CREDIT RISK MODELING IN A SEMI-MARKOV PROCESS ENVIRONMENT

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Credit Risk Modeling in a Semi-Markov Process Environment
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In recent times, credit risk analysis has grown to become one of the most important problems dealt with in the mathematical finance literature. Fundamentally, the problem deals with estimating the probability that an obligor defaults on their debt in a certain time. To obtain such a probability, several methods have been developed which are regulated by the Basel Accord. This establishes a legal framework for dealing with credit and market risks, and empowers banks to perform their own methodologies according to their interests under certain criteria. Credit risk analysis is founded on the rating system, which is an assessment of the capability of an obligor to make its payments in full and on time, in order to estimate risks and make the investor decisions easier.

Credit risk models can be classified into several different categories. In structural form models (SFM), that are founded on the Black & Scholes theory for option pricing and the Merton model, it is assumed that default occurs if a firm's market value is lower than a threshold, most often its liabilities. The problem is that this is clearly is an unrealistic assumption. The factors models (FM) attempt to predict the random default time by assuming a hazard rate based on latent exogenous and endogenous variables. Reduced form models (RFM) mainly focus on the accuracy of the probability of default (PD), to such an extent that it is given more importance than an intuitive economical interpretation. Portfolio reduced form models (PRFM) belong to the RFM family, and were developed to overcome the SFM's difficulties.

Most of these models are based on the assumption of having an underlying Markovian process, either in discrete or continuous time. For a discrete process, the main information is containted in a transition matrix, from which we obtain migration probabilities. However, according to previous analysis, it has been found that this approach contains embedding problems. The continuous time Markov process (CTMP) has its main information contained in a matrix Q of constant instantaneous transition rates between states. Both approaches assume that the future depends only on the present, though previous empirical analysis has proved that the probability of changing rating depends on the time a firm maintains the same rating. In order to face this difficulty we approach the PD with the continuous time semi-Markov process (CTSMP), which relaxes the exponential waiting time distribution assumption of the Markovian analogue.

In this work we have relaxed the constant transition rate assumption and assumed that it depends on the residence time, thus we have derived CTSMP forward integral and differential equations respectively and the corresponding equations for the particular cases of exponential, gamma and power law waiting time distributions, we have also obtained a numerical solution of the migration probability by the Monte Carlo Method and compared the results with the Markovian models in discrete and continuous time respectively, and the discrete time semi-Markov process. We have focused on firms from U.S.A. and Canada classified as financial sector according to Global Industry Classification Standard and we have concluded that the gamma and Weibull distribution are the best adjustment models.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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This work is dedicated to my beloved children Victoria and Pedro Manuel, I hope that one day they will read this work and feel proud of me.

Glossary

Throughout the course of this thesis, we use the following symbols:

Symbol	Description
$\delta(\cdot)$	Dirac delta function
$\varphi_i(t)$	Waiting time probability density function
$G_{ij}(t)$	Waiting time conditional distribution function
$\Phi_i(t)$	Waiting time cumulative function
$\Psi_i(t)$	Survival function
p_i	probability of initial condition
$Q_{ij}(t)$	Semi-Markov probability kernel
h_{ij}	Transition probability
$\rho_{ij}(t)$	Migration probability
$j_i(t)$	probability of arriving at state i exactly at time t
K	Kernel function
$\gamma_i(\tau)$	Hazard function
$\Omega_i(t,\tau)$	Probability of being in state i at time t with residence time τ
$\mathcal{L}[f(t)] = \hat{f}(s)$	Laplace transform
${}^{GL}\mathcal{D}_t^q f(t)$	Grünwald-Letnikov fractional derivative
${}^{RL}\mathcal{D}_t^q f(t)$	Riemann-Liouville fractional derivative
${}^{RL}\mathcal{D}_t^q f(t)$	Caputo fractional derivative

A summary of some standard abbreviations is provided here

Abbreviation	Description
cdf	Cumulative density function
CRA	Credit rating agency
CTMP	Continuous time Markov process
CTSMP	Continuous time semi-Markov process
DTMP	Discrete time Markov process
DTSMP	Discrete time semi-Markov process
FM	Factor models
GCTSMP	Continuous time semi-Markov process with a gamma distribution
Gam	Gamma waiting time distribution
ILRFM	Individual level reduced form models
LGD	Lost given default
MLE	Maximum likelihood estimator
PD	Probability of default
pdf	Probability density function
PLD	Power-law distribution
PRFM	Portfolio reduced form models
RFM	Reduced form models
SFM	Structural form models
Var	Credit value at risk
WCTSMP	Continuous time semi-Markov process with a Weibull distribution

Chapter 1

Introduction

1.1 Motivation

In recent years, estimating the Probability of default (PD) of a given fixed income security has become one of the most common problems tackled in financial mathematics. Thus, a large number of models have been developed based on various statistical and mathematical techniques. Some of these models are based on the assumption that the default occurs when the value of the underlying assets or the stock prices fall below some specific amount. It should be clear that such assumption tends to be nonrealistic. Other models use macroeconomic variables and various internal indicators of the healthiness of the firm; and some models are based on the migration probabilities of a rating system, based on modelling the migration as a random walk.

The discrete time semi-Markov process (DTMP) and continuous time Markov process (CTMP), are widely used techniques to estimate the PD by estimating migration probabilities, given a state space system such as discrete credit ratings. The DTMP is a random walk jump process, where at discrete time intervals the walker has a probability to jump from its current state to another with a prescribed probability. The information about the process is contatined in a transition matrix, which gives the probability of transitioning from one state to another in a certain time. This tends to have embedding problems [LL09] though. It supposes that in a small time interval there is no change of state, which is not necessarily true. We usually consider that the given time horizon in a transition matrix is approximately 1 year; however, states close to the default state can observe a high frequency of change of states. The CTMP allows jumps between states to occur randomly, at any time. A walker in this scheme waits a random time (the waiting time) at their location, before making a jump to another state. An exponential waiting time distribution for the waiting time, with a constant transition rate, preserves the memorylessness of the process. This is an assumption which is difficult to accept [JT95] with respect to applicability to real world problems.

The continuous time semi-Markov process (CTSMP) is the generalization of CTMP which relaxes the exponential waiting time assumption, and considers two random variables running together: waiting times and jumps. The relaxation of the restriction on waiting time distribution means that in general this is not a Markovian process. We will perform a credit rating migration model based on this process by a detailed theoretical and empirical analysis.

1.2 Objectives

This thesis has two objectives:

The first is to derive the Kolmogorov forward and backward equations for the CTSMP, and illustrate specific cases with different waiting time distributions to apply to the credit rating problem.

The second goal is to develop an approximation of our CTSMP credit rating system using Monte Carlo techniques, and to compare the results with Markovian models.

1.3 Outline

This report covers work conducted and progress made during my studies of the PhD. In this introductory Chapter we presented a concise motivation to research the credit risk models founding on the CTSMP and the objectives that pursue this work. Chapter 2 introduces the basic definitions of credit risk analysis, the rating credit system and gives a general framework of legal structure in order to perform a credit risk rating model.

In Chapter 3, we present a literature review which focuses on the advantages of PRFM models over the others.

In Chapter 4, we present an overview of the different Markovian and non-Markovian processes and how they can be applied in order to develop a credit risk model. Here we also introduce the concept of fractional derivatives.

In Chapter 5, we derive the CTSMP forward and backward differential equations. We also give these equations for the particular cases of gamma, exponential, and power-law waiting time distributions.

In Chapter 6, we perform an empirical analysis of our CTSMP approach by approximating it Monte Carlo methods and compare results with DTMP and CTMP.

Finally, some conclusions are drawn and directions for future works and research outlined in the last Chapter.

Chapter 2

Credit Risk Models

In this chapter we will introduce the basic concepts and requirements that a credit risk model must fulfill in order to be applied to the financial system. In section 1, we define the concept of credit, a credit system, and provide an overview of fixed income securities. In Section 2, we define the probability of default (PD), the loss given default (LGD), and describe the different classification of credit risk models. In section 3, we provide an overview of the credit rating system, describe the rating process and introduce the main credit rating agencies and its evaluation system. In section 4 we provide a summary of the regulative framework.

2.1 Credit and Types of Credits

In this section we define some terms necessary for a better understanding of a credit risk system and provide an overview of fixed income securities. A detailed description of this subject is given in [Fab04]

Credit is a means by which one party, the obligor, obtains a resource, in order to get an economic benefit either in the present or in the future. It is provided by another party, the creditor, in exchange of a future promise of payment equal to the amount borrowed plus a additional amount. Such a credit can be a corporate, consumer, investment, public, real state, etc.

One of the most popular credit products are fixed income securities. They are instruments by which a firm or a government obtains funds in exchange for fixed payments, that include regular interest plus the amount borrowed. If the obligor misses making any such payments the borrower may force it into bankruptcy, although this can vary depending on the legal framework and the conditions of the security. A fixed income security is made up of the following elements: the indenture, which is the document that states all the terms of the instrument; the obligor, which is the entity who issues the instrument and is therefore responsible for making payments; the creditor, which is the entity who invests in the instrument; the principal, which is the amount borrowed; the face value, which is the amount to paid to the creditor at maturity; the coupon, which is the nominal rate to pay each year in terms of a percentage of the principal; the maturity, which is the date of the end of the instrument, in which the issuer must return the face value.

Nowadays, there are a variety of such instruments, but the most used are:

- Treasuries: are instruments issued by the U.S. government and represent the safest investment on the market and, therefore, their coupons are the lowest. Their maturities can be up to 30 years.

- Municipal bonds: are issued by county, state or city governments, and their goal is funding to develop or improve public infrastructure, but sometimes they are issued to fund their ongoing operations. They usually pay interest periodically and at maturity pay a face value.

- Corporate bonds: they are issued by public firms to finance their expansion projects or ongoing operations and are traded on public markets. Their coupons mainly depend on the rating given by an agency, although some external factors can influence. They may pay interest periodically and pay the face value at maturity or, in the simplest form, a payment at maturity, this instrument is called a zero coupon bond and its operating mechanism is shown in figure 2.1. Most of this work will focus on these securities.

- Asset-backed securities: They refer to a bundle of assets that are not possible to be sold individually, but used to generate regular income for firms. They are traded in public markets and examples include home equity loans, auto loans, credit card receivables, student loans, etc.

- Preferred stock: are instruments that pay regular interest quarterly and a face

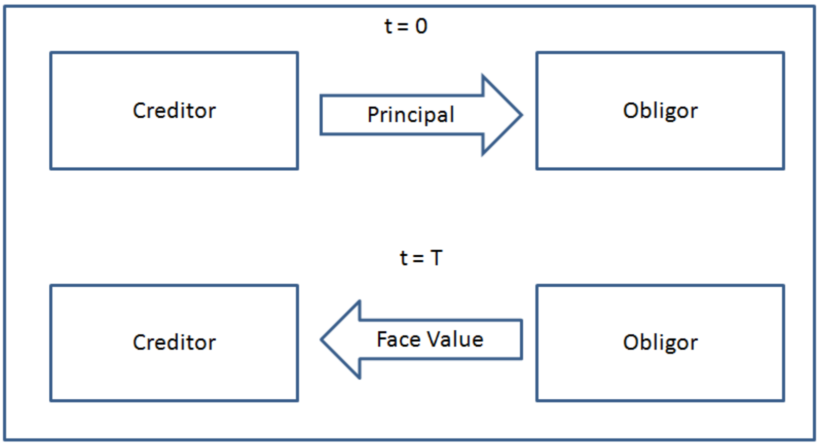

Figure 2.1: Cash flow of a zero-coupon bond. Extracted from [SSS07]

value at maturity but, unlike corporate bonds, in case a firm goes to bankruptcy they can return their investment before a regular bondholder.

The main advantage of these instruments is that they ensure regular income for investors and offer less volatility than other stocks, although sometimes their return and stability may vary. Thus, investors should analayse various aspects of instruments, strategic plans of the obligor, financial statements, macroeconomic conditions.

2.2 Probability of Default and Credit Rating Models

In this section we define the PD, the LGD and the describe the different classifications of credit risk models. The main event concerning the party lending money is the event that the other party defaults on their loan. This can result in severe losses for the lender. Thus its probability of occurrence must be estimated, and from this a risk rate can be obtained, i.e. an interest rate. Similarly, this part must analyze the LGD which, as its name implies, is the percentage lost in the event of default.

Nowadays, there are several methodologies to model a credit risk and they are broadly divided in three categories. Namely, structural form models (SFM), reduced form models (RFM) and factor models (FM). SFM are based upon the Black & Scholes theory for option pricing and the Merton model [BOW02]. This, in turn, can be divided into three generations. The first generation assumes a risky bond in order to approach the PD of a firm, which occurs if its market value is lower than the its liabilities at the maturity time. The second generation relaxes the default at maturity time assumption, such that this event this event can take place at anytime during the bond's lifetime. The third generation represents the VaR and assumes that LGD is a random variable related to PD. However, in some literature considers these models as an independent classification divided in two categories: the default mode models and mark to market models. In the former, default risk is considered a two state system (default and survival), such that the PD can be discussed as being a binomial variable. The latter assumes a multi-state system thought of as a multinomial distribution in which a loss is equivalent to a migration to a lower state.

RFM mainly focuses on the accuracy of the PD, such that it is more important than an intuitive economical interpretation. It can be classified as an individual level reduced form model (ILRFM) and portfolio reduced form model (PRFM). The former is based on a credit scoring system (two-state or multistates), and the latter assumes an intensity jump process. This work is based on these models. FM is an approach based on prediction of the random variable default by modelling a hazard rate based on latent exogenous and endogenous variables, where the former are mainly economical variables and the latter are firm's variables such as asset value. It assumes that the obligor will default if the system falls below a given threshold. Some authors consider these models as a sub-classification of RFM, but it has a structural part, so could be classified as a family of model by itself. In the next chapter we will provide a detailed literature review of each classification.

2.3 Credit Rating Agencies

In this section we provide an overview of a credit rating system, responsible agencies issue a rating and different scales used. A detailed review of this subject is provided in [TCI03, NRS12].

A credit rating is an evaluation of the worthiness of an obligor (firm or government), that issues a fixed income security, to be able to achieve its payments in full and on time. The rating is assigned by a credit rating agency (CRA) which, through a standardized process, provides a rating to an obligor according to a defined scale. This rating is translated into the cost of capital, so that a better rated firm can get a lower interest rate. Thus, CRA play a crucial role in the fixed income securities market because it contributes to the decision of investors, by summarizing a lot of data in a single rating. Therefore, they are forced to have a high accuracy in ratings assigned and to review them in defined periods of time, they also should not involve capital markets transactions in order to maintain investor confidence. They are regulated by a technical committee whose main function is to state the principles of the activities of a CRA. [TCI05] gives the code of conduct fundamental of a CRA.

There exists a number of CRA all around the world, most of them are focused on specific industries or markets, the most important are Standard & Poor's, Moody's Investor Service, and Fitch Group. The Securities and Exchange Commission, in 1975, designated them as the three Nationally Recognized Statistical Rating Organizations [SEC08] in the USA, and are considered the big three credit rating agencies. Currently there are other agencies recognized, namely: A.M. Best Company Inc, DBRS Ltd, Japan Credit Rating Agency Ltd, and Rating and Investment Information Inc. Each agency has its own methodology in order to express its rating opinion of an instrument issued by a firm. Such opinion is typically expressed in a scale of letters grades, for example, from "AAA" to "D" where the former represents the best state and the latter the default state.

In order to assess a firm, a CRA considers current and historical information in order to identify strengths and weaknesses events that could affect the credit quality of a debt issuer. Information can be quantitative or qualitative, such as legal structure, business plans, financial statements and indicators, forecasting, macroeconomic environment, industry trends, position in industry and competitiveness, credit history [SPs11]. This evaluation is usually paid for by the obligor and is made public immediately thereafter, but sometimes it requests a private rating of an instrument. Figure 2.2 shows the typical rating process followed by S&P.

Standard & Poor's Rating Services is a rating agency with offices in 23 countries and is one of the oldest companies of its kind, being more than 150 years old. It is a division of McGraw-Hills, and is focused on providing index information, risk analysis and research in the world markets in order to give confidence to investors. To rate a firm, the agency uses a scale from AAA to D, where the former is best state and the latter is the default state. Every intermediate state has three positions, the upgrade

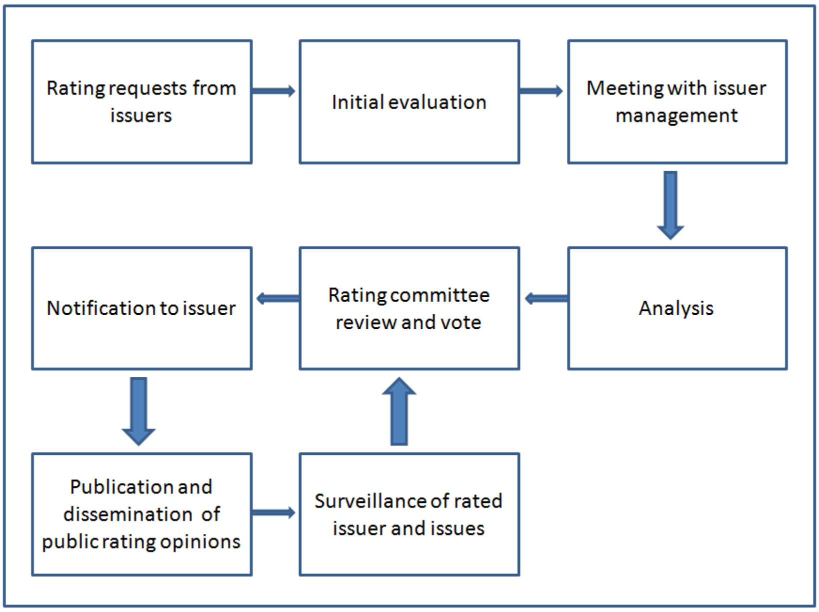

Figure 2.2: Process of assessment, source [SPs11]

(+), downgrade (-), and uncertain, table 2.1 describes the rating scale for long term fixed income securities [SPs11].

Moody's is a credit rating agency founded in 1909 and offers credit ratings services, development of risk analysis tools and research in order to contribute to the efficient operation of fixed income securities market. Moodys rating system is based on the LGD in case of default and its aim is to provide a simple mechanism that can predict the creditworthiness of a security in order to facilitate decision-making of investors. The long term fixed income securities use a scale of states from Aaa to C, where the former is the highest and the latter is the lowest. In turn, intermediate states are added the numeric index 1,2 and 3, where the lowest number represents the highest rating. Table 2.2 shows the Modys scale rating system [Moo11] .

Fitch Group is a firm that provides products and services to the fixed income securities market in order to make more timely and informed business decisions [Fit12], it was founded in 1913, and is divided into Fitch Rating and Fitch solution. It is the smallest of the three big rating agencies, and its main function in the market is that of a tie-breaker if Moody's and S&P ratings differ. In order to evaluate long term income securities, Fitch use the S&P rating scale system (see table 2.1).

Rating	Interpretation	Grade
AAA	Extremely strong capacity to meet its financial commitments.	Investment grades
AA+	Very strong capacity to meet its financial commitments	
AA		
AA-		
A+	Strong capacity to meet its financial commitments but is more susceptible to the adverse effects of changes in circumstances and economic conditions than obligors in higher-rated categories	
A		
A-		
BBB+	Adequate capacity to meet its financial commitments. However, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity of the obligor to meet its financial commitment	
BBB		
BBB-		
BB+	Less vulnerable in the near term than other lower-rated obligors. However, it faces major ongoing uncertainties and exposure to adverse conditions, which could lead to the obligor's inadequate capacity to meet its financial commitments	
BB		
BB-		
B-	More vulnerable than the obligors rated 'BB', but the obligor currently has the capacity to meet its financial commitments. Adverse will likely impair the obligor's capacity or willingness to meet its financial commitments	Speculative grades
B		
B+		
CCC+	Currently vulnerable, and is dependent upon favorable business, financial, and economic conditions to meet its financial commitments	
CCC		
CCC-		
CC+	Currently highly vulnerable	
CC		
CC-		
C	highly vulnerable, perhaps in bankruptcy or in arrears but still continuing to pay out on obligations	
D	has defaulted on obligations and S&P believes that it will generally default on most or all obligations	

Table 2.1: Standard & Poor's credit rating system for long term fixed income securities

2.4 Regulative Framework

In this Section we introduce a general overview of banking regulatory framework. A detailed explanation is provided in [Bas99, Bas05a, Bas06, Bas04a, Bas05b, Bas01, Bas04b].

The Basel Accord is a number of recommendations created through by central banks of a group of 10 countries, with the aim of establishing a set of minimum capital holding requirements in banking. In 1988 the Basel I agreement was created, and subsequently updated in 2004 because of economic changes that occurred during this period. Subsequently, in 2010 the Basel III Accord was created in response to the world financial crisis. Basel Accords are based on three fundamental pillars: minimum capital requirements, supervisory review, and market discipline.

PD plays a very important role in the former since from it estimates the market risk, and the Accords state various rules for its adequate estimation. The Accords

Rating	Interpretation	Grade
Aaa	The highest quality and lowest credit risk.	
Aa1		
Aa2		
Aa3	High quality and very low credit risk.	
A1		Investment grades
A2		
A3	Upper-medium grade and low credit risk	
Baa1		
Baa2	Medium grade, with some speculative elements and moderate credit risk.	
Baa3		
Ba1		
Ba2		
Ba3	There are speculative elements and a significant credit risk.	
B1		
B2		
B2	Speculative and a high credit risk	
Caa1		Speculative grades
Caa2		
Caa2	Poor quality and very high credit risk.	
Ca	Highly speculative and with likelihood of being near or in default, but some possibility of recovering principal and interest.	
C	The lowest quality, usually in default and low likelihood of recovering principal or interest.	

Table 2.2: Moodys credit rating system for long term fixed income securities

require banks to use proper techniques for estimating the long run average PD for each credit rating. Such techniques can be based on their own qualitative reasons, or on statistical models. Additionally, it states that banks must record their estimation techniques, and they must be compared to others and adjustments be made if they are necessary.

When a Bank chooses to draw upon their own experience, it must show the standardization of its criteria of estimations. In case of data being limited or the methodology changing, it should consider a margin of error in the PD. Similarly, the Accord authorizes the generation a joint information system with other banks in order to estimate the PD, as long as the criteria are comparable.

On other hand, when a bank considers a statistical model, it may associate its internal systems of classification with that used by an external credit evaluation institution. This is in order to compare the rating of an obligor, avoiding possible biases or inconsistencies that can exist. Although it is allowed to use a simple average to estimate the PD of an obligor in a given state, the agreement empowers to use a credit risk model as a primary or partial basis for estimating PD.

Thus, a credit risk model must satisfy the following:

- The model must show its predictive accuracy, such that on average it should be accurate for the borrower which is applied. Similarly, it must ensure that the model does not influence the regulatory capital requirements.
- It must consider a system that allows verifying of the completeness and appropriateness of the data, in order to achieve an appropriate estimate of PD.
- It must be shown that the sample used is representative of the whole market.
- There should be a guide that describes the various judgment criteria used for the proper analysis of results of model.
- It should establish standard procedures to ensure the adequate assignment of obligors to the rating system, such that the potential associated error is limited.
- There should be a mechanism of periodic validation of estimated values of the model against observed values.

The Accord also states that historical observation of at least five years should be considered. In case much older observations are taken, it must be ensured it provides relevant information to the estimation.

The Basel III Accord was created because of lack of official supervision to CRAs that caused incorrect rating assignment of various fixed income securities. Thus, in the PD framework, the main change with respect to previous Accords is the creation of three parties to carry out a rating: the regulatory agency, rating agency, and the bank. Other important changes can be found in [Bas11a, Bas11b].

Chapter 3

Literature Review

In this chapter we will provide a historical review of the evolution of the credit risk models. We will discuss structural form models (SFM), reduced form models (RFM), factor models (FM), and other relevant approaches. We will also focus on the difficulties of the real application of these approaches.

3.1 The Structural Form Models

The Merton Model [Mer74] represented the first attempt at a credit risk model and is based on the option pricing theory [BM73]. Merton derived a formula to estimate the probability of default (PD) by assuming firm issues a zero-coupon bond that represents its entire debt. In this model the PD is equivalent to the probability that the value of the firm's assets are less than the face value of a given bond, or similarly, the probability that market value of a firm is lower than its liabilities. Derived from the Merton model, other models were developed subsequently by modifying the original one by relaxing one or more of the assumptions. So, [BC76] provides a more complete model by assuming firms with complex structures of capital and debt. Whilst, [Ges77] extended the model to a bond in which interest is paid periodically and the default events can happen either at maturity time or every time the coupon is paid, and [Vas84] makes a distinction between short and long term debts. These models are known as the first generation of structural form models. [JMR84] found that these approaches are very useful in qualitative aspects of credit risk, but that in practical applications they are not realistic because of the following reasons:

- The default event can occur just at maturity of the bond.
- The default risk of a firm should be estimated according to its composition in financial statements rather than to assuming the issue of a zero coupon bond.

Meanwhile, [JT94] found, with empirical evidences, that the absolute priority rule is often violated and the lognormal distribution assumption tends to overstate the lost given default (LGD) in the event of default. In response, a second generation of SFM were developed by [KRS93, HW95, NSS93, LS95]. These approaches assume that default may occur at anytime during the bond's lifetime, so the event of default happens when the firm's assets value are less than a reference indicator. [EYH01] found that despite these improvements, these approaches have relatively poor empirical performances that are caused by:

- Asset value of a firm is a non-observable variable and we are required to estimate it.
- A fixed income security uses to have downgrades when it is closed to the default event and by SFM it is not possible to model it.
- Most SFM assume that the market value of a certain firm can be seen at any time

As a result, the event of default can be identified just before it occurs, [DL00] concluded that is better to use an approach based on a multistate system that estimates the default event before it occurs. The third generation of the SFM are the credit value-at-risk models, whose development has been encouraged by the greater significance of credit risk analysis. for example, banks and consultants have developed and patented their own model. These include J.P Morgans CreditMetric [JPM97], Credit Suisse Financial Products Credit Risk+ [CF97], Mckinseys CreditPortfolioView [Wil98] and Kamukura Risk Manager [Kam08].

3.2 The Reduced Form Models

3.2.1 Individual Level Reduced Form Models

Indiidual leve reduded form models (ILRFM) were proposed by [Alt68], who used linear or binomial models (such as logit or probit) for differentiating default from non-default firms. The approach focuses on estimating coefficients and assigning a Z-score to an obligor. After its proposal, many models based on credit scoring system have been widely used in credit risk analysis. [AN97] analyzed various predictor variables in order to identify the most significant ones on the event of default and found that indicators related to profitability, liquidity and leverage were the most representative.[JR03] proposed a method to estimate portfolio credit without bias, and [Li09] developed a new approach by three kinds of two stage hybrid models of logistic regression artificial neural network. Finally, [LMS07], showed that traditional accounting based credit scoring had less explanatory power in non-profit firms than in profit-making ones. [AT08] used data from the UK in order to compare RFM and SMF, and they found that there were few differences between SFM and scoring models in terms of predictive power, [DHS09] found similar results. The main weakness of these approaches is they are founded in financial statements and indicators as forecasting variables, such that they do not usually adjust in a fast way to the changing conditions of markets [CZ09]; So, tendencies are focused on dynamic portfolio models more than a purely static and individual level credit scoring model.

3.2.2 Portfolio Reduced Form Models

Portfolio reduced form models (PRFM) are an attempt to overcome the weaknesses of SFM and, in empirical studies, are reported to perform better in capturing the properties of credit risks [CZ09]. These approaches were originally introduced by [JT92], in which they decomposed the credit spread into two components: PD and LGD. So this approach is based on the distributions of such variables. The simplest model was proposed by [JT95] who modelled the default event as a Poisson process with instantaneous transition rate λ and default time τ exponentially distributed. However, it assumed that λ was constant over the time and across the states, which in reality is not easy to accept. In order to relax this assumption, [MU98] defined

$\lambda(t)$ as a function of the excess return on the issuers equity, and [CJL07] worked on duration models. Similarly, [Lan98] modelled the intensity as a Cox process. [JT97] modelled the PD by the DTMP approach, where the default event was considered as an absorbing state, such that from a historical database was obtained a transition matrix in order to estimate migration probabilities at any time. This work served as the basis for other studies, including: [KF06, FS08, KL08, BDK02]. [LKS01] applied both discrete time Markov process (DTMP) and continuous time Markov process (CTMP) models in empirical studies. Meanwhile, [LL09] studied differences between DTMP and CTMP for estimating migration probabilities. The conclusions taken were:

- DTMP contained the embedding problems whereas the CPMP did not.
- DTMP tends to underestimate migration probabilities since it is not consider changes occurred within a year, which can be often in non-investment states.

Meanwhile, some works about the suitability of Markov processes in the credit risk environment have been done. They mainly concern the following:

- The waiting time in a state: the time that a firm will remain in a certain state is related with the time that it has been in such a state. [DS03].
- The rating assignment and time: assigning the rating to a firm is related to the time it is made and in particular with the business cycle [NPV00].
- Dependence of the current state: the current state may depend of various previous states assigned to a firm, not only in the previous one. [CF94].

[DJM05] and [MSL06] dealt with the first problem by means of semi-Markov process, that relaxes the exponential waiting time distribution assumption. The second problem has been solved with the addition of a non-homogeneous environment [DJM09a, VV06]. The third problem was dealt with by the backward recurrence time [DJM06, DJM09b] and by with a hidden semi-Markov process [BL07]. Other PRFM were also performed for the mortality analysis of survival time of a loan, and was introduced by [AS00], who applied actuarial methods to study mortality rates of obligations. Subsequently, the analysis is based on the recovery rates and LGD values when the event of default occurs, and the correlation between these variables and the frequencies of such an event.[NPV00] studied the probability of survival through time, based in a representative set of banks, and [DC06] applied a mortality analysis to LGD.

3.3 Factor Models

The simplest version of a factor models (FM) is considered the one proposed by [Tas04], who assumed a systematic risk factor and an individual error term that follows the standard normal distribution. Alternatively, other single FM were performed. These include [DP02, DP04, RS04, Wit09]. They were developed from the internal rated base approach and use FM to calibrate risk weights. [kup07] considered two vectors of explanatory variables for latent variables: a set of macro-economical variables, such as GDP; and a set of firm-specific variables which account for individual risk, usually for the the return rates. [Ped05, BFL01] considered multifactor models, and [EB07] extended the standard single FM to multi-period framework from a set of return rates in different time periods. On the other hand, [Gor00, DP02] showed a factor model based on a gamma distribution with mean 1 and variance σ^2.

3.4 Other Relevant Approaches

Recently, new approaches have been developed mainly focusing on the correlation between the PD and the LGD, these include [BMZ01, JP03, Fry00a, Fr00b, CG03, SP02]. [BMZ01] took an approach to make more flexible the correlation between the PD and LGD, empirical results found evidence to support a negative correlation between these variables. Previously, Frye [Fry00a, Fr00b] also found an inverse relationship in PD and LGD. Its model is founded on the model developed by [Fin99] and [Gor00], and the approach assumes that certain economical indicators can determine the event of default. Meanwhile, [JP03] created a model that combined features of SFM and RFM by using the option pricing framework and assuming that the borrowing firm's total assets determines the PD, and also found an inverse correlation in PD and LGD. [CG03], based on some historical databases, analysed correlation patterns between LGD measures and PD and found a correlation close to zero. However they were more in line with Frye's results [Fr00b] when the sample was limited to the period 1988-1998. Moreover, [SP02] studied correlations between LGD and PD based on Moodys historical database, by using the probability techniques of extreme value analysis and nonparametric statistics. Similarly, they studied the patterns of correlation on VaR measures.

Chapter 4

Mathematical Background

In this chapter we will present an overview of Markov and semi-Markov processes in order to apply them to credit risk models. In Section 4.1 we introduce the discrete time Markov process (DTMP) and its transition probabilities, in Section 4.2 we describe the Continuous time Markov process (CTMP) concept and obtain the migration probability for two and three states respectively and express a way to obtain it for a n states. In Section 4.3 we define the discrete time semi-Markov process (DTSMP) and set up an equation in order to obtain the migration probability, and in Section 4.4 we define a Continuous time semi-Markov process (CTSMP) and introduce an approximation of migration probability and its characteristics. Finally, we present an overview of fractional derivatives. All these concepts will be used in the following chapters.

4.1 Discrete Time Markov Process

The purpose of this section is to introduce the DTMP concept and its application in credit risk models. [Chu01, JM06] provide a good introduction to this subject. This will be of most use to us in Chapter 6, when we simulate our system.

4.1.1 Definition

Introduce a system S with m possible states with space state $I = [1, 2, \ldots m]$. The system S performs randomly in the discrete time set $T = [t_0, t_1, \ldots]$. Define X_{t_n} as the state of S at time $t_n \in T$. Assuming that in small intervals of time there are no changes, we say that the random sequence $(X_{t_n}, t_n \in T)$ is a Markov process iff for all

$x_{t_o}, x_{t_1}, \ldots, x_{t_n} \in S$

$$P[X_{t_n} = x_{t_n} | X_{t_0} = x_{t_0}, \ldots, X_{t_{n-1}} = x_{t_{n-1}}] = P[X_{t_n} = x_{t_n} | X_{t_{n-1}} = x_{t_{n-1}}]. \tag{4.1}$$

Similarly, the random sequence $(X_{t_n}, t_n \in T)$ is homogeneous iff

$$P[X_{t_n} = x_{t_n} | X_{t_{n-1}} = x_{t_{n-1}}] = P[X_{t_1} = x_{t_1} | X_{t_0} = x_{t_0}]. \tag{4.2}$$

i.e. jumps between states is independent of the time; otherwise it is non-homogeneous. Now, we introduce the migration probability in one step

$$P[X_{t_n} = j | X_{t_{n-1}} = i] = \rho_{ij}, \quad \text{for all } i, j \in I, t_n \in T. \tag{4.3}$$

Consider the transition matrix P defined as $P = [\rho_{ij}]_{mXm}$, for all $i, j \in I$. with the following properties for its elements

- $\rho_{ij} \geq 0$, for all $i, j \in I$, and
- $\sum_{j \in I} \rho_{ij} = 1$, for all $i \in I$.

Define the vector $p = (p_1, p_2, \ldots p_m)$ as the initial condition probabilities, with $p_i = P[X_0 = i], i \in I$. This probability should satisfy the following conditions

- $p_i \geq 0$, for all $i \in I$, and
- $\sum_{i \in I} p_i = 1$.

Thus, a homogeneous DTMP is characterised by the couple (p, P). If $X_0 = i$, i.e. if the probablity that S starts from state i is equal to 1, then the vector p will be $p_j = \delta_{ij}$, for all $j \in I$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{Otherwise.} \end{cases}$$

4.1.2 Transition Probabilities

Define the transition probability as $p_{ij}(n) = P(X_{r+n} = j | X_r = i)$, for all $r, r+n \in T$. Considering the homogeneity assumption and $n = 2$ gives

$$\rho_{ij}(2) = \sum_{k \in I} \rho_{ik} \rho_{kj} \quad \text{for all } i, j \in I. \tag{4.4}$$

These probabilities are contained in the two-step matrix $P^{(2)} = [\rho_{ij}(2)]_{m\times m}$ $i, j \in I$. Notice that this matrix is equivalent to P^2, similarly, by the induction method it can be shown that

$$P^{(n)} = P^n, \quad i, j \in I. \tag{4.5}$$

It implies that $P^n = [\rho_{ij}(n)]_{m\times m}$, $i, j \in I$, i.e. the migration probability $\rho_{ij}(n) = P[X_{r+n} = j|X_n = i]$ is the ij-th element of the matrix P^n.

On the other hand, define the initial state probability of X_n as $p_i(n) = P(X_n = i), i \in I$ and $n \in T$. Which we obtain from $p_i(n) = \sum_{j\in I} p_j\rho_{ji}(n)$, which in matrix notation can be represented as

$$p_i(n) = pP^n. \tag{4.6}$$

4.1.3 Discrete Time Markov Process with and Absorbing State

We say that a state $i \in I$ is absorbing iff

- $\rho_{ij}(n) = 0, \quad$ for all $j \neq i, n \in T$, and
- $\rho_{ii}(n) = 1, \quad$ for all $n \in T$.

So, the system cannot leave when it is in an absorbing state. A DTMP is called absorbent if it has at least one absorbing state and the system can migrate from any non-absorbing state to an absorbing state. See figure 4.1. When we have a process with at least one absorbing state, the main information that we can get is:

- The expected time before the process goes to the absorbing state.
- $\lim_{t\to\infty} \rho_{ij}(t)$, for all $i, j \in I$

So, we must define a sub-matix A of P that contains the migration probabilities between non-absorbing states in one step. Likewise, A^2 gives us the migration probability between non-absorbing states in two steps, and A^n provides the same probability in n steps. Thus, the expected time before being absorbed can be derived as follows

$$I + A + A^2 + A^3 + \ldots = (I - A)^{-1} = B, \tag{4.7}$$

where B is represents the expected time that the system will be in each non-absorbing state before be absorbed, and the sum of every row of B is the average time spent

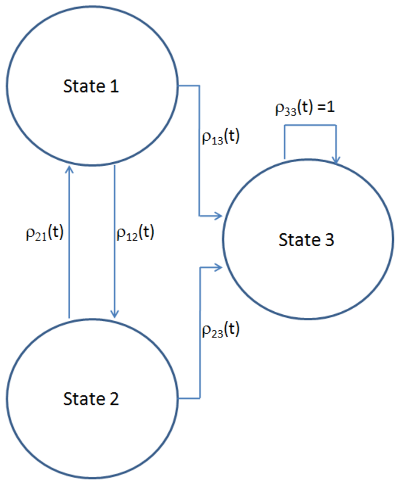

Figure 4.1: Process with one absorbing state

before moving to an absorbing state. Similarly, we can define a sub-matrix C of P from non-absorbing states to absorbing states. Thus, C represents the migration probability from a non-absorbing state to the absorbents in one step, and AC is the probability of jumping from a non-absorbing state to an absorbent in two steps and so on. Therefore, A^nC corresponds to the probability of jumping from a non-absorbing state to an absorbent in n steps. So

$$C + AC + A^2C + \ldots = (I + A + A^2 + A^3 + \ldots)C = (I - A)^{-1}C = BC \tag{4.8}$$

is the expected migration probability of moving from a non-absorbing state to an absorbing state in the long run.

4.1.4 Example

Suppose a credit risk rating system S with three possible states that performs randomly in the discrete set $T = [0, 1, 2, \ldots]$. Assume that S is homogeneous in time and the following probability transition matrix

$$P = \begin{pmatrix} 1-a & a & 0 \\ a & 1-2a & a \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 < a < \frac{1}{2}. \tag{4.9}$$

So, the matrix A of transition probabilities between non-absorbing states is given by

$$A = \begin{pmatrix} 1-a & a \\ a & 1-2a \end{pmatrix}. \tag{4.10}$$

So, from equation (4.7) we obtain

$$B = \frac{1}{a^2}\begin{pmatrix} 2a & a \\ a & a \end{pmatrix}. \tag{4.11}$$

Hence, the expected period before getting absorbed is $\frac{3}{a}$ for state 1 and $\frac{2}{a}$ for the state 2.

We can calculate the probability of moving from a non-absorbing state to an absorbing state in the long term, BC, where

$$C = \begin{pmatrix} 0 \\ a \end{pmatrix}. \tag{4.12}$$

Thus,

$$BC = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.13}$$

Therefore, in the long run with probability one, the system will be in the absorbing state. This result can be easily extended to a n-state model.

4.2 Continuous Time Markov Process

The purpose of this section is to introduce the CTMP and derive a solution for obtaining the migration probability. This will be of most use to us in Chapter 6 when we run our model.

4.2.1 Introduction

Introduce a continuous time random walk $(X_t : t \geq 0)$ with m possible states, $m \in \mathbb{N}$ and define by I the set of possible states with $I = [1, 2, \ldots, m]$. Suppose that the process begins in a state $i_1 \in I$ and stays there for a random time $T_1 > 0$, before jumping into a new state $i_2 \neq i_1, i_2 \in I$ in which the system stays for a random time $T_2 > 0$, and goes into a new state $i_3 \neq i_2, i_3 \in I$ and so on. This process is called a continuous time Markov process if we have the following property: for a system S, with the following waiting times $t_1 < t_2 < \ldots < t_k$, with its corresponding states $X(t_1), X(t_2), \ldots, X(t_k)$ we have

$$P[X(t_n) = i|X(t_1), X(t_2), \ldots X(t_{n-1})] = P[X(t_n) = i|X(t_{n-1})], \quad i \in I. \tag{4.14}$$

Furthermore, we have a homogeneous in time process iff:

$$P(X(t+s)=j|X(s)=i)=P(X(t)=j|X(0)=i),\quad \text{for all } t,s\in\mathbb{R}. \tag{4.15}$$

4.2.2 The Q-matrix

Define the function $f(h)=P(X(t+h)=j|X(t)=i),\ i,j\in I$. The main infomation that we obtain from this is the probability of jumping in small time interval, i.e. in small values of $h>0$. It is clear that $f(0)=0$ for all $j\neq i$ and, by assuming that f is differentiable at $t=0$, we obtain the derivative at $t=0$ ($f'(0)$).

By defining $f'(0)=\lambda_{ij}$ we have

$$\lim_{h\to 0}\frac{P[X(t+h)=j|X(t)=i]}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=f'(0)=\lambda_{ij}. \tag{4.16}$$

On the other hand, for small values of h, $f(h)$ can be expressed as

$$P[X(t+h)=j|X(t)=i]=\lambda_{ij}h+o(h). \tag{4.17}$$

where is $o(h)$ a function with the property that $\lim_{h\to 0}\frac{o(h)}{h}=0$. Since $f(0)=0$ for all $i\neq j$, the above equation can be expressed as

$$\frac{P[X(t+h)=j|X(t)=i]-P[X(t)=j|X(t)=i]}{h}=\lambda_{ij}+o(h). \tag{4.18}$$

Applying lim as $h\to 0$ we obtain

$$\lim_{h\to 0}\frac{P[X(t+h)=j|X(t)=i]-P[X(t)=j|X(t)=i]}{h}=\lambda_{ij}. \tag{4.19}$$

So, $f'(t)=\lambda_{ij}$ for all $j\neq i$. If $j=i$:

$$P[X(t+h)=i|X(t)=i]=1-\sum_{j\neq i}\lambda_{ij}h+o(h). \tag{4.20}$$

Let $\lambda_{ii}=-\sum_{j\neq i}\lambda_{ij}$ so

$$P[X(t+h)=i|X(t)=i]=1+\lambda_{ii}h+o(h). \tag{4.21}$$

It can be shown that $f'(t)=\lambda_{ii}$ for all $i=j$. Now, introduce the Q-matrix defined as $Q=[\lambda_{ij}]_{m\times m}$, for all $i,j\in I$, which characterises a Markov process $(X(t):t\geq 0)$. This matrix has the following properties:

- $\lambda_{ii}\leq 0$ for all $i\in I$.

- $\lambda_{ij} \geq 0$, for all $i, j \in I$
- $\sum_{j \in I} \lambda_{ij} = 0$ for all $i \in I$

Thus, with the assumption that this is constant through time, λ_{ij} is known as the instantaneous rate from i to j.

4.2.3 Exponential times

Assume a random time V before the system jumps, then given the Markov property we have

$$\begin{aligned} P[V > t+h|V > t, X(t) = i] &= P[V > t+h|X(t) = i] \\ &= P[V > h], \quad \text{for all } i, j \in I,\ t \geq 0. \end{aligned} \tag{4.22}$$

This random time V satisfies the following memoryless property: If at a certain time t the system has not jumped, the remaining time will have the same waiting time distribution as the original. Thus, given an exponential waiting time distribution with instantaneous transition rate λ and $P[V > t] = e^{-\lambda t}$, then

$$P[V > t+h|V > t] = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = P[V > h], \tag{4.23}$$

i.e. the exponential distribution has the memoryless property. Likewise, it can be shown that the exponential is the only distribution with such a property.

4.2.4 Kolmogorov Equations

Define the migration probability at time t as $\rho_{ij}(t) = P[X(t) = j|X(0) = i]$, for all $i, j \in I$ and its corresponding migration matrix at time t as $P(t) = [\rho_{ij}(t)]_{m \times m}$, where $\rho_{ij}(t)$ has the following properties:

- $\rho_{ij}(t) \geq 0$ and $\sum_{j \in I} \rho_{ij}(t) = 1$, for all $i \in I,\ t \geq 0$.
- $\lim_{t \to 0} \rho_{ij}(t) = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{Otherwise.} \end{cases}$

Now, define the following joint probability: Let $0 < t_1 < \ldots < t_n$ and $i, j_1, \ldots . j_n \in I$ then

$$P[X(t_1) = j_1, \ldots, X(t_n) = j_n|X(0) = i] = \rho_{ij_1}(t_1)\rho_{j_1 j_2}(t_2 - t_1) \ldots \rho_{j_{n-1} j_n}(t_n - t_{n-1}). \tag{4.24}$$

Similarly, introduce the Chapman-Kolmogorov equation

$$\rho_{ij}(t+s) = \sum_{k \in I} \rho_{ik}(t)\rho_{kj}(s), \quad \text{for all } i, j \in I,\ t, s \geq 0. \tag{4.25}$$

The above equation is proved as follows

$$\rho_{ij}(t+s) = \sum_{k \in I} P[X(t) = k, X(t+s) = j | X(0) = i]. \tag{4.26}$$

By equation (4.24) we obtain

$$= \sum_{k \in I} \rho_{ik}(t) P_i[X(t+s) = j | X(t) = k]. \tag{4.27}$$

Thus

$$= \sum_{k \in I} \rho_{ik}(t)\rho_{kj}(s). \tag{4.28}$$

Expresing the Chapman-Kolmogorov equations in matrix notation gives

$$P(t+s) = P(t)P(s), \quad \text{for all } t, s \geq 0. \tag{4.29}$$

By differentiating the equation with respect to t and s respectively, we have

$$P'(t) = QP(t) \ \text{ and } \quad P'(t) = P(t)Q. \tag{4.30}$$

These equations are known as Kolmogorov backward and forward equations respectively with $P(0) = I$ as initial condition. The above differential equation has the unique solution $P(t) = e^{Qt}$, where

$$e^{Qt} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{K!}, \tag{4.31}$$

with $Q^0 = I$.

4.2.5 Resolvents

Now, in order to find a solution of the Kolmogorov equation, we introduce the Laplace transform of a function f defined as (see [Dav02] for more details).

$$\mathcal{L}[f(t)] = \widehat{f}(s) = \int_0^{\infty} f(t)e^{-st}dt, \tag{4.32}$$

where s can be defined in a way that ensures that this integral can be solved. Now, suppose a CTMP $(X_t : t \geq 0)$ defined by its corresponding Q-matrix and migration matrix at time t, $P(t) = e^{Qt}$. Taking the Laplace transform gives

$$\mathcal{L}[\lambda] = \int_0^\infty e^{-\lambda t} P(t)dt = \int_0^\infty e^{-\lambda t} e^{Qt} dt = \int_0^\infty e^{-(\lambda - Q)t} = (\lambda I - Q)^{-1}, \quad \lambda > 0. \tag{4.33}$$

Notice that in order to derive a solution we define $s = \lambda$. It is clear that $(\lambda I - Q)^{-1}$ can be obtained for any value of λ that is not an eigenvalue of the matrix Q, i.e. for all λ with $\det|\lambda I - Q| \neq 0$, or equivalently where the characteristic polynomial is different to zero. So, to get a solution we must calculate $\mathcal{L}[\lambda] = (\lambda I - Q)^{-1}$ and take Laplace inverse in order to obtain $P(t)$. In terms of the migration probability, the solution is given by $\mathcal{L}_{ij}(\lambda)$ as follows

$$\mathcal{L}_{ij}[\lambda] = \int_0^\infty e^{-\lambda t} \rho_{ij}(t)dt = \hat{\rho}_{ij}(\lambda). \tag{4.34}$$

4.2.6 Interpretation of the Resolvent

The resolvent may have the following probabilistic interpretation. Consider a continuous random variable exponentially distributed and independent of the process of the system S, such that

$$P[Y \geq t] = e^{-\lambda t}, \quad t \geq 0. \tag{4.35}$$

and

$$P[Y \in t, t + dt] = \lambda e^{-\lambda t} dt, \quad t \geq 0. \tag{4.36}$$

Assume that S performs a CTMP. We want to know the probability of being in state $j \in I$ when Y occurs, thus

$$\begin{aligned}\rho_{ij}(Y) &= \int_0^\infty P[X(t) = j, Y \in (t, t + dt)|X(0) = i] = \int_0^\infty \rho_{ij}(t)\lambda e^{-\lambda t} dt \\ &= \lambda \mathcal{L}_{ij}[\lambda], \quad \lambda > 0.\end{aligned} \tag{4.37}$$

4.2.7 Two-State Markov Process Solution

Suppose a two-state CTMP with the following Q-matrix:

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}. \tag{4.38}$$

So, from the Kolmogorov forward equations (4.30) we obtain

$$p'_{11}(t) = -\lambda_1 p_{11}(t) + \lambda_2 p_{12}(t), \tag{4.39}$$

$$p'_{12}(t) = \lambda_1 p_{11}(t) - \lambda_2 p_{12}(t), \tag{4.40}$$

$$p'_{21}(t) = -\lambda_1 p_{21}(t) + \lambda_2 p_{22}(t), \tag{4.41}$$

$$p'_{22}(t) = -\lambda_1 p_{21}(t) + \lambda_2 p_{22}(t). \tag{4.42}$$

It is clear that

$$p_{i1}(t) + p_{i2}(t) = 1, \quad i = 1, 2. \tag{4.43}$$

So, by equation (4.39) and (4.43) we obtain

$$p'_{11}(t) = -\lambda_1 p_{11}(t) + \lambda_2 (1 - p_{11}(t)), \tag{4.44}$$

which we can be written as

$$p_{11}(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_1 + \lambda_2} p'_{11}(t). \tag{4.45}$$

The solution to this differential equation is

$$p_{11}(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}. \tag{4.46}$$

Similarly, we obtain the following equations

$$p_{12}(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}, \tag{4.47}$$

$$p_{21}(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}, \tag{4.48}$$

$$p_{22}(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}. \tag{4.49}$$

Applying the limit as $t \to \infty$, we obtain the stationary states

$$\Pi = \begin{pmatrix} \frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_1}{\lambda_1+\lambda_2} \\ \frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_1}{\lambda_1+\lambda_2} \end{pmatrix}. \tag{4.50}$$

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4.2.8 Three-State Markov Process Solution

Consider a three-state CTMP, where two states are 'good states' and the other represents the 'default state', with the following Q-matrix

$$Q = \begin{pmatrix} -a & a & 0 \\ a & -2a & a \\ 0 & 0 & 0 \end{pmatrix}, \quad a > 0. \tag{4.51}$$

We want to know $\rho_{ij}(t),\ i,j = (1,2,3),\ t \geq 0$, which can be solved it by Laplace transform as seen in Section 4.2.6, but an easier way is by the spectral decomposition. We know, by the the Kolmogorov equations, that $P'(t) = PQ = QP$ and that it has as unique solution $P(t) = e^{Qt}$. By Taylor's theorem

$$e^{Qt} = \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!}. \tag{4.52}$$

Substituting by the spectral decomposition of $Q = CDC^{-1}$

$$\sum_{n=0}^{\infty} \frac{(Qt)^n}{n!} = C \sum_{n=0}^{\infty} \frac{D^n t^n}{n!} C^{-1}, \tag{4.53}$$

where D is a diagonal matrix of eigenvalues of Q, and C is the matrix of by eigenvalues of Q. Thus, by calculating the eigenvector and eigenvalues and using (4.53) we obtain,

$$\rho_{11}(t) = 0.72369e^{-0.3820at} + 0.27623e^{-2.6180at}, \tag{4.54}$$

$$\rho_{12}(t) = 0.44722e^{-0.3820at} - 0.44722e^{-2.6180at}, \tag{4.55}$$

$$\rho_{13}(t) = -1.17089e^{-0.3820at} - 0.17089e^{-2.6180at} + 1, \tag{4.56}$$

$$\rho_{21}(t) = 0.44722e^{-0.3820at} - 0.44722e^{-2.6180at}, \tag{4.57}$$

$$\rho_{22}(t) = 0.27636e^{-0.3820at} + 0.72369e^{-2.6180at}, \tag{4.58}$$

$$\rho_{23}(t) = -0.72356e^{-0.3820at} - 0.27645e^{-2.6180at} + 1. \tag{4.59}$$

We can verify that

$$\rho_{11} + \rho_{12} + \rho_{13} = \rho_{21} + \rho_{22} + \rho_{23} = \rho_{31} + \rho_{32} + \rho_{33} = 1, \text{ and}$$
$$lim_{t\to\infty}\rho_{i3}(t) = 1, i = 1, 2.$$

i.e. in the long term a firm in this system will move, with probability one, to the 'bad state'.

This procedure presents the problem that for a large number of states the final equation is difficult to handle. However, with a good computational algorithm such a procedure is easy to solve it for any number of states.

4.3 Discrete Time Semi-Markov Process

The purpose of this section is to introduce the DTSMP concept in order to be applied in a credit risk system. [JM06] performs a detailed theoretical analysis, meanwhile [JM84] presents the process in a non-homogeneous environment. The equation for the migration probability was developed by [DM84], and [JM01] for homogeneous and non-homogeneous cases respectively. This will be of most use to us in Chapter 6, when we run our model.

4.3.1 Introduction

Define a system S with state space $I = [1, 2, ..., m]$. As with the DTMP, it assumes that there are no changes in small intervals of time, so that it evolves in the indexed time $T = [0, 1, 2, ...]$. The process can be described as a process with two-random variables that perform together known as $(X - T)$ process, where the sequence $(X_n, n \geq 0)$ represents the successive states of S and $(T_n, n \geq 0)$ gives the corresponding waiting times.

4.3.2 The $(X - T)$ process

The $(X - T)$ process is also known as discrete time Markov renewal process, with the following definitions:

$$P(X_0 = i) = p_i,\ i \in I, \quad \text{with} \sum_{i \in I} p_i = 1. \tag{4.60}$$

$$Q_{ij}(t) = P[X_n = j, T_n \leq t | X_0, \ldots, X_{n-1}; T_1, \ldots, T_{n-1}]. \tag{4.61}$$

.

$$= P[X_n = j, T_n \leq t | X_{n-1}] \quad i, j \in I,\ t \in T, \tag{4.62}$$

where $Q_{ij}, i, j \in I$ is a non-decreasing function on [0,1] We can also define the transition probability

$$h_{ij} = \lim_{t \to \infty} Q_{ij}(t), i, j \in I, \quad \text{with} \sum_{i \in I} h_{ij} = 1. \tag{4.63}$$

In matrix notations, these functions can be expressed by

$$Q = [Q_{ij}]_{m \times m},\ H = [h_{ij}]_{m \times m},\ p = [p_1, \ldots, p_m], \tag{4.64}$$

where, H is the transition probability matrix of the $(X-T)$ process, Q the semi-Markov matrix and p the vector of initial conditions. Thus, every couple (p, Q) defines a positive $(X-T)$ process. Now, we will define the unconditional waiting time probability as

$$\Phi_i(t) = P[T_{n+1} \leq t | X_n = i], \quad \text{for all } i, j \in I. \tag{4.65}$$

It is the probability that the system will move from state i at a time t . Now, introduce the conditional waiting time probability as

$$G_{ij}(t) = P[T_{n+1} \leq t | X_n = i, X_{n+1} = j], \quad i, j \in I, \ \ t \in T. \tag{4.66}$$

It can be obtained as follows

$$G_{ij}(t) \begin{cases} \frac{Q_{ij}(t)}{h_{ij}} & \text{if } h_{ij} \neq 0, \\ U_1(t) & \text{Otherwise,} \end{cases}$$

where

$$U_1(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

4.3.3 Semi-Markov Process Definition

Let $Z = (Z_t, t \in T)$ be the DTSMP with the following migration probability

$$\rho_{ij}(t) = P[Z_t = j | Z_0 = i], \quad \text{for all } i, j \in I. \tag{4.67}$$

[DM84] found the following solution

$$\rho_{ij}(t) = dij(t) + \sum_{k \in I} \sum_{\tau=1}^{t} v_{ik}(\tau) \rho_{kj}(t-\tau), \ \text{for all } i, j \in I, \ \ t \in T, \tag{4.68}$$

where $d_{ij}(t) = (1 - \Phi_i(t))\delta_{ij}$ is called the survival function and δ_{ij} represents the Kronecker delta function. Likewise

$$v_{ij}(t_n) = Q_{ij}(t_n) - Q_{ij}(t_{n-1}), \ \text{for all } i, j \in I, \tag{4.69}$$

Equation (4.68) can be written in matrix form as

$$P(t) - \sum_{\tau=1}^{t} V(\tau) P(t-\tau) = \Psi(t). \tag{4.70}$$

with

$$P(t) = [\rho_{ij}(t)]_{m\times m}, \quad V(t) = [\upsilon_{ij}(t)]_{m\times m}, \quad \Psi(t) = [d_{ij}(t)]_{m\times m}, \quad i,j \in I, \ \ t \in T. \tag{4.71}$$

[DM84] showed that the matrix $P(t)$ is stochastic and that the equation (4.71) admits a unique solution. Such an equation can be solved in the finite-time horizon by recursion: At time 0 we have, $D(0) = P(0) = I$; so now, we can calculate $P(1)$, and with these matrices we can compute $P(2)$ and so on.

4.3.4 Waiting Time Distribution Assumption

Assuming that the waiting time distribution does not depend on the state to migrate, gives

$$G_{ij}(t) = \Phi_i(t), \quad \text{for all } i,j \in I, \ \ i \neq j, \ \ t \in T. \tag{4.72}$$

So, the Q-matrix is

$$Q_{ij}(t) = h_{ij}\Phi_i(t), \quad \text{for all } i,j \in I, \ \ i \neq j, \ \ t \in T. \tag{4.73}$$

And equation (4.69) can be written as,

$$\upsilon_{ij}(t) = Q_{ij}(t) - Q_{ij}(t-1) = h_{ij}(\Phi_i(t) - \Phi_i(t-1)), \text{for all } i,j \in I, \ \ i \neq j, \ \ t \in T. \tag{4.74}$$

This equation will be used in Chapter 6 for estimating migration probabilities of a credit risk model under the assumption that the system undergoes a DTSMP.

4.4 Continuous Time Semi-Markov Process

CTSMP represents the extension of CTMP and was defined by [Lev54] and [Smi55] independently. Recently, [JM06] performed a detailed analysis, and [JM07] highlighted different applications in finance. In this section, we present an overview of the CTSMP for credit risk modelling that will be used in Chapters 5 and 6.

4.4.1 Introduction

As usual, define a system S with state space $I = [1, 2, \ldots .m]$, and initial state $X_0 \in I$. The system stays a random time $T_1 > 0$ in this state, and jumps into another state

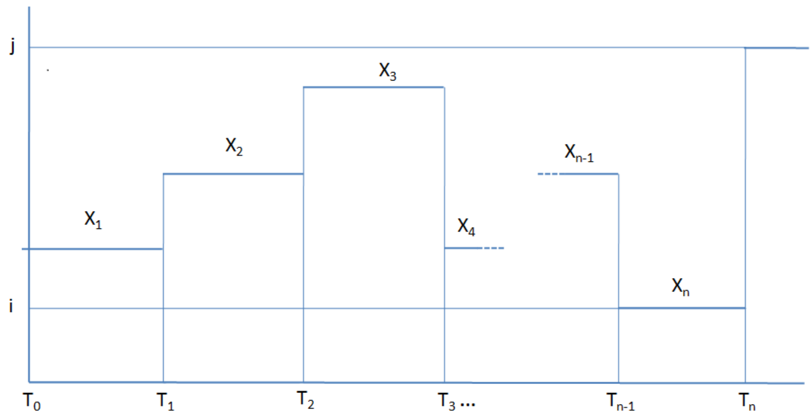

Figure 4.2: Realization of a $(X - T)$ process

$X_1 \in I$, again, stays for random length of time $T_2 > 0$ before jumping into state $X_2 \in I$, and so on. Thus, we have two random variables that perform together known as $(X - T)$ process, where the discrete random variable $(X_n, n \geq 0)$ is the state of S, and the continuous random variable $(T_n, n \geq 0)$ is the waiting time in X_{n-1} before jumping. Figure 4.2 shows a typical realisation of this process.

4.4.2 $(X - T)$ Process

Let the $(X - T)$ process be on the complete probability space (Ω, F, P) such that $X_n : \Omega \to I, \quad T_n : \Omega \to \mathbb{R}^+ + [0]$. with the following characteristics:

- Initial probabilities

$$P(X_0 = i) = p_i, \quad i \in I. \tag{4.75}$$

  with, $\sum_{i \in I} p_i = 1$.

- The semi-Markov kernel.

$$P[X_{n+1} = i, T_{n+1} \leq t | (x_k, t_k), k = 1, 2, \dots n] = Q_{x_n i}(t). \tag{4.76}$$

  According to the Markov property

$$Q_{x_n i}(t) = P[X_{n+1} = i, T_{n+1} \leq t | (x_n, t_n)]. \tag{4.77}$$

- Transition probability: Let $Q_{ij(t)}$ be a non-decreasing real function on [0,1], we define the transition probability from i to j as

$$h_{ij} = \lim_{t \to \infty} Q_{ij}(t), \quad i, j \in I. \tag{4.78}$$

$$\sum_{j \in I} h_{ij} = 1, \quad i \in I. \tag{4.79}$$

In matrix notation, these functions are expressed as:

$$Q(t) = [Q_{ij}(t)]_{m\times m},\ H = [h_{ij}]_{m\times m},\ p = (p_1, \dots, p_m], \quad \text{forall}\, i, j \in I, \quad t \geq 0. \tag{4.80}$$

Thus, an $(X - T)$ process is defining by its semi-Markov kernel Q and its vector of initial probabilities p, and is known as Markov renewal process.

4.4.3 Waiting Time Distributions

Introduce the unconditional probability of the waiting time as

$$\Phi_i(t) = P[T_{n+1} \leq t | X_n] = i, \quad i \in I,\ t \geq 0. \tag{4.81}$$

Now, define the conditional probability of the waiting time T_n as

$$G_{ij}(t) = P[T_{n+1} \leq t | X_n = i, X_{n+1} = j], \quad i, j \in I,\ t \geq 0. \tag{4.82}$$

Thus,

$$G_{ij}(t) = \begin{cases} \frac{Q_{ij}(t)}{h_{ij}} & \text{if } h_{ij} \neq 0, \\ U_1(t) & \text{Otherwise,} \end{cases}$$

where

$$U_1(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

If we assume that the random waiting time is independent of the state to migrate, we obtian: $\Phi_i(t) = G_{ij}(t)\ i, j \in I,$. The probability density function of the waiting time distribution is defined as

$$\varphi_i(t) = \frac{\partial \Phi_i(t)}{\partial t} \quad \text{for all}\, i, j \in I, t \geq 0. \tag{4.83}$$

Now, we introduce the survival function $\Psi_i(t)$, that represents the probability of no migration in a time less than t. Thus,

$$\Psi_i(t) = P[T_{n+1} > t | X_n]. \tag{4.84}$$

It is clear that

$$\Psi_i(t) = 1 - \Phi_i(t), \quad i \in I, t \geq 0. \tag{4.85}$$

4.4.4 Definition of Semi-Markov Process

Define the random variable $\Theta_n = \sum_{r=1}^{n} T_n$ where $(T_n, n \geq 1)$ as the sequence of waiting times, and introduce the N-process $(N(\theta), \theta \geq 0)$ as the number of jumps of the $(X-T)$ process on the interval $(0, \theta)$ for any fixed time θ. Now, we can provide the definition of a semi-Markov process: For all $(X-T)$ process, we associate the following sequence $Z = (Z(\Theta), \theta \geq t)$ with space state I, $Z(\Theta) = X_{N(\Theta)}$ and migration probability at time t

$$P[Z(\Theta + v) = j|Z(\Theta) = i] = P[Z(\Theta) = j|Z(0) = i] = \rho_{ij}(t). \tag{4.86}$$

[CJM04] provides a numerical approximation of the above equation by means of quadrature method. It proves that the DTSMP's probability migration equation tends to the continuous case as the discretization interval tends to 0, thus by defining a migration probability $\rho^h_{ij}(t)$, where h represents the discretization step gives

$$\rho^h_{ij}(kh) = d^h_{ij} + \sum_{r=1}^{m} \sum_{\tau=1}^{k} \upsilon^h_{ir} \rho^h_{rj}((k-\tau)h), \tag{4.87}$$

where

$$d^h_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 - \Phi^h_i(kh) & if\ i = j \geq 0, \end{cases}$$

and

$$\upsilon^h_{ij}(kh)) = \begin{cases} 0 & \text{if } k = 0, \\ Q^h_{ij}(kh) - Q^h_{ij}((k-1)h) & \text{if } k > 0. \end{cases}$$

Equation (4.87) can be written in matrix form as

$$\mathbf{P}^h(kh) - \sum_{\tau=1}^{k} \mathbf{V}^h(\tau h)\mathbf{P}^h(k-\tau)h) = \mathbf{D}^h(kh). \tag{4.88}$$

They also proved:

- The equation (4.87) admits a unique solution.
- The matrix $\mathbf{P}^h(kh)$ is stochastic.

As the discrete case, equation (4.87) can be solved by a recursive method. In the next chapter, we will give a detailed theory of this process.

4.4.5 Credit Risk Models in a Reliability Environment

Credit risk can be analysed by means of semi-Markov process, the idea is to consider it as a reliability problem. Suppose a credit risk rating system S with state space $I = [1, 2, \ldots, m]$ that performs the stochastic process of the successive states $(X_t, t \geq 0)$. The state space I is partitioned in two sets W and D, such that

$$I = W \cup D, \quad W \cap D = \emptyset, \quad W \neq \emptyset, \quad D \neq \emptyset. \tag{4.89}$$

The subset W has all 'good states' (no default states), in which the system is working, and the subset D has all 'bad states' in which the system fails (default state). In our particular case

$$W = [1, 2, \ldots m-1], \quad D = [m]. \tag{4.90}$$

The probability that best describe this system is the reliability function R, that provides the probability that the system is working in the interval of time [0,T].

$$R(t) = P[S_w \in W, \text{ for all } w \in [0, t]]. \tag{4.91}$$

Since in our system there exists a 'bad state' that is absorbent (m)

$$R(t) = P[Z(t) \neq m | Z(0) = i] = 1 - P[Z(t) = m | Z(0) = i], \quad i \in U, m \in D. \tag{4.92}$$

4.5 Fractional Derivatives

In this secction we introduce the main fractional derivative concepts, as the generalization of a regular derivative of order $n \in \mathbb{N}$. A good introduction of this subject is provided in [Pod99, SKM93]. We will deduce the Grünwald-Letnikov fractional derivative as an extension of the regular derivative concepts; subsequently we will derive the Riemann-Liouville and the Caputo fractional derivatives, finally we will obtain the Laplace transform of Caputo and Riemann-Liouville derivatives. This will be of most use to us in Chapter 5 when we define the power-law waiting time distribution.

4.5.1 Grünwald-Letnikov Fractional Derivative

Grünwald-Letnikov fractional derivative is a basic the extension of the regular derivative of n-th order, By this method is possible to take the derivative a non-integer number of time. Consider the first derivative of a function n times differentiable

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}. \tag{4.93}$$

the second order derivative is given by

$$f'(t) = \lim_{h\to 0} \frac{f(t+2h) - 2f(t+h) + f(t)}{h^2}. \tag{4.94}$$

Similarly, the n-th derivative is

$$f^{(n)}(t) = \lim_{h\to 0} \frac{\sum_{r=0}^{n}(-1)^r \binom{n}{r} f(t+rh)}{h^n}. \tag{4.95}$$

We can extend the above equation by relaxing the assumption that n be a positive integer, such that

$$\frac{\partial^q}{\partial t^q} = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{\infty} (-1)^r \binom{q}{r} f(t+rh), \tag{4.96}$$

where $\binom{q}{r} = \frac{\Gamma(q+1)}{\Gamma(r+1)\Gamma(q+1-r)}$, and $\Gamma(x)$ is the gamma function defined as

$$\Gamma(x) = \begin{cases} (x-1)! & If\ x \in \mathbb{N}, \\ \int_0^\infty t^{x-1} e^{-t} dt & \text{Otherwise.} \end{cases}$$

Define $\Delta_h^q f(t) = \sum_{r=0}^{\infty} (-1)^r \binom{q}{r} f(t+rh)$, so we obtain the Grünwald-Letnikov fractional derivative defined as

$$^{GL}\mathcal{D}_t^q f(t) = lim_{h\to 0} \frac{1}{h^q} \Delta_h^q f(t). \tag{4.97}$$

4.5.2 Riemann-Liouville Derivative

Riemann-Liouville derivative is a generalization of regular derivatives widely used. In order to define it, consider the following derivative function in terms of its integral equation

$$\mathcal{D}^{-1} f(t) = \int_0^t f(\tau) d\tau. \tag{4.98}$$

Similarly, the second derivative function in terms of its integral equation gives

$$\mathcal{D}^{-2} f(t) = \int_0^t \int_0^{\tau_1} f(\tau) d\tau d\tau_1 = \int_0^t \int_\tau^t f(\tau) d\tau_1 d\tau. \tag{4.99}$$

$$= \int_0^t f(\tau) \int_\tau^t d\tau_1 d\tau = \int_0^t f(\tau)(t-\tau) d\tau. \tag{4.100}$$

This procedure can be applied repeatedly, and we obtain the n-th derivative of a function in terms of its integral

$$\mathcal{D}^{-n}f(t) = \frac{1}{(n-1)!}\int_0^t f(\tau)(t-\tau)^{n-1}d\tau. \tag{4.101}$$

Generalizing this procedure to a $q \in \mathbb{R}$, we obtain the Riemann-Liouville derivative defined as

$$^{RL}\mathcal{D}_t^q f(t) = \frac{1}{\Gamma(-q)}\int_0^t \frac{f(\tau)}{(t-\tau)^{q+1}}d\tau. \tag{4.102}$$

4.5.3 Caputo Fractional Derivative

Caputo fractional derivative was introduced by M. Caputo [Cap67] and in order to obtain it, define the Grüwald weight as

$$w_r = (-1)^r \frac{\Gamma(q+1)}{\Gamma(r+1)\Gamma(q+1-r)}. \tag{4.103}$$

Applying the Stirling approximation, it can be shown that [Rob55]

$$w_r \approx \frac{r^{-(1+q)}}{\Gamma(-q)}. \tag{4.104}$$

Substituting the above equation into (4.96) and by assuming $0 < q < 1$ gives

$$\frac{\partial^q}{\partial t^q}f(t) \approx \frac{1}{\Gamma(-q)}\sum_{r=0}^{\infty} f(t+rh)(rh)^{-(1+q)}h. \tag{4.105}$$

Define $\tau = rh$, thus

$$\approx \frac{1}{\Gamma(-q)}\int_0^{\infty} f(t+\tau)(\tau)^{-(1+q)}d\tau. \tag{4.106}$$

The right hand side of above equation is defined as Caputo fractional derivative of order q, and it can be solved by integration by parts, so

$$^{C}\mathcal{D}_t^q f(t) = -\frac{1}{\Gamma(-q)q}[f(t+\tau)\tau^{-q}]_0^{\infty} + \frac{1}{\Gamma(1-q)}\int_0^{\infty} \tau^{-q}f'(t+\tau)d\tau. \tag{4.107}$$

$$= \frac{1}{\Gamma(1-q)}\int_0^{\infty} f'(t+\tau)\tau^{-q}d\tau. \tag{4.108}$$

The equation (4.108) is known as Caputo fractional derivative. It can be shown that the Riemann-Louville fractional derivative, can be expressed on Caputo form, as follows (see proof at [Bag07])

$$^{RL}\mathcal{D}_t^q f(t) =^{C} \mathcal{D}_t^q f(t) + \frac{f(0)}{\Gamma(1-q)t^q}. \tag{4.109}$$

4.5.4 Laplace Transforms of Riemann-Liouville and Caputo Derivatives

Notice that the right hand side of equation (4.102) is the convolution of functions $f(t)$ and $\frac{1}{\Gamma(-q)t^{-(q+1)}}$, so the Laplace transform of the Riemann-Liouville derivative is given by

$$\mathcal{L}[{}^{RL}\mathcal{D}_t^q] = \widehat{f}(s)s^q\frac{\Gamma(-q)}{\Gamma(-q)}. \tag{4.110}$$

Thus,

$$\mathcal{L}[{}^{RL}\mathcal{D}_t^q] = s^q\widehat{f}(s). \tag{4.111}$$

By equation (4.109) we obtain

$$\mathcal{L}[{}^{C}\mathcal{D}_t^q] = \mathcal{L}[{}^{C}\mathcal{D}_t^q] - \frac{f(0)}{\Gamma(1-q)}s^{q-1}\Gamma(1-q). \tag{4.112}$$

Thus, the Laplace transform of Caputo fractional derivative is

$$\mathcal{L}[{}^{C}\mathcal{D}_t^q] = s^q\widehat{f}(s) - s^{q-1}f(0). \tag{4.113}$$

These results will be used in Section 5.6.

Chapter 5

Semi-Markov Forward and Backward Equations

In this chapter we present our theoretical results. Its purpose is to derive the forward and backward differential equations respectively and to illustrate some particular cases of waiting time distributions. In section 5.1, we will define our system and the forward and backward concepts. In Section 5.2, we will set up a non-Markovian n-state model where the transition rate is not constant and responds on age, the goal of this section is to derive the forward differential equation, we will also define the memory kernel in terms of its Laplace transform. In Section 5.3, we will derive the backward differential equation from its integral backward equation. In Sections 5.4, 5.5 and 5.6, we will derive the backward and forward differential equations for exponential, gamma and power-law waiting time distributions respectively. We will also consider the two-state case for each distribution and analyse the differences between forward and backward solutions.

5.1 Introduction

Introduce a credit risk system S with state space $I = [1, ...m]$. S is a two random variable process (X_n, τ_n) where the sequence $(X_n, n \geq 0)$ represents the successive states (rankings) of the process and $(\tau_n, n \geq 0)$ the successive waiting times. Thus, the system can be modeled as a continuous time semi-Markov process (CTSMP), as defined in Section 4.4.3, characterized by the probability $\rho_{ij}(t) = P[Z(t) = j|Z(0) =$

$i]$, $i, j \in I$. Such a probability is obtained by the forward and backward equations, the former is focused on the final jump from a state $k \in I$ into a state j at time τ, given that the system has arrived to k in any way at time $t - \tau$, thus the initial state is almost irrelevant. The backward equation focuses on the first jump from the initial state i to a state $k \in I$ at time τ, from there, the system goes to j at time $t - \tau$ by any possible way. Both equations are defined in terms of integral equations, known as master equations, that we will convert into differential equations because it is possible to find an easier solution this way, through a differential equations system or by a numerical method. We will assume that $\Phi_i(t) = G_{ij}(t)$, for all $i, j \in I$, i.e., the time on which the process leaves the state i is independent from the state which it migrates.

5.2 n-State Residence Model

In this Section we will set up the forward differential equation under the standard assumption that the transition rate depends on the residence time. This is the extension of the residence time two-state model developed in [Als11] and represents the main theoretical contribution of this work. Consider the residence time τ as the time the system remains in a given state i before jumping, and its corresponding waiting time probability density function $\Phi_i(\tau)$, for all $i \in I$. Define the survival function i as

$$\Psi_i(\tau) = \int_\tau^\infty \varphi_i(s)ds, \quad i \in I. \tag{5.1}$$

In order to describe the change of state process, we introduce the hazard function $\gamma_i(\tau)$ defined as

$$\gamma_i(\tau) = \frac{\varphi_i(\tau)}{\Psi_i(\tau)}, \quad \tau \geq 0, \ i \in I. \tag{5.2}$$

Consider the product $h\gamma_i(\tau)$ which represents the conditional probability of transition from state i in the small time interval $(\tau, \tau + h)$, given that there is no transtition up time τ. Notice that by equation (5.1) we obtain $\frac{\partial}{\partial\tau}\Psi_i(\tau) = -\varphi_i(\tau)$ and by equation (5.2) we get $\varphi_i(\tau) = \gamma_i(\tau)\Psi_i(\tau)$, thus we derive

$$\frac{\partial}{\partial\tau}\Psi_i(\tau) = -\gamma_i(\tau)\Psi_i(\tau), \quad i \in I. \tag{5.3}$$

This differential equation has the unique solution

$$\Psi_i(\tau) = e^{-\int_0^\tau \gamma_i(s)ds}, \quad i \in I. \tag{5.4}$$

Hence, by the above equation and (5.1) we obtain

$$\varphi_i(\tau) = \gamma_i(\tau)e^{-\int_0^\tau \gamma_i(s)ds}, \quad i \in I. \tag{5.5}$$

On the other hand, introduce $\Omega_i(t,\tau)$ as the probability of being in state $i \in I$ at time t, where the residence time is the interval $(\tau, \tau + d\tau)$, and the migration probability $\rho_i(t) = P[Z(t) = i]$ defined as

$$\rho_i(t) = \int_0^t \Omega_i(t,\tau)d\tau, \quad i \in I. \tag{5.6}$$

Differentiating both sides of equation (5.6) with respect to t and applying the Leibniz integral rule gives

$$\frac{\partial}{\partial t}\rho_i(t) = \Omega_i(t,t) + \int_0^t \frac{\partial}{\partial t}\Omega_i(t,\tau)d\tau, \quad i \in I. \tag{5.7}$$

Introduce the following balance equation

$$\Omega_i(t+h, \tau+h) = (1 - \gamma_i(\tau)h)\Omega_i(t,\tau) + o(h), \quad i \in I. \tag{5.8}$$

It implies that the probability of being in state i at time $t+h$ with residence time $\tau + h$ is the probability of being in state i at time t with residence time τ multiplied by the survival probability $(1 - \gamma_i(\tau)h)$ plus $o(h)$, that is function with the following property: $\lim_{h\to 0} \frac{o(h)}{h} = 0$, as defined in section 4.2.2. By equation (5.8) we obtain

$$\frac{\Omega_i(t+h,\tau+h) - \Omega_i(t,\tau)}{h} = -\gamma_i(\tau)\Omega_i(t,\tau) + \frac{o(h)}{h}, \quad i \in I. \tag{5.9}$$

letting $h \to 0$, gives the following differential equation

$$\frac{\partial}{\partial t}\Omega_i(t,\tau) + \frac{\partial}{\partial \tau}\Omega_i(t,\tau) = -\gamma_i(\tau)\Omega_i(t,\tau), \quad i \in I. \tag{5.10}$$

By integrating both sides of equation (5.10) we have

$$\int_0^t \frac{\partial}{\partial t}\Omega_i(t,\tau)d\tau + \int_0^t \frac{\partial}{\partial \tau}\Omega_i(t,\tau)d\tau = -\int_0^t \gamma_i(\tau)\Omega_i(t,\tau)d\tau, \quad i \in I, \tag{5.11}$$

Notice that

$$\int_0^t \frac{\partial}{\partial \tau}\Omega_i(t,\tau)d\tau = \Omega_i(t,t) - \Omega_i(t,0), \quad i \in I. \tag{5.12}$$

Hence, the equation (5.11) can be expressed as

$$\int_0^t \frac{\delta}{\delta t}\Omega_i(t,\tau)d\tau + \Omega_i(t,t) - \Omega_i(t,0) = -\int_0^t \gamma_i(\tau)\Omega_i(t,\tau)d\tau, \quad i \in I. \tag{5.13}$$

Thus, by equations (5.13) and (5.7) we obtain

$$\frac{\partial}{\partial t}\rho_i(t) = \Omega_i(t,0) - \int_0^t \gamma_i(\tau)\Omega_i(t,\tau)d\tau \quad i \in I. \tag{5.14}$$

On the other hand, consider the boundary condition for $\Omega_i(t,\tau)$ at $\tau = 0$

$$\Omega_i(t,0) = \sum_{j\neq i}\int_0^t \gamma_j(\tau)\Omega_j(t,\tau)h_{ji}d\tau, \quad i \in I, \tag{5.15}$$

where $h_{ij} = \lim_{\tau\to\infty} P[X_{n+1} = i, T \leq \tau | X_n = j]$. It states that the probability of just arriving to state i at time t is equal to the transition rate (hazard rate) from state j at time τ multiplied by the probability of being in state j at time t with residence time τ, and the probability of jumping to state i, given that the system has left the state j, for all $\tau \leq t$ and for all state $j \in I$, with $j \neq i$. Substituting (5.15) into (5.14) gives

$$\frac{\partial}{\partial t}\rho_i(t) = \sum_{j\neq i}\int_0^t \gamma_j(\tau)\Omega_j(t,\tau)h_{ji}d\tau - \int_0^t \gamma_i(\tau)\Omega_i(t,\tau)d\tau, \quad i \in I. \tag{5.16}$$

Notice that in the specific case that $\gamma_i(\tau) = \gamma_i$ for all $i \in I$, we obtain the Markov process backward Kolmogorov equation, as defined in matrix form in Section 4.2.4. Consider $\frac{d\tau}{dt} = 1$, thus τ can be expressed as a function of t as follows

$$\tau(t) = \begin{cases} t + \tau_0 & \text{if } \tau > t, \\ t - t_0 & \text{if } \tau < t, \end{cases}$$

where τ_0 is defined as the initial age of the system at time $t = 0$ and t_0 is the entry time of the system to i. By substituting this equation in (5.10) we obtain

$$\frac{\partial}{\partial t}\Omega_i(t,\tau(t)) = -\gamma_i(\tau(t))\Omega_i(t,\tau(t)), \quad i \in I. \tag{5.17}$$

Consider the case $\tau < t$, i.e. the system accesses to i after $t = 0$, thus equation (5.17) has the following solution

$$\Omega_i(t,\tau(t)) = \Omega_i(t_0,0)e^{-\int_{t_0}^t \gamma_i(\tau(s))ds}, \quad i \in I. \tag{5.18}$$

Notice that $\Omega_i(t_0,0) = \Omega_i(t-\tau,0)$, so

$$\Omega_i(t,\tau(t)) = \Omega_i(t-\tau,0)e^{-\int_0^\tau \gamma_i(s)ds}, \quad i \in I. \tag{5.19}$$

Now, Consider the case $\tau > t$, i.e. the system was in state i at $t = 0$, thus we obtain the following solution for equation (5.17).

$$\Omega_i(t,\tau(t)) = \Omega_i(0,\tau_0)e^{-\int_0^t \gamma_i(\tau(s))ds}, \quad i \in I, \tag{5.20}$$

Now, Define the probability of initial conditions $\Omega_i(0,\tau)$ as

$$\Omega_i(0,\tau) = \rho_{i0}\delta_i(\tau), \quad i \in I, \tag{5.21}$$

where ρ_{i0} is the probability of being in state i at $t = 0$ and $\delta_i(\tau)$ is the residence time distribution of the process in state i at time $t = 0$. Thus, substituting the above equation into (5.20) gives

$$\Omega_i(t,\tau(t)) = \rho_{i0}\delta_i(\tau - t)e^{-\int_{\tau-t}^{\tau}\gamma_i(s)ds}, \quad i \in I. \tag{5.22}$$

By substituting (5.19) and (5.22) into equation (5.6) we obtain

$$\rho_i(t) = \int_0^t j_i(t-\tau)e^{-\int_0^{\tau}\gamma_i(s)ds}d\tau + \int_t^{\infty}\rho_{i0}\delta_i(\tau - t)e^{-\int_{\tau-t}^{\tau}\gamma_i(s)ds}d\tau, \quad i \in I. \tag{5.23}$$

Introduce $j_i(t) = \Omega_i(t,0),\ i \in I$, which is defined as the probability of arriving in state i exactly at time t. $j_i(t)$ is obtained by substituting equations (5.19) and (5.22) into the (5.15), thus

$$\begin{aligned} j_i(t) = & \sum_{j\neq i}\int_0^t h_{ji}\gamma_j(\tau)j_j(t-\tau)e^{-\int_0^{\tau}\gamma_j(s)ds}d\tau \\ & + \sum_{j\neq i}\int_t^{\infty} h_{ji}\gamma_j(\tau)\rho_{j0}\delta_j(\tau - t)e^{-\int_{\tau-t}^{\tau}\gamma_j(s)ds}d\tau, \quad i \in I. \end{aligned} \tag{5.24}$$

Consider $\tau = t - t'$ and substitute (5.5) into equation (5.23) to have

$$\rho_i(t) = \int_0^t j_i(t')\Psi_i(t-t')dt' + \rho_{i0}\Psi_i(t), \quad i \in I. \tag{5.25}$$

The first term of the right hand side is the probability of arriving to state i exactly at time t', and staying there for a time $t - t'$, for all $t' \leq t$. The second term is the probability of staying in state i at time t, given that the system was in this state at the begining of the process. Similarly, we can consider $\tau = t - t'$ and by substituting equation (5.5) into (5.24) we obtain

$$j_i(t) = \sum_{j\neq i}\int_0^t h_{ji}j_j(t')\varphi_j(t-t')dt' + \sum_{j\neq i}\rho_{j0}\varphi_j(t)h_{ji}, \quad i \in I. \tag{5.26}$$

Taking Laplace transforms of equations (5.26) and (5.25) gives

$$\hat{\rho}_i(s) = (\hat{j}_i(s) + \rho_{i0})\hat{\Psi}_i(s), \quad i \in I, \tag{5.27}$$

and

$$\hat{j}_i(s) = \sum_{j\neq i}(\hat{j}_j(s) + \rho_{j0})\hat{\varphi}_j(s)h_{ji}, \quad i \in I. \tag{5.28}$$

From equations (5.28) and (5.27), we obtain

$$\hat{j}_i(s) = \sum_{j \neq i} h_{ji} \hat{K}_j(s) \hat{\rho}_i(s), \quad i \in I, \tag{5.29}$$

where $K(t)$ is the memory kernel and is defined in terms of its Laplace transform as $\hat{K}_j(s) = \frac{\hat{\varphi}_i(s)}{\hat{\Psi}_i(s}$. By taking the inverse Laplace transform to 5.29,

$$j_i(t) = \sum_{j \neq i} \int_0^t h_{ji} K_j(t - t') \rho_j(t) dt', \quad i \in I. \tag{5.30}$$

On the other hand, by equation (5.27) we obtain

$$\frac{\hat{\rho}_i(s)}{\hat{\Psi}_i(s)} = \hat{j}_i(s) + \rho_{i0}, \quad i \in I. \tag{5.31}$$

The above equation can be rewritten as

$$\frac{s\hat{\Psi}_i(s) + \hat{\varphi}_i(s)}{\hat{\Psi}_i(s)} \hat{\rho}_i(s) - \rho_{i0} = \hat{j}_i(s), \quad i \in I. \tag{5.32}$$

So

$$s\hat{\rho}_i(s) - \rho_{i0} = \hat{j}_i(s) - \hat{K}_i(s)\hat{\rho}_i(s), \quad i \in I. \tag{5.33}$$

Taking Laplace inverse, we obtain the forward differential equation of the migration probability $\rho_i(t)$

$$\rho_i'(t) = -\int_0^t K_i(t - t')\rho_i(t')dt' + \sum_{j \neq i} \int_0^t K_j(t - t') h_{ji} \rho_j(t') dt'. \tag{5.34}$$

Since the initial state is irrelevant, we deduce easily that in the particular case when the initial state is given the forward differential equation is

$$\rho_{ij}'(t) = -\int_0^t K_j(t - t')\rho_{ij}(t')dt' + \sum_{k \neq j} \int_0^t K_k(t - t') h_{kj} \rho_{ik}(t') dt'. \tag{5.35}$$

The first term of the right hand side is the probability of migration from j into i in a time t' since the memory of the process has stayed in j for a time $t - t'$; the second term considers the probability that the system migrates from state i into a state $k \in I$ at time t' and staying there for a time $t - t'$ and then jumping into j, this for all $t' \leq t$ and for all state $k \in I$, with $k \neq i$. This process is represented in figure 4.1. The equation (5.35) in terms of its Laplace transforms is

$$s\hat{\rho}_{ij}(s) - \rho_{ij}(0) = -\hat{K}_j(s)\hat{\rho}_{ij}(s) + \sum_{k \neq j} \hat{K}_k(s) h_{kj} \hat{\rho}_{ik}(s). \tag{5.36}$$

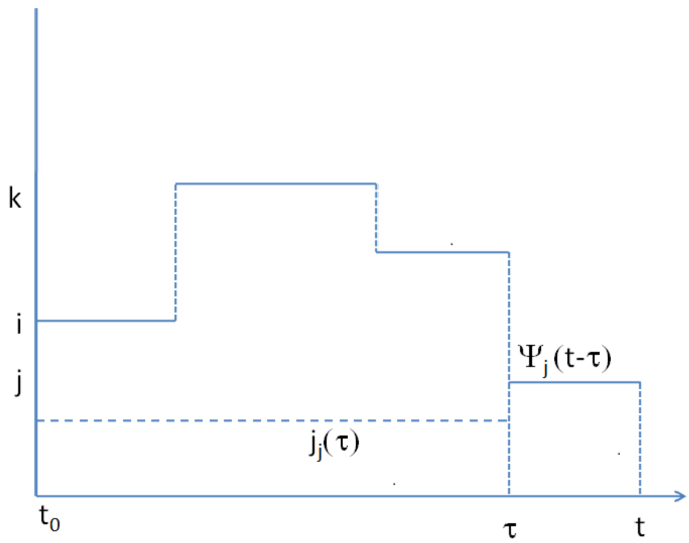

Figure 5.1: Illustration of a forward equation

5.3 Backward Differential Equation

In this section we will derive the backward differential equation for migration probability $\rho_{ij}(t) = P[Z(t) = i|Z(0) = j]$, for all $i, j \in I$. It satisfies the following backward integral equation

$$\rho_{ij}(t) = \delta_{ij}\Psi_i(t) + \Sigma_{k\in I}\int_0^t h_{ik}\varphi_i(t-\tau)\rho_{kj}(\tau)d\tau, \tag{5.37}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{Otherwise.} \end{cases}$$

The first term on the right hand side, is the probability that the system does not leave the state i until the end of the period t, subject to $i = j$. The second term is the sum, for all $k \in I$, of the probability that the system stays at state i for a time τ and at the end of this period jumps to state k, multiplied by the migration probability from k to j in a time $t - \tau$ in any possible way. Taking Laplace transform in equation (5.37) gives

$$\hat{\rho}_{ij}(s) = \delta_{ij}\hat{\Psi}_i(s) + \Sigma_{k\in I}h_{ik}\hat{\varphi}_i(s)\hat{\rho}_{kj}(s). \tag{5.38}$$

We know that

$$\Psi_i(t) = \int_t^{\infty} \varphi_i(r)dr = 1 - \int_0^t \varphi_i(r)dr. \tag{5.39}$$

By Laplace transform integration properties we obtain

$$\hat{\Psi}_i(s) = \frac{1}{s} - \frac{\hat{\varphi}_i(s)}{s}. \tag{5.40}$$

Substituting the above equation into (5.38) we obtain

$$\hat{\rho}_{ij}(s) = \delta_{ij}\left(\frac{1}{s} - \frac{\hat{\varphi}_i(s)}{s}\right) + \Sigma_{k\in I} h_{ik}\hat{\varphi}_i(s)\hat{\rho}_{kj}(s). \tag{5.41}$$

Subtracting $\delta_{ij}\left(\frac{1-\hat{\varphi}_i(s)}{s}\right)$ from both sides of equation

$$\hat{\rho}_{ij}(s) - \delta_{ij}\left(\frac{1-\hat{\varphi}_i(s)}{s}\right) = \Sigma_{k\in I} h_{ik}\hat{\varphi}_i(s)\hat{\rho}_{kj}(s). \tag{5.42}$$

Rearranging the equation, we obtain

$$\frac{\left(\frac{s}{1-\hat{\varphi}_i(s)}\right)\hat{\rho}_{ij}(s) - \delta_{ij}}{s(1-\hat{\varphi}_i(s))^{-1}} = \Sigma_{k\in I} h_{ik}\hat{\varphi}_i(s)\hat{\rho}_{kj}(s). \tag{5.43}$$

Multiplying $s(1-\hat{\varphi}_i(s))^{-1}$ and by adding the term $s\hat{\rho}_{ij}(s) - \frac{s}{1-\hat{\varphi}_i(s)}\hat{\rho}_{ij}(s)$ respectively from both sides of equation gives

$$s\hat{\rho}_{ij}(s) - \delta_{ij} = s\hat{\rho}_{ij}(s) - \frac{s}{1-\hat{\varphi}_i(s)}\hat{\rho}_{ij}(s) + \frac{s}{1-\hat{\varphi}_i(s)}\Sigma_{k\in I} h_{ik}\hat{\varphi}_i(s)\hat{\rho}_{kj}(s). \tag{5.44}$$

This equation can be rewritten as

$$s\hat{\rho}_{ij}(s) - \delta_{ij} = \frac{-s\hat{\varphi}_i(s)}{1-\hat{\varphi}_i(s)}\hat{\rho}_{ij}(s) + \frac{s\hat{\varphi}_i(s)}{1-\hat{\varphi}_i(s)}\Sigma_{k\in I} h_{ik}\hat{\rho}_{kj}(s). \tag{5.45}$$

Consider the memory kernel in terms of its Laplace transform $\hat{K}_j(s) = \frac{\hat{\varphi}_i(s)}{\hat{\Psi}_i(s)}$, as defined in Secction 5.2. It is easy to show that $\hat{K}(s) = \frac{s\hat{\varphi}_i(s)}{1-\hat{\varphi}_i(s)}$. Thus

$$s\hat{\rho}_{ij}(s) - \delta_{ij} = -\hat{K}_i(s)\hat{\rho}_{ij}(s) + \hat{K}_i(s)\Sigma_{k\in I} h_{ik}\hat{\rho}_{kj}(s). \tag{5.46}$$

Taking Laplace inverse of the above equation gives the backward differential equation

$$\rho'_{ij}(t) = -\int_0^t K_i(\tau)\rho_{ij}(t-\tau)d\tau + \sum_{k\in I}\int_0^t K_i(\tau)h_{ik}\rho_{kj}(t-\tau)d\tau. \tag{5.47}$$

The first term of the right hand side above represents the migration probability from i to j at time $t-\tau$ by any possible way, given that the memory of the process has been at i for a time τ, and has been doing the same for all τ possible that $(0 < \tau < t)$. The second term is the sum, for all $k \in I$, of the probability of jumping from i to k at time τ, given the memory of the process has been at state i for such a time, and then migrates to j in the remaining $t-\tau$ time in any possible way, and does the same for all τ with $(0 < \tau < t)$. The figure 5.2 shows the meaning of backward equation.

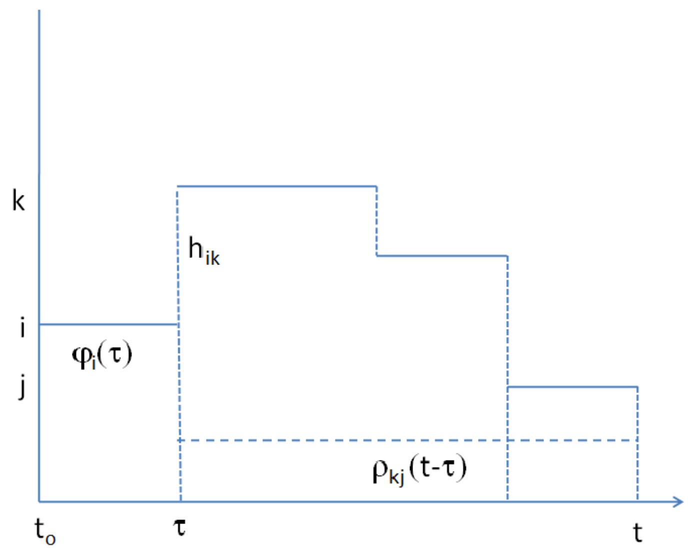

Figure 5.2: Illustration of a backward equation

5.4 Exponential Waiting Time Distribution

Now we will illustrate the backward and forward differential equations for the particular case of exponential waiting time distributions. We will derive the well-known Kolmogorov forward and backward equations seen in Section 4.2.4

5.4.1 Introduction

The exponential distribution is defined as the continuous random time occuring between two independent Poisson events, which is characterized by a constant instantaneous transition rate (λ) . The cumulative density function (cdf) and probability density function (pdf) are respectively given by

$$\Phi_i(\tau) = P[T_{n+1} - T_n \leq \tau | X_n = i] = 1 - e^{-\lambda_i \tau} \quad \text{for all } i, j \in I, \ \lambda_i, \tau \geq 0, \tag{5.48}$$

$$\varphi_i(\tau) = \frac{d\Phi}{d\tau} = \lambda_i e^{-\lambda_i \tau}. \tag{5.49}$$

It can be shown that their corresponding Laplace transforms are given by

$$\hat{\Phi}_i(s) = \frac{1}{s} - \frac{1}{s + \lambda_i}. \tag{5.50}$$

$$\hat{\varphi}_i(s) = \frac{\lambda_i}{s + \lambda_i}. \tag{5.51}$$

By equation (5.51), the Laplace transform of the memory kernel is given by

$$\hat{K}_i(s) = \frac{\frac{s\lambda_i}{s+\lambda_i}}{1 - \frac{\lambda_i}{s+\lambda_i}} = \lambda_i. \tag{5.52}$$

5.4.2 Exponential Forward Differential Equation

We are interested in the migration probability from state i to state j at time t, i.e., $\rho_{ij}(t) = P[Z(t) = j|Z(0) = i]$. Substituting (5.52) into equation (5.36) gives

$$s\hat{\rho}_{ij}(s) - \rho_{ij}(0) = -\lambda_j \hat{\varphi}_{ij}(s) + \sum_{k\neq j} \lambda_k h_{kj} \hat{\varphi}_{ik}(s). \tag{5.53}$$

Taking Laplace transform inverse gives

$$\rho'_{ij}(t) = -\lambda_j \rho_{ij}(t) + \sum_{k\neq j} \lambda_k h_{kj} \rho_{ik}(t). \tag{5.54}$$

5.4.3 Exponential Backward Differencial Equation

Substituting equation (5.52) into equation (5.46), we obtain

$$s\hat{\rho}_{ij}(s) - \delta_{ij} = -\lambda_i \hat{\rho}_{ij}(s) + \lambda_i \Sigma_{k\in I} h_{ik} \hat{\rho}_{kj}(s). \tag{5.55}$$

Applying Laplace inverse,

$$\rho'_{ij}(t) = -\lambda_i \rho_{ij}(t) + \lambda_i \Sigma_{k\in I} h_{ik} \rho_{kj}(t). \tag{5.56}$$

5.4.4 Two-State Model

Consider a two-state system with space state $I = [1, 2]$ that performs a CTSMP with exponential waiting time distributions and average rates (parameters)λ_1 and λ_2 respectively, and transition matrix given by

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.57}$$

This system is known as switching process. By equation (5.54) we obtain

$$\rho'_{11}(t) = -\lambda_1 \rho_{11}(t) + \sum_{k\neq 1} \lambda_k h_{kj} \rho_{ik}(t) = -\lambda_1 \rho_{11}(t) + \lambda_2 \rho_{12}(t). \tag{5.58}$$

Simillarly, we obtain the following forward equations

$$\rho'_{12}(t)) = -\lambda_2 \rho_{12}(t) + \lambda_1 \rho_{11}(t), \tag{5.59}$$

$$\rho'_{21}(t)) = -\lambda_1\rho_{21}(t) + \lambda_1\rho_{11}(t), \tag{5.60}$$

$$\rho'_{22}(t)) = -\lambda_2\rho_{22}(t) + \lambda_1\rho_{21}(t). \tag{5.61}$$

From equation (5.56) we obtain the following backward equation

$$\rho'_{11}(t) = \sum_{k=1}^{2} \lambda_1\rho_{k1}(t)h_{1k} - \lambda_1\rho_{11}(t) = -\lambda_1\rho_{11}(t) + \lambda_1\rho_{21}(t). \tag{5.62}$$

Similarly, we derive

$$\rho'_{12}(t) = -\lambda_1\rho_{12}(t) + \lambda_1\rho_{22}(t), \tag{5.63}$$

$$\rho'_{21}(t)) = -\lambda_2\rho_{21}(t) + \lambda_2\rho_{11}(t), \tag{5.64}$$

$$\rho'_{22}(t)) = -\lambda_2\rho_{22}(t) + \lambda_2\rho_{12}(t). \tag{5.65}$$

Notice that the backward equations take only one parameter, while the forward equations take both of the parameters in each equation. However the forward equation has a much easier solution (just replace $\rho_{ij}(t) = 1 - \rho_{ii}(t),\ i \neq j,\ i,j = [1,2]$ to obtain a differential equation in terms of the given probability and its derivative). In Section 4.2.8 the solution of Kolmogorov backward equation was found. Observe that equation (5.63) can be obtained from equation (5.62) (Just consider $\rho_{21}(t) = 1 - \rho_{22}(t)$, $\rho_{11}(t) = 1 - \rho_{12}(t)$ and $\rho'_{11}(t) = -\rho'_{12}(t)$). Similarly, (5.65) can be derived from (5.64).

5.5 Gamma Waiting Time Distribution

In this section we will derive the forward and backward differential equations for the Gamma waiting time distribution and will illustrate the two-state case.

5.5.1 Introduction

Gamma distribution is a two-parameter continuous time probability distribution given by θ (shape) and β (scale). It is is mainly used to model the time between the occurence of two consecutive events, thus it has many applications in reliability analysis. [AK96, Tri02] provide different aplications of this distribution in this subject. Its pdf is

$$\varphi(t,\theta,\beta) = \frac{\beta^\theta}{\Gamma(\theta)} t^{\theta-1} e^{-\beta t}, \quad \text{for all } t \geq 0,\ \ \theta,\beta > 0. \tag{5.66}$$

where $\Gamma(\theta)$ is known as the gamma function, as defined in section 4.5, thus

$$\Gamma(\theta) = \begin{cases} (\theta-1)! & If\ \theta \in \mathbb{N}, \\ \int_0^\infty t^{\theta-1}e^{-t}dt & \text{Otherwise.} \end{cases}$$

The Laplace transform of the gamma distribution probability density function is defined as

$$\hat{\varphi}(s) = \int_0^\infty \frac{\beta^\theta}{\Gamma(\theta)} t^{\theta-1}e^{-\beta t}e^{-st}dt, \quad \text{for all } t \geq 0,\ \ \theta, \beta > 0. \tag{5.67}$$

It can be shown that by making the following substitution $u = (\beta + s)t$ we obtain

$$\hat{\varphi}(s) = \beta^\theta(s+\beta)^{-\theta}. \tag{5.68}$$

Now, consider the memory kernel in terms of its Laplace transform $\hat{K}(s) = \frac{s\hat{\varphi(s)}}{1-\hat{\varphi(s)}}$ as defined in Section 5.2. Thus by equation (5.68) we obtain

$$\hat{K}(s) = \frac{s\beta^\theta}{(s+\beta)^\theta - \beta^\theta}. \tag{5.69}$$

In the following sections we will obtain the forward and backward differential equations.

5.5.2 Gamma Forward Differential Equation

In order to find a closed solution, assume that $\varphi_i(t) = \varphi(t)$ for all $i \in I$, thus, its Laplace transform is given by $\hat{\varphi}(s) = \beta^\theta(s+\beta)^{-\theta}$ and the corresponding memory kernel defined in term of its Laplace trasform is $\frac{s\beta^\beta}{(s+\beta)^\theta - \beta^\theta}$. Similarly, for simplicity we will assume that $\theta \in \mathbb{N}$. Hence, equation (5.36) can be written as

$$s\hat{\rho}_{ij}(s) - \rho_{ij}(0) = -\frac{s\beta\theta}{(s+\beta)^\theta - \beta^\theta}\hat{\rho}_{ij}(s) + \sum_{k\neq j} h_{ij}\frac{s\beta^\theta}{(s+\beta)^\theta - \beta^\theta}\hat{\rho}_{ik}(s). \tag{5.70}$$

Multiplying both sides of equation by $(s+\beta)^\theta - \beta^\theta$ get

$$(s+\beta)^\theta(s\hat{\rho}_{ij}(s) - \rho_{ij}(0)) - \beta^\theta(s\hat{\rho}_{ij}(s) - \rho_{ij}(0)) = -s\beta^\theta\hat{\rho}_{ij}(s) + \sum_{k\neq j} h_{ik}s\beta^\theta\hat{\rho}_{ik}(s). \tag{5.71}$$

Expanding the left hand side of the above equation gives

$$\left(s^\theta + \theta s^{\theta-1}\beta + \frac{\theta(1+\theta)}{2}s^{\theta-2}\beta^2 + ... + \beta^\theta\right)(s\hat{\rho}_{ij}(s) - \rho_{ij}(0)) =$$
$$- s\beta^\theta\hat{\rho}_{ij}(s) + \sum_{k\neq j} h_{ik}s\beta^\theta\hat{\rho}_{ik}(s) + \beta^\theta(s\hat{\rho}_{ij}(s) - \rho_{ij}(0)). \tag{5.72}$$

We rewrite the above equation as

$$s^{\theta}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0))+\theta\beta s^{\theta-1}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0))+\frac{\theta(1+\theta)}{2}\beta^{2}s^{\theta-2}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0))+...$$
$$+\beta^{\theta}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0))=-s\beta^{\theta}\hat{\rho}_{ij}(s)+\sum_{k\neq j}h_{ik}s\beta^{\theta}\hat{\rho}_{ik}(s)+\beta^{\theta}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0)). \tag{5.73}$$

Assuming that $\rho_{ij}^{(n)}(0)=0$, for all $n\geq 1$, we obtain

$$\mathcal{L}[\rho_{ij}^{(\theta+1)}(t)]+\theta\beta\mathcal{L}[\rho_{ij}^{(\theta)}(t)]+\frac{\theta(1+\theta)}{2}\beta^{2}\mathcal{L}[\rho_{ij}^{(\theta-1)}(t)]+...+\beta^{\theta}\mathcal{L}[\hat{\rho}'_{ij}(t)]$$
$$=-s\beta^{\theta}\hat{\rho}_{ij}(s)+\sum_{k\neq j}h_{ik}s\beta^{\theta}\hat{\rho}_{ik}(s)+\beta^{\theta}\mathcal{L}[\hat{\rho}'_{ij}(t)]. \tag{5.74}$$

Multiplying both side of equation by s^{-1} get

$$\frac{\mathcal{L}[\rho_{ij}^{(\theta+1)}(t)]}{s}+\theta\beta\frac{\mathcal{L}[\rho_{ij}^{(\theta)}(t)]}{s}+\frac{\theta(1+\theta)}{2}\beta^{2}\frac{\mathcal{L}[\rho_{ij}^{(\theta-1)}(t)]}{s}]+...+\beta^{\theta}\frac{\mathcal{L}[\hat{\rho}'_{ij}(t)]}{s}$$
$$=-\beta^{\theta}\hat{\rho}_{ij}(s)+\sum_{k\neq j}h_{ik}\beta^{\theta}\hat{\rho}_{ik}(s)+\beta_i^{\theta_i}\frac{\mathcal{L}[\hat{\rho}'_{ij}(t)]}{s}. \tag{5.75}$$

Taking the Laplace transform inverse gives

$$\int_0^t\rho_{ij}^{(\theta+1)}(\tau)d\tau+\beta\int_0^t\rho_{ij}^{(\theta)}(\tau)d\tau+\beta^{2}\int_0^t\rho_{ij}^{(\theta_i-1)}(\tau)d\tau+...+\beta^{\theta}\int_0^t\rho'_{ij}(\tau)d\tau$$
$$=-\beta^{\theta}\rho_{ij}(t)+\sum_{k\neq j}h_{ik}\beta^{\theta}\rho_{ik}(t)+\beta^{\theta}\int_0^t\rho'_{ij}(\tau)d\tau. \tag{5.76}$$

Thus

$$\rho_{ij}^{(\theta)}(t)+\beta\rho_{ij}^{(\theta-1)}(t)+\beta^{2}\rho_{ij}^{(\theta-2)}(t)+...+\beta^{\theta}\rho_{ij}(t)=\beta^{\theta}\sum_{k\neq j}h_{ik}\rho_{ik}(t). \tag{5.77}$$

Introduce $\mathcal{D}_t$ as the $\frac{\partial}{\partial t}$ operator, thus the above equation can be rewritten as

$$(\beta+\mathcal{D}_t)^{\theta}\rho_{ij}(t)=\beta^{\theta}\sum_{k\neq j}h_{ik}\rho_{ik}(t). \tag{5.78}$$

Notice that this equation is based on the unreal assumption that $\varphi_i(t)=\varphi(t)$, for all $i\in I$, $t\geq 0$ and it was developed for illustrative purposes.

5.5.3 Gamma Backward Differential Equation

By equations (5.46) and (5.69) we obtain

$$s\hat{\rho}_{ij}(s)-\rho_{ij}(0)=-\frac{s\beta_i^{\theta_i}}{(s+\beta_i)^{\theta_i}-\beta_i^{\theta_i}}\hat{\rho}_{ij}(s)+\sum_{k\in I}h_{ik}\frac{s\beta_i^{\theta_i}}{(s+\beta_i)^{\theta_i}-\beta_i^{\theta_i}}\hat{\rho}_{kj}(s). \tag{5.79}$$

Multiplying both sides of equation by $(s+\beta_i)^{\theta_i} - \beta_i^{\theta_i}$ gives

$$(s+\beta_i)^{\theta_i}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0)) - \beta_i^{\theta_i}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0)) = -s\beta_i^{\theta_i}\hat{\rho}_{ij}(s) + \sum_{k\in I} h_{ik}s\beta_i^{\theta_i}\hat{\rho}_{kj}(s). \tag{5.80}$$

Expanding the left hand side of the above equation gives

$$\begin{aligned}\left(s^{\theta_i} + \theta_i s^{\theta_i-1}\beta_i + \frac{\theta_i(1+\theta_i)}{2}s^{\theta_i-2}\beta_i^2 + \ldots + \beta_i^{\theta}\right)(s\hat{\rho}_{ij}(s)-\rho_{ij}(0))\\ = -s\beta_i^{\theta_i}\hat{\rho}_{ij}(s) + \sum_{k\in I} h_{ik}s\beta_i^{\theta_i}\hat{\rho}_{kj}(s) + \beta_i^{\theta_i}(s\hat{\rho}_{ij}(s)-\rho_{ij}(0)).\end{aligned} \tag{5.81}$$

As in the forward case, assume that $\rho_{ij}^{(n)}(0)=0$, for all $n\geq 1$ gives

$$\begin{aligned}\mathcal{L}[\rho_{ij}^{(\theta_i+1)}(t)] + \theta\beta_i\mathcal{L}[\rho_{ij}^{(\theta_i)}(t)] + \frac{\theta_i(1+\theta_i)}{2}\beta_i^2\mathcal{L}[\rho_{ij}^{(\theta_i-1)}(t)] + \ldots + \beta_i^{\theta_i}\mathcal{L}[\rho_{ij}'(t)]\\ = -s\beta_i^{\theta_i}\hat{\rho}_{ij}(s) + \sum_{k\in I} h_{ik}s\beta_i^{\theta_i}\hat{\rho}_{kj}(s) + \beta_i^{\theta_i}\mathcal{L}[\hat{\rho}_{ij}'(t)].\end{aligned} \tag{5.82}$$

Multiplying both sides of equations by s^{-1} we obtain

$$\begin{aligned}\frac{\mathcal{L}[\rho_{ij}^{(\theta_i+1)}(t)]}{s} + \theta\beta_i\frac{\mathcal{L}[\rho_{ij}^{(\theta_i)}(t)]}{s} + \frac{\theta_i(1+\theta_i)}{2}\beta_i^2\frac{\mathcal{L}[\rho_{ij}^{(\theta_i-1)}(t)]}{s} + \ldots + \beta_i^{\theta_i}\frac{\mathcal{L}[\rho_{ij}'(t)]}{s}\\ = -\beta_i^{\theta_i}\hat{\rho}_{ij}(s) + \sum_{k\in I} h_{ik}\beta_i^{\theta_i}\hat{\rho}_{kj}(s) + \beta_i^{\theta_i}\frac{\mathcal{L}[\hat{\rho}_{ij}'(t)]}{s}.\end{aligned} \tag{5.83}$$

Taking the Laplace transform inverse gives

$$\begin{aligned}\rho_{ij}^{(\theta_i)}(t) + \theta\beta_i\rho_{ij}^{(\theta_i-1)}(t) + \frac{\theta_i(1+\theta_i)}{2}\beta_i^2\rho_{ij}^{(\theta_i-2)}(t) + \ldots + \beta_i^{\theta_i}\rho_{ij}(t)\\ = -\beta_i^{\theta_i}\rho_{ij}(t) + \sum_{k\in I}\beta_i^{\theta_i}h_{ik}\rho_{kj}(t) + \beta_i^{\theta_i}\int_0^t \rho_{ij}'(\tau)d\tau.\end{aligned} \tag{5.84}$$

As in the forward case, consider $\mathcal{D}_t$ as the $\frac{\partial}{\partial t}$ operator, thus the above equation can be rewritten as

$$(\beta_i + \mathcal{D}_t)^{\theta_i}\rho_{ij}(t) = \sum_{k\in I}\beta_i^{\theta_i}h_{ik}\rho_{kj}(t). \tag{5.85}$$

5.5.4 Two-State Model

Consider a system with space state $I = [1,2]$ that performs a CTSMP with gamma waiting time distribution. For simplicity assume that both states have the same pdf $\varphi(t)$ with parameters β and θ. As in the exponential case, the transition matrix is given by

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.86}$$

Thus, By equation (5.78) we obtain the following forward equations

$$(\beta + \mathcal{D}_t)^\theta \rho_{11}(t) = \beta^\theta \rho_{12}(t). \tag{5.87}$$

Thus

$$\rho_{11}^{(\theta)}(t) + \theta\beta\rho_{11}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\beta^2\rho_{11}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{11}(t) = \beta^\theta\rho_{12}(t). \tag{5.88}$$

Similarly, we obtain

$$\rho_{12}^{(\theta)}(t) + \theta\beta\rho_{12}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\beta^2\rho_{12}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{12}(t) = \beta^\theta\rho_{12}(t), \tag{5.89}$$

$$\rho_{21}^{(\theta)}(t) + \theta\beta\rho_{21}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\beta^2\rho_{21}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{21}(t) = \beta^\theta\rho_{22}(t), \tag{5.90}$$

$$\rho_{22}^{(\theta)}(t) + \theta\beta\rho_{22}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\beta^2\rho_{22}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{22}(t) = \beta^\theta\rho_{21}(t). \tag{5.91}$$

Likewise, by equation (5.85) we get the following backward equations

$$(\beta + \mathcal{D}_t)^\theta \rho_{11}(t) = \beta^\theta \rho_{21}(t). \tag{5.92}$$

Then

$$\rho_{11}^{(\theta)}(t) + \theta\beta\rho_{11}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\rho_{11}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{11}(t) = \beta^\theta\rho_{21}(t), \tag{5.93}$$

and similarly, we obtain

$$\rho_{12}^{(\theta)}(t) + \theta\beta\rho_{12}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\rho_{12}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{12}(t) = \beta^\theta\rho_{22}(t), \tag{5.94}$$

$$\rho_{21}^{(\theta)}(t) + \theta\beta\rho_{21}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\rho_{21}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{12}(t) = \beta^\theta\rho_{11}(t), \tag{5.95}$$

$$\rho_{22}^{(\theta)}(t) + \theta\beta\rho_{22}^{(\theta-1)}(t) + \frac{\theta(1+\theta)}{2}\rho_{22}^{(\theta-2)}(t) + ... + \beta^\theta\rho_{12}(t) = \beta^\theta\rho_{12}(t). \tag{5.96}$$

Notice that, as in the case of the exponential distribution, the equation (5.94) and (5.95), can be obtained by (5.93) and (5.95) respectively, just applying $\rho_{ii}+\rho_{ij} = 1, \;\; i \neq j$ and $\rho_{ii}^{(n)} = -\rho_{ij}^{(n)}, n \in \mathbb{N}$. Likewise, forward equations provide an easier solution (just consider the complement of the probability of the right hand side equation to get a differential equation as a function of the same variable and their derivatives). On the other hand, observe that in the case of $\theta = 1$, we obtain the forward and backward equations corresponding to the exponential waiting time distributions, as derived in

Section 5.4. Now, we will assume that $\theta = 2$. Thus, we obtain the following forward equations

$$\rho''_{11}(t) + 2\beta\rho'_{11}(t) + \beta^2\rho_{11}(t) = \beta^2\rho_{12}(t), \tag{5.97}$$

$$\rho''_{12}(t) + 2\beta\rho'_{12}(t) + \beta^2\rho_{12}(t) = \beta^2\rho_{11}(t), \tag{5.98}$$

$$\rho''_{21}(t) + 2\beta\rho'_{21}(t) + \beta^2\rho_{21}(t) = \beta^2\rho_{22}(t), \tag{5.99}$$

$$\rho''_{22}(t) + 2\beta\rho'_{22}(t) + \beta^2\rho_{22}(t) = \beta^2\rho_{21}(t), \tag{5.100}$$

and similarly, we can derive the following backward equations

$$\rho''_{11}(t) + 2\beta\rho'_{11}(t) + \beta^2\rho_{11}(t) = \beta^2\rho_{21}(t), \tag{5.101}$$

$$\rho''_{12}(t) + 2\beta\rho'_{12}(t) + \beta^2\rho_{12}(t) = \beta^2\rho_{22}(t), \tag{5.102}$$

$$\rho''_{21}(t) + 2\beta\rho'_{21}(t) + \beta^2\rho_{21}(t) = \beta^2\rho_{11}(t), \tag{5.103}$$

$$\rho''_{22}(t) + 2\beta\rho'_{22}(t) + \beta^2\rho_{22}(t) = \beta^2\rho_{12}(t). \tag{5.104}$$

5.6 Power Law Distribution

In this section we will derive the forward and backward differential equations for a wating time distribution with a power-law tail and will illustrate the two-state case.

5.6.1 Introduction

The power-law distribution (PLD) is a very useful tool for describing many problems in different phenomena in finance related with random growth, optimization, extreme value theory, etc. Thus, many applications have been been developed mainly focused on the analysis of stock markets; [DGE93, GP03, GPG00, IS77, ZF02] provide different applications. Now, we will give an overview of PLD theory, a detailed explanation is found in [MH08]. A waiting time with a power law tail has the general form

$$\varphi(t) \propto f(t)t^{-\gamma}, \quad \gamma > 1,\ t \geq 0, \tag{5.105}$$

where $f(t)$ is any function with the following property

$$\lim_{t\to\infty} \frac{f(ct)}{f(t)} = 1, \text{with c constant.} \tag{5.106}$$

If $f(t)$ is a constant function, the power-law is valid for all values of t. $\varphi(t)$ is scale invariance, i.e. there exists a function g such that $\varphi(ct) = g(c)\varphi(t)$ for all c constant ant $t \geq 0$.

5.6.2 Milttag-Leffler Function

The Milttag-Leffler Function ($E_{\alpha,\beta}$) is defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta > 0, \tag{5.107}$$

where $\Gamma(k\alpha, \beta)$ is the gamma function as defined in Section 4.5. In order to obtain a convenient survival function [MRG00, SGM04] proposed independently the following substitution $x = -t^\alpha$ $t \geq 0$, $\alpha > 0$ with $\beta = 1$,thus, we obtain the following function

$$E_{\alpha,1}(-t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^{\alpha k} t^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{5.108}$$

They also considered the following approximation as $t \to \infty$:

$$E_{\alpha,1}(-t^\alpha) \sim \frac{sin(\alpha\pi)\Gamma(\alpha)t^{-\alpha}}{\pi} \sim t^{-(\alpha+1)}, \quad \alpha > 0. \tag{5.109}$$

[GLM99] performs a detailed analysis of this function and obtains the following Laplace transform

$$\hat{E}_{\alpha,1}(s) = \frac{s^{\alpha-1}}{1 + s^\alpha}. \tag{5.110}$$

It is clear that the equation (5.109) is a monotonic decreasing function, thus we can define $\Psi(t) = E_{\alpha,1}(-t^\alpha)$ and $\hat{\Psi}(s) = \hat{E}_{\alpha,1}(s)$. Hence, the waiting time distribution is given by

$$\varphi(t) \sim \frac{sin(\alpha\pi)\Gamma(\alpha + 1)t^{-(\alpha+1)}}{\pi} \sim t_0^{\alpha+1} t^{-(\alpha+1)}, \quad \alpha > 0 \tag{5.111}$$

Notice that the above equation has the form defined in equation (5.106) with $f(t) = t_0^{\alpha+1}$, i.e., a constant function, with $t^{-\gamma} = t^{-(\alpha+1)}$, so we have a power-law distribution valid for all values of t, and the following Laplace transfom given by [GLM99]

$$\hat{\varphi}(s) = (1 + (t_0 s)^\alpha)^{-1} \approx 1 - (\tau_0 s)^\alpha, \quad \text{as } s \to 0. \tag{5.112}$$

And the Laplace transform of the corresponding memory kernel is:

$$\hat{K}(s) = \frac{s\hat{\varphi}(s)}{1 - \varphi(s)} \approx \frac{1}{\tau_0^\alpha s^{\alpha-1}}. \tag{5.113}$$

5.6.3 Power Law Forward Differential Equation

Define $\hat{K}_i = \frac{\tau_{0i}^{-\alpha_i}}{s^{\alpha_i - 1}}$ for all $i \in I$ and substituting it into equation (5.36) to get

$$s\hat{\rho}_{ij}(s) - \rho_{ij}(0) = -\frac{1}{\tau_{0j}^{\alpha_j} s^{\alpha_j - 1}}\hat{\rho}_{ij}(s) + \Sigma_{k \neq j}\frac{1}{\tau_{0k}^{-\alpha_k} s^{\alpha_k - 1}} h_{kj}\hat{\rho}_{ik}(s). \tag{5.114}$$

Introduce $\alpha_i' = 1 - \alpha_i$ for all $i \in I$. Thus, the above the equation can be rewritten as

$$s\hat{\rho}_{ij}(s) - \rho_{ij}(0) = -\tau_{0j}^{\alpha_j' - 1} s^{\alpha_j'}\hat{\rho}_{ij}(s) + \Sigma_{k \neq j}\tau_{0k}^{\alpha_k' - 1} h_{kj} s^{\alpha_k'}\hat{\rho}_{ik}(s). \tag{5.115}$$

Consider the Laplace transform of the Riemann-Liouville fractional derivative of order q, $\mathcal{L}[{}^{RL}\mathcal{D}_t^q] = s^q \widehat{f}(s)$ as defined in Section 3.7.4, so the forward differential equation is derived by taking the Laplace inverse of the above equation as follows

$$\rho_{ij}'(t) = -(\tau_{0j}^{\alpha_j' - 1})^{RL}\mathcal{D}_t^{\alpha_j'}\rho_{ij}(t) + \Sigma_{k \neq j}\tau_{0k}^{\alpha_k' - 1} h_{kj}^{RL}\mathcal{D}_t^{\alpha_k'}\rho_{ik}(t). \tag{5.116}$$

5.6.4 Power Law Backward Differential Equation

As in the forward case, define $\hat{K}_i = \frac{\tau_{0i}^{-\alpha_i}}{s^{\alpha_i - 1}}$ for all $i \in I$ and substitute it into equation (5.46) to get:

$$s\hat{\rho}_{ij}(s) - \delta_{ij} = -\frac{1}{\tau_{0i}^{\alpha_i} s^{\alpha_i - 1}}\hat{\rho}_{ij}(s) + \Sigma_{k \in I}\frac{1}{\tau_{0i}^{\alpha_i} s^{\alpha_i - 1}} h_{ik}\hat{\rho}_{kj}(s). \tag{5.117}$$

Multiplying both sides of the equation by $s^{\alpha_i - 1}\tau_{0i}^{\alpha_i}$ gives

$$\tau_{0i}^{\alpha_i}(s^{\alpha_i}\hat{\rho}_{ij}(s) - s^{\alpha_i - 1}\delta_{ij}) = -\hat{\rho}_{ij}(s) + \Sigma_{k \in I} h_{ik}\hat{\rho}_{kj}(s). \tag{5.118}$$

Consider the Laplace transform of the Caputo derivative of order q $\mathcal{L}[{}^{C}\mathcal{D}_t^q] = s^q \widehat{f}(s) - s^{q-1} f(0)$, as defined in Section 4.5.4. So the backward differential equation is derived by taking the Laplace inverse of the above equation as follows

$$(\tau_{0i}^{\alpha_i})^{C}\mathcal{D}_t^{\alpha_i}\rho_{ij}(t) = -\rho_{ij}(t) + \sum_{k \in I} h_{ik}\rho_{kj}(t). \tag{5.119}$$

5.6.5 Two-State Model

Consider a system S that performs a CTSMP with space state $I = [1, 2]$. So by the PLD waiting time forward differential equation (5.116) we get

$$\rho_{11}'(t) = -(\tau_{01}^{-\alpha_1})^{RL}\mathcal{D}_t^{1-\alpha_1}\rho_{11}(t) + \Sigma_{k \neq 1}\tau_{0k}^{-\alpha_k} h_{k1}^{RL}\mathcal{D}_t^{1-\alpha_k}\rho_{1k}(t). \tag{5.120}$$

So,

$$\rho'_{11}(t) = -(\tau_{0_1}^{-\alpha_1})^{RL}\mathcal{D}_t^{1-\alpha_1}\rho_{11}(t) + (\tau_{0_2}^{-\alpha_2})^{RL}\mathcal{D}_t^{1-\alpha_2}\rho_{12}(t), \tag{5.121}$$

and similarly, we obtain

$$\rho'_{12}(t) = -(\tau_{0_2}^{-\alpha_2})^{RL}\mathcal{D}_t^{1-\alpha_2}\rho_{12}(t) + (\tau_{0_1}^{-\alpha_1})^{RL}\mathcal{D}_t^{1-\alpha_1}\rho_{11}(t), \tag{5.122}$$

$$\rho'_{21}(t) = -(\tau_{0_1}^{-\alpha_1})^{RL}\mathcal{D}_t^{1-\alpha_1}\rho_{21}(t) + (\tau_{0_2}^{-\alpha_2})^{RL}\mathcal{D}_t^{1-\alpha_2}\rho_{22}(t), \tag{5.123}$$

$$\rho'_{22}(t) = -(\tau_{0_2}^{-\alpha_2})^{RL}\mathcal{D}_t^{1-\alpha_2}\rho_{22}(t) + (\tau_{0_1}^{-\alpha_1})^{RL}\mathcal{D}_t^{1-\alpha_1}\rho_{21}(t). \tag{5.124}$$

By PLD waiting time backward equation (5.119) we obtain

$$(\tau_{0i}^{\alpha_1})^{C}\mathcal{D}_t^{\alpha_1}\rho_{11}(t) = -\rho_{11}(t) + \sum_{k\in I} h_{1k}\rho_{k1}(t) = -\rho_{11}(t) + \rho_{21}(t). \tag{5.125}$$

Similarly, we can easily obtain

$$(\tau_0^{\alpha_1})^{C}\mathcal{D}_t^{\alpha_1}\rho_{12}(t) = -\rho_{12}(t) + \rho_{22}(t), \tag{5.126}$$

$$(\tau_0^{\alpha_2})^{C}\mathcal{D}_t^{\alpha_2}\rho_{21}(t) = -\rho_{21}(t) + \rho_{11}(t), \tag{5.127}$$

$$(\tau_0^{\alpha_2})^{C}\mathcal{D}_t^{\alpha_2}\rho_{22}(t) = -\rho_{22}(t) + \rho_{12}(t). \tag{5.128}$$

In this case, the differences between forward and backward differential equations are even greater than the previous ones. The former is a function of the Riemann-Liouville fractional derivative, the regular derivative and considers both parameters. The latter is a function of Caputo derivative and only considers one parameter. Notice that, as in the previous waiting time distributions, the forward differential equations (5.126) and (5.128) can be obtained from the equations (5.125) and (5.127) respectively, by considering that $\rho_{ii}(t) + \rho_{ij}(t) = 1$, for all $i, j = 1, 2$. $i \neq j$.

Chapter 6

Empirical Analysis

In the previous section we derived semi-Markov forward and backward differential equations with particular cases of exponential, gamma and power-law waiting time distributions. Now we will apply the the different approaches to obtain the probability of migration. We will estimate the semi-Markov transition probabilities by the Monte Carlo method, and compare them with the corresponding discrete time Markov process (DTMP) and continuous time Markov process (CTMP) transition probabilities.

6.1 Monte Carlo Method

6.1.1 Antecedents

Monte Carlo method is a technique widely applied in order to approximate complicated processes by simulating a large number of random realisations of it. It tends to be used when it is not possible to compute an exact result with a deterministic method. It was introduced in 1940s by John von Neumann, Stanislaw Ulam and Nicholas Metropolis, when they were working on Manhattan project. Since then, it has been widely used in physics, biology, chemistry and mathematics.

6.1.2 Monte Carlo Method in Finance

Problems in finance are often highly complex, so, in many cases it is not possible to find a numerical solution by statistical or deterministic methods. Thus, Monte Carlo method has been used to asset price dynamics and option values. In finance, it was introduced by [Her79] to solve problems related to corporate finance, and by [Boy77] for simulating a derivative instrument prices. Nowadays it is used in many fields such as corporate finance [AT02], real option analysis [Raz03], portfolio valuation [DLV03], option pricing [BG12], etc.

In credit risk analysis, Monte Carlo method has been applied mainly to the estimation of the maximum value that will be lost if the counter-party defaults. A detailed analysis is presented in [AC98, GL05, DW07]. It is also used in structural models to estimate the PD [Van10]

6.2 Data Collection

The data was obtained from Compustat, Wharton research Data Service, University of Pennsylvania, which includes historical information of ratings. The database is Standard & Poor's Domestic Issuer Credit Rating. Ratings are given monthly and, to homogenize the analysis, we just focused on companies classified as financial sector according to the Global Industry Classification Standard [MSC02]. The reference period is April 1987 to March 2012 (300 months). The analysis considers $97,599$ data points from 859 firms, 54 of which have data points over the whole period.

6.3 Rating System

The rating scale used is Standard & Poor's credit rating system for long term corporate fixed income securities as defined in Section 2.2.1. This system contains 24 scales, however for simplicity we will consider 8 states classified as follows

$AAA = [AAA]$,.

$AA^* = [AA+, AA, AA-]$,

$A^* = [A+, A, A-]$,

$BBB^* = [BBB+, BBB, BBB-]$,

$BB^* = [BB+, BB, BB-]$,

$B^* = [B+, B, B-]$,

$C^* = [CCC+, CCC, CCC-, CC+, CC, CC-, C]$,

$D = [D]$,

where D represents the default state.
Thus $I = [AAA, AA^*, A^*, BBB^*, BB^*, B^*, C^*, D]$ will be the state space of our credit risk system.

6.4 Definition of the Problem

Consider a credit risk system defined in the above state space I that evolves randomly through time, as observed in figure 4.2. According the theory presented, this random evolution could be Markovian or non-Markovian. We will model this process with a CTRW, with non-Markovian behaviour corresponding to a non-exponential waiting time distribution.

The model will have the following characteristics:

- The D state is an absorbing state
- Any state, apart from D, communicates with the other states, i.e.
  there exists t such that, $\rho_{ij}(t) \neq 0$. $i, j \neq D$.
- The waiting time probability does not depend on the state to migrate to, i.e. $G_{ij}(t) = \Phi_i(t)$, for all $i, j \in I$.

6.5 Assumptions

To obtain the migration probabilities, we will make these assumptions on our data

- Firms are properly rated.
- Firms with no response have the same migration probability as those with responses.
- Transition probabilities are homogeneous in time.

These assumptions will simplify and homogenise the data, without impeding too much on the generality of the conclusions drawn.

6.6 Discrete Time Markov Process Approach

Suppose that our system S evolves randomly in discrete time $T = 0, 1, 2, \ldots$ where we take the unit of time to be one month. We are interested in estimating the following probability

$$\rho_{ij}(1) = P[X_1 = j | X_0 = i], \quad i, j \in I. \tag{6.1}$$

This information is contained in a transition matrix $P = [\rho_{ij}(1)]_{m \times m}$. By the Markov property and the homogeneity assumption, we can estimate the migration probability at time 1 as follows,

$$\rho_{ij}(1) = \frac{\text{Number of transitions from state } i \text{ to state } j}{\text{Number of transitions from state } i \text{ to any other state, including itself}}. \tag{6.2}$$

We are also interested in obtaining the n-th step transition probability $\rho_{ij}(n) = P[X_n = j | X_0 = i]$, which can be expressed as

$$\rho_{ij}(n) = \rho_{ij}^{(n)} \quad i, j \in I, \ n \in \mathbb{N}, \tag{6.3}$$

where $\rho_{ij}^{(n)}$ is the ij-th element of the matrix P^n, previously defined in Section 4.1. The migration probability can then be calculated with simple matrix calculations.

6.7 Continuous Time Markov Process Approach

Suppose now, instead, that our system evolves randomly in continuous time $t \geq 0$. It can be modeled with CTMP, as defined in Section 4.2. The waiting time distribution must be exponential, in order for the process to satisfy the Markov property, and is denoted by

$$\varphi_i(t) = \lambda_i e^{-\lambda_i t}, \quad t \geq 0, \quad \text{for all } i \in I \tag{6.4}$$

For our data, we must estimate the parameter λ_i for all $i \in I$. We will use the maximum likelihood estimator method (MLE). The MLE is a well-known technique of estimating the parameters of a statistical model from a random sample $\{x_1, x_2, \ldots, x_n\}$ of n independent and identically distributed observations coming from a probability density function (pdf) $f(\cdot|\hat{\theta})$, where $\hat{\theta}$ is the vector of unknown parameters.

The log-likelihood functions is defined by

$$Ln[f(x_1, x_2, \ldots x_n|\hat{\theta})] = \sum_{r=1}^{n} Ln(f(x_i|\hat{\theta}), \tag{6.5}$$

where n is the number of observations. The average log-likelihood is given by $\hat{l} = \frac{1}{n}\sum_{r=1}^{n} Ln(f(x_i|\hat{\theta}))$, and the MLE estimates $\hat{\theta}_{MLE}$ by finding a value of θ that maximizes $\hat{l}$.

It can be shown that the MLE estimator of the exponential parameter is given by $\hat{\lambda}_i = (\frac{1}{n}\sum_{n}^{r=1} t_r)^{-1}$, where t is the migration time to any state different to i. Notice that the MLE of λ is the inverse of the average waiting time.

Now consider $\lambda_{ij} = \lim_{h\to 0} \frac{P[X(t+h)=j|X(t)=i]-P[X(t)=j|X(t)=i]}{h}$ as defined in Section 4.2.2. Assuming that as $h \to 0$ is only possible one jump, it is clear that λ_{ij} can be also expressed as

$$\lambda_{ij} = -h_{ij} \lim_{h\to 0} \frac{1 - P[X(t+h) = i|X(t) = i]}{h}, \quad \text{for all} i, j \in I, \ t \geq 0. \tag{6.6}$$

where h_{ij} is the transition probability from i to j as defined in Section 4.4.2. The above equation can be rewritten as follows

$$\lambda_{ij} = -h_{ij} \lim_{h\to 0} \frac{P[X(t+h) = i|X(t) = i] - P[X(t) = i|X(t) = i]}{h}, \tag{6.7}$$

Notice that the above equation is the derivative of $\lambda_{ii} = -\sum_{j\neq i}\lambda_{ij}$ as defined in Section 4.2.2, thus

$$\lambda_{ij} = -h_{ij}\lambda_{ii}, \tag{6.8}$$

where, λ_{ii} is estimated by the MLE defined above and the way to estimate h_{ij} is explained in the next Section. Once we have estimated the parameters λ, for all $i, j \in I$, we can obtain the matrix $\hat{Q} = [\hat{\lambda}]_{m\times m}$. Again, we are interested in obtaining the migration probability, as defined in Section 4.2.5.

$$\rho_{ij}(t) = P[X_t = j|X_0 = i], \quad i, j \in I. \tag{6.9}$$

It can be obtained through the spectral decomposition, as shown in Section 4.2.9.

6.8 Discrete Time Semi-Markov Process Approach

Consider a discrete time semi-Markov process (DTSMP), as defined in Section 4.3, with the following migration probability

$$\rho_{ij}(t_n) = dij(t_n) + \sum_{k\in I}\sum_{\tau=1}^{t_n} \upsilon_{ik}(\tau)\rho_{kj}(t_n - \tau), \text{for all } i, j \in I, t \in T, \tag{6.10}$$

with

$$\upsilon_{ik}(\tau) = h_{ij}(\Phi(t_n) - \Phi(t_{n-1})), \tag{6.11}$$

where h_{ij} is the ij-th element of the transition matrix H, and $\Phi(t_n)$ the waiting time probability at time t_n. The element ij-th of the matrix H will be estimated by

$$\hat{h}_{ij} = \frac{\text{Number of transitions from state } i \text{ to state } j}{\text{Number of transitions from state } i}. \tag{6.12}$$

Notice that this formula is similar to the 1 month migration probability introduced in Section 6.6. In this case, we do not include the initial state since we will assume $h_{ii} = 0$ for all i.

On the other hand, the waiting time probability will be estimated by the following way

$$\hat{\Phi}_i(t) = \frac{\text{Number of transitions from state } i \text{ in a time less than or equal equal to } t}{\text{Number of transitions from state } i}. \tag{6.13}$$

6.9 Continuous Time Semi-Markov Process Approach

We will introduce the transition probability $h_{ij} = \lim_{t\to\infty} Q_{ij}$ as defined in Section 4.4.2, where Q is the semi-Markov kernel and $\hat{h}_{ij}$ was already defined in the previous section. We will use the gamma, Weibull and power-law distributions, that are widely used in reliability analysis.

The gamma distribution, as defined in Section 5.5, is a two parameter distribution with the following pdf

$$\varphi(t, \theta, \beta) = \frac{\beta^\theta}{\Gamma(\theta)} t^{\theta-1} e^{-\beta t}, \quad \text{for all } t \geq 0. \tag{6.14}$$

It can be shown that the MLE parameters are

$$\hat{\beta} = \frac{1}{n\hat{\theta}} \sum_{i=1}^{n} t_i, \quad t \geq 0. \tag{6.15}$$

There is not closed-form MLE solution for the variable θ, but it can be found by the following approximation

$$\hat{\theta} \approx \frac{3 - a + \sqrt{(a-3)^2 + 24a}}{12a}, \tag{6.16}$$

where $a = Ln(\frac{1}{n}\sum_{i=1}^{n} t_i) - \frac{1}{n}\sum_{i=1}^{n} Ln(t_i)$. The Weibull distribution is a continuous probability distribution. Its pdf is defined as,

$$\varphi(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-\frac{t}{\lambda}}. \tag{6.17}$$

There is not closed-form MLE solution for this pdf. [Alf00] presents an approximation by the Newton-Raphson method as follows,

$$\frac{\sum_{i=1}^{n} t_i^{\hat{k}} Ln(t_i)}{\sum_{i=1}^{n} t_r^{\hat{k}}} - \frac{1}{\hat{k}} - \frac{1}{n}\sum_{i=1}^{n} Ln(t_i) = 0. \tag{6.18}$$

Once k is determined, λ can be estimated by

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} t_i^{\hat{k}}. \tag{6.19}$$

Now, suppose a PLD, as defined in Section 5.6, with the following pdf

$$\varphi(t) \sim \left(\frac{t_0}{t}\right)^{\alpha+1}, \quad \alpha > 0. \tag{6.20}$$

In order to estimate the parameters, define the following variable

$$t_m = \frac{t_0^{\alpha+1}}{\alpha}. \tag{6.21}$$

Thus, the pdf the is given by

$$\varphi(t) \sim \frac{\gamma t_m^{\alpha}}{t^{\alpha+1}}. \tag{6.22}$$

It can be shown that the MLE of parameters are given by

$$\hat{t}_m = \min_i \ t_i. \tag{6.23}$$

and,

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}(Ln(t_i) - Ln(\hat{t}_m))}. \tag{6.24}$$

6.10 Calculations

The one month transition probability matrix $P = [X_{n+1} = j | X_n = i]$, corresponding to long term fixed income securities from financial sector firms, is given by

$$\hat{P} = \begin{pmatrix} 0.9933 & 0.0063 & 0.0000 & 0.0004 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0004 & 0.9894 & 0.0101 & 0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0001 & 0.0018 & 0.9937 & 0.0043 & 0.0001 & 0.0001 & 0.0000 & 0.0000 \\ 0.0000 & 0.0003 & 0.0030 & 0.9922 & 0.0040 & 0.0003 & 0.0001 & 0.0001 \\ 0.0000 & 0.0004 & 0.0004 & 0.0080 & 0.9778 & 0.0111 & 0.0017 & 0.0004 \\ 0.0000 & 0.0000 & 0.0011 & 0.0005 & 0.0084 & 0.9736 & 0.0152 & 0.0011 \\ 0.0000 & 0.0000 & 0.0008 & 0.0000 & 0.0032 & 0.0129 & 0.9579 & 0.0251 \end{pmatrix}. \tag{6.25}$$

Clearly, in the short term, the system tends to remain in the same state, and the few observed changes are mostly in the neighborhood. This is shown by the high values along the diagonal, and almost zero probabilities elsewhere. However, closer to the default state we can see that migration is somewhat more likely to occur.

Now, assume that the Q-matrix of transition rates from a CTMP model is given by

$$\hat{Q} = \begin{pmatrix} -0.0067 & 0.0063 & 0.0000 & 0.0004 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0004 & -0.0106 & 0.0101 & 0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0001 & 0.0018 & -0.0063 & 0.0043 & 0.0001 & 0.0001 & 0.0000 & 0.0000 \\ 0.0000 & 0.0003 & 0.0030 & -0.0078 & 0.0040 & 0.0003 & 0.0001 & 0.0001 \\ 0.0000 & 0.0004 & 0.0004 & 0.0080 & -0.0222 & 0.0111 & 0.0017 & 0.00004 \\ 0.0000 & 0.0000 & 0.0011 & 0.0005 & 0.0084 & -0.0264 & 0.0152 & 0.0011 \\ 0.0000 & 0.0000 & 0.0008 & 0.0000 & 0.0032 & 0.0129 & -0.0421 & 0.0251 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}. \tag{6.26}$$

Notice that the states near to the default state have a higher instantaneous rate, it is because of they tend to migrate faster. Similarly, the rate corresponding to the state A^*, is the smallest one. The system tends to be for more time before migrating.

The MLE corresponding to the gamma distribution parameters are

State	$\hat{\theta}$	$\hat{\beta}$
AAA	1.0191	147.4020
AA*	1.0326	91.2239
A*	1.0180	154.7998
BBB*	1.0230	125.8340
BB*	1.0681	42.2537
B*	1.0803	35.0628
C*	1.1234	21.1574.

Notice that the shape parameter θ must be close to one in order to obtain a exponential waiting time distribution. On the other hand, the MLE corresponding to the Weibull distribution parameters are

State	$\hat{k}$	$\hat{\lambda}$
AAA	1.0129	161.0909
AA*	1.0191	103.5800
A*	1.0124	168.6754
BBB*	1.0147	139.1253
BB*	1.0353	52.4494
B*	1.0411	44.7531
C*	1.0606	29.546.

Finally, the estimation of the PLD's parameter is given by

State	$\hat{\alpha}$
AAA	0.2246
AA*	0.2504
A*	0.2222
BBB*	0.2325
BB*	0.3050
B*	0.3216
C*	0.3751.

6.11 Simulation

The DTMP migration probabilities were obtained by multiplying n times the transition matrix by itself. In order to achieve the DTSMP transition probabilities, we obtained equation 6.10 by the computing method defined in [DM84]. The corresponding probability by CTMP was calculated by the spectral decomposition methodology described in Section 4.2.8 (see code in Apendix A). Figure 6.1 shows the flowchart used to estimating the migration probability for a credit rating model approached by CTMP.

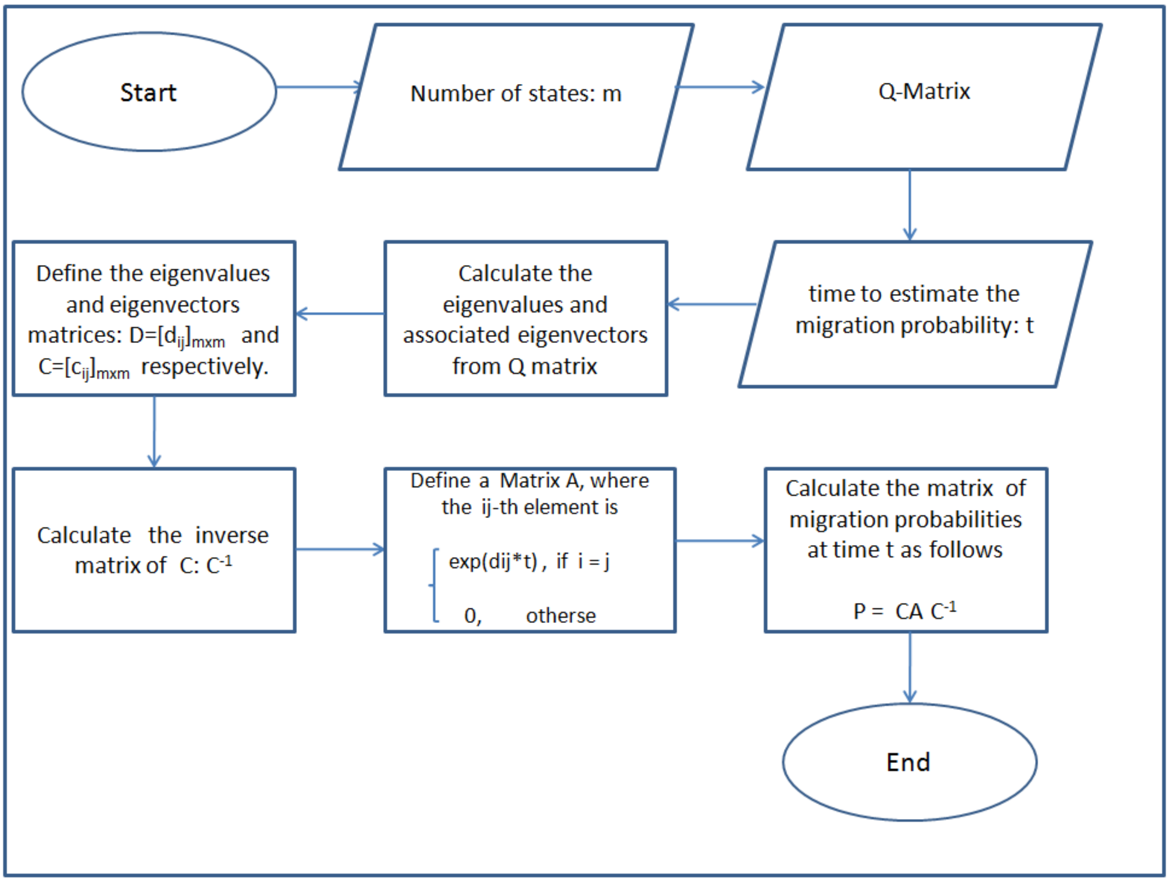

Figure 6.1: Flowchart of Migration Probabilities by CTMP

The migration probability of the CTSMP was estimated by Monte Carlo method. For each state, we simulated 5000 realisations of the process from 0 to a specific time t. A probability was estimated as follows:

$$\hat{\rho}_{ij}(t) = \frac{\text{Number of migrations from state i to state } j \text{ at time } t}{\text{Number of realisations}}. \tag{6.27}$$

Figure 6.2 shows for random realisations at $t = 10$ years of the process, obtained by the Monte Carlo Method. Each realization is given by the following procedure

- Given a system S with default state m and initial state $i_0 \neq m$, we generate a random migration time τ_1, based on the i_0-waiting time distribution.
- If $\tau_1 \leq t$, we generate a random state i_1 with the i_0-th row of the transition matrix H.
- Otherwise, the final state will be i_0, and the procedure ends.
- If $i_1 = m$ the procedure ends and the final state will be m, otherwise, we generate a new random time τ_2 from the i_1 -waiting time distribution.
- if $\tau_1 + \tau_2 \leq t$, we generate a random state i_2 from the i_1-th row of transition matrix H.
- Otherwise, the final state will be i_1, and the procedure ends.
- We apply this procedure until the sum of random times be greater than t or $i = m$, and the final state will be the last random state generated.

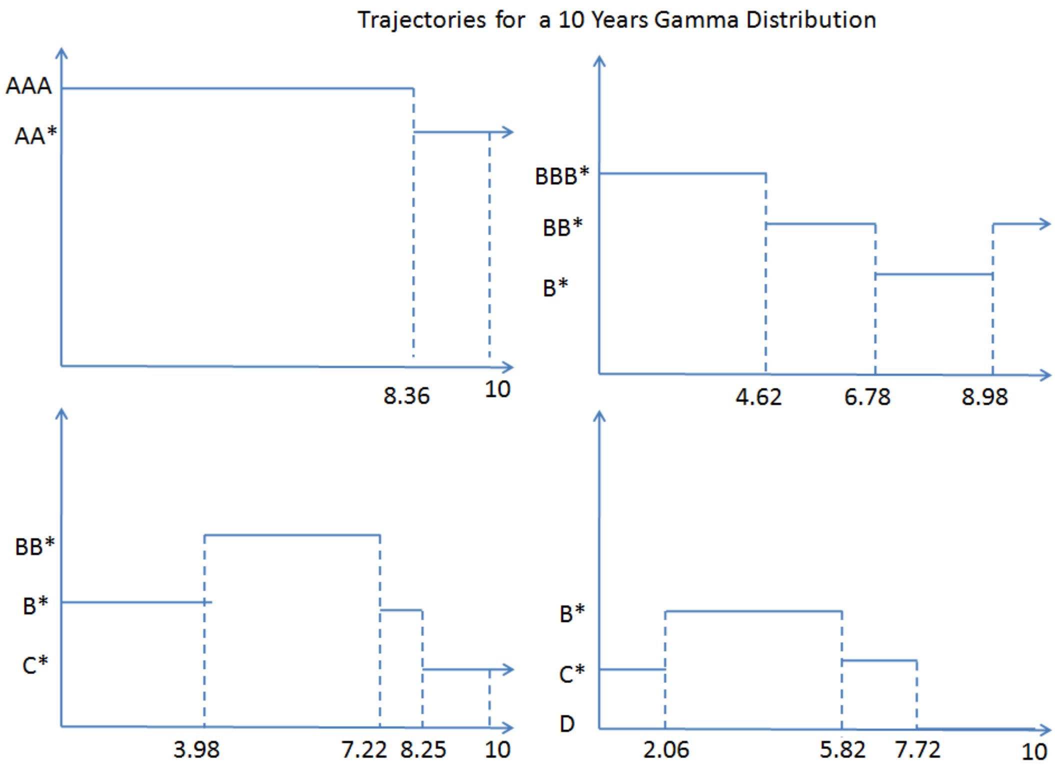

Figure 6.2: Trajectories Simulated by Monte Carlo Method

The simulation was programmed in Matlab (see code in Appendix B). Figure 6.3 shows the flowchart followed in order to elaborate it. This methodology will be applied by computational methods a large number of times in order to estimate the migration probability from state i to state j. So the migration probability will be given by

$$\hat{\rho}_{ij}(t) = \frac{\text{Number of times that a process started at state i, was in state j at time t}}{\text{Number of observations of migrations from state i to state j, including it self at time t}}. \quad (6.28)$$

Figure 6.2 shows some typical realisations of the process at time 10, obtained from the simulations. Notice that the best states tend to be less volatile.

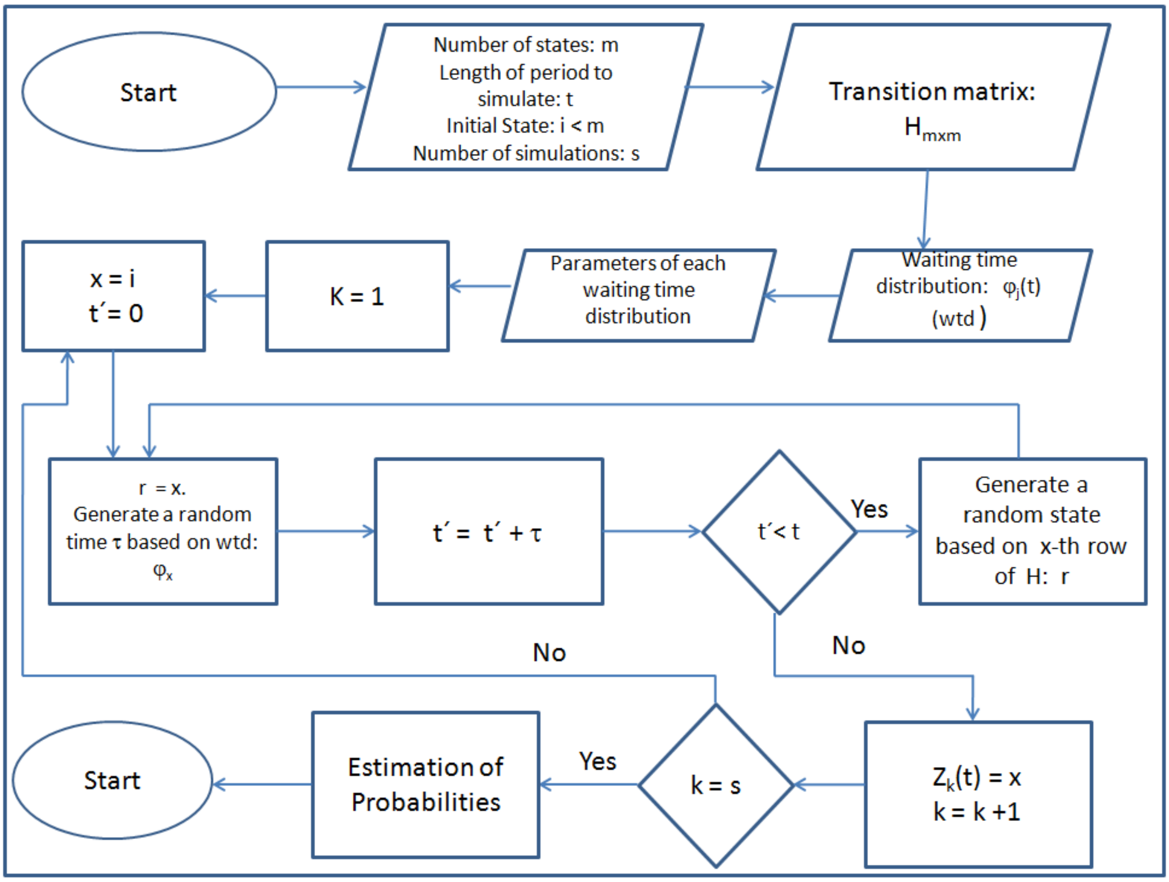

Figure 6.3: Flowchart of Monte Carlo Method

6.12 Results

The results of the different approaches for $t = 1$ year were

$\rho_{AAAj}(1) = P[X(1) = j|X(0) = AAA]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.9503	0.0462	0.0035	0.0000	0.0000	0.0000	0.0000	0.0000
DTMP	0.9227	0.0687	0.0040	0.0045	0.0001	0.0000	0.0000	0.0000
DTSMP	0.9172	0.0786	0.0042	0 .0000	0 .0000	0 .0000	0 .0000	0 .0000
CTMP	0.9195	0.0730	0.0075	0.0000	0.0000	0.0000	0.0000	0.0000
GCTSMP	0.9475	0.0505	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
WCTSMP	0.9550	0.0385	0.0065	0.0000	0.0000	0.0000	0.0000	0.0000
PWCTSMP	0.5752	0.2794	0.1174	0.0234	0.0036	0.0004	0.0004	0.0002

$\rho_{AA^*j}(1) = P[X(1) = j|X(0) = AA^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0092	0.8724	0.1132	0.0052	0.0000	0.0000	0.0000	0.0000
DTMP	0.0044	0.8812	0.1105	0.0037	0.0001	0.0001	0.0000	0.0000
DTSMP	0.0042	0.8780	0.1118	0.0060	0 .0000	0 .0000	0 .0000	0 .0000
CTMP	0.0075	0.8135	0.1625	0.0145	0.0010	0.0005	0.0005	0.0000
GCTSMP	0.0065	0.8725	0.1180	0.0030	0.0000	0.0000	0.0000	0.0000
WCTSMP	0.0055	0.8540	0.1320	0.0085	0.0000	0.0000	0.0000	0.0000
PWCTSMP	0.0126	0.5752	0.3118	0.0794	0.0140	0.0050	0.0014	0.0006

$\rho_{A^*j}(1) = P[X(1) = j|X(0) = A^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0011	0.0388	0.8842	0.0717	0.0031	0.0007	0.0004	0.0000
DTMP	0.0000	0.0198	0.9289	0.0479	0.0021	0.0012	0.0001	0.0000
DTSMP	0.0006	0.0198	0.9344	0.0424	0.0024	0.0002	0.000 2	0 .0000
CTMP	0.0025	0.0530	0.8260	0.1045	0.0095	0.0030	0.0010	0.0005
GCTSMP	0.0000	0.0300	0.8945	0.0710	0.0035	0.0005	0.0005	0.0000
WCTSMP	0.0015	0.0400	0.8805	0.0780	0.0000	0.0000	0.0000	0.0000
PWCTSMP	0.0060	0.0848	0.6392	0.2060	0.0430	0.0144	0.0032	0.0034

$\rho_{BBB^*j}(1) = P[X(1) = j|X(0) = BBB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0003	0.0157	0.0946	0.7512	0.1076	0.0186	0.0042	0.0078
DTMP	0.0000	0.0037	0.0336	0.9130	0.0410	0.0056	0.0017	0.0014
DTSMP	0.0002	0.0032	0.0352	0.9084	0.0444	0.0054	0.0018	0.0014
CTMP	0.0010	0.0135	0.1010	0.7000	0.1405	0.0255	0.0095	0.0090
GCTSMP	0.0005	0.0165	0.0920	0.7455	0.1050	0.0240	0.0070	0.0095
WCTSMP	0.0005	0.0125	0.1025	0.7445	0.1025	0.0235	0.0070	0.0070
PWCTSMP	0.0016	0.0250	0.1264	0.6252	0.1450	0.0458	0.0166	0.0144

$\rho_{BB^*j}(1) = P[X(1) = j|X(0) = BB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0000	0.0092	0.0198	0.1304	0.5963	0.1683	0.0456	0.0304
DTMP	0.0000	0.0042	0.0065	0.0820	0.7711	0.1045	0.0236	0.0083
DTSMP	0 .0000	0.0052	0.0060	0.0842	0.7700	0.1026	0.0234	0.0086
CTMP	0.0000	0.0095	0.0210	0.1255	0.6370	0.1380	0.0450	0.0240
GCTSMP	0.0000	0.0075	0.0215	0.1230	0.6015	0.1615	0.0480	0.0370
WCTSMP	0.0000	0.0065	0.0195	0.1345	0.6020	0.1735	0.0425	0.0215
PWCTSMP	0.0002	0.0160	0.0362	0.1344	0.5330	0.1614	0.0722	0.0466

$\rho_{B^*j}(1) = P[X(1) = j|X(0) = B^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0000	0.0024	0.0224	0.0294	0.1320	0.5226	0.1692	0.1220
DTMP	0.0000	0.0003	0.0120	0.0090	0.0799	0.7398	0.1268	0.0321
DTSMP	0.0000	0.0004	0.0114	0.0090	0.0868	0.7366	0.1234	0.0324
CTMP	0.0010	0.0010	0.0270	0.0220	0.1355	0.5200	0.1705	0.1230
GCTSMP	0.0000	0.0015	0.0260	0.0270	0.1360	0.5175	0.1730	0.1190
WCTSMP	0.0000	0.0005	0.0275	0.0310	0.1415	0.5125	0.1685	0.1185
PWCTSMP	0.0002	0.0054	0.0244	0.0314	0.1200	0.5216	0.1658	0.1312

$\rho_{C^*j}(1) = P[X(1) = j|X(0) = C^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0000	0.0017	0.0192	0.0094	0.0591	0.1437	0.3048	0.4621
DTMP	0.0000	0.0002	0.0082	0.0020	0.0323	0.1080	0.6062	0.2431
DTSMP	0.0000	0.0002	0.0082	0.0020	0.0340	0.1066	0.6010	0.2480
CTMP	0.0000	0.0020	0.0200	0.0065	0.0645	0.1585	0.2765	0.4720
GCTSMP	0.0000	0.0010	0.0180	0.0105	0.0640	0.1465	0.3035	0.4565
WCTSMP	0.0000	0.0010	0.0165	0.0145	0.0695	0.1525	0.2870	0.4590
PWCTSMP	0.0000	0.0022	0.0112	0.0108	0.0498	0.1196	0.4224	0.3840

The discrete time approaches are well fitting models in the first two states mainly due to low frequency of changes that they observe in a period of one year. In the subsequent states, it is clear that they are a poor fit because they tend to overestimate the probability of being in the same state at the end of the period and underestimate the probability of default. On the other hand, CTSMP with power law waiting time distribution tends to underestimate significantly the probability of staying in the same state at the end of the period in the first five states. In state C^* , it tends to overestimate the probability of remaining in the same state at the end of the period and underestimate the probability of default. This distribution has been used in finance, mainly to model high frequency observations such as share and foreign exchange markets [Lux04]. It assumes a high frequency of changes at the beginning of the period, reducing gradually through time, which does not necessarily apply in credit risks. The fit based on CTMP tends to underestimate the probability of being in the same state at the end of the period and overestimate the probability of migration to the next lower state in the first five states. In the last two states they produce a better fit, because they present a higher frequency of changing than the previous ones. In the presence of low frequency of changing observations, CTMP seems not to be a good fit, because it does not consider the memory effect of such states. Meanwhile, CTSMP adjustments based on the Weibull and gamma distributions were consistently the best fitting models, although the former presented some important differences in the fit of state AA^* compared with the observed ones during the analysing period In the subsequent estimations, we present the CTMP and the CTSMP with Weibull and gamma waiting time distributions respectively, in order to focus just on the best fits.

The results of the different approaches for $t = 5$ years were

$\rho_{AAA^*j}(5) = P[X(5) = j|X(0) = AAA]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.6776	0.2323	0.0804	0.0086	0.0003	0.0008	0.0000	0.0000
CTMP	0.6365	0.2380	0.0975	0.0225	0.0040	0.0010	0.0005	0.0000
GCTSMP	0.6880	0.2235	0.0745	0.0105	0.0005	0.0010	0.0000	0.0000
WCTSMP	0.6235	0.1765	0.1380	0.0450	0.0140	0.0020	0.0005	0.0005

$\rho_{AA^*j}(5) = P[X(5) = j|X(0) = AA^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0218	0.4584	0.3984	0.0913	0.0212	0.0049	0.0009	0.0031
CTMP	0.0240	0.4310	0.3820	0.1160	0.0295	0.0105	0.0045	0.0025
GCTSMP	0.0195	0.4585	0.3950	0.0960	0.0190	0.0065	0.0015	0.0040
WCTSMP	0.0210	0.4600	0.4150	0.0755	0.018	0.0080	0.0000	0.0025

$\rho_{A^*j}(5) = P[X(5) = j|X(0) = A^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0011	0.1316	0.4963	0.2335	0.0775	0.0291	0.0078	0.0231
CTMP	0.0080	0.1275	0.4995	0.2300	0.0680	0.0325	0.0080	0.0315
GCTSMP	0.0060	0.1295	0.4965	0.2400	0.0705	0.0280	0.0090	0.0205
WCTSMP	0.0040	0.1395	0.4880	0.2210	0.0870	0.0360	0.0050	0.0195

$\rho_{BBB^*j}(5) = P[X(5) = j|X(0) = BBB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0031	0.0584	0.2699	0.2812	0.1596	0.0841	0.0352	0.1085
CTMP	0.0020	0.0525	0.2480	0.2985	0.1815	0.0780	0.0390	0.1005
GCTSMP	0.0045	0.0545	0.2755	0.2895	0.1615	0.0770	0.0345	0.1030
WCTSMP	0.0035	0.0485	0.2450	0.2675	0.2260	0.0735	0.0345	0.1015

$\rho_{BB^*j}(5) = P[X(5) = j|X(0) = BB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0016	0.0286	0.1164	0.1792	0.2184	0.1394	0.0489	0.2675
CTMP	0.0015	0.0250	0.1005	0.1815	0.2180	0.1420	0.0665	0.2650
GCTSMP	0.0010	0.0300	0.1200	0.1820	0.2120	0.1350	0.0510	0.2690
WCTSMP	0.0005	0.0285	0.1170	0.1765	0.2075	0.1460	0.0670	0.2570

$\rho_{B^*j}(5) = P[X(5) = j|X(0) = B^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0000	0.0135	0.0684	0.0952	0.1368	0.1506	0.0580	0.4775
CTMP	0.0005	0.0140	0.0745	0.0860	0.1510	0.1320	0.0545	0.4875
GCTSMP	0.0000	0.0165	0.0680	0.1015	0.1290	0.1580	0.0520	0.4750
WCTSMP	0.0000	0.0190	0.014	0.0760	0.1170	0.1440	0.1580	0.4720

$\rho_{C^*j}(5) = P[X(5) = j|X(0) = C^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0000	0.0109	0.0352	0.0452	0.0753	0.0685	0.0264	0.7385
CTMP	0.0000	0.0120	0.0360	0.0370	0.0725	0.0630	0.0275	0.7520
GCTSMP	0.0010	0.0100	0.0490	0.0440	0.0630	0.0685	0.0270	0.7375
WCTSMP	0.0000	0.0190	0.0520	0.0415	0.0680	0.0625	0.0250	0.7320

The CTMP adjustment underestimates the probability of being in the same state at the end of the period for the first four states. In the subsequent states, in which there is a higher frequency of changes through the time, it is a well fitting model. The model based on the Weibull distribution CTSMP presented a good adjustment, although with some inconsistencies in the simulations of states AAA, BBB^* and B^*, mainly in the next lower state. The gamma distribution CTSMP was consistent in all cases.

The results of the different approaches for $t = 10$ years were

$\rho_{AAAj}(10) = P[X(10) = j|X(0) = AAA]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.4396	0.2594	0.2178	0.0542	0.0118	0.0086	0.0021	0.0065
CTMP	0.4180	0.2585	0.2145	0.0685	0.0200	0.0075	0.0025	0.0105
GCTSMP	0.4570	0.2570	0.2075	0.0530	0.0135	0.0060	0.0015	0.0045
WCTSMP	0.4250	0.2680	0.2150	0.0540	0.0120	0.0080	0.0095	0.0085

$\rho_{AA^*j}(10) = P[X(10) = j|X(0) = AA^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0254	0.2412	0.4065	0.1780	0.0704	0.0297	0.0098	0.0390
CTMP	0.0275	0.2450	0.3995	0.1620	0.0720	0.0290	0.0170	0.0480
GCTSMP	0.0275	0.2340	0.4180	0.1800	0.0660	0.0305	0.0070	0.0370
WCTSMP	0.0325	0.2455	0.3965	0.1865	0.0585	0.0425	0.0035	0.0345

$\rho_{A^*j}(10) = P[X(10) = j|X(0) = A^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0143	0.1426	0.3527	0.2168	0.0965	0.0564	0.0182	0.1025
CTMP	0.0125	0.1345	0.3490	0.2130	0.1115	0.0560	0.0130	0.1105
GCTSMP	0.0150	0.1480	0.3550	0.2150	0.0935	0.0580	0.0165	0.0990
WCTSMP	0.0005	0.1550	0.3380	0.2265	0.1145	0.0495	0.0225	0.0935

$\rho_{BBB^*j}(10) = P[X(10) = j|X(0) = BBB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0046	0.0912	0.2318	0.2033	0.1156	0.0754	0.0271	0.2510
CTMP	0.0055	0.0850	0.2355	0.2015	0.1225	0.0715	0.0255	0.2530
GCTSMP	0.0060	0.0855	0.2400	0.1930	0.1225	0.0730	0.0265	0.2535
WCTSMP	0.0040	0.0835	0.2345	0.2140	0.1215	0.0710	0.0225	0.2490

$\rho_{BB^*j}(10) = P[X(10) = j|X(0) = BB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0067	0.0537	0.1507	0.1421	0.1098	0.0754	0.0304	0.4312
CTMP	0.0060	0.0500	0.1510	0.1410	0.1085	0.0670	0.0270	0.4495
GCTSMP	0.0060	0.0585	0.1510	0.1415	0.1110	0.0730	0.0300	0.4290
WCTSMP	0.0045	0.0560	0.1585	0.1525	0.1010	0.0790	0.0200	0.4285

$\rho_{B^*j}(10) = P[X(10) = j|X(0) = B^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0021	0.0274	0.0935	0.0962	0.0762	0.0537	0.0187	0.6322
CTMP	0.0030	0.0315	0.0960	0.0975	0.0830	0.0520	0.0195	0.6175
GCTSMP	0.0015	0.0235	0.0915	0.0950	0.0740	0.0525	0.0140	0.6480
WCTSMP	0.002	0.0215	0.0875	0.0985	0.0865	0.0610	0.0185	0.6245

$\rho_{C^*j}(10) = P[X(10) = j|X(0) = C^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0000	0.0111	0.0562	0.0417	0.0342	0.0206	0.0069	0.8293
CTMP	0.0025	0.0160	0.0525	0.0380	0.0415	0.0265	0.0070	0.8160
GCTSMP	0.0005	0.0140	0.0550	0.0405	0.0305	0.0205	0.0085	0.8305
WCTSMP	0.0000	0.0105	0.0530	0.0395	0.0285	0.0280	0.0095	0.8310

The Weibull CTSMP approach presented the closest estimation to the observed values for the probability of migration from state AAA. Similarly, CTMP model had the best adjust from AA^*. The gamma CTSMP approach was consistently the closest estimation to the observed values for the rest of states.

The results of the different approaches for $t = 15$ years were.

$\rho_{AAAj}(15) = P[X(15) = j|X(0) = AAA]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.2762	0.2614	0.2721	0.0957	0.0456	0.0126	0.0075	0.0289
CTMP	0.2710	0.2445	0.2645	0.1120	0.0480	0.0170	0.0075	0.0355
GCTSMP	0.2730	0.2715	0.2800	0.0900	0.0345	0.0150	0.0085	0.0275
WCTSMP	0.2645	0.2605	0.2785	0.1075	0.0485	0.0115	0.0055	0.0235

$\rho_{AA^*j}(15) = P[X(15) = j|X(0) = AA^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0215	0.1742	0.3394	0.2103	0.0862	0.0424	0.0168	0.1092
CTMP	0.0255	0.1725	0.3450	0.1950	0.0900	0.0405	0.0165	0.1150
GCTSMP	0.0270	0.1730	0.3395	0.2040	0.0885	0.0465	0.0155	0.1060
WCTSMP	0.0225	0.1740	0.3415	0.1985	0.0880	0.0495	0.0165	0.1095

$\rho_{A^*j}(15) = P[X(15) = j|X(0) = A^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0152	0.1402	0.2966	0.1919	0.0915	0.0586	0.0212	0.1848
CTMP	0.0160	0.1355	0.2940	0.1800	0.1110	0.0475	0.0180	0.1980
GCTSMP	0.0180	0.1355	0.3030	0.1980	0.0820	0.0595	0.0200	0.1840
WCTSMP	0.0165	0.1325	0.3055	0.1955	0.0795	0.0545	0.0265	0.1895

$\rho_{BBB^*j}(15) = P[X(15) = j|X(0) = BBB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0092	0.0985	0.2093	0.1468	0.1001	0.0506	0.0206	0.3649
CTMP	0.0140	0.0695	0.2330	0.1565	0.0985	0.0495	0.0190	0.3600
GCTSMP	0.0120	0.0950	0.2085	0.1465	0.1020	0.0525	0.0175	0.3660
WCTSMP	0.0110	0.0885	0.2100	0.1520	0.0985	0.0560	0.0155	0.3685

$\rho_{BB^*j}(15) = P[X(15) = j|X(0) = BB^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0069	0.0517	0.1496	0.1103	0.0750	0.0366	0.0193	0.5506
CTMP	0.0075	0.0605	0.1420	0.1015	0.0785	0.0475	0.0165	0.5460
GCTSMP	0.0085	0.0490	0.1500	0.1115	0.0730	0.0365	0.0190	0.5525
WCTSMP	0.0070	0.0480	0.1510	0.1210	0.0795	0.0295	0.0155	0.5485

$\rho_{B^*j}(15) = P[X(15) = j|X(0) = B^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0041	0.0394	0.0879	0.0526	0.0582	0.0356	0.0120	0.7102
CTMP	0.0060	0.0350	0.0915	0.0670	0.0425	0.0265	0.0135	0.7180
GCTSMP	0.0025	0.0370	0.0815	0.0600	0.0585	0.0385	0.0130	0.7090
WCTSMP	0.0025	0.00355	0.0800	0.0595	0.0605	0.0415	0.0120	0.7085

$\rho_{C^*j}(15) = P[X(15) = j|X(0) = C^*]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*	D
Observed	0.0024	0.0148	0.0406	0.0361	0.0312	0.0169	0.0084	0.8496
CTMP	0.0025	0.0140	0.0450	0.0370	0.0215	0.0215	0.0055	0.8640
GCTSMP	0.0030	0.0200	0.0430	0.0385	0.0300	0.0135	0.0065	0.8455
WCTSMP	0.0015	0.0165	0.0395	0.0395	0.0285	0.0150	0.0080	0.8515

There were not significant differences between the three approaches, although the gamma and Weibull CTSMP models presented closer probabilities to the values observed. It is clear that gaps between the different approaches tend be smaller as t increases. This may lead us to conclude that these models are only suitable for short term modelling.

6.12.1 Estimation of Reliability Function

Now, we will estimate the function R, which is the probability that the rating is not default from 0 to t, as defined in section 4.4.5. Since the DTMP, CTSMP and the PL CTSMP approaches were poor fits, as seen in the previous section, we will just focus on the CTMP and the gamma and Weibull CTSMP approaches. The probabilities estimated will be again compared with the probabilities observed from April 1987 to March 2012.

$R_i(1) = 1 - P[Z(1) = D|Z(0) = i]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*
Observed	1	1	1	0.9922	0.9696	0.8780	0.5379
CTMP	1	1	0.9995	0.9910	0.9760	0.8770	0.5280
GCTSMP	1	1	1	0.9905	0.9630	0.8810	0.5435
WCTSMP	1	1	1	0.9930	0.9785	0.8815	0.5410

There are not considerable differences in the three approaches with respect to the observed probabilities. It is clear that the inconsistencies observed in the CTMP adjustment do not affect significantly its estimation of reliability probability.

$R_i(5) = 1 - P[Z(5) = D|Z(0) = i]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*
Observed	1	0.9969	0.9769	0.8915	0.7325	0.5225	0.2615
CTMP	1	0.9975	0.9687	0.8995	0.7350	0.5125	0.2480
GCTSMP	1	0.9960	0.9795	0.8970	0.7310	0.5230	0.2625
WCTSMP	0.0095	0.9975	0.9805	0.8985	0.7430	0.5280	0.2680

There are not considerable differences in the three approaches with respect to the observed probabilities. Although it is noteworthy that in each state, the gamma CTSMP presented a closer estimation to the observed value.

$R_i(10) = 1 - P[Z(10) = D|Z(0) = i]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*
Observed	0.9935	0.9610	0.8975	0.7490	0.5688	0.3678	0.1707
CTMP	0.9895	0.9520	0.8895	0.7470	0.5505	0.3825	0.1840
GCTSMP	0.9955	0.9630	0.9010	0.7465	0.5710	0.3520	0.1695
WCTSMP	0.9915	0.9655	0.9065	0.7510	0.5715	0.3755	0.1690

There appears to be no significant differences between the various adjustments and the observed probability, the CTMP estimation observed a inconsistency in the case of the probability of state C^*. The gamma CTSMP approach seems to be the estimation closer to the observed values, although it presents a big gap in the case of state B^*.

$R_i(15) = 1 - P[Z(15) = D|Z(0) = i]$

State	AAA	AA*	A*	BBB*	BB*	B*	C*
Observed	0.9711	0.8908	0.8152	0.6351	0.4494	0.2898	0.1504
CTMP	0.9645	0.8850	0.8020	0.6400	0.4540	0.2820	0.1360
GCTSMP	0.9725	0.8940	0.8160	0.6340	0.4475	0.2910	0.1545
WCTSMP	0.9765	0.8905	0.8105	0.6315	0.4515	0.2915	0.1485

Estimations of the CTMP approach shows a bigger gap with respect to the observed probability than the CTSMP's approaches. The gamma CTSMP estimations tend to be closer to the observed probabilities.

Chapter 7

Conclusions

Markov processes are widely used in the modeling of credit risk, however this requires certain assumptions that are difficult to accept. Discrete time Markov process (DTMP) assumes no changes occur in small intervals of time. Which, if we consider that the transition matrix is usually based around one year migration probabilities, this seems unrealistic. In reality, states close to the default state may experience a high frequency of changes.

Though the Continuous time Marko process (CTMP) assumes an exponential waiting time distribution, the memoryless property is perhaps unrealistic. Thus, some authors have extended this to continuous time semi-Markov process (CTSMP), which does not possess this memoryless property. So that, it is more flexible and could be argued to be more applicable to real world problems. In recent years, this process has been introduced to estimate the migration probability of a certain credit raint system which is obtained if we discretise the integral backward equation, and let the discretisation parameter go to zero to get the continuous time version.

This work began defining the problem and introducing various useful mathematical and economic concepts required for the credit risk model. In Chapter 3, A literature review was presented, in which we analysed the historical evolution of credit risk models, beginning with the Merton model. In this chapter we mainly focused on structural form models and reduced form models, their assumptions and the difficulties that they face. After that, we provided a general overview of the historical evolution of factor models and other models based on the correlation analysis between probablity of default (PD) and loss given default (LGD).

In Chapter 4, we provided an overview of the different Markovian and non-Markovian models in order to apply them in the following chapters. We derived an expression for the migration probability of a DTSM and analysed some characteristic of a model with an absorbing state. Then we presented the CTMP, derived from the Kolmogorov forward and backward equation and found a solution by using the Laplace transform. We then deduced the solution for a two-state model by a system of differential equations, and for a three-state model by the spectral decomposition. Additionally, we provided the discrete time semi-Markov Process (DTSMP) theory and deduced the migration probability. Finally, we introduced the CTSMP and we defined an $(X-T)$ process, the functions which defined this process and the migration probability.

Throughout, we have assumed that the waiting time distribution and jump distribution are independent. Thus, in Chapter 5 we set up the forward differential equation from the n-state residence model, which is our main theorical contribution in this work. We also derived the backward differential equation from the backward integral equation. Some examples of particular waiting time distributions were provided.

We first considered the exponential distribution and derived the forward and backward differential equations. The result was a memoryless process that corresponded to Kolmogorov equations for the Markov case. We illustrated an example with the particular two states case, known as a Markov switching regime, and the corresponding forward and backward equation were expressed in a system of differential equations. The forward equation provided an easier solution, since we only needed to apply a substitution to obtain a differential equation in terms of the same probability function. On the other hand, in the forward case we obtained a system in which two of equations of the system could be derived from the other two.

Then, we derived the forward and backward differential equations with the gamma waiting time distribution, under the assumption that the shape parameter was an integer. In order to derive a closed solution for the forward equation we assumed that all the states had the same waiting time distribution. A two-state example was provided by assuming the same distribution for all states in order to compare results, as in the exponential case. The forward equation provided an easier solution, since only a substitution was needed to obtain a differential equation in terms of the same probability function. Similarly, two equations could be derived from the other two in the backward case.

We also illustrated the example with the power las distribution (PLD) waiting time. First, we derived the pdf and the memory kernel in terms of its Laplace transform, and later we obtained the backward equation that contained a Caputo fractional derivative. Thereafter we derived the forward equation, expressed in terms of the Riemann-Liouville derivative and finally, a two-state case example was provided.

In Chapter 6, we approached the CTSMP by the Monte Carlo method. We simulated, in a defined interval of time with an initial state given, several trajectories by a nesting process that generated random times and states based on the waiting time distributions and the transition matrix respectively. We continued until a predefined long time horizon, or the system migrated into the absorbing default state. In order to show this process, we provided an example based on the historical database of monthly ratings given by Standard & Poor's for the issue of financial firms' long-term fixed income securities from April 1987 to March 2012.

The waiting time distributions proposed were gamma, Weibull and power-law, which were compared with the Markov process estimations in continuous and discrete time. We also estimated the DTSMP's migration probabilities. Times of reference were set to 1, 5, 10 and 15 years. The results of the simulation showed a poor fit by the discrete-time processes, which tended to overestimate drastically the probability of being in the same state at the end of the period. The CTSMP with power-law distribution approach tended to underestimate significantly the same probability. It is because this distribution assumes a high frequency of changes in the short term, a situation that does not apply in our model.

The CTMP approach had a tendency to underestimate the probability of being in the same state at the end of the period, and overestimate the probability of migration to the next lower state in the early states and short periods of time. In the long term, this weakness was eliminated and estimations were closer to the observed values. Meanwhile, the Weibull waiting time distribution CTSM approach was a good fit to the observed values, however some inconsistencies were found. Finally, the corresponding gamma distribution approach had fitted well, and tended to be consistently the closest estimation to the observed probabilities. Another important result is that the different estimations tend to be closer each other as t increases, this is because they tend to the same stationary states.

It should be mentioned that the results obtained in the empirical analysis of chapter 7 should not be expanded to other classifications of firms, since their behavior can change significantly.

7.1 Future Research

The thesis dealt with the difficulty that the probability of migration usually depends on the time a firm maintains in the same rating. This is not possible to model by CTMP because it is a memoryless process. Thus, we extended the model to the SMCP that relaxes the exponential waiting time distribution assumption. So, we derived the CTSMP forward and backward equations in order to obtain the migration probability and illustrated some specific waiting time distributions.

The future work should be focused on the generalization of the CTSMP, through the non-Homogeneous CTSMP, to perform a rigorous theory (in particular in the forward equation case, since in the backward equation some theory has been developed).

Our process assumed a discrete space state and constant transition probability. Moreover, performing a credit risk model that relaxes one or both of such assumptions could be an interesting work.

We also obtained an approximation of the probability of migration by Monte Carlo simulation and provided the particular cases of exponential, gamma and power law waiting time distributions for firms of the financial sector. This work could be extended to other industries and other waiting time distributions.

Appendix A

Markov Process Code in Matlab

Now, we will introduce the code in order to obtain the migration probability of a Markov process with an absorbing state.

```
clc
clear
disp ('Estimation of the migration probability of a Markov process')
disp ('Markov process')   % Inputs
a = 1.22
while a > floor(a)
    a = input ('Enter the number of states:  ');
    if a > floor(a)  %Validating the input
        disp('The number of states should be a natural number')
    end
end
b = -2
while b < 0
    b = input ('Enter the length of time to calculate:  ');
    if b < 0        %Validating the input
        disp('Time should be a positive number')
    end
end
disp ('Transition Matrix')
for i = 1:a-1      %Transition Matrix from 1 to a-1 as initial states
```

```
     for j= 1:a
        if i==j
            H(i,j) = 0
        else
        disp(i)
        disp(j)
        H(i,j) = 2
        while H(i,j) > 1 || H(i,j) < 0  %Validating the input
            H(i,j) = input ('Introduce the transition probability: ');
            if H(i,j) > 1
                disp ('The probability should not be greater than 1')
             end
             if H(i,j) < 0
                disp ('The probability should be greater than 0')
            end
        end
        end
     end
end
for j=1:a   % Transition Matrix for absorbing state
    H(a,j) = 0
end
    disp (H)
    disp('Exponential waiting time distribution')   % Paremeters
for i=1:a-1
    c(i) = -2
    while c(i) < 0
        disp ('Instantaneous transition rate (lambda):    ')
        disp(i)
        c(i) = input ('Enter the value of the parameter:  ');
        if c(i) < 0
            disp('the instantaneous transition should be a positive')
```

```
        end
     end
end
for i=1:a-1               % Q Matrix
     d(i) = 0
     for j= 1:a
         q(i,j)= H (i,j) * (1/c (i))
         d(i) = d(i) + q(i,j)
     end
end
for i=1:a-1
    q(i,i)= -d(i)
 end
for i=1:a-1
    q(a,i)= 0
end
[F,G] = eig(q)     % Eigenvectors Matrix and eigenvalues
M = inv(F)         % Eigenvector Matrix inverse
for i=1:a
     for j= 1:a
        if i==j
            N(i,j) = exp(G(i,j)*b)  % Eigenvalues exponential matrix
        else
            N(i,j) = 0
        end
     end

end
P = F*N*M      % Migration probabilities probabilities
clc  % Report of results
disp('Q matrix')
disp (q)
```

```
disp ('Eigenvalues matrix')

disp(G)

disp ('Eivectors matrix')

disp (F)

disp('Migration probabiity matrix')

disp (P)
```

Appendix B

Monte Carlo Approach Code in Matlab

The Monte-Carlo approach code is

```
clc
clear
disp ('Monte Carlo simulation')
    a = 1.22        % Number of states
while a > floor(a) ||  a < 0
    a = input ('Enter the number of states:  ');
    if a > floor(a)  % Validating the input
        disp('The number of states should be a natural number')
    end
    if a < 0  %Validating the input
        disp('The number of states should be greater than 0')
   end
end
b = -2  % Time
while b < 0
    b = input ('Enter the length of time to calculate:  ');
    if b < 0        % % Validating the inputalidating the input
        disp('Time should be a positive number')
    end
```

```
end
ae = 2.5   % Number of repetititions of Monte Carlo simulation
while ae > floor(ae) ||  ae < 0
    ae = input ('Enter the number of repetitions:  ');
    if ae > floor(ae)  %Validating the input
        disp('The number of repetitions should be a natural number')
    end
    if ae < 0  %Validating the input
        disp('The number of repetitions should be greater than 0')
   end
end

disp ('Transition Matrix')
for i = 1:a-1      %Transition Matrix from 1 to a-1 as initial states
     for j= 1:a
        if i==j
            H(i,j) = 0
        else
        disp(i)
        disp(j)
        H(i,j) = 2
        while H(i,j) > 1 || H(i,j) < 0  %Validating the input
            H(i,j) = input ('Introduce the transition probability:');
            if H(i,j) > 1
                disp ('The probability should not be greater than 1')
             end
             if H(i,j) < 0
                disp ('The probability should be greater than 0')
            end
        end
        end
     end
```

```
end
for j=1:a   % Transition Matrix for absorbing state
    H(a,j) = 0
end
    disp (H)
ao = 9   %Choice of distribution
while ao ~= 1 && ao ~= 2 && ao ~= 3 && ao ~= 4 && ao ~= 5
    ao= input ('pdf Gam=1, Exp=2, Lognor=3, Ray=4, Weib= 5:');
end
if ao==1  % Gamma waiting time distribution
disp('Gamma waiting time distribution')
    for i=1:a-1
        disp ('State')
        disp(i)
        c(i) = -2
        while c(i) < 0
            c(i) = input ('Enter the shape parameter value:  ');
            if c(i) < 0        % Validating the input
                disp('Shape parameter should be a positive number')
            end
        end
        d(i) = -2
        while d(i) < 0
            d(i) = input ('Enter the scale parameter value:  ');
                if d(i) < 0        % Validating the input
                    disp('Scale parameter should be a positive number')
                end
        end
    end

elseif ao==2   % Exponential waiting time distribution
    disp('Exponential waiting time distribution')
```

```
    for i=1:a-1
        c(i) = -2           % Validating the input
        while c(i) < 0
            disp ('Instantaneous transition rate (lambda):   ')
            disp(i)
            c(i) = input ('Enter the value of the parameter: ');
            if c(i) < 0
                disp('The parameter should be a positive')
            end
        end
    end
elseif ao==3    % Lognormal waiting time distribution
disp('Lognormal waiting time distribution')
    for i=1:a-1
        disp ('State')
        disp(i)
        c(i) = input ('Enter the mu parameter value: ');
        d(i) = -2
        while d(i) < 0
            d(i) = input ('Enter the sigma parameter value:  ');
            if d(i) < 0       % Validating the input
                disp('Sigma parameter should be a positive number')
            end
        end
    end
elseif ao==4         % Rayleigh waiting time distribution
    disp('Rayleigh waiting time distribution')
    for i=1:a-1
        disp ('State')
        disp(i)
        c(i) = -2    % Validating the input
          while c(i) < 0
```

```
        disp ('Instantaneous transition rate (lambda):  ')
        disp(i)
        c(i) = input ('Enter the value of the parameter:');
        if c(i) < 0
            disp('The parameter should be a positive')
        end
    end
    end
else     % Weibull waiting time distribution
 disp('Weibull waiting time distribution')
    for i=1:a-1
        disp ('State')
        disp(i)
        c(i) = -2    % Validating the input
          while c(i) < 0
            disp ('Instantaneous transition rate (lambda):  ')
            disp(i)
            c(i) = input ('Enter the shape parameter: ');
            if c(i) < 0
                disp('the shape parameter should be a positive')
            end
          end
        d(i) = -2      % Validating the input
        while d(i) < 0
            d(i) = input ('Enter the sigma parameter value:  ');
            if d(i) < 0        % Validating the input
                disp('Sigma parameter should be a positive number')
            end
        end
    end
end
f = 3.5   % Validating the input
```

```
while f > floor(f) || f > a || f < 0
    f = input ('Enter  the state you wish to simulate:  ');
end
for i= 1:ae   % Simulation
m=0
n=0
o=f
p=0
q(1)=f
r(1)=0
while m <= b
    n= n+1
    if ao == 1
        p = random ('Gamma',c(o),d(o))   %Random time
    elseif ao == 2
        p = random ('exp' ,c(o))
    elseif ao==3
        p = random ('logn',d(o),c(o))
    elseif ao==4
        p = random ('rayl',c(o))
    else
        p = random ('wbl',c(o),d(o))
    end
    m = m + p
    if m >= b
        r(n+1) = b
    else
        r(n+1) = m
    end
    if m <= b
        o = randsample(a,1,true,H(q(n),:))   % Random state
        if o == a
```

```
            m = b+1
        end
    end
    q(n+1) = o
    disp(m)
end
s(i)=q(n+1)
end
t=[1:a]     %Estimators
u=hist(s,t)
w=mean(s)
x= var(s)
for i=1:a  %Probabilities estimated
    y(i) = u(i)/ae
end
hist(s,t)
clc         %Final Report
disp ('Transition matrix')
disp(H)
if ao==1
    disp('Gamma distribution parameters');
     for i=1:a-1
        disp ('State');
        disp(i);
        disp ('Shape parameter');
        disp (c(i))
        disp ('Scale parameter');
        disp (d(i))
     end
elseif ao==2
    disp('Exponential distribution parameter');
    for i=1:a-1
```

```
        disp ('State');
        disp(i);
        disp (c(i))
     end
elseif ao==3
    disp('Lognormal distribution parameters');
    for i=1:a-1
        disp ('State');
        disp(i);
        disp ('mu parameter');
        disp (c(i))
        disp ('sigma parameter');
        disp (d(i))
    end
elseif ao==4
    disp('Rayleigh distribution parameters');
     for i=1:a-1
        disp ('State');
        disp(i);
        disp (c(i))
     end
else
    disp('Weibul distribution parameters');
     for i=1:a-1
        disp ('State');
        disp(i);
        disp ('Shape parameter');
        disp (c(i))
        disp ('Scale parameter');
        disp (d(i))
     end
end
```

```
disp ('probabilities');
disp(t)
disp(y)
disp('mean value');
disp(w)
disp('variance');
disp(x)
```

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