frac{\alpha}{\lambda} Kh_{eq}(s) \left( 1 - e^{-\lambda T_{av}} \right) \quad (5.3)$$

or, equivalently, with $L/N = s$:

$$T_{av} = -\frac{1}{\lambda} \ln \left( 1 - \frac{\lambda s}{\alpha Kh_{eq}(s)} \right). \quad (5.4)$$

What we have done until now is to evaluate the catch-up time for an average pair of waves, i.e., with average distance and average velocity difference. This is evidently not the same as the lifetime $\tau_N$, which is the catch-up time for the pair that merges first (since this event ends the $N$-wave state and heralds the beginning of the state with $N - 1$ waves). In other words, $\tau_N$ is the minimum catch-up time from the ensemble of all wave pairs.

Now, as long as $N$ is sufficiently large, it is reasonable to assume that the time $T_{av}$ represents the average over a set of statistically independent catch-up times $T_i$, $i = 1, 2, .., N$, which may be assumed to follow an exponentially decaying probability distribution $P(T_i) = T_i^{-1} \exp(-T_i/T_{av})$. A basic property of the exponential distribution is that the expected minimum value (i.e., the smallest element in a set of $N$ randomly chosen numbers obeying this distribution) decreases with the sample length as $1/N$. That is, the expected minimum $T_i$-value in the set (alias $\tau_N$) is $T_{av}/N$. So we obtain:

$$\tau_N = \frac{T_{av}}{N} = -\frac{1}{\lambda N} \ln \left( 1 - \frac{\lambda s}{\alpha Kh_{eq}(s)} \right), \quad (5.5)$$
or equivalently, bringing out the dependence on $N$ and $L$ as explicitly as possible:

$$
\tau_N = -\frac{1}{\lambda N} \ln \left(1 - \frac{\lambda(L/N)}{\alpha K [h_0 + C (L/N)^\eta]}\right),
$$

(5.6)

where we have rewritten the equilibrium height $h_{eq}(s)$ in the form of Eq. (4.23) and have replaced $s$ by $L/N$.

### 5.2. The maximal wavelength $s_{arr}$ and the associated minimum number of roll waves

One of the most striking features of the coarsening law derived above is that $T_{av}$ (and hence also $\tau_N$) exhibits asymptotic behavior. That is, its value becomes infinite at a certain wavelength $s = s_{arr}$ where the argument of the logarithm in Eq. (5.4) becomes zero:

$$
s_{arr} = \frac{\alpha K h_{eq}(s_{arr})}{\lambda}.
$$

(5.7)

By virtue of Eq. (4.23) this can also be written as:

$$
\alpha K C s^{\eta}_{arr} - \lambda s_{arr} + \alpha K h_0 = 0,
$$

(5.8)

which can readily be solved numerically. The number of waves in the arrested state, $N = N_{arr}$, is simply given by $N_{arr} = L/s_{arr}$.

Note that $s_{arr}$ is fully determined by the various coefficients characterizing the two competing mechanisms (overtaking and relaxation) that between themselves govern the progress of the coarsening. Given the values of the various parameters concerning overtaking ($K = 32.8 \text{ s}^{-1}$), relaxation ($\lambda = 0.0021 \text{ s}^{-1}$, $C = 0.0019 \text{ m}^{1-\eta}$ and $\eta = 0.45$) and the estimated spread of the peak heights ($\alpha = 0.07 \pm 0.02$) we find that the maximal wavelength in the arrested state can assume any value between $s_{arr} = 6.789$ (for $\alpha = 0.05$) and $s_{arr} = 14.885$ (for $\alpha = 0.09$). Its value is seen to depend quite sensitively on the fit parameter $\alpha$. It is an increasing function of the noisy parameter $\alpha$, which stands to reason, because a larger spread of the peak height enhances the coarsening process, resulting in an arrested state with less roll waves, or equivalently a larger wavelength $s_{arr}$. For very small spread ($\alpha \to 0$) the
value of $s_{arr}$ tends to zero, and consequently the equilibrium height $h_{eq}$ tends to $h_0$ (as seen in Fig. 10). This implies that the coarsening process will never really get started in this limit, simply because the roll wave formation is suppressed.

The value $s_{arr} = 14.885$, for the relatively large spread $\alpha = 0.09$, is in good agreement with the record wavelength of 15 m (observed in a 15 m chute). The best correspondence with our numerical data for the chutes of 50 m and 25 m (for which $s_{arr}$ is observed to be $L/N = 50/4 = 25/2 = 12.5$ m) is found by choosing a value close to $\alpha = 0.08$, or to be precise $\alpha = 0.07918$. The associated values of $N_{arr}$ are 4 and 2, respectively.

These values yield the vertical asymptotes in Fig. 13, where we compare the theoretical expression (5.6) for the lifetime $\tau_N$ with the numerically obtained lifetimes both for the 50 m chute (Fig. 13a) and the 25 m chute (Fig. 13b). Evidently, $\tau_N$ diverges at the asymptotic values $N = N_{arr}$.

The overall agreement between the theoretical prediction for $\tau_N$ (solid curve) and the numerical data is seen to be quite satisfactory. Interestingly, the accuracy of the prediction is in general better for relatively large values of $N$ than for small ones.

This is partly due to the poorer statistics of the numerical results at small values of $N$. Another reason is that the assumption of the statistical independence of the merging events (a central ingredient of our model, lying at the basis of the step in which we took $\tau_N = T_{av}/N$) becomes increasingly audacious when the number of waves decreases.

Given that the measured lifetimes in Figs. 13a and b follow the trend described by Eq. (5.6), we can infer a deeper connection between the two plots, namely that they are rescaled versions of one another, confirming our earlier observation that the coarsening process on both chutes takes place in essentially the same way.

More specifically, according to our theory, the lifetimes $\tau_N$ for the 25 m chute are twice as large as $\tau_{2N}$ for the 50 m chute. The different spans of the horizontal
Figure 13. Comparison of the numerically observed lifetimes $\tau_N$ (dots with error bars, cf. Fig. 7) with the theoretical prediction given by Eq. (5.6) (solid curve) for periodic chutes of length (a) $L = 50$ m and (b) $L = 25$ m. The values of the various coefficients in the theoretical expression are given in the text. The vertical dashed lines denote the asymptotes where the lifetime $\tau_N$ diverges: for the 50 m chute this happens at $N_{\text{arr}} = 4$ (meaning that the coarsening is arrested, at the very latest, in a 4-wave state) and for the 25 m chute at $N_{\text{arr}} = 2$ (corresponding to a 2-wave state).
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Axes in Figs. 13a and b are chosen such as to allow a direct comparison: the solid curve in Fig. 13b lies twice as high as the corresponding curve in Fig. 13a, and is identical in all other respects. This can be traced back to the fact that Eq. (5.6) for $\tau_N$ is invariant under the joint transformation

$$L \rightarrow kL, \quad N \rightarrow kN, \quad \tau_N \rightarrow \frac{1}{k} \tau_N. \quad (5.9)$$

That is to say, if the domain length is stretched by a factor $k$ and the number of waves is multiplied by the same factor, then the lifetime of the newly constructed $kN$-wave state is $1/k$ times that of the $N$-wave state on the original chute of length $L$. This factor $1/k$ has to do with the aforementioned property for a sample of statistically independent catch-up times $T_i$: if the number of elements in the sample becomes $k$ times as large, without changing the average value $T_{av}$, the expected value of the smallest element is multiplied by $1/k$. Note that the average value $T_{av}$ [Eq. (5.4)] remains indeed unchanged owing to the fact that the average separation distance between the waves, $s = L/N$, is an invariant quantity under the transformation (5.9).

In short, the results for the 50 m chute can be inferred directly from those for the 25 m chute via the above transformation with $k = 2$. One should only keep in mind that this correspondence holds better for large values of $N$ than for small ones, since it hinges again on the statistical independence of the catch-up times. In our system, this independence appears to be well obeyed as long as the average separation distance $s = L/N$ does not exceed a critical value of about $s_* \approx 2.5$ m (corresponding to the state with $N_* = L/s_*$ waves).

As a matter of fact, following this line of thought, the duration $t_*$ of the coarsening process from its beginning until the state with $N_*$ waves, may be anticipated to be the same for any domain length. If the process passes from $N = N_{init}$ to $N_*$ for a chute of domain length $L$, the corresponding process on a stretched domain of length $kL$ passes through all states from $N = kN_{init}$ to $kN_*$. The fact that it has to pass through $k$ times as many states is precisely compensated by the fact
that these states live shorter by a factor $1/k$. The duration $t_*$ is equal to the sum of all lifetimes from the start of the process to the state with $kN_*$ waves:

$$t_* = \tau_{(kN_*)} + \tau_{(kN_*+1)} + \ldots + \tau_{(kN_{\text{init}})} = \sum_{kN_*}^{kN_{\text{init}}} \tau_N \approx \int_{kN_*}^{\infty} \tau_N dN,$$

(5.10)

where in the last step we have used that $kN_{\text{init}}$ is typically quite large (and that the corresponding lifetime $\tau_{kN_{\text{init}}}$ is vanishingly small). Note that the above integral, thanks to the transformation (5.9), is independent of the value of $k$. This makes these early stages of the coarsening process analogous to Christiaan Huygens’ classic tautochrone pendulum, the period of which is independent of the amplitude of its oscillations.

6. Concluding remarks

We have studied the coarsening dynamics of roll waves in granular material flowing down a rough inclined channel of length $L$, with periodic boundary conditions, focusing upon the fact that the coarsening process is arrested before it arrives at the intuitively expected 1-wave state. We have explained this by taking into account two competing mechanisms: (i) the merging of waves, in which larger waves overtake the smaller ones and swallow them, and (ii) the tendency of all waves in the $N$-wave state to relax toward an equilibrium height (and corresponding equilibrium velocity) that depends on the average separation distance $s = L/N$ between the waves. The interplay between these two rival mechanisms was used to derive the expected lifetime $\tau_N$ of the $N$-wave state, Eq. (5.6). This predicted lifetime agrees well with numerical observations (see Fig. 13), including the divergence of $\tau_N$ at a certain value $N = N_{\text{arr}}$, at which point the coarsening process is arrested. As a matter of fact, we found that the state with $N_{\text{arr}}$ waves is attained only when the initial conditions happen to be such that the coarsening process runs its full course, whereas in practice it usually gets stuck in a state with several more waves. In the final state (either with $N_{\text{arr}}$ waves or more) all surviving roll
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Figure 14. Roll wave formation on a non-periodic chute of 5km length, with an inflow thickness \( x = 0 \) km that fluctuates randomly around \( h_0 = 0.42 \) m. This simulation shows that the height of the roll waves scales roughly linearly with the value of \( h_0 \), the value of \( h_0 \) is precisely 100 times as large as in all previous simulations, and the peak heights attained at the exit \( x = 5 \) km are of the order of 1 m, i.e., almost 100 times as large as before. The dimensions of this upscaled version of our system (as well as the fact that we do not use periodic boundary conditions) have been chosen such as to enable a comparison with real geophysical flows.

Waves eventually attain exactly the same height and velocity and chase each other indefinitely around the system. The average distance between them then does not change anymore, and thus the whole coarsening process may be interpreted as a selection mechanism to arrive at the preferential wavelength of the system. The specific wavelength \( s_{arr} = L/N_{arr} \) (corresponding to the minimal number of waves in the system) defines the upper bound of the average separation distance between the roll waves.

Although all our simulations have been carried out on periodic chutes of finite domain length \( L \), the results can be translated to arbitrarily long chutes by making use of the fact that any chute can be regarded as a repetition of elementary cells of length \( s_{arr} \) (which in our system never becomes larger than 15 m). This has to
do with the fact that the equilibrium peak height cannot grow beyond a certain threshold value or, equivalently, that the maximum wavelength cannot grow indefinitely (see Fig. 10); whenever a merging event momentarily produces a wave that exceeds these bounds, the mass of this wave has to be re-distributed by shedding off material that is picked up by the waves that follow. In this way the coarsening process leads to a natural, self-organized length scale in the flow direction.

For instance, the chutes with \( L = 50 \) m and \( 25 \) m can both be regarded as multiple copies of an elementary cell of length 12.5 m. This is clearly illustrated by Fig. 13 where we see that the coarsening is (at the very latest) arrested in a state with 4 and 2 waves, respectively. In the same way, any arbitrarily long chute can be viewed as consisting of repeated copies of an elementary unit cell. The length of the unit cell may vary but (for the parameter values used in the present paper) will never exceed 15 m; the corresponding maximum peak height in the arrested state is 10.4 mm. Even in the limit \( L \to \infty \), where the periodic boundary conditions lose their physical meaning and the system becomes equivalent to an unbounded linear chute, one will never observe larger roll waves. We checked this numerically via a simulation on a chute with length \( L = 5000 \) m (but with all other parameters unaltered) and also here we found that the peak height of a fully developed roll wave never exceeded 10.4 mm.

To find higher roll waves, one has to insert more mass in the system. This is easily achieved by upscaling the value of the uniform flow thickness \( h_0 \). If one does so, the peak height of the roll waves is found to increase proportionally. This is illustrated in Fig. 14, where we see the results of a 5km chute without periodic boundary conditions in which \( h_0 \) equals 0.42 m, i.e., a hundred times higher than before (which has been done by choosing \( \mathcal{L} = 0.1 \) m without changing any other parameters except for the automatic change in \( \nu \), which is related to \( \mathcal{L} \)). The roll waves towards the end of the chute are seen to reach heights in the order of 1 m, i.e., almost a hundred times as large as before (and they reach velocities of the order of
5 m/s). This roughly linear scaling can be traced back to the fact that the governing equations (2.2)-(2.3) are approximately linear in $h$. In fact, the continuity equation (2.2) is exactly linear with respect to $h$, whereas the momentum balance (2.3) is linear in $h$ apart from the last two terms on the right hand side, representing the depth-averaged pressure gradient and the diffusive term. These two terms will give corrections to the linear scaling but apparently do not affect the general trend too much.

The peak heights and velocities in this upscaled, non-periodic version of our system compare well to granular roll waves observed in real geophysical flows, such as in the Jiang-Jia ravine in China or the Moscardo torrent in the Italian Alps (Li, Jianmo, Bi & Luo 1983; Marchi, Arattano & Deganutti 2002; Zanuttigh & Lamberti 2007).

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Chapter 5

Conclusions

This thesis has presented experiments and modelling of roll waves and erosion-deposition waves in granular free-surface flows.

Firstly, in a depth-averaged $\mu(I)$-rheology for shallow granular free-surface flows (chapter 2), an approximate depth-integration of the full $\mu(I)$-rheology is performed in order to derive a viscous-like diffusive term in the governing equations. A diffusive term such as this is required to smooth out the discontinuities that occur naturally for inviscid roll wave solutions. It was shown that the particular form of the new viscous-like term is an improvement on previous efforts in several ways. The linear stability analysis of a steady uniform flow in the new model predicts instability over a critical Froude number $F_c = 2/3$, whilst the growth rate and cut-off frequency of the instability are in good quantitative agreement with existing experiments. Inviscid models fail to predict a cut-off frequency at all and previous depth-averaged rheology models required an arbitrary fitting parameter to achieve good quantitative agreement with experiments, whereas the new model has no adjustable parameters. The form of the new viscous-like term is such that the new model is also able to exactly reproduce the physical inviscid solutions to a propagating front problem, which previous depth-averaged rheology models fail to do.

Secondly, in erosion-deposition waves in shallow granular free-surface flows (chapter 3), it is observed in laboratory-scale experiments that a granular avalanche close to the minimum thickness for a steady uniform flow can develop into a series of discrete surges with stationary layers between them. These surges are labelled erosion-deposition waves, as a result of the way in which they are able to steadily propagate.
downslope by eroding at a static layer ahead of the wave front and depositing material behind it in a delicate balance. These waves have been modelled by incorporating a complex basal friction law which transitions between mobile, intermediate and static regimes, as well as the new depth-averaged $\mu(I)$-rheology model, which was shown to be necessary in order to produce travelling erosion-deposition wave solutions. Numerical simulations showed that it is possible to produce both roll waves and erosion-deposition waves using the same model and the results were in good quantitative agreement with experiments. A comparison shows that roll waves, occurring from a thicker inflow than their erosion-deposition wave counterparts, are generally of a smaller amplitude and growth rate. This implies that erosion-deposition waves are potentially more destructive than roll waves and so further studies into their formation and growth could be very important.

Finally, in arrested coarsening of granular roll waves (chapter 4), the development and coarsening dynamics of roll waves was investigated by performing numerical simulations on periodic domains. The coarsening process is a result of larger, faster waves catching up to and merging with smaller, slower waves and it is shown to be arrested before reaching a one-wave state for sufficiently long domains, implying the existence of a maximum wave height and corresponding wavelength. An upper bound on the wave peak height was determined from the equations, which may be important in understanding the height of waves in natural debris flows. The coarsening mechanisms were studied in order to derive a law to predict the final number of waves or, equivalently, the average wave separation in the final arrested state of a system. The simulation was also scaled up to the order of a natural debris flow, on a non-periodic chute, where it was shown that the whole coarsening process may be upscaled and as such is a possible mechanism for the growth of large amplitude surges in natural debris flows.
Appendix A

Numerical scheme

The numerical simulations in this paper (chapters 3 and 4) are carried out by using the semi-discrete high-resolution non-oscillatory central scheme of Kurganov & Tadmor (2000). The choice of this solver is motivated by the inclusion of viscous-like diffusive terms in the governing equations, which change the hyperbolic system of the standard depth-averaged avalanche equations (e.g. Gray, Tai & Noelle, 2003), that require high-resolution shock capturing numerical methods (e.g. Nessyahu & Tadmor, 1990), into a system of convection-diffusion equations. This chosen scheme also benefits from lower numerical viscosity than the Nessyahu & Tadmor (1990) scheme, whilst retaining its simplicity, thus maintaining high-resolution independent of $\mathcal{O}(1/\Delta t)$ for a timestep $\Delta t$.

In order to prevent negative values of the momentum, $m = h \bar{u}$, the vector of conserved variables, $\mathbf{w} = (h, m)^T$, is set to zero below a chosen minimum threshold value of the flow thickness, $h = h_{\text{min}}$. For all of the results prevented here this value has been set to $h_{\text{min}} = 0.01\text{mm}$, although the results are unaffected (to the order of 0.1mm) for other values that have been tested in the range $10^{-5}\text{mm} \leq h_{\text{min}} \leq 1\text{mm}$.

The grid resolution that has been used throughout (except for the natural debris flow scale simulation) is 2000 grid points per 1m of domain. Various other grid resolutions have also been tested, ranging from 400 to 4000 grid points per metre, and again the results are unaffected (to the order of 0.1mm).

The time evolution is carried out by a 2nd-order Runge-Kutta time-stepper for all of the simulations here, except for those in which it is stated that we have used the 3rd-order adaptive Runge-Kutta time-stepper, ‘DUMKA3’, proposed by Medovikov.
(1998), for validation (chapter 3). The results of simulations with these two time-steppers only differ in the presence of stationary regions of flow. The standard deviation from the mean flow thickness of the stationary layer, $h_\pm$, is of the magnitude $10^{-4}$ times less with the DUMKA3 time-stepper than with the 2nd-order Runge-Kutta method. However, computation times with the adaptive time-stepper are much larger, to the order of 1000 times, when stationary layers cause the time steps to adapt to their minimum possible values, determined by error tolerances. This makes its implementation impractical in the case of erosion-deposition waves.

The DUMKA3 time-stepper is freely available to download from http://dumkaland.org/dumka3.cpp.

The numerical code, written in C++, is listed below.
# ifndef EQUATION_H
#define EQUATION_H

typedef void (*ConvectionFluxFunction)(double *, const double *);
typedef void (*DiffusionFluxFunction)(double *, const double *, const double *, const double *);
typedef void (*SourceTermFunction)(double *, double *, const double *);
typedef double (*WaveSpeedFunction)(const double *);
typedef void (*VectorSpatialFunctionPt)(double *, const double, const double);

class Equation
{
private:
  int nSolvedFields, nSpecifiedFields;

public:
  Equation(int nSolved, int nSpecified) : nSolvedFields(nSolved), nSpecifiedFields(nSpecified) {}
  Equation(int nDimensions) : nSolvedFields(nDimensions), nSpecifiedFields(0) {}

  virtual void FillOutEquations(ConvectionFluxFunction &fx,
                                 DiffusionFluxFunction &dx,
                                 DiffusionFluxFunction &dy,
                                 SourceTermFunction &st,
                                 WaveSpeedFunction &xws,
                                 WaveSpeedFunction &yws) = 0;

  void FillOutDimensions(int &solved, int &specified)
  {
    solved=nSolvedFields;
    specified=nSpecifiedFields;
  }
};
#endif

# ifndef ARRAY2D_H
#define ARRAY2D_H

#include <exception>
#include <stdexcept>

using namespace std;

class VectorArray2d
{
private:

int width, height, dimension;
double *data;

public:
  int size() const {return width*height*dimension;}
  int Width() const {return width;}
  int Height() const {return height;}
  int Dimension() const {return dimension;}
  void SetSize(int d, int w, int h);
VectorArray2d();
VectorArray2d(const VectorArray2d &rhs);
VectorArray2d(int d, int w, int h);
VectorArray2d(int l);
~VectorArray2d();

double L2Norm();
double L1Norm();
double LInfNorm();
inline double operator()(const int i, const int x, const int y) const
{
  #ifndef NDEBUG
  if (i>=dimension || i<0 || x>=width || x<0 || y >=height || y<0) throw "VectorArray2d bounds error";
  #endif
  return data[i+(x+y*width)*dimension];
}
inline double &operator()(const int i, const int x, const int y)
{
  #ifndef NDEBUG
  if (i>=dimension || i<0 || x>=width || x<0 || y >=height || y<0) throw "VectorArray2d bounds error";
  #endif
  return data[i+(x+y*width)*dimension];
}
inline double* Ptr(const int x, const int y) const
{
  #ifndef NDEBUG
  if (x>=width || x<0 || y>=height || y<0) throw "VectorArray2d bounds error";
  #endif
  return &(data[(x+y*width)*dimension]);
}
inline double operator[](const int index) const {
  return data[index];}
inline double &operator[](const int index) {
  return data[index];}
VectorArray2d &operator=(const VectorArray2d &rhs);
VectorArray2d operator+ (const VectorArray2d &a);
VectorArray2d operator− (const VectorArray2d &a);
VectorArray2d operator* (const double &f);
void Zero();
}
Listing 3. VectorArray2d.cpp
#include "VectorArray2d.h"
#include <cstring>
#include <cmath>

VectorArray2d::VectorArray2d() {data=0; width=0; height=0; dimension=0;}

VectorArray2d::VectorArray2d(const VectorArray2d &rhs)
{
data=0;
width=rhs.width; height=rhs.height; dimension=rhs.dimension;
if (rhs.data)
{
data=new double[width*height*dimension];
if (!data) return;
std::memcpy(data, rhs.data, width*height*dimension*sizeof(double));
}

void VectorArray2d::SetSize(int d, int w, int h)
{
if (data) delete[] data;
data=0;
width=w; height=h; dimension=d;
data=new double[width*height*dimension];
if (!data) return;
Zero();
}

VectorArray2d& VectorArray2d::operator=(const VectorArray2d &rhs)
{
if (width!=rhs.width || height!=rhs.height || dimension!=rhs.dimension)
{
if (data) delete[] data;
data=0;
width=rhs.width; height=rhs.height; dimension=rhs.dimension;
}
if (rhs.data)
{
data=new double[width*height*dimension];
if (data) memcpy(data, rhs.data, width * height *
    dimension * sizeof(double));
}
return *this;
}
VectorArray2d::VectorArray2d(int d, int w, int h)
{
    width=w; height=h; dimension=d;
data=0;
data=new double[width*height*dimension];
    if (!data) return;
    Zero();
}
VectorArray2d::VectorArray2d(int l)
{
    width=1; height=1; dimension=l;
data=0;
data=new double[width*height*dimension];
    if (!data) return;
    Zero();
}
double VectorArray2d::L2Norm()
{
    double sum2=0;
    int x,y,d;
    for (y=0; y<height; y++)
        for (x=0; x<width; x++)
            for (d=0; d<dimension; d++)
                sum2+=(*this)(d, x, y) * (*this)(d, x, y);
    return sqrt(sum2/(width*height));
}
double VectorArray2d::L1Norm()
{
    double sum2=0;
    int x,y,d;
    for (y=0; y<height; y++)
        for (x=0; x<width; x++)
            for (d=0; d<dimension; d++)
                sum2+=fabs((*this)(d, x, y));
    return sum2/(width*height);
}
double VectorArray2d::LInfNorm()
{
    double maxmod=0;
    int x,y,d;
    for (y=0; y<height; y++)
        for (x=0; x<width; x++)
            for (d=0; d<dimension; d++)
                if (fabs((*this)(d, x, y))>
                    maxmod) maxmod=fabs((*this)(d, x, y));
    return maxmod;
}
void VectorArray2d::Zero()
{
    for (int y=0; y<height; y++)
        for (int x=0; x<width; x
Listing 4. rk2.cpp

```cpp
#include <stdio.h>
#include <math.h>
#include <iostream>

++ for (int d=0; d<dimension; d++) (*this)(d,x,y) =0;

VectorArray2d::~VectorArray2d ()
{
    if (data) delete [] data;
    data=0;
}

VectorArray2d VectorArray2d::operator +(const VectorArray2d &a)
{
    int x,y;
    if (width!=a.Width() || height!=a.Height() || dimension
    !=a.Dimension()) throw invalid_argument("Incompatible size/dimension in VectorArray2d::operator+");
    VectorArray2d tmp(dimension, width, height);
    for (y=0; y<height; y++) for (x=0; x<width; x++) for (int d=0; d<dimension; d++)
    tmp(d,x,y)=(*this)(d,x,y)+a(d,x,y);
    return tmp;
}

VectorArray2d VectorArray2d::operator -(const VectorArray2d &a)
{
    int x,y;
    if (width!=a.Width() || height!=a.Height() || dimension
    !=a.Dimension()) throw invalid_argument("Incompatible size/dimensions in VectorArray2d::
    operator-");
    VectorArray2d tmp(dimension, width, height);
    for (y=0; y<height; y++) for (x=0; x<width; x++) for (int d=0; d<dimension; d++)
    tmp(d,x,y)=(*this)(d,x,y)-a(d,x,y);
    return tmp;
}

VectorArray2d VectorArray2d::operator *(const double &f)
{
    int x,y;
    VectorArray2d tmp(dimension, width, height);
    for (y=0; y<height; y++) for (x=0; x<width; x++) for (int d=0; d<dimension; d++)
    tmp(d,x,y)=(*this)(d,x,y)*f;
    return tmp;
}
```

1 #include <stdio.h>
2 #include <math.h>
3 #include <iostream>

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#include <fstream>
using namespace std;

/* VectorFunctionBase — interface with three functions, which have to be implemented */

// void compute(Vector& y, double t, Vector& result) — calculates right hand side of the ODE y' = f(y, t)
// double cour(Vector& y, double t) — used if "isComputeEigenValue=false" to calculate h_{euler} — step which you would use in explicit Euler method
// Element GetNonZeroElement() — can be used if "isComputeEigenValue=true" in eigenvalue function, if power method have to be restarted, but y, f(y) equals to zero,
// in this case program need to initialize random non-zero vector, because vector’s Element is unknown in advance
// for example if Vector contains "double"(s) you can return 0.01*(rand()%100)
// but if Vector contains "Point"(s) you can return Point(0.01*(rand()%100),0.01*(rand()%100),0.01*(rand()%100)) */

template <class Vector, class Element>

class VectorFunctionBase
{
    public:
        virtual void compute(Vector& y, double t, Vector& result) = 0;
        virtual double cour(Vector& y, double t) = 0;
        virtual Element GetNonZeroElement() = 0;
};

/* rk2 — solves ordinary differential equations y' = f(y, t) in order to be able to use this function you should implement interface VectorFunctionBase (above) */

// t — time
// tend — end of the integration interval
// f — Class which implement VectorFunctionBase interface, or class, which has 3 functions
// compute(Vector& y, double t, Vector& result) — calculates right hand side
// cour(Vector& y, double t) — used if "isComputeEigenValue=false" to calculate
GetNonZeroElement() — can be used if "isComputeEigenValue=true" in eigenvalue function, if iterations of power method have to be restarted, but $y, f(y)$ equals to zero, program need to initialize random non-zero vector, because Vector's Element can be complicated like Point etc, user should provide random non-zero element $y$ — solution of the ODE $y'=f(y,t)$ isComputeEigenValue — if "true"; function "eigenvalue" will be used to determine maximum eigenvalue and $h_{\{euler\}}=2/eigenvalue$ — if "false"; user should provide $H_{\{euler\}}$ via "cour" function of the VectorFunction $f$

```cpp
template <class VectorFunction, class Vector>
double rk2(double& t, const double& tend, VectorFunction& f, Vector& y)
{
    double timeInterval = tend - t;
    long _size = y.size();
    Vector intermed(y), ddt(y);
    intermed.Zero();
    ddt.Zero();

    if (tend <= t)
    {
        cerr << "End-time Tend is less than initial time" << endl;
        return 0;
    }

    do
    {
        double advisedTimeStep = 0.225 * f.cour(y, t); // 0.475 * f.cour(y, t);
        // if (advisedTimeStep < 0.001) advisedTimeStep = 0.001;
        double nextT = min(tend, t + advisedTimeStep);
        double thisDeltaT = nextT - t;
        // cout << "this t=" << t << " next t=" << nextT << endl;
        f.compute(y, t, ddt);
        for (int i = 0; i < _size; i++) intermed[i] = y[i] + thisDeltaT * ddt[i];
        f.compute(intermed, nextT, ddt);
    } while (t < tend);
}
```

h_{\{euler\}} — step which you would use in explicit Euler method
for (int i = 0; i < size; i++) y[i] = 0.5*y[i] + 0.5*(intermed[i] + thisDeltaT*ddt[i]);

t = nextT;

cerr << "t=" << t << " \r";
cerr.flush();
f.StepEndProcessing(y);
} while (t < tend);

cerr << "t=" << t << " \r";
cerr.flush();
return 0;

Listing 5. RollWaveEquations.h

#include "Equation.h"

inline double sq(double v) {
    return v*v;
}

const double twopi = 8 * atan(1);
const double degrees = twopi / 360.0;

class RollWaveEqns : public Equation {
public:
    static double hzero;
    static double nu, L, d, alpha, beta, gravity, gPrime, epsilon, epsPrime, zetaSlope, delta1, delta2, delta3, deltaSlope;
    static double mu1, mu2, mu3, sinZetaSlope, cosZetaSlope, gamma;

    RollWaveEqns() : Equation(4) {
        gravity = 9.81;
        double slopeAngle = 35.1;
        double lowAngle = 32.9;
        double highAngle = 42;
        double thirdAngle = 33.9;
        hzero = 1e-5; // This is the threshold value of h below which h, hu and hv are all set to 0.
        beta = 0.65;
        alpha = 0;
\[L=0.001;\]
\[\nu=0.0024;\]

\[\zeta_{\text{Slope}} = \text{slopeAngle} \times \text{degrees};\]
\[\delta_1 = \text{lowAngle} \times \text{degrees};\]
\[\delta_2 = \text{highAngle} \times \text{degrees};\]
\[\delta_3 = \text{thirdAngle} \times \text{degrees};\]

\[\epsilon_{\text{Prime}} = \epsilon \times \cos(\zeta_{\text{Slope}});\]
\[g_{\text{Prime}} = g \times \cos(\zeta_{\text{Slope}});\]
\[\delta_{\text{Slope}} = \tan(\zeta_{\text{Slope}});\]
\[\mu_1 = \tan(\delta_1);\]
\[\mu_2 = \tan(\delta_2);\]
\[\mu_3 = \tan(\delta_3);\]

\[\sinZeta_{\text{Slope}} = \sin(\zeta_{\text{Slope}});\]
\[\cosZeta_{\text{Slope}} = \cos(\zeta_{\text{Slope}});\]

\[\gamma = \frac{(\mu_2 - \mu_1)}{(\delta_{\text{Slope}} - \mu_1)} - 1;\]

\[
\text{static double GetU0(double h0) // This is called by}
\text{sdkt.cpp when calculating u0 from the h0 input there}
\{
\text{double u0 = beta \times pow(h0, 1.5) \times sqrt(gPrime) / (L} \\
\text{\times ((mu2 - mu1)/(deltaSlope - mu1)) - 1));} \\
\text{return u0;}
\}
\]

\[
\text{enum Variables}
\{
\text{H = 0, HU = 1, HV = 2, FL = 3 //4th 'variable' FL=3}
\text{is used to keep track of the friction}
\text{coefficient regime}
\};
\]

\[
\text{void FillOutEquations(ConvectionFluxFunction &fx,}
\text{ConvectionFluxFunction &fy,}
\text{DiffusionFluxFunction &dx,}
\text{DiffusionFluxFunction &dy,}
\text{SourceTermFunction &st,}
\text{WaveSpeedFunction &xws,}
\text{WaveSpeedFunction &yws)}
\{
\text{fx = XConvectionFlux;}
\text{fy = YConvectionFlux;}
\text{dx = XDiffusionFlux;}
\text{dy = YDiffusionFlux;}
\text{st = SourceTerms;}
\text{xws = XWaveSpeeds;}
\text{yws = YWaveSpeeds;}
\}
private:

inline static void XConvectionFlux(double *xFlux, const double *u)
{
    if (u[H] <= hzero)
    {
        xFlux[H] = 0;
        xFlux[HU] = 0;
        xFlux[HV] = 0;
        xFlux[FL] = 0;
        return;
    }

    xFlux[H] = u[HU];
    xFlux[HU] = u[HU] * u[H] / u[H] + 0.5 * gPrime * u[H] * u[H];
    xFlux[HV] = u[HU] * u[HV] / u[H];
    xFlux[FL] = 0;
}

inline static void YConvectionFlux(double *yFlux, const double *u)
{
    if (u[H] <= hzero)
    {
        yFlux[H] = 0;
        yFlux[HU] = 0;
        yFlux[HV] = 0;
        yFlux[FL] = 0;
        return;
    }

    yFlux[H] = u[HV];
    yFlux[HU] = u[HU] * u[HV] / u[H];
    yFlux[HV] = u[HV] / u[H] + 0.5 * gPrime * u[H] * u[H];
}

inline static void XDiffusionFlux(double *xFlux, const double *u, const double *dudx, const double *dudy)
{
    if (u[H] <= hzero)
    {
        xFlux[H] = 0;
        xFlux[HU] = 0;
        xFlux[HV] = 0;
        xFlux[FL] = 0;
        return;
    }

    xFlux[H] = 0;
    xFlux[HU] = nu * pow(u[H], 0.5) * (dudx[HU] - u[HU] * dudx[H] / u[H]);
inline static void YDiffusionFlux(double *yFlux, const double *u, const double *dudx, const double *dudy)
{
    if (u[H] <= hzero)
    {
        yFlux[H] = 0;
        yFlux[HU] = 0;
        yFlux[HV] = 0;
        yFlux[FL] = 0;
        return;
    }
    yFlux[H] = 0;
    yFlux[HU] = 0;
    yFlux[HV] = nu*(dudx[HV]-u[HV]*dudx[H]/u[H]);
    yFlux[FL] = 0;
}

inline static double XWaveSpeeds(const double *u)
{
    double modU;
    if (u[H] <= hzero) return 0;
    modU = fabs(u[HU] / u[H]);
    return modU + sqrt(gPrime * u[H]);
}

inline static double YWaveSpeeds(const double *u)
{
    double modU;
    if (u[H] <= hzero) return 0;
    modU = fabs(u[HV] / u[H]);
    return modU + sqrt(gPrime * u[H]);
}

inline static void SourceTerms(double *stvect, double *u, const double *dudx)
{
    double lambda, mu, muS, muL, modU, fr, hstop, FLindex;
    const double kappa = 0.001;
    if (u[H]<=hzero)
    {
        lambda = 0;
    }
    else
    {
        stvect[HU] += gravity*u[H]*sinZetaSlope;
modU = sqrt(sq(u[HU]) + sq(u[HV])) / u[H];

if (modU > 0)
{
  fr = modU / sqrt(gPrime * u[H]);

  if (fr > beta)
  {
    mu = mu1 + (mu2 - mu1) / (1 + (beta * u[H] / (L * fr)));
    FLindex = 2; // Dynamic friction regime
  }

  else if (fr > 0)
  {
    mu = pow(fr / beta, kappa) * (mu1 - mu3) + mu3 + (mu2 - mu1) / (1 + (u[H] / (L)));
    FLindex = 1; // Intermediate friction regime
  }

  else
  {
    mu = mu3 + (mu2 - mu1) / (1 + u[H] / L);
    FLindex = 0; // Static friction regime
  }

  lambda = gravity * cosZetaSlope * mu / modU;
}

else lambda = 1;

stvec[HU] = lambda * u[HU];
stvec[HV] = lambda * u[HV];

};

double RollWaveEqns::hzero;
double RollWaveEqns::L;
double RollWaveEqns::d;
double RollWaveEqns::nu;
double RollWaveEqns::beta;
double RollWaveEqns::epsPrime, RollWaveEqns::epsilon, RollWaveEqns::alpha;
double RollWaveEqns::zetaSlope, RollWaveEqns::delta1, RollWaveEqns::delta2, RollWaveEqns::delta3;
double RollWaveEqns::deltaSlope, RollWaveEqns::gPrime, RollWaveEqns::gravity;
double RollWaveEqns::sinZetaSlope, RollWaveEqns::cosZetaSlope, RollWaveEqns::gamma;
double RollWaveEqns::mu1, RollWaveEqns::mu2, RollWaveEqns::mu3;
Listing 6. SDKTSolver.h

```cpp
1 #ifndef SDKTSOLVER_H
2 #define SDKTSOLVER_H
3 #include "VectorArray2d.h"
4 #include <iostream>
5 #include <vector>
6 #include <fstream>
7 #include "Equation.h"
8 using namespace std;
9
typedef double (*LimiterPt)(const double, const double);
typedef void (*BoundaryConditionPointer)(double *, const double, const double);

void NullBCFunction(double *u, double x, double y);

class SDKTSolver
{
public:
    SDKTSolver(int nx, int ny, Equation *eqn);
    ~SDKTSolver();
    void CalculateTimeDerivative(VectorArray2d &uvect, double t, VectorArray2d &result);
    void SetInitialConditions(VectorSpatialFunctionPt icf);
    void IntegrateTo(double destinationTime);
    void IntegrateFor(double integrationTime);
    void OutputResults(ofstream &out);
    void OutputResults(string filename);
    void OutputResults(const char *filename);
    void SampleFunction(VectorArray2d &soln, VectorSpatialFunctionPt fn);
    double MaxWaveSpeed(VectorArray2d &uvect);
    void GetSolution(VectorArray2d &soln);
    const VectorArray2d& Solution() {return u;}

enum Limiter {MinMod2, WENO, None};
enum GradientBoundary {GradientPeriodic, GradientExtrapolate0, GradientExtrapolate1, GradientExtrapolate2};
enum FluxBoundary {FluxPeriodic, FluxZeroFlux, FluxSetValue, FluxSetFlux, FluxUseExtrapolated};
enum BoundaryID {North=0, South=1, East=2, West=3};
void SetLimiter(Limiter l);
void SetDomainSize(double xsize, double ysize);
void SetDomainOffset(double xo, double yo);
```

void SetDomain(double xo, double yo, double xsize, double ysize);

void SetBoundaryConditionType(BoundaryID bid, int variable, GradientBoundary gradientBC, FluxBoundary fluxBC);

void SetBoundaryConditionFunctions(BoundaryID bid, BoundaryConditionPointer valueFn=NullBCFunction, BoundaryConditionPointer fluxFn=NullBCFunction);

void SetPeriodicBoundaryConditions();

int XPoints() { return nXPoints; }

int YPoints() { return nYPoints; }

int Dimension() { return dimension; }

double GetInstantaneousTime() { return instantaneousTime; }

void InputICs(const char *filename, int d=-1);

void InputICs(ifstream &in, int d=-1);

private:

GradientBoundary *gradientBCs[4];

FluxBoundary *fluxBCs[4];

BoundaryConditionPointer fluxValueBCFunction[4], fluxFluxBCFunction[4];

LimiterPt limiterPtr;

int outputCount;

double instantaneousTime;

void SetInstantaneousTime(double t) { instantaneousTime = t; }

void CalculateLimitedDerivs(VectorArray2d&uvect);

void CalculateLimitedDerivsBoundary(int d, int i, int j, VectorArray2d&uvect, BoundaryID bid, GradientBoundary bType);

void CalculateFluxes(VectorArray2d&uvect);

void CalculateFluxBoundaryConditions();

void SetBoundaryFlux(int i, int j, BoundaryID bid, FluxBoundary *boundaryType);

char startTimeBuffer[80];

double deltaX, deltaY, domainXSize, domainYSize, domainXOffset, domainYOffset;

double deltaXRecip, deltaYRecip;

double simulationTime;

int dimension, nSolvedDimensions, nSpecifiedDimensions, nXPoints, nYPoints;

ConvectionFluxFunction xConvectionFlux, yConvectionFlux;

DiffusionFluxFunction xDiffusionFlux, yDiffusionFlux;

WaveSpeedFunction xWaveSpeedFunction, yWaveSpeedFunction;

SourceTermFunction sourceTermFunction;

VectorArray2d u, uLimX, uLimY, uPlusX, uMinusX, uPlusY, uMinusY;
VectorArray2d hXFlux, hYFlux;
VectorArray2d pXFlux, pYFlux;

static double LimiterMinMod2(const double a, const double b);
static double LimiterNone(const double a, const double b);
static double LimiterWENO(const double a, const double b);

double integrationStepSize;

Listing 7. SDKTSolver.cpp

#include "Equation.h"
#include "SDKTSolver.h"
#include "dumka3.cpp" //uncomment this for dumka3 time-stepper
#include "rk2.cpp" //comment this for dumka3 time-stepper
#include <cstdlib>
#include <iomanip>
#include <iostream>

void NullBCFunction(double *u, double x, double y)
{
}

SDKTSolver::SDKTSolver(int nx, int ny, Equation *eqn) : nXPoints(nx), nYPoints(ny)
{
    eqn->FillOutDimensions(nSolvedDimensions, nSpecifiedDimensions);
    dimension = nSolvedDimensions + nSpecifiedDimensions;
    u.SetSize(dimension, nXPoints, nYPoints);
    uLimX.SetSize(dimension, nXPoints, nYPoints);
    uPlusX.SetSize(dimension, nXPoints + 1, nYPoints);
    uMinusX.SetSize(dimension, nXPoints + 1, nYPoints);
    uPlusY.SetSize(dimension, nXPoints, nYPoints + 1);
    uMinusY.SetSize(dimension, nXPoints, nYPoints + 1);
    hXFlux.SetSize(dimension, nXPoints + 1, nYPoints);
    hYFlux.SetSize(dimension, nXPoints, nYPoints + 1);
    pXFlux.SetSize(dimension, nXPoints + 1, nYPoints);
    pYFlux.SetSize(dimension, nXPoints, nYPoints + 1);
    eqn->FillOutEquations(xConvectionFlux, yConvectionFlux,
        xDiffusionFlux, yDiffusionFlux,
        sourceTermFunction,
        xWaveSpeedFunction, yWaveSpeedFunction);
SetDomain(0, 0, 1, 1);
SetLimiter(MinMod2);
simulationTime = 0;
outputCount = 0;
time_t rawtime;
struct tm * timeinfo;
time(&rawtime);
timeinfo = localtime(&rawtime);
strftime(startTimeBuffer, 80, "%Y/%m/%d %H:%M:%S", timeinfo);

integrationStepSize = 0.0001;

for (int i = 0; i < 4; i++)
{
    gradientBCs[i] = new GradientBoundary[dimension];
    fluxBCs[i] = new FluxBoundary[dimension];
    fluxValueBCFunction[i] = NullBCFunction;
    fluxFluxBCFunction[i] = NullBCFunction;
}

SetBoundaryConditionFunctions(SDLTSolver::North);
SetBoundaryConditionFunctions(SDLTSolver::South);
SetBoundaryConditionFunctions(SDLTSolver::East);
SetBoundaryConditionFunctions(SDLTSolver::West);

SDKTSolver::~SDKTSolver()
{
    for (int i = 0; i < 4; i++)
    {
        delete[] gradientBCs[i];
        delete[] fluxBCs[i];
    }
}

void SDKTSolver::SetLimiter(Limiter l)
{
    switch (l)
    {
    case None: limiterPtr = LimiterNone;
        break;
    case MinMod2: limiterPtr = LimiterMinMod2;
        break;
    case WENO: limiterPtr = LimiterWENO;
        break;
    }
}

void SDKTSolver::InputICs(const char *filename, int dim)
```cpp
ifstream inFile(filename);
if (inFile.fail())
{
    std::cerr << "Error in InputICs(...): could not load " << filename << ". Aborting load." << endl;
    return;
}
InputICs(inFile, dim);
inFile.close();

void SDKTSolver::InputICs(ifstream &in, int dim)
{
    int i, j, startDimension, endDimension;
    try
    {
        in.exceptions ( ifstream::eofbit | ifstream::failbit | ifstream::badbit );
    } catch (ifstream::failure e) // Catch bad ifstream being passed in directly. InputICs(char *filename,...) will catch this earlier.
    {
        std::cerr << "Error in InputICs(...): bad ifstream. Aborting load." << endl;
        return;
    }
    if (dim==-1)
    {
        startDimension=0;
        endDimension=dimension-1;
    }
    else
    {
        startDimension=dim;
        endDimension=dim;
    }
    int fileNDims, fileNXP, fileNYP;
    try
    {
        in >> fileNDims; in.ignore();
        in >> fileNXP; in.ignore();
        in >> fileNYP; in.ignore(10, '\n');
        if (fileNDims!=(endDimension-startDimension+1))
        {
            std::cerr << "Error in InputICs(...): input file specifies " << fileNDims << " variables, whereas solver expects " << (endDimension-startDimension+1) << ". Aborting load." << endl;
            return;
        }
    }
```
if (fileNXP != nXPoints)
{
    std::cerr << "Error in InputICs (...) : input file has " << fileNXP << " cells in the x-direction, whereas solver expects " << nXPoints << ". Aborting load." << endl;
    return;
}
if (fileNYP != nYPoints)
{
    std::cerr << "Error in InputICs (...) : input file has " << fileNYP << " cells in the y-direction, whereas solver expects " << nYPoints << ". Aborting load." << endl;
    return;
}
for (int d=startDimension ; d<=endDimension ; d++)
{
    for (j = 0 ; j < nYPoints ; j++)
    {
        for (i = 0 ; i < nXPoints ; i++)
        {
            u(d, i, j) = 0;
            in >> u(d, i, j);
            in.ignore();
        }
        in.ignore(10, '\n');
    }
}
catch (ifstream::failure e)
{
    std::cerr << "Error in InputICs (...) : could not load data. Aborting load. Initial conditions may be corrupted." << endl;
}

void SDKTSolver::SetPeriodicBoundaryConditions()
{
    for (int d = 0; d < dimension ; d++)
    {
        SetBoundaryConditionType(North, d, SDKTSolver::GradientPeriodic, SDKTSolver::FluxPeriodic);
        SetBoundaryConditionType(South, d, SDKTSolver::GradientPeriodic, SDKTSolver::FluxPeriodic);
        SetBoundaryConditionType(East, d, SDKTSolver::GradientPeriodic, SDKTSolver::FluxPeriodic);
        SetBoundaryConditionType(West, d, SDKTSolver::GradientPeriodic, SDKTSolver::FluxPeriodic);
    }
}
void SDKTSolver::GetSolution(VectorArray2d &soln)
{
    soln = u;
}

void SDKTSolver::OutputResults(std::string filename)
{
    OutputResults(filename.c_str());
}

void SDKTSolver::OutputResults(const char *filename)
{
    ofstream outFile(filename);
    OutputResults(outFile);
    outFile.close();
}

void SDKTSolver::OutputResults(ofstream &out)
{
    int d, i, j;
    char timebuffer[80];
    time_t rawtime;
    struct tm *timeinfo;
    time(&rawtime);
    timeinfo = localtime(&rawtime);
    strftime(timebuffer, 80, "%Y%m%d %H:%M:%S", timeinfo);
    out.precision(16);
    out << "5" << endl;
    out << "1 " << dimension << " " << nXPoints << " " << nYPoints << endl;
    out << "2 " << domainXOffset << " " << domainYOffset << " " << domainXSize << " " << domainYSize << " " << simulationTime << " " << -1 << endl;
    out << "3 " << timebuffer << endl;
    out << "4 " << startTimeBuffer << endl;
    out << "5 " << outputCount++ << endl;
    for (d = 0; d < dimension; d++)
    {
        for (j = 0; j < nYPoints; j++)
        {
            for (i = 0; i < nXPoints; i++) out << u(d, i, j) << " ";
            out << endl;
        }
    }
}

void SDKTSolver::SetDomainSize(double xsize, double ysize)
void SDKTSolver::SetDomainOffset(double xo, double yo)
{
    domainXOffset = xo;
    domainYOffset = yo;
}

void SDKTSolver::SetDomain(double xo, double yo, double xsize, double ysize)
{
    SetDomainOffset(xo, yo);
    SetDomainSize(xsize, ysize);
}

void SDKTSolver::SampleFunction(VectorArray2d &soln, VectorSpatialFunctionPt fn)
{
    int i, j;
    soln.SetSize(dimension, nXPoints, nYPoints);
    // Set initial conditions
    for (i = 0; i < nXPoints; i++)
        for (j = 0; j < nYPoints; j++)
            {
                double xPos = domainXOffset + (i + 0.5) * deltaX;
                double yPos = domainYOffset + (j + 0.5) * deltaY;
                fn(soln.Ptr(i, j), xPos, yPos);
            }
}

void SDKTSolver::SetInitialConditions(VectorSpatialFunctionPt icf)
{
    double dudxVect[4];
    double dudyVect[4];
    int i, j, d;
    // Set initial conditions
    for (i = 0; i < nXPoints; i++)
        for (j = 0; j < nYPoints; j++)
            {
                double xPos = domainXOffset + (i + 0.5) * deltaX;
                double yPos = domainYOffset + (j + 0.5) * deltaY;
            }
icf(u.Ptr(i, j), xPos, yPos);

if (nSpecifiedDimensions > 0) {
    for (j = 0; j < nYPoints; j++)
    {
        int prevYCell = (j - 1 + nYPoints) % nYPoints;
        int nextYCell = (j + 1) % nYPoints;
        for (i = 0; i < nXPoints; i++)
        {
            int prevXCell = (i - 1 + nXPoints) % nXPoints;
            int nextXCell = (i + 1) % nXPoints;
            for (d = nSolvedDimensions - 1; d < dimension; d++)
            {
                int uLimX(d, i, j) = deltaXRecip * 
                    limiterPtr(u(d, nextXCell, j) - u(d, i, j), u(d, i, j) - u(d, prevXCell, j));
                uLimY(d, i, j) = deltaYRecip * 
                    limiterPtr(u(d, i, nextYCell) - u(d, i, j), u(d, i, j) - u(d, i, prevYCell));
            }
        }
    }
}
for (j = 0; j <= nYPoints; j++) for (i = 0; i <= nXPoints; i++)
{
    int thisYCell = j % nYPoints;
    int prevXCell = (i - 1 + nXPoints) % nXPoints;
    int nextXCell = i % nXPoints;
    int thisXCell = i % nXPoints;
    int prevYCell = (j - 1 + nYPoints) % nYPoints;
    int nextYCell = j % nYPoints;
    for (d = nSolvedDimensions - 1; d < dimension; d++)
    {
        uPlusX(d, i, j) = u(d, nextXCell, thisYCell) - uLimX(d, nextXCell, thisYCell) * 0.5 * deltaX;
        uMinusX(d, i, j) = u(d, prevXCell, thisYCell) + uLimX(d, prevXCell, thisYCell) * 0.5 * deltaX;
        dudxVect[d] = (u(d, nextXCell,
\[
\text{thisYCell}) - u(d, \text{prevXCell}, \text{thisYCell}) \ast \text{deltaXRecip;}
\]

\[
uPlusY(d, i, j) = u(d, \text{thisXCell}, \text{nextYCell}) - uLimY(d, \text{thisXCell}, \text{nextYCell}) \ast 0.5 \ast \text{deltaY;}
\]

\[
uMinusY(d, i, j) = u(d, \text{thisXCell}, \text{prevYCell}) + uLimY(d, \text{thisXCell}, \text{prevYCell}) \ast 0.5 \ast \text{deltaY;}
\]

\[
dudyVect[d] = (u(d, \text{thisXCell}, \text{nextYCell}) - u(d, \text{thisXCell}, \text{prevYCell}) \ast \text{deltaYRecip;}
\]

\[
\text{void \text{SDKTSolver}}::\text{SetBoundaryConditionType(BoundaryID bid, int variable, GradientBoundary gradientBC, FluxBoundary fluxBC)}
\]

\[
\{ \text{gradientBCs[bid][variable] = gradientBC;}
\]

\[
\text{fluxBCs[bid][variable] = fluxBC;}
\]

\[
\}
\]

\[
\text{void \text{SDKTSolver}}::\text{SetBoundaryConditionFunctions(BoundaryID bid, BoundaryConditionPointer valueFn, BoundaryConditionPointer fluxFn)}
\]

\[
\{ \text{fluxValueBCFunction[bid] = valueFn;}
\]

\[
\text{fluxFluxBCFunction[bid] = fluxFn;}
\]

\[
\}
\]

\[
\text{inline double min(const double a, const double b)}
\]

\[
\{ \text{if (a < b) return a;}
\]

\[
\text{else return b;}
\]

\[
\}
\]

\[
\text{inline double max(const double a, const double b)}
\]

\[
\{ \text{if (a > b) return a;}
\]

\[
\text{else return b;}
\]

\[
\}
\]

\[
\text{inline double sign(const double a)}
\]

\[
\{ \text{if (a < 0) return -1;}
\]

\[
\text{else if (a == 0) return 0;}
\]

\[
\text{else return 1;}
\]

\[
\}
\]
inline double SDKTSolver::LimiterMinMod2(const double a, const double b)
{
    const double theta = 2; // 1.25;
    if (sign(a) == 1 && sign(b) == 1 && sign(a + b) == 1)
        return min(theta * a, min(theta * b, 0.5 * (a + b)));
    else if (sign(a) == -1 && sign(b) == -1 && sign(a + b) == -1)
        return max(theta * a, max(theta * b, 0.5 * (a + b)));
    else return 0;
}

inline double SDKTSolver::LimiterNone(const double a, const double b)
{
    return 0.5 * (a + b);
}

double weno_func(double a)
{
    double eps = 1e-6;
    return 1.0 / ((eps + a * a)*(eps + a * a));
}

inline double SDKTSolver::LimiterWENO(const double a, const double b)
{
    return (weno_func(a) * a + weno_func(b) * b) / (weno_func(a) + weno_func(b));
}

void SDKTSolver::CalculateLimitedDerivs(VectorArray2d &uvect)
{
    int i, j, d;
    // Over all y–points, but not East and West boundaries, calculate limited X–derivs
    for (j = 0; j < nYPoints; j++)
    {
        for (i = 1; i < nXPoints - 1; i++)
        {
            int prevXCell = (i - 1 + nXPoints) % nXPoints;
            int nextXCell = (i + 1) % nXPoints;
            for (d = 0; d < nSolvedDimensions; d++)
            {
                uLimX(d, i, j) = deltaXRecip * limiterPtr(uvect(d, nextXCell, j) - uvect(d, i, j),
                                                        uvect(d, i, j) - uvect(d, prevXCell, j));
            }
        }
    }
}
// Over all x-points, but not North and South boundaries, calculate limited Y-derivs
for (j = 1; j < nYPoints - 1; j++)
{
    int prevYCell = (j - 1 + nYPoints) % nYPoints;
    int nextYCell = (j + 1) % nYPoints;
    for (i = 0; i < nXPoints; i++)
    {
        for (d = 0; d < nSolvedDimensions; d++)
        {
            uLimY(d, i, j) = deltaYRecip * limiterPtr(
                uvec(d, i, nextYCell) - uvec(d, i, j),
                uvec(d, i, j) - uvec(d, i, prevYCell)) ;
        }
    }
}

// Now fill in boundaries:

// Y-derivs on South boundary
j = 0;
for (i = 0; i < nXPoints; i++) for (d = 0; d < nSolvedDimensions; d++)
    CalculateLimitedDerivsBoundary(d, i, j, uvect, South , gradientBCs[South][d]);

// Y-derivs on North boundary
j = nYPoints - 1;
for (i = 0; i < nXPoints; i++) for (d = 0; d < nSolvedDimensions; d++)
    CalculateLimitedDerivsBoundary(d, i, j, uvect, North , gradientBCs[North][d]);

// X-derivs on West boundary
i = 0;
for (j = 0; j < nYPoints; j++) for (d = 0; d < nSolvedDimensions; d++)
    CalculateLimitedDerivsBoundary(d, i, j, uvect, West , gradientBCs[West][d]);

// X-derivs on East boundary
i = nXPoints - 1;
for (j = 0; j < nYPoints; j++) for (d = 0; d < nSolvedDimensions; d++)
    CalculateLimitedDerivsBoundary(d, i, j, uvect, East , gradientBCs[East][d]);

}
i, int j, VectorArray2d &uvect, BoundaryID bid, GradientBoundary bType)
{
    int prevXCell, prevYCell, jIncr = 0;
    int nextXCell, nextYCell, iIncr = 0;
    if (bType != GradientPeriodic)
    {
        switch (bid)
        {
            case North: jIncr = -1;
                break;
            case South: jIncr = 1;
                break;
            case East: iIncr = -1;
                break;
            case West: iIncr = 1;
                break;
        }
        switch (bid)
        {
            case North:
                case South:
                    switch (bType)
                    {
                        case GradientPeriodic:
                            prevXCell = i;
                            nextXCell = i;
                            prevYCell = (j - 1 + nYPoints) % nYPoints;
                            nextYCell = (j + 1) % nYPoints;
                            uLimY(d, i, j) = deltaYRecip * limiterPtr(uvect(d, nextXCell, nextYCell) - uvect(d, i, j), uvect(d, i, j) - uvect(d, prevXCell, prevYCell));
                            break;
                        case GradientExtrapolate0: uLimY(d, i, j) = 0;
                            break;
                        case GradientExtrapolate1: uLimY(d, i, j) = uLimY(d, i + iIncr, j + jIncr);
                            break;
                        case GradientExtrapolate2: uLimY(d, i, j) = 2.0 * uLimY(d, i + iIncr, j + jIncr) - uLimY(d, i + 2 * iIncr, j + 2 * jIncr);
                            break;
                    }
                    break;
            case East:
                case West:
                    switch (bType)
case GradientPeriodic:
    prevXCell = (i - 1 + nXPoints) % nXPoints;
    nextXCell = (i + 1) % nXPoints;
    prevYCell = j;
    nextYCell = j;
    uLimX(d, i, j) = deltaXRecip * limiterPtr(uvect(d, nextXCell, nextYCell) - uvect(d, i, j), uvect(d, i, j) - uvect(d, prevXCell, prevYCell));
    break;
    case GradientExtrapolate0: uLimX(d, i, j) = 0;
    break;
    case GradientExtrapolate1: uLimX(d, i, j) = uLimX(d, i + iIncr, j + jIncr);
    break;
    case GradientExtrapolate2: uLimX(d, i, j) = 2.0 * uLimX(d, i + iIncr, j + jIncr) - uLimX(d, i + 2 * iIncr, j + 2 * jIncr);
    break;
    }
    break;
}

void SDKTSolver::CalculateFluxes(VectorArray2d&uvect)
{
    int i, j, d;
    // Calculate H and P fluxes at each cell edge
    double uPlusConvectionFlux[4], uMinusConvectionFlux[4];
    double dudxVect[4], xPrevDiffusionFlux[4], xNextDiffusionFlux[4];
    double dudyVect[4], yPrevDiffusionFlux[4], yNextDiffusionFlux[4];
    double wsPlus, wsMinus, a;
    // Do X fluxes
    for (j = 0; j < nYPoints; j++) for (i = 0; i <= nXPoints; i++)
    {
        int thisYCell = j;
        int prevXCell = (i - 1 + nXPoints) % nXPoints;
        int nextXCell = i % nXPoints;
        for (d = 0; d < nSolvedDimensions; d++)
        {
            uPlusX(d, i, j) = uvect(d, nextXCell,
thisYCell) - uLimX(d, nextXCell,
thisYCell)*0.5*deltaX;
507 uMinusX(d, i, j) = uvect(d, prevXCell,
thisYCell) + uLimX(d, prevXCell,
thisYCell)*0.5*deltaX;
508 dudxVect[d] = (uvect(d, nextXCell,
thisYCell) - uvect(d, prevXCell,
thisYCell)) * deltaXRecip;
509
510 wsPlus = xWaveSpeedFunction(uPlusX.Ptr(i, j));
511 wsMinus = xWaveSpeedFunction(uMinusX.Ptr(i, j));
512 a = (wsPlus > wsMinus) ? wsPlus : wsMinus;
513
514 xConvectionFlux(uPlusConvectionFlux, uPlusX.Ptr
(i, j));
515 xConvectionFlux(uMinusConvectionFlux, uMinusX.
Ptr(i, j));
516 xDiffusionFlux(xPrevDiffusionFlux, uvect.Ptr(
prevXCell, thisYCell), dudxVect, uLimY.Ptr(
prevXCell, thisYCell));
517 xDiffusionFlux(xNextDiffusionFlux, uvect.Ptr(
nextXCell, thisYCell), dudxVect, uLimY.Ptr(
nextXCell, thisYCell));
518
519 for (d = 0; d < nSolvedDimensions; d++)
520 {
521 hXFlux(d, i, j) = 0.5* ((
522 uPlusConvectionFlux[d] +
523 uMinusConvectionFlux[d])
524 - a * (uPlusX(d, i, j) - uMinusX(d,
525 i, j)));
526 pXFlux(d, i, j) = 0.5 * (xPrevDiffusionFlux
527 [d] + xNextDiffusionFlux[d]);
528
529 for (j = 0; j <= nYPoints; j++) for (i = 0; i <
nXPoints; i++)
530 {
531 int thisXCell = i;
532
533 int prevYCell = (j - 1 + nYPoints) % nYPoints;
534 int nextYCell = j % nYPoints;
535
536 for (d = 0; d < nSolvedDimensions; d++)
537 {
538 uPlusY(d, i, j) = uvect(d, thisXCell,
539 nextYCell) - uLimY(d, thisXCell,
540 nextYCell)*0.5*deltaY;
541 uMinusY(d, i, j) = uvect(d, thisXCell,
prevYCell) + uLimY(d, thisXCell, prevYCell) * 0.5 * deltaY;

dudyVect[d] = (uvect(d, thisXCell, nextYCell) - uvect(d, thisXCell, prevYCell)) * deltaYRecip;

double wsPlus, wsMinus, a;

wsPlus = yWaveSpeedFunction(uPlusY_Ptr(i, j));
wsMinus = yWaveSpeedFunction(uMinusY_Ptr(i, j));
a = (wsPlus > wsMinus) ? wsPlus : wsMinus;
yConvectionFlux(uPlusConvectionFlux, uPlusY_Ptr(i, j));
yConvectionFlux(uMinusConvectionFlux, uMinusY_Ptr(i, j));
yDiffusionFlux(yPrevDiffusionFlux, uvect_Ptr(thisXCell, prevYCell), uLimX_Ptr(thisXCell, prevYCell), dudyVect);
yDiffusionFlux(yNextDiffusionFlux, uvect_Ptr(thisXCell, nextYCell), uLimX_Ptr(thisXCell, nextYCell), dudyVect);

for (d = 0; d < nSolvedDimensions; d++)
{
    hYFlux(d, i, j) = 0.5 * ((
        uPlusConvectionFlux[d] + uMinusConvectionFlux[d])
    - a * (uPlusY(d, i, j) - uMinusY(d, i, j)));
    pYFlux(d, i, j) = 0.5 * (yPrevDiffusionFlux[d] + yNextDiffusionFlux[d]);
}

void SDKTSolver::SetBoundaryFlux(int i, int j, BoundaryID bid, FluxBoundary *boundaryType)
{
    double uPlusConvectionFlux[4], uMinusConvectionFlux[4];
    double enforcedFlux[4];
    int d;
    double xPos, yPos;
    double wsPlus, wsMinus, a;

    // Get position
    xPos = domainXOffset + i * deltaX;
    yPos = domainYOffset + j * deltaY;
    switch (bid)
    {
        case North: case South:
            xPos += 0.5 * deltaX;
            break;
        case East: case West:
            yPos += 0.5 * deltaY;
            break;
        default:
            break;
    }

    // Compute boundary fluxes
    switch (bid)
    {
        case North:
            break;
        case South:
            break;
        case East:
            break;
        case West:
            break;
    }
}
case East: case West:
    yPos += 0.5 * deltaY;
    break;
}

switch (bid)
{
    case West:
        fluxValueBCFunction[bid](uMinusX.Ptr(i, j), xPos, yPos);
        break;
    case East:
        fluxValueBCFunction[bid](uPlusX.Ptr(i, j), xPos, yPos);
        break;
    case South:
        fluxValueBCFunction[bid](uMinusY.Ptr(i, j), xPos, yPos);
        break;
    case North:
        fluxValueBCFunction[bid](uPlusY.Ptr(i, j), xPos, yPos);
        break;
}

fluxFluxBCFunction[bid](enforcedFlux, xPos, yPos);

for (d = 0; d < nSolvedDimensions; d++)
{
    if (boundaryType[d] != FluxSetValue && boundaryType[d] != FluxPeriodic)
        switch (bid)
        {
            case West: uMinusX(d, i, j) = uPlusX(d, i, j);
                        break;
            case East: uPlusX(d, i, j) = uMinusX(d, i, j);
                        break;
            case South: uMinusY(d, i, j) = uPlusY(d, i, j);
                        break;
            case North: uPlusY(d, i, j) = uMinusY(d, i, j);
                        break;
        }

    // Calculate fluxes
    switch (bid)
    {
        case North: case South:
            wsPlus = yWaveSpeedFunction(uPlusY.Ptr(i, j));
wsMinus = yWaveSpeedFunction(uMinusY_Ptr(i, j))
;
a = (wsPlus > wsMinus) ? wsPlus : wsMinus;
yConvectionFlux(uPlusConvectionFlux, uPlusY_Ptr(i, j));
yConvectionFlux(uMinusConvectionFlux, uMinusY_Ptr(i, j));
for (d = 0; d < nSolvedDimensions; d++)
{
    hYFlux(d, i, j) = 0.5 * ((
        uPlusConvectionFlux[d] +
        uMinusConvectionFlux[d])
    - a * (uPlusY(d, i, j) - uMinusY(d, i, j)));
    if (boundaryType[d] != FluxPeriodic) pYFlux(d, i, j) = 0;
}
break;
}
case East: case West:
    wsPlus = xWaveSpeedFunction(uPlusX_Ptr(i, j));
    wsMinus = yWaveSpeedFunction(uMinusX_Ptr(i, j))
    ;
    a = (wsPlus > wsMinus) ? wsPlus : wsMinus;
xConvectionFlux(uPlusConvectionFlux, uPlusX_Ptr(i, j));
xConvectionFlux(uMinusConvectionFlux, uMinusX_Ptr(i, j));
for (d = 0; d < nSolvedDimensions; d++)
{
    hXFlux(d, i, j) = 0.5 * ((
        uPlusConvectionFlux[d] +
        uMinusConvectionFlux[d])
    - a * (uPlusX(d, i, j) - uMinusX(d, i, j)));
    if (boundaryType[d] != FluxPeriodic) pXFlux(d, i, j) = 0;
}
break;
}
void SDKTSolver::CalculateFluxBoundaryConditions()
{
    int i, j;
    j = 0;
    for (i = 0; i < nXPoints; i++) SetBoundaryFlux(i, j, South, fluxBCs[South]);
    j = nYPoints;
    for (i = 0; i < nXPoints; i++) SetBoundaryFlux(i, j, North, fluxBCs[North]);
i = 0;
for (j = 0; j < nYPoints; j++) SetBoundaryFlux(i, j, West, fluxBCs[West]);
i = nXPoints;
for (j = 0; j < nYPoints; j++) SetBoundaryFlux(i, j, East, fluxBCs[East]);
}

void SDKTSolver::CalculateTimeDerivative(VectorArray2d& uvect, double t, VectorArray2d& result)
{
    // this section is needed for the dudxVect called in sourceTerms
    int i, j, d;

    // Calculate H and P fluxes at each cell edge
    double uPlusConvectionFlux[4], uMinusConvectionFlux[4];
    double dudxVect[4], xPrevDiffusionFlux[4], xNextDiffusionFlux[4];
    double dudyVect[4], yPrevDiffusionFlux[4], yNextDiffusionFlux[4];
    double wsPlus, wsMinus, a;

    // Do X fluxes
    for (j = 0; j < nYPoints; j++) for (i = 0; i <= nXPoints; i++)
    {
        int thisYCell = j;

        int prevXCell = (i - 1 + nXPoints) % nXPoints;
        int nextXCell = i % nXPoints;

        for (d = 0; d < nSolvedDimensions; d++)
        {
            uPlusX(d, i, j) = uvect(d, nextXCell, thisYCell) - uLimX(d, nextXCell, thisYCell) * 0.5 * deltaX;
            uMinusX(d, i, j) = uvect(d, prevXCell, thisYCell) + uLimX(d, prevXCell, thisYCell) * 0.5 * deltaX;
            dudxVect[d] = (uvect(d, nextXCell, thisYCell) - uvect(d, prevXCell, thisYCell)) * deltaXRecip;
        }

        wsPlus = xWaveSpeedFunction(uPlusX.Ptr(i, j));
        wsMinus = xWaveSpeedFunction(uMinusX.Ptr(i, j));

        a = (wsPlus > wsMinus) ? wsPlus : wsMinus;
    }
ConvectionFlux(uPlusConvectionFlux, uPlusX.Ptr(i, j));
ConvectionFlux(uMinusConvectionFlux, uMinusX.Ptr(i, j));
DiffusionFlux(xPrevDiffusionFlux, uvect.Ptr(prevXCell, thisYCell), dudxVect, uLimY.Ptr(prevXCell, thisYCell));
DiffusionFlux(xNextDiffusionFlux, uvect.Ptr(nextXCell, thisYCell), dudxVect, uLimY.Ptr(nextXCell, thisYCell));
for (d = 0; d < nSolvedDimensions; d++) {
  hXFlux(d, i, j) = 0.5 * ((uPlusConvectionFlux[d] + uMinusConvectionFlux[d])
                - a * (uPlusX(d, i, j) - uMinusX(d, i, j)));
pXFlux(d, i, j) = 0.5 * (xPrevDiffusionFlux[d] + xNextDiffusionFlux[d]);
}
}
SetInstantaneousTime(t);
CalculateLimitedDerivs(uvect);
CalculateFluxes(uvect);
CalculateFluxBoundaryConditions();
for (j = 0; j < nYPoints; j++) for (i = 0; i < nXPoints; i++) {
  for (d = 0; d < nSolvedDimensions; d++)
  {
    result(d, i, j) = (hXFlux(d, i, j) - hXFlux(d, i + 1, j) + pXFlux(d, i + 1, j) - pXFlux(d, i, j)) * deltaXRecip
     + (hYFlux(d, i, j) - hYFlux(d, i, j + 1) + pYFlux(d, i, j + 1) - pYFlux(d, i, j)) * deltaYRecip;
  }
  sourceTermFunction(result.Ptr(i, j), uvect.Ptr(i, j), dudxVect);
}
double SDKTSolver::MaxWaveSpeed(VectorArray2d &uvect) {
  double thistc, mintc = 1e30;
  int i, j;
  double wsf;
for (j = 0; j < nYPoints; j++) for (i = 0; i < nXPoints; i++)
{
    wsf=xWaveSpeedFunction(uvect.Ptr(i, j));
    if (wsf>0)
    {
        thistc = deltaX / wsf;
        if (thistc < mintc) mintc = thistc;
    }
    wsf=yWaveSpeedFunction(uvect.Ptr(i, j));
    if (wsf>0)
    {
        thistc = deltaY / wsf;
        if (thistc < mintc) mintc = thistc;
    }
}

return 0.5*mintc;

class VectorFunction : public VectorFunctionBase<VectorArray2d, double>
{
friend class SDKTSolver;
public:

    VectorFunction()
    {
    }

    VectorFunction(SDKTSolver *sp) : solverPtr(sp)
    {
    }

    virtual ~VectorFunction()
    {
    }

    bool isStepAccepted;

    void compute(VectorArray2d& y, double t, VectorArray2d& result)
    {
        solverPtr->CalculateTimeDerivative(y, t, result);
    }

    double GetNonZeroElement()
    {
        return (rand() % 100) / 100.;
    }

    double cour(VectorArray2d& y, double t)
    {
        return 2.0 * solverPtr->MaxWaveSpeed(y);
Listing 8. sdtk.cpp

```c++
#include "SDKTSolver.h"
#include <cmath>
#include <cstdlib>
#include <fstream>
#include <sstream>
#include <iomanip>
#include "RollWaveEquations.h"
#include <fenv.h>

using namespace std;

double frnd(double val) { return val * static_cast<double>(rand()) / static_cast<double>(RAND_MAX); } // generates a random number between 0 and val

double modulo(double a, double b) {
  int result = static_cast<int>(a / b);
  return a - static_cast<double>(result) * b;
}
```
// Tuneable parameters here
double domainLength = 0.6;
int nXPoints = 1200;
double GridBoxSize = domainLength / nXPoints;
int nYPoints = 1;
double domainWidth = GridBoxSize * nYPoints;
double yOffSet = -0.5 * domainWidth;

const double pi = 4 * atan(1);
double omega = 1.0 / 20.0;

double u0;
double Fr0;
double Re0;
double Gamma;
double l;
double Hstop;

const double a = 1.0; // 1.2;
const double h0 = a * 0.003535836060767125; // h0 = a*hstop
const double pert = 0.0001;
double f = 0.47; // frequency of sinusoidal oscillation
SDKTSolver *solverPtr;

void InitialConditions(double *u, double x, double y)
{
    u[RollWaveEqns::H] = a * Hstop - 2 * frnd(pert) + pert; // random perturbations about h0
    u[RollWaveEqns::HU] = a * h0 * u0;
    u[RollWaveEqns::HV] = 0.0;
}

// Set in flow conditions on the 'West' boundary, where
downslope direction is West to East and cross slope is
North to South
// If periodic conditions are imposed this will not be applied
void InflowValueFunction(double *u, double x, double y)
{
    double instTime;
    instTime = solverPtr->GetInstantaneousTime();
    u[RollWaveEqns::H] = h0 + pert * cos(2 * pi * f * instTime); // sinusoidal inflow perturbation
    u[RollWaveEqns::HU] = h0 - 2 * frnd(pert) + pert; // random inflow perturbation
    u[RollWaveEqns::HV] = h0 * u0;
    u[RollWaveEqns::HV] = 0.0;
`/*
// Set in flow conditions with a .txt file of data instead (only 1 of these 2 functions can be used at a time)
void InflowValueFunction(double *u, double x, double y)
{
    int pertNumber = 100;
    int discTimeFactor = 50;
    double instTime;
    instTime = solverPtr->GetInstantaneousTime();
    double discreteTime, fractpart, inptpart;
    discreteTime = floor((solverPtr->GetInstantaneousTime() * discTimeFactor))/discTimeFactor;
    int discTimeIndex;
    discTimeIndex = modulo(discreteTime * discTimeFactor, pertNumber);
    double pertArrayH[pertNumber];
    ifstream in("ICsInFlowH.txt");
    if(!in)
    {
        cout << "Cannot open file \n";
        throw; // Throw an exception (end program)
    }
    for(int i=0;i<=pertNumber-1;i++)
    {
        in >> pertArrayH[i];
    }
    in.close();
    double pertH;
    int i=discTimeIndex;
    pertH = pertArrayH[i];
    double pertArrayHU[pertNumber];
    ifstream in2("ICsInFlowHU.txt");
    if(!in2)
    {
        cout << "Cannot open file \n";
        throw; // Throw an exception (end program)
    }
    for(int j=0;j<=pertNumber-1;j++)
    {
        in2 >> pertArrayHU[j];
    }
    in2.close();
    double pertHU;
    int j=discTimeIndex;
    pertHU = pertArrayHU[j];
`
```cpp
int main()
{
    srand(time(NULL));
    cout.precision(15);
    feenableexcept(FE_INVALID|FE_DIVBYZERO|FE_OVERFLOW);
    RollWaveEqns eqns;
    u0=RollWaveEqns::GetU0(h0);
    SDKTSolver solver(nXPoints, nYPoints, &eqns);
    solverPtr = &solver;
    solver.SetDomain(0, yOffSet, domainLength, domainWidth);
    solver.SetInitialConditions(InitialConditions);
    // solver.InputICs("initialConditions.txt"); // Input
    Initial Conditions from .txt data, overwriting any
    previously set
    solver.SetPeriodicBoundaryConditions();
    // Set in flow and 'free' outflow boundary conditions,
    // overwriting the periodic ones if this section is
    // uncommented
    /*
    solver.SetBoundaryConditionType(SDKTSolver::West,
        RollWaveEqns::H, SDKTSolver::GradientExtrapolate1,
        SDKTSolver::FluxSetValue);
    solver.SetBoundaryConditionType(SDKTSolver::West,
        RollWaveEqns::HU, SDKTSolver::GradientExtrapolate1,
        SDKTSolver::FluxSetValue);
    solver.SetBoundaryConditionType(SDKTSolver::West,
        RollWaveEqns::HV, SDKTSolver::GradientExtrapolate1,
        SDKTSolver::FluxSetValue);
    solver.SetBoundaryConditionType(SDKTSolver::East,
        RollWaveEqns::H, SDKTSolver::GradientExtrapolate1,
        SDKTSolver::FluxUseExtrapolated);
    solver.SetBoundaryConditionType(SDKTSolver::East,
        RollWaveEqns::HU, SDKTSolver::GradientExtrapolate1,
        SDKTSolver::FluxUseExtrapolated);
    solver.SetBoundaryConditionType(SDKTSolver::East,
```
RollWaveEqns::HV, SDKTSolver::GradientExtrapolate1, SDKTSolver::FluxUseExtrapolated);

solver.SetBoundaryConditionFunctions(SDKTSolver::West, InflowValueFunction);

*/
solver.SetLimiter(SDKTSolver::WENO);
// solver.SetLimiter(SDKTSolver::MinMod2);

stringstream oss;
oss << "outputs/output_" << setw(6) << setfill('0') << 0 << ".txt";
solver.OutputResults(oss.str());

int k =0;
int numOutputs = 2001;
double intTimeStep = 0.01;
double outputTimeStep = 0.1;

double jMod = outputTimeStep/intTimeStep;
int iMax = jMod*numOutputs;

for (int i=1; i<iMax; i++)
{
    solver.IntegrateFor(intTimeStep);
    cout.precision(15);

    int j = modulo(i,jMod);

    if (j==0)
    {
        k=k+1;
        stringstream oss;
        oss << "matlab/outputs/output_" << setw(6) << setfill('0') << k << ".txt";
        solver.OutputResults(oss.str());
    }
}
return 0;
Bibliography


