EXPLORING PULSAR–BLACK HOLE BINARIES USING THE NEXT GENERATION OF RADIO TELESCOPES

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Abstract

Binary pulsars are well known for their usefulness in testing gravitational theories. Pulsar–black hole (PSR-BH) binary systems, once discovered, will be the ‘next generation of celestial laboratories’ being able to both probe BH properties and test gravity especially General Relativity’s cosmic censorship conjecture and no-hair theorem. The achievement of these prized goals requires highly precise pulsar timing observations, which are more readily achieved with the next generation of radio telescopes such as the Square Kilometre Array (SKA) and the Five-hundred-metre Aperture Spherical Radio Telescope (FAST). The purpose of work carried out in this thesis is to investigate the limits on precision timing and to explore the potential for tests of theories of gravity by PSR-BH binaries with the future telescopes.

Millisecond pulsars (MSPs) are more stable timers compared with normal pulsars. While the precision of MSP timing with present hardware is mainly limited by radiometer noise, for the brightest few MSPs one can already notice effects from other aspects, which can be separated into three categories: intrinsic noise from the pulsar, variations of the interstellar medium (ISM) effects, and instrumental artefacts. The case study based on the brightest MSP, PSR J0437−4715, demonstrates that most instrument-associated uncertainties in pulse time-of-arrival (TOA) measurement can be corrected by state-of-the-art techniques. The influence on TOAs by the interstellar medium (ISM) is shown to be unimportant for the source, and can potentially be corrected by approaches which are being developed. The TOA uncertainties for most MSPs observed with
the next generation of radio telescopes, will mainly be limited by pulse jitter and radiometer noise. Based on this result, it is predicted that for normal-brightness MSPs a TOA precision of between 80 and 230 ns can be achieved at 1.4 GHz with 10-minute integrations by the SKA.

With the current sensitivity, a further investigation on pulse jitter has been performed, regarding both the shape and central phase variability of several MSPs. No significant shape changes within a few hours observing time have been detected based on $\sim 10$ to $\sim 100$ s integrations. For PSR J0437$-$$4715$ the jitter parameter is quantified as $f_J = 0.067 \pm 0.002$, based on timing on short timescales. Potential instrumental effects on this measurement have also been demonstrated. Jitter noise is found to be independent of observing frequency and bandwidth around 1.4 GHz on frequency scales of $< 100$ MHz, suggesting that the resulting uncertainty might not be mitigated by extending the observing bandwidth.

Through detailed simulations, it has been found that timing a pulsar in orbit around a stellar mass BH (SBH) binary system with the next generation of radio telescopes can lead to a high precision determination of the BH mass and spin in ten years. Especially for MSPs, the measurements of mass and spin can be allowed for wide orbits of orbital period up to roughly ten days. The constrain on BH quadrupole moment is possible only with timing precisions achievable with MSPs, and for systems of either high-mass (e.g. $\sim 80 M_\odot$) SBH or high orbital eccentricity. Meanwhile, timing a normal pulsar orbiting around the Galactic Centre BH, Sgr A*, with the SKA would lead to the extraction of the BH mass, spin and quadrupole moment within five years. Considering the perturbation from other stellar masses in the Galactic Centre region, these measurements could be converted to a test of the no-hair theorem with $\sim 1\%$ precision.
Declaration

No portion of the work referred to in the dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Kuo Liu, September 2011.
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Kuo Liu, Bonn, September 2011.
Chapter 1

Introduction

The subject, radio astronomy, was founded in 1933 when Karl Jansky caught the “first light” from the radio sky (Jansky, 1933). After the Second World War, the newly developed radar technology was soon translated for the usage of radio astronomy in constructing big antennas, which directly led to the blossom of this field in the 1960s. The discovery of the first pulsar was achieved with this specific historic background. In the past forty years with the development of hardware and software, a variety of astonishing results have been achieved in pulsar science, including testing theories of gravities with binary pulsars. The ongoing projects of the next generation of radio telescopes will provide the best chance to find the first pulsar–black hole binary system, and perform gravity tests with this celestial laboratory in the strong field regime. This chapter will provide an introduction into pulsars, including their characteristics, the interstellar medium (ISM) effect on their radio signal, their utilisation in testing gravitational theories, and the pulsar searching technique. The outline for the rest of this thesis can be found in the last subsection.
1.1 Pulsars

Pulsars are commonly believed to be fast rotating magnetised neutron stars, the concept of which in reality was established before the discovery of pulsars. In 1932, Landau first predicted the existence of stars composed of only atomic nuclei (Landau, 1932). In 1934, soon after the discovery of the neutron (Chadwick, 1932), Baade & Zwicky (1934) proposed that a supernova explosion is caused by the core collapse of a star at the end of its main-sequence lifetime, giving birth to a neutron star. Later on, relying on the equation of state for cold Fermi gas, Oppenheimer & Volkoff (1939) calculated the upper limit on the mass of a neutron star to retain gravitational equilibrium. During this time, theoretical studies suggested that neutron stars would be too small, cool, and hence too dim to be observable. However, it was not expected that the rapid development of radio astronomy in the 1960s would provided direct access to neutron stars.

In 1965 Antony Hewish and Samuel Okoye recognised a source of high radio brightness temperature in the Crab Nebula (Hewish & Okoye, 1965), which later turned out to be the nowadays known Crab pulsar. Two years later, Jocelyn Bell and Antony Hewish discovered regular radio pulses from CP 1919 (Hewish et al., 1968), as the first ever pulsar found, now known as PSR B1919+21. Although possibilities including extra terrestrial life were considered to explain the nature of the periodic signal, soon the pulsar was linked to the idea of a rapidly rotating and highly magnetised neutron star provided with the short and stable periodicity (Gold, 1968; Pacini, 1968). The discovery of pulsars and their identification as neutron stars opened up a new subfield of radio astronomy and resulted in a large variety of studies of pulsar science in the past 40 years.

Based on the currently standard star evolutionary scenario, a pulsar (thus a neutron star) is one of the three fates after the death of a star which evolves from the main-sequence phase into the giant branch (Shapiro & Teukolsky, 1983). At the end of its lifetime, the progenitor star will endure a core collapse due to the
lack of fuel for nuclear fusion which otherwise would generate radiation temperature pressure to balance the force of gravity. For stars with cores lighter than $1.4 \, M_\odot$ (Chandrasekhar, 1931), the collapse will be terminated when the pressure from the resulting degenerate electron gas is enough to sustain the gravity. This results in an approximately earth-sized object known as a white dwarf. If the stellar core is beyond the Chandrasekhar threshold, the collapse will continue until the phase when protons and electrons are fused into neutrons and the force of gravity is balanced by neutron degeneracy pressure. This procedure gives birth to a neutron star with a radius of only $\sim 10$ km. When the core is massive enough to break any degeneracy balance, the collapse will continue until a body of infinite density (black hole) is created.

It should be noted that the nature of pulsars is still a matter of some debate. Although the model of neutron star has been commonly accepted, there are still alternative theories involving strange quark matter (e.g. Alcock et al., 1986; Xu, 2002). Some theories may still hold (Özel et al., 2010; Lai & Xu, 2011), even after the recent discovery of PSR J1614$-2230$ with a mass of $2 \, M_\odot$ which places strict constraint on models of the stellar equation of state concerning quark matter (Demorest et al., 2010). In the following content of this thesis, we will refer to a pulsar as a neutron star, being aware of the fact that its internal structure is still an open question.

### 1.2 Dipole model

Emission mechanisms of pulsars have been widely studied since their discoveries. Still, no theory is yet able to construct a complete model that explains most of the observed pulsar properties (e.g. Bahcall & Ostriker, 1997). Nevertheless, the well known dipole radiation model (Pacini, 1967; Gold, 1968) still provides a general picture of pulsar magnetosphere, as shown in Fig. 1.1. Here the neutron star is viewed as a highly magnetised, rotating sphere with co-rotating magnetosphere.
The radio beam is believed to be generated by electric field acceleration from a region above the magnetic pole called inner gap (e.g. Ruderman & Sutherland, 1975). For each rotation when the beam sweeps past the Earth, like a celestial lighthouse, a pulse can be observed. Many pulsars are also visible at high energy, where the radiation is thought to be associated with different processes such as synchrotron emission, curvature emission and inverse Compton scattering (e.g. Daugherty & Harding, 1996). The first two are believed to mostly occur in a higher latitude region called outer gap (Cheng et al., 1986; Romani, 1996). The radius of the light cylinder is defined as the distance to the rotation axis, where the co-rotating velocity of the last closed magnetic field line with the pulsar equals to speed of light. Summaries of pulsar magnetosphere and emission mechanism can be found in both Michel (1991) and Lorimer & Kramer (2005). In the following, we introduce a few indicative parameters of pulsar characteristics based on the dipole assumption.

1.2.1 Spin evolution and braking index

Pulsars commonly exhibit the nature of slowing down in rotation, which can be related to the energy loss due to the rotation of the magnetised body. According to classical electrodynamics, a spinning magnetic dipole will radiate an electromagnetic wave at its rotation frequency, with radiation power of (Jackson, 1962)

$$\dot{E}_{\text{dipole}} = \frac{2}{3c^3} |m|^2 \Omega^4 \sin^2 \alpha.$$  \hfill (1.1)

Here \(m\) is the magnetic moment, \(\alpha\) is the magnetic inclination angle defined in Fig. 1.1, \(\Omega\) is the rotational angular velocity, and \(c\) is the speed of light. Meanwhile, the spin-down loss rate of rotational kinetic energy can be written in

$$\dot{E} = -I\Omega \dot{\Omega},$$ \hfill (1.2)

where \(I\) is the moment of inertia. If the spin-down is assumed to be mainly caused by the dipole emission, the combination of Eqs. (1.1) and (1.2) leads to

$$\dot{\Omega} = -\left(\frac{2|m|^2 \sin^2 \alpha}{3Ic^3}\right) \Omega^3.$$  \hfill (1.3)
1.2. DIPOLE MODEL

Fig. 1.1: A general picture of pulsar magnetosphere based on rotating dipole model. See the main text for a detailed description. The figure is reproduced from Lorimer & Kramer (2005).
Note that in reality, the index of the power-law dependency between \( \dot{\Omega} \) and \( \Omega \) is usually found not to be 3 (e.g. Kaspi & Helfand, 2002). Therefore, Eq. (1.3) is often expressed in a more general way in terms of rotational frequency \( \nu = 1/P \), as

\[
\dot{\nu} = -K\nu^n \quad \text{or} \quad \dot{P} = K\nu^{2-n}.
\] (1.4)

Here \( K \) is a constant and \( n \) is called the “braking index”, predicted to be 3 by the dipole model. In practice it can be determined from a measurement of the second spin frequency derivative by \( n = \nu\ddot{\nu}/\dot{\nu}^2 \) (e.g. Lyne & Smith, 2005).

### 1.2.2 Characteristic age

The spin-down relation in Eq. (1.4) can be integrated to derive the age of a pulsar as (assuming \( n \neq 1 \))

\[
T = \frac{P}{(n-1)\dot{P}} \left[ 1 - \left( \frac{P_0}{P} \right)^{n-1} \right],
\] (1.5)

where \( P_0 \) is the spin period at birth. If the spin-down is sufficient to enable \( P_0 \ll P \), with the application of the dipole radiation model \( (n = 3) \) one can estimate a pulsar characteristic age by

\[
\tau_c \equiv \frac{P}{2\dot{P}} \simeq 15.8 \text{ Myr} \left( \frac{P}{s} \right) \left( \frac{\dot{P}}{10^{-15}} \right)^{-1},
\] (1.6)

which is determined only from \( P \) and \( \dot{P} \). Nevertheless, due to aforementioned assumptions to achieve the relation, this quantity only presents a rough estimate of the true pulsar age.

### 1.2.3 Magnetic field strength

Assuming that dipole emission is dominant in the spin-down process, one can additionally estimate the magnetic field in the nearby region of a pulsar. As the
magnetic moment is associated with the field strength by \( B \approx \frac{|m|}{r^3} \); rearrangement of Eq. (1.3) leads to

\[
B = \left( \frac{3r^3I}{8\pi^2r^6\sin^2\alpha}P\dot{P} \right)^{1/2}.
\] (1.7)

For the canonical neutron star with \( I = 10^{45} \text{g cm}^2 \) and radius \( R = 10 \text{km} \) (e.g. Lorimer & Kramer, 2005), the surface magnetic field strength can be obtained from the known \( P \) and \( \dot{P} \) in the form of (assuming \( \alpha = 90^\circ \))

\[
B|_{r=R} \simeq 10^{12} \text{G} \left( \frac{\dot{P}}{10^{-15}} \right)^{1/2} \left( \frac{P}{1 \text{s}} \right)^{1/2}.
\] (1.8)

Again, given the limited knowledge of the assumed parameters and the uncertainty of the emission model, Eq. (1.8) is essentially an order of magnitude estimate.

### 1.3 Population and the \( P-\dot{P} \) diagram

After the discovery of a pulsar, its rotational period \( P \) and spin-down rate \( \dot{P} \) is precisely determined by following-up observations. These properties can be used to study the population and evolution of pulsars via a \( P-\dot{P} \) diagram, an example of which is shown in Fig. 1.2. Here in general, one can distinguish four different categories of sources whose characteristics and possible evolutionary path are discussed below.

The majority of pulsars (\( \sim 90\% \)) are located in the middle of the diagram, with periods mainly between 50 ms and 5 s, typical characteristic ages of \( 10^7 \text{yr} \), and surface magnetic fields \( 10^{11}-10^{13} \text{G} \). The very young ones on the upper left, including the Crab pulsar, are often associated with visible supernova remnants. Some of them are also identified as \( \gamma \)-ray sources. The binarity percentage for this group is well below 1%.

The millisecond pulsars (MSPs) are identified at the bottom left corner, with \( P \lesssim 30 \text{ms} \), characteristic ages of \( 10^8-10^9 \text{yr} \), and magnetic field of order \( 10^8-10^9 \text{G} \). About 70% of them are found in a binary system (Manchester et al., 2005)
and the companion can be a white dwarf, a neutron star, or a main-sequence star. The formation of MSPs are commonly believed to be associated with the accretion during binary evolution procedure (see Section 1.5 for more details).

Furthermore, on the top right corner there is a small group of sources first observed in X-rays named magnetars\(^1\). Those include both the soft $\gamma$-ray repeaters (SGRs) and the anomalous X-ray pulsars (AXPs). The common features are very high magnetic fields in the range $10^{14}$-$10^{15}$ G, and the long rotational period from 2 s to 8 s. Right below the magnetars the several discovered rotating radio transients (RRATs) forms another small group and some of them are identified as normal pulsars (McLaughlin et al., 2006; Keane et al., 2011).

If the braking index is obtained from a measurement of the period second derivative, the evolutionary trend of a pulsar on the $P$-$\dot{P}$ plane can be identified. This has been achieved already for a few young pulsars and detailed discussions can be found in Kaspi & Helfand (2002); Espinoza (2010). The “death line” in Fig. 1.2 is defined by the cease of rotation powered radio emission predicted by theoretical models (Ruderman & Sutherland, 1975; Chen & Ruderman, 1993).

### 1.4 Propagation through the ISM

Before reaching the earth, pulses from pulsars will firstly pass through the interstellar medium (ISM), which would change the characteristics of the radio signal. In the following a few effects that have been noticed from current pulsar observation are introduced.

#### 1.4.1 Dispersion delay

The ISM contains a bulk of ionised plasma of temperature $\sim 8000$ K and any electromagnetic radiation propagating through it will endure a frequency dependent

\(^1\)A few pulsars can also be found in this region.
1.4. PROPAGATION THROUGH THE ISM

Fig. 1.2: \( P - \dot{P} \) diagram showing discovered radio pulsars, magnetars and radio transients. Lines of constant magnetic field, characteristic age and spin-down luminosity are shown. Red circles are used to specify sources found in globular clusters (GCs). The abbreviations (not explained in the text) DTN and CCO represent dim thermal neutron star (e.g. Alpar, 2001) and central compact object (e.g. de Luca, 2008), respectively. The figure is kindly provided by Y. L. Yue on April 2011 when the data were obtained from the Australia Telescope National Facility (ATNF) catalog (Manchester et al., 2005).
index of refraction, given by (neglecting the magnetic field correction)

\[ \mu = \sqrt{1 - \left( \frac{f_p}{f} \right)^2}, \quad (1.9) \]

where \( f \) is the wave frequency and \( f_p \) the plasma frequency in the form of

\[ f_p = \sqrt{\frac{e^2 n_e}{\pi m_e}}, \quad (1.10) \]

Here \( n_e \) is the free electron number density, while \( e \) and \( m_e \) are the electron charge and mass, respectively. As can be seen from Eq. (1.9), signals with frequency below \( f_p \) will not be able to propagate. Those with frequency above this threshold will have a group velocity \( v_g = c \mu \), less than the vacuum speed of light \( c \). Therefore, assuming a travelling distance of \( d \) from the pulsar to the earth, the corresponding time delay caused by the ISM is (e.g. Lorimer & Kramer, 2005)

\[ \Delta t = D \times \frac{DM}{f^2}, \quad (1.11) \]

Here the dispersion measure

\[ DM = \int_0^d n_e dl \quad (1.12) \]

is the integrated electron density along the line of sight, and the dispersion constant is given by

\[ D = \frac{e^2}{2\pi m_e c}. \quad (1.13) \]

Clearly from Eq. (1.11), the time delay is frequency dependent. Consequently, if measurements of the signal arrival time from two or more different frequencies are obtained, a DM value can then be inferred from the time differences (e.g. You et al., 2007), based on the assumption that the signals have passed through equal amount of ISM. This may not be sufficiently precise if multi-path scattering is considered (see the following subsection).

### 1.4.2 Scattering and scintillation

If the structure of the ISM along the line of sight is not homogenous, the wavefront of the pulsar radiation will be varied during the propagation through the ISM.
Consequently, observers at the earth may receive pulsar signals from a small fraction of solid angle. The multiple rays emitted simultaneously at the pulsar will arrive at slightly different times due to the difference in travelling paths. This effect broadens the pulse profile and effectively shifts the arrival phase. Based on a thin screen model proposed by Scheuer (1968), a picture of the scattering phenomenon is summarised in Fig. 1.3, and the broadening timescale of the profile can be found as (e.g. Williamson, 1972)

\[
\tau_{\text{scat}} = \frac{e^4 d^2}{4\pi^2 m_e^2} \Delta n_e^2 f^{-4},
\]

where \( a \) is the thickness of the ISM slab. Note that in reality the structure of the ISM is much more complicated than a single screen, which can result in different power-law index of the \( \tau_{\text{scat}} \) and \( f \) relationship (e.g. Löhmer et al., 2001). Alternative models concerning more detailed spatial distribution of scattering material along the line of sight, have been widely discussed in previous studies (e.g. Cordes et al., 1986; Cordes & Rickett, 1998; Brogan & Goss, 2003).

In addition, the signals from multiple orientations will interfere differently at the points on the observer’s plane, which produces an interference pattern containing patches with enhanced and reduced intensity. Relative motions between the pulsar, the ISM and the observer will change either the position of the observer in the pattern or the pattern itself, which can result in an intensity variation (in time) of the observed signal known as “scintillation”. The intensity is also varied for different observing frequencies and the frequency scale of such variation can be estimated by following e.g. Cordes (1986) and Kramer et al. (2003).

1.5 Binary systems

Among the \( \sim 2000 \) currently discovered pulsars, \( \sim 4\% \) are found in a binary system. Specially, the binarity percentage of MSPs (\( \approx 70\% \)) is significantly higher than the average level. On the whole, the companions can be either white dwarfs,
Fig. 1.3: A simple screen model of the ISM to demonstrate the effect of scattering. Here the spatially coherent electromagnetic radiation from the pulsar is distorted by a thin slab with irregular matter distribution. Consequently, on the observer’s plane one can receive signals at multiple directions from the distorted wavefront. Interference of these signals produces intensity variation on the plane, which results in the phenomenon of scintillation once there is relative motion in the system. The figure is obtained in Lorimer & Kramer (2005), originally from Cordes (2002).
1.5. BINARY SYSTEMS

Two OB main-sequence stars
More massive star (primary) overfills Roche lobe
Helium-rich WR star with OB-companion
Primary explodes as type IIb Supernova and becomes a neutron star or black hole
Secondary is close to Roche lobe. Accretion of stellar wind results in powerful X-ray emission
Helium core of the secondary with compact companion inside mass-losing common envelope
Components merge. Red (super)giant with neutron star or black hole core (Thorne-Zytkow object)
Single neutron star or black hole
Super Nova explosion disrupts the system. Two single neutron stars or black holes

Fig. 1.4: Picture showing the general evolutionary process that gives birth to a compact binary system, reproduced from Yungelson & Portegies Zwart (1998).

neutron stars, main sequence stars, or even planets. Binary pulsars with low-mass companions ($\lesssim 0.5 \, M_\odot$, mainly white dwarfs) are more likely to have high spin frequencies and circular orbits with eccentricities $e \lesssim 10^{-3}$, while those with high-mass companion ($\gtrsim 1 \, M_\odot$) tend to have slower rotation and eccentric orbits ($e \gtrsim 0.1$).

The formation of binary pulsars can be understood by a simple evolutionary scenario as shown in Fig. 1.4 (Bhattacharya & van den Heuvel, 1991; Yungelson & Portegies Zwart, 1998; Voss & Tauris, 2003), starting with two main-sequence
stars. The initially more massive (primary) star evolves faster and first reaches the red giant phase. When it fills its Roche lobe, mass transfer to the secondary star begins. This process comes to the end when the helium core of the primary is exposed. The naked helium core, called a Wolf Rayet star, finally explodes as a type Ib supernova and becomes a neutron star. The asymmetric nature of the explosion will bring along a kick to the neutron star, which may disrupt the system, or change the orbital parameters and systemic velocity. If the system survives, the secondary star would also evolve to fill its Roche lobe. At this point, due to the large mass ratio, the mass transfer is not conservative and the secondary star will continue extending to engulf the neutron star, which is called common envelop (CE) phase. During this period of time, the matter accreted onto the neutron star would spin it up and dramatically decrease its magnetic field (Bisnovatyi-Kogan & Komberg, 1974), which is referred to as the recycling phase. Due to the loss of angular momentum caused by dynamical collision and friction, the orbit gets circularised and decays significantly, which may lead to the merger of the two objects for most systems. Those who survive until the end of the evolution of the secondary star, will either form a double neutron star binary (if the secondary star is sufficiently massive to undergo core collapse) or a white dwarf-neutron star system (if the mass transfer continues until the secondary star sheds its out layers). For the first case the pulsar would be half-recycled, while the comparably long timescale ($\sim$Gyr) of the second path can enable a spin-up of the pulsar with $P \sim$ ms. The standard evolutionary scenario fails to explain the existence of eccentric MSP binaries, the formation of which may be associated with multiple body interactions (Freire et al., 2004, 2007; Champion et al., 2008; Freire et al., 2011).

Formation of a pulsar and a black hole binary (PSR-BH) is achievable both via the standard evolutionary path and other channels. A detailed discussion on this topic will be followed in Section 5.1.3.
1.6 Rotational stability

On short timescales (∼ days) all pulsars exhibit a smooth rotation as expected from their large and stable moment of inertia. However, as time extends, many, mostly young pulsars begin to show irregular spin behaviour which cannot be explained by the spin-down due to the loss of angular momentum from magnetic dipole emission. The level of this rotational instability is observed to be source dependent and fundamentally different for normal pulsars and MSPs.

In general, two categories of irregularities have been noticed: spectacular changes in rotation speed and speed derivative on very short timescale known as ‘glitches’ (Shemar & Lyne, 1996; Yuan et al., 2010; Espinoza et al., 2011), and low frequency oscillations in rotation speed on timescales of years denoted by ‘spin noise’ (Cordes & Helfand, 1980; Stairs et al., 2000; Hobbs et al., 2004; Hobbs et al., 2010). Most occurrences of these phenomena have been witnessed in normal pulsars, limiting the precision of time keeping above order of 10 µs. The irregularities are thought to be associated with either internal superfluid of the neutron star (Packard, 1972; Anderson & Itoh, 1975; Popov, 2008) or external torques (Link & Epstein, 2001; Qiao et al., 2003; Liu et al., 2007; Lyne et al., 2010).

On the contrary, apart from several exceptions (Cordes et al., 1990; Cognard & Backer, 2004), most MSPs are shown to have regular rotational behaviour on timescales of years (Hotan et al., 2006; Verbiest et al., 2009). The current state-of-art, PSR J0437−4715, has exhibited a stability similar to that achieved by the best atomic clocks, as shown in Fig. 1.5 (Hartnett & Luiten, 2011). Here the performance of different clocks are compared based on two statistical quantities: the square root of the Allan variance ($\sigma_y$), concerning a potential frequency drift of a clock, and the $\sigma_z$ parameter calculated from additional variations in the drift rate\(^2\) (Matsakis et al., 1997). The results are shown for three artificial time

\(^2\)Note that pulsars are showing an intrinsic secular change in period, the comparison is better performed when regarding the $\sigma_z$ statistics.
Fig. 1.5: Stability comparison of the best celestial and laboratory clocks, based on $\sigma_y$ and $\sigma_z$ statistics within ten years’ time span (see the text for more details). The plot is reproduced from Hartnett & Luiten (2011).

standards, the commercial thermal beam cesium clock (Cs 5071A), the microwave frequency standard (FO2-FOM), and the inter-comparison between two terrestrial timescales (TAI-AT1), as well as the one yielded by timing of PSR J0437−4715. It can be noticed that although the pulsar is less stable than the atomic clocks on short timescales, their stability becomes comparable over time spans of years. This demonstrates the strong capability in time keeping of MSPs which is essential in testing theories of gravity (as discussed in the next section).

The explanation of the fact that MSPs show more regular rotation than young pulsars, has not been fully achieved yet. Nevertheless, indications might be drawn from Fig. 1.2 that the intensity of spin-associated torques should be anti-proportional to characteristic age, and proportional to magnetic field strength.
1.7 Celestial laboratory in relativity test

For a test of gravitational theory two components will be ideal: a deep gravitational potential to enable strong interaction between space and time, and an accurate clock to precisely measure the effect on time. Fortunately, they can both be found in binary pulsar systems, especially binary MSPs where the companions are mostly degenerate objects. In the following we introduce the tests of General Relativity (GR) and alternative theory of gravity that can be yielded by studies of three types of binary pulsar systems.

1.7.1 Double neutron star binaries

The first discovered binary pulsar PSR B1913+16 (Hulse & Taylor, 1975), is also the first ever found in a double neutron star (DNS) system. Here two stars move around the mass centre in an eccentric orbit of 7.75-hour period and are separated by only 3.3 light-seconds. The measurement of orbital period variation presented the first evidence for orbital shrinking due to gravitational wave radiation (Taylor & Weisberg, 1982). Fig. 1.6 shows the latest result reported by Weisberg et al. (2010). The determined orbital period decay due to gravitational wave emission was also shown to coincide with the value calculated from quadrupole radiation to better than 0.5%.

PSR J0737−3039A/B is the first and so far the only discovered double pulsar system (Burgay et al., 2003; Lyne et al., 2004), which shows even stronger relativistic effects as the orbital period is only 0.1 day. The tracking of the orbital motion of the A pulsar (with 22.7-ms spinning period) yields the best ever test of GR in the strong field regime. This is fulfilled by modelling the orbital variation on top of Newtonian motion (Kramer et al., 2006), which can be described by a set of post-Keplerian (PK) parameters (see Section 2.4.3 for more details). Each of them can be written as a function of the two masses. Once they are obtained, the results can be collected on a mass-mass plane as demonstrated in
Fig. 1.6: Predicted and observed accumulated shift of periastron passage time, due to the orbital decay caused by gravitational radiation. The figure is reproduced from Weisberg et al. (2010).

Fig. 1.7 which shows the latest measurements from the double pulsar (Kramer & Wex, 2009). For any viable theory of gravity, all curves on the plane yielded by PK determinations are supposed to intersect at a single point. The test in Fig. 1.7 achieves an agreement between observations and GR prediction with an uncertainty of only 0.05% (Kramer & Wex, 2009).

1.7.2 Neutron star-white dwarf binaries

Generally, most neutron star–white dwarf (NS-WD) systems are less relativistic than DNSs due to the lower mass of the white dwarfs. However, the significant mass difference of the two objects, corresponding to an asymmetric distribution of the gravitational self-energy, provides the advantage in testing alternative gravitational theories. In detail, Einstein’s GR is thought to be the only theory of
1.7. CELESTIAL LABORATORY IN RELATIVITY TEST

Fig. 1.7: Mass-mass diagram of the double pulsar system PSR J0737−3039, summarizing all measured PK parameters (periastron advance $\dot{\omega}$, gravitational redshift/time dilation $\gamma$, orbital period decay $\dot{P}_b$, Shapiro delay parameters $r$ and $s$, the rate of spin precession of B pulsar $\Omega_{\text{SO}}$) and the derived mass ratio $R$. This figure is reproduced from Kramer & Wex (2009).

gravity that satisfies the *Strong Equivalence Principle*, which predicts the equivalence of inertial mass and gravitational mass. The violation of this principle, allowed by other gravity theories such as Tensor-Scalar theory, would result in non-zero dipole gravitational emission and a variation in gravitational constant which GR does not predict. Such consequences can be constrained by measuring the variation of orbital period in a NS-WD system and extracting the contributions by other effects (e.g. Damour & Esposito-Farese, 1996; Stairs, 2003). Precision timing of a few MSPs with white dwarf companion has already yielded such tests (Lange et al., 2001; Verbiest et al., 2008; Lazaridis et al., 2009), and so far no announcement of contradiction to GR has been made.


1.7.3 Neutron star-black hole binaries

A pulsar–black hole (PSR-BH) binary system would be the ‘next generation of celestial laboratory’ in testing theory of gravities, surpassing all current and foreseeable competitors in testing GR (Damour & Esposito-Farèse, 1998). Although previous pulsar surveys have not been succeeded in finding such desirable systems, the discovery would be most likely to be yielded by the next generation of radio telescopes (Kramer et al., 2004). Studies on the orbital motion of the pulsar can not only produce tests of GR in a new parameter space, but also lead to the measurement of the BH properties and validation of the ‘Cosmic censorship conjecture’ and the ‘no-hair theorem’ (Wex & Kopeikin, 1999; Kramer et al., 2004). Details of the whole topic will be discussed in Chapter 5.

1.8 Searching for binary pulsars

Finding pulsars involves detecting the periodic signal within a given length of data. In the standard procedure of searching for isolated pulsars, the dataset is firstly summed across the bandwidth regarding a set of DM trials so as to compensate for the ISM delay. Then a Fast Fourier transform (FFT) is performed to the time series and as the pulsar is rotating with a stable period, in the frequency domain a set of sharp features (harmonics) are expected with the zeroth one identified as the rotational frequency. The frequency power can be greatly smeared if the period of the signal is changing within observing time due to the acceleration of the pulsar in a binary orbit. Therefore, the application of an algorithm to compensate for the Doppler shift is required to recover the S/N in detection.

1.8.1 Constant acceleration search

For wide orbit systems where the observing time covers only a small fraction of the orbital phase, a constant acceleration of the source can be assumed and
is the only additional unknown that needs to be determined. The search can
be performed in both time and frequency domains. One approach in the time
domain, is before Fourier transforming the time series to resample the time series
by applying a correction to the time interval in the observed frame, \( t \), to obtain
the corresponding interval in the pulsar frame, \( \tau \), which can be expressed by (e.g.
Lorimer & Kramer, 2005)

\[
\tau(t) \simeq \tau_0(1 + a_l t/c),
\]

where \( a_l \) is the radial acceleration of the pulsar along the line of sight, \( \tau_0 \) is a
normalisation constant, \( c \) is the speed of light and higher order terms in \((v/c)\) have
been neglected. For a given \( a_l \), the new time series can be created by calculating
the new intervals based on Eq. (1.15) and compensating for the relative phase
shift between \( t \) and \( \tau_0 \) due to the earth’s motion in the solar system. The so-
called ‘acceleration search’ can then be carried out for the corrected time series by
assuming different trial values of \( a_l \), so as to cover a range of acceleration values.

Another approach that works in the frequency domain is the ‘stacking’ technique.
Here, the integration is firstly divided up into a number of contiguous subsets,
each of which is then Fourier-transformed, separately. The individual segments
corresponding to different orbital phases are then supposed to show a continuous
change in rotational frequency, which then can be compensated for by applying
a shift before stacking them together. For an observation of length \( T \), the drift
in spin frequency \( \nu_0 \) due to a constant acceleration in unit of Fourier bin \((1/T)\)
is simply

\[
N_d = a_l \nu_0 T^2/c.
\]

So if the integration is split into \( n \) segments, the shift in between is just \( N_d/n \).

While this stacking technique is less sensitive than the coherent acceleration
search (Faulkner, 2004), it brings along benefit from two aspects. One is that
the computational cost is saved during the operation of FFT. The other is that
an accelerated signal has less time to drift within each segment, which may allow
a smaller step size in the acceleration trials.
1.8.2 Phase modulation search

When the observing time is comparable to, or even longer than a whole orbital period, the assumption of a constant source acceleration certainly does not hold. In this case, an alternative approach capable of tracking the signal from the full range of orbital phase is needed. In Jouteux et al. (2002) and Ransom et al. (2003), a technique called ‘phase modulation search’ was developed, which is optimized when the observation covers several orbits. The method of search is based on the fact that the power spectrum of an observation encompassing a full orbit shows a characteristic shape induced by the periodic and orbital phase dependent spin frequency, which is retained and accumulated when the observation extends. The spacing of the feature is found to be simply the orbital period, which can be obtained by performing a Discrete-Fourier-transform of that region. The orbital semi-major axis and epoch of ascending node can then be determined from the width and phase of the feature.

1.8.3 Additional tools

None of the methods mentioned above is optimized when the actual observing time is comparable to one orbital period, which leaves a search sensitivity gap. A potential technique to compensate for this gap is the ‘dynamic power-spectrum search’. As in the stack search, the time series is divided into a number of segments and Fourier-transformed, separately. Then the individual power spectra can be harmonically added and plotted on a frequency versus time plane, where the orbital phase related pulsar signal is expected to show a sinusoidal pattern. The plot can be further investigated both visually and systematically for orbital parameters (e.g. Lyne et al., 2000; Chandler, 2003).

The search for binary pulsars requires additional parameter space investigation, which greatly increases the computational cost. Besides the current computer clusters that have been used to reduce the search data, recently developed
techniques such as the Graphics Processing Units (GPUs) and the global volunteer distributed computing project Einstein@home have already been applied to pulsar data reduction (Barsdell et al., 2010; Knispel et al., 2010). These significant increases in computing power may also solve the sensitivity gap problem, by performing a coherent search into the space of orbital parameters that can fully describe the binary motion. Therefore, the efficiency and effectiveness of binary searches can be expected to be significantly improved in the near future, which may lead to the first discovery of a PSR-BH system.

1.9 Thesis structure

The rest of the thesis is divided into several chapters as presented below:

Chapter 2 introduces the whole procedure of the pulsar timing technique and discusses the improvement that can be yielded by the next generation of radio telescopes.

Chapter 3 reviews most issues that may influence profile shape stability and precision of pulse time-of-arrival (TOA) measurement, and gives prediction for TOA precision by the next generation of radio telescopes.

Chapter 4 presents an analysis on integrated profile stability of several MSPs and estimate the phase jitter of PSR J0437–4715.

Chapter 5 performs a detailed simulation to investigate the measurability of the BH properties by pulsar timing and the potential of testing GR’s comic censorship conjecture and no-hair theorem.

Chapter 6 is a summary of the results with a prospect for future work.
CHAPTER 1. INTRODUCTION
Chapter 2

Pulsar Timing analysis

This chapter introduces the main procedures of the pulsar timing technique, which is used in many aspects of pulsar science and is directly related to the application of binary pulsar studies in testing General Relativity (GR). Timing with high precision by both the current and the next generation of radio telescopes requires a detailed understanding of the entire process so as to either avoid, or correct, for potential systematic errors, and thereby reveal as much information, about the phenomenon being studied as possible.

2.1 Introduction

Pulsars are known to be stable rotators (e.g. Hotan et al., 2006; Verbiest et al., 2009). Their clock-like nature allows investigation of the space-time associated with the object, which can be achieved by measuring the pulse time-of-arrivals (TOAs) and maintaining the counts of pulses across a long time baseline. This timing technique can be performed on all known pulsars, but the highest precision can be achieved for millisecond pulsars (MSPs) because of their rapid and stable rotation. In Fig. 2.1 the major data processing hardware and software stages involved in timing observation and the subsequent data reduction used within the framework of this thesis are presented. In the following sessions detailed
Fig. 2.1: Data path of pulsar timing. Instrumental configuration that produces the data used in this thesis is presented for illustration. The labels in the ellipse show the purpose of each step.

information will be provided regarding: instruments, TOA determination and TOA monitoring. Descriptions of these issues can also be found in van Straten (2003) and Verbiest (2009). Potential improvements of pulsar timing that can be expected with future radio telescopes are discussed at the end.

2.2 Instrumentation

An astronomical radio signal can be described as a plane-propagating, transverse electromagnetic wave with two orthogonal senses of polarisation. After being focused by the radio antenna, the signal is then collected by a receiver designed to have two probes sensitive to both polarisations. The signals received by these probes are converted into complex voltages, and amplified by a cryogenically cooled low-noise amplifier. To illustrate the basic characters of a receiver, details of both the Parkes 21 cm Multibeam (MB, Staveley-Smith et al., 1996b) and the
Table 2.1: Basic characters of MB and H-OH receiver. The parameters $f$, $T_{rec}$ and Pols represent frequency range, receiver temperature and number of polarisations, respectively. Both receivers have a dual-linear feed (L) configuration and the MB consists of 13 feed horns.

<table>
<thead>
<tr>
<th></th>
<th>$f$ (GHz)</th>
<th>$T_{rec}$ (K)</th>
<th>Pols</th>
</tr>
</thead>
<tbody>
<tr>
<td>MB</td>
<td>1.23-1.53</td>
<td>23.5</td>
<td>26×L</td>
</tr>
<tr>
<td>H-OH</td>
<td>1.2-1.8</td>
<td>28</td>
<td>2×L</td>
</tr>
</tbody>
</table>

H-OH receiver, which were used to collect the data in this thesis, are shown in Table 2.1.

In order to minimise the attenuation during data transfer through cables, the signal is now down-converted from the observational radio frequency (RF) $f_{RF}$ to an intermediate frequency (IF) $f_{IF}$. This is achieved by a mixer that beats the RF signal with a monochromatic signal of frequency $f_{LO}$ generated from a local oscillator (LO), and therefore we have $f_{IF} = f_{RF} - f_{LO}$. For a frequency range that satisfies $f_{RF} > f_{LO}$, the down-converted signal will be an exact copy of the original bandpass, and is called upper sideband. For the opposite case, the bandpass is mirrored as well as being transferred in frequency, and is called lower sideband. The signal is then passed through a bandpass filter to remove any harmonics induced by interfering signals outside the band-of-interest. Afterwards, the signal is either directly sent to the data acquisition device, or further down-converted to baseband resulting in a frequency range of $[0, \Delta \nu]$, where $\Delta \nu$ is the bandwidth. Baseband conversion includes an additional step called band limiting, where the signal is sent through a low-pass filter so as to remove power from frequencies higher than the Nyquist frequency to avoid aliasing pollution.

The signal is now suitable for analysis with the pulsar specific back-end, which folds the data with respect to an input source ephemeris containing information of its rotational period, period derivative, binarity, etc, to produce integrated profiles. Two types of pulsar back-end were used for the work of this thesis and a brief description is presented below:

- **CPSR2**: The second generation Caltech-Parkes-Swinburne Recorder, is a
coherent de-dispersion baseband system that simultaneously samples two 64 MHz observing bands with 2-bit resolution. For observations around 1.4 GHz, these two bands are located adjacent to each other, with central frequencies at 1341 and 1405 MHz, respectively. During the digitisation, a dynamic level setting strategy is applied to optimise the sampling threshold (details follows in Chapter 3). In order to phase-coherently correct the dispersion smearing (see Section 1.4.1 for details), the signal is convolved in the Fourier-domain with the inverse of the dispersion impulse response function derived from Hankins & Rickett (1975). The data are finally folded in near-real time, keeping frequency resolution of 128 0.5-MHz frequency channels and 1024 bins per pulse period. More details are contained in Hotan et al. (2006); van Straten (2003); Verbiest (2009).

- DFB: The Pulsar Digital Filter Bank, is a system capable of 8-bit sampling and processing up to 1 GHz of bandwidth in four primary modes (folding, search, spectrometer, baseband). Here the time series are converted into a number of frequency sub-bands by using a polyphase filtering technique, which involves a combination of Finite Impulse Response filters and a FFT to shift each band to baseband. The pulsar-processing unit collects the outputted frequency channels and produces individual integrations. There are eight settings (1024, 512, 256, 128, 64, 32, 16 MHz) for the input bandwidth and four options for the number of frequency channels (512, 1024, 2048, 4096) for the output data. Detailed descriptions can be found in Hampson & Brown (2008).

After integrated profiles are created on-line, follow-up reductions are still required to correct for any instrumental response and to recover the real profile shape. For low-bit (e.g. 2-bit) sampled data digitisation artefacts called “quantisation noise” needs to be removed. The full Stokes information also has to be recovered by polarisation calibration. Details on such issues are discussed in the next Chapter.
2.3 TOA measurement

Once an integrated profile is formed from a particular observing session, the next step is to estimate the equivalent TOA, defined as the fiducial point on the profile. Ideally, the uncertainty in a TOA is mainly caused by the additive noise (from system temperature\(^1\)) on the observed profile, and can be written in the form of (Downs & Reichley, 1983):

\[
\sigma_{rn} = \frac{1}{\beta} \cdot \frac{S}{N_1} \sqrt{\frac{\Delta}{N}}. \tag{2.1}
\]

Here \(S/N_1\) is the averaged single pulse peak signal-to-noise ratio (S/N) and

\[
\beta = \sqrt{\int [U'(t)]^2 dt} \tag{2.2}
\]

is the pulse sharpness parameter, with \(U(t)\) the peak-normalised pulse waveform and

\[
\Delta = \int \frac{\langle n(t)n(t+\tau) \rangle}{\sigma_n^2} d\tau \tag{2.3}
\]

is the noise de-correlation time scale where \(n(t)\) is the noise series. It can be seen that the sharpness parameter basically relates the intrinsic profile shape to the precision of the TOA. The derivation can be found in Appendix B.1.

The TOA can be measured by the so-called template matching technique, where a high S/N template formed from the sum of previous observations, is cross-correlated with the targeted profile and a least-squares fit is then performed to find the best estimate of the phase shift between them. Detailed description and limitations of the method will be discussed in Section 2.3.2.

\(^1\)In radio astronomy, system temperature is the sum of contributions from the sky background radiation, the receiver temperature, ‘spillover noise’ from the ground, and the emission from the atmosphere.
2.3.1 Algorithm

The template matching method is based on the assumption that the profile can be broken into a few components:

\[ P(t) = a + bT(t - \tau) + n(t), \]  

where \( P(t) \) represent the profile, \( T(t) \) stands for the template, \( a \) is an arbitrary offset, \( b \) is the scaling factor and \( \tau \) is a phase shift. The cross-correlation can be performed in either the time-domain or the frequency-domain, and here the description by Taylor (1992) is shown for demonstration. After performing a FFT, Eq. (2.5) can be rewritten as:

\[ P_k e^{i \theta_k} = aN + bT_k e^{i(\phi_k + k\tau)} + n(k), \quad k = 0, \cdots, (N - 1), \]  

where \( N \) is the number of bins in the profile, and \( \theta_k, \phi_k \) are the phase of the transformed profile and template, respectively, in the \( k \)-th bin. The baseline offset is then readily expressed as \( a = (P_0 - bS_0)/N \). For a first estimate of the phase shift, one can calculate the Cross-Correlation Function (CCF) of \( P_k \) and \( T_k \), select several harmonics (e.g., eight) and perform an inverse FFT of them. The location of the main peak of the new function then yields the shift between the template and the profile. Afterwards, a more precise estimation of the phase shift \( \tau \) and scaling factor \( b \) can be obtained by minimizing the goodness-of-fit statistic written as:

\[ \chi^2(b, \tau) = \sum_{k=1}^{N} \left| \frac{P_k - bT_k e^{i(\phi_k - \theta_k + k\tau)}}{\sigma_k} \right|^2, \]  

where \( \sigma_k \) is the root-mean-square intensity of the noise at frequency \( k \). Note that in the frequency-domain the signal \( P_k \) and \( T_k \) normally fall off as a function of \( k \), much faster than the noise \( \sigma_k \) does. Consequently, we can treat the \( \sigma_k \)s as constant in the following calculation. The \( \chi^2 \) expression can then be reduced to:

\[ \chi^2 = \sigma^{-2} \sum_k (P_k^2 + b^2T_k^2) - 2b\sigma^{-2} \sum_k P_k T_k \cos(\phi_k - \theta_k + k\tau). \]  

(2.7)
At the minimum of $\chi^2(\tau, b)$, its partial derivative corresponding to $\tau$ needs to be zero:

$$\frac{\partial \chi^2}{\partial \tau} = \frac{2b}{\sigma^2} \sum_k k P_k T_k \sin(\phi_k - \theta_k + k\tau) = 0. \quad (2.8)$$

This equation can be used to obtain $\tau$ by Brent’s method (Press et al., 1986). The corresponding uncertainty can be yielded by considering where $\Delta \chi^2 = 1$ near its minimum, which to the second order term of a Taylor expansion, leads to

$$\sigma^2_{\tau} = \left(\frac{\partial^2 \chi^2}{\partial \tau^2}\right)^{-1} = \frac{\sigma^2}{2b \sum_k k^2 P_k T_k \cos(\phi_k - \theta_k + k\tau)}. \quad (2.9)$$

### 2.3.2 Limitation

The algorithm produces the TOA uncertainty as calculated using Eq. (2.1), only if the model in Eq. (2.5) is valid which may not be true in reality. We now consider a few cases where this assumption breaks down.

Practically, the pre-formed template is not entirely noise-free. If the S/N level of the profile is close to that of the template, the noise of both will limit
the least-squares fit and make the method less precise. For illustration, an analytic template was created from the profile of PSR J0437–4715, and used to cross-correlate with profiles of identical intrinsic shape and different S/N. Three different noise levels were also applied to the template to demonstrate the loss of precision in Fig. 2.2. The cross-correlation based on the noise-free template yields exactly the same TOA precision as expected by Eq. (2.1). With the application of noisy templates, the resulting TOA precision deviates from the expectation when the S/N of the profile is approaching that of the template, and is hardly improved by increasing profile S/N once the template is more noisy.

The problem can be solved by removing the noise in the template, which can be achieved by a variety of methods such as Guassian components fit (e.g. Kramer et al., 1999) and wavelet smoothing (Percival & Walden, 2000). However, for very high S/N templates one can see remained features after subtracting the best modelled shape. In the example of Fig. 2.3, a 19-component model was fitted to the data, and a least-squares fit yields a reduced $\chi^2 \approx 25$ suggesting a significant insufficiency of the model. Note that the S/N of the template is about 3200, the 5% difference between the model and the data seen in the small panel

Fig. 2.3: The profile of PSR J0437–4715 and the residuals after subtraction of a Guassian component fit to the profile.
in Fig. 2.3 is clearly above noise level. The addition of further components hardly improves the goodness of fit, which indicates that the intrinsic profile cannot be fully described by Guassian functions, and requires more complex modelling.

If the intrinsic profile shape appears to be significantly different from the template, which can happen under a variety of situations discussed in the next chapter, the TOA uncertainty estimated in template matching is worse than expected in Eq. 2.1 for high-S/N profiles. Fig. 2.4 shows a simulated example of this case. Here a template is created as a Gaussian and fake observation profiles are formed by adding white noise to the template, after either broadened or narrowed by 0.5%. It is clear that the calculated TOA errors begin to deviate from the predicted uncertainty once the S/N of the observed profile rises to values beyond 1000. Note that the deviations are seen to be very close to each other for both the broadened and narrowed cases, which indicates that it is not the absolute pulse shape of the observation determining the reliability of the TOA uncertainty, but the relative difference between the observation and the noise-free template. These results stress the importance of using reliable template profiles. The different results in the S/N−σ graphs presented by Verbiest et al. (2010) and Hobbs et al. (2009) already suggested this to be the case.

If the shape difference is stable, the problem can be solved, by either subtracting the difference before cross-correlation, or producing a new template based on the new observations. If the shape difference is switching between modes, multiple templates would be required to perform TOA calculations depending on the state of the observed profile. Then one only needs to fit for the fiducial phase offset when later monitoring the TOAs. However, for the case that the difference tends to vary randomly, further modelling would be necessary for the shape relation between the profile and the template.

When the S/N of the profile is not high enough (e.g., S/N\(\lesssim\) 10), the technique fails to estimate the TOA accurately as expected. For illustration, a set of Monte Carlo simulations were performed to examine the accuracy of TOA estimates,
Fig. 2.4: Simulation showing the difference between ideal template matching (solid line) and two cases with profile distortion (dashed line: profile width increased by 0.5%; dotted line: width decreased by 0.5%). The small inside panel shows an expanded view of part of the dashed line and the dotted line.

where profiles are created by summing a Gaussian template with white noise of different rms. The results are shown in Fig. 2.5, where it can be seen that the confidence of the error produced by template matching deviates greatly from the 1-σ confidence in the low S/N region. To avoid this, integration needs to be performed for long enough to result in a strong enough signal.

2.4 Interpreting TOAs

Pulsar timing needs to be performed with respect to the pulsar’s inertial reference frame. Therefore, after a series of TOAs are collected, they are then transferred from the observatory local time to the pulsar proper time. In a reference frame co-moving with the pulsar, the $N$-th count of the rotation since $T_0$ can be approximated via a Taylor expansion:

$$N = \nu_0 (T - T_0) + \frac{1}{2} \dot{\nu}(T - T_0)^2 + \frac{1}{6} \ddot{\nu}(T - T_0)^3 + \cdots,$$

where $\nu_0$ is the spin frequency at $T_0$. Next we discuss the time transfer process.
Fig. 2.5: Confidence test of template matching for a wide range of profile S/N value. The technique tends to underestimate the TOA uncertainty if the S/N is significantly less than 10.

### 2.4.1 Time standard

The TOAs are typically measured with reference to a local time defined by a hydrogen maser clock at the observatory. The local time is compared to Coordinated Universal Time (UTC) by using the Global Positioning System (GPS). UTC is determined to be an integral numbers of seconds from International Atomic Time (TAI), and the difference is an algebraic sum of leap seconds which is inserted in UTC due to the irregular rotation of the Earth and needs to be removed to obtain TOAs in a smoothly running timescale. The TAI can be further converted to Terrestrial Time (TT) which uses the International System of Units (SI) second as a unit and can be considered as the time of an ideal atomic clock on the geoid. This timescale is used in the timing analysis by correcting the initial TOAs to TT, so as to represent the ideal geocentric time.
2.4.2 Timing model

Note that the topocentric TOAs are not in a stable reference frame with respect to the pulsar, due to the orbital motion of the Earth around the sun and relative motion between the solar system barycentre (SSB) and the pulsar. The first thing that needs to be corrected for is the time-depandant geometric propagation delay, which is simply the Euclidean distance from the observatory to the pulsar divided by the speed of light. Consider $dD_0$ is the pulsar position (line of sight) vector at epoch $t_0$ (where $D_0$ is a unit vector), $r$ is the observer position vector from the SSB, and $k$ is the displacement of the pulsar since $t_0$ due to the relative motion. The geometric distance is given by $D_G = |dD_0 + k - r|$, which can be further written as

$$D_G = (|d|^2 + |k|^2 + |r|^2 + 2dD_0 \cdot k - 2dD_0 \cdot r - 2k \cdot r)^{1/2}$$

(2.11)

$$= d(1 + \frac{1}{2}A - \frac{1}{8}A^2) + \cdots ,$$

(2.12)

where

$$A = \frac{|k|^2}{d^2} + \frac{|r|^2}{d^2} + \frac{2D_0 \cdot k}{d} - \frac{2D_0 \cdot r}{d} - \frac{2k \cdot r}{d^2}.$$  

(2.13)

Dropping the $|d|^{-2}$ and higher order terms, and using subindex “∥” and “⊥” to denote components parallel and perpendicular to the line of sight, respectively, one finds

$$D_G = d[1 + \frac{|k|^2}{2d^2} + \frac{|r|^2}{2d^2} + \frac{D_0 \cdot k}{d} - \frac{D_0 \cdot r}{d} - \frac{k \cdot r}{d^2} - \frac{(D_0 \cdot k)^2}{2d^2} - \frac{(D_0 \cdot r)^2}{2d^2} + \frac{(D_0 \cdot k)(D_0 \cdot r)}{d^2}]$$

$$d^2$$

$$= d + k_\parallel - r_\parallel + \frac{k_\perp^2}{2d} + \frac{r_\perp^2}{2d} - \frac{k_\perp \cdot r_\perp}{d}.$$  

(2.14)

The displacement vector can be written in a time dependent form:

$$k = v(t - t_0) + \frac{1}{2}a(t - t_0)^2.$$  

(2.15)

Note that the acceleration usually contributes only a small fraction to $r$ in 20 yr time (Bell & Bailes, 1996), and therefore can be considered as a higher order
term. Neglecting the time-invariant components and higher order corrections, the time correction can be written as

\[
\Delta_{\text{geo}} = \frac{v_\parallel (t - t_0)}{c} - \frac{r_\parallel}{c} + \frac{r_\perp^2}{2dc} - \frac{v_\perp(t - t_0) \cdot r_\perp}{dc} + \left[ \frac{v_\perp^2(t - t_0)}{2dc} + \frac{1}{2c} a_\parallel(t - t_0)^2 \right].
\]  

(2.16)

The first term is the radial Doppler shift which induces a bias in rotational period estimation. The second is the arrival time correction from the observer to the SSB in far field regime (known as the Römer delay), without considering the curvature of the wavefront. The third is the annual parallax, which is measurable in timing only for a few nearby MSPs. The fourth corresponds to the annual effect of proper motion, and the last is the Shklovskii effect representing the secular change of the redshift, which induces an apparent spin frequency derivative. More detailed descriptions of the geometric delays can be found in Edwards et al. (2006).

Additionally, two further types of GR corrections to the light travel time to compensate for the gravity field of the large objects in the solar system should also be considered. The Shapiro delay due to the curvature of space-time in this case can be written in the form of (Shapiro, 1964)

\[
\Delta_{S\odot} = -2 \frac{GM}{c^3} \ln \left[ \frac{s \cdot r^E + r^E}{s \cdot r^P + r^P} \right].
\]  

(2.17)

Here, \( M \) is the mass of the object, \( r^P \) is the pulsar position relative to that object and \( r^E \) is the telescope position relative to it at the closest approach of the photon. Usually, only the effect of the Sun and in some cases Jupiter needs to be taken into consideration (Backer & Hellings, 1986). The Einstein delay describes the integral effect of time dilation as a result of the motion of the Earth and gravitational redshift by other objects in the solar system. Here it can be expressed as (Backer & Hellings, 1986):

\[
\frac{d\Delta_{E\odot}}{dt} = \frac{GM}{c^2 r^E} + \frac{v_E^2}{2c^2} - \text{constant},
\]  

(2.18)

where \( r^E \) is the distance between the Earth and the object and \( v_E \) is the velocity of the Earth relative to the sun.
Computation of the solar system geometries requires accurate information of the Sun and all major planets, and we usually use a solar system ephemeris (e.g., DE405) published by Jet Propulsion Laboratory (JPL).

Besides the travel time corrections in vacuum, the interstellar medium (ISM) contributes a frequency dependent delay as shown in Section 1.2. The variation of dispersion measure (DM), which can be induced by both relative motion of the pulsar & the Earth and the intrinsic evolution of the ISM, will lead to a time dependent delay in the form of \( \delta t_{\text{DM}} = -D \times \delta DM \), where \( D \) is the dispersion constant. The correction for this effect can be achieved by performing simultaneous multi-wavelength observations and measuring the relative DM change (You et al., 2007). Note that due to the frequency dependence of the profile shape and possibly the fiducial phase, the measurement of the DM value will be biased but not the relative variation if the bias is a constant.

### 2.4.3 Binary correction

All the above effects concern the time transfer to an isolated pulsar frame. If the pulsar is in a binary system, further corrections from the binary barycentre to the pulsar inertial reference frame are still needed. Most commonly, the Römer, Einstein and Shapiro delay which have been introduced before in the SSB time transfer, are also required here to account for the effects induced by the companion. The scale of Römer delay is mainly the orbital size in light seconds (e.g., \( 10^0 \)–\( 10^2 \) s). The values of the Einstein and Shapiro delay depend on the gravitational field strength, binary orbital period, and geometric configuration, and can be of order 1–\( 10^2 \) ms and 1–\( 10^2 \) \( \mu \)s, respectively, for close orbits. The expressions for the three effects are collected below (Damour & Deruelle, 1986):

\[
\Delta_{\text{RB}} = x[\cos E - e(1 + \delta_r)] \sin \omega + x \sin E \sqrt{1 - e^2(1 + \delta_\theta)^2} \cos \omega, \quad (2.19)
\]
\[
\Delta_{\text{EB}} = \gamma \sin E, \quad (2.20)
\]
\[
\Delta_{\text{SB}} = -2r \ln \Lambda, \quad (2.21)
\]
where

\[
\Lambda = 1 - e \cos E - s \left( \sin \omega (\cos E - e) + \sqrt{1 - e^2} \cos \omega \sin E \right).
\] (2.22)

The \( E \) and \( A_T \) terms are the eccentric anomaly and the true anomaly, respectively, which can be obtained by solving the Keplerian orbital equations:

\[
E - e \sin E = \frac{2\pi}{P_b} \left[ (t - T_0) - \frac{1}{2} \frac{\dot{P}_b}{P_b} (t - T_0)^2 \right],
\] (2.23)

\[
A_T(E) = 2 \arctan \left[ \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \right].
\] (2.24)

The \( \omega \), \( x \), \( e \), \( P_b \) and \( T_0 \) are the classic Keplerian orbital parameters: orbital periastron, projected semi-major axis, eccentricity, orbital period and epoch of periastron passage, respectively. Especially, \( \omega \), \( x \), \( e \) and \( P_b \) have a time dependency in GR due to the emission of gravitational waves. The other parameters in Eq. (2.19)-(2.24) represent the post-Keplerian (PK) contributions to the time correction, and for the case of point masses are predicted by GR to the lowest order to be:

\[
\dot{\omega}_{pn} = 3T_\odot^{2/3} \left( \frac{P_b}{2\pi} \right)^{-5/3} \frac{1}{1 - e^2} (m_p + m_c)^{2/3},
\] (2.25)

\[
\gamma = T_\odot^{2/3} \left( \frac{P_b}{2\pi} \right)^{1/3} e^{1/3} \frac{m_c (m_p + 2m_c)}{(m_p + m_c)^{4/3}},
\] (2.26)

\[
\dot{P}_b = -\frac{192\pi T_\odot^5}{5} \left( \frac{P_b}{2\pi} \right)^{-5/3} f(e) \frac{m_pm_c}{(m_p + m_c)^{1/3}},
\] (2.27)

\[
r = T_\odot m_c,
\] (2.28)

\[
s = \sin i = T_\odot^{-1/3} \left( \frac{P_b}{2\pi} \right)^{-2/3} x \frac{(m_p + m_c)^{2/3}}{m_c},
\] (2.29)

\[
\delta_r = T_\odot^{2/3} \left( \frac{2\pi}{P_b} \right)^{2/3} \frac{3m_p^2 + 6m_pm_c + 2m_c^2}{(m_p + m_c)^{4/3}},
\] (2.30)

\[
\delta_\theta = T_\odot^{2/3} \left( \frac{2\pi}{P_b} \right)^{2/3} \frac{7m_p^2 + 6m_pm_c + 2m_c^2}{(m_p + m_c)^{4/3}},
\] (2.31)

\[
\dot{x} = \frac{2}{3} \frac{\dot{P}_b}{P_b},
\] (2.32)

\[
\dot{e} = -\frac{304}{15} e T_\odot^5 \left( \frac{2\pi}{P_b} \right)^{8/3} \frac{1 + (121/304)e^2}{(1 - e^2)^{5/2}} \frac{m_pm_c}{(m_p + m_c)^{1/3}},
\] (2.33)
where

\[ f(e) = \frac{1 + \left(\frac{73}{24}e^2 + \frac{37}{96}e^4\right)}{(1 - e^2)^{7/2}}, \]  
(2.34)

\[ T_{\odot} = \frac{GM_{\odot}}{c^3}. \]  
(2.35)

The determination of the Keplerian parameters will set initial constraints on the masses of the two bodies by giving the mass function as

\[ f(m_p, m_c) = \frac{(m_c \sin i)^3}{(m_p + m_c)^2} = \frac{4\pi^2 x^3}{T_{\odot} P_b^2}. \]  
(2.36)

Assuming \( m_p \approx 1.4 M_\odot \), a lower limit on \( m_c \) can be obtained by letting \( i = 90^\circ \). Moreover, note that all the PK parameters are functions of the mass of the two bodies, in principle measurements of two of them would lead to a determination of the two masses, and further measurements of PK parameters can be used to test the predictions of GR as shown in Section 1.7.

As the timing baseline extends, \( \dot{\omega} \) is usually the first measured PK parameters, especially for compact binary systems. However, the observed \( \dot{\omega} \) can also be affected by other effects, especially the spin and quadrupole field of the companion (Wex, 1998; Wex & Kopeikin, 1999). In this case the mass determination by treating the observed \( \dot{\omega} \) as \( \dot{\omega}_{pn} \) is biased and one should switch to the other PK parameters for clean measurements of the masses.

The Einstein delay parameter \( \gamma \) and secular change in orbital period \( P_b \) are another two PK parameters which are commonly used for mass determination purposes. They are both best constrained in short period and highly eccentric orbits. As can be seen from the expression Eq. (2.20), \( \gamma \) has the same eccentric anomaly dependency as part of the Römer delay, so the Einstein delay cannot be well separated until significant advance of the periastron has occurred. The orbital decay by gravitational wave radiation is the main contribution to the observed \( \dot{P}_b \), apart from special cases such as when the binary is highly accelerated in the Galactic gravitational field, or there is significant mass transfer from the companion star.
The determination of the Shapiro delay parameters \((r\) and \(s)\) are less dependent on the size of the orbit than the other PK parameters discussed above, and so can be used for mass determination in wider orbits. The \(s\) parameter is especially well constrained for edge-on systems, while the \(r\) parameter could be still significant for binaries without high inclination but with significant eccentricity. Note that the measurement of \(r\) and \(s\) are usually highly correlated, an alternative parameterization can be used to characterize the delay signal where the two variables are less correlated (Freire & Wex, 2010).

The relativistic parameters in the Römer delay, \(\delta_r\) and \(\delta_\theta\), describe purely the periodic orbital motion (Damour & Deruelle, 1986). In principle, \(\delta_\theta\) can be extracted only when the orbit has precessed through a significant amount of orbital phase. In this case, the signal from \(\delta_r\) can still be absorbed into the measurement of the pulsar rotational parameters, which means the determination of \(\delta_r\) is even more difficult.

The orbital shrinking induced \(\dot{x}\) and \(\dot{e}\) are less usually determined, because not only their influence is less significant on the timing residuals, but also a variety of other effects can contribute to the observed secular changes, such as proper motion, geodetic precession, Doppler shift variation and so forth. If the companion is a BH, the spin-orbit coupling would produce a significant amount of inclination angle variation and then a secular change in \(x\), which is the case studied in Chapter 5.

### 2.4.4 Further corrections

In addition to the time delays discussed above, there are still other potential effects involved which may become significant under certain circumstance.

The aberration delay describes the correction to the emission time and rotational phase correlation, due to the transfer from a co-rotating frame to inertial frame of the observer. The correction is a constant if there is no relative motion between the pulsar and the observer, but can vary as function of orbital phase for
binary pulsars. Following Smarr & Blandford (1976); Damour & Deruelle (1986), we have

\[ \Delta_{AB} = A[\sin(\omega + A_T(E)) + e \sin \omega] + B[\cos(\omega + A_T(E)) + e \cos \omega], \tag{2.37} \]

where \( A \) and \( B \) are constants related to the rotational frequency of the pulsar, the binary Keplerian parameters and the system geometry. Note that the delay form can be absorbed into the Römer delay through re-parameterisation (Damour & Deruelle, 1986), the measurement of \( A \) and \( B \) is possible only if the system geometry changes on relatively short timescales due to effects like precession.

Another effect associating pulsar emission and orbital motion is the bending delay, which describes the beam deflection due to the gravitational field of the companion. The deflection changes the association between the inertial pulse phase and the observed pulse arrival phase, making the arrival time relatively earlier or later (depending on the system geometry and orbital phase). This effect is an additional correction on top of the Shapiro delay, and would be strong only for very edge-on systems. The expression can be found from Doroshenko & Kopeikin (1995) as:

\[ \Delta_B = \Lambda^{-1}[A_B \cos i \sin(\omega + A_T) + B_B \cos A_T], \tag{2.38} \]

where \( A_B \) and \( B_B \) are constants related to the pulsar rotational frequency, orbital period, masses and system geometry. For a 5-ms pulsar in a 0.1 day orbit with a companion mass of 10 \( M_\odot \), the scale of the delay is found to be \( \lesssim 20 \) ns, which indicates the effect is not significant except for highly edge-on cases.

Furthermore, the spin-induced gravitomagnetic field of the companion can also cause periodic fluctuation in the orbital motion of the pulsar (Thorne et al., 1986), which is called frame dragging effect. Following Wex & Kopeikin (1999), one can have

\[ \Delta_{FD} = \Lambda^{-1}[A_{FD} \cos i \sin(\omega + A_T) + B_{FD} \cos(\omega + A_T)], \tag{2.39} \]

where \( A_{FD} \) and \( B_{FD} \) are constant for a given system. Adopting the same binary parameters as above, the scale of the effect is found to be an order of magnitude
less than the bending delay. Additionally, the expression has the same orbital phase dependency as the bending delay, which makes the detection of the effect even more difficult. However, the frame dragging also induces long-term orbital precession of the pulsar’s orbit, which can be further used to extract information about the BH. Details will be shown in Chapter 5.

2.4.5 Methodology

For a given timing model, an initial guess for the unknowns is required to provide a first phase coherent solution. Then a least-squares fit is performed to the time series to find the global minimum of $\chi^2$ and the best estimates of the variables. The $\chi^2$ is given by

$$
\chi^2 = \sum_i \left( \frac{\Delta x_i}{\sigma_i} \right)^2,
$$

where $\Delta x_i, \sigma_i$ are the $i$th residual after subtracting an estimated timing model, and its corresponding measurement error, respectively. For high precision cases, the uncertainties of the measured parameters can be obtained by the covariance matrix, defined as

$$
C_{\alpha\beta} = \left[ \sum_i \frac{1}{\sigma_i^2} \frac{\partial N(t_i)}{\partial p_\alpha} \frac{\partial N(t_i)}{\partial p_\beta} \right]^{-1},
$$

where $N(t_i)$ is the estimated number of the pulse arriving at time $t_i$ (usually not an integer due to measurement uncertainty), $\sigma_i$ the error in the measurement $N(t_i)$, and $p_k$ the $k$th parameter to fit for. The main diagonal elements of the matrix $C_{\alpha\alpha}$ represent the variances of the fitted parameters, while the other elements $C_{\alpha\beta}$ ($\alpha \neq \beta$) stand for the covariance between them. The whole process can be performed by using the Tempo and Tempo2 software packages (e.g., Hobbs et al., 2006).
2.5 Improvement with future telescopes

Despite the development of the instrumentation in the past forty years since the discovery of pulsars, the precision of current timing observations (especially for MSPs) are still limited by gain. The next generation of radio telescope, providing a dramatic increase of collecting area, will hopefully raise the precision of pulsar timing up to a new height.

2.5.1 Design goals

The Square Kilometre Array (SKA) will be the ultimate aim of future radio telescopes (Schilizzi et al., 2007). The key science of this international project is to address a wide range of questions in astrophysics, fundamental physics, cosmology and particle astrophysics, including gravitational wave background detection and extreme GR tests by using pulsar–black hole binaries. Fig. 2.6 shows an artist’s impression of part of the SKA configuration. In the current design, for the purpose of high sensitivity the antennas are proposed to be densely distributed in the core region of the array, which consists of three sub-arrays: Sparse Aperture Arrays of simple dipole antennas to cover the low frequency range of 70-200 MHz, Dense Aperture Arrays of 3 m×3 m tiles to cover the medium frequency range of 200-500 MHz, and Dish Array of 15 m diameter elements to cover the high frequency range from 500 MHz to 10 GHz. Each of the three will be located a circular region with diameter of about 5 km. Around the central region there is an extended region out to 180 km, which contains more sparsely distributed groups of dishes and pairs of medium and low frequency stations. In order to allow high angular resolution observation of the radio sky, an outer region, which comprises five spiral arms consisting of 20-dish groups, extends up to 3000 km scale. The combination of signals from all antennas will create a telescope with an effective collecting area of about one square kilometre, which will enable an ∼50 times improvement in gain compared with a 100 m dish.
2.5. IMPROVEMENT WITH FUTURE TELESCOPES

The Five-Hundred-Metre Aperture Spherical Radio Telescope will be the largest single dish radio telescope on the Earth (Nan, 2006; Nan et al., 2011). Fig. 2.7 shows its optical configuration and an artist’s impression. The telescope is designed as an Arecibo-type antenna and built on the karst depression, which is sufficiently large to hold the 500-m diameter dish with effective aperture of 300 m. The main reflector consists of $\sim 4400$ triangular elements which allow surface formation from a sphere to a paraboloid in real time via active control. The deep depression and feed cabin suspension system allow a 40° zenith angle, and extended range up to 60° allowing the sky-coverage beyond the galactic center can be achieved by applying feeding technique like Phased Array Feed (PAF) in an upgrade stage (Nan et al., 2011). An order of magnitude improvement in gain can be expected with FAST compared to a 100 m dish. The expected technical specification of the two future telescopes are summarized in Table 2.2.

2.5.2 Benefit

The large collecting area and multibeam capabilities of these two future telescopes will greatly improve the efficiency and effectiveness of pulsar surveys. With the
Fig. 2.7: FAST optical geometry (left) and 3-D model (right), reproduced from Nan et al. (2011).

Table 2.2: Proposed technical specification of the next generation of radio telescopes, SKA and FAST. The $f_{\text{ran}}$, $A_s$, $T_{\text{sys}}$, and $\theta_{\text{res}}$ represents frequency range, sensitivity area, system temperature and angular resolution, respectively.

<table>
<thead>
<tr>
<th></th>
<th>$f_{\text{ran}}$ (GHz)</th>
<th>$A_s$ ($m^2 \cdot K^{-1}$)</th>
<th>$T_{\text{sys}}$ ($K$)</th>
<th>$\theta_{\text{res}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SKA</td>
<td>0.07-10</td>
<td>5000 (70-300 MHz)</td>
<td>$\sim$30</td>
<td>$&lt;1$</td>
</tr>
<tr>
<td>FAST</td>
<td>0.07-3</td>
<td>$\sim$2000 (L-band)</td>
<td>$\sim$20</td>
<td>2.9' (L-band)</td>
</tr>
</tbody>
</table>
SKA it was shown that about 14000 normal pulsars and 6000 MSPs can be found from a whole sky search, with only the 1-km core and a 30-min integration time (Smits et al., 2009a). The FAST 19-beam system can enable a discovery of 5200 pulsars in the Galactic plane, including about 460 MSPs (Smits et al., 2009b). Further deep pointing into M13 and M33 can yield 50-100 extragalactic pulsars (Smits et al., 2009b). The improvement in sensitivity for both these telescopes will also greatly enhance the search for binary pulsars and provide the best ever opportunity to find a pulsar–black hole binary system (Cordes et al., 2004).

The improvement in instrumentation will also allow timing at a significantly higher precision level and for many more MSPs (Smits et al., 2009a; Liu et al., 2011a). This will first greatly improve the precision of current GR tests with binary pulsars (e.g. Kramer et al., 2006). If a pulsar–black hole binary is found with the future telescopes as expected, measurement of the BH properties would also be achievable via precisely monitoring the orbital motion (Wex & Kopeikin, 1999, see Chapter 5 for more details). Furthermore, the sensitivity of a coherent study on a pulsar timing array (PTA), through which upper bounds have already been placed on the stochastic gravitational wave background (GWB, Jenet et al., 2006; van Haasteren et al., 2011; Yardley et al., 2011), can be greatly increased by integrating more sources into the PTA and decreasing the timing uncertainties. This advance would enable even more detailed studies of the GWB, such as its polarisation and velocity (Lee et al., 2008; Lee et al., 2010). Individual gravitational wave sources, such as supermassive black hole binaries at the centre of a Galaxy can also be identified via a future PTA study (e.g. Lee et al., 2011).

Additionally, the wide bandwidth of the future radio telescopes will allow simultaneous observation at multiple frequencies, which enables the determination of DM variation for each observation and the corresponding correction for the TOAs. The increase in gain and frequency range will also greatly improve the current ISM studies (e.g. Bhat et al., 2004; Walker et al., 2008), and achieve more precise corrections to compensate for the scattering and scintillation effects.
in TOA estimation.
Chapter 3

Prospects for High-Precision Pulsar Timing

Timing pulses of pulsars has been proved to be a most powerful technique useful to a host of research areas in astronomy and physics. Importantly, the precision of timing is not only affected by radiometer noise but also by other effects. In this chapter within the framework of current instrumentation we review the known causes of pulse shape variations on short timescales, and assess their effect on the precision and accuracy of a single measurement of pulse time-of-arrival (TOA). Based on this analysis TOA precisions with the next generation of radio telescopes are predicted. This chapter is a revised version of the paper by Liu et al. (2011a).

3.1 Introduction

Pulsars are stable and rapidly rotating radio sources as discussed in Section 1.6. This stability can be exploited through pulsar timing following the procedures presented in Chapter 2. Currently, several MSPs have already been timed at precisions down to a few hundred nanoseconds over time spans of a decade or more (Verbiest et al., 2009).

Profile shape stability is one of the essential factors in achieving precision
timing. Although the integrated profiles of MSPs appear stable over time scales of years, there are a variety of effects that can affect the shape of an integrated profile on short timescales: multi-path propagation in the turbulent ISM, pulse jitter, data processing artefacts and improper calibration, for example. Profile variations from these effects may only change the pulse shape at low levels, but will cause the subsequent TOA calculation to be less accurate and less precise than what is expected if only radiometer noise is contributing to the uncertainty. This will complicate timing with the next generation of radio telescopes since in these cases the timing will be limited by factors other than merely telescope sensitivity.

In order to investigate the level at which short-term instabilities in pulse shape may affect pulsar timing with this new generation of telescopes, we present an analysis of PSR J0437−4715. This pulsar was discovered by Johnston et al. (1993) and is the nearest and brightest MSP known, resulting in outstanding timing precision that has already led to a variety of interesting results (van Straten et al., 2001; Verbiest et al., 2008). Furthermore, the TOA precision of PSR J0437−4715 obtained by current instruments (see e.g. Verbiest et al., 2010) is already comparable to the precision future telescopes may expect to obtain for other less bright MSPs (see Section 3.4), making it a perfect target for investigations of the pulsar timing potential of future telescopes.

The structure of this chapter is as follows. First we describe the observations and data preprocessing in Section 3.2. Next the possible effects involved in profile distortion are reviewed and the results of data reduction are presented in Section 3.3. We conclude with an overview of our main findings and a brief discussion of future research in Section 3.4.
3.2 Observations

The data used in this chapter consist of five long observations of PSR J0437−4715, taken between June 2005 and March 2008 at the Parkes radio telescope. Most observations were taken with the Caltech-Parkes-Swinburne Recorder 2 (CPSR2). On two of the five days the data were taken with the H-OH receiver, on the remaining three days the central beam of the 20 cm multibeam (MB) receiver (Staveley-Smith et al., 1996b) was used, as listed in Table 3.1. During each day of observations the data were folded in near-real time to 16.8 s for the early data and to 67.1 s for the later data (see Table 3.1). Off-source observations of a pulsed noise probe at 45° to the linear feed probes but with otherwise identical set-up, were taken at regular intervals to allow for polarimetric calibration. Details of the instrumentation can be found in Section 2.2.

For the data processing we used the PSRCHIVE software package (Hotan et al., 2004b). We removed 12.5% of each edge of the bandpass to avoid possible effects of aliasing and spectral leakage. Two models named “single axis” and “full reception”, respectively, were used for calibration purposes and details will be presented in Section 3.3.4. Unless otherwise specified in the text, we combined the polarisations into total power (Stokes I) and also the power across the remaining 96 frequency channels. Through the following analysis, TOAs and their uncertainties were determined through the standard cross-correlation approach as described in Section 2.3 (Taylor, 1992), with the fully integrated 2005-07-24 profiles (one for each observing band) unless otherwise stated. Where needed, we used the timing model derived by Verbiest et al. (2008) without fitting for any parameters.

3.3 Issues affecting profile stability

In Fig. 3.1, we summarise the propagation path of pulsar timing data and identify for each stage the phenomena that can affect TOAs and their precision. Some
Table 3.1: Basic features of the selected datasets. Note that the 2005-07-24 dataset was used only to create a timing template and a receiver model.

<table>
<thead>
<tr>
<th>Date</th>
<th>Receiver</th>
<th>Time span (hours)</th>
<th>Number of files</th>
<th>File length (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005-07-24</td>
<td>MB</td>
<td>8.7</td>
<td>1596</td>
<td>16.8</td>
</tr>
<tr>
<td>2005-09-07</td>
<td>MB</td>
<td>9.0</td>
<td>1500</td>
<td>16.8</td>
</tr>
<tr>
<td>2006-12-31</td>
<td>H-OH</td>
<td>7.4</td>
<td>212</td>
<td>67.1</td>
</tr>
<tr>
<td>2007-05-06</td>
<td>H-OH</td>
<td>8.9</td>
<td>152</td>
<td>67.1</td>
</tr>
<tr>
<td>2008-02-24</td>
<td>MB</td>
<td>4.0</td>
<td>180</td>
<td>67.1</td>
</tr>
</tbody>
</table>

Fig. 3.1: Flow chart of the stages involved in the information transition of pulsar signals. The corresponding effects that can lead to profile shape changes are identified for each stage.

of these are instrumental and correctable while others induce a natural limit to timing precision. Detailed discussions of our current knowledge about these issues together with more in-depth investigations based on our data are presented below.

### 3.3.1 Dispersion changes

#### 3.3.1.1 Theory

The dispersion measure (DM) is defined as the integrated free electron density along the line of sight and introduces a delay between two observing frequencies as introduced in Section 1.4.1. There are two ways to correct for this dispersive delay in pulsar timing data. One way is to use a filterbank to split the observing bandwidth up into a finite number of frequency channels, which are subsequently dedispersed with respect to each other. This is called “incoherent dedispersion” because the dispersion is performed post detection when the phase information is no longer available. The alternative approach of “coherent dedispersion” performs
a de-convolution on the Fourier transform of the data stream, without loss of frequency resolution (Hankins & Rickett, 1975), as mentioned in Section 2.2. The coherently dedispersed data is subsequently stored with limited frequency resolution, so any error in the DM value used in the dedispersion, will corrupt the pulse profile permanently.

Because of turbulent motion in the ISM and the relative motions of the pulsar, the Earth and the ISM, the integrated electron density between the pulsar and Earth is continuously changing, implying that ideally the DM value used in on-line coherent dedispersion would be regularly updated to remain close to the current value. Regular updates of the DM values are possible with (near) simultaneous multi-frequency or wide-bandwidth observations (You et al., 2007), but the accuracy of determination by this method is limited by the system sensitivity and complicated by frequency dependent evolution of the profile shape.

3.3.1.2 Discussion

For the PSR J0437−4715 data used for the work here, given 1024 bins across the profile and a DM of $2.644 \text{ cm}^{-3}\text{pc}$, the smearing time (see Eq. 1.11) within a 0.5 MHz wide channel is approximately 0.75 bins. Note that in reality the DM variations of PSR J0437−4715 on timescales of years are below $10^{-3} \text{ cm}^{-3}\text{pc}$ (You et al., 2007), any shape distortion induced by this amount of DM deviation in coherent dedispersion would not be detectable. Therefore, unless the DM variation becomes significantly larger than previously observed, this effect does not affect our current TOA precision. Also note that, since this type of distortion increases the TOA uncertainty by broadening the profile, the uncertainty will still scale following the radiometer equation for future telescopes\footnote{The effect can also be more significant for ultra-broad band receivers where the channel width is wider.}
3.3.2 ISM Scattering and scintillation

3.3.2.1 Theory

As the radio pulses travel through the ionised ISM, multi-path scattering can cause both constructive and destructive interference, observed in the detected signal as apparent brightening and dimming of the pulsar signal (e.g. Cordes, 1986). This scintillation effect, as discussed in Section 1.4.2, is dependent both on observing frequency and on time because of the relative motion of the pulsar, the Earth and the turbulent ISM. When summing frequency channels of an observation that is strongly scintillated, not all frequency channels will contribute equally to the final profile, but effectively a brightness-dependent weighting scheme will be used. In the case where the pulse profile shape varies considerably across the observational bandwidth, such weighting will change the resulting pulse shape as scintles move across the observed bandwidth.

Additionally, as shown in Section 1.4.2, multi-path scattering will cause different delays for signals with different path lengths and thereby effectively broaden the observed profile. For low DM sources observed at high frequency, the mean of this profile broadening is non-zero but usually not significant compared with the limited time resolution of the backend (Cordes & Shannon, 2010). Still, the change of the pulse broadening function (PBF) associated with either fast, stochastic variations in the diffractive delay, or long-term evolution of the refraction angle, will result in instability of the profile. The pulse broadening will be significantly larger for high DM objects observed at low frequency (Cordes & Shannon, 2010). The broadening timescale can be investigated by assuming different scattering models, based on which it is also possible to reveal the intrinsic profile shape by de-convolving the pulse waveform with a theoretical broadening function (Williamson, 1973; Bhat et al., 2004; Walker et al., 2008).
3.3.2.2 Data investigation

The low DM of PSR J0437−4715 (2.644 cm$^{-3}$pc) corresponds to a short broadening timescale $\tau_s (< 10^{-5}$ ms) and a large decorrelation frequency scale (estimated by $\Delta f \sim 1/(2\pi \tau_s) \approx 0.4$ GHz, see Cordes & Lazio, 2001; Bhat et al., 2004). Consequently, given the effective 48 MHz bandwidth\(^2\) of our data, the shape change on short timescales by the PBF variation should be negligible here.

The influence on pulse shape, by the combination of scintillation and profile shape evolution over frequency, can be investigated by dividing the entire bandwidth into sub-bands and carrying out individual template matching. The templates for the sub-bands are produced with the same bandwidth and central frequency as the sub-band averaged profiles. Fig. 3.2 presents an example from the 2005-09-07 dataset. Here the effective bandwidth of 48 MHz is divided into six bands of 8 MHz each. The S/N is calculated from off-pulse bins which show Gaussian statistics. The plot shows that all of the matching processes have identically behaved $S/N-\sigma_{\text{TOA}}$ curves and the same final $\sigma_{\text{TOA}}$ of about 22 ns.

Statistically, the six TOAs with an uncertainty of 22 ns each, are equivalent to a single one of uncertainty $22/\sqrt{6} \simeq 9$ ns. This is the TOA precision we obtain from template matching the fully frequency integrated observation. It means that across the 48 MHz bandwidth of this dataset, TOA precision does not benefit from conducting sub-band template matching. The effect could become significant when the observational bandwidth is comparable to or larger than the scintillation frequency scale. Still, as the effect can be dealt with through the application of frequency-dependent template matching, we therefore conclude that it will not be a limiting factor to TOA precision with either current or future telescopes.

\(^2\)As described in Section 3.2, the total bandwidth is 64 MHz but on either side of the bandpass 12.5% was removed, leaving 48 MHz of effective bandwidth.
3.3.3 Signal digitisation effects

3.3.3.1 Phenomena

Shape distortion induced by instrumental effects obscures the true pulse shape and can therefore be expected to decrease the precision of timing. Two main digitisation effects are pertinent to a low-bit observing system.

The first effect is caused by the underestimation of the undigitised power in a system with low dynamic range (e.g. only 2 bits per sample). As discussed in Jenet & Anderson (1998) the earlier arrival of pulsar emission at the high end of the observing bandwidth causes an increase in undigitised power, and then a decrease of the digitised-to-undigitised power ratio at all frequencies if the output power level is kept constant. As a result, the off-pulse power will be decreased in the rest of the band. This effect can be avoided through dynamically setting the output power levels, which provides the required dynamic output range and therefore does not result in negative off-pulse dips on either side of the pulse profile.
The second artefact is caused by quantisation errors as a second-order distortion, and manifests itself as an increase in white noise uniformly redistributed across all frequency channels which is induced by the increase in pulsed power in one part of the band (Jenet & Anderson, 1998). This scattered power broadens the profile and causes additional pulse shape variations as a function of observing frequency, which decreases the achievable TOA precision.

### 3.3.3.2 Correction

During on-line processing, all CPSR2 data presented were corrected for the low dynamic range artefact by the dynamic output level setting algorithm implemented in dspsr\(^3\) (van Straten & Bailes, 2011). The scattered power was mitigated during off-line processing through application of the correction algorithm implemented in psrchive (Hotan et al., 2004b). Given the uncorrected, mean digitized power \(\hat{\sigma}^2\) in each pulse phase bin, this algorithm inverts Eq. (A5) of Jenet & Anderson (1998) to estimate the mean undigitised power \(\sigma^2\) and the mean scattered power \(A\) via Eqs. (45) and (43) in the same paper, respectively. The effect of correction is demonstrated in Fig. 3.3, which shows the pulse profile formed from the 2005-07-24 dataset with and without application of the algorithm, as well as the difference between the two. The decreased pulse width of the corrected profile allows higher timing precision. Note that the distortion would not change significantly once the back-end settings are stable. This means that the TOA precision would still scale with effective collecting area as described by the radiometer equation. We therefore conclude that this effect does not limit the current TOA precision, and will not limit it for future telescopes either, which are likely to employ digitisers with a higher number of digitisation levels.

\(^3\)http://dspsr.sourceforge.net/
3.3.4 Polarimetric calibration imperfections

3.3.4.1 Theory

When a fixed-linear-feed, alt-azimuth radio telescope tracks a polarised source across the sky, the feed will rotate with respect to the plane of polarisation by the parallactic angle \( \theta \), which is defined as the angle between the object-zenith great circle and the hour circle. The change of \( \theta \) combined with the instrumental response, will result in a variation of the observed Stokes parameters with time. Polarisation calibration, the aim of which is to reveal the intrinsic profile, will correct the time-dependent variation, but this correction will only be partial if any non-orthogonality of the receptors is not fully modelled. In this case, a difference between profiles at different \( \theta \) will be seen even after calibration. In practice, a “single axis” model considering only the differentials in gain and phase for the two linear polarisation probes is usually applied (e.g. Stinebring et al., 1984). The most recent (here mentioned as “full reception”) model, described by van Straten (2004), solves the matrix description of the polarisation measurement equations, accounting for differential gains and phases, as well as for coupling and leakage.
3.3. **ISSUES AFFECTING PROFILE STABILITY**

effects between the receiver feeds.

### 3.3.4.2 Calibration

For comparison, here we used both the “single axis” and the “full reception” model (constructed from the 2005-07-24 dataset) for the calibration of the 2005-09-07 dataset. Fig. 3.4 (a)-(b) shows two hour-long integrated profiles formed from the 2005-09-07 dataset, covering a different range of parallactic angles and calibrated according to the single axis model. The large differences in linear polarisation and position angle demonstrate the imperfection of this calibration model. The difference between the total intensity profile at the two different values of $q$ is shown in subplot (d) of Fig. 3.4. The same profile as in subplot (a), but calibrated with the full reception model, is shown in subplot (c) - this profile is very similar for both observing times, as the difference plot (e) shows. Furthermore, the profiles calibrated with the full reception model compare well with the previously published polarimetry of Navarro et al. (1997), regardless of $q$. The remaining difference is clearly far less than 2% of the total intensity, and is therefore below the uncertainty level of the calibration, as quantified through simulations by Ord et al. (2004). Further simulation shows that the profile difference in plot (e) will induce TOA errors of less than 30 ns. It is therefore clear that the full reception model removes all polarisation calibration artefacts down to the level of our current TOA precision on PSR J0437−4715. We hence conclude that polarimetric calibration does not limit the current TOA accuracy above 30 ns. Though the application of such calibration schemes to future interferometers (such as SKA) remains to be solved, such accuracy would still be achievable for future single-dish telescopes (such as FAST).
Fig. 3.4: Comparison of polarimetric calibration models. Shown are two hour-long observations from the 2005-09-07 dataset at 1405 MHz at different parallactic angles and calibrated with the single axis (subplots a and b) and full reception models (subplot c, identical for both hours of observations). Here the solid, dashed, and dotted line correspond to total intensity, linear polarisation, and circular polarisation profile, individually. The difference of the total intensity profiles for the two hours of observations are shown in subplot (d) for the single-axis calibration scheme and in subplot (e) for the full reception model calibration.
3.3.5 Interference and unknown observing system instabilities

3.3.5.1 Diagnostic theory

Besides the expected effects described above, there are some unpredictable effects that may also affect the data quality. Radio frequency interference (RFI) and instrumental failure are the most important two. Specifically in the case of an observing system with only four digitisation levels (i.e. a 2-bit system), any excess in power (as potentially caused by RFI) or a temporary non-linear response in the system, can be expected to affect the pulse shape.

In order to keep track of any such occurrences, the statistics of the digitised data can be compared to those expected from theory, in the following way. First, the digitised data are divided into consecutive segments of \( L \) samples and, for each segment, the number of low-voltage states \( M \) is counted. The digital signal processing software\(^4\) that is used to process the 2-bit data maintains a histogram of occurrences of \( M \) that is archived with the pulsar data for later use as a diagnostic tool (van Straten & Bailes, 2011). When the voltage input to the digitiser is normally distributed, the ratio \( \Phi = M/L \) has a binomial distribution as in Eq. (A6) of Jenet & Anderson (1998). The difference between this theoretical expectation and the recorded histogram of \( M \) provides a measure of 1) the degree to which the input signal deviates from a normal distribution, and 2) the degree to which the sampling thresholds diverge from optimality. This difference, called the 2-bit distortion, is given by

\[
D = \sum_{M=0}^{L} \left[ P(M/L) - H(M) \right]^2
\]

(3.1)

where \( P(\Phi) \) is the expected binominal distribution and \( H(M) \) is the recorded distribution of \( M \). Separate histograms of \( M \) are maintained for each polarisation,

\(^4\)http://dspsr.sourceforge.net/
and the reported distortion is simply the sum of the distortion in each polarisation.

### 3.3.5.2 Data analysis

The sharpness parameter $\beta$ introduced in Section 2.3 can be used to examine profile stability over short lengths of observing time. In Fig. 3.5, the variations of sharpness and 2-bit distortion (defined in Eq. 3.1) during the observing runs of 2005-09-07 and 2006-12-31 are shown side by side. Note that the S/N of the individual integrations is typically not sufficient to provide a robust calculation of the derivative of profile waveform in Eq. (2.2). The sharpness is therefore indirectly obtained through Eq. (2.1). In doing so, the uncertainty by white noise is estimated from the template matching technique as mentioned in Section 3.2. The S/N is defined by the ratio of peak amplitude and the RMS of off-pulse phase bins. As the profile has a large on-pulse duty cycle, only $\sim 15\%$ of the pulse period can be used to calculate the noise RMS, which results in a $\sim 10\%$ uncertainty of the measured S/N and in the sharpness estimates. The 2-bit distortion is calculated by following the stages laid out in Section 3.3.5.1. As shown in Fig. 3.5 the 2-bit distortion levels were consistently low during the 2005-09-07 observation and the sharpness was close to normally distributed. The observation of 2006-12-31, however, shows significantly larger levels of 2-bit distortion as well as correlated, non-Gaussian variations in the sharpness parameter. Furthermore, the 2-bit distortion has a high degree of correlation with the sharpness levels. The correlation coefficient of the two time series is $-0.73$, suggesting that the profile distortion is related to the digitisation.

To illustrate the effect these digitisation-induced shape changes may have on TOA precision, we provide the $S/N - \sigma_{\text{TOA}}$ plot for the data analysed in Fig. 3.6. Clearly the 2005-09-07 data behave as expected: they follow the theoretical inverse relationship and worsen slightly for $S/N > 1000$. This worsening is caused by noise in the template profile, which was constructed from the 2005-07-24 dataset.
3.3. **ISSUES AFFECTING PROFILE STABILITY**

![Graph showing Sharpness $\beta$ (top) and 2-bit distortion $D$ (bottom) for the observations of 2005-09-07 (left-hand plots) and 2006-12-31 (right-hand plots). Results of the observing band centred at 1405 MHz are shown with solid lines; those of the 1341 MHz observing band with dashed lines. The shape variations on 2006-12-31 identified by the changes in sharpness are clearly correlated with changes in 2-bit distortion, suggesting these variations are instrumental rather than intrinsic to the pulsar. The sharpness variations of the 2005-09-07 data are close to Gaussian as their histogram (inset) shows.](image-url)
Fig. 3.6: S/N-σ_{TOA} relations for both real data and simulations. The solid line represents the result from the 2005-09-07 dataset, the long-dashed line from the 2006-12-31 dataset. The dotted line shows the theoretical prediction from Eq. (2.1), and the short-dashed line shows the relationship for simulated profiles cross-correlated with a noisy standard. All these curves are for the observing band centred at 1405 MHz.

To demonstrate this, we created the S/N-σ_{TOA} curve for simulated data, based on a (simulated) template profile with a S/N identical to that of the 2005-07-24 standard profile. This simulated result is shown as the dotted line in Fig. 3.6 and follows the 2005-09-07 curve well. Ideally, therefore, a noise-free analytic template profile would be used (as in Kramer et al., 1999), but the small-scale features present in the profile of PSR J0437-4715 require advanced modelling, as shown in Section 2.3.2. This may also explain the flattening of the σ_{TOA}-S/N curve in Verbiest et al. (2010). The other three datasets (2005-07-24, 2007-05-06 and 2008-02-24) also yield well-behaved S/N-σ_{TOA} curves like that of 2005-09-07.

The S/N-σ_{TOA} curve for the 2006-12-31 data displays much larger deviations: for equal S/N its TOA uncertainty (σ_{TOA}) is several factors higher than for the 2005-09-07 data and limits the calculated precision to the ∼ 50 ns level.

To investigate the frequency of occurrence for this phenomenon, we performed a long-term profile stability monitoring of PSR J0437-4715 CPSR2 data from the Swinburne pulsar timing database. The results were summarised in Fig. 3.7,
showing again the correlation between profile shape variation and 2-bit distortion $D$. Two continuous periods of time, MJD 53950-MJD 54360 (apart from observations around MJD 54230) and MJD 54650-MJD 55000, were identified when the data were corrupted due to faulty digitisation. This demonstrates the importance to investigate the digitisation distortion in any 2-bit pulsar timing data on long timescales, and to exclude any distorted data from future pulsar timing analyses. However, for systems with more digitisation levels, this type of distortion would not affect profile shape and thus TOA precision as significantly as here (Jenet & Anderson, 1998). It is therefore likely that timing with more state-of-the-art systems on both present and future telescopes, will not be limited by these effects.

### 3.3.6 Pulse jitter

#### 3.3.6.1 Theoretical model

The phases of single pulses vary around an expected, average phase. For some pulsars these phase variations seem random (Ekers & Moffet, 1968) and are called “jitter”. This phase variations slightly broaden the pulse shape and induce additional arrival phase fluctuation of integrated profiles. For the case of pulse jitter, assuming no modulation of the single pulse shape and intensity, the jitter-induced TOA scatter can be written as (Cordes & Shannon, 2010, also see Section B.2):

$$
\sigma_J = \left[ \frac{f_J^2 \int U(t)^2 dt}{N \int U(t) dt} \right]^{1/2},
$$

(3.2)

where $N$ is the number of integrated pulses (assuming no systematic drifting), $f_J$ is the width of the probability density function (PDF) of the phase jitter in units of pulse width, and $U(t)$ is the normalised pulse waveform. Note that this uncertainty is attributed to emission stability of the source and not relevant to the observational hardware. One can see that the higher the system sensitivity is, the more important pulse phase jitter may become.
Fig. 3.7: Profile shape variation and 2-bit distortion $D$ as a function of time. Here for each individual observation we integrated all time dumps to produce an averaged profile, aligned it with the template and calculated the accumulated shape difference between them. The 2-bit distortion $D$ for each epoch is the average of all individual integrations of the observation.
3.3.6.2 Data analysis

Pulse jitter can be studied by a few methods and in the following we describe each approach in detail.

3.3.6.2.1 Timing RMS The first way to evaluate the impact of pulse phase jitter on timing, is to study the random variations of the TOAs after a timing model has been subtracted. When we consider the timing residuals for the 2007-05-06 dataset, we notice that the TOAs are widely scattered and the reduced chi-square is well above unity (shown in Fig. 3.8). This is so because the TOA uncertainties have been determined based purely on the amount of radiometer noise in the pulse profile, while other possible contributions of uncertainty such as phase jitter, timing model imperfections, short-term interstellar instabilities and instrumental effects remain unquantified. Note that a previous single pulse study has shown no evidence for pulse drifting (Jenet et al., 1998). The dataset also has passed through the 2-bit distortion test (mentioned in Section 3.3.5.2) and the residuals satisfy a distribution close to Gaussian, which suggests the insignificance of effects such as faulty ephemeris and improper calibration that can induce non-white noise in timing. The deviation from Gaussian distribution is attributed to both small number statistics, and to differences in measurement precision of the residuals caused by S/N variations in individual integrations. Consequently, we can set up an upper limit on the phase jitter contribution, by assuming that jitter noise is dominant in the additional residual scatter (the amount not quantified by the TOA uncertainties from radiometer noise) and using Eq. (3.2) to derive the parameter $f_J$. We find that in this worst-case scenario, $f_J \simeq 0.08$ and $\sigma_{\text{Rad}}/\sigma_J \simeq 0.3$, which means that the timing residuals are dominated by pulse phase jitter and can therefore hardly be improved by increasing the telescope gain, but only by extending the integration time.
3.3.6.2.2  \( N_{\text{efc}} \)-S/N relation  A second approach to investigate pulse jitter is through the \( N_{\text{efc}} \)-S/N relation introduced in Appendix A. A deviation from its theoretical scaling can be induced by inaccurate folding, non-Gaussian noise in the off-pulse baseline or, more interestingly, by pulse jitter. Based on the model of Eq. (3.2), the effect of pulse phase jitter on the \( \sqrt{N_{\text{efc}}} \)-S/N scaling law has been evaluated through Monte-Carlo simulations, as shown in Fig. 3.9. Here the template of PSR J0437–4715 is used as a single pulse shape, and for each pulse we apply a Gaussianly distributed shift based on a given \( f_J \) value. Clearly, the inclusion of pulse phase jitter introduces an initial deviation that is strongly dependent on the value of \( f_J \). After integration of a sufficiently large number of pulses (of the order of a few to several tens of pulses), the scaling law is recovered while the curves remain at a lower S/N than in the jitter-free case. Fig. 3.10 complements Fig. 3.9 by plotting the same parameters over a range of increasing integration times. While Fig. 3.9 shows the simulated effects of pulse phase jitter at the very shortest integration lengths that are inaccessible to our data (which have a minimum length of 16.8 seconds or just under 3000 pulses), Fig. 3.10 shows the (expectedly stable) behaviour on longer timescales. The S/N_{\text{ref}} used
3.3. ISSUES AFFECTING PROFILE STABILITY

![Simulated N_{efc}-S/N curves for both the jitter free and jittered profile cases. The solid, long dashed, short dashed, and dotted lines correspond to jitter factors f_J (see Eq. 2) of 0, 0.1, 0.3 and 0.5, respectively.](image)

Fig. 3.9: Simulated $N_{efc}$-S/N curves for both the jitter free and jittered profile cases. The solid, long dashed, short dashed, and dotted lines correspond to jitter factors $f_J$ (see Eq. 2) of 0, 0.1, 0.3 and 0.5, respectively.

for the calculation of $N_{efc}$ is 150, which is roughly the mean value for all 67.1 s integrations. The left plot underlines how the S/N does not scale linearly with the real number of pulses averaged while the right plot does follow the theoretical linear relationship as expected. The phenomenon shown is visible only when the shortest integration time can resolve of the order of ten pulses. These plots do not show any evidence of the effects simulated in Fig. 3.9. This is to be expected, as the simulations show that the jitter effect saturates after a relatively short integration time, which is not accessible by the (relatively long) integrations of the data we have.

3.3.6.2.3 Sharpness variation A third diagnostic plot that can be used to analyse pulse phase jitter, is the $\beta - N_{efc}$ plot. An example of this is provided in Fig. 3.11, which shows how $\beta$ converges as more pulses are integrated. This simulation (which was performed in a similar way to that described above) is for the extreme jitter-dominated case. Real data would be a combination of the jitter-induced exponential convergence simulated in Fig. 3.11 and of Gaussian white noise. However, because the intrinsic shape of single pulses for PSR J0437–4715
Fig. 3.10: Real and effective pulse number versus S/N, demonstrating the relations between integration time and S/N. These plots are based on data at 1405 MHz from 2005-07-24 (solid line), 2005-09-07 (long dashed line), 2007-05-06 (short dashed line) and 2008-02-24 (dotted line). The theoretical scaling law is clearly reproduced in the $N_{\text{efc}}$ graph but not in the true $N$ graph.

is not easily defined, we cannot compare Fig. 3.11 easily with the $\beta$ values obtained from our data (as shown in Fig. 3.5), but it can be noted from Fig. 3.6 that $\beta$ does not change significantly as we integrate our data.

3.3.6.3 Conclusions

In summary, we have used timing measurements to derive an initial upper limit of $f_J \leq 0.08$ for pulse phase jitter in PSR J0437–4715. This upper limit was subsequently used in simulations to evaluate the impact such jitter would have on the pulse shape, which showed that at integration times beyond 100 pulses ($\sim 0.57\text{s}$) neither the S/N of the integrated profile, nor the pulse sharpness are affected significantly compared to the radiometer noise. However, as in reality the shape of single pulses does vary which is not considered in the simulation, such a threshold needs to be better defined with single pulse data. A further discussion on pulse jitter will be provided in the next chapter.
3.4 Discussion and consequences for future telescopes

3.4.1 Summary

In this chapter, we have used the brightest and most precisely timed MSP, PSR J0437−4715, to illustrate the most important phenomena known to affect the shape of pulse profiles; to estimate the impact of these phenomena and to evaluate the efficacy of mitigation schemes. By concentrating on the brightest MSP currently known, we are able to cast some light on effects that future telescopes like the FAST or the SKA will come across as a matter of course in any standard MSP.

We find that pulse phase jitter may be dominant in current short-term timing of PSR J0437−4715, though further analysis is required to enable quantification of this effect. The effects of faulty de-dispersion are found to be small, negating the need for frequent updates of the dedispersion DM. The effects of scintillation

Fig. 3.11: Simulated integration process in the existence of pulse jitter. The profile shape estimated by the sharpness parameter is well constrained below the 0.5 % level after folding of a few hundred pulses. Note that given the rotational period of PSR J0437−4715, $10^4$ roughly corresponds to an integration time of 1 minute.
across the bandwidth could not be fully investigated given the limited bandwidth of our data, but usage of frequency-dependent templates can be expected to resolve the aforementioned problem that scintillation might cause. We present the results of the application of correction algorithms for the most important digitisation artefacts and illustrate the importance of full calibration modelling, as opposed to the more traditional calibration for differential gain and phase only. Finally, we report the discovery of PSR J0437−4715 profile instability which is induced by faulty 2-bit digitisation. This demonstrates the importance of adequate monitoring of the digitisation statistics for 2-bit pulsar observing systems, which is not commonly practised.

We further propose a few diagnostic plots to assess the data quality of any pulsar timing data:

- **Time-β curve**: Any non-Gaussian variations in profile sharpness suggests changes in pulse shape and imply the data should be carefully studied before being included in any timing analysis (e.g. Fig. 3.5).

- **S/N-σ_{TOA}** relation: To assess the quality of the standard profile used (Fig. 2.4). Deviations from the theoretical relationship or significant differences between datasets indicate data distortion (e.g. Fig. 3.6).

- **S/N-σ_{TOA}** for sub-bands: As in the previous point, any mismatch between curves of different sub-bands indicates loss of data quality (e.g. Fig. 3.2).

Note that the analysis is applicable not only to future telescopes, but also to existing ones such as Parkes, Arecibo, Effelsberg, and the Large European Array for Pulsars (LEAP, see Ferdman et al., 2010; Kramer & Stappers, 2010), if the system sensitivity is high enough given the brightness of the source.

### 3.4.2 Timing in the new millennium

Since PSR J0437−4715 is two orders of magnitude brighter than most MSPs, current observations of this pulsar provide a good demonstration of the TOA
Table 3.2: Key parameters used in prediction of TOA precision for different instruments at 1.4 GHz.

<table>
<thead>
<tr>
<th></th>
<th>(A_{\text{eff}}) (m²)</th>
<th>(T_{\text{rec}}) (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parkes</td>
<td>(2.2 \times 10^4)</td>
<td>23.5</td>
</tr>
<tr>
<td>FAST</td>
<td>(7.1 \times 10^4)</td>
<td>20.0</td>
</tr>
<tr>
<td>SKA core</td>
<td>(2.5 \times 10^5)</td>
<td>30.0</td>
</tr>
</tbody>
</table>

measurement situation for most of the MSPs by the next generation of radio telescopes. The 21\textsuperscript{st} century radio telescope will significantly increase the S/N of pulsar detections and correspondingly reduce the uncertainty of TOA measurements through vast increases in effective collecting area.

Fig. 3.12 shows the expected MSP TOA precision that can be achieved by SKA and FAST. Here for Parkes and FAST we only consider the uncertainty induced by radiometer noise and pulse jitter at the worst-case level derived in Section 3.3.6.2. For the prediction of SKA we calculate the upper limit by applying the worst-case in jitter and set up the lower limit, by assuming that either the jitter uncertainty is still not significant compared with the one by white noise, or can be corrected by potential methods. The instrumental effects causing profile distortions and phase fluctuations are neglected based on previous analysis and the ISM influence on pulse shape is assumed to be either aggressively corrected (e.g. Walker et al., 2008) or not significant on a short timescale with the given frequency and bandwidth. Concerning the properties of MSPs, we assume a mean flux density of 3.0 mJy at 1.4 GHz, 50 MHz bandwidth, spin period of 5.0 ms and 5% pulse width. In additional, a 10 K sky temperature at 1.4 GHz is applied. Consequently, it is shown that the TOA precision for normal MSPs can be improved by over one order of magnitude with the next generation of radio telescopes, even when assuming a worst-case level for pulse phase jitter. The result indicates that jitter induced uncertainties will be considerable in future timing of MSPs. The instrumental parameters used for these calculations are presented in Table 3.2 (Schilizzi et al., 2007; Nan, 2006).
We conclude that at 1.4 GHz for 10-min integrations, a TOA precision of between 80 and 230 ns can be expected in timing of normal brightness MSPs by SKA. The wide range of this prediction is caused by our limited knowledge about the pulse jitter mechanism. The future radio telescopes will enable deeper investigation of profile stability of single pulses. Presently the comparatively high levels of radiometer noise mean that only a few bright MSPs have yielded a systematic study of their single pulses, where only a subset of the single pulses can be clearly detected (Cognard et al., 1996; Jenet et al., 1998; Edwards & Stappers, 2003). The significant improvement in instrumental sensitivity will both dramatically increase the number of MSPs available for single pulse analysis and reduce the selection effects present in current studies.

### 3.4.3 Limits and future work

It needs to be pointed out that the analysis based on PSR J0437–4715 is limited by the selected sample. Note that the DM and scattering timescale (see Section 3.3.2) of this pulsar are relatively low so that the influence of the ISM on the TOA precision within the provided bandwidth is expected to be negligible.
3.4. DISCUSSION AND CONSEQUENCES FOR FUTURE TELESCOPES

As system sensitivity and observational bandwidth are improved, more high-DM sources will be timed at high precision TOA measurements. The effects of scattering and scintillation on profile shape will become more considerable than shown in this chapter, and need to be accounted for by using correction algorithms (e.g. Hemberger & Stinebring, 2008; Walker et al., 2008), and/or frequency-dependent template profiles.
Chapter 4

Profile shape stability and phase jitter analyses of millisecond pulsars

Millisecond pulsars (MSPs) have been shown to have stable integrated profiles, which enables precision monitoring of the pulse time-of-arrival (TOA). However, for individual pulses the shape and arrival phase can vary dramatically, which can also cause such variations in integrated profiles. This chapter provides a further investigation of this so-called pulse jitter phenomenon, by both evaluating the stability of integrated pulse profiles and estimating the amount of jitter noise in timing. The results are used to analyse the limit on the precision of TOA for timing observations with the next generation of radio telescopes. This chapter is a revised version of the paper Liu et al. (2012).

4.1 Introduction

MSPs have exhibited a highly stable rotational behaviour (e.g. Verbiest et al., 2009). High precision timing is more readily achievable with MSPs because of their short spin periods, regular rotational behaviour and, last but not least,
highly stable average pulse shapes. In general, single pulses from a pulsar show significant shape modulation. Up to now, several types of variable behaviour have been observed within the population of pulsars. These include intrinsic pulse-to-pulse changes caused by random jitter of individual pulses (Cordes & Downs, 1985; Cordes, 1993), systematic position changes of sub-components named “sub-pulse drifting” (e.g. Drake & Craft, 1968; Cordes, 1975; Edwards & Stappers, 2003), switching between two or more profile shapes on both short and long timescales known as “mode-changing” (e.g. Cordes et al., 1978; Ferguson et al., 1981; Lyne et al., 2010), and intrinsic flux density changes in stages referred to as “nulling” (e.g. Backer, 1970; Wang et al., 2007). Studies of bright individual pulses from MSPs have shown only the first type of pulse-to-pulse variation (Cognard et al., 1996; Jenet et al., 1998; Edwards & Stappers, 2003) which can cause fluctuation in the phase of integrated profiles and introduce TOA uncertainty in addition to the radiometer noise (Cordes & Shannon, 2010). This is the effect mainly addressed within the framework of this chapter.

Statistically, as the single pulses are clearly unstable, shape differences are expected to exist between an integrated profile over a short integration time and a standard template formed by averaging many more pulses. Note that the shape difference, if sufficiently large, would influence the accuracy of TOA estimation by the standard cross-correlation approach (Liu et al., 2011a). Consequently, it is important to evaluate whether or not the shape mismatch is significant compared with the system noise level. There have already been studies investigating the shape correlation between integrated profiles (or even single pulses) and a preformed template (Helfand et al., 1975; Rankin & Rathnasree, 1995; Jenet et al., 1998; Jenet et al., 2001). The results show clear shape difference for young pulsars and a less significant or even undetectable difference for MSPs with the given sensitivity.

If the TOA uncertainties can be shown to be mostly due to radiometer noise and pulse phase jitter, based on an assumed model, the amount of jitter noise
can be estimated from timing on a short timescale and such analysis has already
been carried out for PSR J1713+0747 (Cordes & Shannon, 2010). The result can
be used both for the error analysis in timing and as an input for jittered shape
correction approaches (Messenger et al., 2011; Oslowski et al., 2011).

The structure of this chapter is as follows. In Section 4.2 we introduce the
approach used for evaluating profile stability and jitter estimation. The obser-
vations, instruments and data pre-processing techniques used are described in
Section 4.3. In Section 4.4 we present the results before drawing our conclusions
in Section 5.7.

4.2 Profile shape and phase jitter analysis

4.2.1 Stability Analysis

The shape and phase instability of single pulses will induce shape modulation for
integrated profiles, which can be mitigated by increasing the number of pulses
added. The similarity between an observed integrated profile \( p \) and a normalised
standard shape \( s \), obtained from previous observations, is simply evaluated by
the correlation coefficient \( \rho \), which is defined as:

\[
\rho = \frac{\sum_i (s_i - \bar{s})(p_i - \bar{p})}{\sqrt{\sum_i (s_i - \bar{s})^2 \sum_i (p_i - \bar{p})^2}},
\]

where \( i \) stands for the sample number of the data points across the profile. It
is shown in Appendix C.1 that, assuming an identical intrinsic shape for the
observed profile and standard, there is a scaling law between \( \rho \) and the profile
peak signal-to-noise ratio (SNR, defined as pulse peak amplitude divided by the
root-mean-square of the noise) of: \( (1 - \rho) \propto \text{SNR}^{-2} \). Note that this power law is
followed only in the high SNR regime (e.g. for \( \text{SNR} \gtrsim 20 \)). From the scaling law,
we define a shape constant, $C$, related only to the intrinsic profile shape as:

$$C \equiv \text{SNR} \cdot \sqrt{1 - \rho} \simeq \left( \frac{n_{\text{samp}}}{2 \sum_i (s_i - \bar{s})^2} \right)^{1/2} ,$$  \hspace{1cm} (4.2)$$

where $n_{\text{samp}}$ is the number of time samples of a profile and $i = 1, 2, \cdots n_{\text{samp}}$. Once the SNR and $\rho$ are measured, respectively, the shape constant can then be determined and used to compare with the value calculated directly from the waveform of the standard.

Since the profile integration increases the SNR as $\text{SNR} \propto \sqrt{N}$, where $N$ is the number of pulses folded, we can write the relation between $N$ and $\rho$ as: $(1 - \rho) \propto N^{-1}$. However, the scaling is not obeyed if the SNR varies significantly from pulse to pulse, which can be caused by intrinsic flux variation, system temperature changes and diffractive scintillation. So in our analysis we use the effective pulse number $N_{\text{efc}}$, the number of pulses weighted by SNR to account for the variation of the profile SNR which follows $\text{SNR} \propto \sqrt{N_{\text{efc}}}$ and hence $1 - \rho \propto N_{\text{efc}}^{-1}$ (Liu et al., 2011a).

### 4.2.2 Phase Jitter

Single pulse instability can also cause the TOA fluctuations of integrated profiles. This phase jitter could, in principle, be investigated directly from single pulse data, although the sensitivity achieved by the current instruments may not enable sufficient SNR for carrying out the study on all pulses within a narrow band. Another approach is to perform timing using integrated profiles on short timescales and to estimate the amount of jitter from the timing residuals. As a first-order approximation, by assuming an identical shape for single pulses and a Gaussian-distributed central phase probability, the contribution of phase jitter to the uncertainty of TOA can be calculated as in Cordes & Shannon (2010). In brief, the measurement error of TOAs on short timescales (e.g. several hours)
can be summarised as:

\[ \sigma_{\text{total}}^2 = \sigma_{\text{rn}}^2 + \sigma_J^2 + \sigma_{\text{scint}}^2 + \sigma_0^2, \]

(4.3)

where \( \sigma_{\text{rn}} \), \( \sigma_J \), \( \sigma_{\text{scint}} \) and \( \sigma_0 \) correspond to uncertainty induced by radiometer noise, phase jitter, instability of short-term diffractive scintillation, and all other possible contributions (faults in timing model, instrumental digitisation artefacts, polarisation calibration error, etc), respectively. Following Downs & Reichley (1983), Cordes et al. (1990) and Cordes & Shannon (2010), we have

\[ \sigma_{\text{rn}}^2 = \frac{\Delta}{N \times SNR_1^2 \int |U'(t)|^2 dt}, \]

(4.4)

\[ \sigma_J^2 = \frac{f_J^2}{N} \int U(t)^2 dt, \]

(4.5)

\[ \sigma_{\text{scint}}^2 = \frac{t_d^2}{N_{\text{scint}}}. \]

(4.6)

Here \( N \) is the number of pulses, \( SNR_1 \) is the signal-to-noise ratio for a single pulse, \( U(t) \) is the profile waveform, \( \Delta \) is the sampling time, \( f_J \) is the width of the Gaussian probability of single pulse phase in units of the pulse width, \( t_d \) is the pulse broadening timescale, and \( N_{\text{scint}} \) is the number of scintles contained in one integration. We can see that the ratio of \( \sigma_{\text{rn}} \) to \( \sigma_J \) is inversely proportional to the equivalent single pulse SNR. The \( SNR_1 \) of MSPs based on current observations is far less than unity, leading to the case where the white noise term is dominant in the TOA uncertainty. However, the contribution from phase jitter will become more significant when timing is carried out by the next generation of radio telescopes, which will have a significant improvement in sensitivity of \( 1 - 2 \) orders of magnitude over current systems (Nan, 2006; Schilizzi et al., 2007; Liu et al., 2011a). In this case, TOA error prediction based solely on radiometer noise will be incorrect.

The statistics of the timing residuals which are obtained by subtracting a timing model from the measured TOAs, can be evaluated via a reduced \( \chi^2 \) value given by

\[ \chi^2_{\text{rec}} = \frac{1}{N - n - 1} \chi^2 = \frac{1}{N - n - 1} \sum_i \left( \frac{\Delta x_i}{\sigma_i} \right)^2, \]

(4.7)
where \( N \) is the number of residuals, \( n \) is the number of fitted parameters, and \( \Delta x_i, \sigma_i \) are the \( i \)th residual and its corresponding measurement error, respectively. Normally, \( \sigma_i \) accounts for only the uncertainty due to the radiometer noise \( \sigma_{rn} \) in Eq. (4.4) and this uncertainty can be obtained from the template matching technique (Taylor, 1992). Theoretically, if the timing residuals are dominated by radiometer noise, a timing solution with residuals of \( \chi_{\text{rec}}^2 \simeq 1 \) will be expected. The existence of other types of noise would mean that \( \sigma_i \) is underestimated and \( \chi_{\text{rec}}^2 \) will deviate from unity. The contribution from the diffractive scintillation can be estimated by Eq. (4.6). Here \( t_d \) can be obtained from the NE2001 Galactic Free electron Density Model (Cordes & Lazio, 2002), and the number of scintles is assessable via either a dynamic spectrum or more detailed calculations in Cordes et al. (1990). If the additional noise is mostly from phase jitter, one can estimate the jitter noise \( \sigma_J \) by adding its contribution into \( \sigma_i \) to have \( \chi_{\text{rec}}^2 = 1 \). Accordingly, the probability density distribution of \( f_J \) can be derived based on Eq. (4.5), and the probability density distribution of \( f_J \) can be calculated from the standard \( \chi^2 \) distribution, given by

\[
f(x; k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2},
\]

where \( x \) is the \( \chi^2 \) value and \( k = N - n - 1 \) is the degrees of freedom.

### 4.3 Observations

Frequent observations, typically weekly, of more than 20 MSPs are performed at the Parkes 64-m radio telescope (Verbiest et al., 2009). Here we use the data from five sources (PSR J0437−4715, PSR J1022+1001, PSR J1603−7202, PSR J1713+0747, PSR J1730−2304), collected using either the Parkes 20-cm multibeam (MB) receiver (Staveley-Smith et al., 1996a) or the ‘H-OH’ receiver. The data were processed online with the Caltech-Parkes-Swinburne Recorder 2 (CPSR2), a 2-bit coherent de-disperser back-end that records two 64-MHz wide observing bands simultaneously (Hotan et al., 2006). These bands were centred at
4.3. OBSERVATIONS

Table 4.1: Details of selected observations used in this chapter. The symbols $t_0$, $N_0$ and $T_{\text{obs}}$ represent the shortest integration time in the dataset, the number of time dumps and the observational length, respectively. The asterisk (*) denotes that the data were pre-processed by the DFB and the others are from CPSR2.

<table>
<thead>
<tr>
<th>Pulsar Name</th>
<th>MJD</th>
<th>Receiver</th>
<th>$t_0$ (s)</th>
<th>$N_0$</th>
<th>$T_{\text{obs}}$ (hr)</th>
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<td>1498</td>
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<tr>
<td></td>
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<td>MB</td>
<td>67.1088</td>
<td>261</td>
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<td></td>
<td>54095*</td>
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<td>125</td>
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<td></td>
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</tbody>
</table>

observing frequencies of 1341 and 1405 MHz, respectively. Data collected before MJDs 53740 were folded every 16.7772 s, and after that every 67.1088 s. Off-source observations of a pulsed noise probe at 45° to the feed probes, but with otherwise identical set-up, were taken at regular intervals for the purpose of polarimetric calibration. Additionally, we analysed data for PSR J0437$-$4715 from the new Parkes Digital Filterbank (DFB) system, a digital polyphase filterbank capable of 8-bit sampling. In Table 4.1 we present the details for all selected datasets.

The data were then pre-processed with the PSRCHIVE software package (Hotan et al., 2004b). Specifically, for CPSR2 data we corrected the 2-bit digitisation artefact (Jenet & Anderson, 1998) by applying the algorithm in van Straten (2011), and removed 12.5% of each edge of the bandpass to avoid possible effects of aliasing and spectral leakage. A full receiver model was determined and applied to perform the polarisation calibration for the MB receiver data (van Straten, 2004), as the receiver suffers from strong cross-coupling between the orthogonal feeds. For the H-OH data from both back-ends we used the common single-axis model instead, as the coupling was found to be an order of magnitude weaker (e.g. Manchester et al., 2010). The signals from each polarisation were summed into
total power (Stokes $I$), while 0.5 MHz frequency channels were kept for analysis purposes. The template profiles used for the correlation coefficient calculation and the cross-correlation procedure to estimate $\sigma_{rn}$ (Taylor, 1992), were obtained independently from the datasets shown in Table 4.1. All CPSR2 datasets have passed through the test shown in Liu et al. (2011a) to ensure no clear 2-bit digitisation distortion. The test also showed that the template matching produced the radiometer noise uncertainty as expected by Eq. (4.4).

4.4 Results

4.4.1 Measurement of shape constant

The measurement of the shape constant for individual integrations gives an estimate of the profile stability, which here we have carried out for the aforementioned five MSPs. The value of $\rho$ is computed with respect to the on-pulse phase and the root-mean-square (RMS) deviation of the noise is estimated based on the off-pulse region, from which one can derive the value of $C$ from Eq. (4.2). The expected values of $C$ (denoted by $C_0$) are calculated directly from the shape of the high SNR standards formed from previous observations, which are also used in the calculations of $\rho$. The errors in $\rho$ and the RMS of the baseline of the profile are given by Eq. (C.6) and by $\simeq \text{RMS}/\sqrt{2n_{\text{samp}}}$, respectively. For PSR J0437−4715, $C$ was determined using the first 500 time dumps of the MJD 53621 dataset. For PSRs J1022+1001 and J1730−2304 we integrated the time dumps into 1.0 and 1.8 minutes, separately, so as to obtain sufficiently high SNR for calculation of $C$ and statistical analysis (see e.g. Fig. C.1).

We present the shape constant measurement of PSR J0437−4715 for the MJD 53621 dataset in Fig. 4.1 as an example, and show the statistical results in Table 4.2. It is clear for most sources that within the range of estimated error the measured $C$ is in accordance with the analytical value. The RMS of the measured value also matches the mean error bar for each data point. The deviation of the
4.4. RESULTS

Fig. 4.1: Shape constant measurements for the first 500 integrations of the MJD 53621 observation for PSR J0437−4715. The time baseline of these measurements is $\sim 1.9$ hours.

measured $\mathcal{C}$ from the expectation for PSR 1730−2304 may indicate an intrinsic level of profile shape change, but could also be due to an insufficient number of measurements. Note that, in this case, the statistics are only based on 13 data points, while for each of the other MSPs more than 30 measurements were available.

It is worth noting that there has already been previous work regarding the stability of PSR J1022+1001 on both short and long timescales, with inconclusive results (Kramer et al., 1999; Ramachandran & Kramer, 2003; Hotan et al., 2004a). On timescales of roughly an hour, our results do not show profile changes. However, the analysis was carried out using the whole on-pulse phase, and may not be sensitive to variations occurring only around the profile peaks. Any shape variations, if they exist, are likely to affect mostly the peaks. A detailed analysis of the statistics of the peak ratios will be published by Purver et al. (in preparation).

To investigate the scaling relation indicated by Eq. (4.2), for each source we perform incremental integrations along the dataset, adding more pulse periods each time. Our results are presented in Fig. 4.2 which shows the relation between
Table 4.2: Statistical results of the profile shape constant measurements for all five sources. The parameters \(t_{\text{int}}\), \(C_0\), \(C\), RMS, \(\sigma\) and \(k_{\text{NC}}\) represent integration time of the integrated profiles, shape constant calculated from the standard, mean of measured \(C\), RMS of measurements, mean of individual measurement errors, and the fitted slopes of \(N_{\text{efc}}\) versus \((1-\rho)\) curves in Fig. 4.2, respectively.

<table>
<thead>
<tr>
<th>Pulsar Name</th>
<th>(t_{\text{int}}) (s)</th>
<th>(C_0)</th>
<th>(C)</th>
<th>RMS</th>
<th>(\sigma)</th>
<th>(k_{\text{NC}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>J0437−4715</td>
<td>16.8</td>
<td>4.8</td>
<td>4.9</td>
<td>0.32</td>
<td>0.31</td>
<td>-1.00(1)</td>
</tr>
<tr>
<td>J1022+1001</td>
<td>67.1</td>
<td>2.7</td>
<td>2.6</td>
<td>0.22</td>
<td>0.20</td>
<td>-0.98(1)</td>
</tr>
<tr>
<td>J1603−7202</td>
<td>16.8</td>
<td>3.9</td>
<td>4.0</td>
<td>0.33</td>
<td>0.30</td>
<td>-0.99(1)</td>
</tr>
<tr>
<td>J1713+0747</td>
<td>16.8</td>
<td>3.7</td>
<td>3.8</td>
<td>0.23</td>
<td>0.22</td>
<td>-0.94(2)</td>
</tr>
<tr>
<td>J1730−2304</td>
<td>117</td>
<td>2.9</td>
<td>3.2</td>
<td>0.24</td>
<td>0.27</td>
<td>-0.98(1)</td>
</tr>
</tbody>
</table>

the weighted number of integrated pulses and \(\rho\). It is clear that all curves are linear in log-log space for the \(N_{\text{efc}}-(1-\rho)\) relation as expected. The fitted slopes all lie in the range of \((0.94, 1.00)\), as given in the last column of Table 4.2. This result, together with the \(C\) statistics in Table 4.2, indicates no detectable profile shape variations within the integration.

### 4.4.2 Fit of phase jitter

Estimation of the jitter parameter \(f_J\) can be performed only on the brightest source PSR J0437−4715, as single pulse SNR for the other sources is not high enough and radiometer noise is still dominant in the timing residuals. Here we use all PSR J0437−4715 datasets listed in Table 4.1, each of which contains over a hundred individual integrations to yield a stable statistical result. When performing the timing of the datasets, we used the timing models derived by Verbiest et al. (2009). TOAs and their uncertainties were determined through cross-correlation with a pre-formed standard, individually for each side of band. No EFAC or EQUAD value were applied to change the measurement precision as done commonly in timing analysis (e.g. van Haasteren et al., 2011).

The timing solution of PSR J0437−4715 shows a widely scattered series of TOAs over a timescale of several hours, with a reduced \(\chi^2\) far larger than unity, which strongly indicates the existence of extra uncertainty contributions besides
Fig. 4.2: $N_{efc} - (1 - \rho)$ plot for all five sources. The effective pulse number is used to compensate for the varied weight of each integration due to the difference in SNR. All curves show the expected linear behavior with fitted slopes to $-1$ in log-log space.

following the method mentioned in Section 4.2.2, we measure $f_J$ based on 1-min integrations from all PSR J0437$-$4715 datasets in Table 4.1, and the calculated probability density distribution of $f_J$ are shown in Fig. 4.4 for both sidebands. It is clear that the results from datasets collected by different receivers are consistent with each other, which indicates no significant uncertainty contribution by polarimetric calibration error. Results from two types of back-end also
Fig. 4.3: The short-term timing solution of PSR J0437−4715 from the 8-hour MJD 53621 dataset at a central frequency of 1405 MHz with 48 MHz bandwidth.

shows consistency with the given 1-min integrations. Additionally, the same result for fits from both sides of the band suggests that the intensity of phase jitter does not vary significantly on small frequency scales (64 MHz difference between the two bands) at observing frequencies around 1.4 GHz. The combination of all probability density distributions for both sides of the band achieves an estimated $f_J$ of $0.072 \pm 0.003$.

To study the statistics of the residuals, we perform Kolmogorov-Smirnov (K-S) tests on each dataset, on TOAs weighted by the modified uncertainties accounting for phase jitter. The measured p-values\(^1\) are summarised in Table 4.3, and do not suggest a significant deviation of the weighted residuals from a Gaussian distribution. This demonstrates the dominance of Gaussian noise in the residuals. In Fig. 4.5 we show an example histogram of the weighted TOAs from the MJD 53621 dataset at 1405 MHz. The distribution can be well modelled by a Gaussian function with fitted variance of $0.94 \pm 0.04$ and reduced $\chi^2 \simeq 1.1$, close to unity as expected.

\(^1\)The p-value typically ranges from 0 to 1, and unity stands for perfect Guassian. See e.g. Press et al. (1986) for more details.
Fig. 4.4: Normalised probability density functions of $f_J$ from all PSR J0437–4715 datasets based on 1-min integrations. The top and bottom plots represent the results from 1341 MHz and 1405 MHz bands, respectively.
Table 4.3: Results of jitter parameter $f_J$ measurements and K-S tests of all PSR J0437–4715 datasets from two 48-MHz sub-bands. The parameters $f$ and $P$ represent the central frequency and the p-value of K-S test, respectively.

<table>
<thead>
<tr>
<th>Dataset (MJD)</th>
<th>$f$ (MHz)</th>
<th>$f_J$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>53576</td>
<td>1341</td>
<td>0.067 ± 0.004</td>
<td>0.89</td>
</tr>
<tr>
<td></td>
<td>1450</td>
<td>0.067 ± 0.004</td>
<td>0.93</td>
</tr>
<tr>
<td>53621</td>
<td>1341</td>
<td>0.067 ± 0.004</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>1405</td>
<td>0.066 ± 0.004</td>
<td>0.62</td>
</tr>
<tr>
<td>53964</td>
<td>1341</td>
<td>0.066 ± 0.004</td>
<td>0.88</td>
</tr>
<tr>
<td></td>
<td>1405</td>
<td>0.065 ± 0.004</td>
<td>0.92</td>
</tr>
<tr>
<td>54095</td>
<td>1341</td>
<td>0.069 ± 0.006</td>
<td>0.90</td>
</tr>
<tr>
<td></td>
<td>1405</td>
<td>0.073 ± 0.006</td>
<td>0.92</td>
</tr>
<tr>
<td>54222</td>
<td>1341</td>
<td>0.069 ± 0.007</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>1405</td>
<td>0.067 ± 0.007</td>
<td>0.77</td>
</tr>
<tr>
<td>54226</td>
<td>1341</td>
<td>0.068 ± 0.005</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>1405</td>
<td>0.066 ± 0.006</td>
<td>0.64</td>
</tr>
</tbody>
</table>

Fig. 4.5: Histogram of arrival time deviation $\Delta t$ obtained from the MJD 53621 dataset at 1405 MHz, normalised by the measurement uncertainty $\sigma_{\text{TOA}}$ accounting for both radiometer and jitter noise. The distribution can be modelled by a Gaussian function (dashed line) with $\sigma = 0.94 \pm 0.04$ and fitted reduced $\chi^2 \simeq 1.1$. 
To investigate the dependence of the result on the length of individual integrations, we calculate the weighted RMS against integration time $t_{int}$ for datasets with different instrumental combinations. In detail, MJD 53576 (CPSR2+MB), MJD 54095 (DFB+H-OH) and MJD 54226 (CPSR2+H-OH) datasets are chosen to demonstrate different hardware configurations, and the results are shown in Fig. 4.6. Clearly, the curves yielded by CPSR2 data collected by two receivers are consistent, again implying no significant contribution of timing uncertainty by polarisation calibration. The relation obtained from DFB data achieves a fitted slope close to $-0.5$, supporting the idea from Eq. (4.4) and (4.5) that in this case timing residual scales as the square-root of the number of integrated pulses once only the uncertainties due to radiometer noise and phase jitter are significant. The residuals from the CPSR2 data coincide with those from the DFB in 1-min integrations, and then deviate from the $\propto t^{-0.5}$ relation at the level of $\approx 150 \text{ ns}$ as the integration time is extended. This implies that the difference may be due to different digitisation procedures. The saturation corresponds to additional self-correlated noise which contributes $\approx 8\%$ bias in our $f_J$ measurement based on 1-min integrations of CPSR2 data. In Table 4.3 the post-correction $f_J$ measurements from all datasets are summarised, which leads to an estimated $f_J$ of $0.067 \pm 0.002$.

The expression for $\sigma_J$ indicates that the uncertainty due to phase jitter is independent of the observational bandwidth and central frequency. To illustrate this, the MJD 53621 dataset is divided into two, three, four and six sub-bands each time. Then a fit for the jitter parameter is carried out individually on each sub-band and the results are combined incoherently to obtain an estimated value of $f_J$ for a given bandwidth. The procedure is carried out for both side-bands and the result is shown in Fig. 4.7. It is clear that at both central frequencies jitter noise remains once the bandwidth is changed. The result indicates that, on small frequency scales, pulses are jittered in the same way, otherwise jitter noise would not remain the same after summing the entire bandwidth. In Fig. 4.8 we plot
the fitted $f_J$ based on 8 MHz bandwidth against the central frequency for each sub-band. The $f_J$ values do not show a clear dependence on the observational frequency, and yields an average, a weighted RMS and a reduced $\chi^2$ of 0.069, 0.006 and 1.1, respectively. The mean correlation coefficient between residuals of different sub-bands were calculated to be $\simeq 0.51$, which indicates a correlation among the time series and supports the idea of equal jittering on small frequency scales. Such results, if still valid for wider frequency ranges, would suggest that the uncertainty caused by jittering cannot be mitigated by extending the observing bandwidth.
4.4. RESULTS

Fig. 4.7: Fitted amount of jitter against chosen bandwidth for both sub-bands from MJD 53621 dataset. No clear dependency between these two parameters is visible.

Fig. 4.8: Estimated jitter parameter based on an 8-MHz sub-band from MJD 53621 dataset. The $f_J$ shows no clear trend of evolution across a frequency range of $\sim 100$ MHz.
4.5 Conclusions and Discussions

4.5.1 Summary of the results

In this chapter, we investigate the issue of MSP profile stability based on five pulsars in total. A shape constant associated with the correlation coefficient is defined to quantify the stability. No significant shape modulation of integrated profiles beyond the measurement error is found for integration times from $10^1$ to $10^2$ s. For PSR J0437—4715 we estimate the jitter parameter by performing timing on short timescales and comparing the actual timing residual with the amount expected from radiometer noise. The fitted $f_J$ is found to be identical on both sides of the bands (64 MHz difference in central frequency), and the combination of several datasets results in an estimate of $0.067 \pm 0.002$. It is also demonstrated that all the other potential sources of TOA uncertainty, besides radiometer and phase jitter noise, do not strongly influence the measurement. Additionally, we show that jitter noise scales neither with bandwidth within a 50-MHz-band nor with frequency across range of $\sim 100$ MHz at 1.4 GHz, which supports the idea that pulses are equally jittered on small frequency scales. Further investigations based on wide frequency range are still required to see if jitter noise can be mitigated by extending observing bandwidth.

4.5.2 Future telescopes

Apart from bright single pulses and giant pulses (e.g. Cognard et al., 1996; Jenet et al., 1998; Edwards & Stappers, 2003), single pulse jitter of the majority of MSP pulses is still not detectable with currently available sensitivity. However, with the next generation of radio telescopes, the shape modulation of integrated profiles for some of the bright MSPs will become visible. In Fig. 4.9, based on the aforementioned jitter model and assuming $f_J = 0.1$, we perform a Monte Carlo simulation to calculate the relation between the number of integrated pulses and
4.5. CONCLUSIONS AND DISCUSSIONS

Fig. 4.9: $1 - \rho$ versus $N$ curve prediction of PSR J0437−4715 for different instruments and simulation. The pulses are created with identical shape and a jitter parameter of $f_J = 0.1$ is used to simulate the Gaussian-distributed phase variation.

$1 - \rho$ for the case of jitter only, and compare the result with the curves from considering radiometer noise only for a few instruments. We assume a 5-ms period, a 100-µs pulse width, and a 5-mJy flux density at 1.4 GHz for a sample MSP, and 1.4 GHz frequency with 300 MHz bandwidth for the observing parameters. The gains of FAST and SKA are assumed to be 20 m$^2$/K and 100 m$^2$/K, respectively (Nan, 2006; Schilizzi et al., 2007). For the assumed jitter model the scaling also follows $1 - \rho \propto N^{-1}$ as calculated in Appendix C.2. It is shown that an SKA observation of an MSP of typical brightness, single pulse jitter is comparable to radiometer noise in influencing the correlation-coefficient value. Note that in the applied jitter model all single pulses are assumed to be identical, so the simulated jitter curve can potentially move upwards once the shape modulation is also accounted for. Future observations with the SKA of PSR J0437−4715 will be totally dominated by single pulse jitter in shape variation, so it will be ideal for single pulse study.

Once the shape variation of integrated profiles by single pulse jitter becomes significant, the current cross-correlation method for the measurement of TOAs would fail in estimating the TOA uncertainty correctly. Specifically, the model
in the template-matching procedure now needs to be of the form (Taylor, 1992):

\[ P(t) = A_0 \ast S(t) + n(t) + j(t), \]  

(4.9)

where \( P(t) \) is the observed profile, \( S(t) \) is the template, and \( n(t) \) is the noise function. The parameter \( j(t) \) is the jitter-induced shape perturbation, which, referring to MSPs, is mostly negligible compared with \( n(t) \) for the current sensitivity. If this shape difference becomes sufficiently large, the calculated TOA precision would fail to follow the expected SNR scaling (Liu et al., 2011a). In this case, the shape modulation would need to be modelled (e.g., by principal component analysis, see Cordes & Shannon, 2010; Oslowski et al., 2011) and then a global determination of the unknown parameters could be performed to properly estimate the TOA and its uncertainty (Palliyaguru et al., 2011; Messenger et al., 2011).
Chapter 5

Investigating pulsar–black hole binaries with pulsar timing

Pulsars have been shown to be great astrophysical laboratories in the past forty years, especially in testing gravity theories in the strong field regime. Currently, the most relativistic pulsar binary found is the 2.5 hr orbital period double-pulsar system, which presently provides the most precise test of General Relativity (GR) (Kramer et al., 2006; Kramer & Wex, 2009). Meanwhile, it is natural to imagine that if a pulsar is discovered to be accompanied by a black hole (BH), verifications of gravity theories can be improved and extended. Investigating this question is the main purpose of this Chapter. The work presented below is included in two individual papers, Liu et al. (2011b) and Liu et al., in preparation.

5.1 Overview

5.1.1 Black hole properties and testing General Relativity

One of the most intriguing results of GR is the uniqueness theorem for the stationary black hole solutions of the Einstein-Maxwell equations (e.g. Heusler, 1998).
The so-called “no-hair theorem” states, that (under certain conditions) all stationary electrovac\textsuperscript{1} BH spacetimes with a non-degenerate horizon are described by the Kerr-Newman metric. Consequently, in GR all stationary black holes are parametrised by only three parameters which are mass ($M_{\text{BH}}$), spin ($S$) and electric charge, and all uncharged black hole solutions are uniquely determined by $M_{\text{BH}}$ and $S$. Astrophysical BHs are believed to be the result of a gravitational collapse, during which all the properties of the progenitor, apart from mass and spin, are radiated away by gravitational radiation while the gravitational field asymptotically approaches its stationary configuration (Price, 1972a,b).

The outer spacetime of an astrophysical BH is supposed to be described by the Kerr metric. At the center of a BH lies a gravitational singularity, a region where the curvature of spacetime diverges. Penrose’s “Cosmic Censorship Conjecture” states that within GR, such singularities are always hidden within the event horizon (Penrose, 1979). For a Kerr BH, it leads to

$$\chi \equiv \frac{c}{G} \frac{S}{m_{\text{BH}}^2} \leq 1,$$

(5.1)

where $c$ is the speed of light, $G$ the gravitational constant, $S$ the angular momentum and $M_{\text{BH}}$ the BH mass. A measurement of the mass and the spin of a black hole can therefore be used to test this inequality. A measurement of a value for $\chi$ that exceeds 1 would pose a serious problem for our understanding of spacetime, since this would indicate that either GR is wrong or that a region is visible to the outside universe, where our present understanding of gravity and spacetime breaks down (Wald, 1984).

As a result of the no-hair theorem, all higher multipole moments of the gravitational field of an astrophysical black hole can be expressed as a function of $M$ and $S$ (Hansen, 1974). In particular, the quadrupole moment, $Q$, fulfills the relation (Thorne, 1980)

$$q \equiv \frac{c^4}{G^2} \frac{Q}{m_{\text{BH}}^3} = -\chi^2,$$

(5.2)

\textsuperscript{1}Electrovac spacetimes are the solutions of the Einstein-Maxwell equations
where $q$ is the dimensionless quadrupole and $Q$ the quadruple moment. A measurement of $q$, in combination with a $M_{\text{BH}}$ and a $\chi$ measurement, would therefore provide a test of the no-hair theorem with Kerr BHs for the very first time (Wex & Kopeikin, 1999; Kramer et al., 2004).

5.1.2 Measurement approach

As a BH is a region of space that nothing, including light, can ever escape from, the information of the stellar object is more likely to be obtained by investigating the interaction of it with other nearby objects, e.g., an accretion disk or a star.

The mass of the BH can be obtained through a few approaches. The first is by fitting the optical and near-infrared light curves together with the radial velocity measurements to derive masses and the orbital inclination (Greiner et al., 2001; Orosz et al., 2007, 2011), which has already been applied to stellar mass BH binaries. The second is through monitoring the motions of the companion stars via precision astrometric measurements to determine the orbital parameters (Ghez et al., 2008; Gillessen et al., 2009), based on which the mass of the central BH in our Galaxy, Sagittarius A* (Sgr A*), has already been measured as $\sim 4 \times 10^6 M_\odot$. Besides, by modelling the X-ray flares of the accretion disc one can yield a determination of mass and spin in one go (see the following paragraphs for details).

The spin of a BH can be constrained via modelling both the thermal continuum spectrum of the accretion disc, and the broad red wing of the reflection fluorescence Fe-K$\alpha$ line (e.g. Fabian et al., 1989; Zhang et al., 1997; Steiner et al., 2010). These two methods have already yielded spin measurement of several stellar mass BHs in the range from $\approx 0$ to $> 0.98$. Recently it was even proposed by Bambi & Barausse (2010) that independent determination of both spin and quadrupole can be yielded through the continuum-fitting analysis.

Constrains on BH mass and spin are also achievable by extracting the periodicities of the X-ray flares associated with the accretion disc. This method
has been carried out toward Sgr A*, and the results based on data from different epochs by \textit{XMM-Newton} varies from $M_{\text{BH}} \approx 2.7 \times 10^6 M_\odot$ to $M_{\text{BH}} \approx 4.9 \times 10^6 M_\odot$, and $\chi \approx 0.22$ to 0.99 (Aschenbach et al., 2004; Bélanger et al., 2006; Aschenbach, 2010), indicating that further investigation is still required for a robust spin determination.

Pulsars, especially MSPs, are highly accurate astronomical clocks so the monitoring of the pulse arrival time from them will be sensitive to the curvature of spacetime. Therefore, if a pulsar is found to be orbiting around a BH, it is possible to indirectly measure the properties of the BH via modelling the orbital motion of the pulsar and the propagation of its radio signals. The mass of the BH can be well determined by using the measured post-Keplerian (PK) parameters, just as done for binary pulsars (Lorimer & Kramer, 2005). For a Kerr BH, the gravitomagnetic field of the spinning BH will cause the pulsar orbit to precess about the direction of the total angular momentum. In pulsar timing this effect will result in a secular change of the observed projected semi-major axis and longitude of periastron, which can be fitted for and used to derive the spin and system geometry (Wex & Kopeikin, 1999). The classic spin-orbit coupling due to the oblateness of the BH will induce periodic features inside the timing residual, which might be observable with observations by the next generation of radio telescopes, and utilized to constrain the quadruple moment (Wex, 1998; Kramer et al., 2004).

5.1.3 Formation scenario of a PSR-BH system

There are mainly two types of system where a pulsar can be found orbiting a BH. One is a close pulsar and stellar-mass BH (SBH) binary system (PSR-SBH) and the other is a pulsar moving around a central BH of either the Galaxy or a globular cluster. In the following we will present a brief discussion on the formation of PSR-SBH systems.

The first channel to form a PSR-SBH binary is to follow the standard binary
5.1. **OVERVIEW**

evolution procedure, which succeeds in explaining the existence of the majority of currently discovered binary pulsars, as mentioned in Section 1.5. In this case, the pulsar is instead supposed to be formed later than the degenerated companion, because the progenitor of the BH (the primary star) should be comparably more massive (with a Zero Age Main Sequence Mass over the threshold for BH production) and thus have a shorter lifetime. After the supernova explosion of the secondary star which gives birth to the pulsar, the system’s gravitational well may still be deep enough to keep the two objects bound. In this case, a PSR-SBH binary is then born with an eccentric orbit (Yungelson & Portegies Zwart, 1998; Voss & Tauris, 2003).

The pulsar in a PSR-SBH system formed through this first channel is not supposed to experience any recycling phase, as no mass transfer would happen after the formation of the secondary star. Hence, the pulsar would be a normal pulsar (NP), which usually does not yield precision timing as mentioned in Section 1.6. However, there have also been investigations into the “reversal mechanism” where the pulsar is formed first (Sipior et al., 2004). The idea was inspired by the formation scenario study of the binary pulsars B2303+46 and J1141–6545, where the companion white dwarf is believed to be born before the neutron star (van Kerkwijk & Kulkarni, 1999; Tauris & Sennels, 2000). Following this mechanism, in a PSR-SBH system it is possible to form the pulsar first and have it spun up to a MSP during the second phase of mass transfer. However, a number of further conditions need to be satisfied during the formation procedure. Firstly, the two stars need to be close enough to each other for the first mass transfer episode. Secondly, the secondary star must gain enough mass so as to reach the threshold for BH formation, and the primary star still needs to retain enough mass to become a neutron star. Thirdly, the first natal kick induced by the collapse of the primary star into a neutron star must not disrupt the binary system. Fourthly, the stellar separation of the surviving system needs to be sufficiently small to allow the second bout of mass transfer, when the formed pulsar endures recycling
and spun-up. Finally, the system must survive from the second explosion that forms the BH (Portegies Zwart & Yungelson, 1999; Sipior et al., 2004; Pfahl et al., 2005).

A third channel to form a PSR-SBH is through stellar capture, where the pulsar and BH are first formed separately and the PSR-SBH system is formed through a multiple body encounter. This is more likely to occur in regions of high stellar density, such as globular clusters and the Galactic centre (Kulkarni et al., 1993; Faucher-Giguere & Loeb, 2010). Especially, in globular clusters and some “superstellar” clusters it might even be possible to form a pulsar–intermediate mass BH (PSR-IMBH) binary through dynamical capture (Patruno et al., 2005; Devecchi et al., 2007), although up to now there is no strong observational evidence for the existence of this type of BH.

### 5.1.4 Birthrate and population synthesis

There have already been a variety of studies concerning the abundance of PSR-SBH systems in the Galaxy. A summary of the results is presented in Table 5.1. The formation rate by the standard binary evolution scenario is found to be $10^{-6}$-$10^{-7}$ yr$^{-1}$, corresponding to a Galactic population of $\sim$10-100. Considering the number of pulsars beaming towards our earth ($2 \times 10^5$, e.g. Lorimer & Kramer, 2005) and the number that have been discovered (about two thousand), by statistical arguments we could be close to the discovery of the first PSR-SBH. However, one should be aware of the fact that the evolutionary procedure contains a number of crucial parameters that are poorly understood in theory and need to be constrained by observation, such as the stellar wind magnitude, the initial mass threshold for a star to collapse into a BH, the mass ratio distribution of binary stars, the kick velocity during the two supernovas, the efficiency of common-envelope and so forth. The simulations also show a wide spread of results as a consequence of applications of different models and parameter ranges (e.g. Sipior et al., 2004; Pfahl et al., 2005). The discovery of the first PSR-SBH system
will surely help to understand the binary evolutionary process and constrain the model parameters.

The second formation scenario has also been investigated and the birthrate is found to be one to two orders of magnitude lower than that via the first channel. The corresponding population however turns out to be not so low, and may even be comparable due to the longer lifetime of the recycled pulsar. The result is not surprising as pointed out in Section 5.1.3, since several criterions have to be satisfied if the neutron star has to form before the BH. A strong model dependence of the result is also well shown, e.g. by applying a variety of natal kick distributions during the collapse (Sipior et al., 2004).

There has been rarely any work on PSR-SBH formation via dynamic capture in high stellar density regions since the idea was firstly raised (Kulkarni et al., 1993), until recently Faucher-Giguere & Loeb (2010) proposed that 3-body interactions can efficiently produce BH-MSP binaries in the Galactic Centre (GC). The resultant population depends on the number of MSPs in that region, and is found to be stable within a factor of 2 when the fiducial choices of the system parameters are varied by amounts representative of the corresponding uncertainties. The result further contributes to the motivation of performing a deep pulsar search towards the GC region (e.g. Cordes & Lazio, 1997).

Besides, previous analysis has shown that the stellar density in the region around Sgr A* is approximately flat within the region of 10 arcsec (Genzel et al., 2003). Consequently, about 1000 pulsars can be expected with $P_b$ less than 100 yr, and $\sim 100$ may have $P_b \leq 10$ yr orbiting Sgr A* (Pfahl & Loeb, 2004). Some of them may be associated with remnants of the observed main-sequence star population in the neighborhood.

### 5.1.5 Discovery strategy

Finding PSR-BH systems involves searching for binary pulsars with a correction for the orbital acceleration. The strategies discussed in Section 1.8 can be adapted
Table 5.1: Summary of previous studies on the abundance of PSR-SBH systems in the Galaxy. The three types of formation channels refer to three cases discussed in the above section, where the first would result in a binary NP and the next two would give birth to a binary MSP. $N_{\text{MSP}}$ is the number of recycled pulsars around the GC. The ‘∗’ means that the total number is calculated either by assuming $2 \times 10^5$ visible pulsars in the Galaxy, or by multiplying the given birthrate in the corresponding reference with the lifetime of the system ($2 \times 10^7$ yr for NP-SBH systems and $10^8$ yr for MSP-SBH systems, Pfahl et al., 2005).

<table>
<thead>
<tr>
<th>Reference</th>
<th>Formation approach</th>
<th>Birthrate (yr$^{-1}$)</th>
<th>Galactic population</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lipunov et al. (1994)</td>
<td>I+II</td>
<td>−</td>
<td>86*</td>
</tr>
<tr>
<td>Portegies Zwart &amp; Yungelson (1998)</td>
<td>I</td>
<td>$6 \times 10^{-6}$</td>
<td>120*</td>
</tr>
<tr>
<td>Voss &amp; Tauris (2003)</td>
<td>I</td>
<td>$1.1 \times 10^{-6}$</td>
<td>22*</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>$6.3 \times 10^{-7}$</td>
<td>63*</td>
</tr>
<tr>
<td>Sipior et al. (2004)</td>
<td>I</td>
<td>$6.5 \times 10^{-6}$</td>
<td>130</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>$5 \times 10^{-8}$</td>
<td>5</td>
</tr>
<tr>
<td>Lipunov et al. (2005)</td>
<td>I+II</td>
<td>−</td>
<td>133</td>
</tr>
<tr>
<td>Pfahl et al. (2005)</td>
<td>II</td>
<td>$\sim 10^{-7}$</td>
<td>$\sim 10$</td>
</tr>
<tr>
<td>Faucher-Giguere &amp; Loeb (2010)</td>
<td>III</td>
<td>−</td>
<td>$N_{\text{MSP}}/50$</td>
</tr>
</tbody>
</table>

to the search for such systems regarding their formation backgrounds. The PSR-SBH formed via the standard scenario and via the stellar capture would have a relatively wide orbit ($P_b \sim 10 - 100$ days) and high eccentricity (Lipunov et al., 1994; Lipunov et al., 2005; Faucher-Giguere & Loeb, 2010), which indicates that both coherent acceleration and stack searches will be efficient in finding these two types of systems. Meanwhile, the PSR-SBH formed from the reversal mechanism is expected to have a short orbital period ($\sim 1$-10 hours) with a low eccentricity (Sipior et al., 2004), which implies that phase-modulation and dynamic power spectrum searches (see Section 1.8 for details) would be required. These schemes are especially necessary when pointing at globular clusters as the integration time needed to have a strong enough signal is usually several hours, the same timescale as the orbital period of interest.

Finding pulsars moving around Sgr A* requires deep pointings into the GC region, where the radio background is much stronger and the ISM is much denser. Considering the strong frequency dependence of scattering, the pulsar emission
5.1. OVERVIEW

spectrum and the system temperature, it is essential to determine the optimal observing frequency so as to maximize the signal-to-noise ratio (S/N) of detection (Cordes & Lazio, 1997). For instance, in (Macquart et al., 2010) it was found to be 15 GHz for NPs, while the result can be strongly dependant on the model of the ISM. Deep pulsar searches towards the GC region have already been attempted with the largest radio telescopes (Effelsberg, Green Bank, Parkes) and chosen frequencies up to 15 GHz (e.g. Kramer et al., 2000; Macquart et al., 2010). Five pulsars have been found 10-20 arcmin from the Sgr A* (Johnston et al., 2006; Deneva et al., 2009), which are consistent with the estimated large pulsar population within that region, but not close enough to Sgr A* to yield a strong field test. Note that all current searches are only sensitive to slow pulsars. To detect a MSP either in a MSP-BH binary formed via the third channel, or orbiting around Sgr A* within the central region, the optimal observing frequency may reach up to 30 GHz (Cordes & Lazio, 1997; Macquart et al., 2010).

Theoretically, if the ISM, especially the free electron distribution nearby the pulsar and along the line of sight is understood well enough, the scattered profile could be de-convolved with a screen function to recover the original shape and increase the search sensitivity (e.g. Bhat et al., 2004; Walker et al., 2008). This is a technique that needs to be fully developed and carried out in future deep sky searches.

5.1.6 Further application

The discovery of a PSR-BH system would lead to further investigations into the parameter space of strong-field gravity as shown in Fig. 5.1. Here, the area explored by the current tests of gravity lies only at the left bottom corner, and the study based on any type of PSR-BH system would extend tests of GR in the strong field regime.

Finding a PSR-BH system will also enable a variety of tests on alternative
Fig. 5.1: Parameter space of orbital size (in units of Schwarzschild radius) and gravitational mass that can be covered by a PSR-BH system, reproduced from Kramer et al. (2004).

gravity theories. For example, such systems would be ideal for testing the existence of gravitational dipole emission (Damour & Esposito-Farèse, 1998). Besides, recent work also proposes the test of extra spatial dimensions, by measuring the secular changes of the Keplerian parameters due to BH mass loss (Simonetti et al., 2010).

5.1.7 Structure and symbols

The structure of this chapter is as follows. In Section 5.2 we describe the orbital precession in a PSR-BH system due to the BH spin. In Section 5.3 we show the orbital perturbation caused by the BH quadrupole. The scheme of simulations which investigate the measurability of the BH properties via pulsar timing will be introduced in Section 5.4. Section 5.5 and 5.6 include the results for the case of PSR-SBH and PSR-Sgr A*, respectively. We conclude in Section 5.7 with a summary of the measurement time-line and a brief discussion of the other
5.2. Frame dragging and Cosmic Censorship Conjecture

In a binary system, GR predicts the phenomenon of orbital periastron advance and shrinking of the semi-major axis due to the emission of gravitational waves. As mentioned in Section 5.1.1, if the masses have a significant amount of rotation, the resulting gravito-magnetic field will cause an extra precession of the binary
Fig. 5.2: Reference frame based on the invariable plane perpendicular to the system total angular momentum $\mathbf{J}$. The line of sight vector $\mathbf{K}_0$ is fixed to the Y-Z plane, while the orbital momentum $\mathbf{L}$ is supposed to process around $\mathbf{J}$. The definition of angle $\theta_S$, $i_J$, $\Phi$ and $\Psi$ will present a full description of the orbital geometry. The corresponding defined ranges are: $\theta_S$, $\theta_J$, $i_J$, $i \in [0, \pi)$ and $\Phi$, $\Psi$, $\omega \in [0, 2\pi)$.

The precession can be best described in a coordinate shown in Fig. 5.2, based on the invariant plane perpendicular to the total angular momentum $\mathbf{J}$. In orbit (Lense & Thirring, 1918; Barker & O’Connell, 1975). Effectively, it will induce a long-term variation of the periastron and orientation of the orbital plane. Kramer & Wex (2009) have predicted that the current frequent timing observations of the double pulsar system PSR J0737–3039 will yield an accurate (10% uncertainty) estimation of the periastron precession caused by this effect within twenty years. For the case of a PSR-BH system, as the rotation of the BH is dominant in creating the gravito-magnetic field, the separation of the spin-orbit coupling effect from the post-Newtonian (PN) precession will yield an extraction of the BH spin. The details of this approach are shown in this section.

5.2.1 Precession of the orbit

The precession can be best described in a coordinate shown in Fig. 5.2, based on the invariant plane perpendicular to the total angular momentum $\mathbf{J}$. In
5.2. FRAME DRAGGING AND COSMIC CENSORSHIP CONJECTURE

In general, \( J \) can be considered as a conservative quantity and both the orbital and BH rotational angular momentum, \( L \) and \( S \), are supposed to precess around \( J \). Their absolute values are also conserved averaged over a whole orbital period. Briefly, the total angular velocity vector \( \Omega_{\text{tot}} \) of precession consists of three terms: the monopole PN periastron advance \( \Omega_{\text{pn}} \), the relativistic spin-orbit coupling precession \( \Omega_{s} \), and the precession due to the quadrupole moment of the spinning BH \( \Omega_{q} \):

\[
\Omega_{\text{tot}} = \Omega_{\text{pn}} + \Omega_{s} + \Omega_{q}. \quad (5.3)
\]

Following Barker & O’Connell (1975), the three terms can be further expressed by:

\[
\Omega_{\text{pn}} = \Omega_{\text{pn}}^* \dot{L}, \quad (5.4)
\]
\[
\Omega_{s} = \Omega_{s}^*[3(\dot{L} \cdot \dot{S})\dot{L} - \dot{S}], \quad (5.5)
\]
\[
\Omega_{q} = \Omega_{q}^*\{5(\dot{L} \cdot \dot{S})^2 - 1\}\dot{L} - 2(\dot{L} \cdot \dot{S})\dot{S}, \quad (5.6)
\]

where the hat indicates a unit vector and the advance rates are given by

\[
\Omega_{\text{pn}}^* = \frac{3T_{\odot}^{2/3}}{1 - e^2} \left( \frac{2\pi}{P_b} \right)^{5/3} M^{2/3}, \quad (5.7)
\]
\[
\Omega_{s}^* = \frac{\chi T_{\odot}}{2(1 - e^2)^{3/2}} \left( \frac{2\pi}{P_b} \right)^2 \frac{m_{\text{BH}}(3m_p + 4m_{\text{BH}})}{M}, \quad (5.8)
\]
\[
\Omega_{q}^* = \frac{3q T_{\odot}^{4/3}}{4(1 - e^2)^2} \left( \frac{2\pi}{P_b} \right)^{7/3} \frac{m_{\text{BH}}^2}{M^{2/3}}. \quad (5.9)
\]

Note that \( \Omega_{s}^* \) and \( \Omega_{q}^* \) are not the moduli of the corresponding vectors. Here we have \( M \equiv m_{\text{BH}} + m_p \), \( T_{\odot} \equiv GM_\odot/c^3 \), \( c \) the speed of light, \( e \) the orbital eccentricity, \( P_b \) the orbital period, and \( m_p \), \( m_{\text{BH}} \) the mass of the pulsar and the BH in units of solar mass \( M_\odot \), respectively.

In the frame of Fig. 5.2, the long-term precession of the orbit can be fully described by the linear-in-time\(^2\) advance of the angles \( \Phi \) and \( \Psi \), which can be

\(^2\)Strictly speaking, the secular variations in \( \Phi \) and \( \Psi \) are linear functions of the true anomaly (Wex, 1995).
expressed by (Smarr & Blandford, 1976; Wex, 1998; Wex & Kopeikin, 1999):

$$\dot{\Phi} = \dot{\Phi}_s + \dot{\Phi}_q,$$

(5.10)

where

$$\dot{\Phi}_s = \Omega_s^* \frac{\sin \theta_s}{\sin \theta_J},$$

(5.11)

$$\dot{\Phi}_q = \Omega_q^* \frac{\sin 2\theta_s}{\sin \theta_J},$$

(5.12)

and

$$\dot{\Psi} = \dot{\Psi}_{pn} + \dot{\Psi}_s + \dot{\Psi}_q,$$

(5.13)

where

$$\dot{\Psi}_{pn} = \Omega_{pn}^* = \dot{\omega}_{pn},$$

(5.14)

$$\dot{\Psi}_s = -\Omega_s^* \left(2 \cos \theta_s + \sin \theta_s \cot \theta_J\right),$$

(5.15)

$$\dot{\Psi}_q = -\Omega_q^* \left(\frac{1}{2} + \frac{3}{2} \cos 2\theta_s + \sin 2\theta_s \cot \theta_J\right).$$

(5.16)

As can be seen from Fig. 5.2, $\theta_J$ can be expressed by $\theta_S$ as

$$\sin \theta_J = \frac{\Sigma \sin \theta_S}{\sqrt{1 + 2\Sigma \cos \theta_S + \Sigma^2}},$$

(5.17)

$$\cot \theta_J = \cot \theta_S + \frac{1}{\Sigma \sin \theta_S},$$

(5.18)

where

$$\Sigma \equiv \frac{S}{L} = \frac{\chi T^{1/3}_\odot}{\left(1 - e^2\right)^{1/2}} \left(\frac{2\pi}{P_b}\right)^{1/3} \frac{m_{\text{BH}} M^{1/3}}{m_p}. $$

(5.19)

Therefore, the precession rates of $\Psi$ and $\Phi$ can be both written as a function of $\theta_S$. Note that the rate of change is averaged over a whole orbital period, as most of the perturbation actually happens when the pulsar is passing periastron. For the study of long-term variation it is already a sufficient approximation.

In PSR-BH systems the quadrupole contribution to orbital precession is usually far less than that induced by the frame dragging (FD). Assuming an extreme Kerr BH ($\chi = 1$) and neglecting the influence of orbital eccentricity and geometry, the ratio between the precession rates (assuming $m_p \ll m_{\text{BH}}$) from these two
aspects can be written as

\[ \frac{\Omega_s^*}{\Omega_q^*} \approx 1.4 \times 10^3 \left( \frac{P_b}{1 \text{ day}} \right)^{1/3} m_{\text{BH}}^{-1/3}. \]  (5.20)

For a compact PSR-BH binary with \( P_b = 0.4 \text{ day} \) and \( m_{\text{BH}} = 30 \) the ratio is approximately 500. For a pulsar orbiting Sgr A* with \( P_b = 0.1 \text{ yr} \) and \( m_{\text{BH}} = 4 \times 10^6 \) we have \( \Omega_s^*/\Omega_q^* \approx 30 \). So in the following derivations concerning the orbital secular motion we mainly consider the contribution by the Lense-Thirring effect.

### 5.2.2 Long-term evolution of observable quantities

The linear-in-time variation of \( \Phi \) and \( \Psi \) results in a nonlinear-in-time evolution of the projected semi-major axis \( x \) and the longitude of periastron \( \omega \) (Wex, 1998).

In real timing observations, the long-term change of the two parameters can be approximated by Taylor expansion as below:

\[
\begin{align*}
    x &= x_0 + \dot{x}_0 (t - T_0) + \frac{1}{2} \ddot{x}_0 (t - T_0)^2 + \ldots, \quad (5.21) \\
    \omega &= \omega_0 + \dot{\omega}_0 (t - T_0) + \frac{1}{2} \ddot{\omega}_0 (t - T_0)^2 + \ldots, \quad (5.22)
\end{align*}
\]

where \( T_0 \) is the time of periastron passage and \( x_0, \omega_0 \) are initial values at the epoch. Neglecting the precession by quadrupole effect, within the framework of this chapter we can write the derivatives in the following forms

\[
\begin{align*}
    \dot{x} &= \dot{x}_s, \quad (5.23) \\
    \ddot{x} &= \ddot{x}_s, \quad (5.24) \\
    \dot{\omega} &= \dot{\omega}_{\text{pn}} + \dot{\omega}_s, \quad (5.25) \\
    \ddot{\omega} &= \ddot{\omega}_s. \quad (5.26)
\end{align*}
\]

In the coordinates of Fig. 5.2, letting \( \omega = \Psi + \delta \), we can write down the following geometric equations based on the trigonometric relationships (Wex & Kopeikin,


\[
\begin{align*}
\cos i &= \cos i_J \cos \theta_J - \sin \theta_J \sin i_J \cos \Phi, \\
\sin i \sin \delta &= \sin i_J \sin \Phi, \\
\sin i \cos \delta &= \sin \theta_J \cos i_J + \cos \theta_J \sin i_J \cos \Phi \\
&= \frac{\cos i_J - \cos \theta_J \cos i}{\sin \theta_J},
\end{align*}
\]

From the above connections of the angles, the secular change in \( \Phi \) and \( \Psi \) can be linked with the derivatives of the Keplerian parameters. From Eq. (5.27) we have

\[
\begin{align*}
\frac{d}{dt}(\cos i) &= \sin \theta_J \sin i_J \sin \Phi \Omega_s^*, \\
\frac{d^2}{dt^2}(\cos i) &= \sin \theta_J \sin i_J \cos \Phi \Omega_s^*. 
\end{align*}
\]

So the first and second derivative of \( x \) can be written in the form of

\[
\begin{align*}
\dot{x}_s &= a_p \frac{d}{dt}(\sin i) \\
&= -a_p \cot i \frac{d}{dt}(\cos i) \\
&= -a_p \cot i \sin \theta_s \sin i_J \sin \Phi_0 \Omega_s^*. 
\end{align*}
\]

and

\[
\begin{align*}
\ddot{x}_s &= a_p \frac{d^2}{dt^2}(\sin i) \\
&= a_p \left\{ -\cot i \frac{d^2}{dt^2}(\cos i) - \frac{1}{\sin^3 i} \left[ \frac{d}{dt}(\cos i) \right]^2 \right\} \dot{\Phi}_s^2 \\
&= -a_p \sin^2 \theta_s \sin i_J \sin i \sin \theta_J (\cos i \cos \Phi_0 + \frac{\sin \theta_J \sin i_J \sin^2 \Phi_0}{\sin^2 i}) \Omega_s^* \Omega_s^*. 
\end{align*}
\]

The periastron advance by the FD effect can be described as below up to the second time derivative

\[
\begin{align*}
\dot{\omega}_s &= \dot{\Psi}_s + \dot{\delta}, \\
\ddot{\omega}_s &= \ddot{\delta}.
\end{align*}
\]
The first derivative of $\delta$ can be readily obtained from Eq. (5.28):

$$
\dot{\delta} = \frac{1}{\cos \delta} \left[ \sin i_J \cos \Phi_0 \Phi - \frac{\sin i_J \sin \Phi_0}{\sin^2 i} \frac{d(sin i)}{dt} \right] = \frac{\sin i_J \sin \theta_S}{\cos i_J \sin \theta_J + \cos \theta_J \sin i_J \cos \Phi_0 \left( \frac{\cos \Phi_0}{\sin \theta_J} + \frac{\sin i_J \sin^2 \Phi_0 \cos i}{\sin^2 i} \right)} \Omega_s^* \\
= \frac{\sin i_J \sin \theta_S \sin \theta_J}{\cos i_J - \cos \theta_J \cos i} \left( \frac{\cos \Phi_0}{\sin \theta_J} + \frac{\sin i_J \sin^2 \Phi_0 \cos i}{\sin^2 i} \right) \Omega_s^*. 
$$

(5.34)

The expression of $\ddot{\delta}$ (effectively the $\ddot{\omega}_s$) directly obtained from Eq. (5.34) is complicated. Consequently, the result is shown only for the two cases discussed in Section 5.2.3 where $\ddot{\delta}$ can be simplified by approximations.

5.2.3 Spin extraction and geometry determination

Once the secular motion rates caused by the Lense-Thirring effect are known, based on the expressions presented in Section 5.2.2, one can have a full determination of the BH spin together with the system geometry. Two cases that lead to different approximation schemes are discussed below.

5.2.3.1 Pulsar with stellar mass black hole

For a compact PSR-SBH binary, the orbital angular momentum is usually dominant in the total angular momentum. To illustrate, even with $\chi = 1$, $e = 0.6$, $P_b = 0.5$ day, $m_{BH} = 30$, and $m_p = 1.4$, from Eq. 5.19 one has $\Sigma \simeq 7.6 \times 10^{-2}$ and a corresponding small angle $\theta_J < 5^\circ$. Therefore, we can have $i_J \simeq i$ and the derivatives of $\omega$ and $x$ approximated as below:

$$
\dot{x}_s \simeq -x_0 \chi \tilde{\Omega}_s \cot i \sin \theta_S \sin \Phi_0, \\
\ddot{x}_s \simeq -\frac{1}{\Sigma} x_0 \chi \tilde{\Omega}_s \cot i \sin \theta_S \cos \Phi_0, \\
\dot{\omega}_s \simeq -\chi \tilde{\Omega}_s (2 \cos \theta_S + \cot i \sin \theta_S \cos \Phi_0), \\
\ddot{\omega}_s \simeq \frac{1}{\Sigma} \chi \tilde{\Omega}_s^2 \cot i \sin \theta_S \sin \Phi_0,
$$

(5.35) (5.36) (5.37) (5.38)
where

\[ \tilde{\Omega}_s \equiv \frac{\Omega^*_s}{\chi}, \]  
\[ (5.39) \]

\[ \tilde{\Sigma} \equiv \frac{\Sigma}{\chi}. \]  
\[ (5.40) \]

Therefore, there are four equations for three unknown parameters, \( \chi \), \( \theta_S \), \( \Phi_0 \). However, it is clear from the expressions that \( \dot{x}_s \) and \( \dot{\omega}_s \) contain the same spin and geometry information so only one of them can be effective in solving the equations. As will be shown in Section 5.5.2, for PSR-SBH systems \( \dot{x}_s \) is expected to be better measured so in this case we use \( \dot{x}_s \), \( \dot{\omega}_s \) and \( \ddot{x}_s \) for the solutions. By combining the first three equations the BH spin amplitude \( \chi \) and orientation \( \theta_S \) can be obtained in the form of

\[ \chi \simeq \frac{1}{x_0 \tilde{\Omega}_s} \left[ \left( \frac{\Xi_s - x_0 \dot{\omega}_s}{2} \right)^2 + \left( \dot{x}_s^2 + \Xi_s^2 \right) \tan^2 i \right]^{1/2}, \]  
\[ (5.41) \]

\[ \cos \theta_S \simeq \frac{\Xi_s - x_0 \dot{\omega}_s}{\left( \left( \Xi_s - x_0 \dot{\omega}_s \right)^2 + 4 \left( \dot{x}_s^2 + \Xi_s^2 \right) \tan^2 i \right)^{1/2}}, \]  
\[ (5.42) \]

where

\[ \Xi_s \equiv \ddot{x}_s \tilde{\Sigma}/\tilde{\Omega}_s \]  
\[ (5.43) \]

is a spin-independent parameter. The \( \theta_S \) is uniquely determined from \( \cos \theta_S \) with the given range of 0 to \( \pi \).

Once \( \chi \) and \( \theta_S \) are obtained, \( \Phi_0 \) can then be uniquely determined from Eq. (5.35) and Eq. (5.36). The angle between the line of sight (defined in Fig. 5.3, with range of \(-\pi \) to 0) and the BH spin can be derived from

\[ \cos \lambda = K_0 \cdot S_0, \]

\[ = \begin{pmatrix} 0 \\ \sin i_J \end{pmatrix} \cdot \begin{pmatrix} \cos(\Phi_0 + \frac{\pi}{2}) \sin(\theta_S - \theta_J) \\ \sin(\Phi_0 + \frac{\pi}{2}) \sin(\theta_S - \theta_J) \\ \cos(\Phi_0 + \frac{\pi}{2}) \sin(\theta_S - \theta_J) \end{pmatrix}, \]

\[ \simeq \cos \theta_S \cos i + \sin i \sin \theta_S \cos \Phi_0, \]  
\[ (5.44) \]
Consequently, the angle $\Omega$ in Fig. 5.3 can be obtained from
\[
\cos \Omega = \frac{\cos i - \cos \theta_S \cos \lambda}{\sin \theta_S \sin \lambda},
\]
\[
\sin \Omega = \frac{(X_0 \times X_\Omega) \cdot S_0}{|X_0| \cdot |X_\Omega|},
\]
\[
\simeq \frac{-\sin \Phi_0 \sin i}{[\sin^2 \Phi_0 \sin^2 \theta_S + (\cos \Phi_0 \sin \theta_S \cos i - \cos \theta_S \sin i)^2]^{1/2}},
\]
where
\[
X_0 = S_0 \times K_0 \simeq \begin{pmatrix}
\cos \Phi_0 \sin \theta_S \cos i - \cos \theta_S \sin i \\
\sin \Phi_0 \sin \theta_S \cos i \\
-\sin \Phi_0 \sin \theta_S \sin i
\end{pmatrix},
\]
\[
X_\Omega = S_0 \times L_0 \simeq \sin \theta_S \begin{pmatrix}
\cos \Phi_0 \\
\sin \Phi_0 \\
0
\end{pmatrix}.
\]

The longitude of periastron $\omega_{SZ}$ in Fig. 5.3 is determined from
\[
\sin(\omega - \omega_{SZ}) = -\frac{\sin \lambda \sin \Omega}{\sin i},
\]
\[
\cos(\omega - \omega_{SZ}) = \frac{\sin \theta_S \cos \lambda - \cos \theta_S \sin \lambda \cos \Phi_0}{\sin i}.
\]

Note that in general the orbital inclination angle is not completely decided as pulsar timing only provides information of $\sin i$. This may lead to the ambiguity of an alternative binary geometric configuration. Fortunately, the expressions of $\chi$ and $\theta_S$ are only functions of $\sin i$, indicating that the measurements of them are not influenced by the sign of $\cos i$. Meanwhile, $\cos \lambda$ will follow the sign change of $\cos i$ and both sine and cosine of $\Phi_0$, $\Omega$ and $\omega_{SZ}$ do as well. In Table 5.3 the combination of the solutions to $i_J$, $\Phi_0$, $\lambda$, $\Omega$ and $\omega_{SZ}$ are presented. It should be noted that neither of the second derivatives flips sign together with $\cos i$, which indicates that they do not help to identify the real solution. Thus, further information from timing parallax, proper motion (Kopeikin, 1995, 1996), or astrometric approaches is needed to yield a unique determination of the geometric configuration.
Table 5.3: Two possible sets of solutions of the geometric angles in a PSR-SBH binary, for a given value of $s_i$. Note that a $\pm 2\pi$ may be required to ensure that the solution falls in the defined range as mentioned in the caption of Fig. 5.2 and Fig. 5.3.

<table>
<thead>
<tr>
<th>$\cos i$</th>
<th>$i_j$</th>
<th>$\Phi_0$</th>
<th>$\lambda$</th>
<th>$\Omega$</th>
<th>$\omega_{SZ}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 - s_i^2)^{1/2}$</td>
<td>$i_{j0}$</td>
<td>$\Phi_{00}$</td>
<td>$\lambda_0$</td>
<td>$\Omega_0$</td>
<td>$\omega_{SZ0}$</td>
</tr>
<tr>
<td>$-(1 - s_i^2)^{1/2}$</td>
<td>$\pi - i_{j0}$</td>
<td>$\pi + \Phi_{00}$</td>
<td>$-\pi - \lambda_0$</td>
<td>$\pi + \Omega_0$</td>
<td>$\omega_{SZ0} - \pi$</td>
</tr>
</tbody>
</table>

Fig. 5.3: Spin-Z coordinate and the definition of corresponding angles. Here $K$ is the line of sight vector and $\nu$ the orbital true anomaly. The defined ranges of the angles are: $\lambda \in (-\pi, 0]$ and $\Omega, \omega_{SZ} \in [0, 2\pi)$.

5.2.3.2 Pulsar around Sgr A*

In contrast to the situation in PSR-SBH systems, for the case of a pulsar orbiting Sgr A*, the system angular momentum is likely to be dominated by the BH spin. Given the parameters of $P_b = 0.5$ yr, $m_{BH} = 4 \times 10^6$ $e = 0.1$, and $m_p = 1.4$, we have $\Sigma \simeq 3.8 \times 10^4$, which leads to the following approximations:

$$\theta_s \simeq \theta_J, \quad i_J \simeq -\lambda, \quad \Omega \simeq \Phi, \quad \dot{\Phi} \simeq \Omega_s^*, \quad \dot{\Psi} \simeq -3 \cos \theta_s \Omega_s^*.$$ 

Here $\Omega$ is the longitude of orbital ascending node as defined in Fig. 5.3, and $\lambda$ is the angle between line of sight and the BH spin in Fig. 5.3. Then we can readily
5.2. FRAME DRAGGING AND COSMIC CENSORSHIP CONJECTURE

Rewrite the expressions of $\dot{x}_s$, $\ddot{x}_s$, $\dot{\omega}_s$ and the geometry equation Eq. (5.27) as:

\[
\dot{x}_s \simeq a_p \cot i \sin \theta_S \sin \lambda \sin \Phi_0 \Omega^*_s, \quad (5.46)
\]

\[
\ddot{x}_s \simeq a_p \frac{\sin \theta_S \sin \lambda}{\sin i} \left( \cos i \cos \Phi_0 - \frac{\sin \theta_S \sin \lambda \sin^2 \Phi_0}{\sin^2 i} \right) \Omega^*_s. \quad (5.47)
\]

\[
\cos i \simeq \cos \lambda \cos \theta_S + \sin \lambda \sin \theta_S \cos \Phi_0, \quad (5.48)
\]

\[
\dot{\omega}_s = \dot{\Psi} + \dot{\delta} \simeq -3 \cos \theta_S \Omega^*_s - \frac{\sin \lambda \sin^2 \theta_S}{\cos \lambda - \cos \theta_S \cos i} \left( \frac{\cos \Phi_0}{\sin \theta_S} - \frac{\sin \lambda \sin^2 \Phi_0 \cos i}{\sin^2 i} \right) \Omega^*_s. \quad (5.49)
\]

Applying Eq. (5.48) to replace the $\Phi_0$ related term in Eq. (5.49), $\dot{\omega}_s$ can then be represented as

\[
\dot{\omega}_s \simeq \left( \frac{\cos \theta_S - \cos i \cos \lambda}{\sin^2 i} - 3 \cos \theta_S \right) \Omega^*_s. \quad (5.50)
\]

Hence, $\dot{\omega}_s$ is immediately obtained by the time derivative of $\dot{\omega}_s$:

\[
\ddot{\omega}_s \simeq - \frac{2 \cos \theta_S \cos i - (2 - \sin^2 i) \cos \lambda}{\sin^2 i} \sin \lambda \sin \theta_S \sin \Phi_0 \Omega^*_s. \quad (5.51)
\]

Therefore, there are five equations to solve for four unknowns, $\Omega^*_s$, $\theta_S$, $\lambda$, $\Phi_0$. Here, in contrast to the case of compact PSR-SBH binaries, the $\dot{\omega}_s$ is usually better determined than the second derivatives because only one extra PK parameter ($\gamma$ or $\sin i$) is needed to determine the BH mass and subtract the monopole contribution. So in practice the latest two constrained parameters are usually the second derivatives, and only one determination of the two will allow the solution of the equations. As will be shown in Section 5.6.2, for a given observing baseline which of the two parameters will be measured with better precision depends on the system geometry. In Appendix D both situations are considered and discussed. Briefly, the utilisation of $\ddot{\omega}_s$ can yield a unique determination of the spin parameter $\chi$, while that of $\ddot{x}_s$ may lead to multiple solutions. In both cases the unknown sign of $\cos i$ results in two possible sets of solutions to the angles.

By re-parameterizing Eq. (5.46), Eq. (5.47), Eq. (5.48), Eq. (5.50) and Eq. (5.51) in a slightly different way from that in Appendix D, we develop a method to
demonstrate the spin measurement, inspired by the idea of the mass-mass dia-
gram (e.g. Taylor & Weisberg, 1982). Here by defining $c_X \equiv \cos X$, $s_X \equiv \sin X$, $\chi_\theta \equiv c_\theta X$, $\chi_\lambda \equiv c_\lambda X$, $\zeta_3 \equiv -s_is_\theta s_\lambda X$, the aforementioned five equations can be
written as:

$$-\dot{x}_s s^2_s (x_\tilde{\Omega}_s)^{-1} \equiv X_1 = c_i \zeta_3,$$

$$\dot{\omega}_s s^2_s \tilde{\Omega}_s^{-1} \equiv W_1 = (1 - 3s^2_s) \chi_\theta - c_i \chi_\lambda,$$

$$\left(\dot{x}_s x + x^2_s s^2_s c_i^{-2}\right)s^4_s (x_\tilde{\Omega}_s)^{-2} \equiv X_2 = c_i^2 (\chi_\theta^2 + \chi_\lambda^2) - c_i (1 + c_i^2) \chi_\theta \chi_\lambda,$$

$$-\ddot{\omega}_s x_s^{-1} x c^2_i s^2_s \tilde{\Omega}_s^{-1} \equiv W_2 = 2c_i^2 \chi_\theta - c_i (1 + c_i^2) \chi_\lambda,$$

where $\zeta_3$ in the other equations has been eliminated using Eq. (5.52). The $\chi_\theta$ and $\chi_\lambda$ are the projection of the spin-parameter onto the orbital angular momentum and the line of sight direction, respectively. The quantities $X_1$, $W_1$, $X_2$, and $W_2$ are defined such, that they do not change when the sign of $c_i$ is being flipped.

From Eq. (5.52), $\zeta_3$ is readily obtained once $\dot{x}_s$, orbital inclination and BH mass are determined. The best way to represent the solutions to $\chi_\theta$ and $\chi_\lambda$, is to plot the constraints from Eqs. (5.53), (5.54) and (5.55) in the $\chi_\theta$–$\chi_\lambda$ plane. Possible solutions are represented by the region where all three curves meet within the uncertainty given by the measurement error of $\dot{\omega}_s$, $\ddot{x}_s$, and $\ddot{\omega}_s$. With $\chi_\theta$, $\chi_\lambda$ and $\zeta_3$ known, we can calculate the spin parameter of the BH via:

$$\chi = s_i^{-1} \sqrt{\zeta^2_3 + \chi^2_\theta + \chi^2_\lambda - 2c_i \chi_\theta \chi_\lambda}.$$

### 5.3 Quadrupole measurement and no-hair theorem

As outlined in Section 5.1.1, once the BH mass and spin are obtained, the determination of the quadrupole moment will present a direct test for the no-hair theorem. The Newtonian orbit-quadrupole moment interaction will cause secular changes in both periastron and projected semi-major axis. Unfortunately, as the
size of this effect is even a few magnitudes lower than the precession induced by FD (see Eq. 5.20), this contribution is usually not separable from the precession by FD in practical data reduction. Meanwhile, in addition to the secular precession, the orbit of a test particle going around an oblate central object, will also undergo periodic perturbations. Such perturbations have been well studied by previous work, concerning predicting the position of man-made satellites (Sterne, 1957; Garfinkel, 1958, 1959). The solution can also be used to describe the relative motion of two bodies in a binary system. In the PSR-BH case, where GR plays an essential role in the orbit dynamics, the result needs to be modified so as to account for the coupling between quadrupole perturbation and PN motion. The modelling of the periodic feature allows the determination of the BH quadrupole moment, once the masses and geometry of the system have been obtained as shown in Section 5.2.

5.3.1 The Hamiltonian

The orbital dynamics of a binary pulsar with an oblate companion star can be derived from the system Hamiltonian in a centre-of-mass frame (Barker & O’Connell, 1975):

\[ \mathcal{H} = \mathcal{H}_N + \mathcal{H}_{\text{PN}} + \mathcal{H}_{\text{LT}} + \mathcal{H}_Q, \]  

where the indices denote the Newtonian, PN, Lense-Thirring, and quadrupole moment contribution. The oblateness of a massive object will change the distribution of the potential field by a small amount. Within the framework of far field expansion, for the gravitational field of a BH the component of the quadrupole moment can be written in the form of (Thorne et al., 1986)

\[ V_Q = -\frac{GS^2}{2r^3M_{\text{BH}}c^2}(1 - 3 \cos^2 \theta), \]

where \( \theta \) denotes the spherical zenith angle (angle between pulsar position vector and Z-direction) in the spin-Z coordinate of Fig. 5.3. Adopting the relation between the dimensionless spin and quadrupole, \( q = -\chi^2 \) (Thorne et al., 1986),
one can then rewrite Eq. (5.58) as
\[ V_Q = \frac{qG^2M_{BH}^3}{2r^3c^4}(1 - 3\cos^2\theta). \] (5.59)

\section*{5.3.2 The orbital motion}

The orbital motion regarding the Newtonian Hamiltonian plus the quadrupole correction was described in Garfinkel (1959), by splitting the quadrupole effect into three aspects of contributions: secular change in periastron, and short and long-term periodic perturbation. By following Sterne (1957) and Garfinkel (1958), the gravitational potential of the Hamiltonian can be written as
\[ V_{N+Q} = -\frac{GM_{BH}}{r} + \frac{GM_{BH}k(3\cos^2\theta - 1)}{r^3}, \] (5.60)

where \( k \) is readily seen to be associated with the dimensionless quadrupole moment by \( k = -\frac{qG^2M_{BH}^2}{2c^4} \). As \( V \) is not in the form of \( F_1(r) + F_2(\theta)/r^2 \), the corresponding Hamilton-Jacobi equation of this potential field cannot be directly solved by separating the variables (Sterne, 1957). Therefore, Sterne (1957) and Garfinkel (1958) introduced the concept of an intermediary orbit, with six invariables \( a, e, \theta_S, \sigma, \omega_{SZ}, \Omega \), analogous to the usual elliptic elements which are correspondingly semi-major axis, eccentricity, orbit inclination, mean anomaly at epoch, argument of periastron, and longitude of the ascending node. Here the angles are all defined in the spin-Z coordinate system in Fig. 5.3. Letting \( GM_{BH} = 1 \), the potential field is modified to be
\[ V'_{N+Q} = -\frac{\mu}{r} + \frac{3kc_1(\cos^2\theta - c_2)}{r^2}, \] (5.61)

where \( \mu = 1 - 6kc_3 \) and the so-called “disposable parameters” \( c_i \) are introduced to vanish the orbital secular variations of \( \mathcal{O}(k) \). The orbital motion in this gravitational potential is then obtained by separating the Hamilton-Jacobi partial differential equations in the form of the three following equations (Garfinkel,
\[ \int \frac{dr}{p_1} = \beta_1 + t, \]
\[- \int \frac{\alpha_2 dr}{r^2 p_1} + \int \frac{\alpha_2 d\theta}{p_2} = \beta_2, \]
\[ \phi - \int \frac{\alpha_3 d\theta}{p_2 \sin^2 \theta} = \beta_3. \]  
(5.62)

where

\[ p_1 = \left(2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}\right)^{1/2}, \]
\[ p_2 = \left[\alpha_2^2 - \frac{\alpha_3^2}{\sin^2 \theta} - 6kc_1(\cos^2 \theta - c_2)\right]^{1/2}, \]
\[ \alpha_1 = \frac{1}{2}(r^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + V', \]
\[ \alpha_2 = (2\mu r + 2\alpha_1 r^2 - r^2 \dot{r}^2)^{1/2}, \]
\[ \alpha_3 = r^2 \cos^2 \theta \dot{\phi}. \]  
(5.63)

and the \( \alpha_i, \beta_i \) are related to the elliptic elements of the intermediary orbit by:

\[ p = a(1 - e^2), \quad n^2 a^3 = \mu, \quad \epsilon = \frac{6kc_1}{\alpha_2^2}, \]
\[ \alpha_1 = -\frac{\mu}{2a}, \quad \alpha_2 = \mu p, \quad \beta_1 = \frac{\sigma}{n}, \quad \beta_2 = \omega, \quad \beta_3 = \Omega. \]  
(5.64)

Here the secular motions of periastron \((g_{21})\) and the ascending node \((g_{32})\) with respect to the equatorial plane are incorporated within the intermediary orbit and described by the differences in the three fundamental angular frequencies, \(n_i\) (Garfinkel, 1958):

\[ g_{21} \equiv \frac{n_2}{n_1} - 1 = \epsilon (1 - \frac{5}{4} \cos^2 \theta_S), \]
\[ g_{32} \equiv \frac{n_3}{n_2} - 1 = -\epsilon \frac{1}{2} \sin \theta_S. \]

When the orbital declination \(90^\circ - \theta_S\) is close to \(63.4^\circ\), the secular motion of the periastron will vanish and the long-term perturbation be dramatically amplified, which is called orbital resonance. In Garfinkel (1960), another representation of the orbital motion was given for the case where the mean orbit declination is
within the vicinity of the critical value. In our case, where significant relativistic corrections to the Hamiltonian have to be considered, the secular motions are significantly modified. Consequently, there is no critical declination where one has the resonant amplification of the perturbation. Specifically, the relativistic periastron advance is the main contribution and the full solution of Garfinkel (1959) should be modified to account for the effect\(^3\). On collecting the result, after omitting second order perturbations and secular terms, we have the equations of motions as following

\[
E - e \sin E = nt + \sigma, \\
\tan \frac{\nu}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}, \\
r = a(1 - e \cos E), \\
\psi = (1 + \epsilon g_{\text{y}21})(\nu + \omega_{\text{SZ}}), \\
\frac{\delta r_s}{r} = \epsilon \{ -P(a_1 \cos \nu + a_2 \cos 2\nu) + W[\frac{e}{6} \cos(2\psi - \nu) + \frac{1}{3} \cos 2\psi + \frac{e}{6} \cos(2\psi + \nu)] \}, \\
\frac{\delta r_l}{r} = \epsilon \alpha \left[ \frac{e^2}{2} \cos(2\psi - 2\nu) + e \cos(2\psi - \nu) + \frac{e^2}{2} \cos 2\psi \right], \\
r' = r + \delta r = r + \delta r_s + \delta r_l, \\
\delta \Omega_s = \frac{1}{2} \epsilon e \cos \theta_S[ - \sin \nu + \frac{1}{2} \sin(2\psi - \nu) + \frac{1}{6} \sin(2\psi + \nu)], \\
\delta \Omega_l = -\epsilon e^2 \gamma \sin(2\psi - 2\nu), \\
\Omega' = \Omega + \delta \Omega = \Omega + \delta \Omega_s + \delta \Omega_l, \\
\delta \psi_s = \epsilon \{ P(2a_1 \sin \nu + a_2 \sin 2\nu) + W[\frac{2}{3} e \sin(2\psi - \nu) + \frac{1}{6} \sin 2\psi] \} - \cos \theta_S \delta \Omega, \\
\delta \psi_l = \epsilon \left[ e^2 \beta \sin(2\psi - 2\nu) - 2e \alpha \sin(2\psi - \nu) - \frac{e^2 \alpha}{2} \sin 2\psi \right], \\
\psi' = \psi + \delta \psi = \psi + \delta \psi_s + \delta \psi_l, \\
\delta \theta_{s_S} = \frac{ee}{8} \sin 2\theta_S[ \cos(2\psi - \nu) + \frac{1}{3} \cos(2\psi + \nu)], \\
\delta \theta_{s_1} = \frac{1}{2} \epsilon m e^2 \sin 2\theta_S \cos(2\psi - 2\nu),
\]

\(^3\)The Lense-Thirring effect also changes the ascending node but this will only appear in the second order correction.
$$\theta'_S = \theta_S + \delta \theta_S = \theta_S + \delta \theta_{S,1},$$
$$\cos \theta' = \sin \theta'_S \sin \psi',$$
$$\phi' = \Omega' + \arctan(\cos \theta'_S \tan \psi'),$$

where \( r', \theta', \phi' \) denote the real pulsar position in spherical coordinates, and the constants are defined as follows:

$$a_2 = \frac{1}{3}(1 - \sqrt{1 - e^2}), \quad a_1 = a_2 \left( \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}} \right)^{1/2},$$
$$P = \frac{1}{4}(1 - 3 \cos^2 \theta_S), \quad W = \frac{1}{4}(1 - \cos^2 \theta_S), \quad m = -\frac{3}{48} - \frac{1}{96 g'_{21}},$$
$$\alpha = 4mW, \quad \gamma = \cos \theta_S \left( \frac{3}{48} + \frac{1}{96 g'_{21}} \right), \quad \beta = \gamma \cos \theta_S,$$
$$\mu = 1 + \frac{6k \sqrt{1 - e^2} P}{p^2}, \quad \epsilon = \frac{6k}{\mu p^2}, \quad g'_{21} = \frac{1}{4}(5 \cos^2 \theta_S - 1).$$

Note that the PN motion can also be solved by the Hamilton-Jacobi approach in parallel (e.g. Damour & Schäfer, 1988), here we define the new \( g'_{21} \equiv k_{pn}/\epsilon = \dot{\omega}_{pn} P_b/(2\pi \epsilon) \) to account for the coupling between quadrupole periodic perturbation and PN periastron advance. It is clear from the solution that the long period perturbation terms of the Newtonian solution will become infinite when \( g'_{21} = 0 \), corresponding to the resonance angle \( 90^\circ - \theta_S = 63.4^\circ \). While the PN motion is considered, the amplitude parameters of the long period perturbation terms will become:

$$m = -\frac{3}{48} - \frac{\epsilon}{96 k_{pn}}, \quad \gamma = \cos \theta_S \left( \frac{3}{48} + \frac{\epsilon^2}{96 k_{pn}^2} \right).$$

(5.65)

Note that \( \epsilon/k_{pn} \) is usually a small amount for a compact PSR-BH system. For such a binary of \( P_b = 0.2 \) days, \( e = 0.6 \), \( m_{PSR} = 1.4 \) and \( m_{BH} = 40 \), the ratio would be of order \( 10^{-5} \). This implementation of the PN effect does not present an absolute solution to the true orbital motion, but retains the feature of the periodic perturbation by the quadrupole moment and also accounts for the coupling between these two effects.
5.3.3 The Römer delay

The quadrupole moment induced orbital perturbation will change the Römer delay calculation which accounts for the difference in light travelling time along the line of sight at different orbital phases (see Section 2.4.2 for definition). Within the coordinate of Fig. 5.3, the line of sight and pulsar position vector can be written as

\[
\mathbf{K} = \begin{pmatrix} 0 \\ -\sin \lambda \\ \cos \lambda \end{pmatrix}, \quad \mathbf{r}_p = r_p \frac{m_{\text{BH}}}{M} \begin{pmatrix} \sin \theta' \cos \phi' \\ \sin \theta' \sin \phi' \\ \cos \theta' \end{pmatrix}.
\]

The Römer delay is then readily obtained from

\[
\Delta_R = \mathbf{K} \cdot \mathbf{r}_p = r_p \frac{m_{\text{BH}}}{M} (-\sin \theta' \sin \phi' \sin \lambda + \cos \theta' \cos \lambda).
\] (5.66)

At this stage, the parameters needed as an input to model the quadrupole features are already well known from the PN motion and FD effect, in particular the four angles \( \omega, i, \theta_S \) and \( \lambda \) that determine the system geometry. As explained in Section 5.2.3, the first two come from the fit of the PK parameters, and the latter two can be obtained together with the spin measurement. Next we expand Eq. (5.66) with respect to \( \epsilon \), only keeping the zero and first order terms

\[
\Delta_R = \Delta_R^0 + \delta \Delta_R + \mathcal{O}(\epsilon^2),
\] (5.67)

where \( \delta \Delta_R \) is of order \( \epsilon \). Specifically, for \( \theta' \) we have:

\[
\cos \theta' = \sin(\theta_S + \delta \theta_S) \sin(\psi + \delta \psi) \\
\approx (\sin \theta_S \sin \psi) + (\cos \theta_S \sin \psi \delta \theta_S + \sin \theta_S \cos \psi \delta \psi) \\
\equiv A_\theta + \delta A_\theta,
\]

\[
\sin \theta' = (1 - \cos^2 \theta')^{1/2} \\
= (1 - A_\theta^2 - 2 A_\theta \delta A_\theta)^{1/2} \\
\approx \sqrt{1 - A_\theta^2} - \frac{A_\theta}{\sqrt{1 - A_\theta^2}} \delta A_\theta.
\] (5.68)
For $\phi'$ we have:

\[
\phi' = \Omega + \delta \Omega + \arctan[\cos(\theta S + \delta \theta S) \tan(\psi + \delta \psi)] \\
\simeq \Omega + \delta \Omega + \arctan(\cos \theta S \tan \psi) + (\cos \theta S \sin^2 \psi \delta \psi - \sin \theta S \tan \psi \delta \theta S) \\
= \Omega + \delta \Omega + B + \delta B,
\]

and then

\[
\sin \phi' \simeq \sin(\Omega + B + \delta \Omega + \delta B) \\
\simeq \sin(\Omega + B) + \cos(\Omega + B)(\delta \Omega + \delta B) \\
\equiv C_\phi + \delta C_\phi.
\]

Therefore, we have the Römer delay including the quadrupole effect in terms of

\[
\Delta R \simeq (r + \delta r) \frac{m_{\text{BH}}}{M} \left[ -(\sqrt{1 - A^2_\theta} - \frac{A_\theta}{\sqrt{1 - A^2_\theta}} \delta A_\theta)(C_\phi + \delta C_\phi) \sin \lambda + (A_\theta + \delta A_\theta) \cos \lambda \right],
\]

and specifically the periodic perturbations in the form of

\[
\delta \Delta R \simeq r \frac{m_{\text{BH}}}{M} \left[ \frac{C_\phi A_\theta}{\sqrt{1 - A^2_\theta}} \delta A_\theta - \sqrt{1 - A^2_\theta} \delta C_\phi \sin \lambda + \delta A_\theta \cos \lambda \right] \\
+ \frac{m_{\text{BH}}}{M} (-\sqrt{1 - A^2_\theta} C_\phi \sin \lambda + A_\theta \cos \lambda) \delta r.
\]

Comparing with the combinations of the solution to the angles in Table 5.3, the perturbation of the Römer delay will still remain the same once the sign of $\cos i$ is changed. It means that, though by fitting for the quadrupole effect one still cannot uniquely decide the true geometric configuration, the test of no-hair theorem is not influenced by the lack of knowledge of the inclination.

### 5.4 Timing precision and scheme of simulations

Traditionally, pulsar timing analysis involves measuring the pulse times-of-arrival (TOAs) for each observation, and on timescales of years monitoring the residuals after subtracting a best estimated model (see Chapter 2 for details). In this
section, we will discuss the achievable timing precisions for different types of PSR-BH systems and instruments. Technical details of simulations performed in the next section will also be provided.

5.4.1 Timing precisions

As mentioned in Section 1.6, the timing precision of NP is limited by the irregularity of the pulsar’s long-term rotational behaviour. The amplitude of the so-called “spin noise” varies from order of 10 $\mu$s to 100 ms, and some of the residuals can be corrected by following the approach proposed by Lyne et al. (2010) to improve the timing precision by an order of magnitude. Hence, a realistic precision of 100 $\mu$s is applied within the framework of simulation for NPs.

On the contrary, as discussed in Chapter 3, timing precision for most MSPs is currently limited by instrumental gain. Benefiting from the tremendous collecting area, the next generation of radio telescopes will significantly improve the precision especially for these stable timers (also see Section 2.5 for details). Here we consider the application of the Five-hundred-metre Aperture Spherical radio Telescope (FAST) and the Square Kilometre Array (SKA). The gain and TOA precision in comparison with Parkes Radio Telescope are briefly summarised in Table 5.4. It can be seen that ultimately the full SKA will provide roughly two orders of magnitude improvement to the precision.

The TOA precision of future telescopes for NPs near the GC, may still be limited by gain due to the extremely strong scattering of the interstellar medium, which broadens the pulse profile and decreases the observed peak S/N (see Section 1.4.2 for details). Here in Fig. 5.4, we estimate the achievable TOA precision of a NP near the GC for different observing frequencies. Two different spectral indices for the pulsar flux density are applied to typify many of these measured for pulsars (Maron et al., 2000). The scattering timescale is estimated to be $\tau_{\text{scat}} \approx 2.3 \times 10^6$ ms, by using the observed scattering diameter of Sgr A* and the estimated location of the scattering material along the line of sight, the latter
Table 5.4: Effective collecting area, system temperature and TOA precision of an hour integration time for different telescopes (Nan, 2006; Schilizzi et al., 2007). The TOA precisions are calculated with the assumption of 1 mJy flux density, 5 ms period, and 100 µs pulse width. The observation is assumed to be ten minutes long at 1.4 GHz with 300 MHz bandwidth. The 20 ns precision here, depending on the pulsar’s rotational period and profile shape, corresponds to an integrated profile with peak signal-to-noise ratio of $10^2$-$10^3$ in real observations. Note that here for the optimal case we do not account for the contribution of pulse jitter into TOA precision, since this effect, as discussed in Chapter 4, is source dependent and could be corrected by techniques currently being developed. Furthermore, the contribution can also be mitigated by extending the integration time.

<table>
<thead>
<tr>
<th>Gain (K/Jy)</th>
<th>$\sigma_{\text{10min}}$ (µs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parkes</td>
<td>0.64</td>
</tr>
<tr>
<td>FAST</td>
<td>20</td>
</tr>
<tr>
<td>Full SKA</td>
<td>100</td>
</tr>
</tbody>
</table>

Fig. 5.4: Predicted TOA measurement precision of a NP near the GC for two different spectral indices $\alpha$. We assumed a four hour integration time using a 100-meter radio telescope and a one hour integration time using an SKA-like telescope, both with a highest operating frequency of 30 GHz and a bandwidth of 1 GHz. For the pulsar we used a spin period $P = 0.5$ s, an intrinsic pulse width $W_i = 10$ ms, and a period-averaged flux density $S_{1400} = 1$ mJy at 1.4 GHz. It is found that observational frequencies above 15 GHz favors pulsar timing observation, where 100 µs TOA precision seems achievable, in particular with the SKA.
as incorporated in the NE2001 model ($\ell = b = 0$ and $D=8.5$ kpc, Cordes & Lazio, 2002). For this large amount of scattering, we use a scaling of $\tau_{\text{scat}} \propto f^{-4}$ rather than the often used Kolmogorov scaling $\tau_{\text{scat}} \propto f^{-4.4}$ because the dominant length scale is less than the inner scale of the wavenumber spectrum for the electron density (Cordes 2011, private communication). Note that all pulsars with close enough orbits to be of interest for GR tests will be seen along essentially the same line of sight as Sgr A* so one can assume that their lines of sight will have the same scattering characteristics. The system temperature (e.g. $\simeq 40$ K at 15 GHz) is calculated by summing the radio background $T_{\text{BG}}$, receiver temperature and emission of the atmosphere. In this case, the $T_{\text{BG}}$ is dominant at low frequency band (e.g., $\lesssim 5$ GHz) and can be expressed by (Reich et al., 1990; Macquart et al., 2010):

$$T_{\text{BG}} = 340 \times \left( \frac{\nu}{2.7\text{GHz}} \right)^{-2.7} \text{K}$$

(5.73)

The atmosphere is instead the main contribution at high frequency (e.g., $\gtrsim 10$ GHz) and here we apply an example measurement at GBT as presented in Fig. 5.5. Clearly, with a radio telescope like SKA, TOA uncertainties of below 100 $\mu$s seem likely for an observational frequency above 15 GHz, similar to the result of optimized searching frequency. A detection of MSPs in the GC is unlikely as pointed out in Cordes & Lazio (1997) and Macquart et al. (2010), so they are not considered in the following simulations. In addition, we will show that the BH properties can already be extracted by finding and timing a relatively slow pulsar. If a MSP were to be found after all, the experiment would become correspondingly even easier.

### 5.4.2 Simulation schemes

The simulations performed in this chapter mainly contains two steps: creating TOAs and determining the parameters as well as their measurement uncertainties. Firstly, the TOAs are created regularly at the solar system barycentric time and
then combined with the three time delays described in Section 2.4.3 to compensate for the differences in signal arrival time due to the binary motion of the pulsar. The effects of periastron advance and orbital shrink discussed in Section 2.4.3 are also considered in the calculation of the delays. Furthermore, for the simulations of BH spin measurement we account for the secular changes of periastron and projected semi-major axis due to the Lense-Thirring effect to the second order, as discussed in Section 5.2. The description of orbital motion achieved in Section 5.3 is implemented in the simulations of BH quadrupole measurement. Finally, each TOA is added with Gaussian fluctuation created with respect to the assumed measurement precision.

Next the TOAs are passed to the Tempo software package which then performs a least-square fit to determine the Keplerian and PK parameters (see Section 2.4.3 for definitions), and calculate the measurement uncertainty via the covariance matrix (as described in Section 2.4.5). For the spin simulations we use the timing formula described in Wex (1998) which models the secular change of periastron and projected semi-major axis to the second order. A new model
which approximates the perturbation of the quadrupole effect to the first order as shown in Eq. (5.72), is developed for the quadrupole simulations and used to determine the $q$ parameter.

For simulations of NP-SBH systems we assume weekly observation sessions each of which contains 10 TOAs with 100 $\mu$s precision. Weekly observation of four hours with three telescopes (Parkes, FAST and SKA) is applied to the cases of MSP-SBH systems. Each individual observation is then divided up to have 10 TOAs, ensuring a sufficient coverage of orbital phase to better determine the orbital parameters. For simulations of NP-Sgr A* weekly single TOA of 100 $\mu$s precision (achievable with the SKA) is assumed. Except for when mentioned, the uncertainties of measurements calculated in the simulations of this chapter is based on 1-$\sigma$ confidence.

5.5 Simulations for pulsar–stellar mass black hole system

In this section, by following the scheme described in Section 5.4.2 we perform simulations for PSR-SBH systems to study the potential of estimating the BH properties and testing GR’s cosmic censorship conjecture and no-hair theorem through pulsar timing. First of all, a brief guideline of parameter measurements is presented as below.

Generally, for a relativistic binary system, after determining the Keplerian parameters one would first be able to measure the advance of periastron $\dot{\omega}$ as the observing baseline extends. For PSR-SBH systems this contains the contribution from both the monopole ($\dot{\omega}_{\text{pm}}$) and the FD ($\dot{\omega}_{\text{s}}$) effect, and the former is usually 2-3 order of magnitudes larger (as will be shown in Section 5.5.2). Next the PK parameters of $\gamma$, $\dot{P}_b$, $\sin i$, $M_2$, $\dot{x}_s$ would become measurable in a system dependent order as will be shown in Section 5.5.1. Two measurements (not including $\dot{\omega}$ and $\dot{x}_s$) of them can lead to the determination of the masses through
the technique described in Section 1.7.1. Given a long enough timing baseline, the
second derivatives $\ddot{\omega}_s$ and $\dddot{x}_s$ will become measurable. During the same time, the
precision of mass measurements would lead to a precise determination of $\dot{\omega}_p$ and
then a subtraction of $\ddot{\omega}_s$ from the observed $\dot{\omega}$. The combination of the measured
$\dot{x}_s$, $\ddot{x}_s$ and $\dddot{x}_s$ would then provide a determination of the BH spin and system
geometry as shown in Section 5.2.3.1. Finally, one would be able to measure
the BH quadrupole moment when the signal becomes significant in the timing
residuals.

In the following the simulations of BH mass, spin and quadrupole measure-
ments are presented in separate subsections. In each subsection, two sets of
investigations are performed regarding NP-SBH and MSP-SBH systems, respec-
tively.

5.5.1 Mass measurement

Typically, in a binary pulsar system, mass determination of the two objects re-
quires the measurement of two PK parameters (as shown in Section 1.7). In most
cases, the best measured one will be the periastron advance. However, as discuss
in Section 5.2.3, if the companion is a fast rotating BH, the observed $\dot{\omega}$ contains
an extra contribution from the spin field and cannot be used for a precise mass
determination. The next measurable PK parameters would be $\gamma$, $\dot{P}_b$, $\sin i$, and
$M_2$. In Fig. 5.6, by assuming 5 years of observations with the SKA based on the
scheme outlined in Section 5.4.2, the fractional measurement error of the four
PK parameters are calculated. Two cases are considered regarding the formation
scenario discussed in Section 5.1.4: MSP-SBH with medium eccentricity $e = 0.4$
and NP-SBH with high $e = 0.9$. Both plots indicate that the measurement pre-
cision of the Shapiro delay is less orbital size dependent than the Einstein delay
and the orbital decay. In the former case a high precision determination of the
masses can be obtained with $\dot{P}_b$ and $\gamma$ measurements for very compact orbits
($P_b < 0.5$ days), and with the Shapiro delay parameter measurements instead
for wider orbits ($P_b > 1\text{ day}$). In the NP-SBH case where the TOA precision is much worse, mass measurements would still be highly precise for orbits with $P_b < 1\text{ day}$, but less likely when the orbital period is larger than a few days.

### 5.5.2 Spin measurement

#### 5.5.2.1 Normal pulsar–stellar mass black hole systems

As indicated in Wex & Kopeikin (1999), spin measurement of the BH via timing a binary NP is likely to be difficult. In Fig. 5.7, by assuming a 10-year observation and highly eccentric orbit, the measurability of $\dot{x}_s$ and $\chi$ is investigated within the $P_b$ range of 0.4 to 1.0 days, for different masses and spins. The results suggest that for $P_b < 1\text{ day}$ the signal of FD ($\dot{x}_s$) can be seen for a given wide range of BH masses and spins, and is measurable for $P_b$ up to several days when the BH is massive and fast rotating. The full spin determination (magnitude and orientation) is possible for a wide range of $m_{BH}$ and $\chi$ for fairly short orbits ($P_b < 0.5\text{ days}$), and for wide orbits ($P_b \sim 1\text{ days}$) only if the BH is massive and fast rotating.

#### 5.5.2.2 Millisecond pulsar–stellar mass black hole systems

MSPs are usually high precision timers which can dramatically increase the measurability of the binary BH spin. Also note that the timing precision is mainly limited by instrumental gain, the increase in collecting area of future telescopes would further improve the measurements. These systems greatly favour the possibility of a no-hair theorem test which requires extreme precision of pulsar timing.

In the simulation presented in Table 5.5, investigations for a wide range of parameter space are carried out. Based on the constraint of SBH from observations (e.g. Paredes, 2009), we choose 5 and $30\ M_\odot$ as the minimum and maximum. As the latest observations have already shown hints of the existence of extreme rotating SBH (e.g. Mandel & O’Shaughnessy, 2009), a $\chi = 1$ Kerr BH is assumed to
Fig. 5.6: Fractional errors of four PK parameters as a function of \( P_b \) with 5 years of timing observation. The timing precision of the MSP is assumed to be of the SKA level (\( \sigma_{10\text{min}} = 0.02 \mu s \)). The top plot is for a MSP-SBH system with \( e=0.4, \ m_{\text{BH}}=10 \) and the bottom one for NP-SBH with \( e=0.9, \ m_{\text{BH}}=30 \). The other system parameters are assumed to be: \( m_{\text{PSR}} =1.4, \ i = \theta_S = \Phi_0 = \Psi_0 = 45^\circ \). The orbital period in the case of NP-SBH begins from 0.4 days corresponding to a comparably short but reasonable merging time of \( 10^6 \) years, up to 8 days (see Eq. (5.74)).
Fig. 5.7: Measurability of $\dot{x}_s$ (top) and spin (bottom) for a NP-SBH system by assuming a 10-year timing baseline. The other system parameters are assumed to be: $e = 0.9$, $m_{PSR} = 1.4 M_{\odot}$, $i = 45^\circ$, $\theta_S = \Phi_0 = \Psi_0 = 45^\circ$. In short orbits the spin determination is likely to be yielded for a wide parameter range of BH mass and spin. Only massive and rapidly rotating BHs favor the spin measurement when orbital period extends to one day.
5.5. SIMULATIONS FOR PSR-SBH SYSTEM

represent the optimistic case which is still very close to reality. We use a constant binary merger time $T_m = 10^8$ years, which is roughly the gravitational damping timescale of the double pulsar system. Following e.g. Weinberg (1972), it can be written by:

$$T_m = \frac{5}{256} T_\odot^{5/3} \left( \frac{P_b}{2\pi} \right)^{8/3} \frac{M^{1/3}}{m_p m_{BH}} g(e),$$

(5.74)

where $g(e)$ is an approximation to the consistent integration in Peters (1964) within 1% fractional accuracy for $e < 0.9$, in form of (Wex 2009, private communication)

$$g(e) = 1 - 3.6481 e^2 + 5.1237 e^4 - 3.5427 e^6 + 1.3124 e^8 - 0.2453 e^{10}.$$  (5.75)

The binary orbital inclination is assumed to be 45° and for a few cases we also apply 88° to test how an edge-one system may improve the measurement. Both $\theta_S = 20^\circ$ and 70° are used to investigate the influence of spin orientation. We use 45° for the initial precessional phase angles $\Phi$ and $\Psi$.

From the results in Table 5.5, firstly, it can be seen that when the merging time is considered as a constant, low-eccentricity and short-period orbits are better for the determination of spin. Secondly, for the same eccentricity, an increase in the BH mass which decreases the ratio $\dot{\Omega}_p / \dot{\Omega}_S^*$ also improves the measurement precision. Besides, the larger angle between the spin and orbital angular momentum results in a stronger spin-orbit coupling effect, not surprisingly yielding a better determination. Last but not least, a great advance in the spin measurement would be expected when the next generation of radio telescopes are ready to use. For all cases, just within five years of regular timing observations with the SKA would lead to a 1% level of accuracy for measurements of both $\chi$ and $\theta_S$, while in some cases they are still not yet measurable after 10 years of observations with Parkes. Considering $i = 88^\circ$, further simulations show that an extra precision improvement of 35% – 50% for $\chi$ and $\theta_S$ could be expected for $e = 0.1$ cases (not shown in the table). Nevertheless, this highly inclined situation is rather unlikely.
### Table 5.5: Prediction of BH spin measurability for MSP-BH systems of different parameter combinations, with otherwise identical merger time of 10^8 years.

<table>
<thead>
<tr>
<th>MV</th>
<th>(\theta_S)</th>
<th>(P_{\text{b}}) (day)</th>
<th>(\Omega^*)</th>
<th>(\Omega^*_{\text{pn}}) ((^{\circ}/\text{year}))</th>
<th>(\theta^*)</th>
<th>(\sigma_{5\text{yr}}) ((^{\circ}/\text{year}))</th>
<th>(\sigma_{10\text{yr}}) ((^{\circ}/\text{year}))</th>
<th>(\sigma_{20\text{yr}}) ((^{\circ}/\text{year}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>10.0 0.1</td>
<td>0.30</td>
<td>0.28</td>
<td>0.33</td>
<td>0.37</td>
<td>0.17</td>
<td>0.42</td>
<td>0.93</td>
</tr>
<tr>
<td>0.05</td>
<td>10.0 0.1</td>
<td>0.30</td>
<td>0.28</td>
<td>0.33</td>
<td>0.37</td>
<td>0.17</td>
<td>0.42</td>
<td>0.93</td>
</tr>
<tr>
<td>0.025</td>
<td>10.0 0.1</td>
<td>0.30</td>
<td>0.28</td>
<td>0.33</td>
<td>0.37</td>
<td>0.17</td>
<td>0.42</td>
<td>0.93</td>
</tr>
<tr>
<td>0.0125</td>
<td>10.0 0.1</td>
<td>0.30</td>
<td>0.28</td>
<td>0.33</td>
<td>0.37</td>
<td>0.17</td>
<td>0.42</td>
<td>0.93</td>
</tr>
</tbody>
</table>

The three \(\hat{\sigma}\) represent fractional errors of spin determination in 5, 10 and 20 years observation, individually. Any resulting uncertainty larger than 10% would be noted as not well measured (NM).
5.5. SIMULATIONS FOR PSR-SBH SYSTEM

Fig. 5.8: Measurability of $\dot{x}_s$ (top), $\ddot{x}_s$, $\dot{\omega}_s$, $\ddot{\omega}_s$ (middle) and $\chi$ (bottom) of a MSP-SBH system with medium eccentricity of 0.4, by assuming 10 years of timing observations with the SKA. Here we use $m_{PSR} = 1.4$, $i = 60^\circ$, $\theta_s = \Phi_0 = 45^\circ$, $\Psi_0 = 60^\circ$. For the middle plot we apply $m_{BH} = 10$ and $\chi = 1$. 
Furthermore, as mentioned in Section 5.1.3, previous simulations indicate that a MSP-BH system formed from dynamic capture would have a comparably wide and eccentric orbit (Faucher-Giguere & Loeb, 2010). Here, by assuming 10-yr of SKA timing observation, two groups of simulations are performed with regards to medium \((e = 0.4)\) and high \((e = 0.9)\) orbital eccentricities and the results are shown in Fig. 5.8 and Fig. 5.9, respectively.

In the medium eccentricity case, the Lense-Thirring effect can be seen for orbital periods up to 4 days if the BH is massive and rapidly rotating, and is still extractable for orbital periods less than 4 days if the BH is lighter and rotating less rapidly. The \(\dot{\omega}_s\), \(\ddot{\omega}_s\) and \(\dddot{x}_s\) measurements plot indicates that the uncertainty in spin determination is dominated by the accuracy of \(\dot{\omega}_s\) subtraction, and the best constrained value comes from \(\dot{P}_b\) and \(\gamma\) for very compact orbits. As the orbital size extends, \(\sin i\) would be used instead of \(\gamma\) and when \(P_b \gtrsim 1\) day, the measurement error of \(\dddot{x}_s\) will become significant in limiting the precision of the spin. For a few days’ orbit spin-orbit coupling precession would still be constrained by using the Shapiro delay parameters, while the second derivatives are not measurable. Accordingly, the spin can be measured for systems of \(m_{BH} = 10\) and orbital period less than 2 days, and 4 days if the BH is three times more massive.

In the high eccentricity case, the determination of the spin-orbit secular motions is similar, apart from that the orbital range allowing the measurement is larger. Also note here that when the orbital period is several days, the first sign of FD will be from the \(\dot{\omega}_s\) measurement instead of \(\dddot{x}_s\). Consequently, orbits of period up to 10 \(\sim 14\) days can still favor the spin measurement, depending on the mass and spin of the BH. Note that orbits with semi-major axis larger than 0.1 AU and high eccentricity are predicted in the simulation of Faucher-Giguere & Loeb (2010) concerning stellar capture in the dense region near the GC.
Fig. 5.9: The same plots as in Fig. 5.8, but for highly eccentric orbits ($e = 0.9$).
5.5.3 Quadrupole measurement

As described in Section 5.3, the quadrupole field of a BH will induce perturbation to the orbit of its companion pulsar which can be described by slight and periodic variations of the pulsar distance to the barycentre, the orbital ascending node longitude, the periastron longitude and the orbital inclination (all regarding the X-Y plane in Fig. 5.3). The effect would be firstly subtracted by modelling the Römer delay as shown in Section 5.3.3. For PSR-SBH system, due to the weak quadrupole field the delay is of order of a nano-second for extremely compact systems (Wex & Kopeikin, 1999). In Fig. 5.10, assuming an extreme (probably unrealistic) system parameter combination we show the timing residuals achieved by subtracting a timing model which does not model the effect of quadrupole moment. In the pre-fit plot the other timing parameters were not fitted for, while in the post-fit one they were also fitted in order to represent the actual situation of timing analysis. It can be seen that the periodic feature can be greatly absorbed into the other timing parameters (e.g. $e$, $P_b$, $x$), and the remaining signal will be significant only near periastron. In this extreme case, the scale of the feature in the post-fit residual would be only about 10 ns, which indicates that modelling the effect requires extremely high timing precision. However, on a timescale of years due to the strong precession of the orbit the feature would be correspondingly evolving, which would benefit the detection of the signal (see Section 5.3.2).

To investigate the measurability of the BH quadrupole by timing the binary pulsar, we assume 20 years of SKA observations with the scheme presented in Section 5.4.1. Note that previous analysis shows that the timing precision of a NP will not allow this measurement, we will only consider the case of a MSP. Apart from the spin inclination angle $\theta_S$, the angles are assumed to be the same as in Fig. 5.10 for the following simulations of this subsection.

The mass of the SBH is usually found to be within the range of $5 \sim 30 M_\odot$ (Ziółkowski, 2008; Silverman & Filippenko, 2008). Meanwhile, latest studies noticed that stars with very low metallicity can form a more massive SBH from
5.5. SIMULATIONS FOR PSR-SBH SYSTEM

Fig. 5.10: Simulated pre-fit (solid line) and post-fit (dash line) timing residuals of a pulsar with a SBH companion. Here we use: $P_b = 0.2$ days, $e = 0.9$, $\omega_0 = 30^\circ$, $\lambda = 60^\circ$, $\Omega = \theta_S = 45^\circ$, $m_{\text{PSR}} = 1.4 M_\odot$, $m_{\text{BH}} = 80$, and $\chi = 1$. Note that the lifetime of this type of system is only 0.1 Myr. The pre-fit feature shows the actual deviation of the timing from expectation when the quadrupole is not considered in the timing model. The post-fit one is obtained by fitting for the other orbital parameters but the quadrupole. The difference indicates that the quadrupole signal can be greatly absorbed into the estimation of other parameters.
Fig. 5.11: Measurability of the quadrupole as function of orbital period (changing merging time) for different spin inclination angles in medium eccentricity case. Here we use: $e = 0.5$, $m_{\text{PSR}} = 1.4 M_\odot$, $m_{\text{BH}} = 30$ and $\chi = 1$. A determination of $q$ seems to be achievable only for very short orbit with certain geometries.

direct collapse (Belczynski et al., 2010). So here we also extend the parameter space of $m_{\text{BH}}$ up to 80. This type of massive SBH is more likely to be found in globular clusters, very metal-poor environments with high stellar density where frequent dynamic captures and 3-body interactions may allow formation of a MSP-SBH system.

In Fig. 5.11 we assume a system with medium orbital eccentricity $e = 0.5$, of a SBH with $m_{\text{BH}} = 30$ and $\chi = 1$. The measurement errors were estimated as function of $P_b$ corresponding to merging time from 1 to 100 Myr. Three spin inclination angle $\theta_s$ were also tried to demonstrate cases of different geometric configurations. The results indicate that until the system is extremely compact and with a favorable geometry, the measurement of the quadrupole is unlikely to be achieved.

Fig. 5.10 implies that in highly eccentric orbits the quadrupole effect will result in sharp features in the timing residual across the orbital phase near periastron. The existence of such features would help to improve the precision of
Fig. 5.12: Measurability of quadrupole as function of orbital period (change eccentricity from 0.1 to 0.9) for 10 Myr lifetime orbits with different spin inclination angles $\theta_s$. Here we use merging time of 10 Myr and vary the eccentricity from 0.1 to 0.9. The other system parameters are the same as in Fig. 5.11. The measurement precision significantly benefits from the increase of eccentricity as indicated by Fig. 5.10.

The quadrupole measurement, as shown in Fig. 5.12. Here we assume the same system parameters as in Fig. 5.11, but use a merging time of 10 Myr and vary the eccentricity (increases with $P_b$) from 0.1 up to 0.9. The measurability appears to dramatically benefit from the increase of orbital eccentricity and a wide range of spin inclinations would allow the quadrupole measurement for orbits of $e > 0.8$.

The quadrupole field is proportional to $m_{BH}^3$ (see Eq. (5.59)), indicating that the measurement would be more precise for pulsars accompanied by high-mass SBHs. In Fig. 5.13 we keep the same pulsar mass and BH spin as in Fig. 5.11, and choose $e = 0.5$ and explore the range of BH mass up to $80 M_\odot$, the upper bound given by the current formation studies (e.g. Belczynski et al., 2010). The results show that for $m_{BH} > 70$ the measurement is likely for most values of $\theta_s$. 
Fig. 5.13: Measurability of the quadrupole moment as function of BH mass for a 10 Myr lifetime and a medium eccentricity ($e = 0.5$) orbits, with different spin inclination angles, $\theta_s$. Note that the corresponding range of $P_b$ is from 0.16 days (left) to 0.21 days (right). The other system parameters are the same as in Fig. 5.11. MSPs with BHs of above 70 $M_\odot$ favour the quadrupole measurement for a wide range of spin inclination.

5.6 Simulations for pulsar–Sgr A* system

In this section we perform simulations by following the scheme described in Section 5.4.2, to investigate the potential of probing the properties of Sgr A* and performing GR test by timing a pulsar orbiting around it. The pipeline of parameter measurements is the same as mentioned at the beginning of Section 5.5 apart from two points. Firstly, here we only need one PK parameter to determine the mass of Sgr A* (as will be shown in Section 5.6.1). Consequently, the measurement precision of $\omega_s$ is usually much better than the second derivatives ($\ddot{\omega}_s$ and $\ddot{x}_s$). Secondly, note that in this case seen from Eq. (5.51), $\ddot{\omega}_s$ does not have the same geometric dependence as $\ddot{x}_s$, it can also be used for BH spin determination. In the following we only consider the possibility of having a NP, and the simulations of BH mass, spin and quadrupole measurements are provided in separate subsections. A value of $4 \times 10^6 M_\odot$ is used for the mass of Sgr A*.

Concerning stars moving around Sgr A*, Merritt et al. (2010) and Sadeghian
& Will (2011) have already pointed out that for orbits with an orbital period $P_b$ larger than 0.1 yr, it becomes likely that the distribution of stars in the vicinity causes “external” perturbations of the orbital motion of the stars and prevent a clean test of the no-hair theorem or even a measurement of the Lense-Thirring effect via astrometric approaches. In order to evaluate the significance of the perturbation, following the analysis of Merritt et al. (2010) in Fig. 5.14 the relation of precessional timescale against orbit size is presented for four different contributions: the periastron advance, the FD, the black-hole quadrupole, and a surrounding mass distribution. Here we assume $10^3$ (the highest number applied in Merritt et al., 2010) one solar mass objects isotropically distributed within 1 milli-parsec around Sgr A*, and as well do not consider the influence of objects outside the central 1 milli-parsec region. Clearly, for wide orbits where the Lense-Thirring effect is weak, the relativistic precession is still significantly larger than that by the external perturbation up to an orbital period of about 10 yr, which indicates that the measured $\dot{\omega}$ can be used to well constrain the black hole mass. The FD will be dominant over the stellar noise once the orbital period is less than 0.5 yr, while only for orbital periods $\lesssim 0.1$ yr the contribution of the quadrupole moment is expected to be significantly above the external perturbation. However, it should be noted that by frequently measuring the TOAs across timescales of years, the periodic signal of the quadrupole effect can be well tracked, which is mainly used to determine the quadrupole moment, instead of studying the precession rate, which will become clear in Section 5.6.3. Therefore, if periodic features by stellar perturbation are not strongly correlated with the quadrupole effect, the no-hair theorem test can still be achieved with pulsars in a wider orbit.

The assumptions applied to calculate the precessional timescale by stellar perturbation may not be secure as the actual stellar components and distributions within the central pc (especially the central milli-parsec) are still not fully investigated provided with the current published optical contributions. In reality, the precessional torque can be larger if there exists a high fraction of massive objects...
Fig. 5.14: Precessional timescales of an orbit about Sgr A*.

near Sgr A* due to mass segregation process (O’Leary et al., 2009; Keshet et al., 2009; Kocsis & Tremaine, 2011), or a significant anisotropy in the distribution of the surrounding masses. Nevertheless, in reality the orbital precession by Lense-Thirring effect is described by a unique function of true anomaly plus a periodic fluctuation which does not contribute to the long-term secular changes of the angles (Wex, 1995). As will be shown in Section 5.6.3, the quadrupolar field will also result in unique periodic features in the timing residuals. Those features can be tracked well with frequent measurements of TOAs on timescales of years. Consequently, under the circumstance that the perturbing precessional rate is not significantly higher than those due to the spin and quadrupole, and the periodic perturbations from these effects are not strongly correlated\(^4\), one may still be able to separate the signals from these three effects and achieve the measurements of Sgr A* spin and quadrupole moment. This might even allow the measurements with a pulsar in a wider orbit. A full investigation in this direction requires a

\[^4\]A strong correlation is thought to be highly unlikely due to the unique orbital motions induced by BH spin and quadrupole field.
5.6. SIMULATIONS FOR PSR-SGR A* SYSTEM

detailed model of the stellar distribution as an input that would go far beyond the scope of the work in this chapter.

5.6.1 Mass measurement

For a pulsar orbiting Sgr A*, the determination of the pulsar mass from timing would be difficult due to the great mass difference between the two objects. However, as the pulsar can be sufficiently approximated by a test particle, the mass determination of Sgr A* will only require the measurement of one PK parameter. Under this circumstance, the $m_{BH}$ can be expressed by

$$m_{BH} \approx \frac{1}{T_\odot} \left( \frac{\gamma}{2e} \right)^{3/2} \left( \frac{P_b}{2\pi} \right)^{-1/2},$$  \hspace{1cm} (5.76)

$$\approx \frac{1}{T_\odot} \left( \frac{P_b}{2\pi} \right)^{-2} \left( \frac{x}{\sin i} \right)^3. \hspace{1cm} (5.77)$$

In Fig. 5.15 the fractional error of three PKs which can be used for BH mass determination in this case are calculated regarding different orbital size. Five years of observations with the SKA of a NP is assumed based on the scheme described in Section 5.4.1. It is clearly shown that the Shapiro delay parameters are better constrained within the given orbital size range. The measurement precision of the Einstein delay appears a strong dependence on the orbital period, and would be useful only for close orbits to have $P_b < 0.2$ yr.

5.6.2 Spin measurement

Noting from Eq. (5.15), the orbital precessional rate due to the Lense-Thirring effect scales linearly with $m_{BH}$ when $m_{BH} \gg m_{PSR}$. Consequently, this effect is $\sim 10^5$ times stronger for Sgr A* than SBHs, and one would be able to extract the BH spin from it even by timing a NP (a relatively poor timer) orbiting Sgr A* in a wide orbit.

In Table 5.6, assuming 5 years of SKA timing observations following the scheme mentioned in Section 5.4.1, we carry out a set of simulations and calculate
Fig. 5.15: Measurability of three PKs for given orbital period for a NP-Sgr A* system, with 5-year observation. The other system parameters are assumed to be: $e = 0.5$, $m_{\text{PSR}} = 1.4$, $m_{\text{BH}} = 4 \times 10^6$, $i = 60^\circ$, $\theta_S = 80^\circ$, $\Phi_0 = 50^\circ$, $\Psi_0 = 45^\circ$.

Table 5.6: Prediction of Sgr A* spin measurability with 5-yr SKA timing observation, for different system parameter combinations. Here we use $m_{\text{PSR}} = 1.4 M_\odot$, $\Phi_0 = 45^\circ$, $\Psi_0 = 45^\circ$, and $\lambda = 60^\circ$. The $\sigma_2$ denotes 2-$\sigma$ standard fractional error and inside the bracket we note out the parameter ($\dot{\omega}_s$ or $\ddot{x}_s$) used to yield the better precision following the calculations in Appendix D. Any resulting uncertainty larger than 10% would be noted as not well measured (NM).

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$e$</th>
<th>$P_b$ (yr)</th>
<th>$\theta S$ (°)</th>
<th>$\sigma_2$</th>
</tr>
</thead>
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<tr>
<td>0.2</td>
<td>0.1</td>
<td>0.3</td>
<td>$20^\circ$</td>
<td>4.7% ($\ddot{x}_s$)</td>
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<td></td>
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<td>70° NM</td>
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<tr>
<td>0.1</td>
<td>0.3</td>
<td>$20^\circ$</td>
<td>0.36% ($\ddot{x}_s$)</td>
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<td></td>
<td></td>
<td></td>
<td>0.92% ($\ddot{\omega}_s$)</td>
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<tr>
<td>0.9</td>
<td>0.3</td>
<td>$20^\circ$</td>
<td>0.07% ($\ddot{x}_s$)</td>
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<td></td>
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<td></td>
<td>0.18% ($\ddot{x}_s$)</td>
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<tr>
<td>0.1</td>
<td>0.3</td>
<td>$20^\circ$</td>
<td>&lt;0.01% ($\ddot{x}_s$)</td>
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<td></td>
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<td></td>
<td>&lt;0.01% ($\ddot{\omega}_s$)</td>
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</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>0.3</td>
<td>$20^\circ$</td>
<td>0.14% ($\ddot{x}_s$)</td>
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<tr>
<td></td>
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<td></td>
<td>70° 1.1% ($\ddot{x}_s$)</td>
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<tr>
<td>0.1</td>
<td>0.3</td>
<td>$20^\circ$</td>
<td>0.01% ($\ddot{x}_s$)</td>
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<td></td>
<td>0.04% ($\ddot{\omega}_s$)</td>
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<tr>
<td>0.9</td>
<td>0.3</td>
<td>$20^\circ$</td>
<td>&lt;0.01% ($\ddot{x}_s$)</td>
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<td>70° 0.01% ($\ddot{x}_s$)</td>
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<td>0.1</td>
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<td></td>
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<td></td>
<td>70° &lt;0.01% ($\ddot{\omega}_s$)</td>
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</tr>
</tbody>
</table>
the measurement precision of spin for different system parameter combinations. Here \( \chi = 0.2, 1 \) are chosen to cover the entire range of the current Sgr A* spin measurements found in the literature mentioned Section 5.1.2, and \( P_b = 0.1 \) and 0.3 yr are applied to demonstrate different orbital sizes. Eccentricities of \( e = 0.1 \) and 0.9 are both used as the orbital character of the S2 star has already indicated that stars can be found to move around the Sgr A* in a highly eccentric orbit. Besides, we select \( \theta_S = 20^\circ \) and \( 70^\circ \) to represent different geometric configurations. The result shows that apart from systems of a slowly rotating BH and a wide orbit \( (P_b \gtrsim 0.3 \text{yr}) \), after five years of SKA timing the spin measurement can be constrained with better than 1% precision. Pulsars in a highly eccentric orbit are likely to yield a precision better than 0.1%.

To demonstrate the spin determination technique discussed in Section 5.2.3.2, we performed further simulations based on different system geometries. Figs. 5.16 and 5.17 show the \( \chi_\theta-\chi_\lambda \) plane for two different orientations of the black hole and the pulsar orbit. According to GR the solution has to lie within the boundaries of the figures, since \( -1 \leq \chi_\theta, \chi_\lambda \leq 1 \) for a Kerr black hole. Moreover, the solution \( (\chi_\theta, \chi_\lambda) \) has to lie within an ellipse defined by setting \( \chi = 1 \) in Eq. (5.56), in order to be consistent with a Kerr black hole with an event horizon. Once \( \dot{x} \) is measured, one can determine \( \zeta_3 \) from Eq. (5.52) and use this quantity to plot the ellipse defined by setting \( \chi = 1 \) in Eq. (5.56) in the \( \chi_\theta-\chi_\lambda \) plane.

Fig. 5.18 shows a simulation for a Kerr solution with a spin that exceeds the spin of an extreme Kerr black hole. Within general relativity, this would represent a naked singularity. For such an object the cosmic censorship is violated and the predictability of the (classical) theory brakes down.

As discussed in great detail in Merritt et al. (2010), the orbit of a star or pulsar around Sgr A* might be subject to perturbations from other stars in the vicinity of the black hole. Depending on the number density of the stars, this could lead to significant contributions to the precession of the pulsar orbit. As it turns out, the demonstration of measurement presented here will, in general,
Fig. 5.16: Determination of the Sgr A* orientation in the $\chi_\theta$-$\chi_\lambda$ plane. We have used an orbital period of 0.3 yr, an orbital eccentricity of 0.5, $\chi = 1$, and $\Psi_0 = 45^\circ$, $\Phi_0 = 45^\circ$, $\theta = 60^\circ$, and $\lambda = 60^\circ$ for this simulation. A change in the sign of $c_i$ mirrors the figure along the $\chi_\lambda = 0$ line, meaning that the solution for $\theta$ is invariant, but $\lambda$ changes to $\pi - \lambda$. The corresponding spin parameter, as calculated from Eq. (5.56), is $\chi = 0.9997 \pm 0.0010$ (95% C.L.). In all the $\chi_\theta$–$\chi_\lambda$ plots (Fig. 5.16 – 5.19) we plot the “±-one-sigma” lines. However, in most cases the separation between the two lines is below the resolution of the plot. The dotted ellipse is the boundary of the area for Kerr black holes (see text for details).

unveil the presence of any external perturbations, since we have three lines in the $\chi_\theta$–$\chi_\lambda$ plane that need to intersect, provided GR is the correct theory of gravity and Sgr A* is indeed a black hole. In Fig. 5.19 we present a $\chi_\theta$–$\chi_\lambda$ diagram, which is based on timing data that contains, besides the gravitational field of Sgr A*, an external perturbation that causes an additional precession of the periastron. For orbits with $P_b \lesssim 0.3$ yr, even a small (compared to the Lense-Thirring precession) external contribution to the precession of the periastron leads to a situation where the $\dot{\omega}$, $\ddot{\omega}$, and $\dddot{\omega}$ lines fail to intersect in one point within the measurement precision. The same is true, if there is an external contribution to
Fig. 5.17: Like Fig. 5.16, but $\Phi_0 = 105^\circ$ and $\theta = 30^\circ$, and $\lambda = 75^\circ$. The corresponding spin parameter, as calculated from Eq. (5.56), is $\chi = 1.0001 \pm 0.0003$ (95% C.L.).

a change in the inclination of the orbital plane. Hence, if all three lines intersect, we not only have a precise determination of the spin of the black hole, but also a test that this measurement is not contaminated by external perturbations.

### 5.6.3 Quadrupole measurement

Noting the strong dependency of the quadrupole field upon mass from Eq. (5.59), the effect of a super-massive BH will be significantly greater than that of a SBH. Fig. 5.20 illustrates the unique periodic residuals caused by the quadrupole of Sgr A*. The amplitude of the pre-fit feature is about a few milliseconds, more than five orders of magnitude larger than the effect of a SBH shown in Fig. 5.10, even when the orbital period is three orders of magnitude longer. This periodic signal will not only allow the determination of the quadrupole moment of Sgr A* with high precision, its characteristic feature also provides a clear identification of the
Fig. 5.18: Parameters as in Fig. 5.16, but $\chi = 1.2$ (naked Kerr singularity). The dotted ellipse is the (outer) border of the region where, for the measured orbital inclination and $\dot{x}$, the Kerr black holes are located, i.e. where $\chi \leq 1$.

Quadrupolar nature of the gravitational field. Moreover, due to the large advance of periastron the quadrupolar signal will also change in a characteristic way from one orbit to the next. This clearly helps to discover any external “contamination” of the orbital motion of the pulsar. This, as in the spin determination, provides high confidence in the reliability of a no-hair theorem test with a pulsar around Sgr A*.

We have carried out a number of simulations, for various orbital configurations, based on the presumed observational scheme mentioned in Section 5.4.1. The results are collected in Fig. 5.21. It should be mentioned, that the precision of the spin determination is at least one order of magnitude better than the determination of $q$. Hence the uncertainty in the $q$-measurement is the limiting factor for the no-hair theorem test. As a conclusion of our simulations, if the stellar perturbations are negligible, the no hair theorem can be tested with high precision, for orbits with an orbital period of less than 0.5 years and low
Fig. 5.19: Parameters as in Fig. 5.16, but the precession of milli-parsec has an additional contribution from an external perturbation that amounts to 10% of the Lense-Thirring contribution. For a better resolution only the first quadrant of Fig. 5.16 is plotted here.

eccentricities. In case of agreement, this would provide strong evidence that the spacetime of Sgr A* is indeed Kerr-like. If we adopt the precessional rate from stellar perturbation calculated in Fig. 5.14, the test can be achieved with high precision for orbits with $P_b \lesssim 0.1$ yr. This range can be extended if in practice the quadrupolar feature is separable from stellar perturbations.

5.7 Conclusions and discussions

5.7.1 Summary

In this Chapter, we have investigated the possibility of determining the mass, the spin, and the quadrupole moment of a BH by timing its binary pulsar. These measurements can be converted to a test of GR’s cosmic censorship conjecture
Fig. 5.20: Residuals caused by the quadrupole moment of Sgr A* plotted for two orbital periods. We have used the same orbital and black hole parameters as in Fig. 5.16.

Fig. 5.21: Measurement precision for the quadrupole moment of Sgr A* as a function of orbital period for three different eccentricities in absence of any external perturbations. We have used the same orbital and black hole parameters as in Fig. 5.16. This time however the TOAs were equally distributed with respect to the true anomaly, by this accounting for the fact, that timing needs to be done more frequently around periastron, in order to optimize the measurement of the quadrupolar signal in the TOAs.
5.7. CONCLUSIONS AND DISCUSSIONS

and no-hair theorem. The orbital secular motions induced by the FD effect were studied in detail, and used to show how to determine the BH spin and the system geometry. For the orbital motion involving quadrupole effect, we modified the solution to orbital perturbation by a quadrupole field from Garfinkel (1959) to account for relativistic precession, and derived a timing model to measure the BH quadrupole from the perturbation. Then based on the timing precision achievable by both the current and the next generation of radio telescopes, we simulate TOAs accounting for these effects and used the Tempo software package to perform a covariance analysis to estimate the measurability of the BH properties. Two types of systems, PSR-SBH binaries and a pulsar orbiting Sgr A* were concerned in the simulations and the results are summarised below, respectively.

5.7.1.1 Pulsar–stellar mass black hole

Both a NP and a MSP can be found with a SBH companion, and the possibility to form a NP-SBH system is believed to be higher than a MSP-SBH system (Sipior et al., 2004; Pfahl et al., 2005). Nevertheless, a MSP, if found orbiting a SBH, would make the probing of BH properties much easier through high precision timing. Below the results of simulations regarding these two types of system are summarised, respectively.

Timing a NP in a NP-SBH binary can lead to mass measurements of the two objects, for systems with $P_b$ up to several days if the binary was formed with high eccentricity as predicted. The sign of FD in this case would still be seen from $\dot{x}_s$ for orbit of similar range of $P_b$, depending on the mass and spin of the BH, and the measurement of the spin is possible for system with short orbit ($P_b < 1$ day) and fast spinning BH. Note that here the measurement precision would be mainly limited by timing noise of the pulsar, and may hardly be improved by the increase of system sensitivity.

A MSP, on the contrary, if found accompanied with a SBH, will make the probing of the BH properties much easier. Timing of a MSP-SBH with a lifetime
comparable to the double pulsar system ($\sim 10^8$ yr), would yield a high precision mass determination with both the current and the future telescopes. Given a 10-year observational baseline the spin can be well determined even with currently available sensitivity, apart from some unfavorable system parameter combinations. For wider orbits with the SKA timing, depending on BH properties the spin is extractable for orbits of $P_b$ up to 4 days in a medium eccentricity case, and up to $8 \sim 14$ days in a high eccentricity case. Measurement of the BH quadrupole would be possible only with future telescopes, and under the circumstance that either the system is highly eccentric or the companion BH was born from a low metallicity star and then more massive than usual (up to $\sim 80 M_\odot$).

5.7.1.2 Pulsar around Sgr A*

Timing a pulsar orbiting around Sgr A* requires the sensitivity of future telescopes (e.g., SKA) due to the strong scattering of the ISM. It was shown that a TOA precision of order 100 $\mu$s is achievable for observations of a NP with the SKA. Note that this precision may be yielded with the current instruments if the technique of interferometry is utilised (see Section 5.7.4 for details).

With 100 $\mu$s timing precision and a 5-year observational baseline, one can expect to be able to determine the mass, the spin, and the quadrupole moment of Sgr A* with high precision, provided the orbital period of the pulsar is well below one year. For a compact orbit (orbital period of a few months) measurement precision of the spin can be expected to be $10^{-3}$ or even better, which would provide a test of the cosmic censorship conjecture. It is also shown that, in general, our analysis will be able to unveil the presence of external perturbations caused by the presence of distributed mass, therefore providing high confidence in a spin determination based on pulsar timing. Moreover, determination of the quadrupole moment of Sgr A* is likely with a precision of a few percent or even better, depending on the size and orientation of the pulsar orbit, the spin of Sgr A*, and the intensity of external stellar perturbations. In combination with
the accurate spin measurement from the Lense-Thirring effect, this will yield a high precision test of the no-hair theorem of stationary black holes.

Finally, the tests presented are not affected by any uncertainty in the distance to the GC. On the contrary, a mass determination via pulsar timing would give a greatly improved value for \( R_0 \) if combined with the astrometric measurements in the near infrared.

### 5.7.2 Measurement pipeline

Once a pulsar is detected in an orbit with a BH, continuous timing observations will allow more and more measurements and tests as the data baseline grows with time. In the following we summarise the most important steps in this experiment:

- After the discovery of the pulsar, follow-up observations (within a few months) would soon determine the orbital Keplerian parameters. The first PK parameter one would obtain from the timing data, is the periastron advance which in the PSR-Sgr A* case will already provide a good estimate of the mass of Sgr A*.

- As the timing baseline extends to a few years, one should begin to determine additional PK parameters, including the amplitude of the Einstein delay (\( \gamma \)), the Shapiro parameters (\( M_2, \sin i \)), the orbital decay \( \dot{P}_b \) (only for PSR-SBH systems), and the change in the projected semi-major axis by the FD effect \( \dot{x}_s \). These parameters (apart from \( \dot{x}_s \)) allow a robust estimation of the BH mass and the inclination of the pulsar orbit with respect to the line of sight. The measured \( \dot{x}_s \) provides the first evidence of FD and allows a lower limit determination of the BH spin.

- Next the other Lense-Thirring effect parameters, \( \dot{\omega}_s, \ddot{\omega}_s \) and \( \dddot{x}_s \) will become measurable. For PSR-SBH systems note that the masses determination require usage of two PK parameters, the subtraction of \( \dot{\omega}_s \) from the observed \( \dot{\omega} \) usually comes later than the second derivatives. For the PSR-Sgr A* case,
as it requires measurement of only one PK parameter to determine the BH mass, one can usually subtract $\dot{\omega}_s$ before measuring the second derivatives. The determinations of $\dot{x}_s$, $\dot{\omega}_s$, $\ddot{x}_s$ and $\ddot{\omega}_s$ (not effective for PSR-SBH cases) allow a precise determination of the BH spin (magnitude and direction), a description of system geometry and a test of the cosmic censorship conjecture. For the PSR-Sgr A* case we also have a test for the “cleanness” of the system.

- Finally, after five to ten years, the obtained parameters of the masses, the BH spin and the orbit would be used to model the periodic features by BH quadrupole moment in the timing residuals. This leads to a determination of the quadrupole moment and a test of the no-hair theorem.

5.7.3 Other effects

Practically, there are other effects that can influence the measured properties used for the GR tests. If they have been shown to be significant in a system, one has to subtract the contributions from these effects in the timing model parameters so as to have unbiased measurements of the BH properties.

There are mainly four alternative effects involved in the secular change of the orbital projected semi-major axis: shrinking of the orbit due to gravitational radiation, proper motion of mass centre (Kopeikin, 1994, 1996), geodetic precession of the pulsar’s spin (Damour & Taylor, 1992), and the relative acceleration between the binary and the earth (Shklovskii effect, see Shklovskii, 1970). For PSR-SBH binaries, assuming a 1.4 $M_\odot$ pulsar, one can obtain the comparison of the gravitational wave damping contribution with the spin-orbit effect as:

$$\left| \frac{\dot{x}_s}{\dot{x}_{gw}} \right| \simeq 2.1 \times 10^5 \frac{\chi \sin \theta \sin \Phi \cot i}{f(e)} \left( \frac{P_b}{\text{1 day}} \right)^{2/3} \left( \frac{M}{M_\odot} \right)^{1/3},$$

where

$$f(e) = \frac{1 + (73/24)e^2 + (37/96)e^4}{(1 - e^2)^2}.$$
For a $10 M_\odot$ BH, a $1.4 M_\odot$ pulsar, a $0.1$ day orbital period, a high eccentricity $e = 0.8$, $45^\circ$ to the angles and extreme Kerr BH ($\chi = 1$), the ratio appears to be $\sim 3 \times 10^3$, which suggests normally the spin-orbit coupling dominates the secular change of $x$. The influence of proper motion of the mass centre could be significant if the binary is close to the solar system, but can be either measured by other astrometric approaches or properly modelled in pulsar timing (see Section 2.4.2). The contribution of the Galactic gravitational field to $\dot{x}$ will not be important either unless the binary system is within the central region (Wex & Kopeikin, 1999), so it may only be necessary to be considered when timing a pulsar moving around the GC. The contribution by the pulsar’s geodetic precession, following Damour & Taylor (1992) and Wex & Kopeikin (1999), can be compared with $\dot{x}_s$ by

$$\left| \frac{\dot{x}_{\text{geo}}}{\dot{x}_s} \right| \sim 0.006 \left( \frac{P}{1 \text{ s}} \right) \left( \frac{P_b}{1 \text{ day}} \right)^{-2/3} \left( \frac{m_{\text{BH}}}{10} \right)^{-1/3},$$

(5.80)

where $P$ is the pulsar rotational period. For PSR-SBH systems of $P = 0.5$ s, $P_b = 0.1$ days, $m_{\text{BH}} = 10$, we have $|\dot{x}_{\text{geo}}/\dot{x}_s| \sim 0.01$, which indicates that this effect needs to be taken into account only for a slow pulsar in a very compact orbit. For the case of PSR-Sgr A*, assuming the same $P$ and $P_b = 0.1$ yr, the ratio turns out to be $\sim 2 \times 10^{-6}$, which means the influence is negligible.

A potential problem that may also be caused by pulsar spin precession is the resulting variation in the pulse profile (e.g. Kramer, 1998; Manchester et al., 2010), which complicates the high precision timing. For the worst case, it might even turn the pulsar emission away from our line of sight. To leading order the spin precession is given by the de Sitter precession rate, which for $m_{\text{BH}} \gg m_{\text{PSR}}$ reads (Barker & O’Connell, 1975)

$$\Omega_{dS} \approx \frac{(0.5 \text{ deg/yr})}{1 - e^2} \left( \frac{P_b}{1 \text{ day}} \right)^{-5/3} \left( \frac{m_{\text{BH}}}{10} \right)^{2/3},$$

(5.81)

Consequently, for PSR-SBH systems with $P_b = 0.1$ days, $m_{\text{BH}} = 10$, and $e = 0.1$ the precession rate is roughly $22 \text{ deg/yr}$, while for PSR-Sgr A* case where $P_b = 0.1 \text{ yr}$ and $e = 0.5$ we have $\Omega_{dS} \approx 8 \text{ deg/yr}$. Fortunately, the spin geometry of
the pulsar can also be studied from polarimetric information (e.g. Kramer, 1998), which may provide a method to properly model the profile evolution and still enable high precision timing.

The no-hair theorem test in PSR-Sgr A* cases can also be affected by the accretion disc around the BH. In principle, one can estimate the influence by calculating the fraction of the quadrupolar potential of the disc to that of the BH, which turns out to be

$$\mathcal{R} \sim \frac{M_{\text{disc}}}{M_{\text{BH}}} \left( \frac{r_{\text{disc}}}{r_g} \right)^2,$$

where $M_{\text{disc}}$ and $r_{\text{disc}}$ are the mass and outer radius of the disc, and $r_g \equiv GM_{\text{BH}}/c^2$ is the BH gravitational radius. Following the advection-dominated accretion flow model in Yuan et al. (2009) and adopting, as an upper limit, the disc scale of $\approx 1$ arcsec determined from X-ray observations (Baganoff et al., 2001), we obtain $\mathcal{R} \approx 0.4\%$, which indicates that the quadrupole moment measurement of Sgr A* would not be biased by the contribution of the disc above 1% precision level.

5.7.4 Discussions

Once a PSR-BH system is found, a new timing model may be required to describe the orbital motion in the long term. The currently used second order secular change of $\omega$ and $x$ to approximate the orbital precession would not be sufficient, for the case that either the spin field is significantly strong, or the observational baseline is long enough. For illustration, one can compare the Römer delays calculated in two ways: one by assuming a fully precessional model (with $\dot{\Phi}$ and $\dot{\Psi}$) and the other by using an approximated model (with $\dot{\omega}, \ddot{\omega}, \dot{x}, \ddot{x}$). The result is shown in Fig. 5.22, where we use $m_{\text{PSR}} = 1.4 M_\odot$, $m_{\text{BH}} = 10$, $P_b=0.3$ days, $e = 0.1$ and $\chi = 0.5$. It is clear that the difference between the two models is significant compared with the presumed timing precision, and would grow up to micro-second level after several years observation. In addition, the signal of the higher order terms, if not properly modelled, can be absorbed into the fit
of the other orbital parameters due to the strong correlation between, and might induce bias in estimation. Nevertheless, further investigation shows that the predicted measurement precision of the key parameters ($\dot{\omega}, \ddot{\omega}, \dot{x}, \ddot{x}$) does not significantly vary once the higher order effects are taken into account in the simulation, which means that the current model is sufficient to be used to predict the measurement of spin.

When searching for pulsars in the GC region, observations with an interferometer rather than a single dish, would greatly increase the angular resolution, resolve out a significant amount of the ISM, and decrease the influence of the radio background. In Fig. 5.23, the resolutions by three planned baseline configurations of the SKA are plotted against observational frequency. The results are compared with the maximum angular separation of a star away from the Sgr A* with two orbital sizes, and the local scattering screen scales yielded by two models (Bower et al., 2006; Lu et al., 2011). Clearly, with only the core baseline ($D_{\text{max}} = 5\, \text{km}$) one would not be able to separate the target source with the central BH. The 500 km baseline would yield the separation for pulsars with
Fig. 5.23: Angular resolutions of the SKA against observational frequency, with three different designed baselines. Maximum angular separations of stars from Sgr A* with two orbital sizes and two modelled scales of local scattering screen in the GC region are also plotted for comparison.

close orbit (e.g., $P_b < 0.3$ yr) at high frequency ($f > 20$ GHz). The 3000 km baseline, will not only be able to image pulsars nearby the Sgr A* but also resolve into the scattering screen and dramatically recover the S/N from scattering. In this case, optimized timing frequency would be much lower than predicted. Note that timing precision discussed previously regarding the NP-Sgr A* case is limited by the pulsar’s spinning noise, this kind of technique would not be expected to greatly improve the no-hair theorem test with the SKA, but would be very helpful in timing observation with the gain level of current instruments (e.g. the Large European Array for Pulsar). Moreover, a high resolution search based on the longest baseline configuration would be essential in finding MSPs within the GC region.
Chapter 6

Conclusions and future work

The motivation of the work in this thesis is to investigate the limit on timing precision achievable with the next generation of radio telescopes and thereby to evaluate the prospects for future timing observations of a pulsar in particular. High precision pulsar timing can be used to probe the gravitational field of a nearby black hole (BH), which would lead to determinations of the mass, the spin, and even the quadrupole moment of the BH, and thereby a test of General Relativity’s (GR) cosmic censorship conjecture and no-hair theorem. Such investigations in the strong field regime, providing a way to answer the question “Did Einstein have the last word on gravitational theories”, form one of the key projects for the Square Kilometre Array (SKA). A brief summary of the results in previous sections will be given below, followed by a discussion of future efforts.

6.1 Summary

6.1.1 Limits on precision timing

A detailed review of the most important effects that can vary the shape of observed integrated pulse profiles on short timescales was performed in Chapter 3.
These effects include: intrinsic pulse shape changes called pulse phase jitter, instabilities caused by the interstellar medium (ISM) via dispersion measure variations and diffractive scintillation, and instrumental distortions from digitisation and polarisation calibration. Based on a case study of the brightest millisecond pulsar (MSP), PSR J0437−4715, it has been shown that most profile variations can be either corrected or taken into account for the estimation of TOAs. The limiting factor of precision timing with future telescopes on short timescales, will be both the jitter and radiometer noise. Based on 1.4 GHz, 100 MHz bandwidth observations of 10-min length, we predicted a TOA precision of between 80 and 230 ns for a typical brightness MSP observed with the SKA.

Chapter 4 provides a more detailed investigation of pulse jitter, from studying both the shape and the central phase of integrated profiles. Data on five MSPs were used for profile stability analysis, and no shape variations were detected on timescales of hours, based on 10~100 s integrations. PSR J0437−4715 was also used for the investigation of arrival phase fluctuation. Based on short-term timing, the jitter parameter was measured to be $f_J = 0.067 \pm 0.002$, after correcting for instrumental TOA uncertainties. We found no frequency or bandwidth dependency of the measurement within a frequency scale of $\sim 100$ MHz around 1.4 GHz.

### 6.1.2 Timing of a pulsar–black hole binary

The methods and potential for using pulsar timing to probe properties of a nearby BH and test GR, were discussed in Chapter 5, where two types of system were considered for this purpose: a compact binary consisting of a pulsar and a stellar-mass BH (SBH) and a pulsar orbiting around the super-massive BH, Sgr A*.

Given the most widely accepted binary evolutionary scenario, pulsars found orbiting a SBH are more likely to be non-recycled ones. In this case, we found that it is possible to measure the mass of the BH for wide orbit (e.g. orbital period of $P_b \approx 5$ days), but the BH spin measurement is only achievable for tight (e.g.
$P_b \lesssim 1$ day) and eccentric orbits where the BH is rapidly spinning. The formation of a MSP-SBH system is possible through the “reversal mechanism” during binary evolutions (Sipior et al., 2004) or 3-body interactions (Kulkarni & Frail, 1993; Faucher-Giguere & Loeb, 2010). Here the mass and spin measurements tend to be achievable with high precision for various system parameter combinations. Determination of the BH spin can be achieved for wide orbits of $P_b \lesssim 10$ days using ten years of timing observations of the precision allowed by the SKA. The BH quadrupole moment would be measurable for systems of either very high eccentricity, or of high-mass SBH (e.g. $\sim 80 M_\odot$) which might be formed in metally poor regions such as globular clusters.

Timing of a pulsar orbiting Sgr A* is likely to require high frequency observations (i.e. $f > 10$ GHz) due to the strong scattering of the local ISM. For the same reason only normal pulsars are likely to be discovered. Considering the perturbation of surrounding stellar masses, the mass, spin, and quadrupole moment of Sgr A* are measurable with high precision (e.g. $\lesssim 1\%$) when the pulsar is in an orbit shorter than approximately 1 yr, 0.5 yr and 0.1 yr, respectively. These measurements would lead to a test of GR’s no-hair theorem with 1% precision. Note that the spin and quadrupole field of Sgr A* will produce unique features in the timing residuals which can potentially be separated from the stellar perturbations, the measurements of them may be allowed even for wider orbits.

## 6.2 Further research

The next generation of radio telescopes promise to improve pulsar timing to a new level of precision, and to provide the best chance to test gravitational theories by use of a pulsar–black hole binary. In order to achieve this goal, several items of future work are proposed below.

- **Single pulse studies and correction for jitter:** Pulse jitter has been demonstrated to be one of the main limiting effects on TOA precision for
future telescopes. High signal-to-noise ratio observations of single pulses will be very helpful in understanding the behaviour of shape modulation and establishing corresponding models. Ideally, such models will be both implemented in the template matching model to avoid errors in TOA uncertainty estimate, and used to correct the phase fluctuations of integrated profiles (e.g. Cordes & Shannon, 2010; Oslowski et al., 2011). Even if a global model is not achievable, one can still aim to seek for a regulation to classify the single pulses based on shape stability, and time the group of pulses that exhibit the most stable integrated profiles. Such methods might greatly improve timing precision on short timescales.

- **Long-term monitoring of ISM delays:** Timing precision over long periods of time is likely to be limited by variations in the ISM (You et al., 2007) and the irregularity of pulsar’s rotation (Hobbs et al., 2010). While the spin noise is intrinsic to the object and not fully understood, work is already ongoing to try to mitigate the ISM influence on both profile shape and central phase (Walker et al., 2008; Hemberger & Stinebring, 2008; Coles et al., 2010; Demorest, 2011). Such correction techniques, after being tested properly, need to be implemented in future precision timing efforts.

- **Acceleration search with full sensitivity:** The efficiency of current acceleration searches is limited by the design of algorithms but, more intrinsically, by the significant amount of required computational power (Eatough, 2009). Theoretically, a full orbital parameter search over a wide range of values could be performed if the computing cost is manageable. A suitable design of algorithms would improve the efficiency in practical cases where this assumption is far from true. However, application of new Graphics Processing Units (GPU) and the global cooperation of computational power (e.g. Einstein@home) have already achieved a vast improvement in computing power for pulsar surveys (Barsdell et al., 2010; Knispel et al., 2010). A wider utilisation of such techniques would be one major direction
for future binary search efforts. Last but not least, acceleration searches with future telescopes will greatly improve the effectiveness of such surveys and increase the probability of finding a PSR-BH system.

- **Imaging searches for pulsars in the Galactic Centre:** The current pulsar survey approach towards the central parsec region is limited by both the system sensitivity and mostly, by the strong scattering of the high density ISM (e.g. Cordes et al., 2004). The utilisation of long baseline interferometry, however, will significantly increase the angular resolution, resolve out some of the scattering regions, and enable an image mapping of the region which can identify individual radio sources, especially those of steep spectrum and a large fraction of polarisation (hence more likely to be pulsars). It would be the ideal way for an SKA pulsar survey with 3000 km baselines. Moreover, such searches are also helpful in understanding the stellar distribution around Sgr A*, and may even be capable of providing corrections for the perturbed orbital motion of the pulsar.

- **Next generation of timing model:** With the improvement of timing precision, binary systems exhibiting strong relativistic effects (such as PSR-BH systems) may exceed the accuracy of first-order approximations for describing orbital motion. In this case, a more accurate modelling of the binary motion containing higher order corrections (Damour & Schäfer, 1988; Wex & Kopeikin, 1999; Königsdörffer & Gopakumar, 2005), will be required for further precision timing tests. A proper correlation analysis regarding the fitted parameters would be necessary in understanding the contamination between different effects and selecting the appropriate combination of variables (e.g. Freire & Wex, 2010).

The main idea of the work in this thesis is to predict future pulsar timing potential based on our current knowledge and status. Although there are still uncertainties for future issues, the next generation of radio telescopes will for no
doubt remarkably enhance the power of the pulsar timing technique and provide
the best chance so far to test gravitational theories in the ‘next generation of
celestial laboratories’. These prized goals, once achieved, will be at least one of
the most remarkable accomplishments from pulsar science in the next decade.
Appendix A

Effective pulse number

As pulsars are weak radio sources and individual pulses are often not detectable, the signal needs to be folded at the rotation period in order to obtain profiles with sufficiently high S/N to derive precise TOAs. Theoretically, the signal is expected to increase linearly with integration length, while the root-mean-square (RMS) of the noise increases according to a square-root law. Consequently, the corresponding improvement in S/N is expected to be proportional to the square-root of the number of pulses. Given $N$ profiles with peak amplitudes of $A_i$ and noise RMSs of $\sigma_i$ ($i = 1, \ldots, N$), the single pulse S/N is:

$$ (S/N)_i = \frac{A_i}{\sigma_i}, \quad (A.1) $$

and the S/N of a folded profile is:

$$ S/N = \frac{\sum_i A_i}{\sqrt{\sum_i \sigma_i^2}}, \quad (A.2) $$

If the profiles are identical ($A_1 = \ldots = A_N; \sigma_1 = \ldots = \sigma_N$), we have $S/N \propto \sqrt{N}$. Practically, however, effects like intrinsic flux variations, scintillation and system temperature variations cause the S/Ns of profiles with identical integration times to differ. This causes deviations from the scaling rule in the processing of real
data. Therefore, we define the effective number of pulses as:

\[ N_{\text{efc}} = n \left( \frac{S/N}{S/N_{\text{mean}}} \right)^2, \]  

where \( n \) is the number of pulses within an individual integration, \( S/N \) is calculated from Eq. (A.2), and \( S/N_{\text{mean}} \) is the averaged \( S/N \) for all integrations. Effectively, \( N_{\text{efc}} \) is a normalised pulse number, which corrects for the varying \( S/N \) of individual pulsar pulses. Consequently, the measured \( S/N \) of averaged profiles should scale linearly with the calculated \( \sqrt{N_{\text{efc}}} \), regardless of the brightness variations of the pulses involved.
Appendix B

Uncertainty in arrival time

B.1 Radiometer noise

In this section we calculate the pulse time-of-arrival (TOA) uncertainty induced by additive noise on an observed profile. The integrated profile $P(t)$, after subtraction of the baseline due to the average system noise, can be broken into a few components as below:

$$P(t) = (\hat{g} + \Delta g)U(t - \hat{t}_0 + \Delta t) + n(t), \quad (B.1)$$

where $U(t)$ is the normalised pulse template waveform, $g$ is the scaling factor, $t_0$ is the phase offset, $n(t)$ is the noise variation, and “$\Delta$” denotes the small fluctuation due to the existence of noise. Following Taylor expansion, the expression can be further written as

$$P(t) \approx \hat{g}U(t - \hat{t}_0) + \Delta gU(t - \hat{t}_0) + \Delta t\hat{g}U'(t - \hat{t}_0) + n(t). \quad (B.2)$$

Measurements of $\hat{t}_0$ and $\hat{g}$ can be yielded by performing a correlation between the profile and the template, which leads to the minimisation of the following quantity:

$$I = \int [P(t) - \hat{g}U(t - \hat{t}_0)]^2 dt. \quad (B.3)$$
APPENDIX B. UNCERTAINTY IN ARRIVAL TIME

Letting \( \partial I / \partial \Delta t = 0 \), we have

\[
\int [\dot{g}^2 U'^2 (t - \hat{t}_0) \Delta t + \dot{g} U'(t - \hat{t}_0) (\Delta g U (t - \hat{t}_0) + n(t))] dt = 0,
\]

which leads to

\[
\Delta t \simeq - \frac{\int U'(t - \hat{t}_0) n(t) dt}{\int \dot{g} U'^2 (t - \hat{t}_0) dt}.
\]

Here term of order of \( \delta g / \hat{g} \) has been omitted. As \( \langle \Delta t \rangle = 0 \), the variance of \( \Delta t \) is

\[
\sigma_{\Delta t}^2 = \langle \Delta t^2 \rangle = \frac{\langle \int \Delta g U (t - \hat{t}_0) n(t) dt \rangle^2 \int \dot{g} U'^2 (t - \hat{t}_0) dt}{\int [\int \dot{g} U'^2 (t - \hat{t}_0) dt]^2}.
\]

The numerator \( N \) can be written in the form where the profile consists of discrete samples:

\[
N = \langle \sum_i U'_i n_i, \sum_j U'_j n_j \delta t^2 \rangle = \langle \sum_i U'^2 n_i^2 + 2 \sum_{i \neq j} U'_i U'_j n_i n_j \delta t \rangle,
\]

where \( i, j \) denote the sample index and \( \delta t \) is the sampling interval. Note that \( \sigma_n^2 = \langle n_i^2 \rangle = \text{and} \langle n_i n_j \rangle = 0 \), we can further have

\[
N \simeq \sigma_n^2 \delta t \sum_i U'^2 dt = \sigma_n^2 \delta t \int U'^2 dt.
\]

Therefore,

\[
\sigma_{\Delta t} = \frac{\sigma_n}{\hat{g}} \left[ \frac{\delta t}{\int U'^2 dt} \right]^{1/2},
\]

where the ratio \( \hat{g} / \sigma_n \) stands for the measured peak S/N of the profile. Note that this expression is valid for only high S/N cases (e.g., S/N \( \gtrsim 10 \)), and tends to underestimate the TOA uncertainty when S/N approaches unity.

### B.2 Phase jitter

Here we calculate the TOA uncertainty of an integrated profile caused by phase jitter of single pulses. We assume that single pulses are of identical waveform
p_0(\phi) and that the effect of phase jitter is to introduce a random phase to each single pulse, i.e. the i-th single pulse takes waveform of p_0(\phi + \Delta \phi_i). The phase jitter is identical-independent Gaussian noise, i.e. \langle \Delta \phi_i, \Delta \phi_j \rangle = F^2_J \delta_{ij}, where the Kronecker delta \delta_{ij} = 1, if i = j, otherwise \delta_{ij} = 0. The waveform of the template profile p(\phi) is the time average of single pulse, i.e.

\[ p(\phi) = \lim_{N \to \infty} \frac{1}{N} \sum_i p_0(\phi + \Delta \phi_i). \]  

If \( F_J \ll w \) (i.e. \( F_J \lesssim 0.3w \)), where \( w \) is the width of the single pulse, and the higher order terms can be sufficiently neglected, we have

\[ p(\phi) = p_0(\phi) + \frac{1}{2} p_0''(\phi) F^2_J + O(F^4_J), \]  

where "\(^{\prime}\) denotes the second-order derivative respected to \( \phi \). The reason for using such an expansion is that the direct calculation similar to that of the previous section is not possible here.

In real observation, one averages \( N \) single pulses and forms the mean pulse profile \( p_N \), where

\[ p_N(\phi) = p(\phi) + \frac{1}{N} \sum_i p_0'(\phi) \Delta \phi_i + O(F^2_J), \]  

and here the ‘ denotes the first-order derivative with respect to \( \phi \). Now assume the shape perturbation of the integrated profile can be effectively approximated by a phase shift \( \delta \phi \) of the template, which can be expressed by

\[ p_N(\phi) \simeq p(\phi + \delta \phi) \simeq p(\phi) + p'(\phi) \delta \phi. \]  

The measurement of \( \delta \phi \) is yielded by minimising the sum of \( p_N(\phi) - p(\phi + \delta \phi) \) across the entire period, which leads to

\[ \partial \int (\delta \phi p'(\phi) - \frac{1}{N} \sum_i p_0'(\phi) \Delta \phi_i)^2 d\phi = 0 \]  

and then

\[ \delta \phi \simeq \int p'(\phi) \frac{1}{N} \sum_i p_0'(\phi) \Delta \phi_i d\phi \int p'^2(\phi) d\phi. \]  

APPENDIX B. UNCERTAINTY IN ARRIVAL TIME

Since $\langle \Delta \phi \rangle = 0$, we have the variance of $\Delta \phi$ in the form of

$$\sigma_{\Delta \phi}^2 = \frac{1}{N^2} \left[ \frac{\int p'(\phi)p'_0(\phi) d\phi}{\int p'^2(\phi) d\phi} \right]^2 \sum_i \Delta \phi_i \sum_j \Delta \phi_j. \quad (B.16)$$

Note that $\sum_i \Delta \phi_i \sum_j \Delta \phi_j \simeq NF_J^2$ and $p'(\phi)p'_0(\phi) = p'^2(\phi) + O(F_J^2)$, one finally obtains

$$\sigma_{\Delta \phi} \simeq \frac{F_J}{\sqrt{N}}. \quad (B.17)$$

To express the variance with $f_J$ which is in unite of pulse width, we then have

$$\sigma_{\Delta \phi} \simeq \frac{f_J}{\sqrt{N}} \left[ \frac{\int \phi^2 p(\phi) d\phi}{\int p(\phi) d\phi} \right]^{1/2}. \quad (B.18)$$
Appendix C

Correlation-coefficient scaling

C.1 Additive noise

In this section we calculate the relation between $N$ and $1 - \rho$, where $N$ is the number of accumulated pulses in an integrated profile and $\rho$ is the correlation coefficient between the profile and a noise-free template. Assume the observed profile is a superposition of a normalised template $p$ and Gaussian noises $n$, i.e. $p_i = s_i + n_i$. The subscript $i$ goes from 1 to $n$, the number of the profile bins. We regard $p$ and $n$ as $n$-dimensional vectors. Following Eq. (4.1), the correlation coefficient between the perfect template $s$ and observed profile $p$ is

$$\rho = \frac{\sum_i c_i (c_i + n_i)}{\sqrt{\sum_i c_i^2 \sum_i (c_i + n_i)^2}}, \quad (C.1)$$

where $c_i = s_i - \bar{s}$.

Assume that the noise $n$ is a multivariate Gaussian with probability distribution $f(n)$ and covariance matrix $C$, where $C_{ij} = \sigma_n^2 \delta_{ij}$. The expectation value of
correlation coefficient $\langle \rho \rangle$ and its second-order moment $\langle \rho^2 \rangle$ are

$$\langle \rho \rangle = \int \rho f(n) \, dn$$  \hspace{1cm} (C.2)

$$\langle \rho^2 \rangle = \int \rho^2 f(n) \, dn$$  \hspace{1cm} (C.3)

The variance for $\rho$ is then $\sigma^2_\rho = \langle \rho^2 \rangle - \langle \rho \rangle^2$. The Eq. (C.2) and (C.3) can be integrated by transforming into hyper-spherical coordinates (Mathews & Walker, 1970). Although no analytical expression could be found, by using asymptotic technique we derive the results for the case of a large sample number which match both the low and high S/N cases as below:

$$1 - \langle \rho \rangle = 1 - \left( \frac{1}{1 + \chi} \right)^{1/2}$$  \hspace{1cm} (C.4)

$$= \begin{cases} 
1 - \left( \frac{1}{\chi} \right)^{1/2} + \mathcal{O} \left( \frac{1}{\chi^{3/2}} \right), & \chi \gg 1 \\
\frac{1}{2} \chi + \mathcal{O} (\chi^2), & \chi \ll 1 
\end{cases}$$  \hspace{1cm} (C.5)

$$\sigma_\rho = \begin{cases} 
\frac{\chi}{n_{\text{samp}}(1 + \chi)}^{1/2} + \mathcal{O} \left( \frac{1}{\chi^2} \right), & \chi \gg 1 \\
\chi \left( \frac{n_{\text{samp}} - 1}{2n_{\text{samp}}^2} \right)^{1/2} + \mathcal{O} (\chi^2), & \chi \ll 1 
\end{cases}$$  \hspace{1cm} (C.6)

where

$$\chi = \frac{\sigma^2_{n}}{\sum c_i^2}; \quad \sum_i c_i^2 = \frac{\sum_i c_i^2}{n_{\text{samp}}}. \hspace{1cm} (C.7)$$

In Fig. C.1 a set of Monte Carlo simulations are performed so as to test the validity of the derivation. Here a simple Gaussian template shape is assumed and profiles are created by adding normal distributed noise onto the template. It can be seen that for most S/N ranges the results from both approaches coincide with each other, and for high S/N value, $1 - \rho$ scales linearly with the increase of signal.
C.2. PHASE JITTER

In this section, we prove that the relation $1 - \langle \rho \rangle \propto 1/N$. Assuming single pulses are of identical waveform $p_0(\phi)$ and the effect of phase jitter is to introduce a random phase to each single pulse, i.e. the $i$-th single pulse takes a waveform of $p_0(\phi + \Delta \phi_i)$, where the $\Delta \phi_i$ is a random phase. The waveform of the template profile $p(\phi)$ is defined by summing infinite number of single pulses as

$$p(\phi) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} p_0(\phi + \Delta \phi_i). \quad (C.8)$$

Meanwhile, an integrated profile $p_N$ obtained from an observational session is yielded by averaging $N$ single pulses ($N \gg 1$) as

$$p_N(\phi) = \frac{1}{N} \sum_{i=1}^{N} p_0(\phi + \Delta \phi_i). \quad (C.9)$$

Fig. C.1: Theoretical and simulated results presenting the relation between S/N and correlation factors. In each case the dashed line represents the analytic solution and the solid line stands for the numerical result. The parameter $n_{bin}$ is the number of segments for the profiles.
The correlation coefficient between the template $p(\phi)$ and $N$-averaged integrated pulse profile $p_N(\phi)$ is then

$$\rho = \frac{\int_0^1 p(\phi)p_N(\phi) \, d\phi}{\left[\int_0^1 p(\phi)^2 \, d\phi \int_0^1 p_N(\phi)^2 \, d\phi\right]^{\frac{1}{2}}}, \tag{C.10}$$

The evaluation for the statistical expectation of correlation coefficient is already presented in Cordes & Shannon (2010). In this appendix, a slightly different approach is applied. Letting

$$\delta(\phi) = p_N(\phi) - p(\phi), \tag{C.11}$$

we have

$$\rho = \frac{\int_0^1 p^2(\phi) + p(\phi)\delta(\phi) \, d\phi}{\left[\int_0^1 p^2(\phi) \, d\phi \int_0^1 p^2(\phi) + 2p(\phi)\delta(\phi) + \delta^2(\phi) \, d\phi\right]^{\frac{1}{2}}}, \tag{C.12}$$

Since the number of pulses in observations is usually large, we expect that the $p_N(\phi)$ is very close to $p(\phi)$, which leads to $\delta(\phi) \ll p(\phi)$. In this case, one can show that

$$\langle \rho \rangle = 1 + \frac{1}{2} \left\langle \left(\frac{\int_0^1 p(\phi)\delta(\phi) \, d\phi}{\int_0^1 p^2(\phi) \, d\phi}\right)^2 - \frac{\int_0^1 \delta^2(\phi) \, d\phi}{\int_0^1 p^2(\phi) \, d\phi}\right\rangle + O\left(\frac{\delta^4}{p^4}\right), \tag{C.13}$$

which is equivalent to

$$\langle 1 - \rho \rangle \approx \frac{1}{2} \left\langle \frac{\int_0^1 p_N^2(\phi) \, d\phi}{\int_0^1 p^2(\phi) \, d\phi} - \left(\frac{\int_0^1 p(\phi)p_N(\phi) \, d\phi}{\int_0^1 p^2(\phi) \, d\phi}\right)^2\right\rangle. \tag{C.14}$$

To understand how the $1 - \langle \rho \rangle$ depends on the number of pulses $N$, we have to expand $p_N(\phi)$ in the above equations. One can then have (Cordes & Shannon, 2010)

$$\left\langle \int_0^1 p_N^2(\phi) \, d\phi \right\rangle = \frac{N^2 - N}{N^2} \int_0^1 p^2(\phi) \, d\phi$$

$$+ \frac{1}{N} \int_0^1 \langle p_N^2(\phi + \Delta\phi) \rangle \, d\phi, \tag{C.15}$$
C.2. PHASE JITTER

Fig. C.2: The $1 - \langle \rho \rangle$ as function of $N$ and $\sigma_J$, where $\sigma_J^2$ is the variance of the phase jitter probability density function. The dashed lines are from analytical calculation, the Eq. (C.17), while the solid lines are from direct Monte Carlo simulations.

and

$$\left\langle \left( \int_0^1 p(\phi) p_N(\phi) \, d\phi \right)^2 \right\rangle = \frac{N^2 - N}{N^2} \int_0^1 p^2(\phi) \, d\phi + \frac{1}{N} \int_0^2 d\phi \int_0^1 d\phi' p(\phi) p(\phi') \langle p_0(\phi + \Delta \phi) p_0(\phi' + \Delta \phi) \rangle .$$

(C.16)

Here we assume that the phase jitter is independent, i.e. $\langle \Delta \phi_i \Delta \phi_j \rangle = 0$, if $i \neq j$.

Thus

$$\langle 1 - \rho \rangle = \frac{1}{2N} [S - K]$$

(C.17)

where

$$S = \frac{\int_0^1 \langle p_0^2(\phi + \Delta \phi) \rangle \, d\phi}{\int_0^1 p^2(\phi) \, d\phi},$$

(C.18)

$$K = \frac{\int_0^2 d\phi \int_0^1 d\phi' p(\phi) p(\phi') \langle p_0(\phi + \Delta \phi) p_0(\phi' + \Delta \phi) \rangle}{\left( \int_0^1 p^2(\phi) \, d\phi \right)^2}.$$  

(C.19)

Because neither $S$ nor $K$ is $N$-dependant, the Eq. (C.17) clearly show the scaling relation that $1 - \langle \rho \rangle \propto 1/N$.

We also further compare the result of $1 - \langle \rho \rangle$ from Monte Carlo simulations...
and from the Eq. (C.17) in the Fig. C.2. Clearly, the numerical simulation and analytical calculation match each other at larger $N$ limit as we expected.
Appendix D

Spin and geometry solution for pulsar–Sgr A* system

Following Section 5.2.3.2, once the second derivative of either periastron and projected semi-major axis is measured, the BH spin together with the system geometry can be fully determined for the case of pulsar orbiting around Sgr A*.

The order of measurement for the two parameters is geometrically dependent as shown in Section 5.2.3.2. By following Eq. (5.46), Eq. (5.47), Eq. (5.48), Eq. (5.50) and Eq. (5.51), with the definition of $c_X \equiv \cos X$, $s_X \equiv \sin X$, and $s_3 \equiv -s_\theta s_\lambda s_\phi$, the expressions of the derivatives and the geometric relation can be rewritten in the form of

\[
\dot{x}_s = -x_0 \frac{c_i s_3}{s_i^2} \Omega_s,
\]  
(D.1)

\[
\ddot{x}_s = -x_0 \frac{s_3^2 + s_i^2 c_i (c_\theta c_\lambda - c_i)}{s_i^4} \Omega_s^2,
\]  
(D.2)

\[
\dot{\omega}_s = \frac{(1 - 3s_i^2)c_\theta - c_i c_\lambda}{s_i^2} \Omega_s,
\]  
(D.3)

\[
\ddot{\omega}_s = \frac{2c_i c_\theta - (2 - s_i^2)c_\lambda}{s_i^4} s_3 \Omega_s^2,
\]  
(D.4)

\[
s_3^2 = s_i^2 - c_\theta^2 - c_\lambda^2 + 2c_\theta c_\lambda c_i,
\]  
(D.5)

\[
c_i = c_\theta c_\lambda + s_\theta s_\lambda c_\Phi.
\]  
(D.6)
APPENDIX D. SPIN AND GEOMETRY SOLUTION FOR PULSAR–SGR A* SYSTEM

The solutions of the spin and the geometric angles, with utilisation of either second derivative are presented and discussed as below.

D.1 Use of $\dot{\omega}_s$

Clearly, the expressions of $\dot{\omega}_s$ and $\ddot{\omega}_s$ present linear combinations of $c_\theta$ and $c_\lambda$, which enables a unique separation of these two parameters. The four targeted equations can be reduced to the forms of

\begin{align*}
X_1 &= c_i \xi_3, \quad (D.7) \\
W_1 &= (1 - 3s_i^2)\xi_\theta - c_i \xi_\lambda, \quad (D.8) \\
W_2 &= 2\xi_\theta - \frac{2 - s_i^2}{c_i} \xi_\lambda, \quad (D.9) \\
\xi_3^2 &= \frac{\chi^2}{s_i^2} - \xi_\theta^2 - \xi_\lambda^2 + 2\xi_\theta \xi_\lambda c_i. \quad (D.10)
\end{align*}

where

\begin{align*}
X_1 &\equiv -\frac{\dot{x}_s}{x_0 \dot{\bar{\Omega}}_s}, \quad W_1 \equiv \frac{\dot{\omega}_s}{\bar{\Omega}_s}, \quad W_2 \equiv -\frac{\ddot{\omega}_s x_0}{\dot{x}_s \bar{\Omega}_s}, \\
\xi_3 &\equiv \frac{s_3 \chi}{s_i^2}, \quad \xi_\theta \equiv c_\theta \chi \frac{s_i^2}{s_i^2}, \quad \xi_\lambda \equiv \frac{c_\lambda \chi}{s_i^2}, \quad (D.11)
\end{align*}

and $\bar{\Omega}_s$ is as defined in Eq. (5.39). $\xi_\theta$ and $\xi_\lambda$ can be readily solved by combining Eq. (D.8) and Eq. (D.9)

\begin{align*}
\xi_\lambda &= \frac{2W_1 - (1 - 3s_i^2)W_2}{(3s_i^2 - 5)s_i^2} c_i, \quad (D.12) \\
\xi_\theta &= \frac{(2 - s_i^2)W_1 - c_i^2 W_2}{(3s_i^2 - 5)s_i^2}. \quad (D.13)
\end{align*}

Applying the results to Eq. (D.10), one can immediately obtain the expression of the spin

\[
\chi = \left[ \frac{X_1^2 s_i^2}{c_i^2} + \frac{(s_i^2 + 4c_i^2)W_1^2 + c_i^2(1 + 3s_i^2)W_2^2 + 2c_i^2(3s_i^2 - 2)W_1 W_2}{(3s_i^2 - 5)^2} \right]^{1/2}. \quad (D.14)
\]

Note that by definition $\theta_S$ and $\lambda$ are from zero to $\pi$, the cosine value given by $\xi_\lambda$ and $\xi_\theta$ can then yield a unique determination of the two angles, followed by the
determination of $\Phi$ from

$$c_\Phi = \frac{c_i - c_\theta c_\lambda}{s_\theta s_\lambda}, \quad \text{(D.15)}$$

$$s_\Phi = -\frac{W_1^2 s_i^2}{\chi s_\theta s_\lambda c_i}. \quad \text{(D.16)}$$

Following the discussion in Section 5.2.3.1, one can also find out if the unknown sign of $\cos i$ induces multiple solution in this case. Seen from their expressions, $\chi$ and $\theta$ are still uniquely determined as they are only $s_i^2$ related, while for $\lambda$ and $\Phi$ alternative solutions exist. Note that here $\Omega \simeq \Phi_0$ and the determination of $\omega_{SZ}$ remains as in Section 5.2.3.1, so the combinations of solutions to the angles are the same as in Table 5.3. Again, a further measurement of $\ddot{x}_s$ does not help to distinguish between to real and fake solution as the value is not influenced by the sign of $c_i$.

### D.2 Use of $\ddot{x}_s$

Once the equation of $\ddot{x}_s$ is dropped, only the expression of $\ddot{\omega}_s$ provides a linear relation between $c_\theta$ and $c_\lambda$ while the left just contains information of $c_\theta c_\lambda$. So higher power terms are inevitable during the process of derivation, which can lead to multiple solutions. By using the same notation as the above section, from the expression of $\ddot{x}_s$ one can immediately have

$$\chi^2 = \frac{s_i^2}{c_i^2} \left( \frac{X_1^2}{c_i^2} + s_i^2 c_\theta \xi_\theta \xi_\lambda - X_2 \right), \quad \text{(D.17)}$$

where

$$X_2 \equiv -\frac{\ddot{x}_s}{x_\theta \Omega_s^2}. \quad \text{(D.18)}$$

Applying the above expression to Eq. (D.10), we have

$$\xi_\theta^2 + \xi_\lambda^2 - \frac{1 + c_i^2}{c_i} \xi_\theta \xi_\lambda + \frac{X_2}{c_i^2} - \frac{X_1^2 s_i^2}{c_i^4} = 0, \quad \text{(D.19)}$$

which combined with Eq. (D.8), leads to

$$6s_i^4 c_\theta^2 + 5s_i^2 W_1 \xi_\theta + W_1^2 + X_2 - X_1^2 s_i^2 c_i^2 = 0. \quad \text{(D.20)}$$
Fig. D.1: Illustration of using $\tilde{\omega}$ to rule out fake solution in the $\xi_\theta$-$\xi_\lambda$ diagram. The geometric angle are assumed as $\lambda = -30^\circ$, $\theta = 60^\circ$, $i = 45^\circ$. The ellipse is obtained after applying the linear relation in Eq. (D.8) to Eq. (D.19) to substitute the coupling term.

So $\xi_\theta$ is readily solved from the root finder equation,

$$
\xi_\theta = \frac{-5W_1 \pm \sqrt{W_1^2 - 24 \left( X_2 - X_1^2 \frac{s_i^2}{c_i^2} \right)}}{12s_i^2}. \quad (D.21)
$$

and then $\xi_\lambda$ and $\chi$ are immediately obtained from Eq. (D.8) and Eq. (D.19). Though it is possible to have two solutions to the spin, there are a few criterions for excluding the fake one. Firstly, the solved value of $\chi^2$ by Eq. (D.19) should be positive. Secondly, the obtained values of $c_\lambda$ and $c_\theta$ need to fall into the range of from -1 to 1. Besides, once constraint well enough, the $\tilde{\omega}$ can be used to help select the true solution, and an example of this utilisation is shown in Fig. D.1. Here the determination of $\xi_\theta$ and $\xi_\lambda$ is yielded from Eq. (D.8) and Eq. (D.19).

While two possible roots appears within the region allowed by the value range of the cosine function and spin (assuming GR is correct), a 20% precision of $\tilde{\omega}$ measurement at the 1-$\sigma$ level already succeeds in figuring out the true solution.

For either form of root, the solution to $\xi_\theta$ contains only terms of $s_i^2$, which indicates it stays for both cases of the inclination angle. Note from Eq. (D.8) that $c_i\xi_\theta$ is only related to $s_i^2$ as well, the solution of spin is also not influenced by the sign of $c_i$. Again, indicated from Eq. (D.8) the solution of $\lambda$ is proportional
to the linear term $c_i$, and is then not uniquely determined once the sign of $c_i$ is unknown. The determinations of $\Phi$, $\Omega$ and $\omega_{SZ}$ stay the same as the above section.
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