GEOMETRIC STRUCTURES ON THE ALGEBRA OF DENSITIES

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This thesis is concerned with certain differential-geometric properties of the algebra of densities. The main results to be defended within this thesis are as follows:

- We find that the space of (anti-)self adjoint lifts of differential operators, $\mathrm{DO}^n(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) \to \mathrm{DO}^n(\mathcal{F}(M))$, equivariant with respect to the algebra of divergenceless vector fields is $2\lceil n/2 \rceil$ dimensional for non-singular λ, proposition 3.2.23. The explicit form of the maps are:

$$D \mapsto D_\varrho + \left(D_\varrho - (-1)^n D_\varrho^\dagger\right) \frac{\widehat{w} - \lambda}{2\lambda + \delta - 1} + \sum_{j=1}^{n} \left(L_\varrho(1) c_j + L_\varrho^\dagger(1) d_j\right) (\widehat{w} - \lambda)^j,$$

  where the c_js and d_js obey a functional equation determined by proposition 3.1.23. The proof of this result uses theorem 3.2.7, 3.2.19 and 3.2.20, as well as the construction of the vertical Taylor expansion. This work is published in the Journal of Mathematical Physics under the title *Operator pencil passing through a given operator*, [10]. It is coauthored with H. Khudaverdian. I have substantial input in these reults.

- We show that there exists a unique strict projectively equivariant lifting of differential operators in $\mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ for non-resonant δ, corollary 3.3.8. Using this we determine the space of (anti-)self adjoint liftings, corollary 3.3.11 and proposition 3.3.14. This work uses the full symbol calculus developed [22, 59] and [60]. Some of these results are contained in the paper of the author quoted above [10], and other results will be the subject of a seperate paper.

- We describe a canonical lifting of Poisson structures from the base manifold to the algebra of densities, theorem 6.1.27. The proof of this lemma uses the fact that their exists a natural Batalin-Vilkovisky structure on $\Pi T^*\widehat{M}$, and the properties of this operator, lemma 6.1.24. The fact that this map is natural, i.e. $\mathfrak{X}(M)$-equivariant, is proved in proposition 6.1.35. The majority of this work is contained in the paper [9]. It is currently on arXiv and being prepared for submission. Moreover we answer a question left open in the paper and show that the singular point of the lifting corresponds to a non-trivial $C^\infty(\Pi TM)$-cocycle.

- The notion of weighted Poisson structure is introduced in chapter 5. As any bundle of densities is geometric it has a natural action of $\mathfrak{X}(M)$ and a weighted Poisson structure on $\mathcal{F}^\lambda(M)$ is simply an L_∞-structure on this vector space that has derivations in each argument corresponding to this action, see definition 6.2.8. We classify such objects in theorem 6.2.11 as functions on $\Pi T^*\widehat{M}$ that satisfy the Maurer-Cartan equation, $(S, S) = 0$, and are divergenceless with respect to the operator in the canonical Batalin-Vilkovisky structure on $\Pi T^*\widehat{M}$, see above. Upon the completion of my thesis I intend to organise this half of chapter 5 into the format of an article to be submitted for publication.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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I would like to thank my supervisors, H. Khudaverdian and Th. Voronov, for their continual support throughout my PhD. They have not only taught me a vast amount but also helped me discern what is important and fundamental in the questions we ask about mathematics.

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Chapter 1

Introduction

History and Motivation

The algebra of densities can be seen to have origins dating back to the 19^{th} century where densities were used to find invariants of the modular group. Since then they have continued to be a source of projective invariants and cocycles related with the projective group, most notably the Schwarzian derivative. One of the first times that the algebra of densities appears in the literature in a similar guise to the way we shall introduce it, is in the work of T.Y. Thomas. He showed that a projective connection on a manifold allows one to determine a canonical affine connection on the total space of a certain bundle which is now known as Thomas' bundle. More recently they have appeared, with the definition we shall use, by H. Khudaverdian and Th. Voronov when studying second order operators generating certain brackets. Of prime importance in this situation is the case of Gerstenhaber algebras and in particular the Batalin-Vilkovisky operator on the odd cotangent bundle. They have also been used by V.Y. Ovsienko and his group in the area of equivariant quantization which is a topic we shall come across in the text. Densities also regularly appear in physics. For example the correct interpretation of a wavefunction is a half-density on a manifold, and this explains their transformation properties under the Galilean group.

These results motivate a study into the geometric structure of the algebra of densities as an object in their own right. We shall see that by considering them as a whole algebra many classical results have a clear geometrical picture. Moreover one finds that there are a wealth of areas within this algebra still to explore. We focused on two fundamental classes of objects, differential operators and Poisson structures. The results we find lead to interesting formula for certain equivariantly defined differential operators which can be applied to gain a wide class of cocycles similar to the Schwarzian derivative. We also find very intimate links with Batalin-Vilkovisky geometry and the methods we use show that it may be useful to consider the full algebra of densities when entering into this arena.

Outline of Chapters

We shall now give a brief outline of the chapters in the text:

1. In the first chapter we introduce the readers to the algebra of densities and explain how it is related to Thomas' bundle. We introduce additional geometric structures on the algebra, in particular the volume form, which will be of fundamental importance in this text. In the final part of the chapter we calculate the de Rham cohomology of the space. Using this we explain how the algebra of densities is a universal object by pullbacks of differential forms.

2. The second chapter initiates our study of differential operators on the algebra of densities. The first half contains all definitions and concepts that we shall use for discussing the structure of differential operators. Certain methods which shall be used regularly in proofs, for example the vertical Taylor expansion, are also introduced here. In the second half of the chapter we study the notion of equivariant liftings of differential operators. We find a complete classification of such liftings for the projective group and the group of volume preserving diffeomorphisms.

3. Next we turn to the more technical aspects of the structure of differential operators and their relation with brackets and groupoids. We firstly give an explicit description of the form of a differential operator on the algebra of densities of a curve in terms of classical combinatoric data. We then explain why this fails for higher dimensional manifolds. In the second half of the chapter we recall the KV-groupoid and reinterpret it in terms of a quantization map. Applying this we generalize it to higher order differential operators.

4. This is the last chapter focused on the study of differential operators and we look in more detail at the algebra of densities on curves. The situation here is unique as all geometric bundles can be described in terms of densities. We explain how the classical theory of differential operators fits into the algebra of densities. Using these ideas we construct a family of invariant brackets and show how they can be used to determine classical invariants, e.g. the unintegrated Virasoro cocycle. Finally we study modular differential operators in the language we have developed.

5. In the final chapter of this thesis we study the interaction of Poisson structures and the algebra of densities. We find canonical liftings of a Poisson structure on the base which allows us to determine how a volume form should evolve. In the second half of this chapter we introduce weighted Poisson structures and determine a result, analogous to the unweighted case, which considers them as a generalised Poisson structure on the algebra of densities.

Remarks on notation and scope

There are a few caveats that we should alert the reader to during this text. The first is that wherever possible we shall use supermanifold notation. In particular this means that we do not write wedge products between differential forms or functions on the odd cotangent space. A standard amount of familiarity with the language of supermanifolds is therefore assumed, however all prerequisites are within the appendix. One thing that we must emphasize along this line however is that the manifolds considered in chapter 2 and 4 are all purely even unless explicitely stated otherwise. Except for this the reader may assume the results refer to all manifolds even or super.

We also give the following short table which contains the main notations we shall use throughout the text:

Symbol	Description	Reference
$\mathfrak{A}(M)$	Fibrewise polynomial functions on ΠT^*M.	
$\Omega(M)$	Algebra of differential forms on M.	
$\mathcal{S}(M)$	The space of algebraic symbols on M.	
$\mathfrak{X}(M)$	Vector fields on M.	
$\mathcal{F}^{\lambda}(M)$	Densities of weight λ.	$p.15$, $p.155$
$\mathcal{F}_S(M)$	The algebra of densities over S.	$p.17$
$\widehat{M}$	Thomas' bundle.	$pp.19-20$
$\mathcal{S}(-)$	The Schwarzian derivative.	$pp.96-97$
$\Gamma(1)$	The modular group.	$pp.107-108$
$(-,-)$	Schouten-Nijenhuis bracket.	$p.122$
Δ	Batalin-Vikovisky/Khudaverdian's operator.	$p.126$
$\mathrm{Ber}(-)$	The Berizinian of a matrix.	$p.153$

Chapter 2

First Concepts and Results

2.1 Basic definitions and first properties.

The algebra of densities, as we shall discuss it, was first introduced by H. Khudaverdian and Th. Voronov to study certain properties of second order differential operators, [47, 48], which we shall study in later chapters. The algebra has many relations to classical objects that have been studied throughout the 20$^{\text{th}}$ century. In this chapter we explain its relation to Thomas' bundle and later its relation with projective geometry. There are also links with Pauli's unified electromagnetic and gravitational theory. We close the chapter by computing the de Rham cohomology of the algebra.

2.1.1 The algebra of densities.

We begin with a description of the algebra of densities and its relation with some classical notions in projective geometry.

Densities and volume forms.

Let M be a smooth manifold. By $\mathcal{F}_M^\lambda$ we denote the line bundle of densities of weight $\lambda \in \mathbb{C}$. This bundle can be defined using the following data: Take a coordinate atlas of the manifold M, $\{U_\alpha \stackrel{\varphi_\alpha}{\longrightarrow} M\}$. Then the gluing data for $\mathcal{F}_M^\lambda$ is the cocycles:

$$J^\lambda_{\alpha,\beta} = \left|\mathrm{Ber}\frac{\partial \varphi_{\alpha\beta}}{\partial x_\alpha}\right|^\lambda,$$

where we shall take the principal branch of the Jacobian. This is a geometric bundle in the sense that there are natural lifting of both $\mathrm{Diff}(M)$ and $\mathfrak{X}(M)$ to this bundle.

For a smooth orientable manifold these in fact constitute all 1-dimensional geometric bundles whilst for a non-orientable manifold there exists a non-trivial geometric line bundle that squares to $\mathcal{F}^2_M$. We shall write a density of weight λ using the notation $s|Dx|^\lambda$.

The original importance of densities, and a property we shall use throughout this text, is the fact that they can be integrated. To be more precise there is a natural pairing:

$$\mathcal{F}^\lambda(M) \otimes \mathcal{F}^{1-\lambda}_c(M) \xrightarrow{\int} \mathbb{R}, \tag{2.1.1}$$

which is defined via integration using a partition of unity. Moreover the topological dual of $\mathcal{F}^\lambda(M)$ (resp. $\mathcal{F}^{1-\lambda}_c(M)$) is the closure of $\mathcal{F}^{1-\lambda}_c(M)$ (resp. $\mathcal{F}^\lambda(M)$) with respect to the above pairing, [56, 88].

From the above we see that a nowhere vanishing section of $\mathcal{F}^1(M)$, ϱ, also known as a volume form, induces a non-degenerate pairing on the space of compactly smooth functions on M:

$$\langle f, g\rangle = \int fg\varrho.$$

We now give examples of manifolds that have a natural volume form.

Example 2.1.1. Let (M, g) be a manifold with a metric. Then $\sqrt{|\det(g)|}|Dx| \in \mathcal{F}^1(M)$ and is clearly nowhere vanishing. We thus see that all manifolds admit a volume form structure.

Example 2.1.2. The odd tangent bundle ΠTM has a natural volume form that has the local form $|D(x, dx)|$. If M is an even orientable compact manifold and $\omega \in C^\infty(\Pi TM)$ is a differential form we have that

$$\int_{\Pi TM} \omega|D(x, dx)| = \int_M \omega^{top},$$

where $(-1)^{top}$ denotes the projection $\Omega^\bullet_M \to \Omega^{\dim(M)}_M$. On a supermanifold this identification does not go over verbatim but can be considered to be the starting point of one theory of integration on supermanifolds, see [8, 28, 83].

Example 2.1.3. The even cotangent bundle, T^*M, also admits a natural volume form $|D(x, p)|$. In fact this is a specific example of that of an even symplectic manifold (M^{2n}, ω) where the associated volume form is $\frac{1}{n!}\omega^n$.

Example 2.1.4. The final example is probably the least trivial of the above. Fix a Lie group G then up to scale there is a unique left invariant (resp. right invariant) density which is called the left (rep. right) Haar measure, see [35, 67]. It immediately follows that the right and left invariant are multiples of one another. If the Lie group G is compact then we can uniquely determine the Haar measure by the condition that $\int_G \mu = 1$. We shall denote the associated density by μ_g.

We shall also have some interest in densities defined on finer spaces than smooth manifolds, in particular complex manifolds. For a complex manifold we have a bigrading $\mathcal{F}_M^{k,l}$, where $k, l \in \mathbb{Z}$, where a section is locally given by the formula $f(z, \overline{z})|Dz|^k|D\overline{z}|^l$. We shall reserve the notion $\mathcal{F}_M^k$ for $\mathcal{F}_M^{k,0}$ and consider only holomorphic sections. This bundle may no longer be trivial and therefore we cannot take arbitrary powers of it. It is the k^{th} power of the canonical bundle so we can also say what densities are on a smooth variety. These objects do have a notion of integration using Serre duality however we shall not use this throughout but simply the residue calculus associated with complex forms throughout the text.

The algebra of densities.

Although we speak of *the* algebra of densities we shall see that this is in fact a bit of a misnomer as in fact their exists a whole family of such algebras. Let us recall the definition of a semigroup:

Definition 2.1.5. A semigroup is a triple (S, e, m), where S is a set, $e \in S$ and $m : S \times S \to S$, such that m is an associative multiplication with identity e. That is a semigroup is a group without inverses.

Definition 2.1.6. Let $S \subset (\mathbb{C}, 0, +)$ be a non-trivial sub semigroup. We then define the algebra of densities over S, $\mathcal{F}_S(M)$, to be

$$\mathcal{F}_S(M) = \oplus_{\lambda \in S} \mathcal{F}^\lambda(M).$$

The multiplication is induced from the natural map $\mathcal{F}^\lambda(M) \otimes \mathcal{F}^\mu(M) \to \mathcal{F}^{\lambda+\mu}(M)$.

Most of the situations we shall come across it will be sufficient just to consider $S = \mathbb{Q}$. However it also useful to bear in mind the following semigroups: $\mathbb{Z}$, $\mathbb{Z}\lambda$, $\mathbb{R}$,

$\mathbb{C}$. Note that $0 \in S$ for all semigroups $S \subset \mathbb{C}$ and therefore we have that $C^\infty(M) = \mathcal{F}^0(M) \subset \mathcal{F}_S(M)$.

Example 2.1.7. Consider a smooth algebraic variety X, so that $\mathcal{F}^1_X = K_X$. Then if we set $S = \mathbb{N}$ the associated ring, $\mathcal{F}^1_{\mathbb{N}}(X)$ is also known as the pluricanonical ring and has been studied by algebraic geometers in relation to the Mori programme, [33, 42].

An important thing to note is that the presheaf $U \mapsto \mathcal{F}_S(U)$ is never a sheaf of algebras over M for any S as above. This follows from simple properties of the commutation of direct limits and direct sums. The sheafification of the presheaf will still obey many of the results that we shall prove concerning $\mathcal{F}(M)$. We would still like to interpret the algebra of densities as functions of some space over M, we shall do this below after introducing another algebra related to $\mathcal{F}(M)$ which we shall have call to use later in the text.

The Laurent algebra and formal dual.

Let $A = \oplus_\lambda A_\lambda$ be an algebra graded over an ordered semigroup S. We now explain how one can construct a formal (positive and negative) Laurent algebra from this data. We define

$$D_+(S) = \{D \subset S : D \text{ is generated by a finite number of positive elements}\}$$

For example we have that $D_+(\mathbb{R})$ is the sum of a finite number of positive cones of lattices in $\mathbb{R}$. From this we see that for subsemigroups of the reals (with the induced ordering) elements of $D_+(S)$ are closed under addition.

Definition 2.1.8. Let A be a graded algebra indexed over $S \subset \mathbb{R}$. The positive Laurent algebra of A, A_+, is defined to be the set of elements

$$A_+ = \{a = (a_\lambda)_{\lambda \in S} : \exists D(a) \in D_+(S), \forall \lambda \notin D,$$

$$a_\lambda = 0 \text{ except for finitely many elements.}\}$$

One must just check that the above does indeed have an algebra structure. As $A_+ \subset \prod_\lambda A_\lambda$ it inherits addition pointwise. For the multiplicative structure take two elements, a and b, then we may assume that there exists $D \in D_+(\mathbb{R})$ such that only finitely many points of a and b are non-zero outside D. We need to show that the naturally induced multiplication,

$$ab_\lambda = \sum_\mu a_\mu b_{\lambda-\mu},$$

is well defined, that is the sum is finite. This follows immediately from the definition. One can define the negative Laurent algebra, A_-, analogously.

Example 2.1.9. Let $S = \mathbb{Z}$ and A be generated by x, that is $A = \mathbb{R}[x^{-1}, x]$. Then $A_+ = \mathbb{R}[x^{-1}, x]]$ and $A_- = \mathbb{R}[[x^{-1}, x]$.

Notation 2.1.10. Let $S \subset \mathbb{R}$ be a semigroup. Then the Laurent algebra associated to S shall be denoted by $\mathcal{F}_{S+}(M)$.

We shall not need the formal Laurent algebra until the last chapter where it is the correct algebra to use to define weighted Poisson structures.

2.1.2 Thomas' bundle and the algebra of densities.

We now introduce Thomas' bundle and explain how it allows us to interpret the algebra of densities as functions on a smooth space using some general algebraic techniques.

Thomas' bundle.

Thomas' bundle is a classic geometric object that has deep relations with projective geometry see [30, 69, 80]. We shall give a slight variant of the original definition however we shall show below that the spaces are essentially isomorphic.

Definition 2.1.11. Thomas' bundle, $\widehat{M}$, is the positive part of the total space of $\mathcal{F}_M^{-1}$ without the zero section.

Let us explain what we mean by the positive part of this bundle: We have the projection $\mathcal{F}_M^{-1} - 0 \to M$ and a coordinate patch of U lifts to a coordinate patch, $\widehat{U}$, on $\mathcal{F}_M^{-1}$. The chart $\varphi : U \to M$ lifts to

$$\widehat{\varphi} : \widehat{U} \to \widehat{M},$$

$$\widehat{\varphi}_{\alpha,\beta}(x,t) = (\varphi_{\alpha,\beta}(x), |J_{\alpha,\beta}(x)|t). \tag{2.1.2}$$

This is the standard construction for an atlas associated to a vector bundle with the additional condition that the fibre variable is never zero. The positive part is the space only containing those $t > 0$. One can see that this is a coordinate independent notion. We shall keep the notation $(\widehat{U}_\alpha, \widehat{\varphi}_\alpha)$ for the charts of this space.

As mentioned above the classical Thomas bundle is defined to have the gluing data defined via the logarithm of the Jacobian rather than the Jacobian. There does exist an isomorphism from $\widehat{M}$ to this space by sending t to $\log t$.

The weight operator and graded subalgebra.

As we have that $\widehat{M}$ is the subspace of a vector bundle[1] it inherits the fibrewise weight operator,

$$\widehat{w} := t\partial_t. \tag{2.1.3}$$

We now go through some abstract algebra that allows us to recover the algebra of densities. Let $(A, \widehat{w})$ be an algebra with an even vector field $\widehat{w}$ (that is a derivation of the algebra that satisfies any locality conditions inherent in the algebra). We then define $A_\lambda \subset A$ to be the λ^{th} eigenvalue subspace of $\widehat{w}$:

$$A_\lambda = \{a \in A : \widehat{w}a = \lambda a\}.$$

The graded algebra associated to $(A, \widehat{w})$ is then equal to $\bigoplus A_\lambda$ where the sum is taken over all eigenvalues of the $\widehat{w}$.

Lemma 2.1.12. *The full algebra of densities, $\mathcal{F}_{\mathbb{C}}(M)$, is isomorphic to the graded algebra associated to $(C^\infty(\widehat{M}), \widehat{w})$.*

Proof. Take a coordinate chart, $U \to M$, and its lift $\widehat{U} \to \widehat{M}$. Then a local function $f(x,t)$ is an eigenvector for $\widehat{w}$ with eigenvalue λ iff

$$t\partial_t f(x,t) = \lambda f(x,t).$$

For any λ we have that the function t^λ, where we take the principle part of the logarithm to define the power, satisfies the above differential equation. We will have completed the proof if we can show that $f(x)t^\lambda$ generate all eigenvalues. This is simple

[1] It is related with the orbits of the $\mathrm{Diff}(M)$-action. The vector bundle structure ensures that this operator is linear as expressed.

for t^λ is invertible so for an arbitrary eigenvalue of $\widehat{w}$ of weight λ, $f(x,t)$, consider $\frac{f}{t^\lambda}$. Then this has weight zero, but as t is invertible, the weight zero functions are just those independent of t, i.e. those pulled back from the base manifold. This completes the proof. $\square$

Corollary 2.1.13. *For any semigroup S the algebra of densities $\mathcal{F}_S(M)$ has a natural embedding into smooth functions on the Thomas bundle, $C^\infty(\widehat{M})$.*

We can thus consider $\widehat{M}$ with two natural algebra either $\mathcal{F}_S(M)$ or $C^\infty(\widehat{M})$. Where ambiguity may exist we shall refer to these spaces as $(\widehat{M}, \mathcal{F}(M))$ or $(\widehat{M}, C^\infty)$ respectively.

The fact that $\mathcal{F}_S(M)$ contains only the algebraic part of the functions means that it has many similarities to the space $\widehat{M}$ but is not identical. For example $\widehat{M}$ is homotopic to M, $\widehat{M} \simeq M$, hence it has the same cohomology, however the space $(\widehat{M}, \mathcal{F}_{\mathbb{Q}}(M))$ has non-trivial cohomology as we shall see below. In fact the space $(\widehat{M}, \mathcal{F}_{\mathbb{Q}}(M))$ is more similar to $M \times \mathbb{C}^\times$. This is because the functions t^λ correspond to algebraic functions on the complexified line bundle, $\mathbb{C} \otimes \mathcal{F}_M^{-1} - 0$. We can again define a logarithm here but it is no longer an isomorphism and taking Fourier coefficients we have that an arbitrary function may locally be written as:

$$\sum_{n \in \mathbb{Z}} f_n(x) t^n.$$

We can thus think of the functions for a general semigroup S as corresponding to algebraic functions and some transcendental powers.

From now on we shall always assume that our semigroup contains $\mathbb{Z}$. We shall drop reference to it unless it is explicitly needed.

2.1.3 The volume form and inner product.

The spaces $(\widehat{M}, \mathcal{F}(M))$ and hence $(\widehat{M}, C^\infty)$ obey a universal property in terms of smooth inner products on $C_c^\infty(M)$ which are volume forms. In particular we have the following:

Lemma 2.1.14. *There is an isomorphism from sections of the bundle $\widehat{M} \to M$ and volume forms on M: $\Gamma(M, \widehat{M}) \xrightarrow{\sim} \mathcal{F}^1(M)^\times$, that is natural with respect to local diffeomorphisms.*

Proof. Let us take a section $\psi : M \to (\widehat{M}, \mathcal{F}(M))$, i.e. $p \circ \psi = 1_M$. Then, as we have enforced the locality condition, we need consider only the value $\psi^*(t)$. We must have that it is invertible, so nowhere vanishing, and its transformation under coordinates immediately implies that it is the inverse of a volume form. □

As $\widehat{M}$ satisfies this property it is natural to expect that it has a canonical volume form itself. This is indeed the case we shall wish to describe this object both algebraically and using the smooth functions on $\widehat{M}$. The algebraic part follows form the natural pairing between compactly supported densities, that is we extend the inner product from equation (2.1.1) to the full algebra:

$$\langle ft^\lambda, gt^\mu \rangle = \begin{cases} \int_M fg & \text{if } \lambda + \mu = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the above we shall assume that either ft^λ or gt^μ has compact support when considered as sections of $\mathcal{F}^\lambda(M)$ or $\mathcal{F}^\mu(M)$. Our first goal will be to interpret this integration as a residue formula. The main property that an integral should obey is that the divergence of a vector valued density should be zero, [66], so we can interpret the integration as a cohomology class. This is standard for powers of a variable, for example in complex analysis, and we have that the only non trivial integral we have is given by the residue:

$$\int t^\lambda |Dt| = C\delta(\lambda + 1),$$

where C is the constant of integration. This is due to the fact that we can always find an antiderivative for t^λ for $\lambda \neq -1$, while for $\lambda = -1$ the antiderivative is $\log t$ which is not in our algebra. We shall set $C = 1$ without loss of generality. Then if we consider the inner product defined above we see that it is equal to:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_M \operatorname{res}\left(\frac{\mathbf{fg}}{t^2}\right),$$

where the residue is defined as above. In particular we then expect to have a volume form

$$\varrho_0 := \frac{1}{t^2}|D(x,t)|. \tag{2.1.4}$$

Proposition 2.1.15. *The object defined above, equation* (2.1.4)*, is a well defined section of* $\mathcal{F}^1(\widehat{M})$.

Proof. The proof follows by considering the transformation laws of all the objects involved and using that the Jacobian of the transformation splits into an upper block triangular matrix. □

We therefore have that the space $(\widehat{M}, C^\infty_{\widehat{M}})$ has a canonical inner product defined on compactly supported functions. The inner product on $(\widehat{M}, \mathcal{F}(M))$ can also be calculated using the same object by using the residue formula as outlined above. This is another instance where we see that $(\widehat{M}, \mathcal{F}(M))$ corresponds to a space isomorphic to $M \times \mathbb{C}^\times$ rather than $M \times \mathbb{R}^>$. It must be emphasized that although the integrations appear to have the same form they are in practice very different. This is because the inner product on $\mathcal{F}(M)$ only requires compact support with respect to the base coordinates whilst that on $C^\infty(\widehat{M})$ requires compact support on the fibre coordinate as well. In particular we cannot integrate any of the elements of $\mathcal{F}(M) \subset C^\infty(\widehat{M})$ with respect to the volume form on $(\widehat{M}, C^\infty)$.

Remark 2.1.16. The inner product we have defined is not necessarily symmetric. This is because if we have odd coordinates on the base manifold then the integration does not obey this property.

Before we close this section we shall give a nice result concerning the Thomas bundle of a compact Lie group G. Note that the association $M \mapsto \widehat{M}$ is *not* a functor and therefore the following result is in theory non-trivial:

Lemma 2.1.17. *Let G be a compact Lie group then the spaces $(\widehat{G}, \mathcal{F}(G))$ and $(\widehat{G}, C^\infty)$ have the structure of a group.*

Proof. A volume form identifies $(\widehat{M}, C^\infty_{\widehat{M}})$ with the smooth manifold $M \times \mathbb{R}^>$ and the group action is defined as the product for G a Lie group. We shall thus only prove it for the space $(\widehat{G}, \mathcal{F}(G))$ where we shall just go through the algebraic manipulations. Let us denote by $(G; m, a, e)$ the structure of the Lie group G:

$$m : G \times G \to G, \quad a : G \to G, \quad e : * \to G,$$

that satisfy the axioms of a Lie group. Moreover let μ_G denote the canonical Haar measure on the compact Lie group so $\mu_G t \in \mathcal{F}^1(G)$, see example 2.1.4. We then define the group structure $((\widehat{G}, \mathcal{F}(G)); \widehat{m}, \widehat{a}, \widehat{e})$ as follows:

$$\widehat{m}^*(ft^\lambda) = m^*\left(\frac{f}{\mu_G^\lambda}\right)(\mu_G(x_1)t_1\mu_G(x_2)t_2)^\lambda,$$

$$\widehat{a}^*(ft^\lambda) = a^*\left(\frac{f}{\mu_G^\lambda}\right)\frac{1}{\mu_G^\lambda t^\lambda},$$
$$\widehat{e}^*(ft^\lambda) = e^*\left(\frac{f}{\mu_G^\lambda}\right).$$

One can check that this is the exact multiplication structure for $G \times \mathbb{C}^\times$ where we consider only algebraic coordinates on $\mathbb{C}^\times$. It follows that all the axioms hold for a group which can be easily checked. □

2.1.4 Geometric bundles for the algebra of densities and de Rham cohomology.

We shall now begin our study of the algebra of densities by considering the de Rham cohomology. This is a simple invariant of an algebra and our calculations will highlight the fact that $(\widehat{M}, \mathcal{F}(M)) \simeq M \times \mathbb{C}^\times$. As mentioned earlier the space $(\widehat{M}, C^\infty)$ is homotopic to M and thus has equivalent cohomology. It therefore is of interest to us only to study the algebra of densities. Before we do this we shall use the identifications we have made above to give some general geometric objects associated to the algebra of densities. Let us take a geometric bundle, $E(M)$, that is a bundle that admits a lifting of all diffeomorphisms and vector fields from the base manifold, e.g. the (co)tangent bundle with even or odd fibres. Then we can lift the weight operator on $\widehat{M}$ to this bundle.

Definition 2.1.18. We shall write $(E(M), E\mathcal{F}(M))$ to denote the set $E(M)$ with functions the graded algebra associated to the algebra $(C^\infty(E(M)), \widehat{w}_E)$.

In the case when we index over the rationals this definition will agree with the naïve algebraic methods to construct these spaces. For example the de Rham complex of the algebra $\mathcal{F}_\mathbb{Q}(M)$, $(\Omega^\bullet(\mathcal{F}_\mathbb{Q}(M)), d)$, [33, 36], is isomorphic to the graded algebra generated by $(\Omega^\bullet(\widehat{M}), d)$ with respect to

$$\widehat{w}_{\Pi T\widehat{M}} = t\partial_t + dt\partial_{dt}.$$

It is important here that the de Rham differential has weight 0 to actually generate the de Rham complex. This interplay between the algebraic and differential geometric shall be used throughout and explored in more detail for differential operators in the next chapter.

The de Rham cohomology.

We now calculate the de Rham cohomology of the space $(\widehat{M}, \mathcal{F}(M))$. We shall see that it is independent of the indexing semigroup but, as we mentioned earlier, is only isomorphic to the algebraic definition of the de Rham complex if the semigroup is a subsemigroup of $\mathbb{Q}$, see also lemma 3.1.4. Now we also have the fibrewise weight operator on the bundle $\Pi TM \to M$,

$$\widehat{w}_1 = dx^i \partial_{dx^i} + dt\partial_{dt},$$

and this commutes with $\widehat{w}_2 = w_{\Pi T\widehat{M}}$. This means we can form a bigraded complex, $\Omega^{\bullet,\bullet}(\widehat{M})$. The de Rham differential has the commutation relations: $[\widehat{w}_1, d] = \widehat{w}_1$ and $[\widehat{w}_2, d] = 0$ and therefore the de Rham differential acts with weight $(1, 0)$:

$$d : \Omega^{n,\lambda}(\widehat{M}) \longrightarrow \Omega^{n+1,\lambda}(\widehat{M}).$$

Our goal shall thus be to calculate the graded cohomology groups, $H^{\bullet,\bullet}(\widehat{M})$.

Proposition 2.1.19. *We have the following graded cohomology groups:*

$$H^{n,\lambda}(\widehat{M}) \cong \begin{cases} 0 & \lambda \neq 0 \\ H^n(M) \oplus H^{n-1}(M) & \lambda = 0. \end{cases}$$

The isomorphism in the second case is non-canonical.

Proof. Fix a coordinate chart $U \to M$ then the form of an element $\widehat{\omega} \in \Omega^{n,\lambda}(\widehat{U})$ has the form:

$$\widehat{\omega} = t^\lambda\big(\omega + \frac{dt}{t}\theta\big),$$

where $\omega \in \Omega^n(U)$ and $\theta \in \Omega^{n-1}(U)$. The condition that $d\widehat{\omega} = 0$ then becomes:

$$t^\lambda\Big(d\omega + d(\log t)\big(\lambda\omega - d\theta\big)\Big) = 0.$$

Thus we have for $\lambda \neq 0$ there exists a unique choice of ω given a θ such that $d\widehat{\omega} = 0$. As $\Omega^{n,\lambda}_{\widehat{M}}$ does form a sheaf we can glue this condition to show that for $\lambda \neq 0$ any form in $\mathrm{Ker}(d)$ has the following (well defined) expression for $\theta \in \Omega^{n-1}(M) \otimes \mathcal{F}^\lambda(M)$:

$$t^\lambda\Big(\frac{1}{\lambda}d\theta + d(\log t)\theta\Big).$$

This form is in fact a coboundary, namely it is the derivative of

$$\frac{t^\lambda}{\lambda}\theta,$$

which is well defined, it is a horizontal differential form. We have thus showed there is no cohomology if $\lambda \neq 0$ in the bigrading. We now turn to the calculation of $H^{\bullet,0}(\widehat{M})$. Fix a flat connection γ and using the transformation laws of the variable t we define an isomorphism $\Omega^{n,0}(\widehat{M}) \leftarrow \Omega^n(M) \oplus \Omega^{n-1}(M)$ defined as:

$$(\omega, \theta) \longmapsto \omega - \gamma\theta + \tfrac{dt}{t}\theta.$$

If our connection comes from a volume form, $\gamma = -d\log(\varrho)$ then we may rewrite the image of (ω, θ) as:

$$\omega + d\log(\varrho t)\theta.$$

Using the fact that the connection is flat we then find that this lift commutes with the de Rham differential and hence the result follows. □

In fact this result is analogous to a result later in the text however the relation requires applications of the odd Fourier transform which we shall not use in this text. This result again emphasizes the idea that $(\widehat{M}, \mathcal{F}(M)) \simeq M \times \mathbb{C}^\times$ using the product formula for cohomology.

Remark 2.1.20. We can now explain in more detail the universality of the algebra of densities, $(\widehat{M}, \mathcal{F}(M))$, in the case of an even compact oriented manifold. In this case the choice of a nowhere vanishing differential form of top rank on M, ϱ defines an isomorphism $H^{\dim(M)}(M) \to \mathbb{R}$ and this is an abstract integration on the manifold M with respect to ϱ[2]. If our base manifold is orientable then so is $\widehat{M}$ and corresponding to the volume form ϱ_0, equation 2.1.4, we have the form:

$$\varrho_0 = \frac{dtdx^1 \cdots dx^n}{t^2}.$$

This is a form of weight $(n+1, -1)$ and as it is necessarily closed it must be exact, and we have a canonical choice of coboundary, the unique horizontal form:

$$-\frac{dx^1 \cdots dx^n}{t}.$$

Then we pull this form back using ϱ and this gives us the negative of the volume form associated to our original density explaining how we may pull back forms to interpret the universality of the algebra of densities.

[2]This is not equivalent to standard integration as the identification sets $\int_M \varrho = 1$ for any choice of volume form.

Chapter 3

Differential Operators I: Pencils and Lifts.

In this chapter we study of differential operators on the algebra of densities. The results of this section are essentially contained within the paper [10] coauthored with my supervisor Hovhannes Khudaverdian. We begin the chapter by interpreting differential operators on the algebra of densities as pencils of differential operators. Then we shall study the equivariant situation when we have a volume form, i.e. we shall classify the lifts of differential operators that respect this volume form. We shall then turn to the study of lifts invariant with respect to the projective group - here the quantization map of C. Duval, P.B.A. Lecomte and V.Y. Ovsienko will be an essential tool to aid our understanding.

3.1 Basic definitions and properties.

3.1.1 Algebraic properties of differential operators.

The algebra of densities is essentially an algebraic object and thus it is natural to begin our exposition of differential operators by considering the classical algebraic notions of differential operators.

Differential operators on graded algebras.

Let us first recall the algebraic notion of a differential operator on a commutative algebra [34].

Definition 3.1.1. Let A be a commutative $\Bbbk$-algebra then we define the filtered algebra of differential operators on A (over $\Bbbk$), $\mathrm{DO}(A)$, inductively as follows:

- $\mathrm{DO}^0(A) = A$.
- $\mathrm{DO}^n(A) = \{L \in \mathrm{End}_{\Bbbk}(A) | [L, a] \in \mathrm{DO}^{n-1}(A) \ \forall a \in A\}$.
- $\mathrm{DO}(A) = \cup_n \mathrm{DO}^n(A)$.

Many standard results hold for algebraic differential operators, for example: The composition of an operator of order n and one of order m is an operator of order $n+m$, whilst their commutator has order $n+m-1$. Another important definition that holds is that if we have $L \in \mathrm{DO}^n(A)$ then we can define its n^{th} symbol to be the map $\sigma_n(L) : A^{\otimes n} \to A$ given as $(a_1, \ldots, a_n) \mapsto [a_1, \cdots, [a_n, L] \cdots]$. Again one can check that this is well defined and is a tensor, that is an element of $\mathfrak{X}(A)^{\otimes n}$. If the n^{th} symbol of L is zero then it lies in $\mathrm{DO}^{n-1}(A)$ and for all non-zero differential operators there exists a minimum m such that $\sigma_m(L) \neq 0$, this is called the principle symbol of L. We shall return to these properties in later sections, see [34] and section 3.2.1.

Remark 3.1.2. When the algebra A is in fact an algebra of functions on a space then the above definition is not the correct one. In this case we must take into account the sheaf structure of these objects and then we define $\mathrm{DO}^n(A)$ to be the Hom-set with these properties. By Peetre's theorem, [70, 71], one then has that for a smooth manifold these two notions coincide and are correct. We shall assume that whenever we are working with a sheaf of algebras over a manifold this locality condition is implicitly implied.

As the algebra of densities is essentially an algebraic object we should like to use the above definition to define differential operators on this space. We shall see below that this may give us pathological operators that we do not want to deal with. Another important property of the algebra of densities is that it is graded and hence the following lemma will aid us in the understanding of differential operators on $\mathcal{F}(M)$.

Recall that if we have a graded[1] vector space $V = \oplus_\lambda V_\lambda$ then an endomorphism L is said to have weight δ if $L(V_\lambda) \subset V_{\lambda+\delta}\ \forall\lambda$.

Lemma 3.1.3. *Let $A = \oplus_\lambda A_\lambda$ be a graded algebra such that A has finite transcendence degree over A_0. Then the space of differential operators on A split into a sum of its weighted parts: $DO(A) = \oplus_\delta DO_\delta(A)$.*

Proof. Let $A = \oplus A_\lambda$ be a graded vector space then some standard homological algebra results, see [17, 40], imply that:

$$\mathrm{Hom}(A, A) = \mathrm{Hom}(\oplus_\lambda A_\lambda, \oplus_{\lambda'} A_{\lambda'}) =$$

$$\prod_\lambda \mathrm{Hom}(A, \oplus A_{\lambda'}) \subset \prod_\lambda \prod_{\lambda'} \mathrm{Hom}(A_\lambda, A_{\lambda'}) = \prod_\delta \prod_\lambda \mathrm{Hom}(A_\lambda, A_{\lambda+\delta}).$$

Thus the space of linear operators are a subalgebra of the product of all graded linear maps. Now assume that A is of the type outlined in the lemma. Finite transcendence degree means that we have n elements, $x_1, \dots, x_n$, of weights λ_i, such that any element in A satisfies an algebraic equation in $A_0(x_1, \dots, x_n)$. We now claim that it is sufficient to check that the space of symbols splits into a direct sum. To show that this holds take an arbitrary symbol, $S \in \mathfrak{X}^{\otimes n}(A)$. Then S is completely determined by its restriction to $A_0(x_1, \dots, x_n)$ where we can clearly split it into well defined weights. Now to show that it is sufficient to consider symbols take $S = \sum X_1 \otimes \cdots \otimes X_n$ the symbol of an operator L, and consider $L' = \sum X_1 \cdots X_n$ for any choice of representative vector fields. Then L' has the same symbol as L and thus $L - L'$ has lower order and we may split L' into a sum of its weighted components. The result then follows by induction as the base case, $\mathrm{DO}^0(A) = A$ holds by definition. □

The above holds for any algebra of densities defined over S where $\mathbb{Q} \otimes_{\mathbb{N}} S$ is a finite dimensional vector space.

If we have a graded algebra $A = \oplus_\lambda A_\lambda$, graded over a semigroup S such that $S \subset A_0$, then there exists a canonical derivation called the weight operator, $\widehat{w}_A$, defined as $\widehat{w}_A|_{A_\lambda} = \lambda$. We can they say that an arbitrary operator, L, has weight δ iff $[\widehat{w}_A, L] = \delta L$. This in particular holds for the algebra of densities, $\mathcal{F}(M)$, and the weight operator is then defined as $\widehat{w} = t\partial_t$ as in equation (2.1.3).

[1]We may assume that our grading is over an arbitrary semigroup.

Differential operators on the Thomas bundle.

As we mentioned in the introduction the algebra of densities is related to the Thomas bundle and is the graded subalgebra of functions on the bundle $\widehat{M} \to M$, see lemma 2.1.12. Thus we do not only have the algebraic notions of a differential operator as above but we can also ask when a differential operator $L \in \mathrm{DO}(\widehat{M})$ preserves the algebra of densities, that is $L(\mathcal{F}(M)) \subset \mathcal{F}(M)$. Let us take an arbitrary differential operator L then the local form of L is $L = \Sigma_{I,m} L(x,t)\partial_I \partial_t^m$. The condition of preserving the algebra of densities then locally becomes that $L(x,t) \in \mathcal{F}(M)$. We thus have two notions of differential operators on $\mathcal{F}(M)$, the first a purely algebraic definition outlined above and the second coming from regarding $\mathcal{F}(M)$ as a subalgebra of the smooth functions on $\widehat{M}$. The following lemma determines when these two notions of differential operators coincide.

Lemma 3.1.4. *Assume that $\mathcal{F}(M) = \oplus_\lambda \mathcal{F}^\lambda(M)$ is indexed over the rational numbers or some sub-semigroup of them (that is $\lambda \in \mathbb{Q}$). Then the two notions of differential operators coincide.*

Proof. Let us take a coordinate patch of the Thomas bundle $\widehat{U} \subset \widehat{M}$, see equation (2.1.2). Then any element in the algebra of densities has the form, $f(x,t) = \Sigma f_\lambda(x) t^\lambda$ in this coordinate patch. It follows from the algebraic definition of a vector field that if one knows its value on a function f then it is known on f^q $\forall q \in \mathbb{Q}$. In particular this implies that for any differential operator L if we consider its principle symbol, $\sigma_n(L) : \mathcal{F}(U)^{\otimes n} \to \mathcal{F}(U)$, then it is completely determined by the data:

$$S^{i_1 \cdots i_k}(x,t) := \sigma_P(L)(x^{i_1}, \ldots, x^{i_k}, t, \ldots, t), \quad 0 \le k \le n.$$

Therefore we may construct the *local* differential operator naïvely given by this information:

$$L' = \sum_{k=0}^{n} S^{i_1 \cdots i_k}(x,t) \partial_{i_1 \cdots i_k} \partial_t^k.$$

If we are indexed over $\mathbb{Q}$ we have that L' and L have the same principal symbol. Therefore their difference is an operator of order $n-1$ and we can proceed by induction as the base case, differential operators of degree zero, is trivial. Finally we can glue these operators together to prove the result. □

Remark 3.1.5. The lack of equivalence for a general indexing set lies in pathological examples. For example if our indexing set is $\mathbb{R}$ then taking a Hamel basis, $\{\lambda_i : i \in I\}$, we may define a vector field on $\mathcal{F}(M)$ as $Y(t^{q^i\lambda_i}) = q^j f_j(x) t^{q^i\lambda_i}$ for any choice of functions $\{f_i : i \in I\}$ - in some sense these are the only examples of the exotic differential operators that stop a general equivalence from holding. This remark also shows that the lemma is strict. That is if $S - \mathbb{Q} \neq \emptyset$ then the two notions do not agree (recall $\mathbb{Z} \subset S$).

We define the space of differential operators on the algebra of densities, $\mathrm{DO}(\mathcal{F}(M))$, to be those that lie in the intersection of the two notions outlined above. As we shall always assume that $\mathbb{Z}$ is a part of our semigroup we then have that all differential operators of Thomas type lie in the algebra so we essentially exclude those operators that give rise to unnatural examples. In some sense this is equivalent to enforcing some stronger notion of continuity in the fibres of $(\widehat{M}, \mathcal{F}(M))$ than would naïvely be assumed. The space of all algebraically defined differential operators on the algebra of densities shall be denoted by $\mathrm{DO}_{\mathfrak{Alg}}(\mathcal{F}(M))$ - this depends on the exact form of the algebra of densities we are using whilst the differential operators we shall usually work with does not.

We can now give an explicit expression for the local form of a differential operator on $\mathcal{F}(M)$ of weight δ. Using the form of an operator on $\widehat{U} \subset \widehat{M}$, where U is a coordinate basis on M, we write ∂_t as $t^{-1}\widehat{w}$. Then commuting all these ts to the left we get that for $L \in \mathrm{DO}^n_\delta(\mathcal{F}(M))$ its local form is:

$$L|\widehat{U} = t^\delta \sum_{\substack{k=0,1,\ldots,n \\ |I| \leq n-k}} L^I_k(x) \partial_I \widehat{w}^k. \tag{3.1.1}$$

Adjoints of differential operators.

The algebra $\mathcal{F}(M)$ and compactly supported functions in the full algebra $C^\infty(\widehat{M})$ have a canonical scalar product, see section 1.1.3., and although this inner product is not necessarily symmetric (remark 2.1.16) it still makes sense to talk about self adjoint operators. As both algebras $\mathcal{F}(M)$ and $C^\infty_c(\widehat{M})$ have inner products and $\mathrm{DO}(\mathcal{F}(M)) \subset \mathrm{DO}(\widehat{M})$ we should check if they give the same product on $\mathrm{DO}(\mathcal{F}(M))$.

Lemma 3.1.6. *Let $L \in DO(\mathcal{F}(M))$, then its adjoint with respect to the inner product on $\mathcal{F}(M)$ is equal to its adjoint with respect to the inner product on $C^\infty_c(\widehat{M})$.*

Proof. The proof follows immediately from working within a coordinate basis and then gluing the results together. □

This result does not appear surprising when we look at the similarity in the definitions of the inner product in these cases but what is quite interesting is that an element $f \in \mathcal{F}(M)$ never has compact support and hence our test functions in fact lie in completely different subspaces.

Remark 3.1.7. It is possible to define the adjoint of arbitrary algebraic differential operators, $\mathrm{DO}_{\mathfrak{Alg}}(\mathcal{F}(M))$. We can no longer check of course that these operators have the same adjoint as they would on the Thomas bundle - we can only say the trivial fact that their adjoint is equal to the adjoint of their associated pencil, see below.

Example 3.1.8. Recall that we have introduced the weight operator $\widehat{w} = t\partial_t$, equation (2.1.3). We now calculate its adjoint with respect to the above inner product. We have that for $f \in \mathcal{F}^\lambda(M)$ and $g \in \mathcal{F}^\mu(M)$ that:

$$\langle g, \widehat{w}f\rangle = \lambda\langle g, f\rangle = \lambda\delta(\mu+\lambda-1)\int gf = (1-\mu)\delta(\mu+\lambda-1)\int gf = \langle(1-\widehat{w})g, f\rangle.$$

That is $\widehat{w}^\dagger = 1 - \widehat{w}$.

Lemma 3.1.9. *Let L be a differential of weight δ, then so is $L^\dagger$.*

Proof. This again has essentially an algebraic proof. We have that:

$$\langle[\widehat{w}, L^\dagger]g, f\rangle = \langle g, [L, 1-\widehat{w}]f\rangle = \langle g, [\widehat{w}, L]f\rangle.$$

Thus the result follows. □

3.1.2 Pencils and lifts.

Pencils of differential operators.

Let us begin with the definition of a pencil of differential operators:

Definition 3.1.10. A pencil of differential operators of weight δ is an indexed family of operators, $\mathcal{L} = \{L_\lambda | L_\lambda \in \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)\}$. The space of pencils of weight δ shall be denoted by $P_\delta(\mathcal{F}(M))$.

Pencils of differential operators form an algebra that behaves very similarly to the algebra of differential operators on $\mathcal{F}(M)$. That is the full algebra is defined to be the direct sum of all weights, $P(\mathcal{F}(M)) = \oplus_\delta P_\delta(\mathcal{F}(M))$, and we compose two pencils component wise. This makes them into an algebra and moreover we can define the adjoint point wise. They are however *maximally* discontinuous in the fibres of $\widehat{M}$ and correspond to the most general definition of differential operator that can be defined on $\mathcal{F}(M)$.

Lemma 3.1.11. *There exists a natural embedding of graded algebras $DO(\mathcal{F}(M)) \hookrightarrow P(\mathcal{F}(M))$. Moreover if we consider the space of all algebraic differential operators on $\mathcal{F}(M)$, $DO_{\mathfrak{Alg}}(\mathcal{F}(M))$ then this also embeds into $P(\mathcal{F}(M))$.*

Proof. As we have that $\mathrm{DO}(\mathcal{F}(M)) \subset \mathrm{DO}_{\mathfrak{Alg}}(\mathcal{F}(M))$ we shall prove the result for algebraic differential operators and then it will follow for differential operators. So let L be such an operator we may then write $L = \oplus_\delta L_\delta$ where L_δ has weight δ by lemma 3.1.3. We may therefore assume that L has a fixed weight we then define the pencil $P(L)$ as follows: $\mathcal{F}^\lambda(M) \subset \mathcal{F}(M)$ and as L has weight δ we have that $L(\mathcal{F}^\lambda(M)) \subset \mathcal{F}^{\lambda+\delta}(M)$, and by remark 3.1.2 we have that this is in fact a differential operator. We define the λ^{th} term of $P(L)$ to be this differential operator. It is clear that this map respects the algebra structure and grading. □

Notation 3.1.12. We shall also denote $P(L)_\lambda$ by $L|\mathcal{F}^\lambda(M)$.

We thus have three algebras of differential operators:

$$\mathrm{DO}(\mathcal{F}(M)) \subset \mathrm{DO}_{\mathfrak{Alg}}(\mathcal{F}(M)) \subset P(\mathcal{F}(M)),$$

each corresponding to a weakening of the topology in the fibre: the first comes from the smooth topology from the interpretation of $\mathcal{F}(M) \subset C^\infty(\widehat{M})$; the second is a Zariski type topology on the fibre; and the third is essentially an indiscrete topology on the fibres.

Example 3.1.13. Let us take the differential operator L as in equation (3.1.1). Then we have that

$$L|\mathcal{F}^\lambda(M) = |Dx|^\delta \sum_{\substack{k=0,\dots,n \\ |I| \le n-k}} \lambda^k L_k^I(x) \partial_I.$$

One of the major differences between differential operators on $\mathcal{F}(M)$ and pencils of differential operators is the notion of order. We can define the order of a pencil of differential operators, $\mathcal{L} = \{L_\lambda\}$ by the following formula:

$$\operatorname{ord}(\mathcal{L}) := \sup_\lambda \operatorname{ord}(L_\lambda) \in \mathbb{N} \cup \{\infty\}, \tag{3.1.2}$$

where by order in the above equation we mean *minimum* order. If we have a differential operator $L \in \mathrm{DO}_{\mathfrak{Alg}}(\mathcal{F}(M))$ then this order is bounded by the order of order L and hence is not infinite.

Definition 3.1.14. Let L be a differential operator on $\mathcal{F}(M)$ (we could let L be algebraic if we wish). Then the *horizontal order* of L is the order of the pencil associated to L.

As mentioned above the horizontal order always exists as an integer and is less than or equal to the order of L as a differential operator, e.g. the order of $\widehat{w}$ is 1 whilst its horizontal order is 0.

Assume that we have a pencil of differential operators, $\mathcal{L} = \{L_\lambda\}$, such that locally its dependence on λ is polynomial[2]. Then assuming that there exists a global bound on the order of this polynomial we have that it comes form a differential operator on the algebra of densities. Thus we can think of differential operators on $\mathcal{F}(M)$ as those pencils that have a very simple dependance on λ.

Lifting differential operators.

We now introduce a notion that will generate the problems we shall focus on solving in the second half of this chapter. The idea is very simple: if we have an operator $L \in \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ then can we describe the space of operators on $\mathcal{F}(M)$ that pass through this operator, and can we put any conditions on such operators that would make them unique? Let us now formalise this with some definitions. Firstly a *lift of* L , where L is as above, is an operator $\widehat{L} \in \mathrm{DO}_\delta(\mathcal{F}(M))$ such that $P(\widehat{L})_\lambda = L$, that is $\widehat{L}|\mathcal{F}^\lambda(M) = L$. We will be interested in equivariance questions so instead of considering just lifts of single operators it is more useful to consider lifts of all differential operators:

[2]This makes sense for locally the differential operators may be thought of as acting on $C^\infty(U)$.

Definition 3.1.15. A lifting of type (λ, δ) is a linear map

$$\nu : \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) \longrightarrow \mathrm{DO}_\delta(\mathcal{F}(M)),$$

such that $\nu(L)$ is a lift of L $\forall L \in \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$. If ν is not defined on the space of all operators just those of order n then we shall still call ν a lifting of type (λ, δ) but make its domain explicit.

We are interested in liftings that satisfy some additional properties related to differential operators. The most basic property is the filtration on differential operators given by their order. We would like this to be preserved in some way which motivates the following definition:

Definition 3.1.16. Let ν be a lifting of type (λ, δ) for differential operators of order n. Then we shall call the lifting:

1. *regular*; if $\forall L$ the horizontal order of $\nu(L)$ is equal to the order of L,
2. *n-regular*; if it is regular and $\nu(L) \in \mathrm{DO}^n_\delta(\mathcal{F}(M))$ $\forall L$,
3. *n-strict*; if when $L \in \mathrm{DO}^m(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ then $\nu(L) \in \mathrm{DO}^m_\delta(\mathcal{F}(M))$ $\forall m \leq n$.

We clearly have that {strict liftings} $\subset$ {n-regular liftings} $\subset$ {regular liftings}. We will see that all such liftings exist on an arbitrary manifold as a corollary of lemma 3.2.1, however one could use a partition of unity argument as well.

Example 3.1.17. Consider the space of 1st order differential operators on M which naturally splits into a direct sum of vector fields and scalars, $\mathrm{DO}^1(M) = \mathfrak{X}(M) \oplus C^\infty(M)$. Now as $\mathcal{F}^\lambda(M)$ is a geometric bundle it inherits a natural action of $\mathfrak{X}(M)$ by the Lie derivative. Using this to any first order operator on M, $X \oplus f$, we can associate a pencil of operators, $\mathcal{L}^\lambda_X \oplus f$. One can check that this pencil comes from a differential operator:

$$\mathcal{L}_X = X^i \partial_i + (-1)^{\widetilde{i}(\widetilde{X}+1)} {X^i}_{,i} \widehat{w},$$

where we have set $f = 0$ for simplicity. Thus there is a natural strict 1-pencil on any manifold that is clearly diff(M)-equivariant. We shall call this pencil the Lie derivative pencil.

There exists a trivial generalisation of the above. Take an arbitrary first order differential operator on $\mathcal{F}^\lambda(M)$, D. Then its symbol defines a vector field, X, and there exists a unique function f such that $\mathcal{L}^\lambda_X + f = D$. We can thus again define a natural pencil associated to any weight 0 first order differential operators.

Example 3.1.18. Let D be a vector field on the algebra of densities of weight δ:

$$D = t^\delta(X^i\partial_i + Y\widehat{w}).$$

The divergence of this vector field is defined as $-D - D^\dagger$ and denoted $\operatorname{div}(D)$. We find that it is equal to

$$\operatorname{div}(D) = X^i{}_{,i} + (\delta - 1)Y.$$

Therefore if $\delta \neq 1$ for any weighted vector field $|Dx|^\delta X^i\partial_i$ there exists a unique choice of Y that makes D a divergenceless vector field on $\widehat{M}$, see [48]. We now apply this to operators in $\mathrm{DO}^1(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ for $\delta \neq 1$. That is we can define a canonical lifting by taking the divergenceless vector field associated to the symbol of the operator and adding a function to ensure that it passes through the original operator. As an example consider $L \in \mathrm{DO}^1(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$,

$$L = |Dx|^\delta(X^i\partial_i + U).$$

Then the lift of L defined above is given as:

$$\widehat{L} = t^\delta\big(X^i\partial_i + \frac{1}{1-\delta}X^i{}_{,i}\widehat{w} + U - \frac{\lambda}{1-\delta}X^i{}_{,i}\big).$$

One can see that this agrees with the Lie derivative lifts defined above for $\delta = 0$. Moreover this lifting is $\operatorname{diff}(M)$-equivariant. In fact it is the first part of a bigger picture we shall explore in the final chapter, see also [9].

The vertical Taylor expansion.

We now introduce one of the most useful tools to study differential operators and liftings, the vertical Taylor expansion. It is relatively versatile and has equivariant and self adjoint version.

Lemma 3.1.19. *Let $L \in DO^n_\delta(\mathcal{F}(M))$ be such that $L|\mathcal{F}^\lambda(M) = 0$. Then $\widehat{w} - \lambda$ divides L, that is there exists a unique $R \in DO^{n-1}_\delta(\mathcal{F}(M))$ such that $L = R(\widehat{w} - \lambda)$.*

Proof. The proof is done in coordinates. Take an atlas of M, $\mathcal{U} = \{U_\alpha, \varphi_{\alpha,\beta}\}$, and we then get the local form of L by equation (3.1.1):

$$L_\alpha := L|\widehat{U}_\alpha = t^\delta \sum_k L_{k,\alpha}(\widehat{w} - \lambda)^k,$$

where $L_{k,\alpha} \in \mathrm{DO}(U_\alpha)$. By assumption we have that $L_{0,\alpha} = 0$, so we have that locally L is divisible by $(\widehat{w} - \lambda)$ and we set $R_\alpha = L_\alpha/(\widehat{w} - \lambda)$. We need to show that these operators do indeed glue together to give a well defined global operator. This follows immediately as $L_\beta|\widehat{U}_{\alpha\beta} = \widehat{\varphi}^*_{\alpha\beta} \circ L_\alpha|\widehat{U}_{\alpha\beta} \circ \widehat{\varphi}^*_{\beta\alpha}$, and the fact that $\widehat{w} - \lambda$ commutes with all such functions, that is it is a globally defined operator. □

We now apply this to the space of lifts outline in definition 3.1.16. Fix two liftings of type (λ, δ), $\nu_i : \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) \to \mathrm{DO}_\delta(\mathcal{F}(M))$, $i = 1, 2$. Then for any operator L we have, using lemma 3.1.19, that $\nu_2(L) = \nu_1(L) + R_L(\widehat{w} - \lambda)$ for some unique operator R_L. We can restrict R_L to $\mathcal{F}^\lambda(M)$ and thus get an operator $L^{(1)}_{\nu_1,\nu_2} \in \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ (in the sequel we shall usually drop reference to the νs in the subscript unless it is needed). We now continue the process inductively and we define $R_{L,2}$ by the equation $R_L = \nu_1(L^{(1)}_{\nu_1,\nu_2}) + R_{L,2}(\widehat{w} - \lambda)$, and $L^{(2)}_{\nu_1,\nu_2} = R_{L,2}|\mathcal{F}^\lambda(M)$. This process will in general not terminate, however we define the Taylor coefficients of the lifting ν_2 with respect to ν_1, $T^n_{\nu_1,\nu_2}$, to be the series of endomorphisms of $\mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ given as $T^n_{\nu_1,\nu_2}(L) = L^{(n)}_{\nu_1,\nu_2}$. For any finite n and any operator L we have that there exists a unique $R_{L,n+1} \in \mathrm{DO}_\delta(\mathcal{F}(M))$ such that:

$$\nu_2(L) = \nu_1(L) + \nu_1(L^{(1)}) + \cdots + \nu_1(L^{(n)}) + R_{L,n+1}(\widehat{w} - \lambda)^{n+1}. \tag{3.1.3}$$

If both of the liftings are n-regular then the above process does terminate after the n^{th} stage. To show this assume both ν_1 and ν_2 are n-regular in the above equation, then as L is of order n we have that $\nu_i(L)$ is of order n and it immediately follows that $R_{L,n+1} = 0$, so the process terminates, and , moreover, the operators $\nu_1(L^{(m)})$ must have order $\leq n - m$, in particular $L^{(m)}$ must have order $n - m$ due to the fact that the horizontal order is decreasing, see definition 3.1.14.

We can immediately use the vertical Taylor expansion to classify the types of liftings given in definition 3.1.16.

Corollary 3.1.20. *The space of n-regular liftings of type (λ, δ) is an affine space modeled on* $\oplus_{k\geq 1} Hom\big(DO^n(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))\,,\, DO^{n-k}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))\big)$.

Proof. Fix an element $\nu \in \{n\text{-regular liftings of type } (\lambda, \delta)\}$. Then for an arbitrary lifting of this type the Taylor coefficients give us the maps in the space defined in the corollary (the k^{th} Taylor expansion corresponds to the k^{th} direct summand). For the inverse let us take a family of such maps $\Phi = (\Phi_1, \ldots, \Phi_n)$ then define the lifting ν_Φ as:

$$\nu_\Phi(L) = \nu(L) + \sum_{k=1}^{n} \nu\big(\Phi_k(L)\big)(\widehat{w} - \lambda)^k,$$

this clearly defines a n-regular lifting and equation (3.1.3) shows that it is inverse to the Taylor expansion. □

3.1.3 Self-adjoint liftings.

We are interested with properties concerning the adjoint and its effect on the structure of differential operators. In particular we shall focus on liftings that are (anti-)self adjoint. The order of the lift is important as the parity of a differential operator under the adjoint restricts its minimum order. We also need to avoid the singular point which we shall defer till the next section.

Definition 3.1.21. An n-regular lifting of type (λ, δ), $\nu : \mathrm{DO}^n(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) \to \mathrm{DO}^n_\delta(\mathcal{F}(M))$, for $\lambda \neq \frac{1-\delta}{2}$, is called self adjoint if $\nu(L)^\dagger = (-1)^n \nu(L)\ \forall L$.

Note that a self-adjoint lifting of order $n > 0$ can never be a strict lifting, for if it were then take

$$L \in \mathrm{DO}^{n-1}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) - \mathrm{DO}^{n-2}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)).$$

We would have that both $\nu(L) \in \mathrm{DO}^{n-1}(M)$ and $\nu(L)^\dagger = (-1)^n \nu(L) \Rightarrow \nu(L) \in \mathrm{DO}^{n-2}_\delta(\mathcal{F}(M)) \Rightarrow L = \nu(L)|\mathcal{F}^\lambda(M)$ is of order $n-2$ which is a contradiction. We now show that such self-adjoint liftings do indeed exist:

Lemma 3.1.22. *Let ν be an n-regular lifting of type (λ, δ) for $\lambda \neq \frac{1-\delta}{2}$. Then there exists a natural extension of ν to a self adjoint lifting, ν^{sa}. The lift itself is defined by the formula:*

$$\nu^{sa}(L) := \nu(L) + \frac{1}{2\lambda + \delta - 1}\left(\nu(L) - (-1)^n \nu(L)^\dagger\right)(\widehat{w} - \lambda) \tag{3.1.4}$$

Proof. We see from the above equation that the map is indeed an n-regular pencil so we just need to check that it is self adjoint. Recall that $\widehat{w}^\dagger = 1 - \widehat{w}$ and using this we find that:

$$\begin{aligned}\left(\nu^{sa}(L)\right)^\dagger &= \nu(L)^\dagger + \frac{1}{2\lambda+\delta-1}(1-\lambda-\widehat{w})\big(\nu(L)^\dagger - (-1)^n\nu(L)\big)\\ &= \nu(L)^\dagger + \frac{(-1)^n}{2\lambda+\delta-1}\left(\nu(L) - (-1)^n\nu(L)^\dagger\right)(\widehat{w}+\lambda+\delta-1)\end{aligned}$$

$$= \nu(L)^\dagger + \frac{(-1)^n}{2\lambda+\delta-1}\left(\nu(L) - (-1)^n\nu(L)^\dagger\right)\Big((\widehat{w}-\lambda) + (2\lambda+\delta-1)\Big) = (-1)^n\nu^{sa}(L),$$

and thus the proof is complete. □

A simple remark to make is that the map defined on the space of regular n-liftings as $\nu \mapsto \nu^{sa}$ is an idempotent mapping and the fixed points are just the self adjoint liftings, that is the map gives a natural projection of all liftings onto the subspace of self adjoint liftings. We can also use the vertical Taylor expansion to classify the space of all self adjoint liftings of order n:

Proposition 3.1.23. *Assume that $\lambda \neq \frac{1-\delta}{2}$, then the space of self adjoint liftings of type (λ, δ) at order n is an affine space modeled on $\oplus_{k=1}^{\lfloor \frac{n}{2} \rfloor} Hom(DO^n(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)),\, DO^{n-2k}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)))$.*

Proof. Fix an n-regular self adjoint lifting of type (λ, δ), ν_0, for $\lambda \neq \frac{1-\delta}{2}$. Now take an arbitrary n-regular lifting, ν, and we have the Taylor expansion of ν with respect to ν_0:

$$\nu(L) = \nu_0(L) + \sum_{k=1}^{n} \nu_0(L^{(k)})(\widehat{w}-\lambda)^k.$$

Now assume that ν is self adjoint as well, that is $\nu(L)^\dagger = (-1)^n\nu(L)$. Putting this into the above equation and calculating we find that this condition is equivalent to

$$\sum_{k=1}^{n} \nu_0(L^{(k)})(\widehat{w}-\lambda)^k = \sum_{k=1}^{n}(-1)^k\nu_0(L^{(k)})\sum_{l=0}^{k}\binom{k}{l}(\widehat{w}-\lambda)^l(2\lambda+\delta-1)^{k-l}.$$

After some rearranging of this equation and using the fact that ν_0 is linear and 1-1 so we can express everything in terms of the Taylor coefficients , $T^{(k)}$. These maps need to obey the condition:

$$\sum_{k>l}^{n}(-1)^k\binom{k}{l}(2\lambda+\delta-1)^{k-l}T^{(k)} = (1-(-1)^l)T^{(l)} \quad \forall l.$$

We now show that the $T^{(2m)}$, $m = 0, \ldots, \lfloor \frac{n}{2} \rfloor$ completely determine all the other operators. We shall do this for the cases when n is even and odd separately:

- $n = 2N$: In this case we write an expansion of $T^{(2m+1)}$ in terms of the even operators using the following ansatz:

$$T^{(2m+1)} = \sum_{r=1}^{\frac{n}{2}-m} \binom{2(m+r)}{2m+1} (2\lambda+\delta-1)^{2r-1} \mu_r T^{(2m+2r)}.$$

We then find that the condition required for self adjointness translates to the following recursion relation among the μ_rs:

$$\mu_r = \frac{1}{2}\left(1 - \sum_{p=1}^{r} \binom{2r-1}{2p} \mu_{r-p}\right), \qquad \mu_0 = 0.$$

Clearly this can be solved uniquely with the initial conditions so the odd maps are completely determined by the even ones. We shall list the first few of these constants:

$$\mu_1 = \frac{1}{2},\ \mu_2 = -\frac{1}{4},\ \mu_3 = \frac{1}{2},\ \mu_4 = \frac{17}{8},\ \mu_5 = -\frac{163}{2}.$$

- $n = 2N+1$: We shall now perform a similar ansatz to the one in the even case, note that we immediately have that $T^{(n)} = 0$. Then setting the ansatz in exactly the same form that as for the even case we see that we have that the associated constants, μ, must obey the identical recursion relation and we have that the general form of the operator $T^{(2m+1)}$ is given as:

$$T^{(2m+1)} = \sum_{r=1}^{\frac{n-1}{2}-m} \binom{2(m+r)}{2m+1} (2\lambda+\delta-1)^{2r-1} \mu_r T^{(2m+2r)}.$$

□

Self-adjoint liftings at the singular point.

We now describe self adjoint liftings passing through the singular point $\lambda = \frac{1-\delta}{2}$. This point is special with respect to the adjointness as taking the adjoint of an operator of weight δ acting on $\mathcal{F}^{\frac{1-\delta}{2}}(M)$ returns an operator of the same type: Recall that if L is an operator of weight δ acting on $\mathcal{F}^{\lambda}(M)$, then its adjoint is of weight δ acting on densities of weight $1-\lambda-\delta$, which is equal to λ exactly in this case.

This in fact allows us to give a very succinct definition of a self adjoint pencil passing through this point:

Definition 3.1.24. A self adjoint lifting of order n on $\mathrm{DO}^n(\mathcal{F}^{\frac{1-\delta}{2}}(M), \mathcal{F}^{\frac{1+\delta}{2}}(M))$ is a regular n-lifting of type $(\frac{1-\delta}{2}, \delta)$, ν, that commutes with taking adjoints. That is:

$$\nu(L)^\dagger = \nu(L^\dagger) \quad \forall L.$$

We can split the space of differential operators at the singular point into anti/self adjoint operators:

$$\mathrm{DO}^n(\mathcal{F}^{\frac{1-\delta}{2}}(M), \mathcal{F}^{\frac{1+\delta}{2}}(M)) =$$
$$\mathrm{DO}^n_+(\mathcal{F}^{\frac{1-\delta}{2}}(M), \mathcal{F}^{\frac{1+\delta}{2}}(M)) \oplus \mathrm{DO}^n_-(\mathcal{F}^{\frac{1-\delta}{2}}(M), \mathcal{F}^{\frac{1+\delta}{2}}(M)).$$

Definition 3.1.25. For a pair of $\mathbb{Z}_2$ graded vector spaces V and W, $\mathrm{Hom}_+(V, W)$ are those maps that preserve the parity and $\mathrm{Hom}_-(V, W)$ are those maps that reverse it.

We define a natural $\mathbb{Z}_2$ grading on the spaces $\mathrm{DO}_\delta(\mathcal{F}^{\frac{1-\delta}{2}}(M))$ and $\mathrm{DO}_\delta(\mathcal{F}(M))$, by considering self-adjoint and anti-self adjoint operators as the positive and negative parts respectively.

For any self adjoint lifting we must have that these spaces are preserved in the sense that any self adjoint (resp. ant-self adjoint) operator is mapped to a self adjoint (resp. ant self adjoint) operator. The classification of such liftings is also simpler than in the general case, namely we have that:

Proposition 3.1.26. *The space of self adjoint liftings of type $(\frac{1-\delta}{2}, \delta)$ of order n is an affine space modeled on*

$$\oplus_{k=1}^n Hom_{(-1)^k}(DO^n(\mathcal{F}^{\frac{1-\delta}{2}}(M), \mathcal{F}^{\frac{1+\delta}{2}}(M)), DO^{n-k}(\mathcal{F}^{\frac{1-\delta}{2}}(M), \mathcal{F}^{\frac{1+\delta}{2}}(M)).$$

Proof. Let us take a self adjoint lifting ν_0 and take the Taylor expansion of an arbitrary lifting with respect to ν_0, $(T^{(1)}, \ldots, T^{(n)})$. The condition that the new lifting is self adjoint can be easily computed:

$$\Big(\sum_{k=0}^n \nu \circ T^{(k)}(-1)\big(\widehat{w} - \frac{1-\delta}{2}\big)\Big)^\dagger = \sum_{k=0}^n (-1)^k \nu \circ \big(T^{(k)}\big)^\dagger \big(\widehat{w} - \frac{1-\delta}{2}\big)^k,$$

and this is equal to $\sum \nu \circ T^{(k)}\,(\widehat{w} - \frac{1-\delta}{2})$ iff $T^{(k)}(L^\dagger) = (-1)^k (T^{(k)}(L))^\dagger$, which is exactly the claim. □

Notation 3.1.27. Recall the weight operator $\widehat{w} = t\partial_t$. An important differential operator when considering self adjoint operators of weight δ is $\widehat{w}_\delta$ defined as:

$$\widehat{w}_\delta := \widehat{w} - \frac{1-\delta}{2}.$$

The reason for this is that if we express a differential operator of weight δ in terms of $\widehat{w}_\delta$ in equation (3.1.1) we get:

$$L = t^\delta \sum L_k^I(x)\partial_I \widehat{w}_\delta^k,$$

and we find the adjoint has a particularly simple form:

$$L^\dagger = t^\delta \sum (-1)^{|I|+k} \partial_I L_k^I \widehat{w}_\delta^k.$$

3.2 Equivariant pencils I: The algebra of divergenceless vector fields.

In the second half of this chapter we turn to the classification of certain types of liftings of differential operators that obey equivariance conditions with respect to some group. The two groups we shall focus on are that of divergenceless vector fields and the projective group. These two groups are important for the following reasons: firstly the group of divergenceless vector fields corresponds to the group that preserves a natural triviliazation of the algebra of densities; the second, the projective group, is important because it is the 'largest' finite dimensional algebra of vector fields.

3.2.1 Volume form pencils.

Let $\varrho \in \mathcal{F}^1(M)^\times$ be a volume form on M. Take any space of differential operators, $\mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$, then for each operator L, we can define a pencil of operators, $P(L_\varrho)$, that passes through L defined by the formula:

$$P(L_\rho)_\mu = \varrho^{\mu-\lambda} \circ L \circ \varrho^{\lambda-\mu}. \tag{3.2.1}$$

Lemma 3.2.1. *The above formula defines a strict lifting of L to the algebra of densities. Moreover assuming $\delta = 0$ this is a morphism of algebras.*

Proof. If we work in local coordinates we see that we can rewrite the above map by the formula $\partial_i \mapsto \partial_i + \gamma_i(\widehat{w} - \lambda)$, where $\gamma_i = -\varrho^{-1}\partial_i\varrho$, the connection induced from the volume form. It is then clear that it is a strict lifting. Moreover the formula for the pencil immediately shows that if $\delta = 0$ it is a map of algebras using lemma 3.1.11. □

The differential operator associated to the above pencil shall be denoted by L_ϱ. The volume form ϱ identifies all the modules $\mathcal{F}^\lambda(M)$ and hence induces a large number of additional isomorphisms associated with these modules. For example if we take the volume form pencil of an operator L and restrict it to a different weight of densities and then reapply the pencil we have the operator we started with:

$$(L_\varrho|\mathcal{F}^\mu(M))_\varrho = L_\varrho.$$

Moreover the volume form lifting is well behaved with taking adjoints:

Lemma 3.2.2. *The volume form pencil commutes with taking adjoints:* $(L_\varrho)^\dagger = (L^\dagger)_\varrho$.

Proof. Let $u \in \mathcal{F}^\mu(M)$ and $v \in \mathcal{F}^{1-\mu-\delta}(M)$, where δ is the weight of L, and μ is arbitrary. We then have:

$$\langle (L_\varrho)^\dagger v, u\rangle = (-1)^{\widetilde{L}\widetilde{v}}\langle v, L_\varrho u\rangle = (-1)^{\widetilde{L}\widetilde{v}}\int_M v\varrho^{\mu-\lambda}D(\varrho^{\lambda-\mu}u)|Dx|$$

$$= \int_M L^\dagger(\varrho^{\mu-\lambda}v)\varrho^{\lambda-\mu}u|Dx| = \int_M (L^\dagger)_\varrho(v)u|Dx| = \langle (L^\dagger)_\varrho v, u\rangle$$

□

Corollary 3.2.3. *Let us take the volume form lift defined on* $DO(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$, $L \mapsto L_\varrho$. *Then we have that* $L_\varrho^\dagger|\mathcal{F}^\lambda(M) = L^{\dagger_\varrho}$, *where by* $L^{\dagger_\varrho}$ *we mean the adjoint of* L *using the volume form* ϱ.

Proof. We have that $L^{\dagger_\varrho}$ is defined by the formula:

$$\int_M L^{\dagger_\varrho}(s)u\varrho^{1-\delta-2\lambda}|Dx| = (-1)^{\widetilde{L}\widetilde{s}}\int_M sL(u)\varrho^{1-\delta-2\lambda}|Dx|.$$

□

A natural question to ask is how the lift alters when we make a variation of the volume form. That is if we set $\varrho = u\varrho_0$, where $u \in C^\infty(M)^\times$, how is the volume form pencil of ϱ expressed in that of the original volume form ϱ.

Lemma 3.2.4. *Let* $\varrho = u\varrho_0$, $u \in C^\infty(M)^\times$, *then:*

$$L_\varrho = u^{\widehat{w}-\lambda}L_\varrho u^{\lambda-\widehat{w}} = \sum_{k=0}\frac{1}{k!}[\log(u), [\log(u), \cdots, [\log(u), L], \cdots]]_\varrho(\widehat{w}-\lambda)^k.$$

Proof. This follows immediately if we look at the pencil of the two operators:

$$\left(e^{f(\widehat{w}-\lambda)}L_{\varrho_0}e^{-f(\widehat{w}-\lambda)}\right)_{\mu} = e^{(\mu-\lambda)f}\varrho_0^{\mu-\lambda}L\varrho^{\lambda-\mu}e^{(\lambda-\mu)f} = (L_{\varrho})_{\mu}.$$

The second equality comes from the expansion of the exponential and using the fact that the pencil commutes with the algebraic structure. □

Remark 3.2.5. If we are using complex valued volume forms then taking the λ^{th} power for arbitrary λ of such a volume form may not be well defined if the manifold is not simply connected. However one can see from the above expression that it is independent of the choice of the branch made and thus a complex volume form will induce a natural lifting of operators even though for certain values of λ it may not induce an actual map of the underlying modules of densities.

Note that the volume form lift only depends on the connection induced by ϱ. Therefore it is even possible to define the lift for an arbitrary connection on $\mathcal{F}^1(M)$, γ, by simply replacing the occurrence of the connection associated to ϱ by γ. Note that the additional result that this induces a map of algebras in the case that $\delta \neq 0$ then holds iff γ is flat. We shall sometimes use the lifting for a connection rather than for a volume form and in this case we shall denote the lifting by $L \mapsto L_\gamma$, where γ is the associated connection.

Example 3.2.6. Let $\Delta \in \mathrm{DO}^2(\mathcal{F}^{\frac{1}{2}}(\Pi T^*M))$ be the Batalin-Vilkovisky operator, see proposition 6.1.14 and [1, 4, 6, 43, 44, 45, 46, 76] and section 5.1.3 for more details. Any nowhere vanishing half-density on ΠT^*M, $\mathbf{s}$, may be expanded as

$$\mathbf{s} = e^{iS}\varrho,$$

where ϱ is a volume form on M and $S = S(x, x^*)$. Let $\Delta_{\mathbf{s}}$ be the associated pencil of the Batalin-Vilkovisky operator. We then have that $\Delta_{\mathbf{s}}(1) = \Delta_{\varrho}(S) - i(S, S)$, the Batalin-Vilkovisky equation for S.

3.2.2 Mappings of operators equivariant with respect to divergenceless vector fields.

In this section we shall prove the main result which allows us to calculate the class of equivariant pencils of operators.

The group of divergenceless diffeomorphisms.

Let ϱ be a volume form on a smooth manifold M. We define the group of volume preserving diffeomorphisms (w.r.t. ϱ), $\mathrm{SDiff}(\varrho)$, to be the space of diffeomorphisms that preserve ϱ, that is:

$$\mathrm{SDiff}(\varrho) = \{f \in \mathrm{Diff}(M) | f^*\varrho = \varrho\}.$$

This group has an associated algebra, $\mathrm{sdiff}(\varrho)$, which is the space of divergenceless vector fields with respect to ϱ:

$$\mathrm{sdiff}(\varrho) = \{X \in \mathfrak{X}(M) | \mathcal{L}_X \varrho = 0\}.$$

The condition that the Lie derivative of X with respect to ϱ is zero is equivalent to the condition that $(-1)^{\tilde{i}(\tilde{X}+1)} X^i{}_{,i} - X^i \gamma_i = 0$. From this condition one would expect that a divergenceless algebra of vector fields can be defined for an arbitrary connection however this is only possible if γ is flat. To see this let us take an arbitrary connection γ and consider two vector fields, X and$Y$, that are "divergenceless" with respect to this connection. Then we have that (for even vector fields to suppress additional signs):

$$[X,Y]^i{}_{,i} = \left(X^j Y^i{}_{,j} - Y^j X^i{}_{,j} \right)_{,i} = [X,Y]^i \gamma_i + d\gamma(X \wedge Y).$$

Thus we see that locally we need $d\gamma = 0$, i.e. γ is flat.

Note that from example 3.1.17 we see that a vector field X is divergenceless with respect to ϱ iff its Lie derivative pencil is equal to its volume form pencil.

Mappings of operators on $\mathbb{R}^n$.

The goal of this section is to prove the following result:

Theorem 3.2.7. *The space of local endomorphisms of differential operators on $\mathbb{R}^d$ for $d \geq 3$ invariant with respect to the algebra of divergenceless vector fields with respect to the coordinate volume form $|Dx|$ is 4-dimensional. All such maps are of the form:*

$$L \longmapsto aL + bL^\dagger + cL(1) + dL^\dagger(1), \tag{3.2.2}$$

where $a, b, c, d \in \mathbb{C}$.

By $L^\dagger$ in the above theorem we simply mean the adjoint with respect to $|Dx|$. To prove the theorem we shall need some preliminary results. Firstly take a general local endomorphism of $\mathrm{DO}(\mathbb{R}^d)$, $\Phi \in \mathrm{End}(\mathrm{DO}(\mathbb{R}^d))$. We may write this endomorphism as, see [70] and [71]:

$$L^I\partial_I \longmapsto \Phi(L) := V_I^{J;K}\partial_j L^I \partial_K. \tag{3.2.3}$$

As the generators of translations, ∂_i, are all divergenceless the $V_I^{J;K}$ are constants. We now produce the results that will go into the solution of theorem 3.2.7.

Remark 3.2.8. The reader may be curious why the dimension must be at least 3. The reason for this is that for dimensions 1 and 2 the group SDiff is of a very different nature than for higher dimensional manifolds, and we shall see that we require the dimension to be at least three in the proofs. To be more thorough one can see that for dimension 1 manifolds the space is given by a unique vector field, namely translation, and thus there are clearly a far wider space of liftings.

For a 2 dimensional orientable manifold a volume form is equivalent to a symplectic structure. This means that the group is different yet again and the space of liftings is not isomorphic. To show this explicitly recall that on a symplectic manifold (M, ω), there is a natural map $C^\infty(M) \to \mathfrak{X}(M)$ given by $f \mapsto X_f = df^\# = \omega^{-1}df$. This implies that the space of SDiff-equivariant lifitings for a 2 dimensional manifold is at least 6 dimensional:

$$L \mapsto \alpha L + \beta L^\dagger + \gamma L(1) + \delta L^\dagger(1) + \epsilon X_{L(1)} + \zeta X_{L^\dagger(1)}.$$

It is not known if an arbitrary symplecticly equivariant lifting is of this form.

Notation 3.2.9. We need to develop some notation for multi-indices to aid the ease of proofs. Recall that a multi-index on a set S of order n is just an n-tuple of elements from S defined up to permutations. If I is a multi-index then we shall denote its order by $|I|$. We shall also need the following notations:

- For $i \in S$, and I a multi-index $n_i(I) :=$ no. of occurrences of i in I.
- For $i \in S$ and I with order n, Ii is the multi-index of order $n+1$ where we add the element i to the tuple.
- Similarly if $n_i(I) \neq 0$ then we can define Ii^{-1} by removing an occurrence of i in I.

- We extend the above notation to hold between multi-indices as well, so if $J \subset I$ then IJ^{-1} is a multi-index.

Lemma 3.2.10. *Any invariant endomorphism of $DO(\mathbb{R}^d)$ ($d \geq 3$) preserves order for $d \geq 3$. Moreover let $V_I^{J;K}$ be the constants defined above, $V_I^{J;K} = 0$ unless $I = JK$*

Proof. To prove this lemma we shall only need the linear divergenceless vector fields, that is

$$X_j^i = x^i \partial_j \quad \forall i \neq j, \qquad Y_{ij} = x^i \partial_i - x^j \partial_j \quad \forall i, j.$$

The invariance of Φ, as in equation (3.2.3), then implies the following two equations:

$$n_i(I) V_{Ii^{-1}j}^{J;K} = (n_i(J)+1) V_I^{Jj^{-1}i,K} + (n_i(K)+1) V_I^{J,Kj^{-1}i} \quad i \neq j, \tag{3.2.4}$$

$$(n_i(I) - n_j(I)) V_I^{J,K} = (n_i(J) + n_i(K) - n_j(J) - n_j(K)) V_K^{I,J} \quad \forall i, j. \tag{3.2.5}$$

In equation (3.2.4) we have that the first term on the right hand side is zero unless $n_j(J) \neq 0$ and the second term is zero unless $n_j(K) \neq 0$. Our first aim will be to show that a general term $V_I^{J;K}$ is a multiple of one with specified choices for I, J and K. Assume that $V_I^{J;K}$ is non-zero, then using equation (3.2.8) we get that the quantity $n_i(I) - n_i(JK)$ is constant $\forall i$. If it is zero then we have proved the lemma.

Assume that $\exists j$ such that $n_j(I) = |I|$, that is $I = (j, j, \ldots, j)$. Then for any choice of J and K, with the same I, we have that $V_I^{J;K}$ is a multiple of one where $n_j(J) = |J|$. To see this assume that $\exists i \neq j$ and $n_i(J) \neq 0$. Then using this in equation (3.2.4) we set $J \mapsto Ji^{-1}j$ and we have that:

$$n_i(J) V_I^{J,K} + (n_i(K)+1) V_I^{Ji^{-1}j,K} \Rightarrow V_I^{J,K} = -\frac{n_i(K)+1}{n_i(J)} V_I^{Ji^{-1}j,Kj^{-1}i}.$$

We can then proceed by induction over all the elements not equal to i in the multi index. Now in this equation assume that there exists an $i \neq j$ such that $n_i(K) \neq 0$. Then choose $k \notin \{i, j\}$ which is possible from the dimension restriction we have imposed. Then we consider equation (3.2.4) with the vector field X_i^k and $K \mapsto Ki^{-1}k$. We have that the term on the left hand side is immediately zero whilst the first term on the right hand side is as well as $n_i(J) = 0$ and hence we get that

$$(n_k(K)+1) V_I^{J,K} = 0 \Rightarrow V_I^{J,K} = 0.$$

We now return to the case when I is arbitrary and suppose that $I = \{i, \ldots, i, j_1, \ldots, j_r\}$, $j_l \neq i$. We set $I \mapsto Ij_1^{-1}i$ in equation (3.2.4) and then apply it with $X^i_{j_1}$ to find that:

$$(n_i(I)+1)V_I^{J,K} = (n_i(J)+1)V_{Ij_1^{-1}i}^{Jj_1^{-1}i,K} + (n_i(K)+1)V_{Ij_1^{-1}i}^{J,Kj_1^{-1}i}.$$

By repeated application of this process we see that we may write $V_I^{J;K}$ as a linear combination of $V_{I'}^{J';K'}$, where $n_i(I') = |I'|$ and the $(J';K')$ are summed over all possible combinations of the form:

$$(Jj_{\sigma(1)}^{-1}j_{\sigma(2)}^{-1}\cdots j_{\sigma(q)}^{-1}\underbrace{i\cdots i}_{q \text{ times}};Kj_{\sigma(q+1)}^{-1}\cdots j_{\sigma(r)}^{-1}\underbrace{i\cdots i}_{r-q \text{ times}}),$$

where σ is a $(q, r-q)$-permutation and $q = 0, \ldots, r$. Now we may apply the above result to determine that these terms are all zero unless we can find a J' and K' of the above form such that $n_i(J') = |J'|$, $n_i(K') = |K'|$ and $|I'| = |J'K'|$. In particular we can immediately deduce from the structure of J' and K' that we must have that $I = JK$ and hence the lemma. □

An immediate corollary of this result is that we may write a general equivariant mapping as having the form:

$$L^{i_1\cdots i_n}\partial_{i_1\cdots i_n} \longmapsto \textstyle\sum_k V_{i_1\cdots i_n;k}\partial_{i_1\cdots i_k}L^{i_1\cdots i_n}\partial_{i_{k+1}\cdots i_n},$$

by using the symmetry relations among the coefficients L^I. Then applying equivariance of this equation with respect to $x^i\partial_j$, $i \neq j$, we see that in fact the constant $V_{I;k}$ is independent of I and only depends on the difference of orders k, $V_{I,k} = V_k$.

Corollary 3.2.11. *The only endomorphisms of $\mathcal{S}^n(\mathbb{R}^d)$ that are equivariant with respect to sdiff$(|Dx|)$ are multiples of the identity.*

Proof. The linear part of any algebra acts in the same way on both symbols and differential operators. That is if we have the map $Sp_I \mapsto V_I^{J;K}S^I{}_{,J}p_K$, where $|I| = |K| = n$, then we can go through the same analysis as in the previous lemma to get that $J = 0$ and $I = K$. □

Proposition 3.2.12. *Let $n \geq 3$, and let $Q : \mathcal{S}^n(\mathbb{R}^d) \to DO^n(\mathbb{R}^d)$ be a map equivariant with respect to sdiff$(|Dx|)$, then Q is zero. In particular there is no quantization map that is equivariant to the space of divergenceless vector fields.*

Proof. Let us take map, Q, as in the statement of the proposition and consider the quotients $Q^{(r)} : \mathcal{S}^n(\mathbb{R}^d) \to \mathrm{DO}^n(\mathbb{R}^d)/\mathrm{DO}^{n-r}(\mathbb{R}^d)$ from lemma 3.2.10 and the fact that linear vector fields act in the same manner on symbols and operators, we have that Q must be of the form:

$$S^I p_I \longmapsto Q(S) = \sum_{k=0}^{n} a_k S^{i_1\cdots i_n}{}_{,i_k\cdots i_k}\partial_{i_{k+1}\cdots i_n} .$$

If we use the above form of the map Q we find that there is a unique choice of a_1 that will guarantee $Q^{(2)}$ is equivariant, $A_1 = \frac{n}{2}a_0$. When $n \geq 3$ we look at the next prolongation for Q, that is $Q^{(3)}$. If we consider the coefficient of the term $\overline{\partial}_{i_1\cdots i_{n-2}}$ for $[X, Q](S) + \mathrm{DO}^{n-3}(\mathbb{R}^d)$ when X is divergenceless, we find that it is equal to

$$\sum_{\mu\leq n-2}\left(a_0\frac{n(n-1)(n-2)}{6} - a_2\right) X^{i_\mu}{}_{,i_{n-1}i_n i_{n+1}} S^{i_1\cdots\widehat{i_\mu}\cdots i_{n+1}}$$
$$+\left(a_0\frac{n(n-1)}{4} - 2a_2\right) X^{i_\mu}{}_{,i_n i_{n+1}} S^{i_1\cdots\widehat{i_\mu}\cdots i_{n+1}}{}_{,i_{n-1}} - a_2 X^{i_{n+1}}{}_{,i_{n-1}i_n} S^{i_1\cdots i_n}{}_{,i_{n+1}}.$$

It is not possible to set the final two terms equal to zero for this coefficient unless both a_0 and a_2 are equal to zero. As this is the case the map Q has image that lies in $\mathrm{DO}^{n-r}(\mathbb{R}^d)$ where $r \geq 2$ and hence will induce an equivariant, $\overline{Q}^{(r)} : \mathcal{S}^n(\mathbb{R}^d) \to \mathcal{S}^{n-r}(\mathbb{R}^d)$, where $r \geq 2$. We will thus have completed the proof if we show that such a map is necessarily 0. To show this assume such a map has the form $S^I p_I \mapsto W_I^{J;K}\partial_J S^I p_K$, where $I = JK$, $|K| = n - r$, $r \geq 3$. Then calculating the equation of equivariance for this map with respect to the vector field $x^i x^j \partial_k$, we find that we require

$$n_i(J)(n_j(J) - \delta_{ij})W_I^{J;K}\partial_{Ji^{-1}j^{-1}k} S^I p_K = 0 \quad \forall S^I p_I.$$

As $|J| \geq 2$ and $d \geq 3$ we can always make the coefficient $n_i(J)(n_j(J) - \delta_{ij})$ non-zero and thus we must have that all the $W_I^{J;K}$s are zero, hence the map is zero and we have completed the proof. □

This result can also be considered as a cohomological obstruction similar to the results obtained in paper [60] for the projective and conformal group.

Remark 3.2.13. We saw in the above proof that it was essential that the differential operators considered had order greater than or equal to 3. This is by no means superfluous, in fact in the next chapter we shall see that there exists a quantization map that is equivariant with respect to the space of divergenceless vector fields.

Lemma 3.2.14. *The space of maps $DO^2(\mathbb{R}^d) \to DO(\mathbb{R}^d)$, invariant with respect to sdiff($|Dx|$), is given by those maps of the form in equation* (3.2.2)*.*

Proof. We know from lemma 3.2.10 that the V_I^{JK}s are zero unless $JK = I$, hence the most general map that satisfies this condition is

$$S^{ij}\partial_{ij} + A^i\partial_i + B \longmapsto \alpha_0 S^{ij}\partial_{ij} + (\alpha_1 S^{ij}{}_{,j} + \beta_0 A^i)\partial_i + \alpha_2 S^{ij}{}_{,ij} + \beta_1 A^i{}_{,i} + \gamma B.$$

Calculating the invariance of this map with respect to a general divergenceless vector field we find the following constraints on the above constants:

$$\alpha_0 = \alpha_1 + \beta_0 \quad \alpha_2 = -\beta_1.$$

Setting $a = \frac{1}{2}(\alpha_0 + \beta_0)$, $b = \frac{1}{2}(\alpha_0 - \beta_0)$, $c = \gamma - \frac{1}{2}(\alpha_0 + \beta_0)$ and $d = -\beta_1 + \frac{1}{2}(\alpha_0 - \beta_0)$, we find that the above map has the form:

$$L \longmapsto aL + bL^\dagger + cL(1) + dL^\dagger(1).$$

□

Proof of theorem 3.2.7. We finish the proof by induction on the order of the operator. So assume that we have an equivariant map $\Phi : \mathrm{DO}^n(\mathbb{R}^d) \to \mathrm{DO}^n(\mathbb{R}^d)$, for $n \geq 3$, such that $\Phi|\mathrm{DO}^{n-1}(\mathbb{R}^d)$ has the form of equation (3.2.2). Then consider the operator Ψ, defined by

$$\Psi(L) = \Phi(L) - aL - bL - cL(1) - dL^\dagger(1).$$

Now by the inductive step we have that $\Psi|\mathrm{DO}^{n-1}(\mathbb{R}^d) = 0$, and hence Ψ induces a map, $Q : \mathcal{S}^n(\mathbb{R}^d) \cong \mathrm{DO}^n(\mathbb{R}^d)/\mathrm{DO}^{n-1}(\mathbb{R}^d) \to \mathrm{DO}^n(\mathbb{R}^d)$, and as the base case was $n = 2$ we have that $n \geq 3$ in this equation. Then using proposition 3.2.12 we have that $Q = 0 \Rightarrow \Psi = 0$, and thus the map Φ must have this form as well.

The same form holds for arbitrary $\mathrm{DO}(\mathbb{R}^d)$ as this is just the union of all finite order operators and we have thus finished the proof. □

Equivariant mappings on arbitrary manifolds.

Corollary 3.2.15. *Let $U \subset \mathbb{R}^d$, $d \geq 3$, be an open subset of $\mathbb{R}^d$ endowed with a volume form, $\varrho = \varrho(x)|Dx|$. Then the space of local endomorphisms of $DO(U)$ invariant with respect to sdiff(ϱ) is 4-dimensional with maps given as in equation* (3.2.2)*.*

Proof. We can always find a suitably small neighbourhood and a coordinate change that sets the volume form equal to $|Dy|$. For example we may set $y_i = x_i, i > 1$ and $y_1(x_1, \ldots, x_n) = \int_1^x \varrho(\widetilde{x}, x_2, \ldots, x_n) d\widetilde{x}$. This map is well defined on a suitably small ball around the point where we are integrating from and is clearly invertible. We then apply theorem 3.2.7 in the new coordinate system and pull back. Finally we need to show that the map glues together correctly but this is also clear from its explicit form as in equation (3.2.2) □

Corollary 3.2.16. *Let M be a connected manifold of dimension greater than* 2 *and ϱ a volume form. The endomorphism of $DO(M)$ equivariant with respect to sdiff(ϱ)*[3] *is four dimensional and given by the maps in equation* (3.2.2).

We now turn to the extension to supermanifolds. The proofs are essentially of exactly the same nature with some minor alterations required by odd variables and hence the details may be skipped on first reading. Our first goal is thus to calculate the equivariant local mappings on $\mathbb{R}^{n|m}$ with respect to the coordinate volume form, $|Dx|$. We have a priori that such a map must have the form:

$$L^I \partial_I \longmapsto \sum (-1)^{\phi(\widetilde{I},\widetilde{J},\widetilde{K})\widetilde{L}} V_I^{J,K} \partial_J L^I \partial_K.$$

Commuting this operator with respect to the generators of translations, ∂_i, tells us that the $V_I^{J,K}$s must all be constant and that $\phi(\widetilde{I}, \widetilde{J}, \widetilde{K}) = \widetilde{J}$. Our next stage is to consider the linear vector fields and we only need $X_j^i = x^i \partial_j$ $i \neq j$ and $Y_{ij} = x^i \partial_i - x^j \partial_j$ with no summation as before. Before we proceed to study the equivariance with respect to these vector fields let us just say a couple of words concerning the calculus of multi-indices on a super set, $S = S_0 \sqcup S_1$. The first thing to notice is that the ordering does have some affect, essentially what we have to do is employ the sign manifesto:

- For $i \in S$ and I a multi-index we have that Ii is defined as before. Note that if $\widetilde{i} = 1$ and there is an occurrence of i in I then $Ii = 0$.

- For $i \in S$ we have that Ii^{-1} is defined as expected, we push the index i^{-1} with parity $\widetilde{i}$ past the indices on the right until we come across the first occurrence of i in I and then remove it.

[3] *We mean the sheaf of invariant vector fields as there is no guarantee that they will glue together to give global* divergenceless *vector fields.*

- For $i \in S$ and I a multi-index, $n_i(I)$ is defined by the formula: $[\partial_I, x^i] = n_i(I)\partial_{Ii^{-1}}$.
- We can extend the above operation to multi-indices but one must be careful when applying $^{-1}$ to a multi-index.

We now can calculate equivariance with respect to these linear maps and we find that the equations that must be satisfied for equivariance with respect to Y_{ij} and X^i_j are

$$n_i(I)V^{Ji^{-1}j,Ki^{-1}j}_{Ii^{-1}j} = (-1)^{\widetilde{J}(\widetilde{i}+\widetilde{j})}n_i(J)V^{J,Ki^{-1}j}_I + (-1)^{(\widetilde{i}+\widetilde{j})(\widetilde{I}+\widetilde{J}+\widetilde{K}+1)}n_i(K)V^{Ji^{-1}j,K}_I, \tag{3.2.6}$$

$$\big(n_i(I) - n_i(J) - n_i(K)\big)V^{J,K}_I = \big(n_j(I) - n_j(J) - n_i(K)\big)V^{J,K}_I. \tag{3.2.7}$$

The second equation tells us, as before, that $V^{J,K}_I = 0$ unless $n_i(I) - n_i(J) - n_i(K)$ is constant. We now need to proceed using the same techniques as in lemma 3.2.10 but we also require the additional assumption that we have one even variable:

Lemma 3.2.17. *Let Φ be a local sdiff($|Dx|$)-equivariant map of $DO(\mathbb{R}^{d_0|d_1})$ such that $d_0+d_1 \geq 3$ and $d_0 \geq 1$. Then Φ preserves the order of differential operators, moreover the constants $V^{J,K}_I$ are zero unless $I = JK$.*

Proof. Assume that in equation (3.2.6) that $\exists j$ such that $n_j(I) = |I|$. Then we have that the equation becomes after setting $K \mapsto Kj^{-1}i$:

$$0 = (-1)^{\widetilde{J}(\widetilde{i}+\widetilde{j})}n_i(J)V^{J,K}_I + (-1)^{(\widetilde{i}+\widetilde{j})(\widetilde{I}+\widetilde{J}+\widetilde{K})}n_i(Kj^{-1}i)V^{Ji^{-1}j,Kj^{-1}i}_I.$$

As before we may then set $V^{J,K}_I$ proportional to a term where $n_j(J) = |J|$. The techniques is as before by inductively going through all elements such that $n_i(J) \neq 0$, $i \neq j$. Now we show that if $n_j(I) = |I|$, $n_j(J) = |J|$, then $n_j(K) = |K|$ or $V^{J,K}_I = 0$. Firstly we rewrite equation (3.2.6) where we alter J and use X^i_k to get the equation

$$n_i(I)V^{J,Ki^{-1}k}_{Ii^{-1}k} = (-1)^{(\widetilde{i}+\widetilde{k})(\widetilde{J}+1)}n_i(Jk^{-1}i)V^{Jk^{-1}i,Ki^{-1}k}_I + (-1)^{(\widetilde{i}+\widetilde{k})(\widetilde{I}+\widetilde{J}+\widetilde{K})}n_i(K)V^{J,K}_I.$$

Now picking $k \notin \{i,j\}$ we have that $V^{J,K}_I = 0$ unless $n_i(K) = 0$ $\forall i \neq j$, i.e. $n_j(K) = |K|$.

We now need to show that the constant $V^{J,K}_I$ for an arbitrary index I can be written as one proportional to a sum of those of the form $V^{J',K'}_{I'}$ where $n_j(I') = |I'|$

for some j. To do this let $I = (i, \ldots, i, j_1, \ldots, j_r)$ where $\widetilde{i} = 0$ and $j_l \neq i\ \forall l$. Then we take the following reformulation of equation(3.2.6):

$$n_i(Ij^{-1}i)V_I^{J,K} = (-1)^{\widetilde{j}(\widetilde{J}+1)} n_i(Jj^{-1}i)V_{Ij^{-1}i}^{Jj^{-1}i,K} +$$

$$(-1)^{\widetilde{j}(\widetilde{I}+\widetilde{J}+\widetilde{K}+1)} n_i(Kj^{-1}i)V_{Ij^{-1}i}^{J,Kj^{-1}i}.$$

As i is an even element the left hand side of this equation is not trivially zero and thus we can induct over the $j_l \neq i$ in I to get that it is proportional to a sum of $V_{I'}^{J',K'}$ where $n_i(I') = |I|$ and the J' and K' are summed as in the non super version of this proof and so the result holds. □

As before we have that from the above an equivariance with respect to $x^i\partial_j$ for $i \neq j$, the most general equivariant map has the form:

$$L^{i_1\cdots i_n}\partial_{i_1\cdots i_n} \longmapsto \sum_k (-1)^{\widetilde{L}(\widetilde{i_1}+\cdots+\widetilde{i_k})+\psi(\widetilde{i_1},\ldots,\widetilde{i_n})} a_k \partial_{i_1\cdots i_k} L^{i_1\cdots i_n}\partial_{i_{k+1}\cdots i_n}.$$

Proposition 3.2.18. *Let $n \geq 3$, $Q : \mathcal{S}^n(\mathbb{R}^{d_0|d_1}) \to DO^n(\mathbb{R}^{d_0|d_1})$ be a sdiff($|Dx|$)-equivariant map. Then if $d_0 \geq 1$ and $d_0 + d_1 \geq 3$ we have that $Q = 0$.*

Proof. The proof is the identical calculations for the even case. We require that $d_0 \geq 1$ twice here: firstly to apply lemma 3.2.17 and secondly to assure that we have arbitrarily high degree polynomial divergenceless vector fields. □

The proofs of the remaining results are standard using the above, and thus we have the super version of the theorem:

Theorem 3.2.19. *Let M be a connected supermanifold of dimension $d_0|d_1$ such that $d_0 \geq 1$ and $d_0 + d_1 \geq 3$. Fix a volume form $\varrho \in \mathcal{F}^1(M)^\times$. Then the space of linear mappings $DO(M) \to DO(M)$ equivariant with respect to sdiff(ϱ) is four dimensional and a general map has the form as in equation* (3.2.2).

3.2.3 Classification of equivariant pencils and liftings.

We now apply the results of the previous section to calculate all sdiff(ϱ) invariant pencils for a given ϱ. Using corollary 3.2.3 we know that the volume form induces its adjoints on specific differential operators by taking the pencil and the adjoint.

Theorem 3.2.20. *Fix a volume form $\varrho \in \mathcal{F}^1(M)^\times$. The maps $DO(\mathcal{F}^\lambda(M), DO^{\lambda+\delta}(M)) \to DO_\delta(\mathcal{F}(M))$ defined by the formula:*

$$L \longmapsto L_\varrho A(\widehat{w}) + L_\varrho^\dagger B(\widehat{w}) + L_\varrho(1)C(\widehat{w}) + L_\varrho^\dagger(1)D(\widehat{w}) \,, \tag{3.2.8}$$

where A, B, C and D are polynomials such that $A(\lambda) = 1$ and $B(\lambda) = C(\lambda) = D(\lambda) = 0$, are sdiff($\varrho$) equivariant lifts. Moreover if $M^{d_0|d_1}$ is connected with $d_0 + d_1 \geq 3$ and $d_0 \geq 1$ then these constitute all liftings equivariant with respect to this algebra.

Proof. It is clear that the above liftings are sdiff(ϱ) equivariant, so we have to show that these are all such maps when M satisfies the requisite conditions. Let ν be an arbitrary lifting of type (λ, δ) that is equivariant with respect to sdiff(ϱ). Then consider the following 1-parametric family of endomorphisms of DO(M):

$$\Phi_\mu(L) := \varrho^{-\delta} \circ \Big(\big[\nu([\varrho^\delta \circ L_\varrho]|\mathcal{F}^\lambda(M))|\mathcal{F}^\mu(M)\big]_\varrho |C^\infty(M)\Big) \,.$$

Let us explain the above map in words, we take a differential operator on M, then we take its ϱ lifting and post-multiply by ϱ^δ to get a differential operator in $\mathrm{DO}_\delta(\mathcal{F}(M))$. We then restrict this operator to $\mathcal{F}^\lambda(M)$, apply the lifting ν and then restrict to $\mathcal{F}^\mu(M)$. Finally we take the ϱ lifting of this pencil and restrict back to $C^\infty(M)$ before post multiplying by $\varrho^{-\delta}$ to get an operator of weight 0 again.

As all these operations are sdiff(ϱ) equivariant so is their composition, and hence we have that Φ_μ is a 1-parametric family of sdiff(ϱ) equivariant endomorphisms of DO(M). We may now apply theorem 3.2.19 as our manifold obeys the required conditions and we have that the above maps must be of the form:

$$L \longmapsto \Phi_\mu(L) = A(\mu)L + B(\mu)L^\dagger + C(\mu)L(1) + D(\mu)L^\dagger(1) \,,$$

for some functions A, B, C, D. Now using Φ_μ we can reconstruct the operator pencil corresponding to ν. To do this take an operator $L \in \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$, we then find that:

$$\varrho^\delta \Phi(\varrho^{-\delta} L_\varrho|C^\infty(M)) = A(\mu)L_\varrho|C^\infty(M) + B(\mu)L_\varrho^\dagger|C^\infty(M) + C(\mu)L_\varrho(1) + D(\mu)L_\varrho^\dagger(1).$$

This then gives us our pencil of operators as

$$\nu(K)|\mathcal{F}^\mu(M) = A(\mu)L_\varrho|\mathcal{F}^\mu(M) + B(\mu)L_\varrho^\dagger|\mathcal{F}^\mu(M) + C(\mu)L_\varrho(1) + D(\mu)L_\varrho^\dagger(1).$$

Finally as we know that these pencils come from a differential operator we may apply the standard association to determine that the functions A, B, C and D must all be polynomial and then we can make the substitution of $\widehat{w}$ (where the ordering of $\widehat{w}$ is now on the right). Finally using that ν is a lifting we find that these functions must also obey the initial conditions outlined above which completes the proof. □

Remark 3.2.21. Note that the above holds in the more general case for pencils of operators rather than just liftings. In this case we drop the restriction that the functions are polynomials in the weight operators, they could be arbitrary (even non-continuous) functions in $\widehat{w}$, however they do still have to obey the additional condition that $A(\lambda) = 1$ and $B(\lambda) = C(\lambda) = D(\lambda) = 0$, for a lifting.

We now turn to classifying liftings that obey either the additional conditions outlined in definition 3.1.16 or are self adjoint. This is a simple algebraic exercise using the above result - note that *any* sdiff(ϱ) equivariant lifting is necessarily a regular lifting, hence we need only consider the stronger conditions.

Corollary 3.2.22. *1. The space of n-regular sdiff(ϱ)-equivariant liftings of type* (λ, δ) *form an affine space of dimension* $2n + 1$ *when* $n \geq 2$.

2. The space of n-strict sdiff(ϱ)-equivariant liftings of type (λ, δ) *form an affine line in the above space* ($n \geq 2$).

Proof. 1. Consider a general sdiff(ϱ)-equivariant lifting of type (λ, δ) for $n \geq 2$, call it ν. For this degree we must have that this lifting is as in equation (3.2.8):

$$L \longmapsto \nu(L) = L_\varrho A(\widehat{w}) + L_\varrho^\dagger B(\widehat{w}) + L_\varrho(1) C(\widehat{w}) + L_\varrho^\dagger(1) D(\widehat{w}).$$

Then as the polynomials A, B, C and D must obey the required initial conditions we must have that $A = 1 + \Sigma_{i\geq 1} a_i(\widehat{w} - \lambda)^i$, $B = \Sigma_{i\geq 1} b_i(\widehat{w} - \lambda)^i$, $C = \Sigma_{i\geq 1} c_i(\widehat{w} - \lambda)^i$ and $D = \Sigma_{i\geq 1} d_i(\widehat{w} - \lambda)^i$. Then applying the conditions that $\nu(L)$ must have order n, and we find:

$$b_1 + (-1)^n a_1 = 0, \quad a_i = b_i = 0 \ \forall i \geq 3, \quad c_j = d_j = 0 \ \forall j > n.$$

We therefore get an affine space of the dimension claimed in the corollary, and we have that the lifting ν has the form

$$\nu(L) = L_\varrho + a_1 \left(L_\varrho - (-1)^n L_\varrho^\dagger\right)(\widehat{w} - \lambda) + \sum_{j=1}^{n} \left(L_\varrho(1)c_j + L_\varrho^\dagger(1)d_j\right)(\widehat{w} - \lambda)^j. \tag{3.2.9}$$

2. Now consider the case of strict n-liftings that are also equivariant with respect to sdiff(ϱ). Take an arbitrary lifting that is n-regular as in equation (3.2.9) and an operator $L \in \mathrm{DO}^{n-1}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) - \mathrm{DO}^{n-2}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$. Then the minimal horizontal order of $L_\varrho - (-1)^n L_\varrho^\dagger$ is equal to $n-1$ and hence as it is multiplied by a first order operator we must necessarily have that it is equal to zero, that is $a_1 = 0$. Now consider $\mathbf{s} \in \mathcal{F}^\delta(M) \subset \mathrm{DO}^n(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$, then as $\mathbf{s}^\dagger = \mathbf{s}$ we require that $c_j = -d_j \forall j$. Finally consider $D \in \mathrm{DO}^1(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$, generically there is no relation between $D(1)$ and $D^\dagger(1)$ and thus we must have that $c_j, d_j = 0 \ \forall j \geq 1$. Therefore the most general n-strict sdiff(ϱ)-equivariant lifting must be of the form:

$$L \longmapsto L_\varrho + c\left(L_\varrho(1) - L_\varrho^\dagger(1)\right)(\widehat{w} - \lambda).$$

□

Self adjoint equivariant pencils.

We now turn to the question of which sdiff(ϱ)-equivariant lifts are self adjoint, as before we shall split this study into to cases: when $2\lambda + \delta \neq 1$ and when it does. For the moment let us focus on the case when $2\lambda + \delta \neq 1$. Applying the projection $\nu \mapsto \nu^{sa}$ we see that $a_1 = \frac{1}{2\lambda+\delta-1}$. Generic $L_\varrho(1)$ and $L_\varrho^\dagger(1)$ are linearly independent so the constants $\{c_j, d_j : j = 1, \ldots, n\}$ must satisfy the equations:

$$(1 - (-1)^{n+k})c_k - \sum_{l>k} (-1)^{n+l} \binom{l}{k} (2\lambda + \delta - 1)^{l-k} c_l = 0.$$

Just as when we were calculating Taylor expansions for self adjoint operators we find that those k such that $n + k$ is even are completely free variables, whilst those k such that $n + k$ is odd are completely determined by the other k.

Proposition 3.2.23. *The space of self-adjoint liftings of type (λ, δ), $2\lambda + \delta \neq 1$, of order n that are equivariant with respect to sdiff(ϱ) has dimension $2\lceil \frac{n}{2} \rceil$.*

Proof. We may use the constants μ_r as in the proof of proposition 3.1.23 to calculate the explicit form if the map. Indeed as the constants are required to obey the same functional relations we see that those c_k where $n-k$ is odd can be expressed explicitly in terms of the ones for which $n-k$ is even:

$$c_{n-2k-1} = \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - k} \binom{n-2(k-r)}{2r+1} (2\lambda+\delta-1)^{2r-1} \mu_r c_{n-2(k-r)},$$

and an identical expression for the ds. □

Example 3.2.24. Let g^{ij} be an invertible tensor, that is it corresponds to the inverse of a metric. We can then define a volume form which is denoted by $\sqrt{g}$. These two bits of data define two connections, Γ^i_{jk} on the tangent bundle and γ_i on volume forms defined as follows:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{lk,j} - g_{kj,l} + g_{jl,k} \right) \qquad \text{Levi-Civita,}$$

$$\gamma_i = -\frac{1}{\sqrt{g}} \partial_i \sqrt{g} = -\frac{1}{2} g^{jk} g_{jk,i} = -\Gamma^j_{ij}.$$

As γ is a connection on densities the associated covariant derivative can be interpreted as a horizontal vector field on $\widehat{M}$:

$$\nabla_i = \partial_i + \gamma_i \widehat{w}.$$

Note that the $\sqrt{g}$-pencil of a differential operator on $C^\infty(M)$ corresponds to replacing each partial derivative by its covariant counterpart. This extends the normal result about lifting vector fielsd using the covariant derivative to lifting arbitrary operators. Let us go through a specific example, the Laplace-Beltrami operator associated to g, Δ_g. This is defined as follows: take the de Rham differential of a function f, then map this to a vector field by using the inverse of the metric and finally take the divergence of this vector field with respect to $\sqrt{g}$. Symbolically this means that:

$$\Delta_g = \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j = \partial_i g^{ij} \partial_j - \gamma^i \partial_i,$$

where $\gamma^i = g^{ij}\gamma_j$. One can see that the $\sqrt{g}$-pencil of this operator is self adjoint and kills 1, and therefore an arbitrary Diff($\sqrt{g}$)-equivariant pencil is equal to the $\sqrt{g}$-pencil. This is equivalent to:

$$\widehat{\Delta_g} = \nabla_i g^{ij} \nabla_j - \gamma^i \nabla_i.$$

Now consider the equivariant liftings that go through the singular point. We know that the volume form pencil commutes with taking adjoints and hence we need to take the vertical Taylor expansion of an arbitrary lift and check that the Taylor coefficients satisfy the requirements that we found in the section of self adjoint liftings at singular points, see proposition 3.1.26. We find that $\Phi_1(L) = a_1(L - (-1)^n L^\dagger)$ which only satisfies the condition of being equivariant if $a_1 = 0$. This is because we require that $\Phi_1(L)^\dagger = -\Phi_1(L^\dagger)$ and this holds iff $a_1 = 0$. We now turn to the remaining terms: they are all of the same form and given by $L \mapsto c\varrho^{-\frac{1-\delta}{2}} L(\varrho^{\frac{1-\delta}{2}}) + d\varrho^{-\frac{1-\delta}{2}} L^\dagger(\varrho^{\frac{1-\delta}{2}})$. How such a map commutes with adjoints is completely determined by the ratio $(c : d)$. In fact we have that d must be proportional to c by a factor of $+/-1$ and hence we find that the map $\Phi_k(L) = c_k \varrho^{-\frac{1-\delta}{2}} \left(L(\varrho^{\frac{1-\delta}{2}} + (-1)^k L^\dagger(\varrho^{\frac{1-\delta}{2}}) \right)$ satisfies the correct transformations under the adjoint operation. We thus have the following:

Corollary 3.2.25. *The space of self adjoint liftings passing through the singular point that are equivariant with respect to sdiff(ϱ) has dimension n and any such map is given by the formula:*

$$L \longmapsto L_\varrho + \sum_{k=1}^n c_k \left(L_\varrho(1) + (-1)^k L_\varrho^\dagger(1) \right) \left(\widehat{w} - \tfrac{1-\delta}{2} \right)^k .$$

3.3 Equivariant pencils II: The projective algebra.

In this section we study projective invariance, in particular we shall be interested in manifolds with a projective structure. The main tool we use is the projective quantization methods developed by C. Duval, P.B.A. Lecomte and V.Y. Ovsienko which is where we shall firstly turn. We then apply this to the theory of lifts and determine all classes of lifts that are projectively equivariant.

The projective group and algebra.

Let us firstly recall the definition of the projective group $\mathrm{PGL}(d;\mathbb{K})$ and its Lie algebra $\mathfrak{pgl}(d;\mathbb{K})$. $\mathrm{PGL}(d;\mathbb{K})$ is the group of isomorphisms of $\mathrm{P}(\mathbb{K}^d)$ coming from the action of $\mathrm{GL}(d;\mathbb{K})$; that is given an element $l \in \mathrm{P}(\mathbb{K}^d)$, then any $g \in \mathrm{GL}(d;\mathbb{K})$ will map this line onto another line however the action only defines g up to scale. Therefore we have that $\mathrm{PGL}(d;\mathbb{K}) \cong \mathrm{GL}(d;\mathbb{K})/Z(\mathrm{GL}(d;\mathbb{K}))$. If we pick a standard affine neighbourhood of $\mathrm{P}(\mathbb{K}^d)$, say $U_0 \subset \mathrm{P}(\mathbb{K}^d)$ defined as $U_0 = \{(X_0 : \ldots : X_{d-1}) | X_0 \neq 0\}$, then we have the local coordinates, $x_i = X_i/X_0$. Now take an element $\overline{g} \in \mathrm{PGL}(d;\mathbb{K})$, and a representative of the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathrm{Mat}(d-1;\mathbb{K})$, $B \in \mathbb{K}^{d-1}$, $C \in (\mathbb{K}^{d-1})^*$, and $D \in \mathbb{K}$. Then the action of this element on the variables $(x_1, \ldots, x_{d-1})$ is defined as:

$$\underline{x} \longmapsto \overline{g}(\underline{x})^i = \tfrac{A^i_j x^j + B^j}{C_k x^k + D}.$$

The space $\mathrm{P}(\mathbb{K}^d)$ is the natural homogeneous space for $\mathrm{PGL}(d;\mathbb{K})$, corresponding to the subgroup $GL(d-1;\mathbb{K})$ which we embed in $\mathrm{PGL}(d;\mathbb{K})$ as

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 3.3.1. A d-dimensional manifold with a projective structure is a manifold modeled on the pair $(\mathrm{PGL}(d+1;\mathbb{R}), \mathbb{R}\mathrm{P}^d)$, see appendix B.

The projective group, $\mathrm{PGL}(d;\mathbb{K})$, is a Lie group and therefore has a natural Lie algebra, $\mathfrak{pgl}(d;\mathbb{K})$. This Lie algebra is generated by $\{a_i, b^i_j, c^j | j = 1, \ldots, d-1\}$, subject to the commutation relations:

$$[a_i, a_j] = 0,$$

$$[a_i, b^j_k] = \delta^j_i a_k,$$

$$[a_i, c^j] = \delta^j_i b^k_k + b^j_i,$$

$$[b^i_j, b^k_l] = \delta^k_j b^i_l - \delta^i_l b^k_j,$$

$$[b^i_j, c^k] = \delta^k_j c^i,$$

$$[c^i, c^j] = 0.$$

The Casimir for the Lie algebra has the following form:

$$\mathcal{C} = b^i_i b^j_j + b^i_j b^j_i - a_j c^j - c^j a_j. \tag{3.3.1}$$

3.3.1 Projectively equivariant quantization.

We now reproduce the result of the existence and uniqueness of a quantization map on manifold with projective structure that is equivaraint with respect to the projective vector fields[4] on the manifold, see [37]. We need to consider quantization maps of different weights so we will recall the definition:

Definition 3.3.2. A *quantization map* of type (λ, δ) is a linear map $\mathfrak{Q} : \mathcal{S}(M) \otimes \mathcal{F}^\delta(M) \to \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$, such that $\mathfrak{Q}$ is a right inverse to the principle symbol map, $\sigma_P : \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) \to \mathcal{S}(M) \otimes \mathcal{F}^\delta(M)$.

In other words a quantization map associates to any homogeneous symbol, $S \in \mathcal{S}^n(M) \otimes \mathcal{F}^\delta(M)$, a differential operator, $\mathfrak{Q}(S)$, of order n whose principle symbol is S. It is well known that there is no natural way to do this for a general manifold. However the works of V.Y. Ovsienko et al. show that if our manifold has some additional structure then there is a canonical way to construct such a map:

Theorem 3.3.3. *[22, 59] Let M be a manifold of dimension d with a projective structure. Then if $\delta \notin \{1 + \frac{m}{d+1} | m \in \mathbb{N}\}$, there exists a naturally defined quantization map of type (λ, δ) $\forall \lambda$.*

To prove this theorem we shall use the Casimir operator of the projective algebra, equation (3.3.1). We shall find it useful to give the explicit form of the action of the Casimir operator.

Lemma 3.3.4. 1. *The action of the Casimir operator on $\mathcal{S}^n(\mathbb{R}^d) \otimes \mathcal{F}^\delta(\mathbb{R}^d)$ is given as:*

$$\mathcal{C}_{\mathcal{S}^n \otimes \mathcal{F}^\delta(M)} = 2n\big(n - 1 - (d+1)(\delta - 1)\big) + d(d+1)\delta(\delta - 1). \tag{3.3.2}$$

2. *The action of the Casimir operator on differential operators is given by the formula:*

$$\mathcal{C}_{DO(\mathbb{R}^d)}(L^I \partial_I) = \Big(2|I|\big(|I| - 1 - (d+1)(\delta - 1)\big) + d(d+1)\delta(\delta - 1)\Big) L^I \partial_I$$
$$+ 2 \sum_{i=1}^{d} n_i(I)\big(|I| - 1 + \lambda(d+1)\big) L^I{}_{,i} \partial_{Ii^{-1}}. \tag{3.3.3}$$

[4] As in the case of divergenceless vector fields we should consider the sheaf of such vector fields. In this case we have that the existence of global projective fields is dictated by the developing map representation of the fundamental group of the manifold, see appendix B.

The proof follows by direct calculation. We can now turn to the proof of the theorem using the techniques in [18]:

Proof of theorem 3.3.3. If an equivariant quantization exists then it must necessarily commute with the local action of the Casimir. Now by lemma 3.3.4 we have that an symbol $|Dx|^\delta S^I p_I$ of a fixed grading is an eigenvector of the Casimir operator and therefore the quantization of this symbol must also be an eigenvalue. Thus the existence and uniqueness of the quantization will follow essentially by showing that there exists differential operator that is an eigenvector of $\mathcal{C}$ for any choice of principle symbol. Thus fix a symbol, $|Dx|^\delta S^I \partial_I$ and consider an arbitrary differential operator with this symbol: $L = S^I \partial_I + \Sigma_{|J|<|I|} S^J \partial_J$. Forcing this operator to be an eigenvector (with eigenvalue $2|I|\big(|I| - 1 - (d+1)(\delta-1)\big) + d(d+1)\delta(\delta-1))$ gives us the following conditions:

$$(|I| - |J|)\big(|I| + |J| - 1 - (d+1)(\delta-1)\big)S^J = \sum_{i=1}^{d} (n_i(J)+1)\big(|J| + \lambda(d+1)\big) S^{Ji}{}_{,i} \quad \forall J.$$

We thus see that there exists a unique choice for each principle symbol that is linear in the symbol if δ is not on of the resonant values mentioned in the assumptions of the theorem. This then defines our map locally and by uniqueness it glues to give a canonical quantization map on a projective manifold. □

The choices of δ for which the above proof fails to hold are also known as resonant values and we will refer to them as such in the sequel. They are related to the fact that at these points the Casimir has multiple eigenvalues.

Remark 3.3.5. In the proof above one will notice that we did not assume locality of the quantization and this condition fell out in the proof. In fact one can prove directly that any projectively equivariant quantization map must be local, see [59].

Projectively equivariant symbol calculus.

It will also be essential to us to prove not only the existence of the quantization map but the explicit form of the symbol calculus contained in such a theory, although we will only be considering differential operators rather than pseudodifferential operators. The constants were also calculated in the paper [22] and we shall reproduce the results here for use in the next section.

We can calculate the explicit form of the quantisation map, $\mathfrak{Q}$, directly from the proof. Let $\nabla = \frac{\partial^2}{\partial x^i \partial_i}$ be the odd divergence operator (which is only locally defined). We then see from the recurrence relation that we would expect the quantization map to be of the form:

$$|Dx|^\delta S^I p_I \longmapsto |Dx|^\delta \sum_{k=0}^{|I|} q_{|I|,k}(\lambda,\delta)\nabla^k(S)|_{p=\partial},$$

where in the final expression we are using QP-quantization. Setting $I = n$ we find that the relation the constants in the above equation must obey are:

$$q_{n,k}(\lambda,\delta) = \frac{n-k+\lambda(d+1)}{k(2n-k-1-(d+1)(\delta-1))} q_{n,k-1}(\lambda,\delta), \quad q_{n,0}(\lambda,\delta) = 1.$$

We can solve these equations explicitly and we find the general formula:

$$q_{n,k}(\lambda,\delta) = \binom{n}{k} \frac{\binom{n-1+\lambda(d+1)}{k}}{\binom{2(n-1)+(d+1)(1-\delta)}{k}} \tag{3.3.4}$$

Now we turn to the question of the projectively equivariant symbol map, that is the inverse to the quantization map as defined above. There is a simple explanation of how we calculate this operator: If we have a differential operator L, then we take the principal symbol of L, $\sigma_P(L)$ and quantize it to get $\mathfrak{Q}(\sigma_P(L))$. We then have that $L - \mathfrak{Q}\sigma_P(L)$ has order 1 less then L so we can proceed by induction to define the full inverse map. To calculate the general formula we simply compute the conditions on the full symbol map, $\sigma : \mathrm{DO}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) \to \mathcal{S}(M) \otimes \mathcal{F}^\delta(M)$. If we have an operator $L = |Dx|^\delta \Sigma S^I \partial_I$, then we should expect that $\sigma(L) = \Sigma_{I,k}\sigma_{|I|,k}(\lambda,\delta)\nabla^k(S^I|_{\partial=p})$. As before acting with the Casimir on both sides of this equation tells us that the following holds between the constants $\sigma_{n,k}(\lambda,\delta)$:

$$k\big(n+k-1+(d+1)(\delta-1)\big)\sigma_{n,k}(\lambda,\delta) = -(n-1+\lambda(d+1))\sigma_{n-1,k-1}(\lambda,\delta),$$

$$\sigma_{n,0}(\lambda,\delta) = 1 \quad \forall n.$$

We can again solve these expressions and we find that the explicit representation of the constants in the full symbol map are given as:

$$\sigma_{n,k}(\lambda,\delta) = (-1)^k \binom{n}{k} \frac{\binom{n-1+\lambda(d+1)}{k}}{\binom{2n-k-1+(d+1)(1-\delta)}{k}}. \tag{3.3.5}$$

3.3.2 Projectively equivariant pencils and lifts.

We now apply the symbol calculus of the previous section to calculate all projectively invariant lifts of differential operators. Let us fix some choice of δ and then consider quantization and symbol maps of type (λ, δ), $\mathfrak{Q}_{(\lambda,\delta)}$ and $\sigma_{(\lambda,\delta)}$, where we would like to think of λ as a variable. Then for an operator $L \in \mathrm{DO}(\mathcal{F}^{\lambda_0}(M), \mathcal{F}^{\lambda_0+\delta}(M))$, we define the pencil of operators passing through L as $P^{proj}(L)_\lambda := \mathfrak{Q}_\lambda \circ \sigma_{\lambda_0}(L)$ (we have suppressed the index δ as it is constant here). If we look at the explicit form of this pencil, using the coefficients defined in equations (3.3.4) and (3.3.5), we see it is not only locally a polynomial in λ, so that the pencil is locally a differential operator, but that it is also a polynomial in λ of degree equal to the order of the differential operator, so it is a differential operator on $\mathcal{F}(M)$ of the same order as L. This then tells us that for non-resonant values of δ there is a strict projectively equivariant pencil which we shall denote by ν^{proj}.

We now classify all projectively invariant pencils. To do this we use an equivariant version of the vertical Taylor expansion which can be stated succinctly as follows.

Lemma 3.3.6. *Let ν be a G-equivariant (resp. $\mathfrak{g}$- equivariant) n-regular lifting of type (λ, δ) for $G \subset Diff(M)$ ($\mathfrak{g} \subset \mathfrak{X}(M)$). Take the Taylor expansion of an arbitrary n-regular lifting ν', with respect to ν: $T^{(1)}, \dots$. Then ν' is G-equivariant iff $T^{(i)}$ is a G-equivariant mapping (resp. ν' is $\mathfrak{g}$-equivariant iff $T^{(i)}$ is $\mathfrak{g}$-equivariant) $\forall i$.*

Proof. Write the Taylor series out explicitly so that:

$$\nu'(L) = \sum_{k=0}^{n} \nu(L^{(k)})(\widehat{w} - \lambda)^k, \quad L^{(0)} = L,$$

where we have used the same notation as in our earlier exposition of the vertical Taylor expansion. We then simply apply $g \in G$ (or $X \in \mathfrak{g}$) to this equation and use the facts that $g^*\nu = \nu g^*$, and $g^*\widehat{w} = \widehat{w} g^*$ as well as uniqueness of the Taylor expansion to get the result. □

To apply this to the projective group we need to calculate all possible equivariant mappings of $\mathrm{DO}^n(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M)) \to \mathrm{DO}^{n-k}(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ for non-resonant δ. The quantization maps shows that the space of differential operators is isomorphic as a $\mathfrak{pgl}(d+1;\mathbb{R})$ module to the space of symbols. Let L be a differential operator which we split it into its graded eigenvectors, $L = L_n + L_{n-1} + \cdots + L_0$, where L_k has eigenvalue $2k(k-1+(d+1)(1-\delta)+d(d+1)\delta(\delta-1)$. Then, as any equivariant map must preserve the space of eigenvectors, we have that all such mappings are given by $n-k+1$ constants, c_r with the mapping defined as:

$$L=\sum_{r=0}^{n} L_r \longmapsto \sum_{r=0}^{n-k} c_r L_r.$$

Putting this result together with lemma 3.3.6 we classify all regular projectively invariant liftings of differential operators for non-resonant δ.

Proposition 3.3.7. *The space of n-regular projectively equivariant liftings has dimension $\frac{n(n-1)}{2}$ for non-resonant δ. If we let $L=\Sigma_r L_r$ denote its decomposition into eigenvalues then we have that the form of an arbitrary lift is given as:*

$$L \longmapsto \sum_r \nu^{proj}(L_r)A_r(\widehat{w}),$$

where A_r is a polynomial of degree $n-r$ such that $A_r(\lambda)=1$.

Corollary 3.3.8. *The strictly equivariant lift ν^{proj} is unique.*

Proof. Take a regular n-lifting associated to the polynomials A_{n-r}. Then from the above explicit description of the associated pencil we see that it is strict iff all the polynomials are constant, hence the proof. □

Example 3.3.9. Consider the strict projective pencil applied to first order operators. If $D=|Dx|^{\delta}\big(X^i\partial_i+Y\big)\in \mathrm{DO}^1(\mathcal{F}^{\lambda}(M),\mathcal{F}^{\lambda+\delta}(M))$ for $\delta\neq 1$, then we have

$$\nu^{proj}(D)=t^{\delta}\big(X^i\partial_i+X^i{}_{,i}\frac{\widehat{w}-\lambda}{1-\delta}+Y\big).$$

Note that this is the lifting as in example 3.1.18. The is a 1-dimensional space of projectively equivariant liftings is given by

$$\nu_a(D)=t^{\delta}\big(X^i\partial_i+X^i{}_{,i}\frac{\widehat{w}-\lambda}{1-\delta}+Y(1+a(\widehat{w}-\lambda))\big).$$

Self adjoint equivariant pencils.

We now turn to the question of lifts of operators that are self-adjoint and equivariant. We start with the analysis at the singular point $\lambda=\frac{1-\delta}{2}$. Recall that we use the notation $\nu^{proj}_{(\lambda,\delta)}$ to denote the unique projective lift of type (λ,δ) for non-resonant δ.

Lemma 3.3.10. *The strict projective lift commutes with the adjoint, that is*

$$\left(\nu^{proj}_{(\lambda,\delta)}(L)\right)^{\dagger} = \nu^{proj}_{(1-\lambda-\delta,\delta)}(L^{\dagger}).$$

Proof. The proof is equivalent to showing the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{DO}(\mathcal{F}^{1-\lambda-\delta}(M),\mathcal{F}^{1-\lambda}(M)) & \xrightarrow{\nu^{proj}} & \mathrm{DO}_{\delta}(\mathcal{F}(M)) \\ \downarrow{\scriptstyle \dagger} & & \uparrow{\scriptstyle \dagger} \\ \mathrm{DO}(\mathcal{F}^{\lambda}(M),\mathcal{F}^{\lambda+\delta}(M)) & \xrightarrow{\nu^{proj}} & \mathrm{DO}_{\delta}(\mathcal{F}(M)). \end{array}$$

To show this we just need to explain why the composition of the three arrows is a strict lifting by corollary 3.3.8. This follows immediately however by taking the pencils associated to all the maps involved and recalling that we take the adjoint of a pencil pointwise. ☐

Corollary 3.3.11. *Assume δ is non resonant. Then the mapping $\nu^{proj}_{\left(\frac{1-\delta}{2},\delta\right)}$ is a self adjoint lifting.*

To calculate the space of all self adjoint equivariant liftings going through the singular point it will be useful to not use the polynomials A_r but B_r which correspond to replacing $\widehat{w}$ by $\widehat{w}_{\delta}$. The boundary conditions these polynomials must satisfy is $B_r(0) = 1\ \forall r$. Expressing the lifting in terms of these polynomials,

$$\nu(L) = \sum \nu^{proj}(L_r) B_r(\widehat{w}_{\delta}),$$

the condition that it is self adjoint is exactly that all the polynomials are even. Therefore to find the dimension of all such pencils we just need to compute the sum:

$$\sum_{r=0}^{n} \left\lfloor \frac{n-r}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Corollary 3.3.12. *The space of self adjoint liftings of type $\left(\frac{1-\delta}{2},\delta\right)$ that are projectively equivariant for non-resonant δ has dimension $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor$, where the isomorphism is defined as above.*

Example 3.3.13. Consider a second order differential operator of weight 0 on $\mathcal{F}^{\frac{1}{2}}(M)$ that is self adjoint:

$$L = S^{ij}\partial_{ij} + S^{ij}{}_{,j}\partial_i + R.$$

Then the most general self adjoint projectively equivariant pencil passing through this differential operator has the form:

$$\nu(L) = S^{ij}\partial_{ij} + \big(S^{ij}{}_{,j} + 2\frac{d+1}{d+3}S^{ij}{}_{,j}(\widehat{w} - \frac{1}{2})\big)\partial_i +$$
$$S^{ij}{}_{,ij}\Big(\frac{d+1}{d+3} + \frac{(d+1)^2}{(d+2)(d+3)}\big(\widehat{w} - \frac{1}{2}\big)\Big)\big(\widehat{w} - \frac{1}{2}\big) + 2\alpha\big(R - \frac{1}{4}\frac{d+1}{d+2}S^{ij}{}_{,ij}\big)\big(\widehat{w} - \frac{1}{2}\big) + R,$$

where α is a constant. Applying this operator to 1 we get the following 1-parametric family of projective scalars:

$$\frac{1}{4}\frac{(d+1)\big((\alpha-3)d + 3\alpha - 7\big)}{(d+2)(d+3)}S^{ij}{}_{,ij} + (1-\alpha)R.$$

Now consider a general self adjoint lifting of type (λ, δ), ν, for non-resonant δ. Then we must have that for each differential operator L, $\nu(L)|\mathcal{F}^{\frac{1-\delta}{2}}(M)$ is anti/self adjoint. Writing out the pencil as:

$$\nu(L) = \sum \nu^{proj}(L_r)A_r(\widehat{w}),$$

We can now determine a necessary condition for the lifting to be anti/self adjoint that is that the polynomials A_r satisfy the condition $A_r(\frac{1-\delta}{2}) = 0$ for $n - r$ odd. As before we shall thus find it easier to write the polynomials in terms of $\widehat{w}_\delta$. Using this we can prove the following

Proposition 3.3.14. *Assume $\lambda \neq \frac{1-\delta}{2}$, then the space of (anti-)self adjoint equivariant liftings has dimension $\left\lceil \frac{n-1}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Note that the action of the adjoint on the space of symbols, using the symbol calculus, is given as:

$$\dagger : \mathcal{S}(M) \to \mathcal{S}(M), \quad \dagger|\mathcal{S}^k(M) = (-1)^k.$$

Thus an immediate corollary of lemma 3.3.10 is that $q_n\left(\frac{1-\delta}{2}, \delta\right)$ is (anti-)self adjoint iff $n \equiv 0 \mod 2$ ($n \equiv 1 \mod 2$). We now use the fact, that follows because we define our pencil using a symbol calculus, that:

$$\nu^{proj}_{(\lambda',\delta)}\Big(\nu^{proj}_{(\lambda,\delta)}(L)|\mathcal{F}^{\lambda'}(M)\Big) = \nu^{proj}_{(\lambda,\delta)}.$$

We shall use the above equation when $\lambda' = \frac{1-\delta}{2}$. We then write the polynomial lifting the differential operators in terms of B_rs:

$$\nu(L) = \sum_r \nu^{proj}_{(\lambda,\delta)}(L_r)B_r(\widehat{w}_\delta),$$

where $B_r\left(\frac{2\lambda-\delta+1}{2}\right) = 1$. We now write $\nu^{proj}_{(\lambda,\delta)} = \nu^{proj}_{\left(\frac{1-\delta}{2},\delta\right)} \circ q_{\left(\frac{1-\delta}{2},\delta\right)} \circ \sigma_{(\lambda,\delta)}$. We immediately see using the above representation of † on the space of symbols that this lifting is (anti-)self adjoint iff B_r is an even function for $r \equiv n \bmod 2$, and B_r is an odd function for $r \equiv n+1 \bmod 2$. All that is left to do is to calculate this space of polynomials. Let us explain the space more geometrically: fix a point $x \neq 0$ and consider the space of even/odd polynomials of degree $\leq n$ such that $P(x) = 1$. It is immediately clear for linear functions the space is 0-dimensional; for quadratics it is 1-dimensional; and for cubics it is also 1-dimensionsal. In general we have that the space of polynomials of parity equal to their degree, r, that pass through some fixed point has dimension $\left\lceil \frac{r-1}{2} \right\rceil$. Now we need to sum all these polynomials of degree less than or equal to n and the sum is simply equal to $\left\lceil \frac{n-1}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$ as claimed.

□

Chapter 4

Differential Operators II: Brackets and Groupoids.

In this chapter we use the methods developed in the previous section to determine the structure of a general operator on the algebra of densities. We start by focusing on the case of curves where a description is explicit. In the second half of the chapter we shall turn to the question of long brackets, generating differential operators and the Khudaverdian-Voronov groupoid. We shall see that for second order operators the case is singular and holds in any suitable inner product algebra. In the final part of the chapter we shall work with the Khudaverdian-Voronov groupoid. We shall use the notion of Bordemann quantization to generalise the groupoid to higher order operators. We shall see that using this method the complexity of the groupoid increases to an almost untenable position.

4.1 General structure of differential operators.

In the first half of this chapter we shall attempt to reduce the question of the content of differential operators to purely combinatoric data. We shall be completely successful for curves and second order operators. We begin to see the divergence of third order operators away from such a simple classification. This question as to the exact discrepancy will be reserved to the second half of the chapter where we shall use different techniques.

4.1.1 Differential operators on curves.

We focus on the case of curves[1] before we generalise to arbitrary dimension. To this end let C be a curve and take a differential operator $L \in \mathrm{DO}^n_\delta(\mathcal{F}(C))$. We shall use the vertical Taylor expansion with the volume form pencil, see equation (3.1.3) and (3.2.1), to obtain a geometrical description of the terms in the differential operator L.

Let us briefly recall how to apply the vertical Taylor expansion with the volume form pencil: Take an operator $L \in \mathrm{DO}^n_\delta(\mathcal{F}(M)$ and choose some λ. We then consider $L^{(0)} := L|\mathcal{F}^\lambda(M)$. We then lift $L^{(0)}$ using the volume form and we have that $L = L^{(0)}_\varrho + R(\widehat{w} - \lambda)$, for some differential operator R. We then proceed by induction to write

$$L = \sum_{k=0}^{n} L^{(k)}_\varrho (\widehat{w} - \lambda)^k, \tag{4.1.1}$$

where n is the order of L and $L^{(k)} \in \mathrm{DO}^{n-k}(\mathcal{F}^\lambda(C), \mathcal{F}^{\lambda+\delta}(C))$. As we are in particular interested in certain adjointness conditions the case where $\lambda = \frac{1-\delta}{2}$ is particularly important, for then we have that the adjoint of L as written in equation (4.1.1) is simply given as:

$$L^\dagger = \sum_{k=0}^{n} (-1)^k \big(L^{(k)}\big)^\dagger_\varrho (\widehat{w} - \frac{1-\delta}{2}).$$

Now let us suppose that $L^{(k)}$ has the local form $L^{(k)} = |Dz|^\delta \Sigma_{j=0}^{n-k} L^k_j \partial^j$. We can then write the operator as in equation (4.1.1) as:

$$L = t^\delta \sum_{k=0}^{n} \sum_{j=0}^{n-k} L^k_j \Big(\partial + \gamma(\widehat{w} - \frac{1-\delta}{2})\Big)^j (\widehat{w} - \frac{1-\delta}{2})^k,$$

where γ is the connection associated to the volume form[2] ϱ. We have thus reduced the question of the geometric content within the operator L to a combinatorial one, that of finding the expansion on the right hand side of the above equation.

[1]By a curve we shall mean either a real or complex manifold of dimension 1. As there is essentially no difference in the case when we have holomorphic coordinates we can include it here. The main problem is the question as to whether there exists a volume form to apply the methods we shall use. We will be interested in local structure so this shall not be a problem. In the next chapter we shall explore this problem in more detail.

[2]In fact the above expression makes sense even if the connection does not come from a volume form, even locally. In the case of curves the connection is always flat so only the fundamental group determines whether or not the connection is integral. The fact we use is that the connection is always locally integrable.

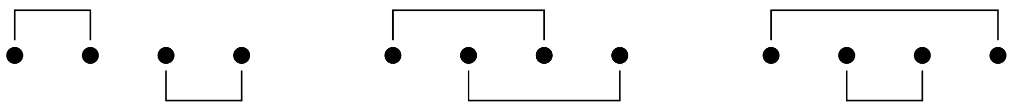
Some combinatorics of partitions.

We shall find that the solution to our problem lies in certain counting functions related with partitions. Let $\mathcal{P}[n]$ denote the set of partitions of n, so that $P \in \mathcal{P}[n]$ is a string of numbers $P = (p_1, \dots, p_k)$, where $p_1 \geq p_2 \dots \geq p_k > 0$ and $p_1 + \dots + p_k = n$. We shall also need to define a function $c : \mathcal{P}[n] \to \mathbb{N}$ which counts the number of ways one can express a partition within n. Let us explain these concepts in more detail with the aid of an example:

Example 4.1.1. Let $n = 4$, we then have that $\mathcal{P}[n] = \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$. We can then calculate the function c as follows:

$$c : \begin{cases} (4) & \mapsto & 1 \\ (3,1) & \mapsto & 4 \\ (2,2) & \mapsto & 3 \\ (2,1,1) & \mapsto & 6 \\ (1,1,1,1) & \mapsto & 1 \end{cases}$$

To be explicit the fact that $c((2,2)) = 3$ is equivalent to the following diagrams:



Using the function c we can express the general formula for the 'gauge' transformation of the derivative, $e^{-f}\partial^n e^f$ - this is the classical Faá di Bruno's formula.

Proposition 4.1.2. *Let G_n be the polynomial in the derivatives of f expressing the following equation: $e^{-f}\partial^n e^f = G_n(f^{(1)}, f^{(2)}, \dots)$. We then have the following expression for G_n:*

$$G_n(f^{(1)}, \dots) = \sum_{P \in \mathcal{P}[n]} c(P) f^{(p_1)} \cdots f^{(p_n)}.$$

We now explain how we use proposition 4.1.2 to express the geometric content contained within a differential operator $L \in \mathrm{DO}^n_\delta(\mathcal{F}(C))$. Our aim is to expand the expression

$$\Big(\partial + \gamma\big(\widehat{w} - \frac{1-\delta}{2}\big)\Big)^j.$$

We have that γ is locally integrable and hence we can find a ϱ such that $\gamma = -\partial \log(\varrho)$. We also can make local sense of the expression $\varrho^{\widehat{w}-\lambda} := \exp(\log(\varrho)(\widehat{w} - \lambda))$. Using this we find that:

$$\big(\partial+\gamma(\widehat{w}-\lambda)\big)^j=\varrho^{\widehat{w}-\lambda}\partial^j\varrho^{\lambda-\widehat{w}}=\sum_{l=0}^{j}\binom{j}{l}G_l\big(-\log(\varrho)(\widehat{w}-\lambda)\big)\partial^{j-l},$$

where G_l denotes the polynomial as in proposition 4.1.2. Putting this together and leaving λ arbitrary we find that a general differential operator $L\in \mathrm{DO}^n_\delta(\mathcal{F}(C))$ has the following expression:

$$L=t^\delta\sum_{k=0}^{n}\sum_{j=0}^{n-k}L^k_j\sum_{l=0}^{j}\binom{j}{l}\sum_{P\in\mathcal{P}[l]}c(P)\gamma^{(p_1)-1}\cdots\gamma^{(p_m)-1}\partial^{j-l}\big(\widehat{w}-\lambda\big)^{i+m}. \tag{4.1.2}$$

Moreover the coefficient of $\partial^p(\widehat{w}-\lambda)^q$ has the following expression:

$$\sum_{k=0}^{n}\sum_{j=0}^{n-k}L^i_j\binom{j}{j-p}\sum_{\substack{P\in\mathcal{P}[j-p]\\|P|=q-i}}c(P)\gamma^{(p_1)-1}\cdots\gamma^{(p_{q-i})-1}, \tag{4.1.3}$$

where by $|P|$ we denote the length of the string representing P. We have thus expressed all differential operators on $\mathcal{F}(C)$ as differential operators on $\mathcal{F}^\lambda(C)$ with some additional geometric structure contained in derivations of connections. On curves a more explicit description is possible. This is due to the fact that we have a natural quantization map on a curve C associated with any connection.

Quantization of symbols on a curve using a connection.

Recall the definition of a quantization map given in definition 3.3.2. We explain how given a volume form on C we have an identification between $\oplus_{m\le n}\mathcal{S}^m(C)\otimes\mathcal{F}^\delta(C)$ and $\mathrm{DO}^n(\mathcal{F}^\lambda(C),\mathcal{F}^{\lambda+\delta}(C))$. To do this take a connection γ then the differential operator

$$D_\gamma=t\big(\partial_z+\gamma\widehat{w}\big), \tag{4.1.4}$$

is well defined on $\mathcal{F}(C)$. This differential operator will define for us our quantization map. Let $L\in\mathrm{DO}^n(\mathcal{F}^\lambda(C),\mathcal{F}^{\lambda+\delta}(C))$ and suppose that $\sigma_n(L)$ is its n^{th} symbol. Then we may consider the differential operator $\sigma_n(L)D^n_\gamma|\mathcal{F}^\lambda(C)$. This operator has the same n^{th} symbol as L and hence we have that $L-\sigma_n(L)D^n_\gamma|\mathcal{F}^\lambda(C)$ is an operator of order $n-1$. We can then proceed by induction to prove the following:

Lemma 4.1.3. *Let γ be a connection on $\mathcal{F}^1(C)$ for a curve C. Then $\forall\delta$ and $\forall\lambda$ there exists a quantization map $\mathcal{S}(C)\otimes\mathcal{F}^\delta(C)\xrightarrow{\sim}DO(\mathcal{F}^\lambda(C),\mathcal{F}^{\lambda+\delta}(C))$.*

Remark 4.1.4. If one had a volume form rather than a connection one could attempt to let δ vary in the above lemma. This is not possible for just a connection as there may exist no volume form generating it see footnote 2 above.

Example 4.1.5. The two real curves, $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z}$, both have a natural affine structure. This affine structure gives rise to the connection which is equal to zero in any chart. In these cases the quantization map is simply QP-quantization.

This is also the case for elliptic curves however here it is in fact slightly stronger. This is because then the canonical bundle $K_C = \mathcal{F}^1_C$ has exactly one global section so there is a unique globally defined volume form connection and this gives a completely canonical connection to use in the above lemma.

Geometric content of differential operators on curves.

We now give a complete classification of the local geometric objects that make up differential operators on curves and their algebra of densities. Firstly we have from lemma 4.1.3 that an arbitrary differential operator $L \in \mathrm{DO}^n(\mathcal{F}^\lambda(C), \mathcal{F}^{\lambda+\delta}(C))$ can be expressed as a string of densities $(a_0, a_1, \ldots, a_n)$ where a_j has weight $\delta - n$ and the identification is given as

$$L = \sum_{i=0}^{n} a_i D^i_\gamma | \mathcal{F}^\lambda(C),$$

where D_γ is the operator in equation (4.1.4). Now note the important fact that if $\lambda = 0$ then D_γ is just the volume form lifting of ∂. This means that if we take a differential operator $L \in \mathrm{DO}(C^\infty(C), \mathcal{F}^\delta(C))$ corresponding to the string $(a_0, \ldots, a_n)$ we have that the volume form extension of L is given by the above equation where we drop the restriction. An immediate corollary of this is the following:

Proposition 4.1.6. *Fix a connection γ and let D_γ be as above. Then there is an SDiff(γ)-equivariant isomorphism between differential operators on $\mathcal{F}(C)$ of weight δ and bi-indexed families of densities, $\{a_{k,l} | k, l \in \mathbb{N}\}$, such that the densities obey the additional conditions that:*

- $\exists N \geq 0$ *and* $a_{k,l} = 0$ *if* $k \geq N$ *or* $l \geq N$.
- *The density* $a_{k,l}$ *has weight* $\delta - k$.

The form of this map is given by:

$$\{a_{k,l}\} \longmapsto \textstyle\sum_{k,l} a_{k,l} D_\gamma^k \widehat{w}^l.$$

We apply this corollary with equation (4.1.3) to give a complete description of the terms in a differential operator on $\widehat{C}$.

Proposition 4.1.7. *Let $L = t^\delta \sum_{i=0}^n \sum_{j=0}^{n-i} L_{i,j} \partial^i \widehat{w}^j$ be a differential of weight δ on the algebra of densities. Then if we fix a connection γ there exists a unique family of densities , $\{a_{k,l}\}$, such that we may express the term $L_{i,j}$ as follows:*

$$L_{i,j} = \sum_{k=0}^{n} \sum_{l=0}^{n-k} a_{k,l} \binom{k}{k-i} \sum_{\substack{P \in \mathcal{P}[k-i] \\ |P| = j-l}} c(P) \gamma^{(p_1)-1} \ldots \gamma^{(p_{j-l})-1}.$$

Example 4.1.8. Let us go through the calculations for a second order operator. We thus need the following list of densities, $\{a_{0,0}, a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, a_{0,2}\}$. The partitions and there counting functions are trivial in both of these cases as well: $\mathcal{P}[1] = \{(1)\}$, $\mathcal{P}[2] = \{(2), (1,1)\}$ and $c|\mathcal{P}[1] \sqcup \mathcal{P}[2] = 1$. We can therefore immediately calculate an arbitrary second order operator as having the form:

$$L = t^\delta \Big(a_{2,0} \partial^2 + \big(2a_{2,0}\gamma + a_{1,1}\big) \partial \widehat{w} + \big(a_{2,0}\gamma^2 + a_{1,1}\gamma + a_{0,2}\big) \widehat{w}^2 +$$

$$a_{1,0}\partial + \big(a_{1,0}\gamma + a_{1,1}\big)\widehat{w} + a_{0,0} \Big).$$

4.1.2 Structure of operators on general manifolds.

We now go through some of the analysis in the previous section for manifolds of arbitrary dimension. As densities contain all geometric data on curves we have a quantization map for a connection on curves, see lemma 4.1.3, but this is not possible for higher dimensions. The first part of the preceding section goes through verbatim which we shall quickly reproduce here: We pick a volume form ϱ, that allows us to take the lifting of operators in $\mathrm{DO}^n(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$ to $\mathrm{DO}^n_\delta(\mathcal{F}(M))$. Then using the vertical Taylor expansion we can write an arbitrary operator $L \in \mathrm{DO}^n_\delta(\mathcal{F}(M))$ as follows, see equation (4.1.1) and (3.1.3):

$$L = \sum_{k=0}^{n} L_\varrho^{(k)} (\widehat{w} - \lambda)^k,$$

where $L^{(k)} \in \mathrm{DO}^k(\mathcal{F}^\lambda(M), \mathcal{F}^{\lambda+\delta}(M))$. If we look at the local form of $L^{(k)}$ then it is of the form $L^{(k)} = |Dx|^\delta \Sigma_{|K|\leq k} L_k^K \partial_K$, and the lift has the form $\partial_i \mapsto \partial_i + \gamma_i(\widehat{w} - \lambda)$. To calculate the expansion we shall need to go through some minor combinatorics as before, however here we shall need to deal with multi-indices. We will find that the constants end up essentially identical to those in the analysis for just partitions:

Combinatorics of multi-indices.

Definition 4.1.9. Let J be a multi-index. A partition of J is a string of sub multi-indices, $\mathcal{K} = (K_1, \ldots, K_r)$, $K_i \subset J$, $K_i \neq \emptyset$, such that $\sqcup_i K_i = J$. We shall denote the set of all partitions of a multi-index J by $\mathcal{P}[J]$.

Recall that in the case the dimension of the manifold is greater than 1 the space of multi-indices only form a partially ordered set rather than a linearly ordered one and thus, without the aid of some unnatural ordering, e.g. lexicographical ordering, there is no natural way to order the elements in a partition of a multi-index. Thus we shall say that two partitions are the same if there exists a permutation of one of the strings so that each part is equal. We similarly define the function $c : \mathcal{P}[J] \to \mathbb{N}$ to be the number of ways of embedding the partition into J.

Example 4.1.10. Let $J = (1, 1, 1, 2)$. We then have the following partitions of J:

$$\mathcal{P}[J] = \{((1,1,1,2)); ((1,1,1),(2)); ((1,1,2),(1)); ((1,1),(1,2)); ((1,1),(1),(2)); \\ ((1,2),(1),(1)); ((1),(1),(1),(2))\}.$$

We therefore see that we have $|\mathcal{P}[J]| = 7$ showing that in general it can be quite hard to determine the structure of such objects if one ignores the numbering of the indices, i.e. $\mathcal{P}[4]$ (we do always know however that $|\mathcal{P}[J]| \geq |\mathcal{P}[|J|]|$). We now compute the function c for these partitions we find that:

$$c : \begin{cases} (1,1,1,2) & \mapsto & 1 \\ ((1,1,1),(2)) & \mapsto & 1 \\ ((1,1,2),1) & \mapsto & 3 \\ ((1,1)(1,2)) & \mapsto & 3 \\ ((1,1),(1),(2)) & \mapsto & 3 \\ ((1,2),(1),(1)) & \mapsto & 3 \\ ((1),(1),(1),(2)) & \mapsto & 1. \end{cases}$$

As we have seen in the above example the size of the set of partitions of a multi-index J can vary wildly from the number of partitions on the set $[|J|]$. However the counting function does have a nice relation with that of the underlying set:

Proposition 4.1.11. *Let c denote the function defined above on arbitrary multi-indices. We then have that for any J:*

$$\sum_{\mathcal{K}\in\mathcal{P}[J]} c(\mathcal{K}) = \sum_{P\in\mathcal{P}[|J|]} c(P),$$

where we consider a set S (or equivalently its cardinal) to be the multi index $(1,1,\ldots,1)$, where there are $|S|$ 1s in the index. In fact we have the stronger result as follows: Given $\mathcal{K}\in\mathcal{P}[J]$ we say that it has the type $|\mathcal{K}|\in\mathcal{P}[|J|]$ where $|\mathcal{K}|$ is the partition defined by identifying all the indices. We then have that:

$$\sum_{\substack{\mathcal{K}\in\mathcal{P}[J]\\ |\mathcal{K}|=P\in\mathcal{P}[|J|]}} c(\mathcal{K}) = c(P).$$

We can now proceed to consider the expansion of $\prod(\partial_i+\gamma_i(\widehat{w}-\lambda))$. We have the following proposition, an exact analogue of proposition 4.1.2 for multi-indices:

Proposition 4.1.12. *Let $G_J(\partial_K f|K\subset J)$ be the polynomial defined by the formula:*

$$e^{-f}\partial_J e^f = S^J G_J(\partial_K f).$$

Then G_J has the following expression:

$$G_J(f) = \sum_{\substack{\mathcal{K}\in\mathcal{P}[J]\\ \mathcal{K}=(K_1,\ldots,K_r)}} c(\mathcal{K})\left(\partial_{K_1}f\right)\cdots\left(\partial_{K_r}f\right).$$

In the Taylor expansion of the operator L we can write the operators $L^{(i)}$ in the form: $L^{(i)} = |Dx|^\delta\Sigma_{|I|\leq|n|-i}S_i^I\partial_I$. The total expression for the operator $L\in\mathrm{DO}_\delta(\mathcal{F}(M))$ may be expanded as follows:

$$L = t^\delta\sum_{i=0}^{n}\sum_{|I|\leq n-i}S_i^I\sum_{J\subset I}\sum_{\substack{\mathcal{K}\in\mathcal{P}[J]\\ \mathcal{K}=(K_1,\ldots,K_r)}}(-1)^r c(\mathcal{K})\partial_{K_1}\log(\varrho)\cdots\partial_{K_r}\log(\varrho)$$
$$\left(\widehat{w}+\frac{\delta-1}{2}\right)^{r+i}\partial_{IJ^{-1}}. \tag{4.1.5}$$

Note the fact that we are contracting our indices with respect to a symmetric tensor S^I and therefore the ordering of the individual terms do not matter. This means that the terms coming from partitions whose strings are the same type, i.e. when we restrict to the diagonal these two partitions become equal, give the same contribution on the far right sum in the above expression. This means we need only pick a single representative and then sum over the associated coefficients which will leave us with a coefficient proportional to $c(|\mathcal{K}|)$. This will become clear in the examples we consider.

Geometric content of second order differential operators.

The theory for second order differential operators is particularly singular and obeys some remarkable properties as we shall see. Let us use the above prescription to write down the structure of a general self adjoint second order operator on the algebra of densities. Consider equation (4.1.5) in the case when $n = 2$. We have that the operator L will be self adjoint if in the expansion we have that $L^{(0)}, L^{(2)} \in \mathrm{DO}^{2\bullet}(\mathcal{F}^{\frac{1-\delta}{2}}(M), \mathcal{F}^{\frac{1+\delta}{2}}(M))$ are self adjoint whilst the first order operator, $L^{(1)}$, is anti-self adjoint. These conditions are simple enough to solve and we find that

$$L^{(0)} = |Dx|^{\delta}\Big(S^{ij}\partial_{ij} + S^{ij}{}_{,j}\partial_i + S\Big),$$

$$L^{(1)} = |Dx|^{\delta}\Big(A^i\partial_{,i} + \frac{1}{2}A^i{}_i\Big), \quad L^{(2)} = |Dx|^{\delta}B.$$

We now substitute these expressions into equation (4.1.5) and we find the general form of a self adjoint second order operator on $\mathcal{F}(M)$ of weight δ has the form:

$$L = t^{\delta}\Bigg(S^{ij}\partial_{ij} + \big(S^{ij}{}_{,j} + \big(2S^{ij}\gamma_j + A^i\big)\,\widehat{w}_{\delta}\big)\,\partial_i +$$

$$S + \left(S^{ij}\gamma_{j,i} + S^{ij}{}_{,j}\gamma_i + \frac{1}{2}A^i{}_{,i}\right)\widehat{w}_{\delta} + \big(S^{ij}\gamma_i\gamma_j + A^i\gamma_i + B\big)\,\widehat{w}_{\delta}^2\Bigg).$$

As we are studying differential operators on an algebra any such operator can be expressed as a direct sum of a normalized operator (an operator that kills 1) and a constant. We find that we can *enforce* the condition of being normalized onto the above operator for any choice of δ. The normalized operator then has the form:

$$\overline{L} = t^{\delta}\Bigg(S^{ij}\partial_{ij} + \big(S^{ij}{}_{,j} + (2S^{ij}\gamma_j + A^i)\widehat{w}_{\delta}\big)\partial_i +$$

$$\left(S^{ij}\gamma_{j,i} + S^{ij}{}_{,j}\gamma_i + \frac{1}{2}A^i{}_{,i}\right)\widehat{w} + \big(S^{ij}\gamma_i\gamma_j + A^i\gamma_i + B\big)\,\widehat{w}(\widehat{w} + \delta - 1)\Bigg). \qquad (4.1.6)$$

This equation is quite remarkable as it shows that the geometric content within a self-adjoint second order operator is only that of three tensors and a connection on *volume forms.* Of course this is to be expected in dimension 1 due to the quantization map, lemma 4.1.3, but this is unexpected in higher dimensions. The following explains the equivariance contained of the equivalent result in [48]:

Proposition 4.1.13. *There is Diff(M)-equivariant covering from the space* $\big(\mathcal{S}^2(M) \oplus\mathcal{S}^1(M) \oplus C^\infty(M)\big) \otimes \mathcal{F}^\delta(M) \oplus Conn(\mathcal{F}^1(M))$ *to self adjoint normalized second order differential operators on* $\mathcal{F}(M)$ *of weight* δ*,* $DO^2_\delta(\mathcal{F}(M))_+$*. The map is given by equation* (4.1.6) *and we shall denote the associated operator by* $\overline{L}(\gamma; S, A, B)$ *for the notation as in the equation.*

Proof. The only thing we need to show is that this map is equivariant with respect to the group of diffeomorphisms of our manifold. Let $\phi \in \mathrm{Diff}(M)$, and consider $\widehat{\phi}^*\overline{L}(\gamma; S, A, B)$, where we use the notation $\widehat{\phi}$ to denote the extension of ϕ to $\widehat{M}$, see equation (2.1.2). Recall that $\widehat{\phi}^*(\widehat{w}) = \widehat{w}$, and that $\widehat{\phi}^*(\partial_i) = \phi^j{}_{,i}\widetilde{\partial}_j + \partial_i(\log J(\phi))\widehat{w}$. As the connection also transforms with this affine part we see that the symbol of $\widehat{\phi}^*\overline{L}(\gamma; S, A, B)$ is equal to the symbol of $\overline{L}(\phi^*\gamma; \phi^*S, \phi^*A, \phi^*B)$. Therefore their difference is a first order operator on $\mathcal{F}(M)$ which is self adjoint $\Rightarrow$ it is a zeroth order operator $\Rightarrow$ it is completely determined by its value on 1 and as we have that $\widehat{\phi}^*\overline{L}(\gamma; S, A, B)(1) = 0$ the two operators must be equal. □

Remark 4.1.14. It is only possible to say that the map is a covering map for their are situations where two different families can be mapped to the same symbol. Essentially this is the case when some of the tensors are degenerate. If however one fixes a connection then the map is a one-one mapping; that is a connection on volume forms in fact provides a partial quantization map for arbitrary dimensions.

Higher order operators.

We now turn to the study of higher order operators and in particular focus on third order operators. We find that there exists no proposition 4.1.13: that is the content of third order operators is inherently more complex than second order operators. To attempt to understand this we shall use quantization and brackets in the next section. For the moment it will suffice to explain where the discrepancy lies. We can go through the same techniques in the last section and we find that a general anti-self adjoint third order operator has the form:

$$L = t^{\delta}\Bigg(S^{ijk}\partial_{ijk} + (\frac{3}{2}S^{ijk}{}_{,k} + (3S^{ijk}\gamma_k + A^{ij})\widehat{w}_{\delta}\partial_{ij} +$$

$$\big(S^i + (3S^{ijk}{}_{,k}\gamma_j + 3S^{ijk}\gamma_{k,j} + A^{ij}{}_{,j})\widehat{w}_{\delta} + (3S^{ijk}\gamma_j\gamma_k + 2A^{ij}\gamma_j + B^i)\widehat{w}_{\delta}^2\big)\partial_i +$$

$$\frac{1}{2}\big(S^i{}_{,i} - \frac{1}{2}S^{ijk}{}_{,ijk}\big) + \big(S^i\gamma_i + \frac{3}{2}S^{ijk}, k\gamma_{j,i} + S^{ijk}\gamma_{k,ji} + A\big)\widehat{w}_{\delta} +$$

$$\big(\frac{3}{2}S^{ijk}{}_{,k}\gamma_j\gamma_i + 3S^{ijk}\gamma_i\gamma_{k,j} + A^{ij}{}_{,j}\gamma_i + A^{ij}\gamma_{j,i} + \frac{1}{2}B^i{}_{,i}\big)\widehat{w}_{\delta}^2 +$$

$$\big(S^{ijk}\gamma_i\gamma_j\gamma_k + A^{ij}\gamma_i\gamma_j + B^i\gamma_i + C\big)\widehat{w}_{\delta}^3\Bigg), \tag{4.1.7}$$

where the expansion of is described in terms of the anti/self adjoint operators:

$$L^{(0)} = |Dx|^{\delta}\Big(S^{ijk}\partial_{ijk} + \frac{3}{2}S^{ijk}{}_{,k}\partial_{ij} + S^i\partial_i + \frac{1}{2}\big(S^i{}_{,i} - \frac{1}{2}S^{ijk}{}_{,ijk}\big)\Big),$$

$$L^{(1)} = |Dx|^{\delta}\Big(A^{ij}\partial_{ij} + A^{ij}{}_{,j}\partial_i + A\Big), \quad |Dx|^{\delta}\Big(B^i\partial_i + \frac{1}{2}B^i{}_{,i}\Big), \quad |Dx|^{\delta}C.$$

We would like to perform a similar analysis for third order operators as we did for second order operators. The first thing we would like to do is normalize the operator. However this is only possible if $\delta \neq 1$. This we shall assume for the moment. We can then express the object A in terms of the rest of the data and we have that:

$$L = t^{\delta}\Bigg(S^{ijk}\partial_{ijk} + (\frac{3}{2}S^{ijk}{}_{,k} + (3S^{ijk}\gamma_k + A^{ij})\widehat{w}_{\delta}\partial_{ij} +$$

$$\big(S^i + (3S^{ijk}{}_{,k}\gamma_j + 3S^{ijk}\gamma_{k,j} + A^{ij}{}_{,j})\widehat{w}_{\delta} + (3S^{ijk}\gamma_j\gamma_k + 2A^{ij}\gamma_j + B^i)\widehat{w}_{\delta}^2\big)\partial_i +$$

$$\frac{1}{1-\delta}\big(S^i{}_{,i} - \frac{1}{2}S^{ijk}{}_{,ijk}\big)\widehat{w} + \big(\frac{3}{2}S^{ijk}{}_{,k}\gamma_j\gamma_i + 3S^{ijk}\gamma_i\gamma_{k,j} + A^{ij}{}_{,j}\gamma_i + A^{ij}\gamma_{j,i} + \frac{1}{2}B^i{}_{,i}\big)\widehat{w}\widehat{w}_{\delta} +$$

$$\big(S^{ijk}\gamma_i\gamma_j\gamma_k + A^{ij}\gamma_i\gamma_j + B^i\gamma_i + C\big)\big(\widehat{w} + \delta - 1\big)\big(\widehat{w} - \frac{1-\delta}{2}\big)\widehat{w}\Bigg). \tag{4.1.8}$$

The only object we have not accounted for is the object S^i appearing in the self adjoint operator $L^{(0)}$. This object is non-tensorial and, moreover, there is no method of expressing it in terms of a connection on volume forms (this can be considered as a corollary of proposition 3.2.12 which states that there is no equivariant quantization of third order symbols). Therefore the geometric content of third order operators lie outside the algebra of densities. We will explore the content in greater detail for $\delta = 0$ below using quantization techniques.

Let us briefly say something about the reason why if the operator L is anti self adjoint and of weight 1 we cannot set $L(1) = 0$ and preserve these properties. If the operator L has domain $C^\infty(M)$ and we consider $\overline{L} = L - L(1)$ then:

$$\overline{L}^\dagger = -L - L(1) = -\overline{L}^\dagger - 2L(1).$$

Therefore normalization is not something that can be enforced but is an integral part of the operator. Let us explore this in slightly more detail. We see from the above expression that $L(1)$ is equal to the following density:

$$L(1) = \frac{|Dx|}{2}\left(S^i{}_{,i} - \frac{1}{2}S^{ijk}{}_{,ijk}\right).$$

This is nothing other than the divergence of the vector valued density, $\frac{1}{2}|Dx|(S^i - \frac{1}{2}S^{ijk}{}_{,jk})$. Therefore for $\delta = 1$ we can express S^i in terms of additional information. Moreover in this case we also have that A is a function. Therefore for $\delta = 1$ the geometric content of the operator is far simpler. We shall express this as follows:

Proposition 4.1.15. *There is a Diff(M)-equivariant covering:*

$$\mathcal{F}^1(M) \otimes \left(\mathcal{S}^3(M) \oplus \mathcal{S}^2(M) \oplus \mathcal{S}^1(M) \oplus \mathcal{S}^1(M)\right) \oplus \mathcal{F}^1(M) \oplus \mathcal{F}^1(M) \oplus Conn(\mathcal{F}^1(M))$$
$$\twoheadrightarrow DO^3_1(\mathcal{F}(M))_-.$$

The explicit form of the map is given as:

$$(S^{ijk}, A^{ij}, X^i, B^i, A, C; \gamma) \mapsto$$
$$t\Bigg(S^{ijk}\partial_{ijk} + (\frac{3}{2}S^{ijk}{}_{,k} + (3S^{ijk}\gamma_k + A^{ij})\widehat{w}\partial_{ij} +$$
$$\big(X^i + \frac{1}{2}S^{ijk}{}_{,jk} + (3S^{ijk}{}_{,k}\gamma_j + 3S^{ijk}\gamma_{k,j} + A^{ij}{}_{,j})\widehat{w} + (3S^{ijk}\gamma_j\gamma_k + 2A^{ij}\gamma_j + B^i)\widehat{w}^2\big)\partial_i +$$
$$\frac{1}{2}X^i{}_{,i} + \big(X^i\gamma_i + \frac{3}{2}S^{ijk}{}_{,k}\gamma_{j,i} + \frac{1}{2}S^{ijk}{}_{,jk}\gamma_i S^{ijk}\gamma_{k,ji} + A\big)\widehat{w} +$$
$$\big(S^{ijk}\gamma_i\gamma_j\gamma_k + A^{ij}\gamma_i\gamma_j + B^i\gamma_i + C\big)\widehat{w}^3_\delta\Bigg).$$

4.2 Long brackets and the Khudaverdian-Voronov groupoid.

We saw in the last section that it was possible to describe differential operators on the algebra of densities in terms of operators on the base manifold, a connection on densities, and some combinatorial data. Moreover for 1 dimensional manifolds and second order operators we could give an even more explicit description. In this section we introduce further methods of determining the structure of these operators as well as trying to explain the vast discrepancy between second and third order operators. In fact as we shall see the case for second order operators is in fact a universal property whilst no such thing exists for third order operators and no additional methods on the algebra of densities can be used to remedy the situation. We firstly recall the notion of derived brackets and generating operators. In the second half we introduce the Khudaverdian-Voronov groupoid to study further properties of differential operators, see [47, 48]. We shall see that it is impossible for the construction of this groupoid to go over verbatim to higher order operators. However interpreting the groupoid in terms of quantizations we can use Bordemann quantization to give a generalisation to higher order operators.

4.2.1 Long brackets and generating operators.

We now explain the relation between operators and symmetric brackets. We reproduce the results contained within [48] concerning second order operators on abstract algebras. The question of whether there exists an extension to higher order operators is then shown to be negative.

Brackets and operators.

Let L be a differential operator on an algebra A. We can then define a family of brackets, $(-,\cdots,-)_{L,n}$, associated with L as follows:

$$(a_1,\ldots,a_n)_{L,n} := [a_1,[a_2,\cdots,[a_n,L]\cdots]](1).$$

These brackets are necessarily symmetric and if L is of order n then we have that $(-,\cdots,-)_{L,n}$ is a derivation in each of its variables.

Example 4.2.1. One of the most classical examples is when one has a Gerstenhaber algebra. Then a generating operator is also known as a Batalin-Vilkovisky operator, see [4, 6, 43, 76]

Lemma 4.2.2. *The map* $L \mapsto \{(-,\dots,-)_{L,n} | n \in \mathbb{N}\}$ *is a 1-1 mapping of differential operators into Hochschild cochains of the algebra* A.

Proof. Essentially what the lemma is saying is that if two differential operators, L and L', are such that all there associated brackets are equal then they are equal. This is very simple for we have that $()_{L,0} = L(1)$, and $(a)_{L,1} = L(a) - aL(1) = (L - L(1))(a)$. □

It is important to note that the above result does not hold if we remove the zeroth and first bracket. We will see an example of this below.

Generating operators.

We can now ask whether or not a bracket is the derived bracket of some differential operator. If the bracket is symmetric then we can always find such an operator however there is no guarantee that the operator will be unique. We will focus on the case when the derived bracket is the symbol of the operator:

Definition 4.2.3. Let $(-,\dots,-) \in \mathrm{Hom}(S^n(A), A)$ be a derivation in each of its arguments. Then we say that a differential operator generates $(-,\dots,-)$ if the bracket is equal to the operators symbol.

We can now produce the following results as in [48]:

Theorem 4.2.4. *Let* $(A; \langle -,- \rangle)$ *be an algebra with an inner product such that*

$$\langle ab, c\rangle = \langle a, bc\rangle \quad \forall a, b, c \in A.$$

Then given an symbol of arity 2 there is a unique normalized differential operator generating this operator that is self adjoint.

Proof. By definition there exists a differential operator generating this bracket, call it L. Then using the property of the inner product we have that $L^\dagger$ is of order 2 and has the same symbol as L. Therefore if we let $\Delta = \frac{1}{2}(L + L^\dagger) - \frac{1}{2}(L(1) - L^\dagger(1))$ we have an operator that satisfies the assumptions and generates the bracket. It is a simple exercise to check that it is unique. □

Remark 4.2.5. Of course we do not need the inner product to be defined on the whole algebra. That is we just require a subspace $B \subset A$ such that $\langle B, - \rangle : A \to \mathbb{K}$ is well defined and non-degenerate. One should bear in mind compactly supported functions on a manifold.

Corollary 4.2.6. *Let N be one of the following manifolds:*

1. $\widehat{M}$,
2. T^*M,
3. ΠTM,

for some manifold M. Then any 2-arity bracket on N has a canonical generating operator.

Consider the case of theorem 4.2.4 in the case that we have a smooth manifold M with a volume form ϱ. Let the bracket be given by the following data $S^{ij} := (x^i, x^j)$. Then the canonical operator generating this bracket has the following form:

$$\Delta := \frac{1}{2}\Big(S^{ij}\partial_{ij} + \big(S^{ij}{}_{,j} + S^{ij}\gamma_j\big)\partial_i\Big),$$

where $\gamma_i = -\partial_i \log(\varrho)$ is the connection induced from the volume form.

By definition a *long bracket* is simply a bracket on the algebra of densities. We thus see that an arbitrary long bracket of arity 2 has a natural generating operator. Recall that the volume form has the form $t^{-2}|D(x,t)|$, and we thus have that the connection is given as:

$$\gamma = \frac{2dt}{t}.$$

Now take a long bracket of weight δ so that it is locally given by the following data:

$$t^\delta S^{ij}(x) := (x^i, x^j), \quad t^\delta \Gamma^i := (x^i, t)t^{-1}, \quad t^\delta \theta := (t,t)t^{-2}.$$

Then using the above we find that the canonical operator has the following form:

$$\Delta = \frac{t^\delta}{2}\Big(S^{ij}\partial_{ij} + \big(S^{ij}{}_{,j} + \Gamma^i(2\widehat{w} + \delta - 1)\big)\partial_i + \big(\Gamma^i{}_{,i} + \theta(\widehat{w} + \delta - 1)\big)\widehat{w}\Big).$$

Generating operators for higher order brackets.

We now turn to the question of whether or not there is any analogous result for higher order operators as there is for second order operators. We shall focus on the algebra of densities - of course the abstract situation fares only worse. Let us take a bracket of arity 3, $(-,-,-) : \mathcal{F}(M)^{\otimes 3} \to \mathcal{F}(M)$, which for simplicity we shall assume has weight 0. The objects comprising this bracket are the following:

$$S^{ijk} := \frac{1}{3!}(x^i, x^j, x^k), \quad \theta^{ij} := \frac{1}{2t}(x^i, x^j, t),$$
$$\theta^i = \frac{1}{t^2}(x^i, t, t), \quad \theta = \frac{1}{t^3}(t,t,t).$$

If we look at the equation determining the structure of a self adjoint third order operator, equation (4.1.8), we then have that an anti-self adjoint normalized operator generating this bracket must have the following form:

$$L = S^{ijk}\partial_{ijk} + \Big(\frac{3}{2}S^{ijk}{}_{,k} + \theta^{ij}(\widehat{w} - \frac{1}{2})\Big)\partial_{ij} + (S^i + \theta^{ij}{}_{,j}(\widehat{w} - \frac{1}{2}) + \theta^i(\widehat{w} - \frac{1}{2})^2)\partial_i +$$
$$(S^i{}_{,i} - \frac{1}{2}S^{ijk}{}_{,ijk})\widehat{w} + \frac{1}{2}\theta^i{}_{,i}\widehat{w}(\widehat{w} - \frac{1}{2}) + \theta(\widehat{w} - 1)(\widehat{w} - \frac{1}{2})\widehat{w}. \tag{4.2.1}$$

We have the term S^i which is non tensorial and cannot be removed by any additional constraint - it relies on richer geometrical data than simply a connection on densities to determine its form, see below and the end of the previous section. We have no chance therefore to find a unique such operator for arity 3 brackets. However we have shown that any two such generating operators differ by a vector field as S^i transforms with an affine representation. For higher order operators the situation cannot be remedied. As the order of the bracket increases the lack of determinacy of the generating operator is equivalent to the operators in $\mathrm{DO}^{n-2}(\mathcal{F}(M))$ that have parity $(-1)^n$ under the adjoint operation.

One way to attempt to circumnavigate this problem is to consider a larger family of brackets, that is if we enforce conditions on both the brackets $(-,-)_{L,2}$ and $(-)_{L,1}$. Firstly the bracket $(-,-)_{L,2}$ is completely determined by the data within the original bracket as the object S^i only appears before a first order operator. Moreover as the differential operator is normalized we have that $L = (-)_{L,1}$ and therefore we have no method of uniquely recovering the third order operator from the brackets (except trivially).

Remark 4.2.7. By considering a larger class of brackets the situation for higher order operators does alter. We shall not prove this here but if one were to consider the family of long brackets, $\{(-,\dots,-)_k : k = 2,\dots,2k,\dots,2n\}$ then these do uniquely determine a $2n^{\text{th}}$ normalized self adjoint operator on the algebra of densities iff they can be generated. The situation for odd order operators is equivalent to third order operators. Determining which family of brackets can be generated by a differential operator is however a non-trivial task and the methods we outline below can be used to find the structure of such brackets. From now on focus solely on operators generating the symbol as generation is trivially satisfied for the algebras we shall consider.

4.2.2 The Khudaverdian-Voronov groupoid.

We now introduce the Khudaverdian-Voronov groupoid. It essentially determines how strongly the self adjoint operator defined by equation (4.1.6) depends on the connection when restricted to densities of weight $\frac{1-\delta}{2}$. One can consider this as determining how strongly a partial quantization depends on the volume form. We will exploit this interpretation when we generalise the groupoid to higher order operators as there is no direct analogue.

Take a tensor $S \in \mathcal{S}^2(M) \otimes \mathcal{F}^\delta(M)$. We have a map $\mathrm{Conn}(\mathcal{F}^1(M)) \to \mathrm{DO}^2_\delta(\mathcal{F}(M))_+$ given by proposition 4.1.13. We denote the associated operator as $\overline{L}(\gamma; S)$.

Definition 4.2.8. Fix a tensor S as above. The Khudaverdian-Voronov groupoid associated to this symbol, denoted $\mathcal{C}(S)$, is the following category:

- $\mathrm{Ob}(\mathcal{C}(S)) = \mathrm{Conn}(\mathcal{F}^1(M))$.
- $\mathrm{Hom}(\gamma,\gamma') = \begin{cases} \{*\} & \text{if } \overline{L}(\gamma; S)|\mathcal{F}^{\frac{1-\delta}{2}}(M) = \overline{L}(\gamma; S)|\mathcal{F}^{\frac{1-\delta}{2}}(M), \\ \emptyset & \text{otherwise.} \end{cases}$

Another way to consider the groupoid is that to each connection and tensor (γ, S) we can define a long bracket. Take the associated differential operator using theorem 4.2.4 and restrict it to the singular point, i.e. $\frac{1-\delta}{2}$. We have an arrow between two connections if this restricted operator is equal.

The operator $\overline{L}(\gamma; S)$ is the quantization of the symbol of the long bracket corresponding to the data, $S^{ij}, \gamma^i := S^{ij}\gamma_j, S^{ij}\gamma_i\gamma_j = \gamma^i\gamma_i$. It has the following form:

$$\overline{L}(\gamma; S) = t^\delta \big(S^{ij}\partial_{ij} + (S^{ij}{}_{,j} + S^{ij}\gamma_j\widehat{w}_\delta)\partial_i + \frac{1}{2}\gamma^i{}_{,i}\widehat{w} + S^{ij}\gamma_i\gamma_j\widehat{w}(\widehat{w} + \delta - 1)\big).$$

Restricting this operator to the singular point, $\widehat{w} = \frac{1-\delta}{2}$, the operator takes form:

$$\overline{L}(\gamma; S)|\mathcal{F}^{\frac{1-\delta}{2}}(M) = |Dx|^{\delta}\Big(S^{ij}\partial_{ij} + S^{ij}{}_{,j}\partial_i + \frac{1-\delta}{2}\big(\gamma^i{}_{,i} - \frac{1-\delta}{2}\gamma^i\gamma_i\big)\Big).$$

Take another connection γ' and write $\gamma' = \gamma + \theta$ for some 1-form θ, and let $X_\theta := S^{ij}\theta_j$ be the associated vector field when we raise the indices using the tensor. Then the condition that there is an arrow between these two connections is equivalent to:

$$\frac{1-\delta}{2}\left(\partial_i X^i_\theta - \frac{1-\delta}{2}X^i_\theta\gamma_i\right) = \frac{(1-\delta)^2}{4}X^i_\theta\theta_i.$$

Example 4.2.9. Batalin-Vilkovisky groupoid. Consider Khudaverdian's operator, see [4, 6, 43, 44]:

$$\Delta = (-1)^{\widetilde{i}}\frac{\partial^2}{\partial x^i \partial x^*_i} : \mathcal{F}^{\frac{1}{2}}(\Pi T^* M) \to \mathcal{F}^{\frac{1}{2}}(\Pi T^* M).$$

Now take a connection generated by a half-density, i.e. $\gamma = -\frac{1}{2}\varrho^{-1}d\varrho$ (we consider a half-density here for the obvious reason). We may write $\varrho = e^{iS(x,x^*)}\varrho_0(x)$, where $\varrho_0(x)$ is a volume form on M. This volume form then quantizes the Schouten-Nijenhuis bracket on $\Pi T^* M$. We find that the associated operator acting on half densities has the form:

$$(-1)^{\widetilde{i}}\frac{\partial}{\partial x^i \partial x^*_i} + \frac{1}{2}\Big(i\Delta_{\varrho_0}(S) + (S,S)\Big).$$

Thus the component of the groupoid that passes through Khudaverdian's operator is equivalent to the Batalin-Vilkovisky equation. This was used by Hovhannes Khudaverdian to describe the operator for P-manifolds, see [43, 44, 45].

Example 4.2.10. Riemannian metric. Let g^{ij} be an invertible symbol, so it is equivalent to a metric on the manifold. We can then study the groupoid for this particular operator. To do this we note that as before we have a canonical component of the groupoid to consider. This is because $\sqrt{g} := \sqrt{|\det(g)|}$, is a nowhere vanishing density of weight 1 so generates a connection Γ defined as $\Gamma_i = -\sqrt{g}^{-1}\partial_i\sqrt{g}$. We then have that the operator generated by this connection is equal to:

$$\Delta_g = \partial_i g^{ij}\partial_j + \frac{1}{2}\Big(\partial_i(g^{ij}\Gamma_j) - \frac{1}{2}g^{ij}\Gamma_i\Gamma_j\Big).$$

Now take a vector field X and suppose that $\Gamma + g(X, -)$ generates the same operator. The condition on X then becomes:

$$\mathrm{div}_\Gamma(X) = \frac{1}{2}\|X\|^2.$$

4.2.3 Bordemann quantization and the associated groupoid.

The above definition of the K-V groupoid is useful because of its lucidity and its links with other parts of geometry. However we cannot generalise it due to the results on the structure of third order operators, see the end of the previous section. To do this we shall consider a vaster category, namely all connections on a manifold, and then consider the groupoid associated to a particular Bordemann quantization.

Bordemann quantization.

We now recall the notion of Bordemann quantization. We interpret the K-V groupoid as determining how dependant a (partial) quantization is on additional data so it is a natural place to start.

Definition 4.2.11. A natural quantization map is an operator from the space of symmetric affine connections on M (that is connections on TM) and symbols to differential operators, $\mathcal{Q}_M : \mathrm{Conn}_M \times \mathcal{S}^\bullet_M \to \mathrm{DO}^\bullet_M$, such that $\mathcal{Q}(\Gamma, -)$ is a quantization map as in definition 3.3.2 (of type $(0,0)$) and $\mathcal{Q}$ is natural with respect to local diffeomorphisms.

Instead of working on the abstract theory behind the quantization maps of Bordemann we develop the necessary tools that will allow us to explicitly compute second and third order partial quantizations.

We are interested in the symmetric properties of connections. Therefore the spaces we shall be concerned with are TM, T^*M and $TM \times_M T^*M$ as we shall see below. The following lemma contains the main tools that we shall need:

Lemma 4.2.12. *Let $\Gamma = [\Gamma^i_{jk}]$ be a connection on M. Then we have the following well defined objects:*

1. *$\nabla \in \mathfrak{X}(\Pi TM)$, a vector field on TM of fibre weight +1. The explicit form of ∇ is given as follows:*
$$\nabla = \dot{x}^i \frac{\partial}{\partial x^i} - \dot{x}^j \dot{x}^k \Gamma^i_{jk} \frac{\partial}{\partial \dot{x}^k}. \tag{4.2.2}$$

2. *Δ_Γ a second order operator on T^*M known as the divergence operator of Γ. It has fibre weight -1 and the explicit form:*
$$\Delta_\Gamma = \frac{\partial^2}{\partial x^i \partial p_i} + p_i \Gamma^i_{jk} \frac{\partial^2}{\partial p_j \partial p_k} + \Gamma_i \frac{\partial}{\partial p_i}. \tag{4.2.3}$$

The final object we need is the operator on $TM \times_M T^*M$ that corresponds to the fact that these two bundles are dual to one another. We shall denote this operator by K, it has the following form:

$$K = \frac{\partial^2}{\partial p_i \partial \dot{x}^i}. \tag{4.2.4}$$

Definition 4.2.13. By a partial Bordemann quantization we shall mean a quantization that is defined only on some linear subspace of symbols. For us this will usually be symbols of a given order.

Theorem 4.2.14. *Let s be an arbitrary smooth function on $\mathbb{R}$ such that $s(0) = 0$. Then we can define a partial Bordemann quantization $\mathcal{Q}_s(-,-) : Conn(M) \times \mathcal{S}^k(M) \to DO^k(M)$. Moreover $\mathcal{Q}_s$ depends only the k^{th} jet of s (so it is sufficient to take a polynomial). The explicit form of this quantization is given by the formula:*

$$\mathcal{Q}_s(\Gamma, S)(f) := i^* \exp(K) \Big((e^{s(\Delta_\Gamma)} S) \, (e^{\nabla} f) \Big), \tag{4.2.5}$$

where $i : M \to TM \times_M T^*M$ denotes the zero section.

Proof. The above map is clearly well defined for all symbols polynomial in the fibre coordinates. Moreover by working in a coordinate system we see it is a map from $\mathcal{S}^k$ to DO^k. Therefore we just need to check that the principal symbol of the associated operator is equal to the symbol we began with and this follows by a direct calculation. □

Remark 4.2.15. We have defined the above theoremfor an arbitrary function as it may be possible to generalise the above formula to hold more generally for pseudo-differential operators when higher order jets will be significant. We shall not explore this avenue.

To define a full quantization we can take a family of functions, $\{f_k : k \geq 1\}$, that send zero to zero. Then apply the above theorem at order k with the function f_k. Let us close this section with the examples of the quantizations at order 2 and 3 which we shall use below.

Example 4.2.16. In this example we shall quantize second order symbols (see also the formulae in [14]). The most general polynomial of the type outline above can be written as $P(x) = \frac{a}{2}x + \frac{b}{2}x^2$. Performing the necessary calculations in this case we see that we have a 2-parametric family of quantisation defined as ($S = S^{ij}p_ip_j$):

$$\mathcal{Q}_{(a,b)}(\Gamma, S) = S^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \big(aS^{ij}{}_{,j} + (a-1)S^{jk}\Gamma^i_{jk} + aS^{ij}\Gamma_j\big)\frac{\partial}{\partial x^j}$$
$$b\Big(S^{ij}{}_{,ij} + S^{ij}{}_{,k}(\Gamma^k_{ij} + \delta^i_k\Gamma_j + \delta^j_k\Gamma_i) + S^{ij}(\Gamma^k_{ij,k} + \Gamma^k_{ij}\Gamma_k + \Gamma_{j,i} + \Gamma_i\Gamma_j)\Big).$$

Note that the point $(a, b) = (1, 0)$ is distinguished as it only relies on the trace of Γ. We will see that this has relation with the Khudaverdian-Voronov groupoid in the next section.

Example 4.2.17. For third order operators the calculations become more involved. We have three constants to consider, $P = \frac{a}{3}x + \frac{b}{6}x^2 + \frac{c}{6}x^3$. The quantization map associated to these constants has the following form:

$$\mathcal{Q}_{(a,b,c)}(\Gamma, S) = S^{ijk}\partial_{ijk} + \Big(aS^{ijk}{}_{,k} + aS^{ijk}\Gamma_k + (a - \frac{3}{2})\big(S^{ikl}\Gamma^j_{kl} + S^{jkl}\Gamma^i_{kl}\big)\Big)\partial_{ij} +$$
$$\Bigg(bS^{ijk}{}_{,jk} + bS^{ijk}{}_{,l}\big(\Gamma^l_{jk} + \delta^l_k\Gamma_j + \delta^l_j\Gamma_k\big) + (2b - a)S^{jkl}{}_{,l}\Gamma^i_{jk} +$$
$$bS^{ijk}\big(\Gamma^l_{jk,l} + \Gamma^l_{jk}\Gamma_l + \Gamma_{k,j} + \Gamma_k\Gamma_j\big) + S^{jkl}\big((b-1)\Gamma^i_{jk,l} + (2b-2a+2)\Gamma^m_{jk}\Gamma^i_{ml} + (2b-a)\Gamma^i_{jk}\Gamma_l\big)\Bigg)\partial_i$$
$$+c\Bigg(S^{ijk}{}_{,ijk} + S^{ijk}{}_{,kl}\big(2\Gamma^l_{ij} + 3\delta^l_j\Gamma_i\big) +$$
$$S^{ijk}{}_{,l}\big(2\Gamma^l_{jk,i} + 2\Gamma^m_{ij}\Gamma^l_{mk} + 3\Gamma^l_{ij}\Gamma_k + 3\delta^l_k\big(\Gamma^m_{ij,m} + \Gamma^m_{ij}\Gamma_m + \Gamma_{i,j} + \Gamma_i\Gamma_j\big)\big) +$$
$$S^{ijk}\big(2\Gamma^l_{ij,kl} + \Gamma_{i,jk} + 2\Gamma^l_{ij,k}\Gamma_l + 3\Gamma^l_{ij,l}\Gamma_k + 2\Gamma^l_{ij}\Gamma_{l,k} + 3\Gamma_i\Gamma_{j,k} + 2\Gamma^m_{ij,l}\Gamma^l_{mk} + 2\Gamma^m_{ij}\Gamma^l_{mk,l} +$$
$$3\Gamma^l_{jk}\Gamma_l\Gamma_i + 2\Gamma^m_{ij}\Gamma^l_{mk}\Gamma_l + \Gamma_i\Gamma_j\Gamma_k\big)\Bigg).$$

One can see the complexity is far greater than for second order operators. We can still see some distinguished points however, for example when $(a, b, c) = (\frac{3}{2}, \frac{3}{4}, 0)$ from which we get the following differential operator:

$$\mathcal{Q}_{\left(\frac{3}{2},\frac{3}{4},0\right)}(\Gamma, S) = S^{ijk}\partial_{ijk} + \frac{3}{2}\Big(S^{ijk}{}_{,k} + S^{ijk}\Gamma_k\Big)\partial_{ij} +$$
$$\frac{3}{4}\Bigg(S^{ijk}{}_{,jk} + S^{ijk}{}_{,l}\big(\Gamma^l_{jk} + \delta^l_k\Gamma_j + \delta^l_j\Gamma_k\big) +$$
$$S^{ijk}\big(\Gamma^l_{jk,l} + \Gamma^l_{jk}\Gamma_l + \Gamma_{k,j} + \Gamma_k\Gamma_j\big) + \frac{1}{3}S^{jkl}\big(\Gamma^i_{jk,l} - 2\Gamma^m_{jk}\Gamma^i_{ml}\big)\Bigg).$$

Remark 4.2.18. Natural quantization maps as defined in theorem 4.2.14 have been used to prove generalisations of the quantization map defined by P.B.A. Lecomte and V.Y. Ovsienko in [59]. In particular M. Bordemann found a unique natural quantization map that is projectively invariant, [13], and T. Leuther and F. Radoux generalised this to certain supermanifolds, [62].

The groupoid via quantization.

We now recover the standard groupoid using quantization methods. One caveat should be mentioned that the groupoids will be defined on connections on the tangent bundle, $\mathrm{Conn}(M)$, whilst the K-V groupoid is on connections on densities, $\mathrm{Conn}(\mathcal{F}^1(M))$. We have a map of sets $-\mathrm{Tr} : \mathrm{Conn}(M) \to \mathrm{Conn}(\mathcal{F}^1(M))$, and this map induces an equivalence of groupoids for the distinguished Bordemann quantization, see example 4.2.16. Firstly we define the generalised groupoid and then show that it has the required property. Recall that we have associated to a polynomial of order k, P, a quantization of the space of symbols of order k:

$$\mathcal{Q}_P(-,-) : \mathrm{Conn}(M) \times \mathcal{S}^k(M) \rightarrowtail \mathrm{DO}^k(M).$$

This quantization map also allows to define the map,

$$\widehat{\mathcal{Q}}_P(-,-) : \mathrm{Conn}(M) \times \mathcal{S}^k(M) \rightarrowtail \mathrm{DO}^k(\widehat{M}),$$

by taking the $-\mathrm{Tr}(\Gamma)$ pencil. We denote by $\mathcal{Q}_P(\Gamma, S; \lambda)$ this pencil on densities of weight λ. Then for $(\Gamma, S) \in \mathrm{Conn}(M) \times \mathcal{S}^k(M)$ we define the following operator:

$$L_P(\Gamma, S) := \frac{1}{2}\Big(\mathcal{Q}_P(\Gamma, S; \frac{1}{2}) + (-1)^k \mathcal{Q}_P(\Gamma, S; \frac{1}{2})^\dagger\Big).$$

Definition 4.2.19. Fix a symbol $S \in \mathcal{S}^k(M)$ and a polynomial P of order k. The Khudaverdian-Voronov groupoid of type (P, S) is the category, denoted $\mathcal{C}_P(S)$, with the following structure:

- $\mathrm{Ob}(\mathcal{C}_P(S)) = \mathrm{Conn}(M)$.
- $\mathrm{Hom}(\Gamma_1, \Gamma_2) = \begin{cases} \{*\} & \text{if } L_P(\Gamma_1, S) = L_P(\Gamma_2, S), \\ \emptyset & \text{otherwise.} \end{cases}$

The operator $L_P(\Gamma, S)$ is the natural differential operator that should be used in the quantization of the symbol S given the data Γ and P. Thus this generalised version is identical to the original groupoid in the sense that it measures how much a quantization relies upon the additional data required. One can trivially extend the definition to non homogeneous symbols.

We now need to show that this method does generalise the original groupoid in definition 4.2.8. We prove that for the distinguished quantization the negative trace induces an equivalence of categories. Recall that second order Bordemann quantizations, of the type we are using, are given by a two parametric family:

$$\mathcal{Q}_{(a,b)}(\Gamma, S) = S^{ij}\partial_{ij} + \Big(aS^{ij}{}_{,j} + aS^{ij}\Gamma_j + (a-1)S^{jk}\Gamma^i_{jk}\Big)\partial_i +$$

$$b\Big(S^{ij}{}_{,ij} + S^{ij}{}_{,k}(\Gamma^k_{ij} + \delta^i_k\Gamma_j + \delta^j_k\Gamma_i) + S^{ij}(\Gamma^k_{ij,k} + \Gamma^k_{ij}\Gamma_k + \Gamma_{j,i} + \Gamma_i\Gamma_j)\Big).$$

Take the $-\Gamma_i$ pencil of this operator and restricting to half densities we get the operator:

$$\mathcal{Q}_{(a,b)}(\Gamma, S; \frac{1}{2}) = S^{ij}\partial_{ij} + \Big(aS^{ij}{}_{,j} + (a-1)S^{ij}\Gamma_j + (a-1)S^{jk}\Gamma^i_{jk}\Big)\partial_i +$$

$$bS^{ij}{}_{,ij} + S^{ij}{}_{,k}\big(b\Gamma^k_{ij} + (2b - \frac{a}{2})\delta^k_j\Gamma_i\big) +$$

$$+S^{ij}\big(b\Gamma^k_{ij,k} + (b - \frac{a-1}{2})\Gamma^k_{ij}\Gamma_k + (b - \frac{1}{2})\Gamma_{j,i} + (b - \frac{a}{2} + \frac{1}{4})\Gamma_i\Gamma_j\big).$$

Our final goal is to take the self adjoint part of this operator. We find that:

$$L_{(a,b)}(\Gamma, S) = S^{ij}\partial_{ij} + S^{ij}{}_{,j}\partial_i +$$

$$\frac{1-a+2b}{2}S^{ij}{}_{,ij} + S^{ij}{}_{,k}\big(\frac{1-a+2b}{2}\Gamma^k_{ij} + \frac{1-2a+4b}{2}\delta^k_j\Gamma_i\big) +$$

$$S^{ij}\big(\frac{1-a+2b}{2}(\Gamma^k_{ij,k} + \Gamma^k_{ij}\Gamma_k) + \frac{2b-a}{2}\Gamma_{j,i} + \frac{1-2a+4b}{4}\Gamma_i\Gamma_j\big).$$

Form this we can read off the general groupoid of type $(a, b; S)$. One sees from the above equation the distinguished point has now become a distinguished line, $a-2b = 1$. From the above expression we have proved the following result:

Lemma 4.2.20. *Consider the generalised K-V groupoid of type* $(1 + 2b, b; S^{ij}p_ip_j)$. *Then the groupoid is independent of* b *and* $-Tr : Conn(M) \to Conn(\mathcal{F}^1(M))$ *induces an equivalence of groupoids.*

It is simpler to find the structure of this groupoid if we write $2c = 1 - a + 2b$, so the distinguished point now corresponds to $c = 0$. We take a connection, Γ_1, and assume that we have an arrow $\Gamma_1 \to \Gamma_2$. Then writing $\Gamma_2 = \Gamma_1 + \Omega$ we find that the differential equation for Ω is the following:

$$\big(c\delta^l_k S^{ij} + (c - \frac{1}{2})S^{il}\delta^j_k\big)\Omega^k_{ij,l} + c\big(S^{ij}{}_{,k} + S^{ij}\Gamma_k\big)\Omega^k_{ij} +$$

$$+\big((2c - \frac{1}{2})\big(S^{ij}{}_{,j} + S^{ij}\Gamma_j\big) + cS^{jk}\Gamma^i_{jk}\big)\Omega^l_{il} + S^{ij}\big(c\Omega^k_{ij}\Omega^l_{kl} + (c - \frac{1}{4})\Omega^k_{ik}\Omega^l_{jl}\big).$$

Example 4.2.21. We now solve the groupoid for a particular choice of c and type of variation. If we let $c = \frac{1}{4}$ then the groupoid takes the following form:

$$\partial_k\big(S^{ij}\Omega^k_{ij}\big) - S^{ij}\Omega^k_{ik,j} + S^{ij}\Omega^k_{ij}\big(\Gamma_k + \Omega^l_{il}\big) + S^{jk}\Gamma^i_{jk}\Omega^l_{il} = 0.$$

We shall write $\Omega^k = S^{ij}\Omega^k_{ij}$ and $\omega_i = \Omega^j_{ij}$. The variation Ω shall be called traceless if $\omega = 0$. From above we see that the traceless variations of this groupoid have a particularly simple expression:

$$\Omega^k{}_{,k} + \Omega^k\Gamma_k = 0,$$

which is a linear differential equation. Thus the non-linearity, which appears in the Khudaverdian-Voronov groupoid, is purely due to transformations that alter the induced connection on volume forms.

The groupoid for third order operators.

We now calculate the form of the groupoid for third order operators. The most general polynomial we could choose for such a quantization is an arbitrary third order polynomial with constant term 1 however the coefficient of x^3, i.e. the coefficient of the constant after the quantization, will be eliminated after we take the anti-self adjoint operator. Therefore it suffices to only consider a second order polynomial and we have the following 2 parametric family:

$$\mathcal{Q}_{(a,b)}(\Gamma, S) = S^{ijk}\partial_{ijk} + \Big(aS^{ijk}{}_{,k} + aS^{ijk}\Gamma_k + (a - \frac{3}{2})\big(S^{ikl}\Gamma^j_{kl} + S^{jkl}\Gamma^i_{kl}\big)\Big)\partial_{ij} +$$

$$\Big(bS^{ijk}{}_{,jk} + bS^{ijk}{}_{,l}\big(\Gamma^l_{jk} + \delta^l_k\Gamma_j + \delta^l_j\Gamma_k\big) + (2b - a)S^{jkl}{}_{,l}\Gamma^i_{jk} +$$

$$bS^{ijk}\big(\Gamma^l_{jk,l}+\Gamma^l_{jk}\Gamma_l+\Gamma_{k,j}+\Gamma_k\Gamma_j\big)+S^{jkl}\big((b-1)\Gamma^i_{jk,l}+(2b-2a+2)\Gamma^m_{jk}\Gamma^i_{ml}+(2b-a)\Gamma^i_{jk}\Gamma_l\big)\Bigg)\partial_i.$$

Instead of going through the calculations in detail we shall simply reproduce the operator:

$$L_{(a,b)}(\Gamma,S)=S^{ijk}\partial_{ijk}+\frac{3}{2}S^{ijk}{}_{,k}\partial_{ij}+$$

$$\Bigg((b-a+\frac{3}{2})S^{ijk}{}_{,jk}+S^{ijk}{}_{,l}\big((b-a+\frac{3}{2})\Gamma^l_{jk}+(b-a+\frac{3}{4})(\delta^l_j\Gamma_k+\delta^l_k\Gamma_j)\big)+$$

$$(2b-2a+\frac{3}{2})S^{jkl}{}_{,l}\Gamma^i_{jk}+S^{ijk}\big((b-a+\frac{3}{2})(\Gamma^l_{jk,l}+\Gamma^l_{jk}\Gamma_l)+(b-a)\Gamma_{j,k}+(b-a+\frac{3}{4})\Gamma_j\Gamma_k\big)+$$

$$S^{jkl}\big((b-a+\frac{3}{2})\Gamma^i_{jk,l}+(2b-2a+2)\Gamma^m_{jk}\Gamma^i_{ml}+(2b-2a+\frac{3}{2})\Gamma^i_{jk}\Gamma_l\big)\Bigg).$$

We see from the above that this two parametric family is in fact a 1 parametric family depending on $c=a-b$. One can see though that we will never be able to get the groupoid to depend solely on $-\mathrm{Tr}(\Gamma)$ and so for third order operators the groupoid is distinctly different for that for second order operators in the sense that there is no natural equivalence of groupoids. This emphasises the work in the first section that showed there is a vast distinction between second order and third order groupoids.

Remark 4.2.22. We can see directly from the structure of the groupoid for third order operators that it is vastly more complex than that for second order operators. Moreover there are no points that appear to completely distinguished. These results emphasise the remarkable interaction of second order operators with inner product spaces that fails for higher order operators.

Chapter 5

Curves

As alluded to in the previous chapter the algebra of densities on a one dimensional manifold plays a particularly important role as all geometric data is contained within them. In this chapter we use some of the results we have obtained so far to study some classical objects that appear in the theory of differential equations. In the first part of this chapter we study how Wronskians fit into the picture we have developed and in the second half we focus on equivariant questions related to the upper half plane.

5.1 Wronskians.

Wronskians are one of the most useful tools in the classical study of differential equations. In this section we review this theory and explain how it can be extended to the algebra of densities. Another use of Wronskians is related with flat bundles and monodromy questions we shall again review this work and see that the algebra of densities are a natural setting for much of these investigations. Later we shall use the quantisation of the last chapter to give a family of invariant differential operators.

5.1.1 Basic definitions and properties.

By a curve we mean a 1 dimensional complex or real manifold. As most of this chapter will be spent working in the category of complex surfaces it will be useful to have notation concerning weighted meromorphic functions:

Notation 5.1.1. We let $\mathcal{M}_C^\lambda$ denote the space of meromorphic densities of weight λ.

That is $\mathcal{M}_C^\lambda = \mathcal{M}_C \otimes \mathcal{F}_C^\lambda$.

Wronskians and differential operators.

Recall that the Wronskian of an n-tuple of functions, $\Phi = (f_1, \dots, f_n)$, is defined as:

$$W(\Phi) = \det \begin{pmatrix} f_1 & \partial f_1 & \cdots & \partial^{n-1} f_1 \\ \vdots & \vdots & & \vdots \\ f_n & \partial f_n & \cdots & \partial^{n-1} f_n \end{pmatrix}.$$

In this equation we shall consider either real or complex domains, that is $f_i \in C^\infty(U)$ for $U \subset \mathbb{R}$, or $f_i \in \mathcal{O}(V)$ for $V \subset \mathbb{C}$, $\forall i$. The Wronskian changes by sign under a permutation of the functions. A large part of the use of Wronskians in the study of linear ordinary differential equations can be summed up in the following theorems, see [11, 12, 38, 39].

Theorem 5.1.2. *Let $\Phi = (f_1, \dots, f_n)$ be an n-tuple of linearly independent smooth functions on $U \subset \mathbb{R}$ or $\mathbb{C}$, and suppose that $W(\Phi)(x) \neq 0$ (resp. $W(\Phi) \in C^\infty(U)^\times$ or $\mathcal{O}(U)^\times$). Then there exists unique linear operator of order n, L_Φ, defined on a neighbourhood of x (resp. defined on U) such that:*

i The symbol of L_Φ, i.e. the coefficient of ∂^n, is 1.

ii The set of solutions of the differential equation $L_\Phi(g) = 0$ is spanned by $\{f_1, \dots, f_n\}$.

An explicit form of the differential operator is given as:

$$L_\Phi(g) = \frac{W(\Phi, g)}{W(\Phi)},$$

where we have written (Φ, g) for $(f_1, \dots, f_n, g)$.

Theorem 5.1.3. *Let $\Phi = (f_1, \dots, f_n)$ be an n-tuple of linearly independent analytic functions. Then $W(\Phi) \neq 0$.*

The second theorem refers to both real analytic and complex analytic functions. In these cases we can use theorem 5.1.2 to show that there exists a meromorphic differential operator defined on the whole space whose zero set is spanned by the elements of Φ. These theorems also reveal that there is a relation between generalised Wronskians and the coefficients of differential operators which we have previously studied, see proposition 4.1.7. This theorem is not coordinate independent and or first task is to remedy this, see also [23].

Lemma 5.1.4. *Let $f_i \in \mathcal{F}^\lambda(U)$, $i = 1, \dots, n$, and $\Phi = (f_1, \dots, f_n)$. Then $W(\Phi)$ is a density of weight $n\big(\lambda + \frac{1}{2}(n-1)\big)$.*

Proof. We shall use a tilde to denote a different coordinate system so that $\widetilde{\Phi}(y) = x_{,y}^{\lambda}\,\Phi(x(y))$ and $\widetilde{\partial} = x_{,y}\partial$. We find that:

$$W(\Phi) = \det(\Phi\,\partial\Phi \cdots \partial^{n-1}\Phi) = \det\big(x_{,y}^{-\lambda}\widetilde{\Phi}\,\partial(x_{,y}^{-\lambda}\widetilde{\Phi}) \cdots \partial^{n-1}(x_{,y}^{-\lambda}\widetilde{\Phi})\big).$$

We have that $\partial^k(x_{,y}^{-\lambda}\widetilde{\Phi}) = \Sigma_{j=0}^{k}\binom{k}{j}\partial^j(x_{,y}^{-\lambda})\partial^{k-j}(\widetilde{\Phi})$. Using bilinearity and antisymmetry of the determinant the only the term not killed in the determinant is $j = 0$, as the other derivatives are proportional to earlier columns. Thus:

$$W(\Phi) = x_{,y}^{-n\lambda}\det(\widetilde{\Phi}\,\partial\widetilde{\Phi} \cdots \partial^{n-1}\widetilde{\Phi}).$$

We have $\partial^k = x_{,y}^{-k}\widetilde{\partial}^k + O(\widetilde{\partial}^{k-1})$, where by $O(\widetilde{\partial}^{k-1})$ we mean a linear operator of order $\leq k-1$. Using the same properties of the determinant we have that only expression that contributes to the Wronskian is that coming from $x_{,y}^{-k}\widetilde{\partial}$. We therefore have that:

$$W(\Phi) = x_{,y}^{-n\lambda}\det\big(\widetilde{\Phi}\,x_{,y}^{-1}\widetilde{\partial}\widetilde{\Phi} \cdots x_{,y}^{1-n}\widetilde{\partial}\widetilde{\Phi}\big) = x_{,y}^{-n\left(\lambda + \frac{n-1}{2}\right)}W(\widetilde{\Phi}).$$

□

Note that in the case that the densities f_i in the tuple Φ are all of weight $\frac{1-n}{2}$ then the Wronskian is in fact a function. As a density is a section of a line bundle it makes sense to say that it is non-zero at some point. We can thus recast theorem 5.1.2 in the following coordinate independent form:

Corollary 5.1.5. *Let $\Phi = (f_1, \dots, f_n)$ where $f_i \in \mathcal{F}^\lambda(C)$ for some curve C. Suppose that $W(\Phi)(x) \neq 0$, (resp. $W(\Phi) \in \mathcal{F}(C)^\times$), then there exists a unique linear differential operator of order n, $L_\Phi : \mathcal{F}_x^\lambda \to \mathcal{F}_x^{\lambda+n}$ (resp. $L_\Phi : \mathcal{F}_C^\lambda \to \mathcal{F}_C^{\lambda+n}$) such that:*

i The principle symbol of L_Φ is 1.

ii The set of solutions to the differential equation $L_\Phi(g) = 0$ is spanned by $\{f_1, \dots, f_n\}$.

Proof. The majority of the proof lies in theorem 5.1.2 by working in a particular coordinate frame. We check that L_Φ does indeed have weight n:

$$L_\Phi(g) = \frac{W(\Phi, f)}{W(\Phi)} = \frac{x_{,y}^{-(n+1)(\lambda+\frac{n}{2})} W(\widetilde{\Phi}, \widetilde{g})}{x_{,y}^{-n(\lambda+\frac{n-1}{2})} W(\widetilde{\Phi})} = x_{,y}^{-\lambda-n} L_{\widetilde{\Phi}}(\widetilde{g}).$$

□

Remark 5.1.6. Corollary 5.1.5 has a simple generalisation to arbitrary line bundles. In this case we have that if we have an n-tuple of sections, $s_i \in \Gamma(L, C)$ for a line bundle L, we have an induced differential operator $L_{\underline{s}} : \mathscr{L} \to \mathscr{L} \otimes \mathcal{F}^n_C$.

Example 5.1.7. Take an embedding of a curve C into projective space $\mathbb{P}^n$ and pull back the tautological bundle:

$$\begin{array}{ccc} \iota^*\gamma_n & \longrightarrow & \gamma_n \\ \downarrow & & \downarrow \\ C & \xrightarrow{\iota} & \mathbb{P}^n. \end{array}$$

Then as γ_n is generated by its $n+1$-linearly independent global sections we have that $\iota^*\gamma_n(C)$ has $n+1$-linearly independent sections, see [33] and [69] for the real case. We can then take the differential operator of these sections to get a differential operator, $L_\iota : \iota^*\gamma_n \to \mathcal{F}^n_C \otimes \iota^*\gamma_n$. For a more specific example consider the embedding $\mathbb{P}^1 \to \mathbb{P}^2$, $(x_0 : x_1) \mapsto (x_0^2 : x_0 x_1 : x_1^2)$. Then $\iota^*\gamma_2 \cong O(2) \cong \mathcal{F}^{-1}_{\mathbb{P}^1}$. This bundle is generated by the sections of the local form $\partial, z\partial$ and $\frac{1}{2}z^2\partial$. We get the associated differential operator:

$$L : \mathcal{F}^{-1}_{\mathbb{P}^1} \to \mathcal{F}^2_{\mathbb{P}^1}, \quad L = |Dz|^3\partial^3.$$

The Schwarzian of a connection.

Let γ be a connection on $\mathcal{F}^1(M)$ we then define the *Schwarzian of the connection*, $\mathcal{S}(\gamma)$, by the following formula:

$$\mathcal{S}(\gamma) = \gamma_{,x} + \frac{1}{2}\gamma^2. \tag{5.1.1}$$

We have to justify this definition as the Schwarzian is of course an extremely classical object, see [55]. Firstly we shall show that this object transforms as one would expect.

Lemma 5.1.8. *Let $U \subset \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let γ be a connection on U. Then under a coordinate transformation $\mathcal{S}(\gamma)$ transforms as a quadratic differential up to* *the Schwarzian of the transformation:*

$$\mathcal{S}(\widetilde{\gamma})\phi_{,x}^2 = \mathcal{S}(\gamma) + \mathcal{S}(\phi),$$

where $\mathcal{S}(\phi)$ denotes the Schwarzian of the map ϕ.

Proof. We have that $\widetilde{\gamma} = \phi^*\gamma(y) = \frac{1}{\phi_{,x}}\gamma + \frac{\phi_{,xx}}{\phi_{,x}^2}$. Then a simple calculation shows that the lemma holds. □

An immediate corollary of this proof is that the Schwarzian of a connection is a projective density of weight 2, just as the Schwarzian of a local isomorphism. In fact they are essentially the same thing to see this let us take a connection on C and consider the differential equation:

$$D_\gamma^2|_{C^\infty(M)}(f) = f'' + \gamma f' = 0.$$

We can locally always find a non-trivial solution to this equation and we have that such a solution must be such that $f_{,x} \neq 0$ around some point x_0. Then we can take the Schwarzian of such a solution and we see that it is equal to the Schwarzian of the connection. Conversely given a diffeomorphism $\phi : U \to V \subset \Bbbk$ and pulling back the flat connection on V to U we find that its Schwarzian is equal to the Schwarzian of ϕ. Hence there is an equivalence between the two objects, at least locally.

We shall find it easier to work with the Schwarzian of a connection rather than with that of a local isomorphism as the connection appears more regularly in the study of linear differential operators.

Proposition 5.1.9. *The operator $D_\gamma^n|\mathcal{F}^{\frac{1-n}{2}}(C)$ depend only on $\mathcal{S}(\gamma)$. That is if $\mathcal{S}(\gamma) = \mathcal{S}(\widetilde{\gamma})$ then $D_\gamma^n|\mathcal{F}^{\frac{1-n}{2}}(C) = D_{\widetilde{\gamma}}^n|\mathcal{F}^{\frac{1-n}{2}}(C)$.*

Proof. First we show that it is a necessary condition. We have that the differential operator $D_\gamma^n|\mathcal{F}^\lambda(M)$ is equal to:

$$|Dx|^n\Big((\partial + (\lambda + n - 1)\gamma)\cdots(\partial + (\lambda + 1)\gamma)(\partial + \lambda\gamma)\Big).$$

Therefore the coefficient of ∂^{n-1} is equal to $n\big(\lambda + \frac{n-1}{2}\big)$ and is zero for $\lambda = \frac{1-n}{2}$. Looking at the next term for this choice of λ we find it is equal to

$$-\frac{(n-1)n(n+1)}{12}\Big(\gamma_x + \gamma^2\Big).$$

Therefore it is a necessary condition, and we are left to show it is sufficient. Now pick a connection γ on C then there exists $f : U \to \mathbb{P}^1$, where $U = \widetilde{C}$, such that $\mathcal{S}(f) = \mathcal{S}(\gamma)$. Such an f is only defined up to the action of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$ and if $\mathcal{S}(\gamma) = \mathcal{S}(\widetilde{\gamma})$ then $\widetilde{f}$ is equivalent to f.

On $\mathbb{P}^1$ there is a canonical operator $D(n) : \mathcal{F}^{\frac{1-n}{2}}(\mathbb{P}^1) \to \mathcal{F}^{\frac{1+n}{2}}(\mathbb{P}^1)$, known as Bol operators, see citeBO1. This operator can be defined by a generalisation of the embedding as in example 5.1.7. It corresponds to the fact that $\mathcal{F}_{\mathbb{P}^1}^{\frac{1-n}{2}} \cong O(n-1)$ and hence is generated by n-sections for $n \geq 0$. We can then apply the Wronskian of these sections to get the corresponding differential operator.

As f, defined to have its Schwarzian equal to that of γ, is a local isomorphism $U \to \mathbb{P}^1$ we can pull back and locally push forward densities and hence the differential operator $D(n)$. We claim that $f^*D(n) = D_\gamma^n|\mathcal{F}^{\frac{1-n}{2}}(U)$. This follows immediately for $df = f_{,z}|Dz|$ is a volume form on U that generates the connection γ and thus the pull back is exactly of the form required. □

Example 5.1.10. The calculations for the square of the operator D_γ are as follows:

$$D_\gamma^2 = t^2\Big(\partial^2 + 2\gamma\widehat{w}\partial + \gamma^2\widehat{w}^2 + \gamma\partial + \gamma_{,x}\widehat{w}\Big),$$
$$D_\gamma^2|\mathcal{F}^{-\frac{1}{2}}(C) = |Dz|^2\Big(\partial^2 + \frac{1}{2}\mathcal{S}(\gamma)\Big).$$

Example 5.1.11. Going through the complete machinations for fourth order operators already becomes more involved. We can however employ some of the information we know about the differential operator $D_\gamma^n|\mathcal{F}^{\frac{1-n}{2}}(C)$ to determine its general structure. Firstly we have that it is self adjoint and therefore using the results in the above proof, elucidated in more detail in the corollary below, we have that it must have the form:

$$D_\gamma^4|\mathcal{F}^{-\frac{3}{2}}(C) = |Dz|^2\Big(\partial^4 - 5\mathcal{S}(\gamma)\partial^2 - 5\mathcal{S}(\gamma)_{,z}\partial + U(\gamma)\Big).$$

This makes the calculations far simpler and we have that:

$$U(\gamma) = -\frac{3}{2}\mathcal{S}(\gamma)_{,zz} + \frac{9}{4}\mathcal{S}(\gamma)^2.$$

Corollary 5.1.12. *Let $L \in DO^n(\mathcal{F}^{\frac{1-n}{2}}(C), \mathcal{F}^{\frac{1+n}{2}}(C))$ have symbol 1. Then if L is (anti-)self adjoint the coefficient of ∂^{n-2} transforms as the Schwarzian multiplied by $-\frac{(n-1)n(n+1)}{12}$.*

In reference to the above corollary see also [15] and [59].

5.1.2 Flat bundles.

Wronskians have another classical application which is in the theory of flat bundles. We firstly go through the requisite definitions before we recall how Wronskians can be applied in this situation. Flat bundles are vector bundles that are defined using only the fundamental group of the manifold. Standard references for this section include [19, 21, 79]. Let us begin with the formal definition:

Definition 5.1.13. A flat vector bundle on a manifold M is a pair (E, d_E) where E is a vector bundle on M and d_E is a connection on E such that $d_E^2 = 0$.

Two flat bundles, (E_i, d_i), are called equivalent if there exists an isomorphism $f : E_1 \to E_2$ such that $d_2 \circ f = f \circ d_1$. In particular we have that two flat bundle structures on a vector bundle E are isomorphic if one of the connections is a gauge transformation of the other.

We shall now give the archetypal example of a flat bundle. It relies on some concepts that we have to define first:

Definition 5.1.14. A locally constant sheaf on X is a sheaf of abelian groups, S, such that S is locally isomorphic to A_X for some abelian group A.

Let us explain the data contained in a locally constant sheaf: we have a cover of the space X, $\mathcal{U} = \{U_i | i \in I\}$, such that $S|U_i \cong A_{U_i}$. The gluing data is then defined as morphisms of abelian groups, $\varphi_{ij} \in \mathrm{Aut}(A_{U_{ij}}, A_{U_{ij}}) = \mathrm{Aut}(A)^{\prod \pi_0(U_{ij})}$ that satisfy the cocycle condition:

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}.$$

Let S be a locally constant sheaf on a connected manifold M. We pick a point $x_0 \in U_0$ where U_0 is a trivializing neighbourhood. Now consider a loop $\gamma : (S^1, 0) \to (M, x_0)$, and take a cover of $\gamma(S^1)$, $\{U_0, U_1, \ldots, U_n\}$ such that $U_i \cap U_{i+1} \neq \emptyset \neq U_n \cap U_0$ and moreover assume all the U_i are contractible. We then define an element of $\mathrm{Aut}(A)$, λ_γ, by the formula $\lambda_\gamma := \varphi_{U_n,U_0} \circ \varphi_{U_{n-1},U_n} \circ \cdots \circ \varphi_{U_0,U_1}$.

Proposition 5.1.15. *Once the choices $x_0 \in U_0$ have been made then the above map, λ_γ, is well defined, i.e. independent of the other elements in the cover of the loop. Moreover λ_γ only depends on the homotopy class of the loop.*

We have thus associated to any locally constant sheaf a representation of the fundamental group of the manifold. Conversely let us take a representation of the fundamental group of M, $\rho : \pi_1(M) \to \mathrm{Aut}(A)$. Then if we let $\widetilde{M} \to M$ denote the universal cover of M we have the locally constant sheaf $\widetilde{M} \times_{\pi_1(M)} A$. We have the following classical result.

Theorem 5.1.16. *The two maps define an equivalence between locally constant sheaves on M and representations of the fundamental group of M.*

We shall be interested not just in abelian groups but in $\Bbbk_X$-modules, where $\Bbbk = \mathbb{C}$ or $\mathbb{R}$. Our first aim is to show that the underlying bundle of a flat bundle is isomorphic to the trivial bundle twisted by a locally constant sheaf of $\Bbbk$-modules, S: $\mathcal{E} \cong \mathcal{O}_X \otimes \mathbb{S}$. To do this we consider the time ordered exponential of the connection to get a map $U_\gamma \in \mathrm{Hom}^\times(E_{x_0}, E_x)$, where $\gamma : I \to M$ is a path that begins at x_0 and ends at x. We finally need to show that this only depends on the homotopy class of the path. Recall the differential equation for the time ordered exponential:

$$\frac{d}{dt}U(t) = \langle A(\gamma(t)), \dot{\gamma}(t)\rangle U(t).$$

If we now take a homotopy of paths, $\gamma(t; s)$, then the difference between the time ordered exponential is over $\gamma(-; 0)$ and $\gamma(-; 1)$ is proportional to the integral of the curvature of the connection over the surface $\gamma(I \times I)$ and hence is zero. Thus the map doesn't depend on the homotopy class of the path. It follows, as above, that we get a representation $\rho_E : \pi_1(M) \to \mathrm{GL}_n(\mathbb{K})$ and the bundle E is isomorphic to the quotient $\widetilde{M} \times \mathbb{C}^n/\pi_1(M)$, where the action is given by $g \cdot (x, v) = (gx, \rho_E(g)v)$.

We have thus defined a map from flat bundles to locally constant $\Bbbk_M$-modules using the representation defined above and theorem 5.1.16. Conversely if we take a locally constant $\Bbbk_M$-module of dimension n, S, then we define a flat bundle structure on $\mathcal{E}_S = \mathcal{O}_M \otimes_{\Bbbk_M} S$ as follows: Take a trivializing neighbourhood U with local generating sections s_i $i = 1, \ldots, n$. Then define $d|U(\sum f_i s_i) = \sum df_i s_i$. One can check that this glues together to define a flat bundle structure. We have the other classical theorem in this subject:

Theorem 5.1.17. *The two maps defined above define an equivalence between flat bundles on M and locally constant $\Bbbk_M$-modules.*

Example 5.1.18. The fundamental group of an elliptic curve is isomorphic to $\mathbb{Z}^2$ or more abstractly Λ, where $E = \mathbb{C}/\Lambda$. We now wish to define a flat bundle associated to this data. To do this consider the dual lattice $\Lambda^* := \{u \in \mathbb{C} | \langle u, \lambda \rangle \in \mathbb{Z} \; \forall \lambda \in \Lambda\}$. Let $w \in \mathbb{C}/\Lambda^*$, then we define the representation:

$$\varrho_w : \Lambda \to \mathbb{C}^\times \qquad \lambda \mapsto \exp(2\pi i \langle w, \lambda \rangle).$$

Clearly this corresponds to all non-equivalent 1-dimensional representations. Therefore flat line bundles on $E = \mathbb{C}/\Lambda^*$ are naturally equivalent to elements of $E^* := \mathbb{C}/\Lambda^*$. Flat bundles can be tensored together and they form the connected component of the Picard group under this identification. We have therefore shown that the Picard group of E is naturally isomorphic to E^* as *groups*. Over the complex numbers one can show that if the elliptic curve E corresponds to $\tau \in \mathbb{H}$ then E^* corresponds to $-\tau^{-1}$, in particular $E \cong E^*$. This allows us to interpret the group structure of E as coming from the connected component of the Picard group. An abstraction of this allows us to define elliptic curves over an algebraically defined space see [41].

Wronskians and flat bundles.

We now associate to a linear differential operator on a simply connected curve, U, that is invariant with respect to some group, Γ, acting properly discontinuously on U, a flat bundle on U/Γ. To do this let us recall the definition of an invariant differential operator:

Definition 5.1.19. Let $L : \mathcal{F}_C^\lambda \to \mathcal{F}_C^{\lambda+\delta}$ be a differential operator on a curve C of weight δ, and let $G \subset \mathrm{Aut}(C)$. Then L is G-invariant if

$$g^* \circ L = L \circ g^*, \qquad \forall g \in G.$$

Assume that L is a linear differential operator of order n on U that is invariant under $G \subset \mathrm{Aut}(U)$. Then assuming that U is simply connected we take the solution set of this differential operator, $\{f_1, \ldots, f_n\}$, and $L = \frac{W(f_1,\ldots,f_n,-)}{W(f_1,\ldots,f_n)}$. We have the following simple lemma that explains the relation with flat bundles:

Lemma 5.1.20. *Let $V = \mathbb{C}\{f_1, \ldots, f_n\}$, that is the vector space of solutions to the differential equation $Lf = 0$. Then G-induces a representation on V, that is we get $\rho_L : G \to GL(V)$.*

Proof. Let $f \in \mathrm{Ker}(L)$. We then have that $Lg^*f = g^*Lf = 0$, hence $g^*f \in \mathrm{Ker}(L)$. ☐

Lemma 5.1.21. *Let $\underline{f} = (f_1, \ldots, f_n)$ be a linearly independent set of analytic functions on U. Let $\Gamma \subset Aut(U)$ and suppose that $g^*\underline{f} = \rho_g(z)\underline{f}$, for $\rho : \Gamma \to GL(n) \otimes Hom(U, \mathbb{C}^\times)$ some representation. Then we have that $W(\underline{f}) \circ g = \det(\rho_g)W(\underline{f})$.*

Proof. The calculations are equivalent to those for determining the weight of the Wronskian of n densities of weight λ, see lemma 5.1.4. ☐

Corollary 5.1.22. *Let L be a $\Gamma-$invariant differential operator acting on $\mathcal{O}_U$. Then it induces a flat bundle structure on U/Γ.*

Proposition 5.1.23. *Let (E, d_E) be a flat vector bundle on $C = U/\Gamma$ and assume that the underlying locally constant bundle comes from an irreducible representation of the fundamental group. Then E is isomorphic to a bundle induced from a linear differential operator.*

Proof. Consider the space of solutions to the linear differential equation $d_E\underline{s} = 0$. This can locally be solved on the manifold and by analytic continuation it lifts to a solution on $\widetilde{C} = U$. We claim that these solutions are linearly independent. Assume not so that there existed a non-trivial relation among them, say:

$$\sum c_i s_i = 0.$$

Then let $V = \mathrm{Ker}(c) \subseteq \mathbb{C}^n$. As $c \neq 0$ V is a proper subspace and a simple calculation shows that it is preserved under the action of ρ_E which contradicts the irreducibility of the representation. Now as the functions are linearly independent we can take the Wronskian to get a differential operator of order n on U. ☐

5.1.3 Invariant brackets of densities.

We now apply the results in the above section and in previous chapters to study how these ideas relate to the algebra of densities. We firstly show that for differential operators defined between non-singular weights there exists a canonical extension of this operator to a (meromorphic) differential operator on the algebra of densities. We then use this to give a family of brackets of densities that are invariant under arbitrary isomorphism.

Extension of differential operators.

Proposition 5.1.24. *Let $L \in DO^n(\mathcal{F}^\lambda(C), \mathcal{F}^{\lambda+\delta}(C))$ such that $\lambda \neq \frac{1-n}{2}$. Then there exists a natural meromorphic extension of L to $\widehat{L} \in DO^n_\delta(\mathcal{F}(C))$.*

Proof. Fix a contractible neighbourhood and take a set of solutions on this neighbourhood, $\underline{f} = (f_1, \dots, f_n)$, $f_i \in \mathcal{F}^\lambda(U)$. Then we have that the Wronskian of $\underline{f}$, $W(\underline{f})$, is a density of weight $n(\lambda + \frac{1}{2}(n-1))$ by lemma 5.1.4. By the assumptions in the proposition this weight is not equal to zero (unless $n = 0$ where the extension is trivial) and thus

$$\gamma := -\frac{1}{n\left(\lambda + \frac{1}{2}(n-1)\right)} \frac{\partial W(\underline{f})}{W(\underline{f})},$$

is a well defined meromorphic connection on $\mathcal{F}^1(U)$. We see that these connections glue together to give a unique flat connection, γ_L, on $\mathcal{F}^1(M)$. We may then apply the quantisation map as in lemma 4.1.3 to define an extension of L to the algebra of densities. □

There is a geometric explanation of the above proposition. We have seen that a differential operator on a curve C corresponds to a map from the universal cover of C, $U = \widetilde{C}$, to $\mathbb{P}^n$, . If $C \ncong \mathbb{C}P^1$ then the universal cover is contractible and therefore, using standard results in obstruction theory, there exists a lift of the mapping:

$$\begin{array}{ccc} & & \mathbb{C}^{n+1} - \{0\} \\ & \nearrow & \downarrow \\ U & \longrightarrow & \mathbb{P}^n. \end{array}$$

Now we can twist the lifted map by any function $u : U \to \mathbb{C}^\times$ without changing the projection. A natural way to think of different lifts are as operators acting on densities of different weights: a pencil of maps $\Phi_\lambda : U \to \mathbb{C}^n - \{0\}$ can be thought of as a pencil of differential operators on $\mathcal{F}(U)$. The above proposition then states that if one is allowed to consider meromorphic functions then there exists a natural 1-parametric family of such lifts.

Example 5.1.25. Consider the second order operator L on $\mathbb{C}^\times$ acting on functions of the local form:

$$L = |Dz|^2 \Big(\partial^2 + \frac{1}{z}\partial\Big).$$

Then The solution set is spanned by 1 and $\log(z)$, where by $\log(z)$ we mean the function u in the universal cover $\mathbb{C}$ where $e^u = z$. The extension of the operator is then given as:

$$\widehat{L} = t^2\Big(\partial_z^2 + \frac{2\widehat{w}+1}{z}\partial_z + \frac{1}{z^2}\big(\widehat{w}+z-1\big)\widehat{w}\Big).$$

Invariant brackets.

We now combine the ideas contained in proposition 5.1.24 and the earlier part of this chapter to define brackets between densities that have good equivariance properties. We then explicitly describe some of these brackets in some examples and see some classical objects, e.g. the unintegrated Virasoro cocycle. A good refernce for this section is the book of D.B. Fuks citeFuk1.

Recall that above we associated to a tuple of densities of weight $\lambda \neq \frac{1-n}{2}$, $\underline{f} = (f_1, \ldots, f_n)$, two differential operators: firstly the differential operator $L_{\underline{f}} : \mathcal{F}^\lambda(U) \to \mathcal{F}^{\lambda+n}(U)$, and secondly $d_{\underline{f}}$ a vector field on the algebra of densities. The brackets are then defined to be the coefficients in the extension of the differential operator $L_{\underline{f}}$ defined in proposition 5.1.24. That is for the tuple $\underline{f}$ there exists unique densities a_k such that:

$$L_{\underline{f}} = \sum_{k=0}^{n} a_k d_{\underline{f}}^k | \mathcal{F}^\lambda(U),$$

and we define the brackets $r_{n,k}^\lambda(\underline{f}) := a_k$ via the above equation.

Proposition 5.1.26. *Let $\phi \in Hom^\times(U,V)$ then the brackets $r_{n,k}^\lambda$ satisfy the following equivariance property:*

$$r_{n,k}^\lambda(\phi^*\underline{f}) = \phi^* r_{n,k}^\lambda(\underline{f}),$$

that is the map $r_{n,k}^\lambda : \mathcal{M}^\lambda(U)^{\Pi n} \to \mathcal{M}^{n-k}(U)$ is well defined.

Moreover assume that the operator $L_{\underline{f}}$ is G-invariant for some group G. Then so are the densities $r_{n,k}^\lambda$. In particular this holds if $f_i \in \mathcal{F}^\lambda(U)^G$ $\forall i$.

Proof. We just need to apply lemma 5.1.21 in the particular case where we consider densities of a fixed weight. □

Lemma 5.1.27. *We have that:*

- $r_{n,0}^\lambda = 1$,

- $r^{\lambda}_{n,1} = 0$.

Proof. The first part is trivial. For the second part note that the coefficient of ∂^{n-1} in the expansion of $d^n_{\underline{f}}|\mathcal{F}^\lambda(U)$ is equal to:

$$-\sum_{k=0}^{n} \frac{\lambda + k}{n\left(\lambda + \frac{1}{2}(n-1)\right)} = -1.$$

□

Example 5.1.28. The Virasoro cocycle. In this example we shall see that $r^{-1}_{2,2}$ essentially corresponds to the Virasoro cocycle, [29, 82]. Let us calculate the explicit form of $r^{\lambda}_{2,2}$. Recall that the expressions for $L_{(f,g)}$ and $d_{(f,g)}$ are given as follows:

$$L_{(f,g)} = |Dz|^2 \left(\partial^2 - \frac{W^{0,2}(f,g)}{W(f,g)}\partial + \frac{W^{1,2}(f,g)}{W(f,g)}\right),$$

$$d_{(f,g)} = t\left(\partial - \frac{1}{2\lambda+1}\frac{W^{0,2}(f,g)}{W(f,g)}\widehat{w}\right).$$

To find the form of $r^{\lambda}_{2,2}$ we calculate the square of the vector field:

$$d^2_{(f,g)} = t^2\Bigg(\partial_z^2 - \partial_z(\log(W(f,g)))\frac{2\widehat{w}+1}{2\lambda+1}\partial +$$

$$\left[\left(\frac{\partial W(f,g)}{W(f,g)}\right)^2\frac{\widehat{w}+2\lambda+2}{2\lambda+1} - \frac{\partial^2 W(f,g)}{W(f,g)}\right]\frac{\widehat{w}}{2\lambda+1}\Bigg).$$

Comparing these two operators on $\mathcal{F}^\lambda(U)$ we find the following expression of $r^{\lambda}_{2,2}$:

$$r^{\lambda}_{2,2}(f,g) = \frac{\lambda}{2\lambda+1}\frac{W^{0,3}(f,g)}{W(f,g)} + \frac{3\lambda+1}{2\lambda+1}\frac{W^{1,2}(f,g)}{W(f,g)} - \frac{\lambda(3\lambda+2)}{(2\lambda+1)^2}\left(\frac{W^{0,2}(f,g)}{W^{0,1}(f,g)}\right)^2.$$

We focus on the case of vector fields, that is when $\lambda = -1$. Here the Wronskian corresponds to the Lie bracket and hence preserves the space: if $f, g \in \mathcal{F}^{-1}(U)$, $W(f,g) \in \mathcal{F}^{-1}(U)$. Using the above formula we find the following explicit form of this function:

$$r^{-1}_{2,2}(f,g) = \frac{fg_{,zzz} - f_{,zzz}g + 2f_{,z}g_{,zz} - 2f_{,zz}g_{,z}}{fg_{,z} - f_{,z}g} - \left(\frac{fg_{,zz} - f_{,zz}g}{fg_{,z} - f_{,z}g}\right)^2. \qquad (5.1.2)$$

This is a quadratic differential, i.e. an element of $\mathcal{F}^2(U)$ that is symmetric in its arguments. We now define a map $C : \mathfrak{X}(U)\times\mathfrak{X}(U) \dashrightarrow \mathcal{F}^1(U)$ as $C(f,g) = W(f,g)r^{-1}_{2,2}(f,g)$. The bracket C is antisymmetric and homogeneous of degree 1 in each variable. Moreover as its image is a density of weight 1 it makes sense to integrate the associated density and thus define the object $c(f,g) = \int C(f,g) \in \mathbb{C}$. Let us focus on the case of polynomial vector fields so that we have a basis for $\mathfrak{X}(U)$ given by $L_n = z^{n+1}\partial_z$. Note that L_n and L_m are linearly independent iff $n \neq m$. This Lie algebra is also known as the Witt algebra and has the following commutation relations:

$$[L_n, L_m] = (m-n)L_{n+m}.$$

We can now calculate $C(L_m, L_n)$ in terms of $r_{2,2}^{-1}(L_m, L_n)$ using equation (5.1.2):

$$C(L_m, L_n) = (n-m)nmz^{n+m-1}|Dz|.$$

Note that this operation is now well defined on the entire algebra and we thus have an antisymmetric bilinear map $C : \mathfrak{X}^{\wedge 2}(U) \to \mathcal{F}^1(U)$. We now claim that this is in fact a cocycle , $C \in C^2(\mathfrak{X}, \mathcal{F}^1)$. The calculation is relatively standard:

$$C([L_{m_1}, L_{m_2}], L_{m_3}) = (m_2 - m_1)(m_3 - m_2 - m_1)(m_1 + m_2)m_3 z^{m_1+m_2+m_3-1}|Dz|,$$

$$L_{m_1}C(Lm_2, L_{m_3}) = (m_3 - m_2)m_2 m_3(m_1 + m_2 + m_3)z^{m_1+m_2+m_3-1}|Dz|,$$

and subtracting these two expressions from each other and summing over permutations the result follows. Recall that the integral, $\int : \mathcal{F}^1(M) \to \mathbb{C}$, is a morphism of $\mathfrak{X}(M)$ modules and therefore we can integrate this cocycle. If we denote by $c(-,-)$ the associated cocycle it is given by the formula:

$$\begin{aligned} c(L_m, L_n) &= (2\pi i)^{-1}(n-m)nm|_{n+m=0} \\ &= -\frac{m^3}{\pi i}\delta_{n+m,0}. \end{aligned}$$

This is not the standard representation so define the cochain, $B : \mathcal{F}^{-1}(U) \to \mathcal{F}^1(U)$, by the formula $B(L_m) = z^{m-1}|Dz|$. Then $V = C + \delta B$ has the following expression:

$$V(L_n, L_m) = (m-n)(nm+1)z^{n+m-1}|Dz|.$$

Integrating this gives the Virasoro cocycle - that is the two cocycles are cohomologous. For this reason it is natural to call V the unintegrated Virasoro cocycle.

5.2 Invariant operators on the upper half plane.

In this section we study certain invariant operators on the upper half plane that are invariant with respect to the modular group and obey some regularity conditions at infinity. These operators form a natural extension of the algebra of modular forms and we shall be able to classify all such operators as well as giving some results and further applications of these operators. To begin this section we shall recall the language of modular forms and then interpret them as certain elements in the algebra of densities. We shall then define modular differential operators and study some of their properties.

5.2.1 Modular forms.

We now recall the classical notions of modular forms and their basic properties, see [51, 57, 77, 87] for any of the results in this section. The one difference is that we shall take from our starting point the algebra of densities.

The modular group.

The space of holomorphic automorphism of the upper half plane, $\mathbb{H}$, is given by real Möbius transformations, that is the action of $\mathrm{PGL}_+(2,\mathbb{R})$ as:

$$g \cdot \tau = \frac{a\tau + b}{c\tau + d}, \qquad g = \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in \mathrm{PGL}_+(2,\mathbb{R}).$$

Note that this action is transitive and thus models $\mathbb{H}$ as a homogeneous space. For example we may choose $i \in \mathbb{H}$ and $\mathrm{Stab}_{\mathrm{PGL}_+(2,\mathbb{R})}(i) = \mathrm{PSO}(2)$ and thus $\mathbb{H} \cong \mathrm{PGL}_+(2,\mathbb{R})/\mathrm{PSO}(2)$.

We have a natural cocompact arithmetic subgroup, $\Gamma(1) := \mathrm{PSL}(2,\mathbb{Z})$, which is known as the modular group. This group appears naturally in the study of complex elliptic curves. Let us recall some basic properties of the modular group, see [51, 57].

Proposition 5.2.1. *A fundamental domain for the modular group is given by the set* $D = \{\tau | \Re(\tau) \in (-\frac{1}{2}, \frac{1}{2}),\, |\tau| > 1\}$.

Proposition 5.2.2. *The modular group is isomorphic to the free group generated by* $\mathbb{Z}/2\mathbb{Z}$ *and* $\mathbb{Z}/3\mathbb{Z}$*:* $\Gamma(1) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

The standard generators for the modular group are defined to be T, $[\tau \mapsto \tau + 1]$, and S, $[\tau \mapsto -\tau^{-1}]$. In the above proposition a particular choice of the isomorphism can be given by sending S to the generator of $\mathbb{Z}/2\mathbb{Z}$ and ST to the generator of $\mathbb{Z}/3\mathbb{Z}$.

The modular group has a natural extension of its action to the $\mathbb{H}^+ := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$. The topology on this space is defined as follows: A system of neighbourhoods for $\tau \in \mathbb{H}$ is just that induced form the upper half plane. A system of neighbourhoods for a point $q \in \mathbb{Q}$ corresponds to the interior of all circles and $\{q\}$ such that the circle has centre in $\mathbb{H}$ and its tangent coincides with the real line at . Finally at $i\infty$ we can choose the system defined by the sets $\{\tau \in \mathbb{H} | \Im(\tau) > r\} \cup \{i\infty\}$. The points in $\mathbb{H}^+ - \mathbb{H}$ are called cusps and we have that the modular group preserves the space of cusps.

Subgroups of the modular group.

We are also be interested in finite index subsets of the modular group. The most standard of these are congruence subgroups, that are those who contain some $\Gamma(N) := \mathrm{PKer}(\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z}))$. Moreover particular examples of these groups, $\Gamma_1(N)$ and $\Gamma_0(N)$ are those that have had a big influence in number theory:

$$\Gamma_1(N) = \mathrm{P}\left\{ g \in \mathrm{SL}(2,\mathbb{Z}) | g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \mathbb{Z}/N\mathbb{Z} \right\},$$

$$\Gamma_0(N) = \mathrm{P}\left\{ g \in \mathrm{SL}(2,\mathbb{Z}) | g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \mathbb{Z}/N\mathbb{Z} \right\}.$$

It has been well known for over a century, [24], that the modular group does not satisfy the congruence subgroup problem: that is there exist subgroups of finite index that do not contain $\Gamma(N)\ \forall N$. These groups are harder to understand, however in our analysis they shall not be distinct from congruence subgroups.

Modular forms.

The modular group has a natural extension to the algebra of densities as it corresponds to automorphism of the base manifold, $\mathbb{H}$. Let us calculate the action of the generators S and T on the algebra of densities:

$$T^* f(\tau, t) = f(\tau + 1, t), \quad S^* f(\tau, t) = s\left(-\frac{1}{\tau}, \frac{t}{\tau^2}\right).$$

In particular invariance with respect to T implies that we may expand f into its Fourier coefficients. This is classically called the q-expansion, we set $q = e^{2\pi i \tau} \in \mathbb{D}^\times$ and we have:

$$f(q,t) = \sum_{n\in\mathbb{Z}} f_n(t)q^n.$$

We call f regular at $i\infty$ if the above expression extends holomorphically to the whole disc, that is if $f_m(t) = 0\ \forall m < 0$. We may also consider meromorphic modular forms where we require that it extends to a meromorphic function on the disc. In the algebra of densities this condition is equivalent to $f_n = 0$ for $n << 0$.

Definition 5.2.3. A modular form is a $\Gamma(1)$-invariant holomorphic density, $s \in \mathcal{F}(M)^{\Gamma(1)}$, such that it is regular at $i\infty$. The algebra of modular forms is denoted $M(\Gamma(1))$. A modular form is called a *cusp form* if it vanishes at $i\infty$, the space of cusp forms shall be denoted by $S(\Gamma(1))$.

As usual we refer to the graded subalgebra, i.e. those modular forms that also live in the algebra of densities, by $M_*(\Gamma(1))$, and $S_*(\Gamma(1))$ the graded cusp forms. We now give the principal example of a modular forms:

Example 5.2.4. Let $n \geq 2$ and define the following function:

$$G_{2n}(\tau,t) = {\sum_{p,q}}' \frac{t^n}{(p\tau+q)^{2n}},$$

where the prime denotes the fact the sum is taken over all p and q not both zero. This sum is absolutely convergence (as $n \geq 2$) and hence a simple calculation shows that it is indeed invariant under the modular group. One can check regularity at $i\infty$ formally, see [51], however we will be content with noticing that when we set $\tau = i\infty$ in the definition the only contributions come when $n = 0$ and hence:

$$G_{2n}(i\infty,t) = 2\zeta(2n)t^n,$$

so G_{2n} does indeed satisfy the regularity condition and is a modular form. These modular forms are called the Eisenstein series. We normalize them to $E_{2n} = \frac{1}{2\zeta(2n)}G_{2n}$. The E_{2n} have one of the most famous q-expansions:

$$E_{2n}(q,t) = \left(1 - \sum_{k\geq 1} \frac{4k}{B_{2k}}\, \sigma_{2k-1}(n)q^n\right) t^n, \tag{5.2.1}$$

where B_l denotes the l^{th} Bernoulli number and $\sigma_l = \sum_{d|n} d^l$ is known as the divisor sum function.

We shall also denote by Δ the modular form $E_6^2 - E_4^3$. It is a cusp form for the modular group and known as Ramanujan's tau function. The q-expansion of this form is also particularly beautiful:

$$\Delta(q,t) = q \prod_{n \geq 1} (1-q^n)^{24} t^6.$$

The following is the first fundamental result in the classification of modular forms:

Theorem 5.2.5. *The graded algebra of modular forms, $M_*(\Gamma(1))$, is isomorphic to $\mathbb{C}[X,Y]$ where X has weight 2 and Y has weight 3. An explicit choice of this isomorphism is given by the map $E_4 \mapsto X$, $E_6 \mapsto Y$. Moreover the ideal of cusp forms, $S_*(\Gamma(1))$, is principal and generated by Δ.*

Modular forms for subgroups.

Let $\Gamma \subset \Gamma(1)$ be a subgroup of finite index, then there exists a notion of modular forms for these subgroups. The first thing we need to do is define the cusps of Γ: these are a subset of $\mathbb{H}^+ - \mathbb{H}$ whose orbits under Γ can be added to $Y(\Gamma) := \mathbb{H}/\Gamma$ to compactify the space. We have the following definitions, see [57, 78]:

Definition 5.2.6. A parabolic element of $\mathrm{SL}(2,\mathbb{Z})$ is a matrix with trace ± 2.

Definition 5.2.7. A cusp for Γ is an element of $\mathbb{Q} \cup \{i\infty\}$ that is fixed by a parabolic element of Γ.

Before we say what we mean for a function invariant under Γ to be regular at the cusps we shall give an alternate description of regularity at $i\infty$ which does not explicitly use the Fourier modes:

Lemma 5.2.8. *Let φ be a function invariant under T. Then φ is regular at $i\infty$ iff $\exists K, L > 0$ such that:*

$$\sup_{\Im(\tau) > L} |\varphi(\tau)| < K.$$

Proof. If φ is regular at $i\infty$ this follows immediately from the Fourier expansion. Conversely assume φ is T^* invariant and satisfies this bound. Then take the Fourier expansion of φ and apply this bound to show that it must have a holomorphic extension over 0. $\square$

We now define what it means for a density $s \in \mathcal{F}^k(\mathbb{H})^\Gamma$ to be regular at a cusp c. Fix an element $g \in \Gamma(1)$ such that $g(c) = i\infty$. Then we say that s is regular at c if g^*s satisfies the above bounds at $i\infty$, i.e. there exists $K, L > 0$ such that

$$\sup_{\Im(\tau)>L} |(g^*\varphi)(\tau)| < K.$$

One can check that this is independent of the choices made. For example if one picked g' such that $g'(c) = i\infty$ then the difference of the two elements must be a stabilizer of $i\infty$ that is a translation which has no affect on the conditions of boundedness. Alternatively if one picked a different representative of the cusp, say $c' \in \Gamma c$, then take $h \in \Gamma$ such that $hc' = c$. Acting by h on φ will return φ up to a multiple $(c\tau + d)^{-2k}$. This is bounded by $c^{-2k}L^{-2k}$ for any choice of L and hence the existence of such bounds is equivalent.

We can also define cusp forms using these uniform bound techniques. Namely we say that s vanishes at the cusp c if $\forall \epsilon > 0$, $\exists L > 0$ such that:

$$\sup_{\Im(\tau)>L} |g^*\varphi(\tau)| < \epsilon,$$

where g is a matrix as before.

Definition 5.2.9. Let $\Gamma \subset \Gamma(1)$ have finite index. Then a modular form is a $\Gamma-$invaraint density that is regular at all cusps of Γ. The algebra of such densities is denoted $M(\Gamma)$. A cusp form is a modular form that vanishes at all cusps of Γ. The space of cusp forms shall be denoted by $S(\Gamma)$.

An important remark that we shall use later is the fact that $i\infty$ is a cusp for any finite index subgroup. This follows from some very simple group theory: namely we wish to show that if $\Gamma \subset \Gamma(1)$ has finite index then $|\langle T\rangle \cap \Gamma| = \infty$. This follows by considering the action of T on the coset space $\Gamma(1)/\Gamma$. An immediate corollary of this result is that if $\varphi(\tau, t)$ is a cusp form for an arbitrary finite index subgroup then $\forall L > 0$ $\exists K_t > 0$ and

$$\sup_{\Im(\tau)>L} |\varphi(\tau, t)| < K_t.$$

In particular if φ is an element of the graded algebra then we can choose the K_t to be a polynomial in t. We shall use the following result later in the text:

Lemma 5.2.10. *If $\varphi \in M_*(\Gamma)$ for $\Gamma \subset \Gamma(1)$ then $g^*\varphi$ is a modular form for $g^{-1}\Gamma g$.*

Example 5.2.11. One of the most famous modular forms is the theta function, Θ, which has the following q expansion:

$$\Theta(z,t) = \big(\sum_n q^{n^2}\big)\sqrt[4]{t}.$$

One can see from the above expression that as it is not a multiple of $\sqrt{t}$ we cannot describe its transformation properties using the simple techniques above. However we can take its square, and we have the following classical result:

$$\Theta^2 \in M_1(\Gamma_1(4)).$$

The coefficient of q^m in the q-expansion of Θ^n is the number of ways m can be represented as the sum of n squares (including zero). In particular note that Θ^4 has no non-zero coefficient by Lagrange's 4-square theorem.

5.2.2 Modular differential operators.

Modular differential operators are described in the same manner as modular operators are. We should emphasize that we are working with differential operators rather than with pseudo-differential operators as in the papers of [18, 90, 61].

Recall the notion of an invariant differential operator, definition 5.1.19. We shall be interested in differential operators that are invariant with respect to the modular group, $\Gamma(1)$. If we consider such an invariant differential operator, $L = \Sigma_{k=0}^{n} a_k(\tau)\partial_\tau^{n-k}$, then the condition that $T^*L = LT^*$ is equivalent to $a_k(\tau+1) = a_k(\tau)\ \forall k$. We may thus take a q-expansion of a differential operator invariant with respect to $\Gamma(1)$. The one important thing to note here is that the differential ∂_τ goes to $2\pi i q\partial_q$.

Definition 5.2.12. A modular differential operator is a differential operator invariant under $\Gamma(1)$ such that its q-expansion is regular, that is it extends to a holomorphic differential operator on the disc. The space of modular forms shall be denoted by $M\mathrm{DO}(\Gamma(1))$ and the graded subalgebra as $M\mathrm{DO}_*(\Gamma(1))$.

From the definition of modular differential operator it is clear that they preserve the algebra of modular forms. As the transformation S^* and T^* have weight 0 the weight operator, $\widehat{w}$, is a modular differential operator.

The second Eisenstein series.

We have seen in example 5.2.4 that for $n \geq 2$ the Eisenstein series, $E_{2n}(\tau, t)$, gives us a modular form of weight n. The reason why this fails for E_2 is because the series defining it does not converge uniformly and thus we cannot simply switch the order of summation to prove invariance under S^*. However this situation is almost remediable, namely one can find a formula that 'agrees' with the definition of E_2 and converges on the upper half plane and even has a regular q-expansion. The $q-$expansion is equivalent to that for the other Eisenstein series (compare with equation (5.2.1)):

$$E_2(\tau, t) = \left(1 - \sum_{n=1}^{\infty} \frac{4}{B_2} \sigma_1(n) q^n\right) t.$$

We know from theorem 5.2.5 that it cannot be a modular form however we do have the following, see [51, 87].

Theorem 5.2.13. *The second Eisenstein series has the following transformation property:*

$$E_2\left(\frac{a\tau + b}{c\tau + d}, \frac{t}{(c\tau + d)^2}\right) = E_2(\tau, t) + \frac{6}{\pi i} \frac{ct}{c\tau + d}. \tag{5.2.2}$$

The transformation law for E_2 should remind one of a connection and this is indeed the case.

Corollary 5.2.14. *There is a non-trivial vector field of weight one on the algebra of densities of the upper half plane that is also a modular differential operator. This operator has the following expression:*

$$D = t\Big(\partial_\tau + \frac{\pi i}{3} E_2(\tau, t) \partial_t\Big).$$

It is known as the Ramanujan-Serre differential.

Proof. We need to check that this operator is invariant with respect to $\Gamma(1)$ and regular at $i\infty$. Invariance under the modular group follows immediately from theorem 5.2.13 as can easily be checked. Moreover regularity at $i\infty$ follows from the explicit description of the $q-$expansion of E_2. □

In the next section we see that this differential operator and $\widehat{w}$ are essentially the only modular vector fields. To close this section we shall turn to Ramanujan's differential equations:

Example 5.2.15. Ramanujan's differential equations. The differential equations which Ramanujan studied were the following relating the first three Eisenstein series, [72]:

$$DE_4 = \frac{2\pi i}{3} E_6 \tag{5.2.3}$$

$$DE_6 = \pi i E_4^2 \tag{5.2.4}$$

$$\partial_\tau E_2 - \frac{\pi i}{6} E_2^2 = -\frac{\pi i}{6} E_4 \tag{5.2.5}$$

The first two of these equations follow from the fact that modular differential operators preserve the algebra of modular forms and the classification in theorem 5.2.5 and the explicit form of the q-expansion to get the coefficient. The third equation is more interesting, set $A(\tau) = \frac{\pi i}{3} E_2(\tau, t)t^{-1}$, then we have that A is a connection on $\mathcal{F}^1(\mathbb{H})$. We see that the left hand side of equation (5.2.5) then has the form:

$$\frac{3}{\pi i}\left(\partial_\tau A - \frac{1}{2}A^2\right) = \frac{3}{\pi i}\mathcal{S}(A),$$

that is a multiple of the Schwarzian of the connection associated to E_2, see equation (5.1.1). We have seen earlier that the Schwarzian of a connection is a projective quadratic differential and as A is invariant under $\Gamma(1)$ it follows that $\mathcal{S}(A)$ must be as well and it must also be regular at $i\infty$. Therefore we have no choice but for the right hand side to be defined up to a scale of proportionality (which we know cannot by 0). It is a natural question to ask what is the local diffeomorphism with the equivalent Schwarzian, that is we are looking for the solution to the differential equation:

$$f'' + \frac{\pi i}{3} E_2 f' = 0.$$

It is a function that is holomorphic on $\mathbb{H}$, invaraint under T^6 and has a pole of order 1 at $i\infty$. Its expansion in $u = q_6$ has the following form:

$$f(u) = u^{-1}\left(1 + \frac{1}{45}u^6 + \frac{68}{495}u^12 + \frac{2081}{213435}u^{18} + \cdots\right).$$

Classification of modular differential operators.

We now classify all modular differential operators, compare this ring to the one introduced in [90].

Proposition 5.2.16. *The space of modular differential operators, $MDO_*(\Gamma(1))$, is isomorphic to the non commutative algebra $\mathbb{C}\langle X, Y, W, \partial\rangle/I$, where I is generated by the relations:*

$$[X, Y] = 0,$$

$$[W, x] = 2X,$$

$$[W, Y] = 3Y,$$

$$[W, \partial] = \partial,$$

$$[\partial, X] = \frac{1}{3}Y,$$

$$[\partial, Y] = \frac{1}{2}X^2.$$

An explicit isomorphism is given by $E_4 \mapsto X$, $E_6 \mapsto Y$, $\widehat{w} \mapsto W$ and $(2\pi i)^{-1}D \mapsto \partial$.

Proof. We know from above that these operators generate a subring that contains all modular forms that has the structure of the abstract algebra in the proposition by Ramanujan's equations. Our goal is to show that it contains the whole ring of differential operators. Now we know that $\frac{\pi i}{3}E_2$ is a connection and hence we can use the quantization map, see lemma 4.1.3, that an *arbitrary* differential operator of order n and weight δ, L, on $\mathcal{F}(\mathbb{H})$ can be expanded as:

$$L = \sum_{k \leq n} \sum_{l \leq n-k} a_{k,l} D^k \widehat{w}^l,$$

for a unique choice of $a_{k,l} \in \mathcal{F}^{\delta-k}(\mathbb{H})$. Now assume that L is a modular differential operator, if we can show that all the $a_{k,l}$ are modular forms we are done by appealing to theorem 5.2.5. Let $g \in \Gamma(1)$ then using that $g^*D = Dg^*$ and $g^*\widehat{w} = \widehat{w}g^*$, and by uniqueness of the coefficients in the above expansion, we must have that $\forall k, l$ $g^*a_{k,l} = a_{k,l}$, that is all the coefficients are $\Gamma(1)$-invariant. To do this let us take the q-expansion of L, so that:

$$L(q, t) = \sum_{k,l} a_{k,l}(q) t^{\delta-k} \left(2\pi i q \partial_q\right)^k \widehat{w}^l.$$

We now prove that all the $a_{k,l}$ must be regular at $i\infty$. Firstly consider $L(q^n)$ for $n \in \mathbb{N}$: as this function must be regular at 0 we have that, if we write $a_{k,l}(q) = \Sigma_r a_{k,l;r} q^r$, then:

$$\sum_k \binom{l}{k} a_{k,0;r} q^{l-k-r},$$

and as this must be regular $\forall l \geq 0$ we have that $a_{k,0;r} = 0\ \forall r < 0$. Therefore we have that all the $a_{k,0}$ are regular at $i\infty$. Now consider $a_{n,0}D^n$ which is thus a modular operator in the subring given in the proposition, and $L - \Sigma_k a_{k,0}D^n$ is divisible by $\widehat{w}$. We also have that $\left(L - \Sigma_k a_{k,0}D^k\right)\widehat{w}^{-1}$ has order $n-1$. We can thus proceed by induction to reduce to the case when we only have vertical derivatives, $\Sigma_l a_{0,l}\widehat{w}^l$. We then can apply this to the family of functions t^m $m \in \mathbb{N}$ (as we only need to consider these weights it follows that this result holds for an arbitrary indexing set). Using an identical argument we have that all the $a_{0,l}$ must be regular and hence the result follows. □

Remark 5.2.17. A similar argument can be used in the case of meromorphic modular forms. If we consider meromorphic forms of weights in $\mathbb{Z}$ then we have that the associated algebra of meromorphic differential operators is simply that given by inverting the elements X and Y.

Remark 5.2.18. Using the fact that $\frac{\pi i}{3}E_2$ is a connection we can then take pencils of any modular differential operator between fixed weights. Therefore we can say that a differential operator between fixed weights is modular if its pencil is.

Example 5.2.19. As the space of modular forms of fixed weight form a finite dimensional vector space we can find a modular differential operator whose kernel is exactly this space. Let us work out the first non trivial example of this, i.e. for modular forms of weight 6. We have that $M_6\Gamma(1) = \mathbb{C}\langle E_4^3, E_6^2\rangle$. Using Ramanujan's differential equations, example 5.2.15, the Wronskians we need can be calculated and the differential operator, $\mathfrak{m}_6 : \mathcal{F}^6_{\mathbb{H}} \to \mathcal{F}^8_{\mathbb{H}}$ has the form:

$$\mathfrak{m}_6 = \partial^2 - \pi i\left(\frac{E_4^4}{E_6} + \frac{4}{3}\frac{E_6}{E_4} - \frac{13}{3}E_2\right)\partial + 2\pi^2\left(\frac{1}{6}E_4 + \left(\frac{E_4^4}{E_6} + \frac{4}{3}\frac{E_6}{E_4}\right)E_2 - \frac{5}{2}E_2^2\right).$$

Then extending this differential operator using the connection pencil, see lemma 3.2.1, we get the meromorphic modular differential operator that extends this operator to the whole space:

$$\widehat{\mathfrak{m}}_6 = D^2 - \pi i\left(\frac{E_4^4}{E_6} + \frac{4}{3}\frac{E_6}{E_4}\right)D.$$

Solutions to modular differential equations.

To finish this chapter we shall use Wronskians to study what additional invariant properties are contained within the ring of modular differential operators. As we have noted above the kernel of a $G-$invaraint differential operator inherits a representation of G which may not decompose into trivial 1-dimensional submodules. We would thus like to understand what the kernel of modular differential operators look like.

Proposition 5.2.20. *Let $\Gamma \subset \Gamma(1)$ be a subgroup with finite index. Then if $s \in M_*(\Gamma)$ there exists a modular differential operator L such that $L(s) = 0$.*

Proof. Let us take a coset representation of Γ in $\Gamma(1)$: $\{g_1, \dots, g_n\}$. We now consider the subspace $V = \mathbb{C}\langle g_i^* s | i = 1, \dots, n\rangle$, note that the dimension of this vector space need not n. Now this space is preserved by the action of $\Gamma(1)$, it essentially corresponds to the action of $\Gamma(1)$ on $\Gamma(1)/\Gamma$. We can then define the meromorphic differential operator $L_V = \frac{W(V,-)}{W(V)}$, and it follows from lemma 5.1.21 that this operator is indeed $\Gamma(1)$-invariant. If we assume that s has weight k and that $\dim_{\mathbb{C}}(V) = N$, then $W(V)$ is a density of weight $N\big(k + \frac{N-1}{2}\big)$ and transforms with a 1-dimensional representation of $\Gamma(1)$. We know from that all 1-dimensional representations to the sixth power are trivial and hence we have that $L = W(V)^5 W(V, -)$ is a holomorphic differential operator that kills s and is invariant under $\Gamma(1)$.

We now need to check regularity of this operator at $i\infty$. We shall use our definition of regularity in terms of finding bounds on the associated functions. Pick some $\epsilon > 0$, then we know that all the $g_i^* s$ are bounded by a constant, K_i on the upper half plane. Then we have that all there derivatives are also bounded in this domain and choosing the maximum of all these we have that the Wronskian of all these functions is regular. This argument also holds for all generalised Wronskians and hence the differential operator is regular at $i\infty$.

□

Remark 5.2.21. The above proposition shows that the ring of modular differential operators is surprisingly rich. It contains some information of modular forms for arbitrary subgroups. Unfortunately it is not clear whether or not these constitute all functions that are in the kernel of modular differential operators. Naïvely looking at the differential equations it does appear as if there could be solutions with logarithmic sigularities at $i\infty$ (which obviously cannot be modular) but no explicit examples of this have been found.

Chapter 6

Poisson Structures.

In this chapter we study the odd analogue of what we have looked at in the previous chapters, that is we consider Poisson structures on the algebra of densities and Thomas' bundle. The situation is vastly different to that of differential operators. In particular we have theorem 6.1.27 which states that we can canonically lift an arbitrary generalised Poisson structure on M to $\widehat{M}$. This half of the chapter is essentially contained in the article [9]. In the second half of this chapter we shall introduce the concept of a weighted Poisson structure and their generalisations and see how they interact with the algebra of densities.

6.1 Lifting Poisson structures.

In this section our goal will be to classify all Poisson structures of weight 0 on the algebra of densities. Our results hold in the generality of P_∞-structures and so we shall develop the language of odd Poisson geometry to describe these objects. With this language we are able to prove that any Poisson structure on M can be lifted canonically to the algebra of densities. This will allow us to classify the space of all Poisson structures.

6.1.1 Even Poisson structures.

Let us firstly recall the definition of a Poisson structure on a smooth manifold:

Definition 6.1.1. An even *Poisson structure* on a smooth manifold M is a bracket, $\{-,-\} : C^\infty_M \otimes C^\infty_M \to C^\infty_M$ such that the following conditions are obeyed $\forall f, g, h \in C^\infty_M$:

- Antisymmetry: $\{f, g\} = (-1)^{(\tilde{f}+1)(\tilde{g}+1)}\{g, f\}$,
- Jacobi identity: $\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{\tilde{f}\tilde{g}}\{g, \{f, h\}\}$,
- Derivation property: $\{f, gh\} = \{f, g\}h + (-1)^{\tilde{f}\tilde{g}}g\{f, h\}$.

Take $H \in C^\infty(M)$, this then defines a vector field, X_H, by the formula $f \mapsto \{H, f\}$. This defines a map $C^\infty(M) \to \mathfrak{X}(M)$ and moreover this is a morphism of Lie algebras (the Poisson structure induces a Lie algebra structure on $C^\infty(M)$ using the first two properties in the above definition). An element in the kernel of the above map is called a *Casimir function.* Clearly $\mathbb{C} \subset \mathrm{Ker}(C^\infty(M) \to \mathfrak{X}(M))$ so we will only be interested in non-constant Casimir functions.

Fix a function $H \in C^\infty(M)^{even}$, which we shall call a Hamiltonian. We can then define the differential equation for all functions $f \in C^\infty(M)$:

$$\frac{df}{dt} = X_H(f).$$

In particular we can do this locally for the coordinate functions on the manifold M: $\dot{x}^i = X^i_H$. The solutions of these equations will be called integral curves (we of course require certain compactness conditions for these curves to exist for all times). Given any function $g \in C^\infty(M)$ such that $X_H(g) = 0$ the value of g on any integral curve is constant. In particular we find that all Casimir functions and H itself are constant on such curves. We shall now give some examples of such structures.

Example 6.1.2. Let (M, ω) be a symplectic manifold and then define a bracket on functions by $\{f, g\} := \omega^{-1}(df \otimes dg)$. One can check that the Jacobi identity is equivalent to the closedness of the form ω and hence defines a Poisson structure. By Darboux's theorem, [20], we can find a coordinate patch around any point with coordinate functions $(x^1, \dots, x^n, p_1, \dots, p_n)$ such that the bracket has the form:

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i}.$$

For a given Hamiltonian H we get that the integral curves are given by the classical equations of motion:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{H}{\partial x^i}.$$

Example 6.1.3. We shall now twist the above example when our symplectic manifold is T^*M and we fix a closed 2-form $B \in \Omega^2(M)$. We then define the Poisson bracket in the standard coordinates by:

$$\{p_i, x^j\} = \delta_i^j, \quad \{x^i, x^j\} = 0, \quad \{p_i, p_j\} = eB_{ij}.$$

This corresponds to a manifold with an electro-magnetic tensor, and the closedness of B corresponds to the Jacobi-identity. Locally we can find a 1-form $A = A_i dx^i$ such that $dA = B$, and we then find that we get the local canonical coordinates, $P_i = p_i - eA_i$, $Q^i = x^i$, in the sense that $\{P_i, Q^j\} = \delta_i^j$, whilst all other brackets between coordinates are zero.

Example 6.1.4. Let L be a Lie algebra with structure constants given as $[e_i, e_j] = C_{ij}^k e_k$, check appendix A for sign conventions. We can then endow the space L^* with a natural Poisson structure coming from this bracket and the fact that $C^\infty(L^*)$ is essentially defined by its subalgebra $S(L)$. The bracket is defined as:

$$\{f, g\} = (-1)^{\tilde{i}} C_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Any Poisson structure is completely determined by its Poisson tensor, $\pi^{ij} x_i^* x_j^* \in \Gamma(M; TM^{\wedge 2})$ as it is an anti-symmetric derivation in each of its variables. Below we consider such a section as a function on ΠT^*M which then allows us to generalise the notion.

6.1.2 Odd Poisson geometry.

We now use the language of odd symplectic geometry to generalise the notion of a Poisson structure to include a wider variety of objects, P_∞-structures, that include Q-manifolds. We then introduce the Batalin-Vilkovisky operator which will be of prime importance to us.

The Schouten-Nijenhuis bracket.

We saw above that any Poisson structure gives an anti-symmetric multivector field of degree 2. This tells us that the natural manifold for us to look at is ΠT^*M. One of the fundamental properties of this space is that it has the Schouten-Nijenhuis bracket, see [25, 26, 75]:

Proposition 6.1.5. *There is a canonical odd operator,*

$$(\, , \,) : C^\infty(\Pi T^*M) \otimes C^\infty(\Pi T^*M) \to C^\infty(\Pi T^*M),$$

such that the following conditions hold $\forall f, g, h \in C^\infty(\Pi T^*M)$*:*

- *Symmetry:* $(f,g) = (-1)^{\widetilde{f}\widetilde{g}}(g,f)$*,*
- *Jacobi identity:* $(f,(g,h)) = ((f,g),h) + (-1)^{(\widetilde{f}+1)(\widetilde{g}+1)}(g,(f,h))$*,*
- *Derivation property:* $(f,gh) = (f,g)h + (-1)^{(\widetilde{f}+1)\widetilde{g}}g(f,h)$*.*

The local form of this operator, in coordinates induced from the base, is given by:

$$(f,g) := (-1)^{\widetilde{i}(\widetilde{f}+1)}\left(\frac{\partial f}{\partial x_i^*}\frac{\partial g}{\partial x^i} + (-1)^{\widetilde{f}}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x_i^*}\right). \tag{6.1.1}$$

This bracket is called the Schouten-Nijenhuis bracket.

Remark 6.1.6. One can give an abstract notion of an odd symplectic manifold, that is a manifold with a bracket that satisfies all the above axioms and the associated tensor is invertible. It can however be shown, using an odd Darboux theorem, that the above proposition is the archetypal example in the sense that any (smooth) odd symplectic manifold is isomorphic to ΠT^*M for some M. One can also give the notion of an odd Poisson structure where we take the above three requirements but remove the non-degeneracy condition.

As we have a vector bundle we have that the algebra $C^\infty(\Pi T^*M)$ has a graded subalgebra, those functions that are polynomial in the fibre coordinates x^* (on an ordinary manifold this is of course the whole algebra). We thus would expect to have an associated DGLA however to do this we need to alter the weight operator to $\widehat{w}' = \widehat{w} - 1 = x_i^*\partial_{x_i^*} - 1$. This then defines a differential graded Lie algebra with the standard weight and hence all the usual problems can apply (of course this is the fundamental one in the theory of deformations). We shall keep the standard weight operator $\widehat{w}$ and just deal with the fact that the bracket has weight -1 and the notion of a Maurer-Cartan element etc. are just redefined with respect to this weighting.

Generalised Poisson structures.

Definition 6.1.7. A generalised Poisson structure is an even Maurer-Cartan element in $C^\infty(\Pi T^*M)$, that is an element $S \in C^\infty(\Pi T^*M)^{even}$ such that

$$\frac{1}{2}(S,S) = 0.$$

We need to show that this definition is well defined in the sense that it covers the classical notion of a Poisson structure. Denote by $i : M \to \Pi T^*M$ the zero section of the vector bundle. Then for $S \in C^\infty(M)$ we can define a family of brackets, $l_n^S : C^\infty(M)^{\otimes n} \to C^\infty(M)$, by the formula:

$$l_n^S(f_1, \ldots, f_n) := i^*(f_1, (f_2, \cdots, (f_n, S)\cdots)).$$

As the bracket $(,)$ is odd we have that the parity of $\{\cdots\}_n^S$ is equal to $\widetilde{S} + n$, in particular if S is even it has parity n. Moreover we have in this case that, using the symmetry properties and Jacobi identity in proposition 6.1.5 and $\widetilde{S} = 0$, we find that:

$$l_n^S(f_1, \ldots, f_i, f_{i+1}, \ldots, f_n) = (-1)^{(\widetilde{f_i}+1)(\widetilde{f_{i+1}}+1)} l_n^S(f_1, \ldots, f_{i+1}, f_i, \ldots, f_n)$$

$$i = 1, \ldots, n-1.$$

$$l_n^S(f_1, \ldots, f_{n-1}, g_1 g_2) =$$

$$= l_n^S(f_1, \ldots, f_{n-1}, g_1)g_2 + (-1)^{\widetilde{g_1}(\widetilde{f_1}+\cdots+\widetilde{f_{n-1}}+n-1)} g_1 l_n^S(f_1, \ldots, f_{n-1}, g_2).$$

The one thing we haven't used so far is the fact that $(S,S) = 0$. One can check that this enforces a variety of constraints on the brackets that are equivalent to the fact that the family $\{l_n^S : n \geq 0\}$ induce an L_∞-structure on $C^\infty(M)$. Such a structure shall also be referred to as a P_∞-structure, that is an L_∞-structure that obeys the strong condition of being a derivation in each of its variables. Note that if S is quadratic in the fibre variables then the only non-zero bracket is l_2^S and the fact that these structures are L_∞ is equivalent to saying that l_2^S is a Poisson structure. The fundamental result is the following well known theorem which we shall state in the most general form possible:

Theorem 6.1.8. *Let $\{l_n : n \geq 0\}$ be a P_∞-structure on $C^\infty(M)$. Then there exists a function $S \in C^\infty(\Pi T^*M)^{even}$ such that $(S,S) \equiv 0 \, mod \, (x^*)^\infty$, and $l_n^S = l_n$.*

Remark 6.1.9. In general the S in the above theorem is neither unique nor polynomial unless our original manifold is even. The reason is because the brackets are induced by Taylor series of the function S about the zero section and hence if the fibre coordinates are even they could all be zero with $S \neq 0$. The only 'honest' functions (see below) we can consider are smooth as opposed to polynomial or analytic as arbitrarily high order l_n may be non zero with no convergence condition.

Let us give some explicit examples of how these two situations may arise. For non-uniqueness consider $\mathbb{R}^{0|1}$ with coordinate ξ, and anti momentum ξ^*. Then the smooth function $S = e^{-\frac{1}{(\xi^*)^2}}$ trivially satisfies the Maurer-Cartan equation. However $l_n^S = 0 \; \forall S$, so the choice of S is not unique.

Now consider the brackets $\{l_n\}$ on $\mathbb{R}^{0|1}$ defined as

$$l_n(\xi, \ldots, \xi) = \begin{cases} n! & \text{for } n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

These generate a P_∞-structure however it is clear from the Taylor series that one requires a smooth, rather than analytic, function to generate these brackets.

The theorem states that any P_∞-structure can be generated by a function that is a generalised Poisson structure up to some smoothing functions. Note that in the case that we have only a finite number of brackets or that the base manifold is even then we can choose the S so that it is a genuine generalised Poisson structure. We shall usually drop reference to the $O((x^*)^\infty)$ term in (S, S) - note that if one uses the formal or pro space of functions on ΠT^*M, $\mathcal{O}(\Pi T^*M) := \prod \mathfrak{A}^n(M)$, then we do not have to worry about the non-convergence of Taylor coefficients. We thus have the algebraic version of the above theorem:

Corollary 6.1.10. *There exists a 1-1 correspondence between P_∞ structures on M and Maurer-Cartan elements of $\mathcal{O}(\Pi T^*M)$.*

Example 6.1.11. Any Poisson structure naturally comes from a generalised Poisson structure. In fact in the end of the first part of this section we showed that any Poisson structure naturally gives an element $\pi \in \mathfrak{A}^2(M) \subset C^\infty(M)$. The theorem then states in this case that the space of all functions $\pi \in \mathfrak{A}^2(M)$ such that $(\pi, \pi) = 0$ are equivalent to Poisson structures on M using the derived bracket formalism.

Example 6.1.12. Another very important class of generalised brackets is those coming form functions that are homogeneous of weight 1 in the fibre coordinates, that is $\mathfrak{A}^1(M)$. There is an isomorphism $\mathfrak{X}(M) \cong \mathfrak{A}^1(M)$, however the isomorphism is odd in the sense that it sends even vector fields to odd functions on ΠT^*M and vice versa. Therefore a generalised Poisson structure in this space is, by definition, even and hence corresponds to an odd vector field on the base. The Schouten-Nijenhuis is, and was originally defined to be, an extension of the standard commutator of vector fields on a manifold, and hence we have that for $X, Y \in \mathfrak{X}(M)$, if we let $\Pi X, \Pi Y \in \mathfrak{A}^1(M)$ denote their image then

$$(\Pi X, \Pi Y) = (-1)^{\widetilde{X}} \Pi[X, Y].$$

Therefore if $Q \in \mathfrak{X}(M)$ the condition that ΠQ defines a generalised Poisson structure is equivalent to $\frac{1}{2}[Q, Q] = Q^2 = 0$. Therefore Q is a homological vector field on M, and so Q-manifolds can be interpreted in the language of Poisson structures.

Example 6.1.13. Let us take an element $S = Q^i x_i^* + \frac{1}{2}\pi^{ji} x_i^* x_j^*$, and assume that it is a Poisson structure. Now recalling that $(,)$ reduces fibre weight by 1 we see that both the terms (Q, Q), and (π, π) must vanish identically in this expression. Thus such an S naturally endows the space M with an odd homological vector field and a Poisson bracket. We also require that $(Q, \pi) = 0$ and this is equivalent to the fact that Q derivates the bracket induced by π as can easily be checked:

$$Q\{f, g\} = (Q, ((\pi, f), g)) = ((Q, (\pi, f)), g) + (-1)^{\tilde{f}}((\pi, f), (Q, g))$$

$$= (((Q, \pi), f), g) + ((\pi, (Q, f)), g) = \{Qf, g\} + (-1)^{\tilde{f}}\{f, Qg\}.$$

Therefore we have that this corresponds to a differential Lie algebra structure on the base manifold (with the additional constraint that the bracket is a derivation).

Similarly to the case of an even Poisson structure we have that the Schouten-Nijenhuis bracket induces a map, $C^\infty(\Pi T^*M) \to \mathfrak{X}(\Pi T^*M)$, however in this case the map is odd. We denote by d_S the image of a function under this map. It follows, using the properties of the Schouten bracket, that this induces a morphism of Lie algebras in the following sense[1]:

$$[d_f, d_g] = d_{(f,g)}.$$

[1] We must be careful about exactly what we mean as the Schouten-Nijenhuis bracket is odd and hence doesn't induce a Lie algebra structure on the space of functions but rather an odd Lie algebra.

The requirement that S be a generalised Poisson structure then becomes the condition that $d_S^2 = \frac{1}{2}[d_S, d_S] = 0$, that is $(\Pi T^*M, d_S)$ is a Q-manifold.

6.1.3 Batalin-Vilkovisky geometry.

The core part of the proof of the main result in this section will rely not on the Schouten-Nijenhuis bracket but on the divergence operator, also known as the Batalin-Vilkovisky operator or Khudaverdian's operator. In this section we introduce this operator and develop the basic properties that we need.

The divergence operator.

Recall that $\mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M) \cong C^\infty(\Pi T^*M) \otimes_{C^\infty(M)} \mathcal{F}^\lambda(M)$, see [45]. The divergence operator is defined on half-densities on the odd cotangent bundle, or on functions on ΠT^*M with values in densities on M using the above identification. We make the following proposition:

Proposition 6.1.14. *There is a canonical odd second order differential operator on $\mathcal{F}^{\frac{1}{2}}(\Pi T^*M)$, Δ, such that:*

- $\Delta^2 = 0$,
- $\Delta = \Delta^\dagger$.

This differential operator can be called either the divergence operator, the Batalin-Vilkovisky operator or Khudaverdian's operator. The local form of this operator is:

$$\Delta \mathbf{s} = (-1)^{\tilde{i}} \frac{\partial^2 s(x, x^*)}{\partial x_i^* \partial x^i} \sqrt{|D(x, x^*)|}.$$

The divergence operator determines the Schouten-Nijenhuis bracket in the following sense: As Δ is a second order operator on the module $\mathcal{F}^{\frac{1}{2}}(\Pi T^*M)$ we get a bracket, $(\,,\,)_\Delta : C^\infty(\Pi T^*M) \otimes C^\infty(\Pi T^*M) \to \mathrm{End}_{C^\infty(\Pi T^*M)}(\mathcal{F}^{\frac{1}{2}}(M)) \cong C^\infty(\Pi T^*M)$. A simple calculation in local coordinates shows that this is indeed the Schouten-Nijenhuis bracket.

The fact that the operator is naturally defined on $\mathcal{F}^{\frac{1}{2}}(\Pi T^*M)$ is very important, both in the Batalin-Vilkovisky formalism in quantum mechanics and also for a geometrical understanding of this operator. Let us fix an even element $\varrho \in \mathcal{F}^{\frac{1}{2}}(\Pi T^*M)^\times$

such that $\Delta\varrho = 0$. This then defines a differential operator on *functions* on the odd cotangent space, $\Delta_\varrho = \varrho^{-1} \circ \Delta \circ \varrho : C^\infty(\Pi T^*M) \to C^\infty(\Pi T^*M)$, such that $\Delta_\varrho(1) = 0$ and $\Delta_\varrho^2 = 0$. We need some algebraic properties of such algebras.

Batalin-Vilkovisky algebras.

Definition 6.1.15. A (non-graded) Batalin-Vilkovisky algebra is a triple (A, d, D), where A is an algebra, d is an odd differential on A, and D is an odd second order differential operator such that $d^2 = D^2 = [d, D] = D(1) = 0$.

In the graded case, i.e. when A is a graded algebra, we additionally require that d has weight $+1$ and D has weight -1.

Batalin-Vilkovisky algebras are closely related to Gerstenhaber algebras, see [31, 32, 48, 54]. A choice of nowhere vanishing half-density that is also in the kernel of Δ gives $C^\infty(\Pi T^*M)$ the structure of a Batalin-Vilkovisky algebra.

Lemma 6.1.16. *Let (A, d, D) be a Batalin-Vilkovisky algebra. Then D generates an odd Poisson structure on A, $(\ ,\ ) : A \otimes A \to A$, and moreover d and D derivates this bracket:*

$$d(a, b) = -(da, b) - (-1)^{\tilde{a}}(a, db), \qquad D(a, b) = -(Da, b) - (-1)^{\tilde{a}}(a, Db).$$

Proof. We define the bracket as $(a, b) := [[D, a], b]$. To check that this does indeed give the structure of an odd Poisson structure is an exercise in algebraic manipulation:

1. Symmetry: $(a, b) = [[D, a], b] = [D, [a, b]] + (-1)^{\tilde{a}\tilde{b}}[[D, b], a]$.

2. Jacobi identity: $(a, (b, c)) = [[D, a], [[D, b], c]] =$

$$[[[D, a], [D, b]], c] + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}[[D, b], [[D, a], c]] =$$

$$[[[D, a], [D, b]], c] + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}(b, (a, c)).$$

   We also have that $[[[D, a], [D, b]], c] = [[[[D, a], D], b], c] + ((a, b), c)$. So we are left to show that $[[[[D, a], D], b], c] = 0 \ \forall a, b, c$. We can do this again from simple algebraic manipulations (recall that for D we have $D^2 = \frac{1}{2}[D, D] = 0$):

$$[[[[D, a], D], b], c] = [[[D, [a, D]], b], c] + (-1)^{\tilde{a}}[[[[D, D], a], b], c] =$$

$$-(-1)^{\tilde{a}}[[[D, [D, a]], b], c] = -[[[[D, a], D], b], c].$$

3. Derivation property: The last thing we need to check is that this bracket is a derivation in each variable which follows as it is the symbol of a second order operator.

□

Corollary 6.1.17. *Let us take* $\varrho \in \mathcal{F}^{\frac{1}{2}}(\Pi T^*M)^\times \cap Ker(\Delta)$. *Then the operator* Δ_ϱ *derivates the Schouten-Nijenhuis bracket.*

In particularly we can apply the above corollary locally with the half -density $\sqrt{|D(x,x^*)|}$. We shall denote this local operator by Δ_0. We then have the equation which appeared in the original works of Batalin and Vilkovisky, see [4]:

$$\Delta_0(f,g) = -(\Delta_0 f, g) - (-1)^{\tilde{f}}(f, \Delta_0 g). \tag{6.1.2}$$

6.1.4 Lifting Poisson structures.

In this section we shall together results from the last section as well as some from the introduction to find a canonical odd Laplacian on the space $\Pi T^*\widehat{M}$; in the terminology of the previous section it means that $C^\infty(\Pi T^*\widehat{M})$ has the natural structure of a Batalin-Vilkovisky algebra. We will then show that this operator allows us to lift nearly all weighted multi-vector fields. We then show that this lifting in fact preserves the space of (generalised) Poisson structures, and hence allows a complete classification of weight zero Poisson structures on the algebra of densities.

Projections of multi-vector fields.

Before we proceed to the proof we study bundles which allow for the question of lifting Poisson structures to be developed. Let $U \to M$ be a fibre bundle so we get the associated sequence of vector bundles:

$$VU \hookrightarrow TU \twoheadrightarrow U \times_M TM,$$

where VU denotes the space of vertical tangent vectors. We take the dual of this sequence to get the injection, $i : U \times_M T^*M \hookrightarrow T^*U$, and then taking the odd space of these bundles we get a natural map:

$$i : U \times_M \Pi T^*M \to \Pi T^*U.$$

We have the map of algebras $i^*: C^\infty(\Pi T^*U) \to C^\infty(U \times_M \Pi T^*M) \cong C^\infty(U) \otimes_{C^\infty(M)} C^\infty(\Pi T^*M)$ (for the completed tensor product), which is the map defined as above, and for smooth spaces is surjective using a partition of unity argument.

Remark 6.1.18. If we wish to find a splitting of this sequence as algebras, that is a cosection of the inclusion of spaces, $U \times_M \Pi T^*M \hookrightarrow \Pi T^*U$, we require a connection. However if we require a section of the projection of algebras $i^*: C^\infty(\Pi T^*U) \twoheadrightarrow C^\infty(U) \otimes_{C^\infty(M)} C^\infty(\Pi T^*M)$, then there are many more possible such maps, and some are natural as we shall see below.

Lemma 6.1.19. *The map $i^*: \Pi T^*U \to U \times_M \Pi T^*M$ preserves weight in the fibre coordinates. In particular we have that i^* induces a map $i^*: \mathfrak{A}^k(U) \to C^\infty(U) \otimes \mathfrak{A}^k(M)$.*

Proof. The map is induced from a map of vector bundles and any such map necessarily must preserve the weight of fibre coordinates. □

One thing that may at first appear a problem is the fact that the map i^* is not a morphism of Lie algebras. The next proposition, which shall be used by us, shows that this is to be expected. Suppose that the fibre bundle $U \to M$ is graded in the following sense: There exists an even vector field $\widehat{w}$ on U, such that $C^\infty(M) \subseteq \operatorname{Ker}(\widehat{w})$. We then have that $\widehat{w}$ corresponds to an odd function w on ΠT^*U. We may split the functions on ΠT^*U into graded components corresponding to this weight operator, $\widehat{w}_1 := d_w$, keeping consistency with the above notation. Moreover the weight operator $\widehat{w}$ also induces a weight operator on $U \times_M \Pi T^*M$, where we use the formula $\widehat{w}(f \otimes s) = \widehat{w}(f) \otimes s$, which is well defined as $C^\infty(M) \subseteq \operatorname{Ker}(\widehat{w})$.

Proposition 6.1.20. *Let $U \to M$ be a graded bundle in the above sense. Then the map i^* preserves weight in the sense that $i^*(C^\infty(\Pi T^*U)^\lambda) \subset C^\infty(U)^\lambda \otimes C^\infty(\Pi T^*M)$.*

Proof. Let us take an element $F \in C^\infty(\Pi T^*U)^\lambda$, such that $(w, F) = \lambda F$. We then apply Taylor's formula to F around the zero section in y^* to get that:

$$F(x, y; x^*, y^*) = F(x, y; x^*, 0) + y^*_a R^a_F(x, y; x^*, y^*).$$

Next we consider w which is a function of the form $w^a(x, y) y^*_a$, in particular in the kernel of i^*. The result will then follow if we can show that, in this local expansion, $F(x, y; x^*, 0)$ itself has weight λ. Applying d_w to this equation we get:

$$\lambda F(x, y; x^*, y^*) = \widehat{w}(F(x, y; y^*, 0)) + (w, y^*_a R^a_F(x, y; x^*, y^*)) \Rightarrow$$

$$(\lambda - \widehat{w})\, F(x, y; x^*, 0) = (d_w - \lambda)\, y_a^* R_F^a(x, y; x^*, y^*).$$

The left hand side of the above equation does not depend on y^* whilst that on the right either does or is identically zero. In either situation we can conclude that both terms must be zero and hence the result follows. □

In particular the above proposition holds in the case of the bundle $\widehat{M} \to M$. We shall denote by i_λ^* the restriction of the map i^* to $C^\infty(\Pi T^* U)^\lambda$. Note that the Schouten-Nijenhuis bracket has weight zero in the sense that $\widehat{w}_1(F, G) = (\widehat{w}_1 F, G) + (F, \widehat{w}_1 G)$, which follows from the fact that $\widehat{w}_1$ is derived from an odd function and the Jacobi identity. Therefore we have that $C^\infty(\Pi T^* U)^0$ is a sub Lie algebra of $C^\infty(\Pi T^* U)$. This motivates the following result in the case when we have a far stronger condition that $\mathrm{Ker}(\widehat{w}) = C^\infty(M)$:

Proposition 6.1.21. *Let $U \to M$ be a graded bundle as above with the additional condition that $Ker(\widehat{w}) = C^\infty(M)$. Then the map $i_0^* : C^\infty(\Pi T^* U)^0 \to C^\infty(\Pi T^* M)$ is a morphism of Lie algebras.*

Proof. Let us take two functions $F, G \in C^\infty(\Pi T^* U)^0$, we may then Taylor expand both functions and, using proposition 6.1.20, we have that $F(x, y; y^*, 0) = f(x, x^*)$, and the same for G, as it must be in the kernel of $\widehat{w}_1$. We then calculate the bracket of these two functions

$$(F, G) = (f + y_a^* R_F^a, g + y_b^* R_G^b) = (f, g) + (f, y_a^* R_G^a) + (y_a^* R_F^a, g) + (y_a^* R_F^a, y_b^* R_G^b).$$

We have that $(i^* F, i^* G) = (f, g)$, $(y_a^* R_F^a, y_b^* R_G^b) \in \mathrm{Ker}(i^*)$ simply because the Schouten-Nijenhuis bracket has fibre weight -1. Therefore the statement holds if we show that $(f, y_a^* R_G^a) = (y_a^* R_F^a, g) = 0$. This follows because f does not depend on y so the y^* term is not differentiated in either of these terms and hence comes out of the bracket and is thus killed by i^*. □

Corollary 6.1.22. *Let $U \to M$ satisfy the conditions of proposition 6.1.21 and $S \in C^\infty(\Pi T^* M)$ be a generalised Poisson structure that is of weight 0. Then S induces a Poisson structure on the base manifold M, $i^* S$.*

Now all the above applies to the case $\widehat{M} \to M$, which is the case we now focus on more explicitly. It is a natural question to ask in the above situation when we can lift a Poisson structure from the base and what the fibre of each Poisson structure is.

Lemma 6.1.23. *Consider the map* $i^*_\lambda : C^\infty(\Pi T^*\widehat{M})^\lambda \to C^\infty(\Pi T^*M) \otimes \mathcal{F}^\lambda(M)$. *Then* $Ker(i^*_\lambda) \cong C^\infty(\Pi T^*M) \otimes \mathcal{F}^\lambda(M)$.

Proof. Let us find the local form of an element in $C^\infty(\Pi T^*\widehat{M})^\lambda$: the weight operator $\widehat{w}_1$ in this case has the form $\widehat{w}_1 = t\partial_t - t^*\partial_{t^*}$, which is the derived vector field of the odd function $w = tt^*$. Now as t^* is odd we may locally expand any function on $\Pi T^*\widehat{M}$ as

$$F(x, t; x^*, t^*) = \sum_\lambda f_\lambda(x, x^*)t^\lambda + g_\lambda(x, x^*)t^\lambda t^*,$$

where $\widetilde{f_\lambda} = \widetilde{g_\lambda} + 1 = \widetilde{F}$. Now applying $\widehat{w}_1$ we see that F has weight λ iff $F = f(x, x^*)t^\lambda + g(x, x^*)t^{\lambda+1}t^*$. If $i^*_\lambda F = 0$ we then have that $f = 0$ in the expansion above. We now claim that the map $\mathrm{Ker}(i^*_\lambda) \to C^\infty(\Pi T^*M) \otimes \mathcal{F}^\lambda(M)$, given by $F \mapsto g$ is firstly well defined and an isomorphism. Under a general coordinate transformation we have that $x^* \mapsto \overline{x}^* + J^{-1}\partial J\overline{t}^*$, and thus we may expand the transformed g around $\overline{x}^* = 0$. Then the t^* term from this Taylor expansion is killed by the external t^* hence the transformation of g is as expected. □

The canonical Laplacian.

Recall from proposition 2.1.15 that the space $\widehat{M}$ has a natural volume form given in local coordinates by:

$$\varrho_0 = \frac{1}{t^2}|D(x, t)|,$$

which corresponds to an invertible half-density on $\Pi T^*\widehat{M}$. As this half-density only depends on coordinates form the base it is clear that $\Delta(\varrho_0) = 0$. We can apply corollary 6.1.17 and we get a canonical Batalin-Vilkovisky structure on $C^\infty(\Pi T^*\widehat{M})$. We shall let δ denote this operator, that is

$$\delta := \varrho_0^{-1} \circ \Delta \circ \varrho_0. \tag{6.1.3}$$

In local coordinates we can compute this operator and it is given by the formula:

$$\delta = \Delta_0 + \partial_{t^*}(\partial_t - \frac{2}{t}) = (-1)^{\widetilde{i}}\partial^2_{x^i x^*_i} + \partial_{t^*}(\partial_t - \frac{2}{t}). \tag{6.1.4}$$

Lemma 6.1.24. *1. The operator* δ *has weight zero with respect to* $\widehat{w}_1$*, that is* $[\widehat{w}_1, \delta] = 0$.

2. *The kernel of δ at a fixed $\widehat{w}_1$ weight λ such that $\lambda \neq 1$, is given as:*

$$Ker(\delta|_{\widehat{w}_1=\lambda\neq 1}) \cong C^\infty(\Pi T^*M) \otimes_{C^\infty(M)} \mathcal{F}^\lambda(M) \cong \mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M).$$

Proof. 1. $\delta\widehat{w}_2(s) = \delta(w,s) = -(\delta w, s) + (w, \delta s)$, hence we just need to check that $(\delta(w), -) = 0$, which follows immediately in local coordinates.

2. A general element in $C^\infty(\Pi T^*\widehat{M})$ of weight λ has the local form $F = t^\lambda(f(x,x^*) + tt^*g(x,x^*))$. The equation $\delta F = 0$ is locally equivalent to the two equations:

$$\Delta_0 f + (\lambda - 1)g = 0$$

$$\Delta_0 g = 0.$$

We see that if $\lambda \neq 1$ there is a unique choice of g that satisfies these equations, that is $g = \frac{1}{1-\lambda}\Delta_0 f$. We thus see that any such f determines a unique solution to this equation. We have that $t^\lambda f(x,x^*)$ is exactly the image of F under $i^*_\lambda : C^\infty(\Pi T^*M)^\lambda \twoheadrightarrow C^\infty(\Pi T^*M) \otimes_{C^\infty(M)} \mathcal{F}^\lambda(M)$, hence the result.

□

Remark 6.1.25. This lemma and the following theorem bear a close resemblance to the existence of the Frölicher-Nijenhuis bracket. Take a vector field on ΠTM, $X = X^i\partial_{x^i} + Y^i\partial_{dx^i}$, then there exists a unique Y^i, determined from $X^i\partial_i \in \Omega(M) \otimes \mathfrak{X}(M)$, such that $[d, X] = 0$. The proof follows in a coordinate frame:

$$[d, X] = (dX^i - (-1)^{\widetilde{X}} Y^i)\partial_{x^i} + dY^i\partial_{dx^i},$$

hence the condition of commuting with d is equivalent to $Y^i = (-1)^{\widetilde{X}} dX^i$. One then defines the Frölicher-Nijenhuis bracket of two differential forms with values in vector fields by lifting them to the unique vector field on ΠTM in the kernel of $\mathrm{ad}(d)$, taking their commutator and then taking the term corresponding to a vector field valued form, see [25, 26]. We shall follow a similar course of action in defining our lifts below.

Example 6.1.26. Although the lift does not hold naturally for $\mathcal{F}^{\frac{1}{2}}(\Pi T^*M)$, we do have the following: Consider a nowhere vanishing density of weight $\frac{1}{2}$ on ΠT^*M, $\varrho \in \mathcal{F}^{\frac{1}{2}}(\Pi T^*M)^\times$. We may then define $\varrho_0 \in \mathcal{F}^1(M)$ given locally by the formula

$$\varrho_0(x) = \varrho(x, 0).$$

It follows that there exists a unique function $S(x, x^*) \in C^\infty(\Pi T^* M)$ such that

$$\varrho(x, x^*) = e^{\frac{i}{\hbar}S}\varrho_0(x),$$

we include $\hbar$ here to associate the above with the standard formalism in the literature, [4, 5, 6]. We know the structure of a general element in $C^\infty(\Pi T^* \widehat{M})$ and thus we can use the connection, $\gamma_a := -\varrho_0^{-1}\partial_a \varrho_0$, to lift the density to a weight 1 element in $C^\infty(\Pi T^* \widehat{M})$:

$$\widehat{\varrho} := (1 + \frac{itt^*}{\hbar}\gamma_a \partial^a S)e^{\frac{i}{\hbar}S}\varrho_0 t.$$

Lifting Poisson structures and their classification.

We can now prove the main theorem we need concerning lifting generalised Poisson structures on manifolds:

Theorem 6.1.27. *Let $S \in C^\infty(\Pi T^* M)^{even}$ be a generalised Poisson structure. There exists a canonical lifting of S to $C^\infty(\Pi T^* \widehat{M})^{even}$, $\widehat{S}$, such that $\widehat{S}$ is a generalised Poisson structure. In other words there exists a natural splitting of the following sequence of differential graded Lie algebras:*

$$C^\infty(\Pi T^* \widehat{M})^0 \twoheadrightarrow C^\infty(\Pi T^* M).$$

The local formula of this map is

$$\widehat{S}(x, x^*; t, t^*) = S(x, x^*) + tt^* \Delta_0 S(x, x^*). \tag{6.1.5}$$

Proof. Using lemma 6.1.24 we see that for the map $i_0^* : C^\infty(\Pi T^* \widehat{M})^0 \twoheadrightarrow C^\infty(\Pi T^* M)$, and some fixed element $f \in C^\infty(\Pi T^* M)$, there exists a unique element $\widehat{f} \in p^{-1}(f)$, such that $\delta \widehat{f} = 0$. Thus the result will follow if we show that for this map $(f, g)^\wedge = (\widehat{f}, \widehat{g})$. We have that the first of these $(f, g)^\wedge$ is equal to $(f, g) + t^* t \Delta_0 (f, g)$ by definition, whilst the second is equal to

$$(\widehat{f}, \widehat{g}) = (f, g) - tt^* \left((\Delta_0 f, g) + (-1)^{\tilde{f}}(f, \Delta_0 g) \right).$$

Thus equality will hold iff we have that $\Delta_0(f, g) = -(\Delta_0 f, g) - (-1)^{\tilde{f}}(f, \Delta_0 g)$. This equation is none other than the derivation property of the Batalin-Vilkovisky operator, equation (6.1.2), and hence the statement holds. □

Using this theorem we can now classify all Poisson structures on $\widehat{M}$ that lie above a fixed Poisson structure on the manifold M.

Theorem 6.1.28. *Fix $S_0 \in C^\infty(\Pi T^*M)^{even}$ such that $(S_0, S_0) = 0$. Then $\mathcal{P}(S_0) = \{$Poisson structures on $\widehat{M}$ that project onto $S_0\}$ is isomorphic to $Ker(d_{S_0})^{odd}$, that is the odd elements in the kernel of $(S_0 \cdot)$.*

Proof. Let us take an arbitrary Poisson structure S that lies above S_0. We then have that $S - \widehat{S_0}$, where $\wedge$ denotes the lift in theorem 6.1.27, lies in the kernel of i_0^*. Hence using lemma 6.1.23 we have that there exists $Q \in C^\infty(\Pi T^*M)^{odd}$ such that $S = \widehat{S_0} + Qtt^*$. We now calculate $(S, S) = 0$ in this presentation. We find that:

$$(S, S) = (\widehat{S_0}, \widehat{S_0}) + 2(\widehat{S_0}, Q)tt^* = 2(S_0, Q)tt^* = 0,$$

where the last equality follows from using local coordinates whilst the first and second can be derived from the algebraic properties of $\wedge$ and $(\ ,\ )$. Therefore given any $S \in \mathcal{P}(S_0)$ and $(S, S) = 0$ we have associated an element in $\mathrm{Ker}(d_{S_0})^{odd}$. For the converse pick $Q \in \mathrm{Ker}(d_{S_0})^{odd}$, then consider the function $\widehat{S} + Qtt^*$, which is well defined from lemma 6.1.23 and satisfies the Maurer-Cartan equation and hence is an element of $\mathcal{P}(S_0)$ completing the proof. □

Corollary 6.1.29. *Let $S \in \mathfrak{A}^k(M)$ be a generalised Poisson structure. Then the set of Poisson structures in $\mathfrak{A}^k(\widehat{M})$ that lie over S is isomorphic to $Ker(d_S) \cap \mathfrak{A}^{k-1}(M)^{odd}$.*

Example 6.1.30. The simplest case is to consider those manifolds which are endowed with a canonical volume form. So consider ΠTM which has the volume form $|D(x, dx)|$, see example 2.1.1. This space also has a canonical odd differential operator, d, the de Rham differential. The function corresponding to d on $\Pi T^*\Pi TM$ is $dx^\mu x^*_\mu$. This function is clearly in the kernel of $\Delta_0 = \Delta_{|D(x,dx)|}$ and thus the canonical lift of d is just d itself. Using corollary 6.1.29 we can easily calculate the space of homological vector fields lying over d. It is simply isomorphic to the set of odd closed differential forms. Restricting this differential to densities of weight 1 we get the well known formula that for a closed odd form θ, $d + \theta$ is a differential, the so called twisted de Rham differential.

Example 6.1.31. In this example we explore the case of a (super) Lie algebra $\mathfrak{g}$. The Lie algebra structure on $\mathfrak{g}$ induces a Poisson structure on $\mathfrak{g}^*$ and thus on the space of densities on $\mathfrak{g}^*$. It is more natural to consider the algebra that is polynomial in the variable t rather than letting it be invertible. Then the space associated to the algebra of densities, $\widehat{\mathfrak{g}}^*$, is isomorphic to $\mathfrak{g}^* \times \mathbb{K}$. This induces a Lie algebra structure on $\widehat{\mathfrak{g}}$, as the Poisson bracket is of the correct weight, and therefore we have the following sequence of Lie algebras

$$0 \longrightarrow \mathfrak{g} \longrightarrow \widehat{\mathfrak{g}}$$

This can be extended to a sequence of $\mathfrak{g}$-modules by taking the quotient of the extended Lie algebra by $\mathfrak{g}$. The final term in the sequence is a one dimensional vector space, and hence isomorphic $\mathbb{K}$. Recall that extensions of Lie algebras, as above, is equivalent to an action of $\mathfrak{g}$ on $\mathbb{K}$, we now determine this action. As everything is canonical one would expect this action to be as well and that is the case.

Let us pick a basis on $\mathfrak{g}$, say $\{x^i\}$ with structure constants $C^{ij}{}_k$. This basis then allows to make an identification $\widehat{\mathfrak{g}}^* \cong \mathfrak{g}^* \times \mathbb{K}$. The Poisson bracket and its lift have the form

$$\pi = \frac{1}{2} C^{ij}{}_k x^k x_j^* x_i^*$$

$$\widehat{\pi} = \frac{1}{2} C^{ij}{}_k x^k x_j^* x_i^* + tt^* (-1)^{\widetilde{i}} C^{ji}{}_i x_j^*$$

Thus the action on $\mathbb{K}$ has the from, $\mathfrak{g} \to \mathfrak{gl}(1)$, given by $x \mapsto \mathrm{str}(x)$.

Example 6.1.32. Let us take the standard symplectic structure on $\mathbb{R}^{2n}$, $\omega = p_*^i x_i^*$, then $\widetilde{\omega} = 0$. A vector field on X is symplectic iff $(\omega, \Pi X) = 0$. This is the case if $X^i{}_{,j} = -X_j{}^{,i}$, $X^{i,j} = X^{j,i}$, and $X_{i,j} = X_{j,i}$. From the above proposition we see that the space of lifts of ω is classified exactly by the space of symplectic vector fields. Let us take a symplectic vector field X, we then have the lift $\Omega_X = \widehat{\omega} + tt^* X^i x_i^*$. The extra term in the bracket is given by:

$$\{H, t\} = ((\Omega_X, H), t) = -X(H)t.$$

On $\mathbb{R}^{2n}$ every symplectic vector field is Hamiltonian and hence there exists a function f such that $X = X_f$ and hence we may rewrite the above as $\{H, t\} = \{H, f\}|Dx|$. Let us consider in detail how this defines a time evolution of λ-densities. Fix a Hamiltonian H on M and take $\Psi = \psi t^\lambda$. We find that the time evolution of this density is given by

$$\frac{d}{d\tau}\Psi(\tau) = ((\Omega_X, H), \Psi) = (\{H, \psi\} + \lambda\{H, f\}\psi)\, t^\lambda = (\dot{\psi} + \lambda \dot{f} \psi) t^\lambda$$

We therefore have the following differential equations for the coordinates:

$$\dot{q}^i = \tfrac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\tfrac{\partial H}{\partial \partial q^i}$$
$$\tfrac{\dot{t}}{t} = \{H, f\} = \langle X_H, df \rangle.$$

We can clearly extend this to T^*M for an arbitrary manifold M where we replace df by an element in $Z^1_{dR}(M)$. The coordinate t in this case transforms as if it were the gauge coordinate of electromagnetism in a *flat* electromagnetic field, $F = 0$.

For a more concrete example of the above consider the simplest case where our symplectic manifold is $\mathbb{C}^\times$. We choose the 1-form

$$gd\theta = g\frac{xdy - ydx}{x^2 + y^2} = \frac{g}{2i}\frac{\overline{z}dz - zd\overline{z}}{z\overline{z}},$$

and the standard harmonic oscillator Hamiltonian, $H = \frac{1}{2}z\overline{z}$. The differential equations become:

$$\dot{x} = y$$
$$\dot{y} = -x$$
$$\tfrac{\dot{t}}{t} = -\tfrac{g}{2}$$

We thus find the solutions:

$$x(s) = \alpha \sin(s) + \beta \cos(s)$$
$$y(s) = -\beta \sin(s) + \alpha \cos(s)$$
$$t(s) = \exp(-\tfrac{gs}{2}),$$

for α and β not both zero. We see that letting s go from 0 to 1 the coordinates of the base return to their original position but t picks up an additional 'phase factor' proportional to $e^{-\frac{g}{2}}$, see the similarity with the Aharonov-Bohm effect, [2].

6.1.5 A diffeomorphism cohomology group.

To complete the first half of this chapter we study the lift defined by lemma 6.1.24 in more detail. Our goal will be to show that the obstruction to this lifting for $\lambda = 1$ comes from a cohomology class. Firstly we shall extend the Lie algebra of $\mathfrak{X}(M)$ to those of all smooth functions on ΠT^*M and then show that for $\lambda \neq 1$ the lifting is $C^\infty(\Pi T^*M)$-equivariant. We then show that the lack of a lifting at the singular point defined above has its origins in a cohomology class of $C^\infty(\Pi T^*M)$.

The action of multi-vector fields.

Consider the (odd) Lie algebra $C^\infty(\Pi T^*M)$ with the Schouten-Nijenhuis bracket. Using lemma 6.1.24 we can show that this algebra has a natural extension of its action.

Proposition 6.1.33. *The Lie algebra $C^\infty(\Pi T^*M)$ has a natural action on the space $C^\infty(\Pi T^*\widehat{M})$.*

Proof. We define the action as follows: take $f \in C^\infty(\Pi T^*M)$, then using lemma 6.1.24 we take its unique lift lying in the kernel of δ, $\widehat{f}$. We then define the actions as $f \mapsto X_{\widehat{f}}$, the vector field induced by $\widehat{f}$ from the Schouten-Nijenhuis bracket on $\widehat{M}$. We can then apply theorem 6.1.27 directly to verify that this is indeed a map of Lie algebras. □

In particular we can apply this action to any of the graded parts of the algebra as the action has weight 0. That is there is a natural $C^\infty(\Pi T^*M)$-module structure on $C^\infty(\Pi T^*\widehat{M})^\lambda$. In local coordinates it has the following expression:

$$f \cdot \big(t^\lambda u(x,x^*) + t^{\lambda+1}t^* v(x,x^*)\big) = t^\lambda\big((f,u)_M + \lambda\Delta_0(f)u\big) +$$

$$t^{\lambda+1}t^*\big((\Delta_0 f, u)_M - (-1)^{\tilde{f}}\big((f,v)_M + \lambda\Delta_0(f)v\big)\big).$$

The kernel of i^*_λ is preserved by this operation and thus we have that this action descends to one on $\mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M)$. In the cases we are looking at we are interested in the extension of these modules by themselves. The cohomology groups that will crop up shall be those of $\mathrm{End}(\mathcal{F}^{\frac{\lambda}{2}}_{\Pi T^*M}) = \mathrm{DO}(\mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M))$. The 0^{th} order cohomology groups for such modules can be easily be found:

Proposition 6.1.34. *We have the following cohomology spaces:*

$$H^0(C^\infty(\Pi T^*M); DO(\mathcal{F}^{\frac{\lambda}{2}}_{\Pi T^*M})) = \begin{cases} \mathbb{C}[\Delta] & \text{for } \lambda = 1, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

To close this section we consider the splitting of the sequences:

$$0 \longrightarrow \mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M) \longrightarrow C^\infty(\Pi T^*\widehat{M})^\lambda \underset{i^*_\lambda}{\overset{j_\lambda}{\rightleftarrows}} \mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M) \longrightarrow 0, \qquad (6.1.6)$$

for $\lambda \neq 1$ where j_λ is defined as in lemma 6.1.24.

Proposition 6.1.35. *The splitting given by lemma 6.1.23 and equation* (6.1.6) *for $\lambda \neq 1$ is a $C^\infty(\Pi T^*M)$-equivariant splitting.*

Proof. We shall denote the splitting $u \mapsto j_\lambda(u)$ simply as $s \mapsto \widehat{s}$. Our goal is to then show that for a function $f \in C^\infty(\Pi T^*M)$, $f \cdot \widehat{s} = \widehat{f \cdot s}$. We note that the action of f on an arbitrary element, ψ, in $C^\infty(\Pi T^*\widehat{M})$ can simply be described by the formula:

$$f \cdot \psi = [\delta, \widehat{f}]\psi.$$

We thus have for $\psi = \widehat{s}$ that $\widehat{f} \cdot \widehat{s} = (\widehat{f}, \widehat{s})$. We now wish to show that this function is $\widehat{f \cdot s}$. As δ derivates the Schouten-Nijenhuis bracket we have that it lies in the kernel of δ so it is the lift of some object and a calculation in local coordinates shows immediately that it is of $f \cdot s$. $\square$

The singular point.

We now turn to the singular point and show that the splitting corresponds to a non-trivial cocycle in $Z^1(\mathrm{DO}(\mathcal{F}^{\frac{1}{2}}(\Pi T^*M))$. It is useful to study the full pencil of liftings to see where the singularity originates. Recall that for a general function $S \in C^\infty(\Pi T^*\widehat{M})^\lambda$, $S = t^\lambda u(x, x^*) + t^{\lambda+1} t^* v(x, x^*)$, the term v transforms as an upper connection over $\partial^i u$. Fix a connection, Γ, then we define a splitting of the sequence in equation (6.1.6) for *arbitrary* λ. j_Γ, defined as:

$$j_\Gamma\big(u(x, x^*)|D(x, x^*)|^{\frac{\lambda}{2}}\big) = t^\lambda u(x, x^*) + t^{\lambda+1} t^* \Gamma_i \partial^i u(x, x^*).$$

We calculate the cocycle, $c_{\Gamma,f} \in \mathrm{DO}(\mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M))$, by considering $[j_\Gamma, f\cdot]$. The cocycle has the form:

$$c_{\Gamma,f} = (-1)^{\widetilde{f}}(\Delta_0 f, -) - (-1)^{\widetilde{k}}\Gamma_k(\partial^k f, -)$$
$$+\lambda\Gamma_k\big(\partial^k(\Delta_0 f) - (-1)^{\widetilde{f}}\Delta_0 f \partial^k\big) + (-1)^{\widetilde{f}}\Gamma_{k,i}\partial^i f \partial^k.$$

We now determine whether or not this cocycle is a coboundary. We denote by d_f the vector field generated by f, so that

$$d_f = (f, -) + \lambda\Delta_0(f).$$

If we take an arbitrary operator one can argue on the order of the corresponding operators that the most general possible operator that could generate the above cocycle would locally be of the form:

$$b = \mu\Delta_0 + A^i(x, x^*)\partial_i + B_i(x, x^*)\partial^i + C(x, x^*), \quad \mu \in \mathbb{C}.$$

Determining the conditions for which $c_{\Gamma,f} = [d_f, b]$ we find:

$$[d_f, b] = (-1)^{\widetilde{f}}\mu(\lambda - 1)(\Delta_0 f, -) + (f, A^i)\partial_i - (-1)^{\widetilde{A}(\widetilde{f}+1)+\widetilde{i}}A^i(\partial_i f, -)$$

$$+(f, B_j)\partial^j + (-1)^{\widetilde{B}(\widetilde{f}+1)+\widetilde{j}}B_j(\partial^j f, -) + \lambda(A + B)(\Delta_0 f) + (f, C).$$

Comparing the terms we see that we have to solve the following equations:

$$A^i = 0,$$

$$B_i = (-1)^{\widetilde{i}}\Gamma_i,$$

$$C \in \mathbb{C},$$

$$\mu(1 - \lambda) = 1.$$

We thus see that there exists an essentially unique coboundary for $\lambda \neq 1$ and for $\lambda = 1$ no such coboundary can exists. We have thus shown that the singular point corresponds to a non trivial $C^\infty(\Pi T^*M)$ cocycle:

Theorem 6.1.36. *The $C^\infty(\Pi T^*M)$ cohomology class generated by the extensions:*

$$0 \longrightarrow \mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M) \longrightarrow C^\infty(\Pi T^*M)^\lambda \longrightarrow \mathcal{F}^{\frac{\lambda}{2}}(\Pi T^*M) \longrightarrow 0,$$

is non trivial iff $\lambda = 1$.

We have no canonical representation of the extension however if we consider the cocycle c_Γ as defined above and we further assume that Γ is flat then we have the following:

Lemma 6.1.37. *The cocycle c_Γ restricts to a coboundary on the subalgebra sdiff$(\Gamma) = \{f|\Delta_\Gamma(f) = 0\}$.*

Proof. If we look at the coboundary we see that we may replace all the $\Delta_0 f$ terms with $-(-1)^{\widetilde{k}}\Gamma_k\partial^k f$. The cocycle then becomes zero. □

6.2 Weighted Poisson structures.

In this section we define and study weighted Poisson structures. We shall see that they have a very close link with Poisson structures on $\widehat{M}$ of a fixed weight in the sense that lemma 6.1.23 establishes a 1-1 correspondence between these objects and those on the algebra of densities for weights not 0. We then define the ∞ version of these objects and explain the 1-1 correspondence between weighted P_∞-structure and generalised Poisson structures to be defined below.

6.2.1 Weighted Poisson structures.

We firstly give a definition of a weighted Poisson structure of arity 2 that is equivalent to that for standard Poisson structures as in definition 6.1.1. It is simple enough to have the same anti-symmetry and Jacobi identity properties however the derivation property shall need to be explained in more detail. One could naïvly just consider the situation where we have a first order operator in each variable. However we want a stricter notion - we shall use the fact that the space of first order differential operators on $\mathcal{F}^\lambda(M)$ naturally splits as $\mathfrak{X}(M) \oplus C^\infty(M)$. Define the isomorphism as $(X, f) \mapsto \mathcal{L}_X + f$, with inverse given by $D \mapsto (\sigma(D), D - \mathcal{L}_{\sigma(D)})$, see example 3.1.17.

Definition 6.2.1. A pairing, $\{-,-\} : \mathcal{F}^\lambda_M \otimes \mathcal{F}^\lambda_M \to \mathcal{F}^\lambda_M$, is called an even Poisson structure of weight $-\lambda$ on M if it gives $\mathcal{F}^\lambda_M$ the structure of a Lie algebra and the image of the map $\mathcal{F}^\lambda_M \to \mathrm{DO}(\mathcal{F}^\lambda_M)$, $\mathbf{s} \mapsto \{\mathbf{s}, -\}$, lies in the space of vector fields.

One of the fundamental results allowing us to interpret a Poisson structure as a function was the fact that we could construct the Poisson tensor associated to a weight 0 Poisson structure. The same holds for weighted Poisson structures but requires a little more analysis:

Lemma 6.2.2. *Let $\{-,-\} : \mathcal{F}^\lambda_M \otimes \mathcal{F}^\lambda_M \to \mathcal{F}^\lambda_M$ be a bracket that satisfies the anti-symmetry and derivation property as outlined in definition 6.2.1. Then if $\lambda \neq -1$ the bracket is equivalent to a section of the bundle $TM^{\wedge 2} \otimes \mathcal{F}^{-\lambda}_M$.*

Proof. Locally an arbitrary map $\Phi : \mathcal{F}^\lambda_M \to \mathfrak{X}(M)$ has the form $\Phi = |Dx|^{-\lambda}(A^{ij}\partial_i \otimes \partial_j + B^j 1 \otimes \partial_j)$. This sends the density $\mathbf{s} = s(x)|Dx|^\lambda$ to the vector field

$$\Phi(s) = (-1)^{\widetilde{s}\widetilde{j}}\big(A^{ij}\partial_i s + B^j s\big)\partial_j.$$

Take the Lie derivative of this vector field and apply it to another section $\mathbf{u} \in \mathcal{F}^\lambda(M)$. Recall, by definition, that the Poisson structure is even so $\widetilde{S^{ij}} = \widetilde{i} + \widetilde{j}$, $\widetilde{B^j} = \widetilde{j}$. We find the following local formula for the bracket generated by Φ:

$$\{\mathbf{s}.\mathbf{u}\} = \Big((-1)^{\widetilde{s}\widetilde{j}} A^{ij}(\partial_i s)(\partial_j u) + (-1)^{\widetilde{s}\widetilde{i}} B^i s(\partial_i u)$$

$$+\lambda\big((-1)^{\widetilde{j}}(\partial_j A^{ij})\partial_i s + (-1)^{\widetilde{i}\widetilde{j}} A^{ij}\partial_{ij} s + (-1)^{\widetilde{i}}(\partial_i B^i) + B^i \partial_i s\big)u\Big)|Dx|^{\lambda}.$$

Now as the operator must be first order we immediately have that $A^{ij} = -(-1)^{\widetilde{i}\widetilde{j}} A^{ji}$. This also gives the leading term in the above equation the correct anti-symmetry properties. We are left to show the remaining terms obey the antisymmetry condition. This is equivalent to:

$$\lambda(-1)^{\widetilde{j}}\partial_j A^{ij} + (\lambda+1)B^i = 0, \quad (-1)^{\widetilde{i}}\partial_i B^i = 0.$$

If $\lambda \neq -1$ the first of these equations has a unique solution which also solves the second equation. Thus the bracket is completely determined by the tensor A^{ij}. □

Given a tensor $A^{ij} x_i^* x_j^* |Dx|^{-\lambda}$ for $\lambda \neq -1$ we generate a bracket:

$$\{\mathbf{s},\mathbf{u}\}_A = \Big((-1)^{\widetilde{s}(\widetilde{j}+1)} A^{ij} s_{,j} u_{,i} - \frac{\lambda}{\lambda+1}(-1)^{\widetilde{j}+\widetilde{s}} A^{ji}{}_{,j}\big((-1)^{\widetilde{s}\widetilde{i}} s u_{,i} - s_{,i} u\big)\Big)|Dx|^{-\lambda}. \quad (6.2.1)$$

This is closely linked to the lifting defined in the previous section as we shall see below. Using the methods outlined in the proof of lemma 6.2.2 we can also show that a Poisson structure on $\mathcal{F}^{-1}(M)$, that is of weight 1, can be defined as follows:

Lemma 6.2.3. *Let $\{-,-\} : \mathcal{F}_M^{-1} \otimes \mathcal{F}_M^{-1} \to \mathcal{F}_M^{-1}$ be a bracket that satisfies the antisymmetry and derivation property. Then the bracket is equivalent to a pair (A, B), where $A \in \mathfrak{A}^2(M) \otimes \mathcal{F}^{\frac{1}{2}}(\Pi T^* M)$, B is an upper $\partial^i A$ connection that satisfies the condition:*

$$\delta(A + tt^* B) = 0.$$

The association is given as follows:

$$\{\mathbf{s},\mathbf{u}\} = \Big((-1)^{\widetilde{s}(\widetilde{j}+1)} A^{ij} s_{,i} u_{,j} - B^i\big(s_{,i} u - (-1)^{\widetilde{s}\widetilde{i}} s u_{,i}\big)\Big)|Dx|^{-1}.$$

Example 6.2.4. This example contains the main example of a weighted Poisson structure. It is analgous to symplectic manifolds in the realm of Poisson geometry. The results in this example are adapted from [16] as to make them coherent with the algebra of densities, and we believe it makes some of the results more geometrically clear.

Recall that a contact manifold is a manifold of odd dimension $2m+1$ with a subbundle ξ of rank $2m$ that is maximally unintegrable in the following sense: if $\alpha \in \Omega^1_M$ is a (locally defined) 1 form such that $\xi = \mathrm{Ker}(\alpha)$ then

$$\rho_\alpha := \alpha (d\alpha)^{2m}$$

is a nowhere vanishing section of Ω^{2m+1}_M. One can check that this does not depend on the choice of α. Following [16] we shall not use α but an object that is in fact uniquely defined by ξ. This is actually an important object to introduce as it allows us to perform many differential geometric operations as we shall see.

We shall say that $\alpha \sim \overline{\alpha}$ if $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\overline{\alpha})$. Thus if $\alpha \sim \overline{\alpha}$ then $\overline{\alpha} = u\alpha$ for some nowhere vanishing function. We then have that $\rho_{\overline{\alpha}} = u^{m+1}\rho_\alpha$. For the following also see [16]:

Proposition 6.2.5. *The object $\theta_\xi \in \Omega^1(M) \otimes \mathcal{F}^{-\frac{1}{m+1}}(M)$ defined to be equal to $\alpha \otimes \rho_\alpha^{-\frac{1}{m+1}}$ is well defined independetly of the choice of α.*

Now using the algebra of densities we have a simple interpretation of when a form with this weight comes from a contact structure. To see this note that for an arbitrary weighted form we may consider it as a horizontal form on the algebra of densities. In particular we can do this for θ_ξ. The following construction is essentially Arnold's symplectification of a contact manifold.

Proposition 6.2.6. *Let θ be a horizontal 1 form on $\widehat{M}$ of weight $-\frac{1}{m+1}$ and assume that M is orientable. Then $d\theta$ is a symplectic form on $\widehat{M}$ if θcomes from a contact structure on M.*

Proof. Assume that $\theta = \theta_\xi$ for some ξ a contact structure. Then we may pick some 1 form α such that $\xi = \mathrm{Ker}(\alpha)$ so that $\theta = (\rho_\alpha(x)t)^{-\frac{1}{m+1}}\alpha_i(x)dx^i$. A simple calculation then shows that:

$$(d\theta)^{m+1} = -\frac{1}{m+1}\frac{dt}{t}\frac{\alpha(d\alpha)^{2m}}{\rho_\alpha t}.$$

□

Thus to any contact manifold (M, ξ) we have associated a symplectic structure on $\widehat{M}$. In particular this symplectic structure defines a weighted Poisson structure of weight $\frac{1}{m+1}$, denoted $\{-,-\}_\xi$. This endows the space $\mathcal{F}^{-\frac{1}{m+1}}(M)$ with a Lie lagebra structure and the map

$$\mathbf{s} \mapsto X_{\mathbf{s}} := \{\mathbf{s}, -\} \in \mathfrak{X}(M)$$

corresponds tp acoordinate independent version of Arnold's isomorphism between "functions" and contact vector fields.

Classification of weighted Poisson structures.

We classify weighted Poisson structures for $\lambda \neq 0$ in terms of Poisson structures in the extended manifold.

Proposition 6.2.7. *Let $\pi = \pi^{ij} x_i^* x_j^* |Dx|^\lambda \in \mathfrak{A}^2(M) \otimes \mathcal{F}^\lambda(M)$ for $\lambda \neq 0, 1$. Then π generates a Poisson structure of weight λ, via equation* (6.2.1)*, iff the lift of π as defined in lemma 6.1.23 is a Poisson structure.*

The proof of this is a simple corollary of theorem 6.2.11 and so we shall not prove this proposition explicitly here.

6.2.2 Generalised weighted Poisson structures.

We now turn to the question of generalised weighted Poisson structures. If we consider a even manifold M then it is possible to stay in the standard algebra of densities to describe these objects. However if our base manifold has odd coordinates we shall need to use one of the Laurent versions of the algebra of densities (depending on whether or not we have that the weight is positive or negative). It shall in fact be easier for us to describe the notion of a weighted P_∞-structure first:

Definition 6.2.8. A P_∞-structure of weight $-\lambda$ is a family of brackets, $\{l_n : (\mathcal{F}_M^\lambda)^{\Pi n} \to \mathcal{F}_M^\lambda\}$, that endow densities of weight λ with the structure of an L_∞-algebra that is a first order differential operator in each variable and moreover the map $(\mathbf{s}_1, \ldots, \mathbf{s}_{n-1}) \mapsto l_n(\mathbf{s}_1, \ldots, \mathbf{s}_{n-1}, -)$ lies in the space $\mathfrak{X}(M) \subset \mathrm{DO}^1(\mathcal{F}^\lambda(M))$.

We want a result analogous to theorem 6.1.8 that tells us that a weighted P_∞-structure is simply a function on some space.

Generating functions for weighted P_∞-structures.

The first object we wish to construct is a generalisation of the map that associates to a graded algebra its Laurent algebras. Our goal will be to describe that situation when we have a bigrading, i.e. $\widehat{w}_1$ and $\widehat{w}_2$ even vector fields such that $[\widehat{w}_1, \widehat{w}_2] = 0$, on an algebra A graded over a subsemigroup of $\mathbb{C}$. Assume that the space of eigenvalues of $\widehat{w}_i$ form an ordered semigroup $\Gamma_i \subset \mathbb{C}$. We then define an element in $\mathcal{O}_{(+,+)}(A)$ to be algebras given by formal products:

$$a = \prod_{(\lambda_1,\lambda_2)\in S} a_{\lambda_1,\lambda_2},$$

where $S \subset \Gamma_1 \times \Gamma_2$ is a closed discrete subset of $\mathbb{R}^2$ and $S \cap \left(\mathbb{R}^2 - \mathbb{R}^{\geq} \times \mathbb{R}^{\geq}\right)$ is finite. One can check that the multiplication on A does indeed induce a multiplication on the space $\mathcal{O}_{(+,+)}(A)$. We can similarly define $\mathcal{O}_{(\pm,\pm)}(A)$ by looking at the different quartiles.

Recall on the space $\Pi T^*\widehat{M}$ we have two weight operators, one coming from the vector bundle and one induced from that on $\widehat{M}$:

$$\widehat{w}_1 = x_i^*\partial_{x_i^*} + t^*\partial_t^*, \qquad \widehat{w}_2 := (tt^*, -) = t\partial_t - t^*\partial_t^*.$$

The local form of an element of biweight (n, λ) is given as:

$$s(x,t;x^*,t^*) = t^\lambda\left(s^{i_1\cdots i_n}(x)x_{i_1}^*\cdots x_{i_n}^* + s^{j_1\cdots j_{n-1}}(c)tt^*x_{j_1}^*\cdots x_{j_{n-1}}^*\right).$$

Definition 6.2.9. We define the function spaces $\mathcal{O}_\pm(\Pi T^*\widehat{M})$ to be the Laurent algebras $\mathcal{O}_{(+,\pm)}(\Pi T^*\widehat{M})$ with respect to the weight operators $\widehat{w}_1$ and $\widehat{w}_2$.

Note that as the differential operator δ has a fixed weight we have that it descends to a differential operator on $\mathcal{O}_\pm(\Pi T^*\widehat{M})$. In particular its symbol, the Schouten-Nijenhuis bracket, also descends to a well defined bidifferential operator on $\mathcal{O}_\pm(\Pi T^*\widehat{M})$. These algebras are the anti pseudomomenta associated to the Laurent algebras $\mathcal{F}_\pm(M)$.

For a fixed choice of λ we introduce the new weight operator $\widehat{w}_{t;\lambda} = \lambda\widehat{w}_1 - \widehat{w}_2$. This operator will allow us to define generalised Poisson structures:

Definition 6.2.10. A generalised Poisson structure of weight $\lambda \neq 0$ is an element $S \in \mathcal{O}_{\mathrm{sgn}(\lambda)}(\Pi T^*\widehat{M})$ such that:

1. $\widehat{w}_{t;\lambda}S = \lambda S$,

2. $\delta S = 0$,

3. $(S, S) = 0$.

We can give a local description of all elements that satisfy the eigenvalue condition in the above definition. We look at all possible points in the product and we find that such an element is equal to:

$$S = \sum_{n \geq 0} S_n(x; x^*, tt^*) t^{(n-1)\lambda}, \quad \text{such that } \widehat{w}_1 S_n = n S_n.$$

Given such an element we define a family of brackets, $l_S^n : \mathcal{F}^{-\lambda}(M)^{\otimes n} \to \mathcal{F}^{-\lambda}(M)$. The following theorem is the analogue of corollary 6.1.10:

Theorem 6.2.11. *There is a 1-1 correspondence between weighted Poisson structures and weighted P_∞-structures.*

Proof. The case when the weight is equal to zero is just theorem 6.1.27 so we shall only focus on the case when the weight is non-zero.

($\Rightarrow$) Our first goal is to show that the brackets $\{l_S^n : n \in \mathbb{N}\}$ form a P_∞-structure. By construction the brackets are anti-symmetric so we need to show that they are derivations in each argument and form an L_∞-structure. The L_∞ structure condition follows immediately from the fact that we require that $(S, S) = 0$ and the fact that $\mathcal{F}^{-\lambda}(M) \subset \mathcal{F}_\pm(M)$. Therefore we just need to show that they are derivations in each argument. To prove this it will be enough to check for a single argument using the antisymmetry property. Let $X = X^i(x) x_i^* + tt^* Y \in \mathfrak{A}^1(\widehat{M})_0$ and let X_λ denote the associated operator on $\mathcal{F}^{-\lambda}(M)$:

$$X_\lambda(s|Dx|^{-\lambda}) = \big(X^i s_{,i} - \lambda Y s\big)|Dx|^{-\lambda}.$$

Thus for $\lambda \neq 0$ X_λ is a derivation iff $\delta X = 0$. We have that l_S^n is given by the formula

$$l_S^n(f_1|Dx|^{-\lambda}, \ldots, f_n|Dx|^{-\lambda}) = (\cdots(S_n, f_1 t^{-\lambda}) \cdots, f_n t^{-\lambda})|_{t=|Dx|}.$$

Therefore we need to show that the element $(\cdots((S_n, f_1 t^{-\lambda}), f_2 t^{-\lambda}), \cdots f_{n-1} t^{-\lambda}) \in \mathfrak{A}^1(M)_0$ is divergence less $\forall f_k |Dx|^{-\lambda} \in \mathcal{F}^{-\lambda}(M)$. We then apply the differential operator δ to this element and using the fact that it derivates the Schouten-Nijenhuis bracket, see corollary 6.1.17, and that $\delta|\mathcal{F}^{-\lambda}(M) = 0$, we get

$$\delta(\cdots((S_n, f_1 t^{-\lambda}), f_2 t^{-\lambda}), \cdots f_{n-1} t^{-\lambda}) = (\cdots((\delta S_n, f_1 t^{-\lambda}), f_2 t^{-\lambda}), \cdots f_{n-1} t^{-\lambda}).$$

Now as δ is a weight 0 operator we have that $\delta S = 0 \Leftrightarrow \delta S_k = 0\ \forall k$. Hence the vector field is divergenceless and so it does indeed define a weighted P_∞-structure.

($\Leftarrow$) We now need to show the converse, that is we need to associate to a P_∞-structure the data of a generalised Poisson structure of weight λ. Let $\{l^n : n \in \mathbb{N}\}$ be a P_∞-structure, then we have the symbols of the operators l^n which correspond to a section $S_n \in \mathfrak{A}^n(M) \otimes \mathcal{F}^{(n-1)\lambda}(M)$. We claim that l^n induces a unique extension of this section to an element of $\mathfrak{A}^n(M)_{(n-1)\lambda}$. By the symmetry conditions on the l^n we must have that it locally it derives the argument of at least $n-1$ terms and thus l^n defines another term which locally is given by the formula

$$A_n(\mathbf{f}_1, \ldots, \mathbf{f}_n) = l^n(\mathbf{f}_1, \ldots, \mathbf{f}_n) - (-1)^{\widetilde{f_1}(\widetilde{i_2}+\cdots+\widetilde{i_n}+n-1)+\cdots+\widetilde{f_{n-1}}(\widetilde{i_n}+1)} S^{i_1\cdots i_n} f_{1,i_1}\cdots f_{n,i_n}.$$

We now claim that $-\frac{1}{\lambda}A_n$ transforms as an upper S_n connection. To see this consider how the bracket changes under a transformation:

$$l^n(f_1, \ldots, f_n) = J^\lambda l^n(\widetilde{f}_1 J^{-\lambda}, \ldots, \widetilde{f}_n J^{-\lambda}) \Rightarrow$$
$$J^{(n-1)\lambda}\widetilde{A}_n(f_1, \ldots, f_n) = A_n - \lambda S_n^{i_1\cdots i_n}\sum_{k=1}^{n} \partial_{i_1} f_1 \cdots \widehat{\partial_{i_k} f_k} \cdots \partial_{i_n} f_n\, \partial_{i_k}\log(J).$$

Therefore we have that

$$\widehat{S}_n t^{(n-1)\lambda} = \big(S_n^{i_1\cdots i_n}(x) x^*_{i_1}\cdots x^*_{i_n} - \frac{1}{\lambda}A_n^{i_1\cdots i_{n-1}} t t^* x^*_{i_1}\cdots x^*_{i_{n-1}}\big) t^{(n-1)\lambda}$$

is a well defined element of $\mathfrak{A}^n(\widehat{M})_{(n-1)\lambda}$. Moreover we see that l^n is the derived bracket of this element. Doing this for all n we have that the P_∞-structure induces an element $S \in \mathcal{O}_\pm(\Pi T^*\widehat{M})$ of weight λ with respect to $\widehat{w}_{t;\lambda}$. Thus we need only to show that it is a Maurer-Cartan element and moreover that it is in the kernel of δ. In some sense this is equivalent to the sufficient conditions on S of $\delta S = (S, S) = 0$ to generate a P_∞-structure to be necessary. For example the condition that the $\{l^n : n \in \mathbb{N}\}$ form an L_∞-structure becomes equivalent to the fact that the derived bracket of $(\widehat{S}, \widehat{S})$ is identically zero on $\mathcal{F}^{-\lambda}(M)$. So take an arbitrary element $U_n t^{(n-1)\lambda}$, an eigenvalue of $\widehat{w}_{t;\lambda}$ of weight λ, and suppose that the derived bracket is identically zero on $\mathcal{F}^{-\lambda}(M)$. Then it must necessarily have no symbol in the sense that $U_n|_{t^*=0} = 0$. Then the fact that $\lambda \neq 0$ implies that $U_n = 0$. Thus we must have that $\widehat{S}$ is a Maurer-Cartan element. The fact that $\widehat{S}$ is divergenceless follows directly from the first half of the proof where we see that it is clearly necessary that $\delta\widehat{S} = 0$ for the bracket to be a derivation in each variable. □

Example 6.2.12. Weighted Poisson brackets. In this example we shall work out the details of a Poisson structure of arity 2. If Π has weight λ with respect to $\widehat{w}_{t;\lambda}$ and weight 2 with respect to $\widehat{w}_1$ then it is of the form:

$$\Pi = t^\lambda \left(\frac{1}{2} \pi^{ij} x_j^* x_i^* + tt^* \pi^i x_i^* \right).$$

We know from lemma 6.1.24 that the divergenceless condition of such a structure depends on whether or not $\lambda = 1$. For the moment let us assume not so that

$$\delta\Pi = 0 \Leftrightarrow \pi^i = \frac{1}{\lambda - 1} \partial_j \pi^{ji}, \quad \text{for } \lambda \neq 1.$$

Let us assume for simplicity that M is a purely even manifold. Then the condition that this defines a Poisson structure of weight λ is equivalent to:

$$\pi^{il} \pi^{jk}{}_{,l} + \text{cyc.}(ijk) = \frac{\lambda}{1-\lambda} \Big(\pi^{li}{}_{,l} \pi^{jk} + \text{cyc.}(ijk) \Big).$$

We now turn to the case when $\lambda = 1$. There is no longer a canonical choice for the upper connection π^i. The condition that $(\Pi, \Pi) = 0$ is equivalent to:

$$\pi^{il} \pi^{jk}{}_{,l} + \text{cyc.}(ijk) = -\pi^i \pi^j k + \text{cyc.}(ijk),$$

$$\pi^k \pi^{ij}{}_{,k} = \pi^{ik} \pi^j{}_{,k} - \pi^{jk} \pi^i{}_{,k}.$$

Whilst the divergence condition splits the terms and becomes that:

$$\pi^{ij}{}_{,j} = 0, \qquad \pi^i{}_{,i}.$$

Remark 6.2.13. We have shown that there is no universal algebra that is in 1-1 correspondence with weighted Poisson structures. This is different from the classical situation where one can take $\mathcal{O}(\Pi T^* M)$, see 6.1.10. Our algebra needs to contain both $\mathcal{O}_+(\Pi T^* \widehat{M})$ and $\mathcal{O}_-(\Pi T^* \widehat{M})$ and as this essentially corresponds to arbitrary formal sums there can be no multiplicative structure. We can remedy the situation if we work with the algebra of smooth functions on $\Pi T^* \widehat{M}$. In this case we still have the $O((x^*)^\infty)$ discrepancy to consider as in theorem 6.1.8.

To close this section we make the following remark: If we ignore the weight condition used to define a weighted P_∞-structure then we see that such a structure provides a trivial solution to the Batalin-Vilkovisky equation on $\Pi T^* \widehat{M}$:

$$\delta S + \frac{1}{i\hbar}(S, S) = 0,$$

where it is trivial in the sense that it splits the terms in the above equation and solves them separately. It is thus a natural question to ask what are the general solutions to the Batalin-Vilkovisky equation on $\widehat{M}$ and how do they relate to such questions on the base manifold.

Chapter 7

Conclusions

We have introduced the reader to the algebra of densities and studied certain concepts surrounding this algebra. In the scond chapter we emphasised not only the algebraic but also the geometric origins of this algebra. In particular we explained the relation of the space $\widehat{M}$ with the bundle $M \times \mathbb{C}^{\times}$. We calculated the de Rham cohomology of $(\widehat{M}, \mathcal{F}(M))$ which allowed us to explain the universality of this bundle.

In the second chapter we turned to the study of lifts of differential operators in an attempt to determine the space of differential operators on $\widehat{M}$ whose pencil passes through a fixed operator in $\mathrm{DO}(\mathcal{F}^{\lambda}(M), \mathcal{F}^{\lambda+\delta}(M))$. We focused on equivariant lifts, as in these cases it is possible for the space of lifts to be finite dimensional. We calculated all such lifts for equivariance with respect to deivergenceless vector fields, and all lifts for non-resonant δ for projective equivariance.

In the fourth chapter we looked at differential operators under a different light, in terms of brackets and groupoids. This brought us to the work of H. Khudaverdian and Th. Voronov on odd Laplacians. In the first half of the chapter we explicitly explained the structure of third order operators using techniques from chapter 2. The results in this section showed that, if the dimension is greater than 1, there is no chance of an identical construction of the Khudaverdian-Voronov groupoid for higher order operators. In the second half of this chapter we gave a construction that allowed us to generalise the groupoid.

In the fifth chapter we studied of the algebra of densities on curves. Here densities constitute all geometrical bundles and hence there are results that cannot possibly be attained for manifolds of other dimensions. In the first half of this chapter we showed that to a connection we can associate a self adjoint differential operator that only depends on the Schwarzian of this connection. We also introduced a family of brackets that are equivariant with respect to arbitrary isomorphisms. In the second half of the chapter we studied differential operators on the upper half plane that are invariant under the modular group. We showed that these operators contain the information of modular forms for arbitrary finite index subgroups.

In the sixth chapter we studied antisymmetric rather than symmetric structures. The situation is completely different and here we find canonical lifts of Poisson structures at all orders. This allows us to classify all Poisson structures on the algebra of densities of weight 0. In the second half of this chapter we introduced weighted Poisson structures and showed that there is a uniform way to describe such structures akin to the classical theory of P_∞-structures.

Appendix A

Supermanifolds and Signs.

The adjective super refers to a $\mathbb{Z}_2$ grading on the associated structures. Many of the results that hold for the purely even case pass over to the super case. In particular there are well defined notions of manifolds, Lie algebras etc.

Definition A.0.14. A super vector space is a $\mathbb{Z}_2$ graded vector space, $V = V_0 \oplus V_1$. If $v \in V$ is homogeneous we write $\widetilde{v}$ to denote its parity. A super algebra is a super vector space, A, with an associative product: $A \otimes A \to A$, such that $A_i A_j \subseteq A_{i+j}$.

Example A.0.15. We shall denote by $\Bbbk^{n|m}$ the super vector space with $\left(\Bbbk^{n|m}\right)_0 = \Bbbk^n$ and $\left(\Bbbk^{n|m}\right)_1 = \Bbbk^m$.

Example A.0.16. Let V be a super vector space. We then define the superalgebra $T(V)$, the tensor algebra, which as a vector space is $\oplus_n V^{\otimes n}$, and

$$T(V)_i = \bigoplus_{\substack{n \\ i_1+\cdots+i_n=i}} V_{i_1} \otimes \cdots \otimes V_{i_n}.$$

We shall call a super algebra commutative if for homogeneous elements v, u we have that:

$$uv = (-1)^{\widetilde{u}\widetilde{v}} vu.$$

Given a super vector space V we can associate a natural commutative algebra $S(V^*)$ which should be thought of as the polynomial functions on V. Firstly note that as $V^* = \mathrm{Hom}(V, \Bbbk)$, and $\Bbbk = \Bbbk^{1|0}$, there is a natural $\mathbb{Z}_2$-grading on V^*. We then quotient out the superalgebra $T(V^*)$ by the ideal generated by $\phi\psi - (-1)^{\widetilde{\phi}\widetilde{\psi}}\psi\phi \ \forall \phi, \psi \in V^*$.

A.1 Lie algebras and signs.

Symmetry.

There are 4 notions of symmetry that n morphism, $A : V^{\otimes n} \to W$, of super vector spaces can have:

i Symmetric 1:

$$A(u_1, \cdots, u_n) = (-1)^{\widetilde{u_i}\widetilde{u_{i+1}}} A(u_1, \cdots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \cdots, u_n), \, \forall i.$$

ii Symmetric 2:

$$A(u_1, \cdots, u_n) = -(-1)^{(\widetilde{u_i}+1)(\widetilde{u_{i+1}}+1)} A(u_1, \cdots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \cdots, u_n), \, \forall i.$$

iii Antisymmetric 1:

$$A(u_1, \cdots, u_n) = (-1)^{(\widetilde{u_i}+1)(\widetilde{u_{i+1}}+1)} A(u_1, \cdots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \cdots, u_n), \, \forall i.$$

iv Antisymmetric 2:

$$A(u_1, \cdots, u_n) = -(-1)^{\widetilde{u_i}\widetilde{u_{i+1}}} A(u_1, \cdots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \cdots, u_n), \, \forall i.$$

Proposition A.1.1. *There is a 1-1 correspondence between linear maps that are symmetric of type 1 and 2, and a 1-1 correspondence between antisymmetric maps of type 1 and 2.*

For example assume that $L : V^{\otimes 2} \to V$ is symmetric of type 1, $L(u, v) = (-1)^{\widetilde{u}\widetilde{v}} L(v, u)$. Then consider L' defined by $L'(u, v) = (-1)^{\widetilde{u}} L(u, v)$. We then have that

$$L'(u, v) = (-1)^{\widetilde{u}} L(u, v) = (-1)^{\widetilde{u}(\widetilde{v}+1)} L(v, u) = -(-1)^{(\widetilde{u}+1)(\widetilde{v}+1)} L'(v, u).$$

We shall thus use the adjective symmetry/antisymmetry to refer to either of the symmetric brackets as one can trivially transform one into another. In certain situations it is useful to be able to switch from one to the other.

A.1.1 Lie algebras.

Definition A.1.2. A lie super algebra is a vector space, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an antisymmetric (of type 2) even bracket, $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

$$[a,[b,c]] = [[a,b],c] + (-1)^{\tilde{a}\tilde{b}}[b,[a,c]].$$

Example A.1.3. Let A be a commutative super algebra. Then $\mathrm{Der}(A)$ has the structure of a Lie superalgebra where the commutator corresponds to the bracket.

The superalgebra $\mathfrak{gl}(n|m)$ is the set of linear endomorphisms of $\mathbb{R}^{n|m}$. It is associative and therefore the supercommutator endows it with the structure of a super Lie algebra. This space has a natural universal morphism to the abelian Lie algebra $\mathbb{R}^{1|0}$ called the supertrace. If $L = l^i_j e^j \otimes e_i$ then the supertrace of L is given by the formula:

$$\mathrm{str}(L) = (-1)^{\tilde{i}} l^i_i.$$

This algebra is the Lie algebra of the Lie group $\mathrm{GL}(n|m, \mathbb{R})$, see below for the definition of a manifold. This is a supermanifold with body $\mathrm{GL}(n;\mathbb{R}) \times \mathrm{GL}(m;\mathbb{R})$. The supertrace exponentiates to define a function called the Berezinian:

$$\mathrm{Ber} : \mathrm{GL}(n|m;\mathbb{R}) \to \mathbb{R}^{1|0}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{\det(A - BD^{-1}C)}{\det D}.$$

Example A.1.4. Consider a diagonal matrix, $L = \mathrm{diag}(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)$. This then exponentiates to $e^L = \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\mu_m})$. We have that

$$\mathrm{str}(L) = \sum_i \lambda_i - \sum_j \mu_j, \qquad \mathrm{Ber}(e^L) = \frac{\prod_i e^{\lambda_i}}{\prod_j e^{\mu_j}},$$

so $\mathrm{Ber}(e^L) = e^{\mathrm{str}(L)}$ as defined.

A.2 Manifolds.

Another important algebra that we shall use more than the polynomial algebra is the smooth algebra, $C^\infty_{\mathbb{R}^{n|m}}$, which is a sheaf on $\mathbb{R}^n$ defined as:

$$C^\infty_{\mathbb{R}^{n|m}}(U) = C^\infty(U) \otimes \Lambda[\theta^1, \ldots, \theta^n],$$

that is smooth functions with values in a finite dimensional Grassman algebra. One can check that this makes $(\mathbb{R}^n, C^\infty_{\mathbb{R}^{n|m}})$ into a locally ringed space.

Definition A.2.1. A supermanifold of dimension (n, m) is a (Hausdorff) locally ringed space, $(M, \mathcal{O})$, such that $\mathcal{N} \subset \mathcal{O}$, the subsheaf of nilpotent elements defines the following short exact sequence:

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{O} \longrightarrow C^\infty_M \longrightarrow 0.$$

Moreover the pair $(M, \mathcal{O})$ is locally modeled on $(\mathbb{R}^n, C^\infty_{\mathbb{R}^{n|m}})$.

Example A.2.2. Let $E \to M$ be a vector bundle of rank r over an even manifold of dimension n. We define the space ΠE^* as follows: Fix a coordinate trivializing atlas of M, $\{U_i \to M\}$, $U_i \subset \mathbb{R}^n$ and $E|U_i \cong U_i \times \mathbb{R}^r$. Let $\phi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ denote the coordinate transformation and $\psi_{ij} : U_{ij} \to \mathrm{GL}(r, \mathbb{C})$ be the gluing data for E. We then define the gluing data for ΠE^*:

$$\Phi^*_{ij}(x^\mu) = \phi^\mu_{ij}, \qquad \Phi^*_{ij}(\theta^a) = \big(\psi_{ij}\big)^a_{b'} \theta^{b'}.$$

The sheaf can also be thought of as sections of the bundle, $\mathbb{R} \times E \times E^{\wedge 2} \times \cdots \times E^{\wedge r}$. One can in fact show that in the category of smooth supermanifolds all supermanifolds are isomorphic to one of the above type.

All standard (differential) geometric constructions hold for supermanifolds as for even manifolds, e.g. tangent bundle, cotangent, etc.

Part of the definition of a super manifold is that $\mathcal{O} \twoheadrightarrow C^\infty_M$ so we have an embedding of locally ringed spaces, $(M, C^\infty_M) \to (M, \mathcal{O})$, and we call (M, C^∞_M) the *body* of M and denote it M_0.

A.2.1 Integration on manifolds

One of the first major divergences of supermanifolds from purely even manifolds is that of integration. We shall only be interested with integration over the whole manifold so many of the intricate parts of the theory can be ignored, see [83]. As usual we would like to define the integral as some cohomology group, however it is not possible to use the de Rham sequence as there is no top term. Instead we employ the fact that the divergence of vector field should be zero. Let us now consider $\mathbb{R}^{n|m}$, so that an arbitrary function has the form:

$$f(x, \theta) = f_0(x) + f_i(x)\theta^i + \cdots + f_{1\cdots m}\theta^1 \cdots \theta^m,$$

where x denotes the even coordinates and θ the odd coordinates. As the odd part has no topology the manifold f has compact support if $f_{i_1\cdots i_k}$ has compact support for $k = 0, \ldots, m$. Now applying the divergence condition with respect to the vector field, ∂_{θ^i}, we get that:

$$\int \partial_i f|D(x,\theta)| = 0 \Rightarrow \int f = \int f_{1\cdots m}\theta^1 \cdots \theta^m |D(x,\theta)|.$$

Therefore we have a natural local notion of integration on supermanifolds:

$$\int_{\mathbb{R}^{n|m}} f|D(x,\theta)| = \alpha \int_{\mathbb{R}^n} f_{1\cdots m}|Dx| = \alpha \int_{\mathbb{R}^n} \left.\frac{\partial^m f}{\partial\theta^m \cdots \partial\theta^1}\right|_{\mathbb{R}^n} |Dx|,$$

for some constant $\alpha = \alpha(n,m) \in \mathbb{C}^\times$, which we shall set to 1. This gives us the local description of integration and all that is left is to find the transformation property of $|D(x,\theta)|$:

Proposition A.2.3. *Let* $(x',\theta') = (x'(x,\theta), \theta'(x,\theta))$*. Then*

$$|D(x',\theta')| = Ber\left(\frac{\partial(x',\theta')}{\partial(x,\theta)}\right)|D(x,\theta)|.$$

This defines a line bundle on a general super manifold which we shall denote by $\mathcal{F}^1_M$ and call the bundle of densities.

Appendix B

Structures on Manifolds.

B.1 Pseudogroups.

Let us consider the category of locally ringed spaces, $\mathfrak{Spaces}$ and its groupoid of isomorphisms, $\mathfrak{Spaces}^\times$.

Definition B.1.1. A *pseudogroup* $\mathfrak{G}$ is a subgroupoid of $\mathfrak{Spaces}^\times$ such that:

1. It is closed under restrictions: If $(V, \mathcal{O}_V) \in \mathfrak{G}$ then for any $U \subset V$ open, $(U, \mathcal{O}_V|U) \in \mathfrak{G}$.

2. It is closed under gluing of maps: If $f \in \mathrm{Hom}(U,V)^\times$, $U, V \in \mathfrak{G}$, and there exists a covering $\{U_i \to U\}$ such that $f|U_i \in \mathfrak{G}(U_i, f(U_i))$ $\forall i$ then $f \in \mathfrak{G}(U,V)$.

Example B.1.2. Take a space $X \in \mathfrak{Spaces}$. Then we define $\mathfrak{G}(X)$ to be the pseudogroup whose objects are the open subsets of X and the maps are all the isomorphisms in $\mathfrak{Spaces}$.

Fix a pseudogroup $\mathfrak{G}$ we call the objects of $\mathfrak{G}$ models. We can construct more spaces from the pseudogroup by gluing together not just morphisms but spaces. We shall that $(X, \mathcal{O}_X)$ is a *modeled on* $\mathfrak{G}$ if there exists a covering $\{U_i \xrightarrow{f_i} X\}$ such that $U_i \in \mathfrak{G}$ and $f_{ij} : U_{ij} \to U_{ji} \in \mathfrak{G}(U_{ij})$. One can define a space modeled on $\mathfrak{G}$ by the cocycle data $\{f_{ij}\}$ which determine the associated space up to isomorphisms that locally come from $\mathfrak{G}$.

Example B.1.3. Let X be medelled on $\mathfrak{G}(\mathbb{R}^n, C^k_{\mathbb{R}^n})$ for $k = 0, \ldots, \infty, \omega$. Then if X is Hausdorff and second countable it is a C^k-manifold.

Example B.1.4. Consider the psuedogroup generated by Spec(A) for all R-algebras. Then the category of spaces modeled on this pseudogroup is equivalent to $\mathfrak{Sch}/\mathrm{Spec}(R)$.

B.2 G-manifolds.

In a lot of situations we are interested not when the pseudogroup is large, i.e. when $\mathfrak{G} \to \mathfrak{Spaces}^{\times}$ is full, but when it is defined by a smaller class of isomorphisms generated by a bona fide group. We will now focus on the smooth case, so take a Lie group G and a manifold X upon which G acts. We define the pseudogroup $\mathfrak{G}(G; X)$ as follows:

- $\mathfrak{G}(G; X) = \{U \subset X : U \text{ open}\}$.
- $\mathfrak{G}(G; X)(U, V) = \{f \in \mathrm{Hom}(U, V)^{\times} : \exists \{U_i \subset U\}$ and $\gamma_i \in G$ that locally define the morphism $f.\}$

In the situations we are interested in one can usually define sharper pseudogroups but this will be sufficient for our work. Spaces that are modeled on such a groupoid (and are Hausdorff and second countable) are called (G, X)-manifolds

Example B.2.1. Consider $\mathbb{R}^n$ acting on itself. Manifolds that are modeled on such a space are called flat. For example the torus, T^n.

The following is the main example we shall use in the tex:

Example B.2.2. Consider $\mathrm{PGL}(n + 1; \mathbb{R})$ acting on $\mathbb{P}^n$. Manifolds modeled on this pair are also called real projective manifolds or manifolds with a projective structure if the atlas is explicitly chosen

B.2.1 The developing map.

To close this appendix we shall define a map called the developing map that is one of the classical tools in studying G-manifolds. It is only defined in the case when the group G obeys the additional property that for any open set U and $g, g' \in G$:

$$g|U = g'|U \Rightarrow g = g'.$$

The pseudogroup is simpler to describe now, essentially any morphism must be locally constant: if $\pi_0(U) = *$ then $\mathfrak{G}(G;X)(U,V) = \{g \in G | gU = V\}$. This assumption holds for the pairs $(\mathrm{GL}(n;\mathbb{R}), \mathbb{R}^n)$ and $(\mathrm{PGL}(n+1); \mathbb{R}, \mathbb{P}^n)$.

Let M be a path connected manifold with a (G,X)-structure, $\{U_i \xrightarrow{f_i} M\}$ and we can assume that $U_i \simeq *$. Now fix a point $m_0 \in M$ and a coordinate patch around m_0, U_0. Let $\gamma : (I, \partial I) \to (M, m_0)$ be a loop in the space M. Take a finite covering of the path $\{(U_0, \ldots, U_n, U_{n+1} = U_0) : U_i \cap U_{i+1} \simeq *, \gamma(I) \subset \cup_m U_m\}$. Then by our assumption on the group G we have that there exists a unique element $g_i = f_i^{-1} \circ f_{i-1} \in G$. Therefore to the loop γ we can associate an element $g(\gamma) \in G$. The classical result is

Proposition B.2.3. *The map $g : \Omega M \to G$ is well defined once U_0 is chosen. Moreover the map only depends on the homotopy class of γ and induces a group homomorphism $\pi_1(M) \to G$.*

Moreover we see that if we take the universal cover of M, $\widetilde{M} \to M$, that the above map induces a local isomorphism of G-spaces $\widetilde{M} \to X$. Thus we have associated to a (G,X)-structure a local diffeomorphism $\phi : \widetilde{M} \to X$, and a representation $\rho : \pi_1(M) \to G$. This pair is called the developing map associated to M. Moreover two developing maps, (ϕ, ρ) and $(\widetilde{\phi}, \widetilde{\rho})$, define the same G-structure on M if there $\exists g \in G$ such that $g \circ \phi = \widetilde{\phi}$ and $\mathrm{Ad}_g(\rho) = \widetilde{\rho}$. The geometry of (G,X)-structures is contained within the geometry of the developing map and this is the starting point of many studies into geometrical structures.

To close this chapter we shall give a result concerning vector fields on a (G,X)-manifold. Let us take the developing map $(\phi, \rho) : (\widetilde{M}, \Gamma) \to (X, G)$. The Lie group G induces an action of $\mathrm{Lie}(G) = \mathfrak{g}$ on functions on X, that is we have a map of Lie algebras $\mathfrak{G} \to \mathfrak{X}(X)$. We have the following diagram of local diffeomorphisms:

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & X \\ \downarrow & & \\ M. & & \end{array}$$

We can pull back any local maps along local diffeomorphisms, in particular we can pull back all the vector fields $\mathfrak{g} \subset \mathfrak{X}$. We cannot however pushforward vector fields but any vector field on M is one on $\widetilde{M}$ that is invaraint under the action of $\Gamma = \pi_1(M)$.

We have that this action is equivalent to the action of $\rho(\Gamma)$ on $\mathfrak{g}$. Summing this up we have the following:

Lemma B.2.4. *Let M be a (G, X) manifold associated to the developing map (ϕ, ρ). Then $X \in \mathfrak{g}$ induces a vector field on M iff $X = Ad_{\rho(\gamma)}(X)$ $\forall \gamma \in \Gamma$.*

For this reason we have to use the sheaf of $\mathfrak{g}$-vector fields when considering equivariance questions which is always suitably rich.

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