K-THEORY, CHAMBER HOMOLOGY AND BASE CHANGE FOR THE p-ADIC GROUPS $SL(2)$, $GL(1)$ AND $GL(2)$

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The thrust of this thesis is to describe base change $BC_{E/F}$ at the level of chamber homology and K-theory for some p-adic groups, such as $SL(2,F)$, $GL(1,F)$ and $GL(2,F)$. Here F is a non-archimedean local field and E is a Galois extension of F. We have had to master the representation theory of $SL(2)$ and $GL(2)$ including the Langlands parameters.

The main result is an explicit computation of the effect of base change on the chamber homology groups, each of which is constructed from cycles. This will have an important connection with the Baum-Connes correspondence for such p-adic groups. This thesis involved the arithmetic of fields such as E and F, geometry of trees, the homology groups and the Weil group $\mathcal{W}_F$.

Declaration

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Chapter 1

Introduction

Let F be a non-archimedean local field, so that F is either a finite extension of $\mathbb{Q}_p$ or is a local function field $\mathbb{F}_q((x))$. The cardinality of the residue field k_F will be denoted by q_F. If $F = \mathbb{Q}_p$ then $q_F = p$. If $F = \mathbb{F}_q((x))$ then $q_F = q$. Now let $GL(n) = GL(n,F)$ and $SL(n) = SL(n,F)$. In [9], Brodzki and Plymen, working directly with L-parameters, equipped the smooth dual of $GL(n)$ with a complex structure. In the smooth dual of $GL(n)$, especially important are the representations with Iwahori fixed vectors. Let E/F be a finite Galois extension of F. We recall that the domain of an L-parameter of G is the local Langlands group $\mathcal{L}_F = \mathcal{W}_F \times SL(2,\mathbb{C})$ where $\mathcal{W}_F$ is the Weil group of F. Base change is defined by restricting the L-parameter from $\mathcal{L}_F$ to $\mathcal{L}_E$. An L-parameter ϕ is tempered if $\phi(\mathcal{W}_F)$ is bounded. Base change therefore determines a map of tempered dual. In this work, we investigate this map at the levels of K-homology and K-theory.

A central result of local class field theory is the identification of the characters of the multiplicative group of a local non-archimedean field F with the characters of the *Weil group* $\mathcal{W}_F$. Langlands conjectured a generalisation of this 'one dimensional representation' theory: a correspondence between smooth irreducible representations of $GL_n(F)$ and certain n-dimensional representations of $\mathcal{W}_F$, called the local Langlands correspondence. For $n = 2$ the proof of the conjecture was completed by [31] and [28]. It has now been proved in all dimensions, first in positive characteristic by [32], then in characteristic zero by [24] and also by [25].

Let G be a second-countable locally compact group (for instance a countable discrete group). One can define a morphism $\mu\colon K_j^G \beta G \to K_j C_r^* G$, called the assembly map, from the equivariant K-homology with G-compact supports of the classifying space of proper actions βG to the K-theory of the reduced C^*-algebra of G. Paul Baum and Alain Connes introduced the following conjecture (1982) about this morphism: The assembly map is an isomorphism. The right hand side tends to be more easily accessible than the left hand side. Therefore, the Baum-Connes correspondence suggests a link between the K-theory of the reduced C^*-algebra of a group and the K-homology of the corresponding classifying space of proper actions of that group.

In the case where G is discrete and torsion-free, the left hand side reduces to the equivariant K-homology with compact supports of the ordinary classifying space βG of G. There is also more general form of the conjecture, known as the Baum-Connes conjecture with coefficients, where both sides have coefficients in the form of a C^*-algebra A on which G acts by C^*-automorphisms. It says in KK-language that the assembly map :

$$\mu_{A,G}\colon KK_j^G(\beta G, A) \to K_j(A \rtimes_r G),$$

is an isomorphism, containing the case without coefficients as the case $A = \mathbb{C}$.

However, counterexamples to the conjecture with coefficients were found in 2002 by Nigel Higson, Vincent Lafforgue and George Skandalis, basing on not universally accepted, as of 2008, results of Gromov on expanders in Cayley graphs. Even assuming the validity of Higson, Lafforgue and Skandalis, the conjecture with coefficients remains an active area of research, since it is, not unlike the classical conjecture, often seen as a statement concerning particular groups or class of groups.

For example, let G be the integers. Then the left hand side is the K-homology of which is the circle. The C^*-algebra of the integers is by the commutative Gelfand-Naimark-transform, which reduces to the Fourier transform in this case, isomorphic to the algebra of continuous functions on the circle. So the right hand side is the topological K-theory of the circle. One can then show that the assembly map is *KK-theoretic Poincaré duality* as defined by Guennadi Kasparov, which is an isomorphism. Another simple example is given by compact groups. In this case, both sides identify naturally with the complex representation ring $\mathfrak{R}(K)$ of the group in such a way that the assembly map becomes the identity.

Using Kasparov's KK-theory we form the equivariant K-homology groups $K_j^G(\beta G)$, with G-compact supports. A class in $K_j^G(\beta G)$ is represented by an abstract G-equivariant elliptic operator on βG which is supported on a G-invariant subset $X \subset \beta G$ with $G \setminus X$ compact. Following Kasparov we define a homomorphism of abelian groups

$$\mu : K_j^G(\beta G) \to K_j(C_r^* G), \quad (j = 0, 1).$$

If A is unital, and if the kernel and cokernel of T are finitely generated projective A -modules, then

$$Index(T) = [kernel(T)] - [cokernel(T)] \in K_0(A),$$

where T is a bounded operator. $Index(T) \in K_1(A)$, if T is self-adjoint. We return now to general G-compact, proper G-spaces X. Kasparov defines

$$K_0^G(X) = \{\textit{homotopy classes of G-equivariant abstract elliptic operators on } X\}$$

his notation for $K_0^G(X)$ is $KK_G^0(C_0(X), \mathbb{C})$. This is an abelian group, and there is a companion group

$$K_1^G(X) = \{\textit{ homotopy classes of self-adjoint G-equivariant abstract elliptic operators on } X\}.$$

If X is a G-compact, proper G-space then associating to each abstract elliptic operator its G-index we obtain a map

$$\mu_X : K_j^G(X) \to K_j(C_r^*(G)), \quad (j = 0, 1), \quad \mu_X(H, \psi, T) = Index_G(T).$$

In the next chapter we introduce the basic definitions from the local fields and their extensions that are essential in this work. This chapter is fairly elementary. However it is important since it sets the background we will be working on.

Chapters 3 and 4 are a review of basic C^*-algebra with K-theory and the enlarged buildings with chamber homology for some p-adic groups $SL(2,F)$, $GL(1,F)$ and $GL(2,F)$ that will be the backbone background of all of our work.

In Chapter 5 we compute the cycles of the chamber homology groups for $SL(2,F)$ in terms of representation theory (principal series, Steinberg, and the discrete series) and we apply the base change map in order to achieve the change of the explicit cycles which leads to an explicit computation of the effect of base change on the chamber homology and K-theory groups. This led us to the following result (Theorems 5.1.4, 5.1.5, 5.2.4, 5.3.1, 5.3.2, 5.3.3, 5.3.4 and 5.3.5):

- the base change of the *principal series* representation remains a *principal series*,
- the base change of the *Steinberg* representation is also a *Steinberg representation* and
- the base change of *cuspidal* representations is either a *cuspidal* or a *principal series*. Also, we have constructed the Baum-Connes correspondence before and after applying the base change map.

In Chapter 6 we use the basic K-homology results to compute the K-homology groups for $GL(1,F)$ (Theorem 6.2.1) and compute the Baum-Connes correspondence (Theorem 6.2.2). We also introduce a way to compute the chamber homology groups for $GL(1,F)$ by determined the quotient

$$\beta GL(1,F)/GL(1,F)$$

which is in this case would be a loop, that led us to the following result $H_0 = H_1$ (§6.2.3). We also apply the base change map on both sides K-theory and chamber homology (Theorem 6.2.3).

Chapter 7 delves into new ways to compute the cycles of the chamber homology groups for $GL(2,F)$ by restricting to the original quotient $\beta G/G$ by using a sub-bicomplex and then *totalising* this sub-bicomplex to get what is called the *little complex*. We have shown that there is a little complex for each type of the representations of $GL(2,F)$ which led us explicitly to compute the homology groups for $GL(2,F)$ (§7.1.1, 7.1.2, 7.1.3 and 7.1.4).

This chapter is devoted to showing that the Baum-Connes correspondence holds for $GL(2,F)$ and the base change map applies for both chamber homology level and the K-theory level (Theorems 7.2.1, 7.2.3, 7.2.4, 7.2.6 and 7.2.7). Consequently, this led us to the following result (Theorem 7.1.7): *the base change of a principal series representations is always a principal series. Similarly, the base change of a twist of Steinberg representation is again a twist of Steinberg. However, an irreducible Galois representation can certainly restrict to a reducible one. So it is possible for base change of cuspidal representations to be a principal series.*

Chapter 2

Local fields

In this chapter we will remind the reader the basic definitions and results from local fields and their extenstions that we will use. We recommend, for proofs and more detailed discussion, some sources like [16], [19] and [44].

2.1 Valuation fields

Let F be a field. Let Γ be a totally ordered discrete abelian group. Add to Γ a formal element $+\infty$ with the properties $a \leq +\infty$, $a + (+\infty) = +\infty$, and denote $\Gamma^{'} = \Gamma \cup \{+\infty\}$. A map $v : F \to \Gamma^{'}$ with the properties:

1. $v(\alpha) = +\infty \Leftrightarrow \alpha = 0$
2. $v(\alpha\beta) = v(\alpha) + v(\beta)$,
3. $v(\alpha + \beta) \geq min(v(\alpha), v(\beta))$

is said to be a discrete valuation on F; F is said to be a valuation field. The map v induces a homomorphism of $F^{\times}$ to Γ and its value group $v(F)$ is a totally ordered subgroup of Γ. If $v(F) = 0$, then v is called the trivial valuation. We get

$$0 = v(1) = v(-1) + v(-1) = 2v(-1)$$

hence $v(-1) = 0$. If $v(\alpha) < v(\beta)$ then

$$v(\alpha) \geq min(v(\alpha+\beta), v(-\beta)) \geq min(v(\alpha), v(\beta)) = v(\alpha).$$

Therefore, if $v(\alpha) \neq v(\beta)$ then

$$v(\alpha+\beta) = min(v(\alpha), v(\beta)).$$

Associated to v are the following objects: the ring of integers

$$\mathcal{O}_v = \{\alpha \in F : v(\alpha) \geq 0\}.$$

Its group of units

$$\mathcal{U}_v = \mathcal{O}_v^\times = \{\alpha \in F : v(\alpha) = 0\}.$$

Its maximal ideal

$$\mathcal{M}_v = \{\alpha \in F : v(\alpha) > 0\}.$$

The residue field

$$\overline{F}_v = \mathcal{O}_v / \mathcal{M}_v.$$

The most important case of discrete valuations are discrete valuations of rank 1 in which case $\Gamma = \mathbb{Z}$. We can assume that $v(F^\times) = \mathbb{Z}$. We often drop rank 1. In this case any element π of F which goes to 1 is called a prime element (a uniformizer) of F with respect to v. Then the ring of integers $\mathcal{O}_v$ is a principal ideal ring, and every proper ideal of $\mathcal{O}_v$ can be written as $\pi^n \mathcal{O}_v$ for some $n > 0$. In particular, $\mathcal{M}_v = \pi \mathcal{O}_v$. The intersection of all proper ideals of $\mathcal{O}_v$ is the zero ideal. Let I be a proper ideal of $\mathcal{O}_v$. Denote $n = \min\{v(\alpha) : \alpha \in I\}$. Then $I = \pi^n \mathcal{O}_v$. If α belongs to the intersection of all proper ideals $\pi^n \mathcal{O}_v$ in $\mathcal{O}_v$, then $v(\alpha) = +\infty$, i.e. $\alpha = 0$.

Example 2.1.1 Let $F = \mathbb{Q}$. For a prime p and a non-zero integer m let $\mathrm{k} = v_p(m)$ be the maximal integer such that p^k divides m. Extend v_p to rational numbers putting

$$v_p(m/n) = v_p(m) - v_p(n); \ \ v_p(0) = +\infty.$$

The map v_p is a discrete valuation. It is called the p-adic valuation. It has ring of integers

$$\{\mathcal{O}_{v_p} = \frac{m}{n} : m, n \in \mathbb{Z}, \ n \ \ is \ \ relatively \ \ prime \ \ to \ \ p\}.$$

A prime element with respect to v_p is pm/n with nonzero integers m,n relatively prime to p. The residue field $\overline{\mathbb{Q}}_{v_p}$ is a finite field of order p. A theorem of Ostrowski tells that every nontrivial norm on $\mathbb{Q}$ is equivalent to either the absolute value or to the norm associated with the p-adic valuation for some prime p.

Example 2.1.2 Let $F = K(X)$ be the field of rational functions over a field K. Let $p(X)$ be an irreducible polynomial. Similarly to the p-adic valuation v_p define the discrete valuation $v_{p(X)}$ on F. The ring of integers

$$\mathcal{O}_{v_{p(X)}} = \{\frac{f(X)}{g(X)} : \ f(X), g(X) \in K[X], \ g(X) \ is\ relatively\ prime\ to\ p(X)\}.$$

A prime element with respect to $v_{p(X)}$ is $p(X)f(X)/g(X)$ with nonzero polynomials $f(X), g(X)$ relatively prime to $p(X)$. The residue field is

$$K[X]/p(X)K[X]$$

which is a finite extension of K. Put

$$v_\infty(f(X)/g(X)) = -degf(X) + degg(X).$$

This is well defined. Note that $v_\infty(\frac{1}{X}) = 1$, so $1/X$ is a prime element with respect to v_∞. The ring of integers is $K[1/X]$ and the residue field is K. The discrete valuations $v_{p(X)}$, v_∞ are all discrete valuations on $K(X)$ which are trivial on $K^\times$.

Definition 2.1.3 Let F be a field. The *characteristic* of F or shortly *Char(F)* is the least positive integer n for which the result of adding any element to itself n times yields 0, we say that *char(F)* $= n$. Otherwise, *Char(F)* is defined to be 0.

Assume that the characteristic of F differs from the characteristic of the residue field $\overline{F}_v$. Then the former is 0 and the latter is a prime p. Indeed, if the characteristic of F is positive p, then $p = 0$ in F and therefore in $\overline{F}_v$. Hence p is the characteristic of $\overline{F}_v$. If the characteristic of F and its residue field coincide we call F an equal characteristic discrete valuation field. Example: $K(X)$ and any discrete valuation $v_p(X)$, v_∞. If the characteristic of F and its residue field do not coincide we call F a mixed characteristic discrete valuation field.

Example 2.1.4 $\mathbb{Q}$ and v_p.

2.2 Completion

Let F be a field with a discrete valuation v. The field F is then a metric topological space with respect to the norm $|\alpha| = (1/2)^{v(\alpha)}$. A sequence $(\alpha_n)_n \geq 0$ of elements of F is called a Cauchy sequence if for every real $c > 0$ there is $n_0 \geq 0$ such that $|\alpha_n - \alpha_m| \leq c$ for $m, n \geq n_0$. Equivalently, for every real $c > 0$ there is $n_0 \geq 0$ such that $v(\alpha_n - \alpha_m) \geq c$ for $m, n \geq n_0$. The set A of all Cauchy sequences forms a ring with respect to componentwise addition and multiplication. The set of all Cauchy sequences $(\alpha_n)_n \geq 0$ with $\alpha_n \to 0$ as $n \to +\infty$ forms a maximal ideal M of A. The field A/M is a discrete valuation field with its discrete valuation $\hat{v}$ defined by $\hat{v}((\alpha_n)) = lim v(\alpha_n)$ for a Cauchy sequence $(\alpha_n)_n \geq 0$. A discrete valuation field F is called a complete discrete valuation field if every Cauchy sequence $(\alpha_n)_n \geq 0$ converges, i.e., there exists $\alpha = lim \alpha_n \in F$ with respect to v. A discrete valuation field L with a discrete valuation w is called a completion of F if it is complete, $w|_F = v$, and F is a dense subfield in L with respect to w. Every discrete valuation field has a completion which is unique up to an isomorphism over F.

The completions of examples 2.1.1 and 2.1.2 will be as follows:

1. The completion of $\mathbb{Q}$ with respect to v_p is denoted by $\mathbb{Q}_p$ and is called the field of p-adic numbers. Certainly, the completion of $\mathbb{Q}$ with respect to the absolute value is

$\mathbb{R}$. Embedding of $\mathbb{Q}$ in $\mathbb{Q}_p$ for all prime p and in $\mathbb{R}$ is a tool to solve various problems over $\mathbb{Q}$.

An example is the *Minkowski- Hasse* Theorem: an equation

$$\sum a_{ij} X_i X_j = 0$$

for $a_{ij} \in \mathbb{Q}$ has a nontrivial solution in $\mathbb{Q}$ if and only if it admits a nontrivial solution in $\mathbb{R}$ and in $\mathbb{Q}_p$ for all prime p. A generalization of this result is the so-called Hasse local-global principal which is of great importance in algebraic number theory. The ring of integers of $\mathbb{Q}_p$ is denoted by $\mathbb{Z}_p$ and is called the ring of p-adic integers. The residue field of $\mathbb{Q}_p$ is the finite field $\mathbb{F}_p$ consisting of p elements.

2. The completion of $K(X)$ with respect to v_X is the formal power series field

$$K((X)) = \sum_{-\infty}^{+\infty} \alpha_n X^n$$

with $\alpha_n \in K$ and $\alpha_n = 0$ for almost all negative n. The ring of integers with respect to v_X is $K[[X]]$, that is, the set of all formal series $\sum_0^{+\infty} \alpha_n X^n$, $\alpha_n \in K$. Its residue field is K.

2.2.1 Local fields

If a field F is a complete discrete valuation field with respect to two valuations v, w and $v(F) = w(F)$, then $v = w$.

Definition 2.2.1 A complete discrete valuation field with finite residue field (more generally perfect residue field of positive characteristic) is called a *non-archimedean local field.*

Definition 2.2.2 A locally compact non-discrete topological field is called a *Local Field.*

Let F be a discrete valuation field with ring of integers $\mathcal{O}$, maximal ideal M, residue field $\overline{F}$. Choose and fix a prime element π of $\mathcal{O}$. Choose a set R of representatives in $\mathcal{O}$ of the residue field $\overline{F}$. So the set R is in one-to-one correspondence with the set $\overline{F}$ via the residue map $\mathcal{O} \to \overline{F} = \mathcal{O}/M$. Denote the inverse bijection *rep*: $\overline{F} \to \mathcal{O}$. We will assume that $0 \in R$. We have a filtration on F:

$$\{0\} ... \subset \pi^n \mathcal{O} \subset \pi^{n-1}\mathcal{O} \subset ... F.$$

For each n the quotient $\pi^n\mathcal{O}/\pi^{n+1}\mathcal{O}$ is isomorphic to $\mathcal{O}/\pi\mathcal{O} = \overline{F} \quad (\pi^n\alpha \mapsto \overline{\alpha})$. We will write $\alpha \equiv \beta \, mod \, \pi^n$, if $\alpha - \beta \in \pi^n\mathcal{O}$. If the residue field $\overline{F}$ is finite, we can choose a set of multiplicative representatives $R : R$ is the union of $\{0\}$ and all roots of unity of order prime to p in F (which is isomorphic to the multiplicative group $\overline{F}^{\times}$).

The group $1 + \pi\mathcal{O}$ is called the group of principal units U_1 and its elements are called principal units. Higher groups of units are $U_i = 1 + \pi^i\mathcal{O}$, for $i \geq 1$. We get the filtration

$$\{1\} ... \subset U_n \subset U_{n-1} \subset ... U_1 \subset U \subset F^{\times},$$

where $U = U_v$. The quotient $F^{\times}/U$ is isomorphic to $\mathbb{Z}$

$$\alpha \mapsto v(\alpha).$$

A choice of a prime element π gives

$$F^{\times} \simeq U \times \mathbb{Z}, \ \ \alpha = u\pi^n \mapsto (u, n).$$

The quotients

$$U/U_1 \cong \overline{F^{\times}}, \ \ \alpha \mapsto \overline{\alpha}, \quad U_n/U_{n+1} \cong \overline{F}, \ \ 1 + \alpha\pi^n \mapsto \overline{\alpha}.$$

Let $char(F) = 0$, $char(\overline{F}) = p > 0$. As $p = 0$ in the residue field $\overline{F}$, we conclude that $p \in \mathcal{M}$ and, therefore, for the surjective discrete valuation v of F we get $v(p) = e \geq 1$. The number $e = e(F) = v(p)$ is called the *absolute ramification index* of F. For example, $e(\mathbb{Q}_p) = 1$.

2.2.2 $\mathbb{Z}_p$-module U_1

Let F be a local field and let $a \in \mathbb{Z}_p$. Write a as the limit of $a_n \in \mathbb{Z}$ with respect to the p-adic topology. For a principal unit $u \in U_1$ define $u^a := lim\ u^{a_n}$. This does not depend on the choice of a_n, since $u^{p^n} \to 1$ when $n \to \infty$. Thus U_1 is a multiplicative $\mathbb{Z}_p$-module. It is a propostition that if $char(F) = p$ then U_1 is a free topological $\mathbb{Z}_p$-module with infinite topological basis $1+\theta\pi^i$ where i runs through all positive integers prime to p, θ runs through representatives of a basis of $\overline{F}$ over $\mathbb{F}_p$.

If $char(F) = 0$ then U_1 is a topological $\mathbb{Z}_p$-module which is the direct sum of the free $\mathbb{Z}_p$-module with topological basis $1+\theta\pi^i$ where $1 \le i < pe/(p-1)$, $(i, p) = 1$, θ runs through representatives of a basis of $\overline{F}$ over $\mathbb{F}_p$, plus, if the p-primary torsion of $F^\times$ is nontrivial of order p^m, the finite cyclic group of order p^m generated by a primitive p^mth root of unity in F. One can replace the last generator by an element $1+\theta_*\pi^{pe/(p-1)}$ which is not in the image of (2) (this element is not of finite order in general).

Let F be a valuation field and L an extension of F with a valuation $w : L \to \Gamma'$. Then w induces the valuation $w_0 = w|_F : F \to \Gamma$ on F. In this context L/F is said to be an extension of valuation fields. The group $w_0(F^\times)$ is a totally ordered subgroup of $w(L^\times)$ and the index of $w_0(F^\times)$ in $w(L^\times)$ is called the ramification index $e(L/F, w)$. The ring of integers $\mathcal{O}_{w_0}$ is a subring of the ring of integers $\mathcal{O}_w$ and the maximal ideal $\mathcal{M}_{w_0}$ coincides with $\mathcal{M}_w \cap \mathcal{O}_{w_0}$. Hence, the residue field $\overline{F}_{w_0}$ is a subfield of the residue field $\overline{L}_w$. The degree of the extension $\overline{L}_w/\overline{F}_{w_0}$ is called the inertia degree or residue degree $f(L/F, w)$.

2.2.3 Extension of discrete valuation

Assume that L/F is a finite extension and w_0 is a discrete valuation. Choose elements $\alpha_1, ..., \alpha_e$ in $L^\times$, $e \le e(L/F, w)$ such that $w(\alpha_1)+w(F^\times), ..., w(\alpha_e)+w(F^\times)$ are distinct in $w(L^\times)/w(F^\times)$. If $\sum_{i=1}^e c_i\alpha_i = 0$ for some $c_i \in F$. Then, since $w(c_i\alpha_i)$ are all distinct, we get $w(\sum_{i=1}^e c_i\alpha_i) = min_{1\le i\le e} w(c_i\alpha_i)$ and so $c_i = 0$ for $1 \le i \le e$. This shows that $\alpha_1, ..., \alpha_e$ are linearly independent over F and hence $e(L/F, w)$ is finite. Let π be a prime element with respect to w_0. Then we deduce that there are only a finite number of positive elements in $w(L^\times)$ which are $\le w(\pi)$. Consider the smallest positive element in $w(L^\times)$. It generates the group $w(L^\times)$, and we conclude that w is a discrete valuation.

Now let F and L be fields with discrete valuations v and w respectively and $F \subset L$. The valuation w is said to be an extension of the valuation v, if the discrete valuation $w_0 = w|_F$ is proportional to v. We shall write $w|v$ and use the notations $e(w|v)$, $f(w|v)$ instead of $e(L/F, w)$, $f(L/F, w)$.

2.3 Local fields

Now denote by v the p-adic valuation on $F = \mathbb{Q}_p$ and the X-adic valuation on $F = \mathbb{F}_p((X))$. These v extend uniquely to finite extensions L of F and the extended valuation w is given by the formula $w = (1/f)v \circ N_{L/F}$. The residue field of L is a finite extension of the residue field of F, hence L is a local field. In turn, let L be a local field with discrete valuation w. Assume L is of characteristic zero. Then it contains $\mathbb{Q}$ and the restriction of w on $\mathbb{Q}$ cannot be the trivial valuation, since the rational number p belongs to the maximal ideal with respect to w where p is the residue characteristic of L. Hence the restriction of w is proportional to the p-adic valuation. Then L contains the completion $\mathbb{Q}_p$. Since the ramification index and inertia degree are finite, L is a finite extension of $\mathbb{Q}_p$. Similarly one shows that every local field in positive characteristic is a finite extension of $\mathbb{F}_p((X))$. This gives the description of all local fields and their discrete valuations.

2.3.1 Tamely, totally and wildly ramified extensions of local fields

From now on let F be a local field. We shall write

$$\mathcal{O} \text{ or } \mathcal{O}_F$$

for the ring of integers $\mathcal{O}_v$

$$\mathcal{M} \text{ or } \mathcal{M}_F$$

for the maximal ideal $\mathcal{M}_v$

$$U \text{ or } \mathcal{U}_F$$

the group of units U_v

$$\pi \text{ or } \pi_F$$

for a prime element π_v with respect to v

$$\overline{F}$$

for the residue field $\overline{F}_v$.

Let L/F be a finite extension. If v_L is the unique discrete valuation on L which extends the valuation $v = v_F$ on F, then we write $e(L|F)$, $f(L|F)$ instead of $e(v_L|v_F)$, $f(v_L|v_F)$ respectively. An extension L/F is called *unramified* if it has the same degree as the degree of the residue field extension $\overline{L}/\overline{F}$. Then $e(L|F) = 1$. An extension L/F is called *totally ramified* if $f(L|F) = 1$. Then $e(L|F) = |L : F|$. An extension L/F is called *tamely ramified* if p does not divide $e(L|F)$ where $p > 0$ is the characteristic of $\overline{F}$. Every unramified extension is tamely ramified. An extension L/F is called *wildly totally ramified* if it is totally ramified and its degree is a power of p. An algebraic extension L of a complete discrete valuation field F is called *unramified* if L/F, $\overline{L}/\overline{F}$ are separable extensions and $e(w|v) = 1$, where v is the discrete valuation on F, and w is the unique extension of v on L. The compositum of all finite unramified extensions of F in a fixed algebraic closure of F is unramified. This extension is a Henselian discrete valuation field, since the Henselian property is preserved in compositum of fields. It is called the maximal unramified extension F^{ur} of F.

2.3.2 Ramification groups

Let L/F be a Galois extension of local fields. Let $\sigma \in Gal(L/F)$ and let v_L be the discrete valuation of L, $v_L(L^\times) = \mathbb{Z}$. Then $v_L \circ \sigma$ is a discrete valuation of L, and $v_L \circ \sigma(L^\times) = \mathbb{Z}$. Thus, $v_L = v_L \circ \sigma$ for every Galois automorphism. In particular, $\sigma\mathcal{O}_L = \mathcal{O}_L$, $\sigma\mathcal{M}_L^i = \mathcal{M}_L^i$.

Definition 2.3.1 Let L/F be a finite Galois extension of local fields, $G = Gal(L/F)$. For an integer i define $G_i = \{\sigma \in G : \sigma\alpha - \alpha \in \mathcal{M}_L^{i+1} \text{ for all } \alpha \in \mathcal{O}_L\}$.

Lemma 2.3.2 *G_i are normal subgroups of G.*

2.4 The norm map for cyclic extensions

Let L/F be a finite cyclic extension of local fields of degree n. We aim to study the action of the norm map $N_{L/F} : L^\times \to F^\times$

The following property will be useful.

Lemma 2.4.1 *Let $\gamma \in \mathcal{M}_L$. Then $N_{L/F}(1+\gamma) = 1 + N_{L/F}(\gamma) + Tr_{L/F}(\gamma) + Tr_{L/F}(\delta)$ with some $\delta \in \mathcal{O}_L$ such that $v_L(\delta) \geq 2v_L(\gamma)$.*

Proposition 2.4.2 *Let L/F be an unramified extension of degree n. Choose a prime element π_F in F, then it is a prime element in L. Let $\lambda_{i,L}, \lambda_{i,F}$ be associated to π_F as prime element of L and of F. Then the following diagrammes are commutative:*

$$
\begin{array}{ccc}
\mathrm{L}^\times & \xrightarrow{v_L} & \mathbb{Z} \\
{\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle \times n} \\
F^\times & \xrightarrow{v_F} & \mathbb{Z}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{U}_L & \xrightarrow{\lambda_{0,L}} & \overline{L}^\times \\
{\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle N_{\overline{L}/\overline{F}}} \\
U_F & \xrightarrow{\lambda_{0,F}} & \overline{F}^\times
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{U}_{i,L} & \xrightarrow{\lambda_{i,L}} & \overline{L} \\
{\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle Tr_{\overline{L}/\overline{F}}} \\
U_{i,F} & \xrightarrow{\lambda_{i,F}} & \overline{F}
\end{array}
$$

Corollary 2.4.3 *If L/F is unramified of degree n, then $N_{L/F}U_L = U_F$ and the quotient $F^\times/N_{L/F}L^\times$ is a cyclic group of order n generated by the coset of π_F.*

The norm map for tamely totally ramified cyclic extensions

Proposition 2.4.4 *Let L/F be a totally tamely ramified cyclic extension of degree n. Then for some prime element π_L in L, the element $\pi_F = \pi_L^n$ is prime in F. Let $\lambda_{i,L}$, $\lambda_{i,F}$ be associated to π_L and π_F. Then the following diagrammes are commutative:*

$$
\begin{array}{ccc}
\mathrm{L}^\times & \xrightarrow{v_L} & \mathbb{Z} \\
{\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle id} \\
F^\times & \xrightarrow{v_F} & \mathbb{Z}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{U}_L & \xrightarrow{\lambda_{0,L}} & \overline{L}^\times \\
{\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle \hat{n}} \\
U_F & \xrightarrow{\lambda_{0,F}} & \overline{F}^\times
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{U}_{ni,L} & \xrightarrow{\lambda_{ni,L}} & \overline{L} = \overline{F} \\
{\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle \times\overline{n}} \\
U_{i,F} & \xrightarrow{\lambda_{i,F}} & \overline{F}
\end{array}
$$

where $\hat{n}$ is raising to the nth power, $\times\overline{n}$ is the multiplication by $\overline{n} \in \overline{F}$, $i \geq 1$. Moreover, $N_{L/F}U_{i,L} = N_{L/F}U_{i+1,L}$ if n does not divide i.

Corollary 2.4.5 *Let L/F be cyclic tamely totally ramified of order n. Then $N_{L/F}U_{1,L} = U_{1,F}$ and the quotient group $F^\times/N_{L/F}L^\times$ is cyclic of order n generated by the coset of a primitive root ζ of unity of order $|\overline{F}|-1$ in F. If F is algebraically closed then $N_{L/F}L^\times = F^\times$.*

Proposition 2.4.6 *Let L/F be a totally ramified Galois extension of degree p. Let π_L be a prime element in L. Then $\pi_F = N_{L/F}\pi_L$ is a prime element in F. Let $\lambda_{i,L}, \lambda_{i,F}$ be associated to π_L and π_F. Then the following diagrammes are commutative:*

$$\begin{tikzcd}
L^\times \arrow[r, "v_L"] \arrow[d, "N_{L/F}"'] & \mathbb{Z} \arrow[d, "id"] \\
F^\times \arrow[r, "v_F"] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
U_L \arrow[r, "\lambda_{0,L}"] \arrow[d, "N_{L/F}"'] & \overline{L}^\times \arrow[d, "\hat{p}"] \\
U_F \arrow[r, "\lambda_{0,F}"] & \overline{F}^\times
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{i,L} \arrow[r, "{i,L}"] \arrow[d, "N_{L/F}"'] & \overline{L} = \overline{F} \arrow[d, "\hat{p}"] \\
U_{i,F} \arrow[r, "\lambda_{i,F}"] & \overline{F}
\end{tikzcd}$$

$$\begin{tikzcd}
U_{s,L} \arrow[r, "\lambda_{s,L}"] \arrow[d, "N_{L/F}"'] & \overline{L} = \overline{F} \arrow[d, "\overline{\theta}\mapsto\overline{\theta}^p-\overline{\eta}^{p-1}\overline{\theta}"] \\
U_{s,F} \arrow[r, "\lambda_{s,F}"] & \overline{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{s+pi,L} \arrow[r, "\lambda_{s+pi,L}"] \arrow[d, "N_{L/F}"'] & \overline{L} = \overline{F} \arrow[d, "\times(-\overline{\eta}^{p-1})"] \\
U_{s+i,F} \arrow[r, "\lambda_{s+i,F}"] & \overline{F}
\end{tikzcd}$$

if $i > 0$ Moreover, $N_{L/F}(U_{s+i,L}) = N_{L/F}(U_{s+i+1,L})$ for $i > 0$ prime to p. Hence $U_{s+1,F} = N_{L/F}U_{s+1,L}$. The quotient group $F^\times/N_{L/F}L^\times$ is cyclic of order p and is generated by the coset of $1+\theta_0\pi_F^s$ where the residue of θ_0 does not belong to the image of the homomorphism $\overline{\theta} \mapsto \overline{\theta}^p - \overline{\eta}^{p-1}\overline{\theta}$. If $\overline{F}$ is algebraically closed then $N_{L/F}L^\times = F^\times$.

Let L/F be a finite Galois totally ramified extension. Then $L^{ur} = LF^{ur}$, $\mathcal{L} = L\mathcal{F}$ and the extensions $L^{ur} = F^{ur}$, $\mathcal{L}/\mathcal{F}$ are totally ramified Galois with the group isomorphic to that of L/F.

Proposition 2.4.7 *Let $\gamma \in \mathcal{L}^\times$ be such that $\gamma^{\varphi-1} \in U(\mathcal{L}/\mathcal{F})$. Then $N_{\mathcal{L}/\mathcal{F}\gamma}$ belongs to the group $N_{L/F}L^\times$.*

Chapter 3

Basic C^*-algebra and K-theory

Some definitions and exposition will be covered in this chapter, more details and proofs can be found in [1], [6], [18], [26], [37], [38], [39], [40] and [48].

3.1 C^*-Algebra

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. An *involution* on $\mathcal{A}$ is a map $*: \mathcal{A} \to \mathcal{A}$; $a \mapsto a^*$ such that:

1. $(a^*)^* = a$
2. $(\lambda_1 a + \lambda_2 b)^* = \overline{\lambda}_1 a^* + \overline{\lambda}_2 b^*$
3. $(ab)^* = b^* a^*$

for $a, b \in \mathcal{A}$ and λ_1, $\lambda_2 \in \mathbb{C}$. The algebra $\mathcal{A}$ with this involution called $*$-algebra and a^* is the adjoint (image) of a. A $*$-algebra $\mathcal{A}$ is called *Banach* $*$-algebra if it is associated with a norm $\|.\|$ such that $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$ and if the algebra $\mathcal{A}$ is *complete*.

Definition 3.1.1 Any Banach $*$-algebra which has the property

$$\|a\|^2 = \|a^* a\|; \ a \in \mathcal{A}$$

is called a C^**-algebra*, this property is called the C^* condition, where

$$\|a\|^2 = sup\{|\lambda|; \ aa^* - \lambda \ \ is \ \ not \ \ invertible\}.$$

An element $a \in \mathcal{A}$ is called *self-adjoint* if $a^* = a$ and *normal* if $aa^* = a^*a$. If $\mathcal{A}$ has an identity e then $u \in \mathcal{A}$ is called *unitary* if $uu^* = u^*u = e$.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\psi : \mathcal{A} \to \mathcal{B}$ be a morphism, then $\psi(\mathcal{A})$ is a sub-C^*-algebra of $\mathcal{B}$.

3.1.1 C^*-module

Let $\mathcal{A}$ be a C^*-algebra.

Definition 3.1.2 A C^*-*module* (or pre-Hilbert $\mathcal{A}$-module) over $\mathcal{A}$ is a right $\mathcal{A}$-module $\mathcal{X}$ equipped with a sesquilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{A}$$

with the following properties:

1. $\langle a, a \rangle^* = \langle a, a \rangle$ for all $a \in \mathcal{X}$
2. $\langle a, b \rangle^* = \langle b, a \rangle$ for all $a, b \in \mathcal{X}$
3. $\langle a, b\lambda \rangle = \langle a, b \rangle \lambda$ for all $a, b \in \mathcal{X}$ and $\lambda \in \mathcal{A}$.

The map $\langle \cdot, \cdot \rangle$ is called an $\mathcal{A}$-valued inner product. The norm of a is defined by

$$\|a\| = \|\langle a, a \rangle\|^{1/2}.$$

3.1.2 Strong Morita equivalence

An endomorphism $\wp$ of $\mathcal{X}$ is a continuous endomorphism of the right $\mathcal{A}$-module $\mathcal{X}$ which admits an adjoint $\wp^*$. This adjoint is uniquely determined for each endomorphism, and equipped with the adjoint operation as an involution. The algebra $End_{\mathcal{A}}(\mathcal{X})$ of endomorphisms of C^*-module $\mathcal{X}$ is a C^*-algebra. If $\|\wp\|$ is the norm of $\wp$, then we have

$$\langle \wp x, \wp x \rangle \leq \|\wp\| \langle x, x \rangle, \;\; x \in \mathcal{X}.$$

Now let $\mathcal{X}_1$ and $\mathcal{X}_2$ be C^*-modules over the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively and let $\hbar$ be a $*$-homomorphism from $\mathcal{A}$ to $End_{\mathcal{B}}(\mathcal{X}_2)$. Then $\langle x_1 \otimes y_1, x_2\rangle = \langle \hbar(\langle x_2, x_1\rangle)y_1, y_2\rangle \in \mathcal{B}$, where $x_i \in \mathcal{X}_1$ and $y_i \in \mathcal{X}_2$. This gives a structure of C^*-module $\mathcal{N}$ over $\mathcal{B}$ on the algebraic tensor product $\mathcal{X} = \mathcal{X}_1 \otimes_{\mathcal{A}} \mathcal{X}_2$. If $\wp \in End_{\mathcal{A}}(\mathcal{X}_1)$, we can define an endomorphism $\wp \otimes 1 \in End_{\mathcal{B}}(\mathcal{X}_1 \otimes_{\mathcal{A}} \mathcal{X}_2)$ by $(\wp \otimes 1)(x \otimes n) = \wp x \otimes n$, $x \in \mathcal{X}_1, n \in \mathcal{X}_2$.

Definition 3.1.3 A C^*-module $\mathcal{X}$ over $\mathcal{B}$ together with $*$-homomorphism $\hbar$ from $\mathcal{A}$ to $End_{\mathcal{B}}(\mathcal{X})$ is called an $\mathcal{A}-\mathcal{B}$ C^*-bimodule.

Let $\mathcal{A}$ be a C^*-algebra, and $1_{\mathcal{A}}$ is the $\mathcal{A}-\mathcal{A}$ C^*-bimodule given by $\mathcal{X} = \mathcal{A}$ and then the $\mathcal{A}$-valued inner product given by $\langle a, \acute{a}\rangle = a^*\acute{a}$.

Definition 3.1.4 Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are *strong Morita equivalent* and denoted by $\mathcal{A} \cong \mathcal{B}$ if there exist $\mathcal{A}-\mathcal{B}C^*$-bimodules $\mathcal{X}_1, \mathcal{X}_2$ such that $\mathcal{X}_1 \otimes_{\mathcal{B}} \mathcal{X}_2 = 1_{\mathcal{A}}$, $\mathcal{X}_2 \otimes_{\mathcal{A}} \mathcal{X}_1 = 1_{\mathcal{B}}$.

Any two strong Morita equivalent C^*-algebras have the same space of classes of irreducible representations. If a C^*-algebra $\mathcal{A}$ is strong Morita equivalent to a commutative C^*-algebra, then the commutative C^*-algebra is unique and it is the C^*-algebra of continuous functions vanishing at infinity on the space of irreducible representations of $\mathcal{A}$.

Theorem 3.1.5 *[17] If two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent, then they have the same K-theory groups, i.e. $K_i(\mathcal{A}) \simeq K_i(\mathcal{B})$.*

Let G be a group equipped with left invariant Haar measure μ, and let $\mathrm{L}^1(G), \mathrm{L}^2(G)$ be the Lesbegue integrable and square integrable functions on this group. The left regular representation λ of $\mathrm{L}^1(G)$ in $\mathrm{L}^2(G)$ is given by $(\lambda(f))(h) = f * h$, $f \in \mathrm{L}^1(G)$, $h \in \mathrm{L}^2(G)$, where $*$ denotes convolution. Here we can define the reduced C^*-algebra of a group, $C_r^*(G)$ as $C_r^*(G) = \overline{\lambda(\mathrm{L}^1 G))} \subset \mathfrak{L}(\mathrm{L}^2(G))$ (where $\overline{A}$ is the topological closure of A).

3.2 Introduction to K-theory

K-theory is a branch of mathematics which brings together ideas from algebraic geometry, linear algebra and number theory. There are two main types of K-theory: Algebraic K-theory and topological K-theory. Topological K-theory is the K-theory that came first and it has to do with vector bundles over a topological space. Elements of K-theory are stable equivalence classes of vector bundles over a topological space. In this work, we will be dealing only with topological K-theory, so from now and on we will use the expression K-theory instead of topological K-theory. Now we shall bring some basic definitions and concept needed to have an idea about K-theory.

3.2.1 Projection

Definition 3.2.1 A linear map $j : V \to V$ is called a *projection* if it acts like the *identity* on its image $(j)^2 = j$, where V be a vector space over a field F.

Proposition 3.2.2 *Let $j : V \to V$ be a projection, then its* image *and the* kernel *are complementary subspace, namely*

$$V = ker j \oplus Img j.$$

Let suppose that $F = \mathbb{R}$ or $\mathbb{C}$ the ground field and let V supposed to be equipped with positive-definite inner product $\langle .,. \rangle$. We call an endomorphism $P : V \to V$ an *orthogonal projection* if it is self-dual $P^* = P$ in addition to satisfying the projection condition $P^2 = P$.

Definition 3.2.3 A category $\mathcal{C}$ is a collection $Ob(\mathcal{C})$ of objects of $\mathcal{C}$ and a collection $Hom(A,B)$ of morphisms from A to B, for each order pair (A,B) of objects of $\mathcal{C}$ with a function $\circ : Hom(A,B) \times Hom(B,C) \to Hom(A,C)$ called composition.

Definition 3.2.4 A covariant *functor* $\Upsilon : \mathcal{C} \to \mathcal{D}$ consists of an assignment for each object X of $\mathcal{C}$ an object $\Upsilon(X)$ of $\mathcal{D}$ (i.e. a function $\Upsilon : Ob(\mathcal{C}) \to Ob(\mathcal{D})$ together with an assignment for every morphism $f \in Hom_{\mathcal{C}}(A,B)$, to a morphism $\Upsilon(f) \in Hom_{\mathcal{D}}(\Upsilon(A),\Upsilon(B))$, such that:

1. $\Upsilon(i_A) = i_{\Upsilon(A)}$ where i_A denotes the identity morphism on the object of X (in the respective category).

2. $\Upsilon(g \circ f) = \Upsilon(g) \circ \Upsilon(f)$ whenever the composition $g \circ f$ is defined.

Definition 3.2.5 Any covariant functor $\Upsilon : \mathcal{C}^{op} \to \mathcal{D}$ from the opposite category of $\mathcal{C}$ to the category $\mathcal{D}$ is called contravariant functor. In other words, the assignment reverses the direction of maps. If $f \in Hom_{\mathcal{C}}(A,B)$, then $\Upsilon(f) \in Hom_{\mathcal{D}}\big(\Upsilon(A),\Upsilon(B)\big)$ and

$$\Upsilon(f \circ g) = \Upsilon(g) \circ \Upsilon(f)$$

whenever the composition is defined (the domain of f is the same as the codomain of g).

If $\mathcal{C}$ be a category and X be an object, we always have the functor $\Upsilon : \mathcal{C} \to Cat(Sets)$ "the category of sets" defined on objects by $\Upsilon(A) = Hom(X,A)$. If $f : A \to B$ is a morphism of $\mathcal{C}$, then we define $\Upsilon(f) : Hom(X,A) \to Hom(X,B)$, $g \mapsto f \circ g$. This is a covariant functor, denoted by $Hom(X,...)$, in a similar way we can define a contravariant functor $Hom(...,X)$.

Now, let $\mathcal{Q}$ be an abelian semigroup. The *Grothendieck* group of $\mathcal{Q}$ is

$$K(\mathcal{Q}) = \mathcal{Q} \times \mathcal{Q}/\sim,$$

where $\sim$ is the equivalence relation $(h,g) \sim (n,m)$ if there exists $r \in \mathcal{Q}$ such that $h+m+r = g+n+r$. This indeed an abelian group with zero element (h,h) and an inverse $-(h,g) = (g,h)$. The Grothendieck group construction is a functor from the category of abelian semi groups to the category of abelian groups. A morphism $f : \mathcal{Q} \to \mathcal{T}$ induces a morphism $K(f) : K(\mathcal{Q}) \to K(\mathcal{T})$.

If Y is any given space, then the set $V(Y)$ of vectors on Y has the structure of an abelian semigroup, where the additive structure is define by direct sum. If $\mathcal{Q}$ is an abelian semigroup, we can associate to $\mathcal{Q}$ an abelian group $K(\mathcal{Q})$ with the following property:

There is a semigroup homomorphism $\alpha : \mathcal{Q} \to K(\mathcal{Q})$ such that if G is any group and $\gamma : \mathcal{Q} \to G$ any semigroup homomorphism, there is a unique homomorphism $\varsigma : K(\mathcal{Q}) \to G$ such that $\gamma = \varsigma\alpha$. If such $K(\mathcal{Q})$ exists, it must be unique. This group usually referred to the *Grothendieck group* of $\mathcal{Q}$. Now let $\mathcal{I}(\mathcal{Q})$ be the abelian group generated by the elements of the semigroup $\mathcal{Q}$, and let $\mathcal{J}(\mathcal{Q})$ be the subgroup of $\mathcal{I}(\mathcal{Q})$ generated by an element of the form $h + \acute{h} - (h \oplus \acute{h})$, where $\oplus$ is the addition in $\mathcal{Q}$ and $h, \acute{h} \in \mathcal{Q}$. Then $K(\mathcal{Q}) = \mathcal{I}(\mathcal{Q})/\mathcal{J}(\mathcal{Q})$ has the universal property described above with $\alpha : \mathcal{Q} \to K(\mathcal{Q})$ being the obvious map.

3.2.2 The functor K_0

Let ψ be a group-valued function from the projections of the C^*-algebra $\mathcal{A}$ to an abelian group G, such that $\psi(j_1) = \psi(j_2)$ whenever $j_1 \sim j_2$ and $\psi(j_1 + j_2) = \psi(j_1) + \psi(j_2)$ whenever $j_1 \perp j_2$ with the property that any function from the projections of $\mathcal{A}$ to an abelian group factors through ψ. The group $K_0(\mathcal{A})$ is the appropriate range group of the function on $\mathcal{A}$. If $\mathcal{A}$ is unital then $K_0(\mathcal{A})$ elements are formal differences of equivalence classes of projections in matrix algebras over $\mathcal{A}$. By the functorality of the definition of K_0 any homomorphism $\xi : \mathcal{A} \to \mathcal{B}$ induces a homomorphism $\xi^* : K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

3.2.3 The functor K_1

The $K_1(\mathcal{A})$ will be taken to be the range group of the index function, and it is a covariant functor from C^*-algebras to the abelian groups. There are two initial connections between K_0 and K_1. For any C^*-algebra $\mathcal{A}$ we have $\mathcal{HA} = C_0(\mathcal{A}, \mathbb{R})$ is the algebra of continuous real-valued functions on $\mathcal{A}$ which vanish at infinity. By *Bott Periodicity* we have

$$K_1(\mathcal{A}) \cong K_0(\mathcal{HA}) \textit{ and } K_0(\mathcal{A}) \cong K_1(\mathcal{HA}).$$

Now let J be a closed two-sided ideal in $\mathcal{A}$, then we have the connecting homomorphism from $K_1(\mathcal{A}/J)$ to $K_0(J)$ and from $K_0(\mathcal{A}/J)$ to $K_1(J)$, this called index or exponential maps (both denoted δ), which make the following sequences exact

$$\begin{tikzcd}
K_0(J) \arrow[r, "i^*"] & K_0(\mathcal{A}) \arrow[r, "\pi^*"] & K_0(\mathcal{A}/J) \arrow[d, "\delta"] \\
K_1(\mathcal{A}/J) \arrow[u, "\delta"] & K_1(\mathcal{A}) \arrow[l, "\pi^*"'] & K_1(J) \arrow[l, "i^*"']
\end{tikzcd}$$

This is the fundamental exact sequence of K-theory.

Chapter 4

Enlarged building and Chamber Homology

A background summary of buildings has been given in this chapter. First of all we recall the affine building (*tree*) of $SL(2) = SL(2,F)$ over the p-adic field F which appears as a very special case of a *Bruhat-Tits building* [10] and [21]. It begins with the definition and the main properties of the tree X attached to a vector space V over a non-archimedean local field F (As it appears in [45] section 2). Also, we present in this chapter a study of chamber homology. We will start with giving quick definitions of homology in general, equivariant homology groups and cyclic homology. Homology is a way of attaching abelian groups (or more elaborate algebraic objects) to a topological space so as to obtain algebraic invariants. In some sense it detects the presence of "*holes*" of various dimensions in the space. The methods developed to handle this led to what is now called *homological algebra* and homological invariants can be calculated for many purely algebraic structures. This chapter finishes with some basic information about the *Weil* group.

A graph $\digamma$ consists of a set X of *vertices*, a set Y of *edges* and two maps:

$$Y \to X \times X; \;\; y \longmapsto \big(\psi(y), \tau(y)\big), \qquad Y \to Y; \quad y \mapsto \bar{y}$$

satisfying these two conditions, for each $y \in Y$:

1. $\bar{\bar{y}} = y, \quad y \neq \bar{y},$

2. $\psi(y) = \tau(\bar{y})$.

Each $v \in X$ is called a *vertex* of Γ, each $y \in Y$ an *edge* and $\bar{y}$ is called the *inverse edge* of y. The vertex $\psi(y) = \tau(\bar{y})$ is called the *origin* of y and the vertex $\tau(y) = \psi(\bar{y})$ is called *terminus*. These two vertices are called *extremities* of the edge y. Any two vertices which are extremities of some edge are called *adjacent*. A subset Y_+ of $Y = edge\,\Gamma$ such that

$$Y = Y_+ \bigsqcup \overline{Y}_+$$

is called an *orientation* of Γ. An *oriented graph* is defined up to isomorphism, by given a set X and Y_+ and a map $Y_+ \longrightarrow X \times X$. The corresponding set of edges is $Y = Y_+ \sqcup \overline{Y}_+$ where $\overline{Y}_+$ is a copy of Y_+.

Definition 4.0.6 A tree is a connected non-empty graph without circuits.

4.1 The tree of $SL(2)$ over a local field

Let F be a local field, a discrete valuation v is a homomorphism from $F^\times$ onto $\mathbb{Z}$ such that

$$v(x+y) \geq \inf\big(v(x), v(y)\big),\ \ x, y \in K \qquad and\ \ v(0) = +\infty.$$

Let $\mathcal{O}$ be the valuation ring of F;

$$\mathcal{O} = \{x \in F : \quad v(x) \geq 0\}.$$

Choose a *uniformizer* ϖ ($\varpi \in F$ such that $v(\varpi) = 1$ with residue field $\mathrm{k} = \mathcal{O}/\varpi\mathcal{O}$).

If $x \in F$

$$x\mathcal{O} = \mathcal{O}x = \varpi^{v(x)}\mathcal{O} = \{y \in F : v(y) \geq v(x)\}.$$

An *$\mathcal{O}$-lattice* L of the vector space V is a finitely generated $\mathcal{O}$-submodule of V which generates the F-vector space V [12], this module is free of rank 2. Let $x \in F^\times$ and L is a lattice of V then Lx is also a lattice of V, this shows that $F^\times$ acts on the set of lattices. An *orbit* of a lattice is the class of this lattice under this action. Two lattices are called *equivalent* if they are belong to the same class Λ. We shall denote $\mathfrak{l}$ to be the set of lattices. Let L and $L^{'}$ be two lattices of V and let $\{e_1,e_2\}$ and $\{e_1^{'},e_2^{'}\}$ be $\mathcal{O}$-bases of L and $L^{'}$ respectively. There are two integers a and b such that $\{e_1\varpi^a, e_2\varpi^b\}$ is an $\mathcal{O}$-basis of $L^{'}$ by the invariant factor theorem. These two integers do not depend on the choice of the $\mathcal{O}$-basis of L and $L^{'}$. We have $L \subset L^{'}$ if and only if $a,b \geq 0$, in this case we have $L/L^{'} \cong (\mathcal{O}/\varpi^a\mathcal{O}) \oplus (\mathcal{O}/\varpi^b\mathcal{O})$. Let $x,x^{'} \in F^{\times}$, if we replace the latices L and $L^{'}$ by the lattices Lx and $L^{'}x^{'}$ we get the same effect if we replace a and b by $a+c$ and $b+c$ and $c = v(x^{'}/x)$. The integer $\mid a-b \mid$ therefor depends only on the classes Λ and $\Lambda^{'}$ of L and $L^{'}$ respectively. We denote the distance between Λ and $\Lambda^{'}$ by $d(\Lambda,\Lambda^{'}) = c$. The length $\ell(L/L^{'}) = d(\Lambda,\Lambda^{'})$ if $L^{'} \subset L$ represent $\Lambda^{'}$ and Λ respectively. Serre in [45] proved that two vertices belong to the same class only if the distance between them was even. The following theorem shows that $\mathfrak{l}$ is a connected graph without circuits, where each vertex is a lattice class and any two vertices joint by an edge if the length between them is 1.

Theorem 4.1.1 *The Graph $\mathfrak{l}$ is a tree.*

PROOF: See [45]. ❒

Remark 4.1.2 The tree $\mathfrak{l}$ is special case of a *Bruhat-Tits building*.

If $\widehat{F}$ is a completion of F with respect to v and

$$\widehat{\mathcal{O}} = \varprojlim \mathcal{O}/\varpi^n\mathcal{O}$$

its valuation ring. Put $\widehat{V} = V \otimes_F \widehat{F}$, if we assigned for each lattice L of V the lattice $\widehat{L} = L \otimes_{\mathcal{O}} \widehat{\mathcal{O}}$ of $\widehat{V}$, then we will get a *bijection* between the set of lattices of V onto the set of lattices of $\widehat{V}$. This induces an *isomorpism* of the tree of V onto the tree of $\widehat{V}$.

Let $GL(V)$ be the group of nonsingular linear maps of V, this group is isomorphic to $GL(2,F)$. Also, let $SL(V)$ be the subgroup of $GL(V)$ consisting of all unipotent elements. Let $End(V)$ be the F-algebra of all F-linear endomorphisms of V. Let $g \in GL(V)$ we have the homomorphism

$$v(det) : GL(V) \longrightarrow \mathbb{Z},$$

where v is a valuation on $F^{\times}$.

Let $GL(V)^0$ be the kernel of the homomorphism $v(det)$ and $GL(V)^+$ be the kernel of the composite

$$GL(V) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Then we have the following:

$$SL(V) \subset GL(V)^0 \subset GL(V)^+ \subset GL(V).$$

Now, let G be a subgroup of $GL(V)$ and let L be a lattice represents the class Λ. Put $G_L = \{g \in G : gL = L\}$ and $G_\Lambda = \{g \in G : g\Lambda = \Lambda\}$ and we call them the stabilizer of G by L and Λ. Also, we can talk about the stabilizer of G by the edge $\Lambda\Lambda^{'}$ in the same way.

4.1.1 Iwahori subgroups

Let L and $L^{'}$ be lattices represented by Λ and $\Lambda^{'}$ respectively with $L^{'} \subset L$ and $\ell(L/L^{'}) = 1$, let $\Lambda\Lambda^{'}$ be an edge of $\mathfrak{l}$. If G be a subgroup of $GL(V)^0$, then we have the stabilizer $G_{\Lambda\Lambda'}$ is $G_L \cap G_{L'}$ [45], and this is a subgroup of G_L consisting of all elements g whose images are in

$$GL(L/L\varpi) \cong GL(2, \mathrm{k})$$

which leaves the line $L/L^{'}$ stable. This subgroup is called the *Iwahori subgroup* of G with respect to the edge $\Lambda\Lambda^{'}$. Let

$$L = e_1\mathcal{O} \oplus e_2\mathcal{O} \;\; and \;\; L' = e_1\mathcal{O} \oplus e_2\varpi\mathcal{O}$$

then the group

$$G_{\Lambda\Lambda'} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \quad \gamma \equiv 0 \, mod \, \varpi \right\},$$

it is the inverse image of the triangular subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ of $GL(2,\mathrm{k})$.

4.1.2 Buildings

The basic definitions and theorems of simplicial (polysimplicial) complexes shall be introduced in this section to be able to give a construction of the building for $GL(n,F)$ where F be a non-archimedean field.

Definition 4.1.3 [42] An *abstract simplicial complex* is a finite set V together with a family of subsets of V say $\mathcal{S}$ satisfying the following conditions:

1. If $\sigma \in \mathcal{S}$ and $\delta \subset \sigma$ then $\delta \in \mathcal{S}$,
2. If $v \in V$ implies $\{v\} \in S$. This condition can be expressed as $\bigcup_{\sigma\in\mathcal{S}} \sigma = V$.

We denote this simplicial complex by $(V,\mathcal{S})$.

A *simplex* of $(V,\mathcal{S})$ is an element σ in $\mathcal{S}$. The rank of any simplex is define to be the cardinality of this set. The dimension of simplex σ is $dim\sigma = card\sigma - 1$. Now we can define a set $\mathcal{S}^n = \{\sigma \in S : dim\sigma = n\}$. Also, we have the disjoint union

$$\mathcal{S} = \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \ldots.$$

The 0-dimensional simplices are the vertices of an abstract simplicial complex, so the vertices are all elements of the set $\mathcal{S}^0 = V$. The 1-dimensional simplices are the edges of an abstract simplicail complex, so the edges are all elements of the set $\mathcal{S}^1$. Any automorphism of V that preserve $\mathcal{S}$ is called an automorphism of the abstract simplicial complex $(V,\mathcal{S})$.

Definition 4.1.4 A topological space Δ together with an abstract simplicial complex $(V,\mathcal{S})$ equipped with a homeomorphism $\bar{h} :| (V,\mathcal{S}) | \to X$ is called a *simplicial complex*. A closed simplex of Δ is a subset of Δ of the form $\bar{h}(\bar{\sigma})$, for

$$\sigma \in \mathcal{S} \ \ and \ \ \bar{\sigma} = \{u \in | (V,\mathcal{S}) | : sup(\sigma) \subset u\}$$

The *rank* of a simplicial complex is defined to be the supremum of the ranks of its simplies, which always should be finite. For any simplex $\sigma \in \Delta$, we associate a simplicial complex $lk_\Delta(\sigma)$, it is called the link of σ [47] and it is consisting of all simplices of Δ containing σ. Since Δ can be considered as a poset, then we can have

$$lk_\Delta(\sigma) = \{\delta \in \Delta : \sigma \leq \delta\}.$$

A *chamber* of rank r is a simplex of a simplicial complex Δ of rank r. A *panel* is a simplex of a simplicial complex Δ of rank $r-1$. Two chambers are adjacent if their intersection is a panel.

Definition 4.1.5 An abstract polysimplicial complex is an r-tuple $((V_1,S_1),(V_2,S_2),...,(V_r,S_r))$ such that each (V_i,S_i) is an abstract polysimplicial complex.

If $\mathfrak{P} = ((V_1,S_1),(V_2,S_2),...,(V_r,S_r))$ is an abstract polysimilicial complex then the set of polysimplices is simply the cartesian product of the set of simplices for each (V_i,S_i),

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times ... \times \mathcal{S}_r.$$

Definition 4.1.6 A topological space Δ together with an abstract polysimplicial complex $\mathfrak{P}$ and a homeomorphism

$$\bar{h} : \mid \mathfrak{P} \mid \to X$$

is called a polysimplicial complex.

A *chamber complex* is a finite-dimensional simplex complex Δ such that all maximal simplices have the same dimension and the maximal simplices are called *chambers*. A *thick*, *weak* and *thin* chamber complex is a chamber complex which its panels are contained in respectively at least three chambers, at least two chambers and exactly two chambers. An endomorphism ϕ of Δ such that $\phi^2 = \phi$ and the preimage of any chamber $\mathcal{C} \in \Delta$ consists of 0 or 2 chambers is called a *folding* of a thin chamber complex. A *Coxeter complex* is the thin chamber complex Δ such that for any two adjacent chambers $\mathcal{C},,\mathcal{D} \in \Delta$ there is a folding ϕ of Δ satisfying $\phi(\mathcal{C}) = \mathcal{D}$. A *root* of a Coxeter complex Δ is a subcomplex of Δ

which is an image of a folding of Δ.

Definition 4.1.7 A *Building* is a simplicial complex (or polysimplicial complex) Δ which can be written as the union of subcomplexes Σ (called apartments) and satisfying the following axioms:

1. Each apartment Σ is a Coxeter complex.
2. For any two simplices (polysimplices) A and B in Δ there exists an apartment Σ such that $A, B \in \Sigma$.
3. If $\Sigma, \Sigma^{'}$ are two apartments containing the simplices (polysimplices) A and B then there exists an isomorphism $\Sigma \to \Sigma^{'}$ which fixes A and B pointwise.

We denote the building of G by βG if G is a (reductive p-adic) group. We say that the group G acts *strongly transitively* on its building βG if G acts transitively on the set of pairs $(\Sigma, \mathcal{C})$ where $\mathcal{C}$ is a maximal simplex in Σ. The *Iwahori subgroup* I is defined to be the intersection of all maximal compact subgroup of G.

4.2 Affine Building for $SL(n)$

Consider the group $SL(n)$ over a non-archimedean field F. A construction of the affine building $\beta SL(n)$ of $SL(n)$ will be given in this section. Now, let the compact open subgroup

$$SL(n, \mathcal{O}_F) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ . & . & \dots & . \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix} \in SL(n) : \alpha_{ij} \in \mathcal{O}_F, 1 \leq i, j \leq n \right\}.$$

The group $SL(n, \mathcal{O}_F)$ is a maximal compact open subgroups of $SL(n)$. The group $SL(n)$ has more than one conjugacy class of maximal compact subgroups. They are given by the following description. Let

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ \varpi & 0 & 0 & ... & 0 \end{pmatrix},$$

where ϖ is the uniformizer, Π is an element of $GL(n)$ since $det(\Pi) = \varpi_F$. However we want to use this element to establish a map

$$\beth\Pi : SL(n) \to SL(n)$$

$$x \longmapsto \beth\Pi(x) = \Pi x \Pi^{-1}.$$

Since $det(x) = 1$, then

$$det(\Pi x \Pi^{-1}) = det(\Pi) det(x) det(\Pi)^{-1} = det(x) = 1,$$

this shows that $\beth\Pi(x) \in SL(n)$. Let $J_0 = SL(n, \mathcal{O}_F)$ and for $0 \leq i \leq n-1$ we can define a compact open subgroup of $SL(n)$ by applying $\beth\Pi$ on J_0 an i times, so

$$J_i = (\beth\Pi)^i (J_0) = \Pi^i J_0 \Pi^{-i}$$

are up to conjugacy the maximal compact open subgroups of $SL(n)$. Therefore any maximal compact open subgroup J in $SL(n)$ can be expressed as $J = x^{-1} J_i x$ for some $0 \leq i \leq n-1$ and some element $x \in SL(n)$.

We define an abstract simplicial complex (V, S) as follows. Let's form the vertex set V to be the maximal compact open subgroups of $SL(n)$, we note that V is a countable set since the map $(x, J_i) \to x J_i x^{-1}$ gives bijective correspondence

$$(SL(n)/J_0) \cup (SL(n)/J_1) \cup ... \cup (SL(n)/J_{n-1}) \leftrightarrows V.$$

Therefor, in our abstract simplicial complex the conjugacy classes of compact open subgroup of $SL(n)$ are the vertices. We can now form the set S in the following way: A nonempty finite subset $\{L_0, L_1, L_2, ..., L_r\}$ is an element of S if and only if there is $x \in SL(n)$ such that

$$\{xL_0x^{-1}, xL_1x^{-1}, xL_2x^{-1}, ..., xL_rx^{-1}\} \subset \{J_0, J_1, J_2, ..., J_{n-1}\}.$$

4.2.1 The affine building $\beta SL(2)$ for $SL(2)$

Let $J_0 = SL(2, \mathcal{O}_F)$. As in the previous section we consider $\Pi \in GL(2)$ given by $\Pi = \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}$. Use the map $\beth\Pi$ to get the conjugacy classes of maximal compact open subgroups of $SL(2)$. So, we have two conjugacy classes of maximal compact open subgroups J_0 and $J_1 = \Pi x \Pi^{-1}$ which they take the following form

$$J_0 = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F \end{pmatrix} \cap SL(2), \quad J_1 = \begin{pmatrix} \mathcal{O}_F & \varpi_F^{-1}\mathcal{O}_F \\ \varpi_F\mathcal{O}_F & \mathcal{O}_F \end{pmatrix} \cap SL(2).$$

The Iwahori subgroup I for $SL(2)$ is defined to be

$$I = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \varpi_F\mathcal{O}_F & \mathcal{O}_F \end{pmatrix} \cap SL(2) = J_0 \cap J_1.$$

Now, define the vertex set V to be the set all maximal compact open subgroups of $SL(2)$. These groups will be J_0, J_1 and their conjugates. So, we have the bijective correspondence

$$(SL(2)/J_0) \cup (SL(2)/J_1) \leftrightarrows V.$$

The set S of nonempty finite subsets of V can be defined as follows:

$$S = S_0 \cup S_1$$

where $S_0 = V$ and S_1 is defined to be the set of simplices of cardinality two. So

$$S_1 = \{\{xJ_0, xJ_1\} : x \in SL(2)\}.$$

Now we note that the bijective correspondence between V and $SL(2)/I$ is given by

$$\{xJ_0, xJ_1\} \leftrightarrows xI.$$

Therefore the set of edges is denoted by $SL(2)/I$. Thus the abstract simplicial complex (V,S) has been defined. For p-adic group $SL(2,\mathbb{Q}_p)$, the affine building is a $p+1$-regular tree, and is denoted by βG. The affine building of $SL(2,F)$ where F is a finite extension of $\mathbb{Q}_p$ is also a tree. If xI is an edge of the tree $\beta SL(2)$, then it has the vertices xJ_0 and xJ_1. The action of $SL(2)$ on $\beta SL(2)$ is by conjugation, extended from its action on V. The fundamental of this action is an edge with two vertices, the edge stabilizer is the Iwahori group I, and the two vertex stabilizers are J_0 and J_1. Each apartment in $\beta SL(2)$ has the form of a line isomorphic to the real line $\mathbb{R}$, which divided into *chambers* take the form of an edge with two faces, namely the vertices.

4.3 The enlarged building for the p-adic group $GL(n)$

The action of $SL(n)$ on the set of vertices V can be extended to an action of $GL(n)$

$$GL(n) \times V \to V$$

$$(x,J) \to xJx^{-1}.$$

Theorem 4.3.1 *Let $\sigma = \{J_0, J_1, J_2, ..., J_r\} \in S$ be a simplex, where (V,S) is the abstract simplicial complex underlying the affine building $\beta SL(n)$ for $SL(n)$. Then for all $x \in GL(n)$ we have*

$$x\sigma = \{xJ_0x^{-1}, xJ_1x^{-1}, xJ_2x^{-1}, ..., xJ_rx^{-1}\} \in S.$$

PROOF: See [42]. ❒

Theorem 4.3.1 shows that the action of $SL(n)$ on the abstract simplicial (polysimplicial) complex (V,S) can be extended to an action of $GL(n)$ on the abstract simplicial (polysimplicial) complex by automorphisms. This action extends to an action of $GL(n)$ on the geometric realization. Thus we obtain $GL(n) \times \beta SL(n) \to \beta SL(n)$. It is not the case that $\beta SL(n) = \beta GL(n)$. We can view the real line $\mathbb{R}$ as simplicial complex, with vertex set $\mathbb{Z}$ and an edges are the intervals $[n,n+1]$ for $n \in \mathbb{Z}$. We now define an action

$$GL(n) \times \mathbb{R} \to \mathbb{R}, \quad (x,t) \to t + v_F(detx)$$

where v_F is the valuation on F. We have an action of $GL(n)$ on $\beta SL(n)$ and on $\mathbb{R}$ as we considered it as a simplicial complex. The affine building $\beta^1 GL(n)$ associated to $GL(n)$ can be defined to be $\beta^1 GL(n) = \beta^1 SL(n) \times \mathbb{R}$ with the group action

$$GL(n) \times \beta GL(n) \to \beta GL(n), \quad (x,y,t) \to (xy, t + v_F(detx)).$$

The building $\beta^1 GL(n)$ is also the enlarged building of $GL(n)$. The p-adic group $GL(n)$ acts by automorphisms on the underlying polysimplicial complex.

4.3.1 The enlarged building $\beta^1 GL(2)$ for $GL(2)$

The affine building $\beta SL(2,\mathbb{Q}_p)$ of $SL(2,\mathbb{Q}_p)$ is a $p+1$-regular tree. Consider the action of $GL(2)$ on $\beta SL(2)$. Let I be Iwahori subgroup and $xI \in S$ is an edge in $\beta SL(2)$. Theorem 4.3.1 shows that the group $GL(2)$ acts on this edge, for $x' \in GL(2)$ we have $x'xI$ is also an element of $\beta^1 GL(2)$. This also true for an element of $GL(2)$ acting on a vertex xJ_0 or xJ_1. The enlarged building of $GL(2)$ is $\beta^1 GL(2) = \beta SL(2) \times \mathbb{R}$, with the action

$$GL(2) \times \beta^1 GL(2) \to \beta^1 GL(2)$$

$$(x,y,t) \to (xy, t + v_F(detx)).$$

The enlarged building $\beta^1 GL(1)$ associated to the number field $F^\times$ is simply a copy of the real line $\mathbb{R}$, with the action $(x,t) \to t + v_F(x)$ is a picture of the building $\beta^1 GL(2)$. The dimension of an apartment in $\beta SL(n)$ is given by the rank of $SL(n)$, namely $n-1$ then the rank of $GL(n)$ is n. An apartment in the enlarged building $\beta^1 GL(n)$ is given by an apartment in $\beta^1 SL(n)$ and a copy of the real line. Thus the apartment in $\beta^1 GL(2)$ is given by an apartment in $\beta^1 SL(2)$ and a copy of the real line.

4.4 Homology group

This section discusses the basic notations of the *homology* in simple geometric cases where the homology groups arise from *boundary operator*. We will use the *chain complexes* to consider the algebraic process of constructing the homology groups. An abelian group G together with endomorphism $h : G \to G$, such that $h^2 = 0$ is called a *differential group*. The endomorphism h is called the boundary operator of G. The elements of G are often called *Chains*, the elements of $\ker h$ are called *cycles* and elements of *Im h* are called *boundaries*. The required condition $h^2 = 0$ is equivalent to the inclusion $Imh \subset \ker h$.The homology group of G is defined as $H_*(G) = \ker h/Imh = \ker h/h(G)$.

4.4.1 Complexes

Let $\mathfrak{A}$ be a ring, by an *open complex* of $\mathfrak{A}$-modules we mean a sequence of modules and homomorphisms $\{(M_i, h_i)\}$

$$\cdots \longleftarrow M_{i+1} \xleftarrow{h_i} M_i \xleftarrow{h_{i-1}} M_{i-1} \longleftarrow \cdots$$

i runs over all integers, and $h_i \circ h_{i-1} = 0$. A *closed complex* of $\mathfrak{A}$-modules is a sequence of modules and homomorphisms $\{(M_i, h_i)\}_{i\geq 1}$, i runs over all integers mod n and $n \geq 2$, closed complex look like this

$$M_1 \xleftarrow{h_n} M_n \xleftarrow{h_{n-1}} \cdots \xleftarrow{h_3} M_3 \xleftarrow{h_2} M_2 \xleftarrow{h_1} M_1$$

We call n the length of the closed complex.

4.4.2 Homology Sequence

Let $\{(M,h)\}$ be complex, define $Z_i(M) = \ker(h_i)$ and $B_i(M) = Im(h_{i-1})$, Z_i is the module of *i-cycles* and B_i is the modules of *i-boundaries*. Note that

$$B_i = B_i(M) \subset Z_i = Z_i(M)$$

since $h_i \circ h_{i-1} = 0$. Now, let $H_i(M) = Z_i/B_i = \ker h_i / Im\, h_{i-1}$, H_i is called the i-th homology group. We denote the graded module associated with the family H_i by $H(M)$ or $H_*(M)$ and it is called the *homology* of M.

If $f : M^{'} \to M$ is a zero-degree morphism of complexes, then we get an induced canonical homomorphism of zero-degree $f^* : H(M^{'}) \to H(M)$ on the homology. Thus H is a functor from the category of complexes into the category of graded modules. We could write $H(f)$ instead of f^* and also $H_i(f)$ instead of f_i^* for the induced map on $H_i^{'}$.

4.5 Chamber Homology

Let $\sigma \in X$ be a simplex in a simplicial complex X. An *ordering* of a simplex σ is a bijection between σ and the set $\{0,1,2,3,...,n\}$. Denote the set of all ordering of σ by $\widehat{\sigma}$. For any bijections

$$\varphi : \{0,1,2,3,...,n\} \longrightarrow \{0,1,2,3,...,n\}$$

and

$$\psi : \sigma \longrightarrow \{0,1,2,3,...,n\}$$

we can define an orientation of σ as a function

$$\varsigma : \widehat{\sigma} \to \{-1,1\}, \quad \varphi \circ \psi \mapsto \varsigma(\varphi \circ \psi) = \varepsilon(\varphi)\varsigma(\psi)$$

where $\varepsilon(\varphi)$ is the sign of φ. An orientation of the simplicial complex X is a function Λ which assigns to each simplex in X an orientation $\Lambda(\sigma)$ of σ.

Definition 4.5.1 [8], [43]

A cosheaf $\mathcal{A}$ on X consists of:

1. An abelian group A_σ for each polysimplex σ.

2. For each inclusion of polysimplices $\eta \subseteq \sigma$, there exists a homomorphism of abelian groups $\psi_\eta^\sigma : \mathrm{A}_\sigma \to \mathrm{A}_\eta$ such that $\psi_\tau^\sigma = \psi_\tau^\eta \psi_\eta^\sigma$ whenever $\tau \subseteq \eta \subseteq \sigma$, and such ψ_σ^σ is the identity of A_σ for each polysimplex σ.

Let G be a discrete or totally disconnected group acts simplicially on X in a way that the stabilizer of each vertex in X is a compact open subgroup of G, where X be finite-dimensional simplicial or polysimplicial complex. Therefore, X is proper G-polysimplicial complex.

Definition 4.5.2 Let $\sigma \in X$ with vertices $\mathrm{v}_0, \mathrm{v}_1, ...\mathrm{v}_p$ (written in order) and let η be a codimension 1 face of σ with vertices $\mathrm{v}_0, \mathrm{v}_1, ..., \widehat{\mathrm{v}_\mathrm{i}}, ..., \mathrm{v}_p$, where $\widehat{\mathrm{v}_i}$ means that the vertex v_i is not included. The incidence number of these vertices is denoted by $(-1)^{\langle \eta:\sigma \rangle}$ and is defined as follows:

$$(-1)^{\langle \eta:\sigma \rangle} = \begin{cases} +1, & \text{if listing of the vertices of } \eta \text{ agrees with the given orientation of } \eta;, \\ -1, & \text{otherwise.} \end{cases} \tag{4.1}$$

4.5.1 Homology group of simplicial (polysimplicial) complex

Let $\mathcal{C}_n(X, \mathcal{A})$ be the abelian group of the elements of the form

$$\sum_{dim\sigma = n} \alpha_\sigma[\sigma],$$

σ runs over all polysimplices of dimension n and $\alpha_\sigma \in \mathrm{A}_\sigma$. Define a homomorphism of the abelian groups

$$\partial : \mathcal{C}_{n+1}(X, \mathcal{A}) \to \mathcal{C}_n(X, \mathcal{A})$$

$$\alpha_\sigma[\sigma] \to \partial(\alpha_\sigma[\sigma]) = \sum (-1)^{\langle \eta:\sigma \rangle} \psi_\eta^\sigma(\alpha_\sigma[\sigma])[\eta],$$

η runs over all polysimplices of codimension 1 in σ, and $(-1)^{\langle \eta:\sigma \rangle}$ is the incidence number of orientated polysimplices η and σ.

Definition 4.5.3 The *homology group with coffecients* in a cosheaf $\mathcal{A}H_*(X,\mathcal{A})$ is the homology group of the complex

$$0 \xleftarrow{\partial} \mathcal{C}_0(G;X) \xleftarrow{\partial} \mathcal{C}_1(G;X) \xleftarrow{\partial} \mathcal{C}_2(G;X) \xleftarrow{\partial} \cdots$$

and so

$$H_n(X,\mathcal{A}) = \frac{\ker \partial_{\mathcal{C}_n(X,\mathcal{A})}}{Im\,\partial_{\mathcal{C}_{n+1}(X,\mathcal{A})}}, \quad for \quad n = 0,1,2,\cdots \quad .$$

Let G be a reductive p-adic group acts on the polysimilicial complex X and suppose that G maps polysimplices to polysimplices. An action of G on the cosheaf $\mathcal{A}$ is a family of maps $\Lambda_g : \mathrm{A}_\sigma \to \mathrm{A}_{g\sigma}$ compatible with composition in G and with the maps $\psi^\sigma_\eta(\eta \subset \sigma)$. The *equivariant homology group* of polysimplicial complex X with coefficients in $\mathcal{A}$ is the homology group of the complex obtained by factoring each $\mathcal{C}_n(X,\mathcal{A})$ out by subgroup generated by elements of the form $\alpha_\sigma[\sigma] - \Lambda_g(\alpha_\sigma)[\sigma]$. This leads us to define the complex of coinvariants, the group of this complex is denoted by $\mathcal{C}_n(X,\mathcal{A})_G$.

Let $\beta^1 G$ be the enlarged building of G, and let σ be a polysimplex in this building. Define the subgroup

$$G_\sigma = \{g \in G : g\sigma = \sigma\},$$

G_σ is compact open subgroup of G. The representation ring of G_σ is denoted by $\mathfrak{R}(G_\sigma)$. Now, let $\eta \subset \sigma$ we have $G_\sigma \subset G_\eta$ and since both of G_σ and G_η are compact open subgroups of G, then G_σ is finite index in G_η. Using the induction or representations we can define the homomorphism

$$\psi^\sigma_\eta : \mathfrak{R}(G_\sigma) \to \mathfrak{R}(G_\eta)$$

$$\varphi \to \psi^\sigma_\eta(\varphi) = Ind^{G_\eta}_{G_\sigma} \varphi.$$

We have formed a cosheaf $\mathcal{R}$ on the $\beta^1 G$. Now, we can form the complex given by the groups $\mathcal{C}_n(\beta^1 G, \mathcal{R})$, the action of the group G on this complex is by automorphisms and is given by:

$$\Lambda_g(\sum_{dim\sigma=n}\varphi_\sigma[\sigma]) = \sum_{dim\sigma=n}\iota(g,\sigma)g_*(\varphi_\sigma)[g\sigma],$$

where $\iota(g,\sigma)$ here is the incidence number, $\iota(g,\sigma)=1$ if g preserves the orientation of σ and $=-1$ otherwise, and $g_*:\mathfrak{R}(G_\sigma)\to\mathfrak{R}(G_{g\sigma})$, mapping a representation $\varphi\in\mathfrak{R}(G_\sigma)$ to a representation $g_*(\varphi)\in\mathfrak{R}(G_{g\sigma})$, where $g_*(\varphi)(\rho)=\varphi(g\rho g^{-1})$ for $\rho\in G_{g\sigma}$. Now, we are able to define the complex of coinvariants given by the groups $\mathcal{C}_n(\beta^1G,\mathcal{R})_G$. Therefore we can calculate the equivariant homology groups of β^1G which is denoted by $H_*^G(\beta^1G;\mathbb{Z})$. Set

$$H_n^G(\beta^1G;\mathbb{C}) = H_*^G(\beta^1G;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}.$$

Kasparov defines the group $H_0^G(X)$ to be the group of homotopy classes of G-equivariant abstract elliptic operators on X and the group $H_1^G(X)$ to be the group of homotopy classes of self-adjoint G-equivariant abstract elliptic operators on X, where X be a proper G-compact space. Both groups $H_0^G(X)$ and $H_1^G(X)$ are abelian groups.

4.5.2 Equivariant homology group $H_*(G;X)$

Let G be a discrete or totally disconnected group acting on a finite-dimesional simplicial (polysimplicial) complex X in a such away that the stabilizer of each vertex in X is a compact open subgroup of G. Also, suppose that if $g\in G$ fixes the simplex σ then G fixes all the vertices of σ. Define the equivaraint homology group $H_*(G;X)$ in a such way:

1. If X is one-point space and if G is profinite group then $H_0(G;X)$ is the space of locally constant class functions on G, while $H_p(G;X)=0$ (for $p>0$). By the character theory we note the space of locally constant class functions on G identifies with $\mathfrak{R}(G)\otimes_{\mathbb{Z}}\mathbb{C}$, the tensor product of the representation ring of the group G with $\mathbb{C}$.

2. If G is a discrete and if the action of G on X is free, then $H_*(G;X)$ identifies with the ordinary homology $H_*(X/G)$ of the quotient space X/G with complex coffecients. In particular if G is trivial group then $H_*(G;X)$ is the ordinary homology of X with complex coffecients.

Let $\sigma \in X$ and let G_σ its isotropy group in G, we know that G_σ is a compact open subgroup of G. Let $\mathcal{H}(G_\sigma)$ be the vector space of locally constant complex-valued functions on G_σ which forms the vector space

$$\mathcal{C}_p(G;X) = \bigoplus \mathcal{H}(G_\sigma)$$

σ runs over X^p of p-simplices in X. If $\phi \in \mathcal{C}_p(G;X)$ then ϕ can be written as finite formal sums

$$\phi = \sum_{\sigma \in X^p} \varphi_\sigma[\sigma]; \quad \varphi_\sigma \in \mathcal{H}(G_\sigma).$$

Now, if G is trivial group then φ_σ is just a complex number and $\mathcal{C}_p(G;X)$ is the space of simplicial p-chains in X with complex coffecients. Let $\sigma, \eta \in X$ where $\eta \subset \sigma$, we already know that $G_\sigma \subset G_\eta$ and we can extend every locally constant function on G_σ by zero to a locally constant function on G_η. So

$$\eta \subset \sigma \Rightarrow \mathcal{H}(G_\sigma) \subset \mathcal{H}(G_\eta).$$

Now we can define $\partial : \mathcal{C}_p(G;X) \to \mathcal{C}_{p-1}(G;X)$

$$\varphi_\sigma[\sigma] \to \partial(\varphi_\sigma[\sigma]) = \sum_{\eta \in X^{p-1}} (-1)^{\langle \eta : \sigma \rangle} \varphi_\sigma[\eta]$$

the sum runs over the codimension 1 of the faces of σ. The map ∂ constitute the differentials in a chain complex:

$$\mathcal{C}_0(G;X) \xleftarrow{\quad\partial\quad} \mathcal{C}_1(G;X) \xleftarrow{\quad\partial\quad} \mathcal{C}_2(G;X) \xleftarrow{\quad\partial\quad} \cdots$$

G acts on this complex as follows:

$$g\Big(\sum_{\sigma \in X^p} \varphi_\sigma[\sigma]\Big) = \sum_{\sigma \in X^p} g\varphi_\sigma[g\sigma]$$

where $g\varphi_\sigma(\gamma) = \varphi_\sigma(g^{-1}\gamma g)$. Also, $g\varphi_\sigma \in \mathcal{H}(G_{g\sigma})$, since $G_{g\sigma} = gG_\sigma g^{-1}$. From each vector space $\mathcal{C}_p(G;X)$ we can form the vector space of coinvariants $\mathcal{C}_p(G;X)_G$ which is a quotient of $\mathcal{C}_p(G;X)$ by the vector subspace spanned by elements of the form $g(a)-a$ with $g \in G$ and $a \in \mathcal{C}_p(G;X)$. If $\mathcal{H}(G)$ is a Hecke algebra of G then $\mathcal{C}_p(G;X)_G$ is the tensor product

$$\mathcal{C}_p(G;X)_G = \mathcal{C}_p(G;X) \otimes_{\mathcal{H}(G)} \mathbb{C}$$

Definition 4.5.4 The chamber homology group of X is defined to be the homology of the coinvariants-complex

$$\mathcal{C}_0(G;X)_G \xleftarrow{\quad\partial\quad} \mathcal{C}_1(G;X)_G \xleftarrow{\quad\partial\quad} \mathcal{C}_2(G;X)_G \xleftarrow{\quad\partial\quad} \cdots$$

Denote this homology group by $H_*(G;X)$.

More details and computes for the chamber homology groups will be showing in the next three chapters depends on which p-adic group will be the group G.

4.6 Weil group and representations

We start recalling some features of the Galois theory of F. In the first five paragraphs, the proofs are all standard or reasonably straightforward, so we have omitted them. Throughout, p means the characteristic of the residue class field $\mathfrak{K} = \mathfrak{o}/\mathfrak{p}$.

We choose a separable algebraic closure $\overline{F}$ of F. We set $\Omega_F = Gal(\overline{F}/F)$, and view it as a profinite group with its natural topology. Thus $\Omega_F = \lim_{\leftarrow} Gal(E/F)$, where E/F ranges over the finite Galois extensions with $E \subset \overline{F}$.

Now, let K/F be a finite field extension of F, $K \subset \overline{F}$. Then

1. Ω_K is an open subgroup $Gal(\overline{F}/K)$ of Ω_F.

2. F_∞ is the composite of all the fields K/F.

3. $\mathcal{I}_F = Gal(\overline{F}/F_\infty)$ is the inertia group of F.

4. $\mathcal{W}_F$ is the inverse image in Ω_F of the cyclic subgroup $\langle \Phi_F \rangle$ of $Gal(F_\infty/F)$ generated by Φ_F. $\mathcal{W}_F$ is the normal dense subgroup in Ω_F generated by the Frobenius elements.

Definition 4.6.1 The Weil group $\mathcal{W}_F$ of F for $\overline{F}/F$ is a topological group, so that:

1. $\mathcal{I}_F$ is an open subgroup of $\mathcal{W}_F$, and
2. the topology on $\mathcal{I}_F$, as a subspace of $\mathcal{W}_F$, coincides with its natural topology as $Gal(\overline{F}/F_\infty) \subset \Omega_F$.

Thus, $\mathcal{W}_F$ is a locally profinite group and the identity map $\iota_F : \mathcal{W}_F \subset \Omega_F$ is a continuous injection. And $v_F : \mathcal{W}_F \to \mathbb{Z}$ is the canonical map taking a geometric Frobenius element to 1 and $\|x\| = q^{-v_F(x)}$, $x \in \mathcal{W}_F$.

In another words, if E/F is a finite cyclic extension of F then $\Phi_F \in \mathcal{W}_F \hookrightarrow Gal(\overline{F}/F)$, where $\Phi_F|_E$ is a generater of $Gal(E/F)$. Therefore we have a short exact sequence

$$1 \to \mathcal{I}_F \to \mathcal{W}_F \xrightarrow{d_F} \mathbb{Z} \to 0 \ .$$

$$\Phi_F \mapsto 1.$$

Let E/F be finite extension, $E \subset \overline{F}$. The canonical inclusion map $\Omega_E \to \Omega_F$ induces a bijection $\mathcal{W}_E \to \mathcal{W}_F \cap \Omega_E$. This map induces a homeomorphism of $\mathcal{W}_E$ with an open subgroup of $\mathcal{W}_F$ of finite index.

Proposition 4.6.2

1. *Let E/F be a finite field extension, $E \subset \overline{F}$.*
   (a) *The group $\mathcal{W}_F$ has a unique subgroup $\mathcal{W}_F^E$ such that $\iota_F(\mathcal{W}_F^E) = \mathcal{W}_F \cap \Omega_E$.*
   (b) *The subgroup $\mathcal{W}_F^E$ is open and of finite index in $\mathcal{W}_F$; it is normal if and only if E/F Galois.*
   (c) *The canonical map $\mathcal{W}_F^E \backslash \mathcal{W}_F \to \Omega_E \backslash \Omega_F$ is a bijection.*
   (d) *The canonical map $\iota_E : \mathcal{W}_E \to \Omega_E$ induces a topological isomorphism $\mathcal{W}_E \cong \mathcal{W}_F^E$.*
2. *The map $E/F \mapsto \mathcal{W}_F^E$ is a bijection between the set of finite extensions E of F inside $\overline{F}$ and the set of open subgroups of $\mathcal{W}_F$ of finite index.*

.

Lemma 4.6.3 *Let E/F be a finite separable field extension, $E \subset \overline{F}$*

1. *Let ρ be a smooth representation of $\mathcal{W}_F$; then ρ is semisimple if and only if ρ_E is semisimple.*

2. *Let τ be a smooth representation of $\mathcal{W}_E$; then τ is semisimple if and only if $Ind_{E/F}\tau$ is semisimple.*

Let $\mathcal{G}_n^{ss}(F)$ be the set of isomorphism classes of semisimple smooth representations of $\mathcal{W}_F$ of dimension n. We denoted by $\mathcal{G}_n^0(F)$ the set of isomorphism classes of irreducible smooth representations of $\mathcal{W}_F$ of dimension n. The Lemma shows that we have induction and restriction maps

$$Ind_{E/F} : \mathcal{G}_n^{ss}(E) \to \mathcal{G}_{nd}^{ss}(F), \;\; Res_{E/F} : \mathcal{G}_n^{ss}(F) \to \mathcal{G}_n^{ss}(E),$$

where E/F is a finite extension, $E \subset \overline{F}$ and $d = [E : F]$.

Definition 4.6.4 The group $\mathcal{W}_F^{ab} := \mathcal{W}_F/[\overline{\mathcal{W}_F, \mathcal{W}_F}]$ is a locally profinite abelian group.

Theorem 4.6.5 *Local Class Field Theory Let $a_F : \mathcal{W}_F \to F^\times$ be a canonical continuous group homomorphism such that*

1. *The map a_F induces a topological isomorphism $\mathcal{W}_F^{ab} \simeq F^\times$.*

2. *An element $x \in \mathcal{W}_F$ is a geometric Frobenius element if and only if $a_F(x)$ a prime element of F.*

3. *$a_F(\mathcal{I}_F) = \mathcal{U}_F$ and $a_F(\mathcal{P}_F) = U_F^1$, where $\mathcal{P}_F = Gal(\overline{F}/E_\infty)$ is called the* Wild Inertia group.

4. *If E/F is a finite separable extension, then the diagram*

$$\begin{array}{ccc} \mathcal{W}_E & \xrightarrow{a_E} & E^\times \\ \downarrow & & \downarrow N_{E/F} \\ \mathcal{W}_F & \xrightarrow[a_F]{} & F^\times \end{array}$$

is commutative.

5. *Let* $\alpha : F \to F^{'}$ *be an isomorphism of fields. The map* α *induces an isomorphism* $\alpha : \mathcal{W}_F^{ab} \to \mathcal{W}_{F'}^{ab}$, *and the diagram*

$$\begin{tikzcd} \mathcal{W}_F^{ab} \arrow[r, "\alpha"] \arrow[d, "a_F"'] & \mathcal{W}_{F'}^{ab} \arrow[d, "a_{F'}"] \\ F^{\times} \arrow[r, "\alpha"'] & F^{'\times} \end{tikzcd}$$

is commutative.

Note that we have a short exact sequence

$$1 \to \mathcal{I}_F \to \mathcal{W}_F \xrightarrow{d_F} \mathbb{Z} \to 0$$

with $\mathcal{W}_F/\mathcal{I}_F \cong \mathbb{Z}$ in the discrete topology. For proofs and more details see [12].

Chapter 5

Base Change and K-theory for SL(2,F)

In this chapter we will describe chamber homology and K-theory for $SL(2,F)$. Also, we will apply base change on both sides and see how it works. Let $\beta SL(2)$ be the tree of $SL(2)$ as in Chapter 4 and Δ be any edge of the tree $\beta SL(2)$. The isotropy group of an edge is an Iwahori group $I \subset G$, and the isotropy groups of the vertices of the edge are two maximal compact subgroups J_0, J_1 of G which contain I, so that we have a segment of groups. The representation rings $\mathfrak{R}(I), \mathfrak{R}(J_0), \mathfrak{R}(J_1)$ then create a chain complex in which the differential is given, once an orientation has been chosen, by $\partial(v) = (Ind_I^{J_0}(v), -Ind_I^{J_1}(v))$

This creates two homology groups:

$$H_0 = \frac{\mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)}{\partial(\mathfrak{R}(I))},$$

and

$$H_1 = \ker \partial.$$

5.1 K-theory for SL(2,F)

We are going to compute the K-theory groups of the reduced C^*-algebra of $SL(2,F)$. The two K-theory groups K_0, K_1 are free abelian of infinite rank. Let's start with the reduced dual of $SL(2,F)$.

Let F be a p-adic non-archimedean local field with $p \neq 2$. We have $F^\times \cong \mathcal{U}_F \times \mathbb{Z}$, where $\mathcal{U}_F$ is the group of p-adic units, and the dual of $F^\times$ is $\widehat{F^\times} \cong \widehat{\mathcal{U}_F} \times \mathbb{T}$, where $\mathbb{T}$ is the circle group. The group $\mathbb{Z}$ has a unique character of order 2, sending n to $(-1)^n$. The group $\mathcal{U}_F$ has a unique character of order 2 the Legendre character λ which sends u to the Legendre symbol of the mod p reduction of u.

The principal series (or simply P. S.) of unitary representations of $SL(2,F)$ is given by $\pi_\sigma = Ind_B^G \sigma$, where σ is a unitary character of $F^\times$. The representations π_σ are irreducible unless σ is a character of order 2.

Two representations π_σ and π_ψ are unitarily equivalent if and only if $\sigma = \psi$ or $\sigma = \psi^{-1}$. The *Weyl* group of $SL(2,F)$ is $\mathbb{W} = \mathbb{Z}/2\mathbb{Z}$. The *Weyl* group acts on the diagonal subgroup of $SL(2,F)$, hence on $F^\times$ and its unitary dual. Let $w \in \mathbb{W}$, if $\sigma = (\varsigma, \mu) \in \widehat{\mathcal{U}_F} \times \mathbb{T}$ then $w\sigma = (\varsigma^{-1}, \mu^{-1})$.

We have two cases:

1. The discrete parameter ς is fixed by $\mathbb{W}$: in which case $\varsigma = \mathbb{1}$ or λ, where λ is the Legendre character. Then $\pi_\sigma \cong \pi_\psi$ where $\sigma = (\varsigma, \mu)$ and $\psi = (\varsigma, \mu^{-1})$.

2. The discrete parameter ς is not fixed by $\mathbb{W}$. Then $\pi_\sigma \cong \pi_\psi$ with $\sigma = (\varsigma, \mu)$,

   $\psi = (\varsigma^{-1}, \mu^{-1})$ and $\varsigma \neq \varsigma^{-1}$.

The parameter space of the P.S. is $(\widehat{\mathcal{U}_F} \times \mathbb{T})/\mathbb{W}$ which is a Hausdorff space in which each connected component is either a circle or a copy of closed unit interval. There are countably many circles and two intervals.

On the other hand, the special representation of $SL(2,F)$ may be viewed as a natural representation of $SL(2,F)$ on the space of square integrable harmonic 1-forms on the tree of $SL(2,F)$. The special representation is the *Steinberg* representation.

The construction of the cuspidal representations of $SL(2,F)$ has been described by *Gelbart* in [22] in terms of the *Weil* representation of symplectic group over a quadratic extension E of F, we will denote this representation by $\aleph(\psi)$. Since the matrix coefficient functions of this representation have compact support (this is the definition of cuspidal representation) then $\aleph(\psi)$ is integrable. Therefore it is an isolated point in the reduced dual, by *Dixmier* [18] 18.4.2. If ψ is a trivial character, then the representations $\aleph(\psi)$ are equivalent to the irreducible components of the reducible representations in the principal series for $SL(2,F)$; see [41].

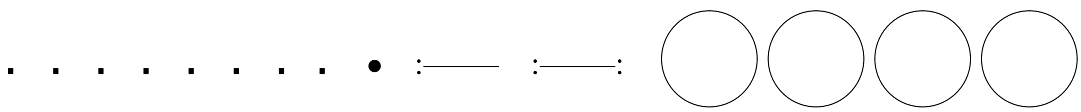
5.1.1 The reduced C^*-algebra of $SL(2,F)$

A generator of $K_0 C_r^*(G)$ can be created by a square integrable representation of $G = SL(2,F)$. We are going to consider the contribution to $K_0 C_r^*(G)$ made by the principal series. Let Υ be the parameter space of this principal series. Thus

$$\Upsilon = \widehat{F^\times}/\mathbb{W} = (\widehat{\mathcal{U}_F} \times \mathbb{T}) \Big/ (\mathbb{Z}/2\mathbb{Z}).$$

The group $\widehat{F^\times}$ is given the Pontryagin dual topology, so that $\widehat{F^\times}$ has countably many connected components: and each connected component is a copy of the circle $\mathbb{T}$ in its Euclidean topology. There are two copies of the closed unit interval and countably many circles. Let $\widehat{G}$ be the union of the irreducible components of reducible principal series and the irreducible principal series. If $\sigma_n \to \sigma$ in Υ and σ is not of order 2 then $\pi_{\sigma_n} \to \pi_\sigma$ in the natural topology on $\widehat{G}$. If $\sigma_n \to \sigma$ in Υ and σ of order 2 then $\pi_{\sigma_n} \to \pi_\sigma^-$ and $\pi_{\sigma_n} \to \pi_\sigma^+$. This topology, so defined, is non-Hausdorff with 3 double points.

The reduced dual of $SL(2,F)$ is the non-Hausdorff space:



where both sides of this diagram are infinite. Now, let's consider the K-theory group K_0 in both cases *(discrete and principal)*.

1. The K_0-group of $C^*_{prin}(G)$ contains one generator for each character of $\mathcal{U}_F$. Also, there are three additional generators associated to the elliptic representations in the principal series which appear as irreducible components of the reducible principal series representations which are labeled by the three non-trivial characters of order two of the diagonal subgroup of G.

2. The K_0- group of $C^*_{disc}(G)$ has one generator for each discrete series representation.

These consist of the Steinberg representation St_G and the cuspidal representations.

5.1.2 Local Langlands correspondence and Base change

Let F be a non-archimedean local field, and $G = SL(2,F)$. Let $\mathcal{L}_F$ be the local Langlands group:

$$\mathcal{L}_F := \mathcal{W}_F \times SL(2,\mathbb{C}).$$

A Langlands parameter is a continuous homomorphism

$$\phi : \mathcal{L}_F \to G^\vee = PGL(2,\mathbb{C}),$$

where $G^\vee = PGL(2,\mathbb{C})$ is the Langlands dual group. We say that two Langlands parameters are equivalent if they are conjugate under the group $PGL(2,\mathbb{C})$. Let $\Phi(G)$ be the set of equivalence classes of the Langlands parameters. Now, the Local Langlands correspondence is defined to be the surjective map

$$Irr(G) \longrightarrow \Phi(G),$$

$$A_\phi \longmapsto \phi$$

where A_ϕ is the pre-image of ϕ which is called the L-packet. The base change map is defined by the restriction of L-parameter from $\mathcal{L}_F$ to $\mathcal{L}_E$, where E is a finite extension of F

$$\phi|_{\mathcal{W}_E} : \mathcal{W}_E \times SL(2,\mathbb{C}) \to PGL(2,\mathbb{C}).$$

Lemma 5.1.1 *Let $\alpha_E = \gamma_E \circ \beta_E : \mathcal{W}_E \to E^\times$, where $\gamma_E : \mathcal{W}_E^{ab} \to E^\times$ and $\beta_E : \mathcal{W}_E \to \mathcal{W}_E^{ab}$ then we have:*

1. $N_{E/F}(\alpha_E(w)) = \alpha_F(w)$, $w \in \mathcal{W}_E \subset \mathcal{W}_F$.

2. $f.val_E = val_F \circ N_{E/F}$.

3. $d_E = -val_E \circ \alpha_E$.

4. *Let $w \in \mathcal{W}_E \subset \mathcal{W}_F$. Then we have $f.d_E(w) = d_F(w)$.*

PROOF: See [7] 1.2.2 for 1, [49] p.139 for 2, and see [34] for 3 and 4. ❐

Now, an unramified character ψ of $\mathcal{W}_E$ is given by the following simple formula:

$$\psi(w) = z^{d_E(w)}, z \in \mathbb{C}^{\times}.$$

The base change formula for a character χ of $\mathcal{W}_F$ is given by

$$BC(\chi) = \chi \mid_{\mathcal{W}_E} .$$

Lemma 5.1.2 *Under base change we have*

$$BC(\psi)(w) = (z^f)^{d_E(w)}$$

for all $w \in \mathcal{W}_E$.

PROOF: The result follows directly from part 4 of lemma 5.1.1. ❐

Lemma 5.1.3 *Let $\phi = \mathbb{1} \otimes \tau(2)$ and $\phi^{'} = \psi \otimes \tau(2)$ be two L-parameters, where ψ is an unramified character of $\mathcal{W}_F$. Then*

$$\phi = \phi^{'}$$

in $PGL(2,\mathbb{C})$.

PROOF: Let

$$d_F : \mathcal{W}_F \longrightarrow \mathcal{W}_F^{ab} \simeq F^{\times} \xrightarrow{val_F} \mathbb{Z} .$$

We have $\psi(w) = z^{d(w)}$ where $z \in \mathbb{C}^{\times}$, ψ unitary character if and only if $z \in \mathbb{T}$. Let

$$\phi = \mathbb{1} \otimes \tau(2) : \mathcal{W}_F \times SL(2,\mathbb{C}) \to PGL_2(\mathbb{C})$$

and

$$\phi^{'} = \psi \otimes \tau(2) : \mathcal{W}_F \times SL(2,\mathbb{C}) \to PGL_2(\mathbb{C})$$

such that

$$\phi(w,A) = \mathbb{1} \cdot \tau(2)\big(A\big) = \tau(A)$$

and

$$\phi^{'}(w,A) = \psi(w) \cdot \tau(2)\big(A\big) = z^{d(w)} \cdot \tau(A).$$

We see that $\tau(A)$ and $(z^{d(w)} \cdot \tau(A))$ are both in the same group $PGL(2,\mathbb{C})$ and this means that $\phi = \phi^{'}$. ❒

Theorem 5.1.4 *Let $\phi = 1 \otimes \tau(2)$ be the L-parameter of the Steinberg representation, then we have $BC(St_G(F)) = St_G(E)$.*

PROOF: Let

$$\mathcal{L}_F = \mathcal{W}_F \times SL(2,\mathbb{C}) \quad \textit{and} \quad \mathcal{L}_E = \mathcal{W}_E \times SL(2,\mathbb{C})$$

be the local Langlands groups and let

$$\phi : \mathcal{L}_F \xrightarrow{\mathbb{1}_{\mathcal{W}_F} \otimes \tau(2)} PGL(2,\mathbb{C})$$

be the L-parameter, this parameter works as follows

$$(w,Y) \longmapsto [Y].$$

We know that $\mathcal{L}_E \subset \mathcal{L}_F$. The base change works by restriction the L-parameter to $\mathcal{W}_E$, in another words

$$\phi|_{\mathcal{W}_E} : \mathcal{L}_E \xrightarrow{\mathbb{1}_{\mathcal{W}_E} \otimes \tau(2)} PGL(2,\mathbb{C})\ .$$

Since the restriction works only on the Weil group side which in our case is the trivial representation of $\mathcal{W}_F$ and since the restriction of the trivial representation of $\mathcal{W}_F$ is also the trivial representation of $\mathcal{W}_E$, then the resulting representation is also the Steinberg representation, i.e

$$BC(St_G(F)) = St_G(E).$$

$$\begin{array}{ccc} \phi : \mathcal{W}_F \times SL(2,\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) \\ \downarrow & & \downarrow \| \\ \phi|_{\mathcal{W}_E} : \mathcal{W}_E \times SL(2,\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) \end{array}$$

❐

Theorem 5.1.5 *Let $\mathbb{T}$ be one of the circles in the unitary principal series of $SL(2,F)$, then we have $\mathbb{T} \to \mathbb{T}$, $z \mapsto z^f$, under base change E/F.*

1. *At the level of the K-theory group K^1, BC induces the map $\mathbb{Z} \to \mathbb{Z}$, $\alpha_1 \mapsto f.\alpha_1$, where f is the residue field degree and α_1 denotes a generator of $K^1(\mathbb{T}) = \mathbb{Z}$.*

2. *At the level of K-theory group K^0, BC induces the identity map $\mathbb{Z} \to \mathbb{Z}$, $\alpha_0 \mapsto \alpha_0$, where α_0 denotes a generator of $K^0(\mathbb{T}) = \mathbb{Z}$.*

PROOF: We know that the principal series of $SL(2,F)$ can be defined as follows:

$$Ind_B^{SL(2,F)}(\chi) \quad where \quad \chi \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} = \chi(x).$$

Now we have

$$\begin{array}{ccc} \mathcal{W}_F \times SL(2,\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) \\ \downarrow & & \downarrow \| \\ F^\times \times SL(2,\mathbb{C}) & \xrightarrow{\phi} & PGL_2(\mathbb{C}) \end{array}$$

This means the above map ϕ works as follows:

$$(x,\tau) \longmapsto \begin{bmatrix} \chi(x) & 0 \\ 0 & 1 \end{bmatrix}.$$

Here $\begin{bmatrix} \chi(x) & 0 \\ 0 & 1 \end{bmatrix}$ is the coset of $\begin{pmatrix} \chi(x) & 0 \\ 0 & 1 \end{pmatrix} \in PGL_2(\mathbb{C})$. If we twist χ by an unramified character we get a circle $\mathbb{T}$ embedded in $PGL_2(\mathbb{C})$. Also, the Weyl group $\mathbb{Z}/2\mathbb{Z}$ acts on character of $F^\times$, character of $F^\times = \mathcal{U}_F \times \langle \varpi_F \rangle$ splits into $\{$*ramified character of* $\mathcal{U}_F$ *say* $\chi_1\}$ and $\{$*an unramified character of* $\langle \varpi_F \rangle$ *say* $\chi_0(\varpi) = z \in \mathbb{T}\}$. The generator w of $\mathbb{Z}/2\mathbb{Z}$ sends z to z^{-1}, it sends χ_1 to χ_1^{-1}. Suppose that

$$\chi_1 \neq \chi_1^{-1}, \quad i.e. \quad \chi_1^2 \neq 1.$$

For such χ, the representation $Ind_B^{SL(2,F)}\chi$ is irreducible. Define the L-parameter ϕ as follows: $\phi = \rho \otimes \mathbb{1}$ where ρ is a unitary character of $\mathcal{W}_F$ such that

$$\rho : \mathcal{W}_F \longrightarrow \mathcal{W}_F^{ab} \simeq F^\times \xrightarrow{\quad \chi \quad} \mathbb{T}\,.$$

Also, we have

$$\rho \longmapsto Ind_B^{SL(2,F)}\chi$$

The unitary characters $(\rho^2 \neq 1)$ of $\mathcal{W}_F$ factor through $F^\times$ and we have

$$\widehat{F^\times} = \widehat{\langle \varpi \rangle} \times \widehat{\mathcal{U}_F},$$

ρ is a unitary character of $\widehat{\mathcal{U}_F}$. The group $\widehat{\mathcal{U}_F}$ admits countably many such characters ρ. Therefore, the compact orbit is the circle $\mathbb{T}$:

$$\mathfrak{O}^t(\phi) \cong \mathfrak{O}^t(BC(\phi)) \cong \mathbb{T}.$$

After restriction and using the local class functions theory we get that this map has degree f. Therefore, if $\chi^2 \neq 1$ this means by lemma 5.1.1 and theorem 5.1.2 in each circle the base change formula is $z \mapsto z^f$. ❒

5.2 Representatives in the chamber homology H_1

A description of the cycles in the group H_1 will be introduced in this section. Let $G = SL(2,F)$ be the group of unimodular 2×2 matrices with entries in the field F. It is a locally compact totally disconnected topological group [23],[46].

Let $I = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2)$. This is a compact open subgroup of G, called the Iwahori subgroup.

Let $w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $w_1 = \begin{pmatrix} 0 & -\varpi^{-1} \\ \varpi & 0 \end{pmatrix}$. These elements appear in the Tits system associated to G, which plays an important role in what follows [10] and [45]. Let

$$J_0 = I \cup Iw_0 I = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2) \quad and \quad J_1 = I \cup Iw_1 I = \begin{pmatrix} \mathcal{O} & \varpi^{-1}\mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2),$$

these are compact open subgroups of G which already are described in Chapter 4. Also, we see that $J_0 \cap J_1 = I$. The tree for $G = SL(2)$ is the graph βG, the group G acting on βG by multiplication on the left. As in Chapter 4, I is the stabilizer of the fundamental edge, and that J_0, J_1 are the stabilizer of the vertices of this edge, respectively.

Now if I, J_0 and J_1 are the compact subgroups of $G = SL(2)$ defined in the previous two paragraphs, then we have this chain complex

$$0 \longleftarrow \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) \xleftarrow{Ind_I^{J_0} \oplus -Ind_I^{J_1} = \partial} \mathfrak{R}(I) \longleftarrow 0$$

So that

$$H_0 = \frac{\mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)}{\partial \mathfrak{R}(I)}$$

and

$$H_1 = \ker \partial.$$

For more details see [3].

Definition 5.2.1 A character χ of $F^\times$ is called *tame* character if $\chi|_{\mathcal{U}_F^1}$ is trivial.

Lemma 5.2.2 *Let χ be a tame character of I then $\chi - \chi^{-1} \in H_1$.*

PROOF: Let χ be character of I, i.e.

$$I \xrightarrow{mod p} \mathcal{B} \subset SL_2(\mathbb{F}_p)$$

$$\chi : SL(2) \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p\mathcal{O} & \mathcal{O} \end{pmatrix} \longrightarrow \mathbb{T}, \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \longmapsto \chi(x).$$

Now let

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2),\ w \in \mathbb{W}$$

where $\mathbb{W}$ is the Weyl group of $SL(2)$. We have $w\chi(x) = \chi(wxw^{-1})$.

To prove that $\chi - w\chi \in H_1$ it is enough to show that $\chi - w\chi \in \mathfrak{R}(I)$, i.e. $\partial(\chi - w\chi) = 0$. In another words we need to show that

$$Ind_I^{J_0}(\chi - w\chi) = 0$$

and

$$Ind_I^{J_1}(\chi - w\chi) = 0.$$

Now choose $\chi \neq w\chi$, so

$$w\chi \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} := \chi \begin{pmatrix} x^{-1} & 0 \\ -y & x \end{pmatrix}.$$

This means $w\chi = \chi^{-1}$.

Therefore we only need to prove $Ind_I^{J_0}(\chi - \chi^{-1}) = 0$ and $Ind_I^{J_1}(\chi - \chi^{-1}) = 0$. But $Ind_I^{J_0}(\chi - \chi^{-1}) = 0$ if and only if $Ind_I^{J_0}\chi \cong Ind_I^{J_0}\chi^{-1}$. Since χ and χ^{-1} are distinct, then they are determine the the same representation and this representation is irreducible if and only if $\chi^2 \neq 1$ [20]. This means $Ind_I^{J_0}\chi \cong Ind_I^{J_0}\chi^{-1}$. Therefore $Ind_I^{J_0}(\chi - \chi^{-1}) = 0$. Same results will be shown if we take J_1. This means we have $\chi - \chi^{-1} \in H_1$. ❒

Now let k be a positive integer, $I(k) = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi^k\mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2,F)$. Consider now the subgroup $I(k)_w = I(k) \cap w^{-1}I(k)w = \begin{pmatrix} \mathcal{O} & \varpi^k\mathcal{O} \\ \varpi^k\mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2)$. Let ψ be an invariant function on I and let

$$\alpha : SL(2) \to SL(2), \quad \alpha : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & \varpi^{-1}c \\ \varpi b & a \end{pmatrix}.$$

Define $\psi^\alpha(g) = \psi(\alpha(g))$, then ψ induces to zero on J_1(resp. J_0) if and only if ψ^α induces to zero on J_0(resp. J_1). Therefore $\psi \in H_1$ if and only if $\psi^\alpha \in H_1$.

Fix a character $\chi : \mathcal{U}_F \to \mathbb{T}$ not of order two, and let k be the least positive integer such that $\chi[1+\varpi^k\mathcal{O}] = 1$. The character χ extends to the group $I(k)$ using the formula

$$\chi : \begin{pmatrix} a & b \\ \varpi^k c & d \end{pmatrix} \mapsto \chi(a).$$

Lemma 5.2.3 *Let $\psi_\chi = Ind_{I(k)}^I \chi$ then*

1. *ψ_χ is an irreducible character.*

2. *$\psi_\chi^\alpha = \psi_{\overline{\chi}}$, where α is the automorphism of $SL(2)$.*

Now let $c_\chi = \psi_\chi - \psi_{\overline{\chi}}$. Then $c_\chi \in H_1$, and the cycle c_χ, one selected from each pair of characters $\{\chi, \overline{\chi}\}$, constitute a basis for H_1. Therefore all cycles will be of this form.

Theorem 5.2.4 *1. The base change on K_1-theory level works as follows:*

$$K_1C_r^*SL(2,E) \xrightarrow{K_1(BC)} K_1C_r^*SL(2,F)\ , \quad \alpha_{\sigma\circ N_{E/F}} \longmapsto f\cdot\alpha_\sigma$$

2. The base change on H_1-level works as follows:

$$H_1(\beta SL(2,E)) \xrightarrow{BC} H_1(\beta SL(2,F))\ ,\ c_{\sigma\circ N_{E/F}} \longmapsto f\cdot c_\sigma$$

where

$$\alpha_{\sigma\circ N_{E/F}}(resp.\alpha_{\sigma}) \text{ and } c_{\sigma\circ N_{E/F}}(resp.c_{\sigma})$$

are the K_1 and H_1 generator of $SL(2,E)\big(SL(2,F)\big)$ respectively.

PROOF: By theorem 5.1.5 this map has degree f. The base change on the K_1-theory level takes the K_1-generator of the reduce C^*-algebra of $SL(2,E)$ to the K_1-generator of the reduce C^*-algebra of $SL(2,F)$ multiplying by the residue field degree f, so (1) has been proved. We also know that the base change on the chamber homology side works by sending each unramified unitary character of the Iwahori subgroup of $SL(2,F)$ to itself composed with the norm map. So, the base change map on the chamber homology side takes the generator of the chamber homology group of $SL(2,E)$ (labeled by this composite) to the generator of the chamber homology group of $SL(2,F)$ (labeled by unramified unitary character) multiplying by the residue field degree f. ❐

Corollary 5.2.5 *The assembly map* $\mathrm{H}_1(SL(2,F)) \xrightarrow{\mu_1^F} K_1C_r^*SL(2,F)$ *under the base change works as follows:*

$$f\cdot c_\sigma \longmapsto f\cdot\alpha_\sigma$$

where c_σ and α_σ are H_1 and K_1 generators for $SL(2,F)$ respectively.

This corollary shows that the Baum-Connes conjecture under base change takes the base change's effect on the homology group side to the base change's effect on the K-theory side, i.e. the multiplication between the generator of the chamber homology group for $SL(2,F)$ and the residue field degree to the multiplication between the K_1-generator of the reduce C^*-algebra for $SL(2,F)$ by the residue field's degree.

$$\begin{array}{ccccccc}
\mathrm{H}_1(SL(2,E)) & \xrightarrow{\mu_1^E} & K_1C_r^*SL(2,E) & \qquad & \mathrm{c}_{\sigma\circ N_{E/F}} & \xrightarrow{\mu_1^E} & \alpha_{\sigma\circ N_{E/F}} \\
\Big\downarrow{\scriptstyle BC} & & \Big\downarrow{\scriptstyle K_1(BC)} & & \Big\downarrow{\scriptstyle BC} & & \Big\downarrow{\scriptstyle K_1(BC)} \\
\mathrm{H}_1(SL(2,F)) & \xrightarrow{\mu_1^F} & K_1C_r^*SL(2,F) & & f\cdot c_\sigma & \xrightarrow{\mu_1^F} & f\cdot\alpha_\sigma
\end{array}$$

5.3 Representatives in the chamber homology H_0

In this section we will investigate the case H_0. Since we have two types of representations for $SL(2,F)$ which are: the discrete series and the principal series representations, so we need to describe each case individually. The unitary principal series representation are as same as described in H_1. We need to deal with reducible principal series, the special representation and the discrete series. Let's start with the special representation. This means we are going to deal with the Steinberg representation. We recall the maximal compact subgroups J_0 and J_1, which were described in the previous section as the stabilizer subgroups of the vertices of the edge of the tree $\beta SL(2,F)$.

Theorem 5.3.1 *Let J_0 and J_1 be the two maximal compact open subgroups of $SL(2)$ and let I the Iwahori subgroup of $SL(2)$. There are only three generators for H_0 which are $\mathbb{1}_{J_0}$, $\mathbb{1}_{J_1}$, and the induced representation of $\mathbb{1}_I$ to J_0 or J_1.*

PROOF: Let $\mathbb{1}_{J_0}\big(resp.\mathbb{1}_{J_1}\big)$ be a representation in $\mathfrak{R}(J_0)\big(resp.\mathfrak{R}(J_1)\big)$, so $[\mathbb{1}_{J_0},0]$ and $[0,\mathbb{1}_{J_1}] \in H_0$. We have

$$[Ind_I^{J_0}\mathbb{1}_I,0] = [0,Ind_I^{J_1}\mathbb{1}_I] \Longleftrightarrow \exists v \in \mathfrak{R}(I)$$

such that $(Ind_I^{J_0}\mathbb{1}_I, -Ind_I^{J_1}\mathbb{1}_I) = \partial(v)$.

This means we have only one possibility which is $v = \mathbb{1}_I$. Therefor three possibilities for H_0-generators are $\mathbb{1}_{J_0}$, $\mathbb{1}_{J_1}$, and $Ind_I^{J_0}\mathbb{1}_I\big(resp.Ind_I^{J_1}\mathbb{1}_I\big)$. ❐

The question here is which combination of these three generators correspond to the Steinberg representation St_G of $SL(2)$?

Theorem 5.3.2 *The 0-cycle corresponding to St_G of $SL(2)$ in K_0 is $(Ind_I^{J_0}\mathbb{1}_I - \mathbb{1}_{J_0},0)$.*

PROOF: Let $G = SL(2,F)$ and $J_0 = SL(2,\mathcal{O})$. According to the *Anh Reciprocity Theorem* in [33] p.57, if $d\mu$ is a Haar measure then we have the following:

1. $Ind_I^G\mathbb{1}_I = \int_X \pi d\mu(\pi), \quad X = \{\pi \in \widehat{G}_r : \pi|_I \supset \mathbb{1}_I\}$.

2. $Ind_{J_0}^{G}\mathbb{1}_{J_0} = \int_Y \pi d\mu(\pi), \quad Y = \{\pi \in \widehat{G}_r : \pi|_{J_0} \supset \mathbb{1}_{J_0}\}.$

Now,

$$Ind_I^G \mathbb{1}_I = Ind_{J_0}^G \mathbb{1}_{J_0} \oplus St_G$$

$$\Longleftrightarrow Ind_{J_0}^G (Ind_I^{J_0} \mathbb{1}_I) - Ind_{J_0}^G \mathbb{1}_{J_0} = St_G$$

$$\Longleftrightarrow Ind_{J_0}^G (Ind_I^{J_0} \mathbb{1}_I - \mathbb{1}_{J_0}) = St_G.$$

Therefore the 0-cycle corresponding to St_G is $(Ind_I^{J_0}\mathbb{1}_I - \mathbb{1}_{J_0}, 0)$. We also see that the Baum-Connes conjecture (map) in this case is $Ind_{J_0}^G$. ❐

The proof of the above theorem shows that the map $Ind_{J_0}^G$ takes $[Ind_I^{J_0}\mathbb{1}_I - \mathbb{1}_{J_0}, 0]_F \mapsto [St_G]_F$, i.e. it takes the generator of H_0^F to the generator of K_0^F labeled by St_G. This means we have three independent elements.

In the same way this map works on the E-sides by taking the

$$[Ind_I^{J_0}\mathbb{1}_I - \mathbb{1}_{J_0}, 0]_E \mapsto [St_G]_E.$$

From now on we will replace the notation of St_G by St_2^F and St_2^E to refer for the Steinberg representation of $SL(2,F)$ and $SL(2,E)$ respectively.

Theorem 5.3.3 *The base change on K_0-theory level takes the K_0-generator of the reduce C^*-algebra of $SL(2,E)$ labeled by St_2^E to the K_0-generator of the reduce C^*-algebra of $SL(2,F)$ labeled by St_2^F and the K-theory group $K_0 C_r^* SL(2,F) = \mathbb{Z}^3$.*

PROOF: From theorems 5.3.1 and 5.3.2 we have only three generators and this implies that $K_0 C_r^* SL(2,F) = \mathbb{Z}^3$.

$$\begin{array}{ccc} \mathrm{H}_0(SL(2,E)) & \xrightarrow{\mu_0^E} & K_0 C_r^* SL(2,E) \\ {\scriptstyle BC}\downarrow & & \downarrow{\scriptstyle K_0(BC)} \\ \mathrm{H}_0(SL(2,F)) & \xrightarrow{\mu_0^F} & K_0 C_r^* SL(2,F) \end{array}, \qquad \begin{array}{ccc} \mathrm{c} & \xrightarrow{\mu_0^E} & \alpha_{St_2^E} \\ {\scriptstyle BC}\downarrow & & \downarrow{\scriptstyle K_0(BC)} \\ c & \xrightarrow{\mu_0^F} & \alpha_{St_2^F} \end{array}$$

❐

On the chamber homology level, the base change map works by taking the generator of the group H_0 of $SL(2,E)$ to the generator of H_0 of $SL(2,F)$.

On the other hand, if we deal with the reducible principal series (the intervals), this means we are going to induce the *Legendre* character to one of the maximal compact subgroups J_0, J_1 or both.

Theorem 5.3.4 *There are three generators for H_0 which they are constructed by inducing a representation of the* Legendre *character from I to the maximal subgroups J_0 and J_1.*

PROOF: We know that if $\lambda^2 = 1$ then $Ind_B^{SL(2,\mathbb{F}_p)}\lambda = \lambda_B^+ \oplus \lambda_B^-$. This means our induced representation can be written as decomposition of two representations. So if we induced to the maximal compact subgroups J_0, J_1 we would have three multiple choices. Let λ_I be any representation in $\mathfrak{R}(I)$, then $Ind_I^{J_0}\lambda_I \left(resp. Ind_I^{J_1}\lambda_I \right)$ is the induced representation of the *Legendre* character from I to $J_0 \left(reps. J_1 \right)$. Now, we have

$$Ind_I^{J_0}\lambda_I = \lambda_{J_0}^+ \oplus \lambda_{J_0}^-$$

and

$$Ind_I^{J_1}\lambda_I = \lambda_{J_1}^+ \oplus \lambda_{J_1}^-.$$

This means we have three generators for H_0 which are:

$$\lambda_{J_1}^+, \quad \lambda_{J_0}^+ \quad and \quad \lambda_{J_0}^- \qquad or \qquad \lambda_{J_1}^-, \quad \lambda_{J_0}^+ \quad and \quad \lambda_{J_0}^-.$$

This also shows that the assembly map $Ind_{J_0}^G$ works as follows

$$[Ind_I^{J_0}\lambda_I - \lambda_{J_0}^+, 0]_F \mapsto [\lambda_{J_0}^-]_F$$

i.e. it takes the generator of H_0^F to the generator of K_0^F labeled by $\lambda_{J_0}^-$. In the same way this map works on the E-sides by taking the

$$[Ind_I^{J_0}\lambda_I - \lambda_{J_0}^+, 0]_E \mapsto [\lambda_{J_0}^-]_E.$$

This means we have three independent elements. ❐

Theorem 5.3.5 *The base change on K_0-theory level takes the K_0-generator of the reduced C^*-algebra of $SL(2,E)$ labeled by $\lambda_{J_0(E)}^-$ to the K_0-generator of the reduce C^*-algebra of $SL(2,F)$ labeled by $\lambda_{J_0(F)}^-$. The K-theory group $K_0 C_r^* SL(2,F)$ in this case is $\mathbb{Z}^3$.*

PROOF: Theorem 5.3.4 shows that we have three generators for K_0 and this means $K_0 = \mathbb{Z}^3$. ❐

On the other hand, we introduce the cuspidal representations as follows:

let

$$\aleph = \rho \otimes \mathbb{1} : \mathcal{W}_F \times SL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$$

then we have

$$\aleph\ |_{\mathcal{W}_E} : \mathcal{W}_E \times SL(2,\mathbb{C}) \to PGL(2,\mathbb{C}).$$

Now let $\aleph$ be an irreducible representation.

1. If $\aleph\ |_{\mathcal{W}_E}$ remains irreducible after restriction, then this determines a cuspidal representation of $SL(2,E)$. Base change in this case, will send one cuspidal representation of $SL(2,F)$ to a cuspidal representation of $SL(2,E)$.

2. If $\aleph\ |_{\mathcal{W}_E}$ is reducible, then this representation split into two 1-dimensional representations. i.e. $\aleph = \aleph_1 \oplus \aleph_2 = (\rho_1 \otimes \mathbb{1}) \oplus (\rho_2 \otimes \mathbb{1})$, where ρ_1 and ρ_2 are two characters of $\mathcal{W}_E$.

This means on the K-theory level there is one generator for each cuspidal representation and the $K_0 = \mathbb{Z}$.

5.4 Conclusion

We emphasized finding the explicit cycles in the chamber homology groups and the K-theory groups in term of each representation for $SL(2,F)$. This led to an explicit computing of chamber homology and the K-theory groups. We have identified the base change effect on each of these cycles. The base change map on the homology group level works by sending a generator of the homology group of $SL(2,E)$ labeled by a character of $E^{\times}$ to the generator of the homology group of $SL(2,F)$ labeled by a character of $F^{\times}$ multiplied by the residue field degree. Whilst, it works by sending the K-theory group generator of the reduce C^{*}-algebra of $SL(2,E)$ labeled by the 1-cycle (resp. 0-cycle) to the multiplication of the residue field degree with a generator of the K-theory group of $SL(2,F)$ labeled by the base changed effect on 1-cycle (resp. 0-cycle).

Consequently, we showed that the base change of *Steinberg* is again a *Steinberg*, the base change of a *principal series* is always a principle series and the base change of a *cuspidal* can certainly be either another cuspidal or a principal series.

We have found that whilst the Baum-Connes correspondence takes the homology group generator of $SL(2,E)$ to a generator of the K-theory group of C^{*}-algebra of $SL(2,E)$ by induction, it takes the effect of the base change map on the homology side to the base change effect on the K-theory side by induction as well.

Chapter 6

Base Change and K-theory for $GL(1)$

In this chapter we will consider the KK-theory and the chamber homology for $GL(1)$. Also, we will consider the base change effect. We start our chapter with recalling the general definitions of K-homology and the equivariant K-homology, afterward we will use it to calculate the K-homology group for $GL(1)$.

6.1 K-homology

Let A be a separable C^*-algebra and let $\mathcal{H}$ be a separable Hilbert space. Define

$$\mathfrak{B}(\mathcal{H}) := \{T : \mathcal{H} \to \mathcal{H} | \, T \; bounded\},$$

$$\|T\| = \sup_{u \in \mathcal{H}, \|u\|=1} \|Tu\|, \; \|u\| = \sqrt{\langle u, u \rangle},$$

$$\langle Tu, v \rangle = \langle u, T^* v \rangle \, for all \, u, \, v \in \mathcal{H}.$$

with operations

$$(T+S)u = Tu + Su,$$

$$(TS)u = T(Su), \; and \; (\lambda T)u = \lambda(Tu), \; \lambda \in \mathbb{C}.$$

Also, we can define

$$\mathfrak{K}(\mathcal{H}) = \{T \in \mathfrak{B}(\mathcal{H}) \, | \; T \;\; is \; compact \; operator\}.$$

Then $\mathfrak{K}(\mathcal{H})$ is a sub-C^*-algebra of $\mathfrak{B}(\mathcal{H})$ and C^*-ideal in $\mathfrak{B}(\mathcal{H})$. A representation of C^*-algebra A is a $*$-homomorphism $\varphi : A \to \mathfrak{B}(\mathcal{H})$.

Definition 6.1.1 A generalized odd elliptic operator over A is a triple $(\mathcal{H}, \psi, T)$ such that

1. $\mathcal{H}$ is a separable Hilbert space,
2. $\psi : A \to \mathfrak{B}(\mathcal{H})$ is a $*$-homomorphism,
3. $T \in \mathfrak{B}(\mathcal{H})$

and

$$T = T^*, \;\; \psi(a)T - T\psi(a) \in \mathfrak{K}(\mathcal{H}), \;\; \psi(a)(1-T^2) \in \mathfrak{K}(\mathcal{H})$$

for all $a \in A$.

We will denote the set of such triples by $\mathcal{E}^1(A)$. If $\varphi : A \to B$ is a $*$-homomorphism then there is an induced map

$$\varphi^* : \mathcal{E}^1(B) \to \mathcal{E}^1(A);$$

$$\varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T).$$

We will define odd K-homology of A by

Definition 6.1.2 Let $\xi = (\mathcal{H}, \psi, T)$, $\eta = (\mathcal{H}', \psi', T')$ be elements of $\mathcal{E}^1(A)$. We say that ξ is isomorphic to η, $\xi \simeq \eta$ if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ with commutativity in the diagrams

$$\begin{tikzcd} \mathcal{H} \arrow[r, "U"] \arrow[d, "T"'] & \mathcal{H}' \arrow[d, "T'"] \\ \mathcal{H} \arrow[r, "U"'] & \mathcal{H}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[r, "U"] \arrow[d, "\psi(a)"'] & \mathcal{H}' \arrow[d, "\psi(a)'"] \\ \mathcal{H} \arrow[r, "U"'] & \mathcal{H}' \end{tikzcd}$$

for all $a \in A$

Definition 6.1.3 We say that $\xi = (\mathcal{H}, \psi, T)$, $\eta = (\mathcal{H}', \psi', T') \in \mathcal{E}^1(A)$ are strictly homotopic if there exist a continuous function $[0,1] \to \mathfrak{B}(\mathcal{H})$, $t \mapsto T_t$ such that

1. $T_0 = T$,
2. for all $t \in [0,1]$, $(\mathcal{H}, \psi, T_t) \in \mathcal{E}^1$,

3. $(\mathcal{H},\psi,T_1)\simeq(\mathcal{H}',\psi',T')$.

Definition 6.1.4 We say that a generalized odd elliptic operator $(\mathcal{H},\psi,T)\in\mathcal{E}^1(A)$ is degenerate if and only if

$$\psi(a)T-T\psi(a)=0,\quad \psi(a)(I-T^2)=0,\quad for\ all\ a\in A.$$

Definition 6.1.5 We say that $\xi=(\mathcal{H},\psi,T)$, $\eta=(\mathcal{H}',\psi',T')\in\mathcal{E}^1(A)$ are homotopic, $\xi\simeq\eta$, if and only if there exists degenerate generalized elliptic operators $\tilde{\xi}$, $\tilde{\eta}$ with $\xi\oplus\tilde{\xi}$ strictly homotopic to $\eta\oplus\tilde{\eta}$.

Definition 6.1.6 The odd K-homology of a C^*-algebra A is defined as the group of homotopy classes of generalized odd elliptic operators,

$$K^1(A):=\mathcal{E}^1(A)/\sim.$$

It is an abelian group with respect to

$$(\mathcal{H},\psi,T)+(\mathcal{H}',\psi',T')=(\mathcal{H}\oplus\mathcal{H}',\psi\oplus\psi',T\oplus T')$$

with inverse defined by

$$-(\mathcal{H},\psi,T)=(\mathcal{H},\psi,-T).$$

If $\varphi:A\to B$ is a $*$-homomorphism, then there is an induced map

$$\varphi^*:K^1(B)\to K^1(A),\quad \varphi^*(\mathcal{H},\psi,T)=(\mathcal{H},\psi\circ\varphi,T).$$

Definition 6.1.7 A generalized even operator over A is a triple $(\mathcal{H},\psi,T)$ such that

1. $\mathcal{H}$ is a separable Hilbert space,

2. $\psi:A\to\mathfrak{B}(\mathcal{H})$ is a $*$-homomorphism,

3. $T\in\mathfrak{B}(\mathcal{H})$

and

$$\psi(a)T - T\psi(a) \in \mathfrak{K}(\mathcal{H}),\ \psi(a)(1-TT^*) \in \mathfrak{K}(\mathcal{H}),\ \psi(a)(1-T^*T) \in \mathfrak{K}(\mathcal{H})$$

for all $a \in A$.

We will denote the set of such triples by $\mathcal{E}^0(A)$

Definition 6.1.8 Even K-homology of a C^*-algebra A is defined as the group of homotopy classes of generalized even elliptic operators,

$$K^0(A) := \mathcal{E}^0(A)/\sim.$$

It is an abelian group with respect to

$$(\mathcal{H},\psi,T) + (\mathcal{H}',\psi',T') = (\mathcal{H}\oplus\mathcal{H}',\psi\oplus\psi',T\oplus T')$$

with inverse defined by

$$-(\mathcal{H},\psi,T) = (\mathcal{H},\psi,-T).$$

If $\varphi : A \to B$ is a $*$-homomorphism, then there is an induced map

$$\varphi^* : K^0(B) \to K^0(A), \quad \varphi^*(\mathcal{H},\psi,T) = (\mathcal{H},\psi\circ\varphi,T).$$

6.1.1 Equivariant K-homology

Let G be a locally compact Hausdorff second countable group, and $\mathcal{H}$ a separable Hilbert space. Denote the set of unitary operators on $\mathcal{H}$ by

$$\mathcal{U}(\mathcal{H}) := \{U \in \mathfrak{B}(\mathcal{H}) | UU^* = U^*U = I\}.$$

Definition 6.1.9 A unitary representation of G is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ such that for each $v \in \mathcal{H}$ the map $G \to \mathcal{H}$, $g \mapsto \pi(g)v$ is a continuous map from G to $\mathcal{H}$.

Definition 6.1.10 A G-C^*-algebra is a C^*-algebra A with a given continuous action

$$G \times A \to A$$

by automorphisms.

Example 6.1.11 Let X be a locally compact G-space. Then G acts on $C_0(X)$ by

$$(g\alpha)(x) = \alpha(g^{-1}x), \quad g \in G, \;\; \alpha \in C_0(X), \;\; x \in X.$$

This makes $C_0(X)$ a G-C^*-algebra.

Let A be a (separable) $G - C^*$-algebra.

Definition 6.1.12 A covariant representation of A is a triple $(\mathcal{H}, \psi, \pi)$ such that

1. $\mathcal{H}$ is a separable Hilbert space,
2. $\psi : A \to \mathfrak{B}(\mathcal{H})$ is a $*$-homomorphism,
3. $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G,
4. and $\psi(ga) = \pi(g)\psi(a)\pi(g^{-1})$ for all $g \in G$, $a \in A$.

Definition 6.1.13 Equivariant odd K-homology $K^1_G(A)$ of a G-C^*-algebra A is the group of homotopy classes of quadriples $(\mathcal{H}, \psi, T, \pi)$, where $(\mathcal{H}, \psi, \pi)$ is a covariant representation of A, and $T \in \mathfrak{B}(\mathcal{H})$ is such that

$$T = T^*, \;\; \pi(g)T - T\pi(g) \in \mathfrak{K}(\mathcal{H}), \;\; \psi(a)T - T\psi(a) \in \mathfrak{K}(\mathcal{H}), \;\; \psi(a)(1 - T^2) \in \mathfrak{K}(\mathcal{H})$$

for all $g \in G$, $a \in A$.

$$K^1_G(A) = \{(\mathcal{H}, \psi, \pi, T)\}/\sim.$$

Definition 6.1.14 Equivariant even K-homology $K^0_G(A)$ of a G-C^*-algebra A is the group of homotopy classes of quadriples $(\mathcal{H}, \psi, T, \pi)$, where $(\mathcal{H}, \psi, \pi)$ is a covariant representation of A, and $T \in \mathfrak{B}(\mathcal{H})$ is such that

$$\pi(g)T - T\pi(g) \in \mathfrak{K}(\mathcal{H}),\ \ \psi(a)T - T\psi(a) \in \mathfrak{K}(\mathcal{H}),\ \ \psi(a)(1 - T^*T) \in \mathfrak{K}(\mathcal{H}),$$

$$\psi(a)(1 - TT^*) \in \mathfrak{K}(\mathcal{H})$$

for all $g \in G$, $a \in A$.

$$K_G^0(A) = \{(\mathcal{H}, \psi, \pi, T)\}/\sim .$$

If A, B are G-C^*-algebras, and $\varphi : A \to B$ is a G-equivariant $*$-homomorphism, then

$$\varphi^* : \mathcal{E}_G^j(B) \to \mathcal{E}_G^j(A)$$

for $j = 0, 1$ is given by

$$\varphi^*(\mathcal{H}, \psi, \pi, T) \mapsto (\mathcal{H}, \psi \circ \varphi, \pi, T).$$

Addition in $K_G^j(A)$ is direct sum

$$(\mathcal{H}, \psi, \pi, T) + (\mathcal{H}', \psi', \pi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', \pi \oplus \pi', T \oplus T'),$$

and the inverse is

$$-(\mathcal{H}, \psi, \pi, T) = (\mathcal{H}, \psi, \pi, -T).$$

For more details see [6] and [27].

6.2 KK-theory for $GL(1)$

We want to describe base change as a map of the locally compact Hausdorff spaces

$$BC : A_1^t(F) \to A_1^t(E),$$

where $A_1^t(F)$ denotes $Irr^t GL(1, F)$.

6.2.1 K-theory

By describing the functorial base change map $K_j(BC)$ we want to study the effect of the base change on the K-theory groups. Let $\chi = |x|_F^s \chi_0$ be a character of $F^\times$, where χ_0 is the restriction of χ to $\mathcal{U}_F$. Put $\chi = z^{v_F(x)}\chi_0$, the character χ_0 is trivial on some $\mathcal{U}_F^m$ if it is a character of $\mathcal{U}_F$. We define the conductor of χ, denoted by $c(\chi)$ to be the least m such that $\chi_0 = \chi\,|_{\mathcal{U}_F}$ is trivial on $\mathcal{U}_F^m$. We have $F^\times \cong \langle \varpi_F \rangle \times \mathcal{U}_F$ where ϖ_F is a uniformizer. It follows that $A_1^t(F) \cong \mathbb{T} \times \widehat{\mathcal{U}_F}$.

If E/F be a finite Galois extension and $\mathcal{W}_E \hookrightarrow \mathcal{W}_F$ denote the inclusion of Weil groups, then the Langlands functoriality predicts the existence a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_1(F) & \longrightarrow & \mathcal{A}_1^t(F) \\ {\scriptstyle Res_{E/F}}\downarrow & & \downarrow{\scriptstyle BC_{E/F}} \\ \mathcal{G}_1(E) & \longrightarrow & \mathcal{A}_1^t(E) \end{array}$$

where $\mathcal{G}_1(F)$(resp. $\mathcal{G}_1(E)$) is the group of characters of $\mathcal{W}_F$(resp. $\mathcal{W}_E$) and BC is the base change map. On the admissible side base change is given by

$$\mathcal{A}_1^t(F) \to \mathcal{A}_1^t(E), \quad \chi_F \mapsto \chi_F \circ N_{F/E}.$$

Mendes and Plymen in [34] showed that the unitary dual of $GL(1)$ is a countable disjoint union of circles and so has the structure of a locally compact Hausdorff space. The base change map

$$BC : \bigsqcup \mathbb{T}_\chi \to \bigsqcup \mathbb{T}_\eta$$

with $\chi \in \widehat{\mathcal{U}_F}$, $\eta \in \widehat{\mathcal{U}_E}$, is a proper map. Each K-theory group is a countably generated free abelian group:

$$K^j(\mathcal{A}_1^t(F)) \cong \bigoplus \mathbb{Z}_\chi, \;\; K^j(\mathcal{A}_1^t(E)) \cong \bigoplus \mathbb{Z}_\eta$$

with $\chi \in \widehat{\mathcal{U}_F}$, $\eta \in \widehat{\mathcal{U}_E}$, $j = 0, 1$, where $\mathbb{Z}_\chi$ and $\mathbb{Z}_\eta$ denote a copy of $\mathbb{Z}$. There is a functorial map at the level of K-theory groups

$$K^j(BC) : \bigoplus \mathbb{Z}_\eta \to \bigoplus \mathbb{Z}_\chi,$$

base change selects among the character of $\eta \in \widehat{\mathcal{U}_E}$ those of the form $\chi_{_E} = \chi_{_F} \circ N_{E/F}$, where $\chi_{_F}$ is a character of $\widehat{\mathcal{U}_F}$.

6.2.2 K-homology for $GL(1)$

Now if we return to the KK-theory, we can construct the K-homology $K_j^G(\beta G)$ as follows:

Theorem 6.2.1 *Let $G = GL(1,F)$, $\beta G(1,F) = \mathbb{R}$ is the tree of $GL(1,F)$. Then*

$$K_j^G \beta G(F) = \mathcal{E}_G^1(\mathbb{R})/\sim .$$

PROOF: The group G acts on $\beta G(1,F) = \mathbb{R}$ by

$$G \times \beta G \to \beta G$$
$$x.t = t + val_F(x).$$

Now, we know that $F^\times \cong \mathbb{Z} \times \mathcal{U}_F$, let $A = C_0(\mathbb{R})$ and let $\mathcal{H} = L^2(\mathbb{R})$. We can define

$$\psi : A \to \mathfrak{B}(\mathcal{H}), \;\; \psi(\alpha)u = \alpha u,$$

$$\alpha u(t) = \alpha(t)u(t), \;\; \alpha \in C_0(\mathbb{R}), \;\; u \in L^2(\mathbb{R}), \, t \in \mathbb{R}.$$

Also, the representation

$$\pi : G \to \mathcal{U}(L^2(\mathbb{R}))$$

is defined by

$$(\pi(g)u)(t) := u(t-g).$$

As an operator on $L^2(\mathbb{R})$ we take $-i\frac{d}{dx}$. It is not a bounded operator on $L^2(\mathbb{R})$, but we can"normalize" it to obtain a bounded operator T. Since $-i\frac{d}{dx}$ is self-adjoint there is functional calculus, and T can be taken to be the function $\frac{x}{\sqrt{1+x^2}}$ applied to $-i\frac{d}{dx}$

$$T := (\frac{x}{\sqrt{1+x^2}})(-i\frac{d}{dx}).$$

Equivalently, T can be constructed using the Fourier transform. Let S_x be the operator of "multiplication by x" $(S_x f)(x) = xf(x)$. Taking the Fourier transform converts $-i\frac{d}{dx}$ to S_x i.e. there is a commutativity in the diagram

$$\begin{tikzcd} \mathrm{L}^2(\mathbb{R}) \arrow[r, "\mathcal{F}"] \arrow[d, "-i\frac{d}{dx}"'] & \mathrm{L}^2(\mathbb{R}) \arrow[d, "S_x"] \\ \mathrm{L}^2(\mathbb{R}) \arrow[r, "\mathcal{F}"'] & \mathrm{L}^2(\mathbb{R}) \end{tikzcd}$$

where $\mathcal{F}$ denotes the Fourier transform. Let $S_{\frac{x}{\sqrt{1+x^2}}}$ be the operator of multiplication by $\frac{x}{\sqrt{1+x^2}}$. Then $(S_{\frac{x}{\sqrt{1+x^2}}} f)(x) = \frac{x}{\sqrt{1+x^2}} f(x)$, and $S_{\frac{x}{\sqrt{1+x^2}}}$ is a bounded operator

$$S_{\frac{x}{\sqrt{1+x^2}}} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Now, T is the unique bounded operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that there is commutativity in the diagram

$$\begin{tikzcd} \mathrm{L}^2(\mathbb{R}) \arrow[r, "\mathcal{F}"] \arrow[d, "T"'] & \mathrm{L}^2(\mathbb{R}) \arrow[d, "S_{\frac{x}{\sqrt{1+x^2}}}"] \\ \mathrm{L}^2(\mathbb{R}) \arrow[r, "\mathcal{F}"'] & \mathrm{L}^2(\mathbb{R}) \end{tikzcd}$$

Then $(L^2(\mathbb{R}), \psi, \pi, T) \in \mathcal{E}_G^1(\mathbb{R})$. ❒

Let

$$\eth = (L^2(\mathbb{R}), \psi, \pi, -i\frac{d}{dx}),$$

then we have

$$[\eth] \in K_G^1(\mathbb{R}).$$

This means the Dirac operator in 1-dimension determines an element in K_G^1, then we have this theorem:

Theorem 6.2.2 *The K-homology $K_G^1 \simeq [\eth] \otimes \mathfrak{R}(U)$, where $\mathfrak{R}(U)$ is the representation ring* *of $\mathcal{U}$ (free abelian group on the set of characters) and the assembly map μ works as follows:*

$$K_G^1(\mathbb{R}) \xrightarrow{\mu} K_1C^*GL(1)$$

$$[\eth] \otimes v \longmapsto [v].$$

PROOF: There are finite elements $[\eth]$ but we need to make sure there are countably many in the K-theory side, so to get that condition we can easily get the countably many by tensoring it with the representation ring of the set of characters, $\mathfrak{R}(\mathcal{U})$. Let $\gamma = [\eth] \otimes v$ be the generator of K-homology group. Now, the assembly map μ can be define as follows

$$\gamma = [\eth] \otimes v \longmapsto [v].$$

It takes the generator of the homology group to one generator of K-theory group. ❐

Theorem 6.2.3 *The base change map on the chamber homology level works as follows:*

$$\gamma \longmapsto f \cdot \gamma,$$

where γ is the homology group generator and f is the degree of the residue field.

$$\begin{array}{ccc} K_G^1\mathbb{R} & \longrightarrow & K_1C^*E^\times \simeq K^1\widehat{GL(1,E)} \\ \downarrow & & \downarrow \\ K_G^1\mathbb{R} & \longrightarrow & K_1C^*F^\times \simeq K^1\widehat{GL(1,F)} \end{array}$$

6.2.3 Chamber homology for $GL(1)$

We are going to introduce the way to compute the chamber homology group for $GL(1,F)$. Let $G = GL(1,F)$ and $\beta^1 G = \mathbb{R}$ is the tree of G, G acts on $\mathbb{R}$ by $x \cdot t = t + val(x)$ with $t \in G$ and $x \in \mathbb{R}$, but the quotient $\mathbb{R}/G$ is a loop [45].

Now, since $x \cdot t = t + val(x)$, then

$$x \cdot t = t$$

$$\Longleftrightarrow \{val(x) = 0,\ x \in F^{\times}\}$$

$$\Longleftrightarrow x \in \mathcal{U}_F.$$

Let $\mathfrak{R}(\mathcal{U}_F)$ be the representations ring of $\mathcal{U}_F$(free abelian group on the set of characters). Now we have the following short chain complex

$$0 \longleftarrow \mathfrak{R}(\mathcal{U}_F) \xleftarrow{\ \delta\ } \mathfrak{R}(\mathcal{U}_F) \longleftarrow 0$$

$$0 \longleftarrow\!\shortmid\ v$$

So our homology groups are $H_0 = \mathfrak{R}(\mathcal{U}_F)$ and $H_1 = \mathfrak{R}(\mathcal{U}_F)$, this means $H_0 \cong H_1$. An element in $\mathfrak{R}(\mathcal{U}_F)$ is a linear combination

$$n_1\chi_1 + \ldots + n_r\chi_r,$$

where $n_j \in \mathbb{Z}$ and $\chi_j \in \widehat{\mathcal{U}_F}$. This mean every character is a generator of the homology group.

Thus the base change map in the homology level works by taking the generator of the group $H_1(\beta^1 GL(1,E))$ to the multiplication of the generator of $H_1(\beta^1 GL(1,F))$ by the residue field degree.

$$\begin{array}{ccc} H_0(\beta^1 GL(1,E)) & \longrightarrow & K_0 C_r^* GL(1,E) \\ \downarrow & & \downarrow \\ H_0(\beta^1 GL(1,F)) & \longrightarrow & K_0 C_r^* GL(1,F) \end{array} \qquad \begin{array}{ccc} \gamma_0 & \xrightarrow{\mu_0} & \alpha_0 \\ {\scriptstyle BC}\downarrow & & \downarrow{\scriptstyle K_0(BC)} \\ \gamma_0 & \xrightarrow{\mu_0} & \alpha_0 \end{array}$$

$$\begin{array}{ccc} H_1(\beta^1 GL(1,E)) & \longrightarrow & K_1 C_r^* GL(1,E) \\ \downarrow & & \downarrow \\ H_1(\beta^1 GL(1,F)) & \longrightarrow & K_1 C_r^* GL(1,F) \end{array} \qquad \begin{array}{ccc} \gamma_1 & \xrightarrow{\mu_1} & \alpha_1 \\ {\scriptstyle BC}\downarrow & & \downarrow{\scriptstyle K_1(BC)} \\ f \cdot \gamma_1 & \xrightarrow{\mu_1} & f \cdot \alpha_1 \end{array}$$

6.3 Conclusion

We had the advantage of using a basic result of K-homology to compute the K-homology group for $GL(1)$ and construct the Baum-Connes correspondence by using the fact that the K-homology group is isomorphic to the chamber homology group. We also computed the chamber homology group for $GL(1)$. The groups H_0 and H_1 are identical. We also applied the base change map and described its effect on both sides.

Chapter 7

Base change and K-theory for $GL(2)$

Let $G = GL(n,F)$ and let C_r^*G denote the reduced C^*-algebra of G. According to the Baum-Connes correspondence, we have a canonical isomorphism [4]

$$\mu_F : K_j^{top}\beta^1 G \to K_j C_r^* G,$$

where $\beta^1 G$ denotes the enlarged building of G. In noncommutative geometry, isomorphisms of C^*-algebras are too restrictive to provide a good notion of isomorphisms of noncommutative spaces, and the correct notion is provided by strong Morita equivalence of C^*-algebras. The noncommutative C^*-algebra C_r^*G is strongly Morita equivalent to the commutative C^*-algebra $C_0(Irr^t G)$ where $Irr^t G$ denotes the tempered dual of G [39]. Consequently, we have

$$K_j C_r^* G \cong K^j Irr^t\, G$$

and this leads to the following formulation of the Baum-Connes correspondence:

$$K_j^{top}\beta^1 G \cong K^j Irr^t G.$$

This in turn leads to the following diagram

$$\begin{tikzcd} K_j^{top}(\beta^1 G(E)) \arrow[r, "\mu_E"] \arrow[d] & K^j(Irr^t G(E)) \arrow[d, "K^j(BC_{E/F})"] \\ K_j^{top}(\beta^1 G(F)) \arrow[r, "\mu_F"'] & K^j(Irr^t G(F)) \end{tikzcd}$$

where the left-hand vertical map is the unique map which makes the diagram commutative. The rest of this chapter will be concerned with presenting an explicit construction of the local Langlands correspondence between so-called *cuspidal* representations of $GL_2(F)$ and certain 2-dimensional representations of $\mathcal{W}_F$, where the residue characteristic $char \neq 2$. We will focus on the classification and the construction of the cuspidal representations for $GL_2(F)$, it was originally treated by [29] and [30].

7.1 Chamber homology for $GL(2)$

Consider the pair (L,κ) where L is a Levi subgroup of a parabolic subgroup of G, and κ is an irreducible cuspidal representation of L. Two pairs $(L_1,\kappa_1),(L_2,\kappa_2)$ are called inertially equivalent if there exist $g\in G$ and an unramified character χ of L_2 such that

$$L_2 = L_1^g \quad and \quad \kappa_1^g = \kappa_2 \otimes \chi,$$

where $L_1^g := g^{-1}L_1 g$ and $\kappa_1^g(x) = \kappa_1(gxg^{-1})$ for all $x \in L_1^g$. Let $[L,\kappa]_G$ be the inertial equivalence class of the pair (L,κ) and let $\mathfrak{B}(G)$ be the set of all inertial equivalence classes. This set is called the *Bernstein spectrum* of G.

Definition 7.1.1 Let $s \in \mathfrak{B}(G)$, an s-type is a pair (J,σ) consisting of a compact open subgroup J of G and an irreducible smooth representation σ of J such that for any irreducible smooth representation π of G, the restriction of π to J contains σ if and only if π is an object of $\mathfrak{R}^s(G)$, [14]. It has been proved by [13] and [15] that there exists an s-types for each point $s \in \mathfrak{B}(G)$.

Now, let $\mathcal{O}_F$ denote the ring of integers of F, $\varpi \in F$ be a uniformizer, and $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_F$. Also, let

$$\Pi = \Pi_n = \begin{pmatrix} 0 & I_{n-1} \\ \varpi_F & 0 \end{pmatrix}, \ and \ s_i = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix},$$

for every $i \in \{1,...,n-1\}$, and $s_0 = \Pi s_1 \Pi^{-1}$ denotes the standard involutions in G. The finite Weyl group is $\mathbb{W}_0 = \langle s_1, s_2, ..., s_{n-1} \rangle$, and the affine Weyl group defined as follows $\mathbb{W} = \langle s_0, s_1, ..., s_{n-1} \rangle$. The extended affine Weyl group is denoted by $\overline{\mathbb{W}} = \mathbb{W} \rtimes \langle \Pi \rangle$. Its clear that $\overline{\mathbb{W}} \cap GL(n, \mathcal{O}_F) = \mathbb{W}_0$.

The standard Iwahori subgroup is

$$I = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F & \dots & \mathcal{O}_F \\ \mathfrak{p}_F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O}_F \\ \mathfrak{p}_F & \dots & \mathfrak{p}_F & \mathcal{O}_F^\times \end{pmatrix}.$$

Let Σ be the apartment attached to the diagonal torus and let Δ be the unique chamber in this apartment which is stabilized by $\langle \Pi \rangle I$. Let J_i be the maximal standard parahoric subgroups of G,

$$J_i = I \langle s_0, s_1, ... s_{i-1}, s_{i+1}, ..., s_{n-1} \rangle I$$

where $J_0 = GL(n, \mathcal{O}_F)$. We see that J_i are the stabilizers of the vertices, the stabilizer of the facets of dimension $n-1$ of Δ are $K_0, K_1, ..., K_{n-1}$, where $K_i = I\langle s_i \rangle I$. The enlarged building $\beta^1 G$ is labelled, this means there exists a simplicial map

$$\mathfrak{I} : \beta^1 G \to \Delta,$$

this map is dimensions preserver. This labelling is unique and it allows us to fix an orientation of the simplices.

The chamber homology groups are obtained by totalizing the bicomplex:

$$\begin{array}{ccccccc}
0 \longleftarrow & \mathfrak{R}(J_0)\oplus\mathfrak{R}(J_1)\oplus\ldots\oplus\mathfrak{R}(J_{n-1}) & \longleftarrow & \cdots & \longleftarrow & \mathfrak{R}(K_0)\oplus\mathfrak{R}(K_1)\oplus\ldots\oplus\mathfrak{R}(K_{n-1}) & \longleftarrow \mathfrak{R}(I) \\
 & \downarrow & & \downarrow & & \downarrow & \downarrow \\
0 \longleftarrow & \mathfrak{R}(J_0)\oplus\mathfrak{R}(J_1)\oplus\ldots\oplus\mathfrak{R}(J_{n-1}) & \longleftarrow & \cdots & \longleftarrow & \mathfrak{R}(K_0)\oplus\mathfrak{R}(K_1)\oplus\ldots\oplus\mathfrak{R}(K_{n-1}) & \longleftarrow \mathfrak{R}(I)
\end{array}$$

the vertical maps are given by $1-\sqsupset\Pi$.

We assume that $\mathcal{C}=\mathfrak{R}(J_0)\oplus\mathfrak{R}(J_1)\oplus\ldots\oplus\mathfrak{R}(J_{n-1})$, $\mathcal{C}'=\mathfrak{R}(K_0)\oplus\mathfrak{R}(K_1)\oplus\ldots\oplus\mathfrak{R}(K_{n-1})$ and $\mathcal{C}''=\mathfrak{R}(I)$. By totalizing the above bicomplex, we obtain this chain complex

$$0 \longleftarrow \mathcal{C} \longleftarrow \cdots \longleftarrow \mathcal{C}'_{i-1}\oplus\mathcal{C}'_i \longleftarrow \mathcal{C}'_i\oplus\mathcal{C}'_{i+1} \longleftarrow \cdots \longleftarrow \mathcal{C}'' \longleftarrow 0\,.$$

Definition 7.1.2 [5] The homology groups of this totalized complex are the chamber homology groups.

Now, for each point $s\in\mathfrak{B}(G)$ let

$$\mathcal{C}(s)=\mathfrak{R}(J_0(s))\oplus\mathfrak{R}(J_1(s))\oplus\ldots\oplus\mathfrak{R}(J_{n-1}(s)),$$

$$\mathcal{C}'(s)=\mathfrak{R}(K_0(s))\oplus\mathfrak{R}(K_1(s))\oplus\ldots\oplus\mathfrak{R}(K_{n-1}(s)),$$

and

$$\mathcal{C}''(s)=\mathfrak{R}(I(s)).$$

We associate a sub-bicomplex

$$\begin{array}{ccccccc}
0 \longleftarrow & \mathcal{C}(s) & \longleftarrow \cdots \longleftarrow & \mathcal{C}'(s) & \longleftarrow & \mathcal{C}''(s) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
0 \longleftarrow & \mathcal{C}(s) & \longleftarrow \cdots \longleftarrow & \mathcal{C}'(s) & \longleftarrow & \mathcal{C}''(s)
\end{array}$$

in which each vertical map is 0. The homology groups of the chain complex

$$0 \longleftarrow \mathcal{C}(s) \longleftarrow \cdots \longleftarrow \mathcal{C}''(s) \longleftarrow 0$$

is denoted by $h_j(s)$, we call this complex the *little complex*. When we totalize the associated bicomplex, we get the chain complex

$$0 \longleftarrow \mathcal{C}(s) \longleftarrow \cdots \longleftarrow \mathcal{C}'_{i-1}(s) \oplus \mathcal{C}'_i(s) \longleftarrow \mathcal{C}'_i(s) \oplus \mathcal{C}'_{i+1}(s) \longleftarrow \cdots \longleftarrow \mathcal{C}''(s) \longleftarrow 0 .$$

Theorem 7.1.3 *[2] The homology groups $H_j(s)$ of this complex are given by*

$$H_0(s) = h_0(s), \;\; H_n(s) = h_{n-1}(s)$$

$$H_{i+1}(s) = h_i(s) \oplus h_{i+1}(s), \quad 0 \leq i \leq n-2$$

$$H_{ev}(s) = h_0(s) \oplus h_1(s) \oplus ... \oplus h_{n-1}(s) = H_{odd}(s)$$

The even (resp. odd) chamber homology is precisely the total homology of the little complex.

Now if we back to our case, let F be non-archimedean p-adic local field, $G = GL(2,F)$ and $\beta^1 GL(2)$ be the enlarged building of G. The enlarged building of G can be defined as

$$\beta^1 G = \beta SL(2) \times \mathbb{R}$$

with an action

$$GL(2) \times \beta^1 GL(2) \longrightarrow \beta^1 GL(2)$$

$$GL(2) \times \beta SL(2) \times \mathbb{R} \longrightarrow \beta^1 GL(2)$$

$$(x,y,t) \longmapsto (xy, t + val_F(detx)).$$

The enlarged building $\beta^1 GL(2)$ has the structure of polysimplicial complex, but we have $\beta^1 G = \beta SL(2) \times \mathbb{R}$. In Chapter 4 we saw that the action of $SL(2)$ on its tree could be extended to an action of $GL(2)$. We will investigate the chamber homology $H_j(\beta^1 GL(2))$ of G acting on its enlarged building properly. The quotient $\beta^1 GL(2)/GL(2)$ is a Mobius band (an identification space of a chamber) which is a compact space.

Let

$$\Pi = \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad and \quad s_0 = \begin{pmatrix} 0 & \varpi_F^{-1} \\ \varpi_F & 0 \end{pmatrix}$$

be the standard involutions in $GL(2)$. Restricted to the affine line $\mathbb{R}$ in the enlarged building $\beta^1 GL(2) = \beta SL(2) \times \mathbb{R}$, Π sends t to $t+1$. It is a bit hard to calculate the chamber homology group for $GL(2)$ a Mobius band but its not that difficult to construct a complex to compute the chamber homology of $GL(2)$ if we restrict to the original quotient space before taking the real line copy which is an edge of the tree of $SL(2)$.

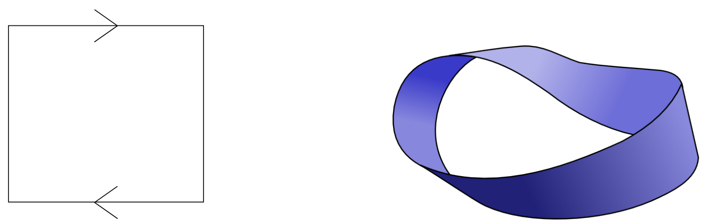
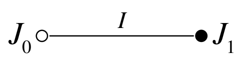


Figure 7.1: The identification square of Mobius band and the Mobius band

Let I, J_0 and J_1 are the stabilizer groups of the edge and the two vertices in the above chamber which they have been defined in Chapter 5.



7.1.1 The trivial type$(I, \mathbb{1}_I)$

Let

$$T = \begin{pmatrix} F^\times & 0 \\ 0 & F^\times \end{pmatrix}$$

be the diagonal subgroup of $G = GL(2,F)$ and let $\mathbb{1}$ be the trivial representation of T. Then the pair $(T, \mathbb{1})$ is a cuspidal pair. Now, let us discuss the special case when $s = [T, \mathbb{1}]_G$, the s-type in this case will be the trivial type $(I, \mathbb{1})$. We will construct the little complex created by $(I, \mathbb{1}_I)$.

Theorem 7.1.4 *Let I be the Iwahori subgroup of $GL(2)$, and let St_2 be the Steinberg representation of $GL(2,F)$, and let χ_1, χ_2, χ be unramified unitary characters. Then the unramified unitary representation of $GL(2)$ can be written as follows:*

1. $Ind_B^G(\chi_1 \times \chi_2) \simeq Ind_B^G(\chi_2 \times \chi_1)$.
2. $\chi \otimes St_2$.

PROOF: see [39] ❒

We have $Ind_I^{J_0} \mathbb{1}_I = \mathbb{1}_{J_0} \oplus St_2^{J_0}$ and $Ind_I^{J_1} \mathbb{1}_I = \mathbb{1}_{J_1} \oplus St_2^{J_1}$. Then the little complex determined by this type is

$$\begin{array}{ccccc} 0 & \longleftarrow & \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) & \longleftarrow & \mathfrak{R}(I) \\ & & \big\downarrow & & \big\downarrow \\ 0 & \longleftarrow & \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) & \longleftarrow & \mathfrak{R}(I) \end{array}$$

where $\mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)$ is the free abelian group on the two elements

$$(\mathbb{1}_{J_0}, \mathbb{1}_{J_1}),\ (St_2^{J_0}, St_2^{J_1}) \in \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)$$

and $\mathfrak{R}(I)$ is the free abelian group on the single generator

$$\mathbb{1}_I \in \mathfrak{R}(I).$$

The above bicomplex chain implies that we need to consider only invariant elements. Therefore, we need to restrict to the invariant elements so we have the following:

$$\mathbb{1}_{J_0} \oplus St_2^{J_0} \sim 0 \ \ i.e.\ \ \mathbb{1}_{J_0} \sim -St_2^{J_0}$$

$$\mathbb{1}_{J_1} \oplus St_2^{J_1} \sim 0 \ \ i.e.\ \ \mathbb{1}_{J_1} \sim -St_2^{J_1}$$

This means we have one element

$$(\mathbb{1}_{J_0}, \mathbb{1}_{J_1}) \sim -(St_2^{J_0}, St_2^{J_1}).$$

By totalizing the above little complex we get

$$0 \longleftarrow \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) \longleftarrow \mathfrak{R}(I) \longleftarrow 0 ,$$

hence by theorem 7.1.3 we have

$$H_0 = h_0,\ H_1 = h_0 + h_1,\ H_2 = h_1.$$

Now, $h_0 = \mathbb{Z}$, $h_1 = \mathbb{Z}$, therefore $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}^2$ and $H_2 = \mathbb{Z}$. i.e.

$$H_{even} = \mathbb{Z}^2 = H_{odd}.$$

Now, let λ be a unitary character of $GL(1,F) \simeq F^{\times}$, and let $\tau = \lambda \circ \det : I \to \mathcal{U}(\mathbb{1})$. Theorem 9 in [2] shows that the totalised little complex created by the (I,τ) is isomorphic to the totalized little complex created by the trivial type $(I, \mathbb{1}_I)$. Therefore, the homology groups $H_{even} = \mathbb{Z}^2 = H_{odd}$

7.1.2 The irreducible components of reducible principal series

Let $s = [T,\sigma]_G$. Consider the s-type (J,τ) where J is compact open subgroup of G and τ is an irreducible smooth representation of J.

Lemma 7.1.5 *Let $\chi = Ind_J^I \tau$. Then χ is irreducible.*

PROOF: See [2] ❐

Lemma 7.1.6 *We have*

$$Ind_I^{J_0} \chi = \alpha_0 \oplus \gamma_0$$

$$Ind_I^{J_1} \chi = \alpha_1 \oplus \gamma_1.$$

Let $\mathfrak{R}(J_0(\tau)) \oplus \mathfrak{R}(J_1(\tau))$ be the free abelian group on the element $(\alpha_0,\alpha_1) \sim (\gamma_0,\gamma_1) \in \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)$ and let $\mathfrak{R}(I(\tau))$ be the free abelian group on the element χ. The little complex is then

$$0 \longleftarrow \mathfrak{R}(J_0(\tau)) \oplus \mathfrak{R}(J_1(\tau)) \longleftarrow \mathfrak{R}(I(\tau)) \longleftarrow 0 .$$

Hence

$$H_0(\tau) = h_0(\tau) = \mathbb{Z}, \quad H_1(\tau) = h_0(\tau) + h_1(\tau) = \mathbb{Z}^2, \; H_2(\tau) = h_1(\tau) = \mathbb{Z},$$

i.e. $H_{even} = \mathbb{Z}^2 = H_{odd}$.

7.1.3 Cuspidal representations

Let $s = [G, \pi]$, where $G = GL(2,F)$ and π is an irreducible cuspidal representation of G. Let (J, σ) be the maximal simple type contained in π [13]. It follows that $J(s) = GL(2, \mathcal{O}_F) = J_0$ [2]. By lemma 7.1.5, it follows that $\lambda = Ind_J^{J_0} \sigma$ is irreducible. Now, the pair (J_0, λ) is an s-type. The restriction of a smooth irreducible representation ρ of G to J_0 contains λ if and only if

$$\rho \cong \pi \otimes \chi \circ \det$$

where χ is an unramified character of $F^\times$, i.e. π contains λ with multiplicity 1. Therefore, the representation λ is the unique smooth irreducible representation τ of J_0 such that (J_0, τ) is an s-type [36].

Now, the little complex determined by λ is

$$0 \longleftarrow \mathcal{C}(s) \longleftarrow 0$$

where $\mathcal{C}(s)$ is the free abelian group on the invariant 0-cycle τ. The total homology of the little complex is given by $h_0(s) = \mathbb{Z}$. Therefore,

$$H_{even} = \mathbb{Z} = H_{odd}.$$

7.1.4 The principal series

Let (J,τ) be s-type, $J_0 = GL(2,\mathcal{O}_F)$. If $J \subset J_0$ then the only double J-coset representative which G-intertwines τ is $\mathbb{1}_G$. Therefore,

$$Ind_J^I \tau \text{ is irreducible}$$

$$Ind_J^{J_i} \tau \text{ is irreducible.}$$

Now, let $\gamma = Ind_J^I \tau$ and $\sigma = Ind_J^{J_i} \tau$. Let $\mathcal{C}(\tau)$ be the free abelian group on the generator σ, and let $\mathcal{C}^{'}(\tau)$ be the free abelian group on the generator γ. The totalized little complex is

$$0 \longleftarrow \mathcal{C}(\tau) \longleftarrow \mathcal{C}^{'}(\tau) \longleftarrow 0\ .$$

Then

$$H_0 = \mathbb{Z},\ H_1 = \mathbb{Z}^2,\ H_2 = \mathbb{Z}\ \text{ and so }\ H_{even} = \mathbb{Z}^2 = H_{odd}.$$

Theorem 7.1.7

1. *The base change of a twist of Steinberg representation is again a twist of Steinberg.*
2. *The base change of a principal series representation is always a principal series.*
3. *The base change of a cuspidal representation will never be a twist of Steinberg representation. It is possible for the base change of a cuspidal representation to be principal series representation.*

PROOF:

Let $\mathcal{L}_F = \mathcal{W}_F \times SL(2,\mathbb{C})$ and $\mathcal{L}_E = \mathcal{W}_E \times SL(2,\mathbb{C})$ be the local Langlands groups and let the two L-parameters corresponding to these groups respectively be

$$\phi : \mathcal{W}_F \times SL(2,\mathbb{C}) \longrightarrow G^\vee = GL_2(\mathbb{C})\ , \quad \phi|_{\mathcal{W}_E} : \mathcal{W}_E \times SL(2,\mathbb{C}) \longrightarrow G^\vee = GL_2(\mathbb{C})\ .$$

1. Let $\phi_F = \psi \otimes St_2^F$, where $\psi \in \Psi^t(\mathcal{W}_F)$. Since the base change works by restricting the L-parameter to $\mathcal{W}_E$ and the restriction is apply only on the Weil group part,

therefore the base change of St_2^F is $BC(St_2^F) = St_2^E$ and hence the the base change works on the unitary twist of Steinberg as follows:

$$BC(\phi_F) = \phi_E,\ BC(\psi \otimes St_2^F) = BC(\psi) \otimes St_2^E = \psi \circ N_{E/F} \otimes St_2^E.$$

2. Let $\phi_F = (\psi_1 \otimes \mathbb{1}) \oplus (\psi_2 \otimes \mathbb{1})$ be the L-parameter, where $\psi_1,\ \psi_2 \in \Psi^t(\mathcal{W}_F)$ then the base change map works on the reducible principal series as follows:

$$BC(\phi_F) = \phi_E$$

$$BC(\psi_1 \otimes \mathbb{1} \oplus \psi_2 \otimes \mathbb{1}) = BC(\psi_1 \otimes \mathbb{1}) \oplus BC(\psi_2 \otimes \mathbb{1}) = (\psi_1 \circ N_{_{E/F}} \otimes \mathbb{1}) \oplus (\psi_2 \circ N_{_{E/F}} \otimes \mathbb{1}).$$

3. Let $\phi_F = \psi\sigma \otimes \mathbb{1}$, where σ is irreducible representation of $\mathcal{W}_F$ and $\psi \in \Psi^t(\mathcal{W}_F)$.

   (a) If the L-parameter ϕ_E remains irreducible after restriction, then this determines a cuspidal representation of $GL(2,E)$. Base change in this case will send one cuspidal representation of $GL(2,F)$ to a cuspidal representation of $GL(2,E)$. Therefore, the map BC works as follows:

   $$BC(\phi_F) = \phi_E$$

   $$BC(\psi\sigma \otimes \mathbb{1}) = BC(\psi)\sigma^* \otimes \mathbb{1} = \psi \circ N_{E/F}\sigma^* \otimes \mathbb{1}.$$

   (b) If the L-parameter ϕ_E is reducible after restriction, then this representation split into two one-dimensional representations say σ_1 and σ_2. This means that the restriction of the cuspidal representation is a principal series. Therefore, the map BC works as follows:

   $$BC(\phi_F) = \phi_E$$

   $$BC(\psi\sigma \otimes \mathbb{1}) = \psi \circ N_{E/F}(\sigma_1 \oplus \sigma_2) \otimes \mathbb{1}.$$

❒

In fact, if π is any irreducible admissible representation of $GL(2,F)$ then one can find an extension E/F such that $BC(\pi)$ is either unramified or Steinberg.

7.2 K-theory for GL(2)

Let F be a non-archimedean local field with characteristic 0 and $p \neq 2$. Such a field has a norm, denoted by mod_F [49]. The representations in $GL(2,F)$ can be view as a one of the following:

1. The irreducible admissible representations of G fall into three classes: principal series, twists of Steinberg and cuspidal.

2. The unramified representations of G are exactly the principal series representations coming from unramified characters. These are parameterized by (unordered) pairs of complex numbers.

Let E/F be a finite Galois extension, and let the corresponding Weil groups be denoted $\mathcal{W}_E, \mathcal{W}_F$. Let $\mathbb{T}$ denote the circle group

$$\mathbb{T} = \{z \in \mathbb{C} : \mid z \mid = 1\}$$

and let $\Psi^t(\mathcal{W}_F)$ denote the group of unramified unitary characters of $\mathcal{W}_F$. Then we have

$$\Psi^t(\mathcal{W}_F) \cong \mathbb{T}, \ \psi \mapsto \psi(\varpi)$$

where ϖ_F is a uniformizer in F.

Now, let $\mathcal{L}_F$ denote the local Langlands group: $\mathcal{L}_F := \mathcal{W}_F \times SL(2,\mathbb{C})$. A Langlands parameter (or L-parameter) is a continuous homomorphism

$$\phi : \mathcal{L}_F \to GL(2,\mathbb{C}),$$

($GL(2,\mathbb{C})$ is given the discrete topology) such that $\phi(\Phi_F)$ is semisimple, where Φ_F is a geometric Frobenius in $\mathcal{W}_F$. Two Langlands parameters are equivalent if they are conjugate under $GL(2,\mathbb{C})$. The set of equivalence classes of Langlands parameters is denoted by $\Phi(GL(2))$. Now the base change is defined by the restriction of L-parameter from $\mathcal{L}_F$ to $\mathcal{L}_E$. Consider first the single L-parameter $\phi = \rho \otimes \tau(j_1) \oplus \rho \otimes \tau(j_2)$. In this formula, ρ is an irreducible representation of $\mathcal{W}_F$, $\tau(j)$ is the j-dimensional complex representation of $SL(2,\mathbb{C})$. We define the compact orbit of ϕ as follows:

$$\mathfrak{O}^t(\phi) = \{\bigoplus_{r=1}^{2} \psi_r \otimes \rho \otimes \tau(j_r) : \ \psi_r \in \Psi^t(\mathcal{W}_F), \ 1 \leq r \leq 2\}/\sim,$$

where as before, $\sim$ denotes the equivalence relation of conjugacy in $GL(2,\mathbb{C})$. Each partition $j_1 + j_2 = 2$ determines an orbit. The disjoint union of these orbits, one of each partition of 2, creates a complex affine algebraic variety with finitely many irreducible components. This variety is smooth by [9].

Also, let $\mathcal{G}_2^0(F)$ be the set of equivalence classes of irreducible 2-dimensional smooth (complex) representations of $\mathcal{W}_F$. Let $\mathcal{A}_2^0(F)$ be the subset of $\mathcal{A}_2^t(F)$ consisting of equivalence classes of irreducible cuspidal representations of $GL(2,F)$. The local Langlands correspondence gives a bijection,

$$\tau : \mathcal{G}_2^0(F) \longrightarrow \mathcal{A}_2^0(F) \ .$$

We will use the local Langlands correspondence for $GL(2)$ [11], [24], [25] and [32]:

$$\pi_F : \Phi(GL(2)) \to Irr(GL(2)).$$

Lemmas 5.1.1 and 5.1.2 explain the formula of the base change. Now let $z_j = \psi_j(\varpi_F)$, we have the map:

$$\psi_1 \otimes \tau(j_1) \oplus \psi_2 \otimes \tau(j_2) \longmapsto (z_1, z_2).$$

This map gives a bijection

$$\mathfrak{O}^t(\phi) \longrightarrow sym^2(\mathbb{T}).$$

So we will write the L-parameter

$$\phi = \psi_1 \otimes \tau(j_1) \oplus \psi_2 \otimes \tau(j_2)$$

as

$$z_1.\tau(j_1) \oplus z_2.\tau(j_2).$$

After base change has been applied, this L-parameter becomes

$$z_1^f.\tau(j_1) \oplus z_2^f.\tau(j_2).$$

The Steinberg representation St_2 has L-parameter $1 \otimes \tau(2)$.

Theorem 7.2.1 *Let $\phi = 1 \otimes \tau(2)$ and let $\mathfrak{O}^t(\phi)$ be the compact orbit of ϕ. Then we have*

$$BC : \mathbb{T} \to \mathbb{T},\ z \mapsto z^f.$$

1. *This map has degree f, and so at the level of the K-theory group K^1, BC induces the map*
$$\mathbb{Z} \to \mathbb{Z},\ \alpha_1 \to f \cdot \alpha_1$$
of multiplication by the residue degree f, where α_1 denotes a generator of the group $K^1(\mathbb{T}) \cong \mathbb{Z}$.
2. *At the level of the K-theory group K^0, BC induces the identity map*
$$\mathbb{Z} \to \mathbb{Z},\ \alpha_0 \mapsto \alpha_0,$$
where α_0 denotes a generator of $K^0(\mathbb{T}) \cong \mathbb{Z}$.

PROOF: Since this map has degree f then 1 has been proved. Since α_0 is the trivial bundle of rank 1 over $\mathbb{T}$ then 2 has been showed. ❒

$$\begin{array}{ccc} \mathbb{1} \otimes \tau(2) & \longmapsto & St_2^F \\ \downarrow & & \downarrow \\ \mathbb{1} \otimes \tau(2) & \longmapsto & St_2^E \end{array}$$

We can generalize this in the following form:

$$\begin{array}{ccc} \sigma \otimes \tau(2) & \longmapsto & St_2^F \\ \downarrow & & \downarrow \\ \sigma \otimes \tau(2) & \longmapsto & St_2^E \end{array}$$

Next we define the L-parameter ϕ to be : $\phi = \rho \otimes 1 \oplus \rho \otimes 1$ where ρ is a unitary character of $\mathcal{W}_F$. The unitary characters of $\mathcal{W}_F$ factor through $F^\times$ and we have $F^\times \cong \langle \varpi_F \rangle \times \mathcal{U}_F$. We will take ρ to be trivial on $\langle \varpi_F \rangle$, and then regard ρ as a unitary character of $\mathcal{U}_F$. The group $\mathcal{U}_F$ admits countably many such characters ρ. In this case the compact orbit is symmetric square of the circle $\mathbb{T}$:

$$\mathfrak{O}^t(\phi) \cong \mathfrak{O}^t(BC(\phi)) \cong Sym^2(\mathbb{T}) := \mathbb{T}^2 \Big/ \mathbb{Z}/2\mathbb{Z} = \mathbb{T}^2/\mathbb{W}.$$

Lemma 7.2.2 *The symmetric square $\mathbb{T}^2/\mathbb{W}$ has the homotopy type of a circle*

$$\mathbb{T}^2/\mathbb{W} \sim \mathbb{T}$$

$$(z_1, z_2) \longmapsto z_1 z_2.$$

PROOF: By sending the pair $z = (z_1, z_2)$ to a unique monic polynomial

$$(z_1, z_2) \longmapsto z^2 + a_1 z + a_0, \quad a_0 \neq 0$$

with roots z_1, z_2. It follows that

$$Sym^2(\mathbb{T}) \cong \{z^2 + a_1 z + a_0 : a_0 \neq 0\} \sim_h \mathbb{T},$$

since the space of coefficients a_1, a_0 is contractible. Therefore,

$$Sym^2(\mathbb{T}) \sim_h \mathbb{T}$$

using the map $(z_1, z_2) \mapsto z_1 \cdot z_2$. ❐

Let π_F be the local Langlands correspondence

$$\pi_F : \Phi(GL(2)) \to IrrGL(2)$$

and let $x = diag(t_1, t_2)$ be a diagonal element in the standard maximal torus T of $GL(2)$. Then

$$\chi : x \mapsto \pi_F(\rho)(t_1, t_2)$$

is a unitary character of T. Let σ be an unramified unitary character of T, and form the induced representation $Ind_{TU}^{G}(\sigma \otimes \chi)$ which is an irreducible unitary representation of G. Let σ vary over all unramified unitary characters of T, then we obtain a subset of the unitary dual of G. This subset has the structure of a symmetric square of $\mathbb{T}$. The consequence for $\mathcal{U}_F$ admits countably many unitary characters is the unitary dual of G contains countably many subspaces (in the Fell topology) each with the structure $Sym^2(\mathbb{T})$. We are concerned with the effect of base change E/F on each of these compact spaces.

Theorem 7.2.3 *Let $\mathbb{T}^2/\mathbb{W}$ denote one of the compact subspaces of the unitary principal series of $GL(2)$. Then we have*

$$BC : \mathbb{T}^2/\mathbb{W} \to \mathbb{T}^2/\mathbb{W},\ (z_1, z_2) \mapsto (z_1^f, z_2^f)$$

1. *At the level of the K-theory group K^1, BC induces the map*

$$\mathbb{Z} \to \mathbb{Z},\ \alpha_1 \mapsto f \cdot \alpha_1$$

of multiplication by f, where f is the residue degree and α_1 denotes a generator of $K^1(\mathbb{T}) = \mathbb{Z}$.

2. *At the level of the K-theory group K^0, BC induces the identity map*

$$\mathbb{Z} \to \mathbb{Z},\ \alpha_0 \mapsto \alpha_0,$$

where α_0 denotes a generator of $K^0(\mathbb{T}) = \mathbb{Z}$.

PROOF: From lemma 7.2.2 we have this commutative diagram:

$$\begin{tikzcd} Sym^2(\mathbb{T}) \arrow[r, "h"] \arrow[d, "BC"'] & \mathbb{T} \arrow[d, "BC^*"] \\ Sym^2(\mathbb{T}) \arrow[r, "h"'] & \mathbb{T} \end{tikzcd}$$

where $BC(z_1,z_2) = (z_1^f, z_2^f)$, $BC^*(z) = z^f$ and $h(z_1,z_2) = z_1 \cdot z_2$. Since

$$(z_1 \cdot z_2)^f = z_1^f \cdot z_2^f$$

we have $K^j(BC) = K^j(BC^*)$, but BC^* is a map of degree f. Therefore,

$$K^1(BC)(\alpha_1) = f \cdot \alpha_1 \quad and \quad K^0(BC)(\alpha_0) = \alpha_0$$

where α_1 is a generator of $K^1(\mathbb{T}) = \mathbb{Z}$ and α_0 is a generator of $K^0(\mathbb{T}) = \mathbb{Z}$. ❐

Therefore, the K-theory for the trivial type $(I, \mathbb{1}_I)$ would be as follows:

Theorem 7.2.4

$$K_j C_r^*(s) = K^j(Sym^2(\mathbb{T}) \bigsqcup \mathbb{T}) \cong \mathbb{Z}^2$$

PROOF: Proof immediately from theorems 7.2.1 and 7.2.3. ❐

Definition 7.2.5 Let E/F be a quadratic extension and let χ be a character of $E^\times$. The pair $\vartheta = (E/F, \chi)$ is called admissible if

1. χ does not factor through the norm map $N_{E/F} : E^\times \to F^\times$ and,
2. if $\chi \mid \mathcal{U}_E^1$ does factor through $N_{E/F}$, then E/F is unramified.

Let $\mathcal{P}_2(F)$ be the set of isomorphism classes of admissible pairs ϑ. The map

$$\mathcal{P}_2(F) \to \mathcal{G}_2^0(F),$$

$$\vartheta \mapsto Ind_{E/F}\chi$$

is bijection according to [12] p. 215, where χ is a character of $\mathcal{W}_E$ via the class field theory isomorphism $\mathcal{W}_E^{ab} \cong E^\times$ and $Ind_{E/F}$ is the functor of induction from representations of $\mathcal{W}_E$ to representations of $\mathcal{W}_F$.

The tempered dual of $GL(2)$ consists of the cuspidal representations with unitary central character, the unitary twists of the Steinberg representation, and the unitary principal series. It is clear that in the admissible pairs we can describe what is happening so we further restrict ourselves to admissible pairs ϑ for which E/F is totally ramified and χ is a unitary character. This ensures that $\pi := Ind_{E/F}\chi$ is unitary. Therefore $det(\pi)$ is unitary and $\tau(\pi)$ has unitary central character. The cuspidal representations of $GL(2)$ with unitary central character arrange themselves in the tempered dual as a countable union of circles. For each circle $\mathbb{T}$, we select an admissible pair ϑ for which $\tau(\pi) \in \mathbb{T}$ and label this circle as $\mathbb{T}_\vartheta$.

Theorem 7.2.6 *Let E'/F be an unramified extension of odd degree. Then we have:*

1. *Base change is a proper map.*
2. *When we restrict base change to one circle we get the following:*

$$BC : \mathbb{T}_\vartheta \to \mathbb{T}_{(EE'/E', \chi_{E'})}, \quad z \mapsto z^{f(E'/F)}$$

with

$$\chi_{E'} = \chi \circ N_{EE'/E}.$$

PROOF: Since we are considering circles indexed by characters of $\widehat{\mathcal{U}_F}$, then the base change maps each circle into one precise circle.

Let $\mathbb{D}$ be a compact subset of $\mathbb{T}_{\chi_{E'}}$ which is a closed arc in $\mathbb{T}_{\chi_{E'}}$. Then we may write

$$\mathbb{D} = \{e^{i\theta} \in \mathbb{T}_{\chi_{E'}} : \theta_0 \leq \theta \leq \theta_1, \theta \in [0, 2\pi]\},$$

and we have the pre-image of this arc

$$BC^{-1}(\mathbb{D}) = \{e^{i\theta} \in \mathbb{T}_{\chi_F} : \theta_0/f \leq \theta \leq \theta_1/f, \theta \in [0, 2\pi]\}$$

which is closed arc in $\mathbb{T}_{\chi_F}$. It follows that $BC^{-1}(\mathbb{D})$ is compact. Therefore, the base change map BC is a proper map and then (1) has been proved. Now, let $\rho \in \mathcal{G}_2^0(F)$, then the order of the cyclic group of all unramified characters χ such that $\chi\rho \simeq \rho$ is called a torsion number of ρ and denotes by $\nu(\rho)$. Put $\sigma = Ind_{E/F}\chi$, $\pi = \tau(\sigma)$ and $\sigma_{E'} = Ind_{EE'/E'}\chi_{E'} = \sigma|_{\mathcal{W}_{E'}}$. The proof of Theorem 3.3 in [11] shows that the representation σ is totally ramified, in the sense that $\nu(\sigma) = 1$. Theorem 4.6 in the same reference shows that the pair $(EE'/E', \chi_{E'})$ is admissible. Also, we have the map

$$\tau(\sigma_{E'}) = BC_{E'/F}\pi.$$

By Proposition 7.2 in [35], EE'/E is unramified, whenever the extension E'/F is unramified and

$$e_{EE'/F} = e_{EE'/E'} \times e_{E'/F} = e_{EE'/E} \times e_{E/F}.$$

and it follows that

$$e_{EE'/E'} = e_{E/F} = 2.$$

Since EE'/E' is quadratic extension, EE'/E' is totally ramified. Therefore $\sigma_{E'}$ is totally ramified, in another words $\nu(\sigma_{E'}) = 1$. Therefore, the base change maps each circle to another circle and its given by

$$z \longmapsto z^{f(E'/F)}.$$

❒

If the extension E'/F is a finite unramified Galois extension, then the cuspidal part of the tempered dual of $GL(2)$ is a countable disjoint union of circles and has the structure of a locally compact Hausdorff space. The base change map

$$BC : \bigsqcup \mathbb{T}_\vartheta \to \bigsqcup \mathbb{T}_\zeta$$

is a proper map, where ϑ an admissible pair, E/F totally ramified, χ unitary and $\zeta = (EE'/E', \eta)$. Therefore, there is a functorial map at the level of K-theory groups

$$K^j(BC) : \bigoplus \mathbb{Z}_\zeta \to \bigoplus \mathbb{Z}_\vartheta.$$

Each K-group is a countably generated free abelian group:

$$K^j(\bigsqcup \mathbb{T}_\vartheta) \cong \bigoplus \mathbb{Z}_\vartheta,\ K^j(\bigsqcup \mathbb{T}_\zeta) \cong \bigoplus \mathbb{Z}_\zeta,$$

where $\mathbb{Z}_\vartheta$ and $\mathbb{Z}_\zeta$ denote a copy of $\mathbb{Z}$, $j = 0, 1$. The base change map selects among the admissible pairs ζ those of the form $(EE'/E', \chi_{E'})$, where

$$\chi_{E'} = \chi \circ N_{EE'/E}.$$

Theorem 7.2.7 *When we restrict $K^1(BC)$ to the direct summand $\mathbb{Z}_{(EE'/E', \chi_{E'})}$ we get the following map:*

$$\mathbb{Z}_{(EE'/E', \chi_{E'})} \to \mathbb{Z}_\vartheta,$$

$$x \mapsto f(E'/F) \cdot x.$$

On the remaining direct summands, $K^1(BC) = 0$. When we restrict $K^0(BC)$ to the direct summand $\mathbb{Z}_{(EE'/E', \chi_{E'})}$ we get the following map:

$$\mathbb{Z}_{(EE'/E', \chi_{E'})} \to \mathbb{Z}_\vartheta,$$

$$x \mapsto x.$$

On the remaining direct summands, $K^0(BC) = 0$.

7.3 Conclusion

In our work on $GL(2)$ we have found that its a bit hard to compute the chamber homology groups from the quotient space $\beta^1 GL(2)/GL(2)$ (Mobius band), so we introduced a new way to compute the chamber homology groups by restricting to the original quotient space (edge) before taking the real line $\mathbb{R}$. We have not yet given a full description of what happening under base change when we work on the cuspidal representation but, we somehow, gave a way to compute the base change effect of some type of cuspidal representations which are the admissible pairs. The base change of a principal series representations is always a principal series. Similarly, the base change of a twist of Steinberg representation is again a twist of Steinberg. However, an irreducible Galois representation can certainly restrict to a reducible one. Thus it is possible for the base change of a cuspidal to be principal series. In fact, if π is any irreducible admissible representation of $GL(2,F)$ then one can find an extension E/F such that $BC(\pi)$ is either unramified or Steinberg.

Below is a summary of all the cases in this chapter :

On the admissible side, if we have the following:

1. The twist of Steinberg : $\{\psi \otimes St(2) : \psi \in \Psi^t(\mathcal{W}_F)\} \cong \mathbb{T}$.
2. The cuspidal : $\{\psi \otimes \pi : \pi \in \mathcal{A}^0 GL(2)\} \cong \mathbb{T}$.
3. U.P.S : $\{Ind_B^G \begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto \psi_1 \cdot \psi_2 : \psi_j \in \Psi^t(\mathcal{W}_F)\} \cong \mathbb{T}^2$, when $\psi_1 \neq \psi_2$.
4. U.P.S : $\{Ind_B^G \begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto \psi_1 \cdot \psi_2 : \psi_j \in \Psi^t(\mathcal{W}_F)\} \cong \mathbb{T}^2/(\mathbb{Z}/2\mathbb{Z})$, when $\psi_1 = \psi_2$

Then, the K-theory groups of each of these cases are as follows:

1. $K_j C_0(\mathbb{T}) = \mathbb{Z}$.
2. $K_j C_0(\mathbb{T}) = \mathbb{Z}$.
3. $K_j C_0(\mathbb{T}^2) = K^j(\mathbb{T}^2) = \mathbb{Z} \bigoplus \mathbb{Z}$.
4. $K_j C_0\big(\mathbb{T}^2/(\mathbb{Z}/2\mathbb{Z})\big) = K_j C_0(\mathbb{T}) = K^j(\mathbb{T}) \cong \mathbb{Z}$.

Also, the homology groups of these cases are as follows:

1. $H_{even} = \mathbb{Z}^2 = H_{odd}$.

2. $H_{even} = \mathbb{Z} = H_{odd}$.

3. $H_{even} = \mathbb{Z}^2 = H_{odd}$.

Bibliography

[1] M. F. Atiyah. *K-theory*. Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[2] A.-M. Aubert, S. Hasan, and R. Plymen. Cycles in the chamber homology of gl(3). *K-Theory*, 37:341–377, 2006. 10.1007/s10977-006-9001-y.

[3] P. Baum, N. Higson, and R. Plymen. Equivariant homology for $SL(2)$ of a p-adic field. In *Index theory and operator algebras (Boulder, CO, 1991)*, volume 148 of *Contemp. Math.*, pages 1–18. Amer. Math. Soc., Providence, RI, 1993.

[4] P. Baum, N. Higson, and R. Plymen. A proof of the Baum-Connes conjecture for p-adic $\mathrm{GL}(n)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(2):171–176, 1997.

[5] P. F. Baum, N. Higson, and R. J. Plymen. Representation theory of p-adic groups: a view from operator algebras. In *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, volume 68 of *Proc. Sympos. Pure Math.*, pages 111–149. Amer. Math. Soc., Providence, RI, 2000.

[6] B. Blackadar. *K-theory for operator algebras*. Springer-Verlag, New York, 1986.

[7] A. Borel and W. Casselman, editors. *Automorphic forms, representations, and L-functions. Part 2*. Proceedings of Symposia in Pure Mathematics, XXXIII. American Mathematical Society, Providence, R.I., 1979.

[8] G. E. Bredon. Cosheaves and homology. *Pacific J. Math.*, 25:1–32, 1968.

[9] J. Brodzki and R. Plymen. Complex structure on the smooth dual of $GL(n)$. *Documenta Mathematica*, 7:91–112 (electronic), 2002.

[10] K. Brown. *Buildings*. Springer-Verlag, New York, 1989.

[11] C. J. Bushnell and G. Henniart. The essentially tame local Langlands correspondence. I. *J. Amer. Math. Soc.*, 18(3):685–710, 2005.

[12] C. J. Bushnell and G. Henniart. *The local Langlands conjecture for GL(2)*. Springer, 2006.

[13] C. J. Bushnell and P. C. Kutzko. *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.

[14] C. J. Bushnell and P. C. Kutzko. Smooth representations of reductive p-adic groups: structure theory via types. *Proc. London Math. Soc. (3)*, 77(3):582–634, 1998.

[15] C. J. Bushnell and P. C. Kutzko. Semisimple types in GL_n. *Compositio Math.*, 119(1):53–97, 1999.

[16] J. W. S. Cassels. *Local fields*, volume 3 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1986.

[17] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.

[18] J. Dixmier. *C^* -algebras.* Amsterdam North-Holland Publishing Company, 1977.

[19] I. B. Fesenko and S. V. Vostokov. *Local fields and their extensions*, volume 121 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, second edition, 2002. With a foreword by I. R. Shafarevich.

[20] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[21] P. Garrett. *Buildings and classical groups*. Chapman & Hall, London; New York, 1st ed. edition, 1997.

[22] S. S. Gelbart. *Automorphic forms on adle groups*. Princeton University Press, Princeton, N.J., 1975. Annals of Mathematics Studies, No. 83.

[23] Harish-Chandra. *Harmonic analysis on reductive p-adic groups*. Lecture Notes in Mathematics, Vol. 162. Springer-Verlag, Berlin, 1970. Notes by G. van Dijk.

[24] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[25] G. Henniart. Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p-adique. *Inventiones Mathematicae*, 139:439–455, 2000.

[26] M. Karoubi. *K-theory*. Springer-Verlag, Berlin, 1978. An introduction, Grundlehren der Mathematischen Wissenschaften, Band 226.

[27] M. Khalkhali. *Basic noncommutative geometry*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zurich, 2009.

[28] P. Kutzko. The Langlands Conjecture for GL_2 sub of a Local Field. *Annals of Mathematics*, 112(2):381, 1980.

[29] P. C. Kutzko. On the supercuspidal representations of GL_2. *American Journal of Mathematics*, 100(1):43–60, 1978.

[30] P. C. Kutzko. On the supercuspidal representations of GL_2. II. *American Journal of Mathematics*, 100(4):705–716, 1978.

[31] P. C. Kutzko. The exceptional representations of GL_2. *Compositio Mathematica*, 51(1):3–14, 1984.

[32] G. Laumon, M. Rapoport, and U. Stuhler. D-elliptic sheaves and the Langlands correspondence. *Inventiones Mathematicae*, 113(2):217–338, 1993.

[33] R. L. Lipsman. *Group representations*. Lecture Notes in Mathematics, Vol. 388. Springer-Verlag, Berlin, 1974. A survey of some current topics.

[34] S. Mendes and R. Plymen. Base change and K-theory for $GL(n)$. *Journal of Noncommutative Geometry*, 1(3):311–331, 2007.

[35] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[36] V. Paskunas. Unicity of types for supercuspidal representations of GL_N. *Proc. London Math. Soc. (3)*, 91(3):623–654, 2005.

[37] R. J. Plymen. K-theory of the reduced C^*-algebra of $SL_2(Q_p)$. In *Operator algebras and their connections with topology and ergodic theory (Bu\c steni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 409–420. Springer, Berlin, 1985.

[38] R. J. Plymen. Reduced C^*-algebra for reductive p-adic groups. *Journal of Functional Analysis*, 88(2):251–266, 1990.

[39] R. J. Plymen. Reduced C^*-algebra of the p-adic group $GL(n)$. II. *Journal of Functional Analysis*, 196(1):119–134, 2002.

[40] M. Rrdam. *An introduction to K-theory for C^*-algebras*. Cambridge University Press, Cambridge UK ; New York NY, 2000.

[41] P. J. Sally, Jr. Character formulas for sl_2. In *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, pages 395–400. Amer. Math. Soc., Providence, R.I., 1973.

[42] P. Samir K. Hasan University of Manchester. Department of Mathematics and Applied. *Chamber homology and K-theory for the p-adic group $GL(3)$*. PhD thesis, Maths, 2004.

[43] J.-P. Schneiders. Cosheaves homology. *Bull. Soc. Math. Belg. Sér. B*, 39(1):1–31, 1987.

[44] J.-P. Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[45] J. P. Serre. *Trees*. Springer, Mar. 2003.

[46] A. J. Silberger. *Introduction to harmonic analysis on reductive p-adic groups*, volume 23 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1979. Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971-1973.

[47] J. Tits. *Buildings of spherical type and finite BN-pairs*. Springer, 1974.

[48] N. E. Wegge-Olsen. *K-theory and C^*-algebras*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1993. A friendly approach.

[49] A. Weil. *Basic number theory*. Springer, 1995.