On the existence and structure of equilibrium in price-setting games

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Abstract

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In this work the problematic issue of price determination in economic theory is re-examined. In the first chapter a state-of-the-art survey regarding the existence of equilibrium and the structure of the equilibrium set in price-setting games is provided. In chapter two a new core concept, the Bertrand core, is introduced and characterized. In chapter three a revealed preference perspective upon the Nash equilibria in price-setting games is provided. In chapter four, the issue of Bayesian equilibrium existence is addressed when traders have incomplete information regarding each others’ types. Finally, a summary of possible future avenues for research in this area is provided.
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Chapter 1

Introduction

On the other hand, when explaining how competitive equilibrium is reached, one is used to refer to an auctioneer who would propose and revise the price that buyers and sellers then have to accept. This again is somewhat inadequate, since auctioneers actually exist only on special markets, such as those concerning stock exchange or commodity future exchange.

Edmond Malinvaud (1985, pp.174-5)

There is one aspect of economic theory which has been, and remains, particularly problematic: the study of price-making behaviour and the determination of market prices. Given the centrality of prices in economic theory, this may seem somewhat surprising. However, economic theory has tended to circumvent the direct problem of price determination by assuming the existence of an agent, a deus ex machina, whose sole job is to aggregate together agents’ decisions, taken in various contexts, and to establish the required market prices. This agent is usually referred to as the auctioneer. So, for example, in the canonical model of general equilibrium exchange it is the auctioneer who prevents trade from taking place at disequilibrium prices where some agent must find their economic choices are rationed. Alternatively, in the model of strategic market behaviour based upon quantity-setting decisions, the Cournot model, it is the auctioneer who sends back the required market-clearing price after traders have decided what quantity to send to the market. Although the auctioneer is a convenient fiction which has permitted much elegant modeling of economic decisions, the
invocation of the auctioneer is a serious problem for economic theory and deserves deeper understanding.

Despite these problems regarding how market prices are determined, there is a line of thought, dating back almost to the establishment of the auctioneer, which takes a quite different approach. In Bertrand (1883) it was considered what would happen if sellers were to directly compete by setting prices. In this case there is no need for an auctioneer as trading prices are chosen by sellers. At this early stage, it was erroneously concluded that there could be no market equilibrium if sellers were to set prices. Instead, market prices would display an inexorable spiral downwards. Since this important insight, there has been substantial work which has formalized Bertrand’s insight using game theory, and which has shown the assumption of the non-existence of a price equilibrium to be incorrect under certain circumstances. Moreover, even if markets are far from competitive price-making behaviour may result in the perfectly competitive market outcome.

In this book we analyze and extend the classical model of Bertrand price competition. The theory of price competition in a perfectly homogeneous good market is often associated with one striking result: the Bertrand paradox. This result, which we state formally later in this chapter, says that only two sellers is necessary for price competition to result in the perfectly competitive outcome. For a long time, this has been the only result regarding Bertrand competition which theoretical economists have been aware of and the simplicity of the model has resulted in researchers not dedicating more time to understand the nuances of the result and under what precise conditions this may hold. Indeed, most of the research on partial equilibrium price games between the period 1960-1990 analyzed Bertrand-Edgeworth games where sellers were either capacity constrained or were permitted to ration the market demand.\footnote{\textsuperscript{1}Vives (1999, Ch.5) provides a comprehensive survey of partial equilibrium price-setting games.} However, in the last twenty years there has been a revival of interest in price-setting games where the form of market contracts which sellers state prices \textit{and} a commitment to supply all the demand forthcoming at this price. This is what distinguishes Bertrand competition from Bertrand-Edgeworth competition.
It is worth noting that in his original review of Cournot’s oligopoly model Bertrand (1883) did not anticipate that price competition between a small number of sellers could result in the perfectly competitive outcome. Instead, he thought that if a seller were to lower its price below the prices of its rivals and obtain all the market demand prices would cycle and there would be no equilibrium. This much mentioned, but rarely quoted passage, says:

“Cournot conjectures that one of the competitors will lower his price to attract buyers, and that the other, in order to bring them back, will lower his more. They will continue until each of them would no longer gain anything more by lowering his price. A peremptory objection arises: with this hypothesis a solution is impossible; the price reduction would have no limit. In fact, whatever jointly determined price were adopted, if only one of the competitors lowers his, he gains, disregarding all unimportant exceptions, all the sales, and he will double his returns if his competitor allows him to do so.” (Daugherty, 1988, p.77)

In this book we present a unified treatment of the model of Bertrand competition. The reader will find that once a grasp of the benchmark model has been acquired it is easier to understand some of the original results presented in later chapters. In this chapter we introduce the general model and provide an up-to-date survey of recent advances. Specifically, we consider the existence of both pure and mixed strategy Bertrand equilibria; the existence of Bayesian equilibrium and other problems caused by incomplete information; the structure of sharing rules at price ties; and the possibility of coalitional deviations by groups of traders. Almost all the results in this section draw upon papers published within the last decade.

In chapter two we return to a classical question in economic theory: under what conditions does price-taking behaviour prevail in a market? As noted at the beginning, most early work on price-setting games studied Bertrand-Edgeworth games where sellers offer market contracts which provide no commitment to supply any particular demand to the market. In Bertrand competition the market contracts are such that sellers quote prices with a commitment to supply all demand forthcoming from the market. In this chapter we admit the possibility that sellers may be able to make contracts with specific buyers in the market, and as a result, the market contracts may be somewhere between the extremes of Bertrand
and Bertrand-Edgeworth contracts. This new type of competition also admits the possibility that a group of traders may leave the grand coalition and improve their outcomes by trading amongst themselves. To study which prices might emerge in the market we introduce a new core concept, which we term the Bertrand core. Under weak conditions we establish that the Bertrand core is non-empty. Moreover, we are able to derive a price-making analogue of the well-known Debreu-Scarf (1963) result by showing that as the market becomes large the only price which cannot be blocked by some subset of traders is the price-taking equilibrium.

In chapter three we take a slightly different tack by considering what structure the equilibrium concept imposes upon observable outcomes in the market. We analyze the following theoretical problem: a finite number of observations of a homogeneous-good oligopoly market are made, and in each observation we observe sellers’ prices, outputs and possibly cost information. What restrictions must the observations satisfy for each observation to be rationalized as a pure strategy Bertrand equilibrium? We provide a complete characterization of price/output/cost observations which can be Bertrand rationalized and some partial characterizations of price/output observations. The conditions which characterize the sets are economically intuitive and take the form of linear inequalities. Moreover, together with recent results established by Carvajal et al. (2010), we can characterize which homogeneous-good market observations are consistent with either the Bertrand or Cournot oligopoly models.

In chapter four we continue by analyzing the existence of a Bayesian equilibrium in a market with a finite number of types, and find, under quite general conditions, that an equilibrium in pure strategies fails to exist. Nevertheless, we are able to establish that a mixed strategy equilibrium does exist and we provide a complete characterization of this equilibrium. It is worth noting that the introduction of incomplete information in Bertrand games results in Bertrand’s original insight, that there might not exist any equilibrium point, being correct, albeit if one restricts attention to pure strategies. In the final chapter we make some tentative conclusions regarding future theoretical research on price-setting games.
1.1 Notation

The following notation is used throughout the rest of the book.
\[ \mathbb{R}^n \] denotes n-dimensional Euclidean space.
\[ \mathbb{R}_+^n \] is the non-negative orthant of \( \mathbb{R}^n \).
\( 2^X \) denotes all the subsets of \( X \).
\( |X| \) denotes the cardinality of \( X \).
\( \setminus \) denotes set theoretic subtraction.
\( \emptyset \) denotes the emptyset.
\( \text{supp}(\mu) \) denotes the support of measure \( \mu \).
\( \inf(X) \) denotes the infimum of \( X \subset \mathbb{R} \).
\( \sup(X) \) denotes the supremum of \( X \subset \mathbb{R} \).
\( \mathbb{N} \) denotes the set of natural numbers.
\( \mathbb{Q} \) denotes the set of rational numbers.
\( \mathbb{Q}^+ \) denotes the set of positive rational numbers.
\( \subset \) denotes weak subset inclusion.

1.2 A general framework for Bertrand games

We now introduce a general model of price competition and bring together most of the existing results in the literature together with some new ones. By developing a good understanding of the model the reader will be better placed to appreciate the results developed in the later chapters. Consider a market with \( N = \{1, ..., n\}, \ n \geq 2 \), sellers which produce a perfectly homogeneous good. Each seller has a cost function \( C_i(Q) \) which gives the cost of producing any quantity of output. The market demand for the good is given by \( D(P) \).

Unless otherwise stated these primitives will be taken to satisfy the following assumptions.

**Assumption 1** The market demand \( D : \mathbb{R}_+ \to \mathbb{R}_+ \) possesses strictly positive real numbers \( \bar{P} \) and \( \bar{Q} \) such that \( D(P) = 0 \) for all \( P \geq \bar{P} \) and \( D(0) = \bar{Q} \). Also, the market demand is \( C^2 \) on \((0, \bar{P})\) and \( D'(P) < 0 \) for all \( P \in (0, \bar{P}) \).
Assumption 2 Each seller’s cost function $C_i(Q)$ is $C^2$ with $C_i(0) = 0$ and. Also, $C_i(Q)$ is either strictly convex, linear or strictly concave. If $C_i(Q)$ is convex we shall assume that $C'_i(0) = 0$.\(^2\)

Assumption 3 When sellers quote prices to buyers they are committed to supplying all the demand forthcoming at that price. Therefore the market demand is not rationed and all trade takes place at the minimum price quoted in the market.

If a seller quotes the unique minimum price in the market then, from A3, the seller must serve all the market demand and obtains its monopoly profit at the quoted price. However, if a seller ties with other sellers at the minimum price then a sharing rule describes how the market demand is split between the sellers tying at the minimum price. Let $\pi_i(P)$ denote the monopoly profit of seller $i$:

$$\pi_i(P) = PD(P) - C_i(P).$$  \hspace{1cm} (1.1)

We shall consider a number of sharing rules which have been studied in the literature, and in section 1.4 we consider arbitrary properties of sharing rules which guarantee the existence of equilibrium. Therefore we shall introduce some general notation. Given a vector of quoted prices $(P_1, ..., P_n)$ let $P^{\text{min}} = \min \{P_1, ..., P_n\}$. That is, $P^{\text{min}}$ is the minimum price quoted in the market. Let $W = \{i \in N : P_i = P^{\text{min}}\}$ so $W$ is the set of sellers which tie at the minimum price. Finally, let $\chi^W = \{M : M \in 2^N \setminus \{\emptyset\}\}$ so $\chi^W$ is all the non-empty subsets of sellers in the market. If seller $i$ ties with $W \setminus \{i\}$ other sellers at the minimum price then we shall let $\pi_i(P_i, W)$ denote the shared payoff which seller $i$ obtains.

Given this setup we can define the Bertrand game. Let $S_i = [0, \bar{P}]$ denote the strategy space of seller $i$. That is, each seller simultaneously and independently chooses a $P_i \in S_i$. Therefore the joint strategy space is $S = \times_{i \in N} S_i$. The payoff which seller $i$ receives is a

\(^2\)We shall make it clear what type of cost function we consider in the results below.
mapping \( u_i : S \rightarrow \mathbb{R} \). Given the assumptions made above the payoffs can be summarized as:

\[
u_i(P_i, P_{-i}) = \begin{cases} 
\pi_i(P_i) & \text{if } P_i < P_k \forall k \neq i; \\
\pi_i(P_i, W) & \text{i ties with } W \setminus \{i\} \text{ sellers at min price}; \\
0 & \text{if } P_i > P_k \text{ for some } k.
\end{cases}
\] (1.2)

**Definition 1.1** A pure strategy Bertrand equilibrium is a vector of prices \((P_i^*, P_{-i}^*)\) such that \(u_i(P_i^*, P_{-i}^*) \geq u_i(P'_i, P_{-i}^*)\) for all \(P'_i \in S_i\) and \(i \in N\).

Now we can consider the mixed extension of the pricing game. Let \(\Delta S_i\) denote the set of Borel probability measures on \(S_i\). A strategy for seller \(i\) is a \(\sigma_i \in \Delta S_i\). Given that the sellers play a vector of probability measures \((\sigma_i, \sigma_{-i})\) the expected payoff for seller \(i\) is:

\[
U_i(\sigma_i, \sigma_{-i}) = \int_{S_1} \int_{S_i} \ldots \int_{S_n} u_i(P_1, \ldots, P_i, \ldots, P_n) d\sigma_1, \ldots, d\sigma_i, \ldots, d\sigma_n.
\] (1.3)

**Definition 1.2** A mixed strategy Bertrand equilibrium is a vector of probability measures \((\sigma_i^*, \sigma_{-i}^*)\) such that \(U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma'_i, \sigma_{-i}^*)\) for all \(\sigma'_i \in \Delta S_i\) and \(i \in N\).

We shall consider the existence of equilibrium, pure and mixed, under a number of specific sharing rules: equal sharing, capacity sharing and winner-takes-all sharing. The most commonly encountered sharing rule used in Bertrand games is equal sharing, and as the name suggests, the rule assumes that the market demand is split equally between the sellers tying at the minimum price. Formally, if \(i \in W\) then the payoff which seller \(i\) receives is:

\[
\pi_i(P_i, W) = \frac{1}{|W|} P_i D(P_i) - C_i \left( \frac{D(P_i)}{|W|} \right) \quad \text{with} \quad 2 \leq |W| \leq n.
\] (1.4)

Alternatively the capacity sharing rule assumes that sellers tying at the minimum price split the market demand in proportion to their competitive supplies. If we let \(\pi_i(Q) = PQ - C_i(Q)\) then the competitive supply of each seller, as a function of price, is \(h_i(P) = \arg\max_{Q \in [0,\hat{Q}]} \pi_i(Q)\). Also, let \(\pi_i^*(P) = Ph_i(P) - C_i(h_i(P))\) so \(\pi_i^*(P)\) is the value function of the maximization problem. If the set of sellers tied at the minimum price is \(W \in \chi_W\) then let \(\beta_i(P) = h_i(P_i)/\sum_{j \in W} h_j(P_j)\). If \(i \in W\) then the profit which seller \(i\) receives under the capacity sharing rule is:

\[
\pi_i(P_i, W) = \beta_i(P_i) P_i D(P_i) - C_i(\beta_i(P_i) D(P_i)).
\] (1.5)
Finally, under the winner-takes-all sharing rule a single seller is selected at random from the sellers tying at the minimum price to serve all the market demand. Therefore if \( i \in W \) the expected profit which seller \( i \) receives under the winner-takes-all sharing rule is:

\[
\pi_i(P_i, W) = \frac{1}{|W|} \pi_i(P_i).
\]

(1.6)

It will be useful to introduce the following lemmas regarding the competitive supplies and value functions of the sellers.

**Lemma 1.1** If \( C_i(\cdot) \) is strictly convex then \( h_i(0) = 0 \) and \( h'_i(P) > 0 \).

*Proof.* See the Appendix.

**Lemma 1.2** If \( C_i(\cdot) \) is strictly convex then \( \pi^*_i(P) > 0 \) for all \( P > 0 \).

*Proof.* See the Appendix.

In deriving some of the results we shall also use the concept of price-taking equilibrium. We now define this concept.

**Definition 1.3** A price-taking equilibrium for a market is a \( P_C \in \mathbb{R}_+ \) such that \( \sum_{i \in N} h_i(P_C) = D(P_C) \).

### 1.3 Existence of pure and mixed strategy equilibrium

We now analyze under what conditions the general Bertrand game introduced here possesses an equilibrium. First consider the equal sharing rule and suppose that the sellers in the market are symmetric.\(^3\) That is, \( C_i(Q) = C(Q) \) for all \( i \in N \). Define the following \( P_{\text{mon}} = \arg\max_{P \in [0, \bar{P}]} \pi(P) \) and \( P^J(W) = \arg\max_{P \in [0, \bar{P}]} \pi(P, W) \). The price \( P_{\text{mon}} \) is the monopoly price for each seller and \( P^J(W) \) is the join profit maximizing price for the sellers. Therefore \( P^J(W) \) is the collusive price if the sellers choose to collude and maximize the sum of their profits. There is a final assumption which we shall employ.

\(^3\)Many, but not all, of the results in this section can be found in Dastidar (1995, 2006).
Assumption 4 The functions $\pi(P)$ and $\pi(P,W)$, assuming demand is shared equally at prices ties, are strictly concave in price and $\pi(P_{\text{mon}}) > \pi(P^J(W),W) > 0$.

The next set of lemmas state important properties of the model which are crucial to understanding the results. The proofs of the next six lemmas are contained in the appendix.

Lemma 1.3 $\exists$ a unique $\tilde{P} \in [0,\bar{P})$ such that $\pi(\tilde{P}) = 0$.

Lemma 1.4 For all $W \in \chi^W$ with $|W| \geq 2 \exists$ a unique $\hat{P}(W) \in [0,\bar{P})$ such that $\pi(\hat{P}(W),W) = 0$.

Lemma 1.5 For all $W \in \chi^W$ with $|W| \geq 2 \exists$ a unique $P^*(W) \in [0,P_{\text{mon}})$ such that $\pi(P^*(W)) = \pi(P^*(W),W)$.

Lemma 1.6 If $C''(\cdot) > 0$ then $\hat{P}(W) < \tilde{P} < P^*(W)$.

Lemma 1.7 If $C''(\cdot) < 0$ then $\hat{P}(W) > \tilde{P} > P^*(W)$.

Lemma 1.8 If $C''(\cdot) = 0$ then $\hat{P}(W) = \tilde{P} = P^*(W)$.

1.3.1 Convexity and concavity of the cost function

We now turn to the issue of equilibrium existence in the price-setting game. The results presented in this section, particularly with regards to the existence of a mixed strategy equilibrium, have only recently been established. As we shall see, the existence of equilibrium depends on the structure of the cost function and the way in which the market demand is shared at price ties.

Proposition 1.1 In a market with symmetric sellers with strictly convex costs and equal sharing at price ties, if A1-A4 are satisfied, then a vector of prices is a pure strategy Bertrand equilibrium iff all sellers quote the same price, $P'$, with $P' \in [\hat{P}(N),P^*(N)]$. 

17
Proof. First we prove the sufficiency part of the result. Suppose all sellers quote the same price $P' \in [\hat{P}(N), P^*(N)]$. The uniqueness of $P^*(N)$ implies that $\pi(P, N) > \pi(P)$ for all $P \in [0, P^*(N))$. The strict concavity of $\pi(P, N)$ and $\pi(\hat{P}(N), N) = 0$ imply that $\pi(P', N) \geq 0$. There are two possible price deviations. First a seller could deviate to a price $P'' < P'$. Then they would obtain $\pi(P'', N)$. As $P^*(N) < P_{mon}$ we have $\pi(P'', N) < \pi(P')$. Therefore $\pi(P'', N) < \pi(P', N)$ and this is not a profitable deviation. Second a seller could raise their price. Then they lose all the demand and earn zero profit. This is not an improvement on $\pi(P', N) \geq 0$. Therefore all sellers quoting $P' \in [\hat{P}(N), P^*(N)]$ is a pure strategy Bertrand equilibrium.

We now establish the necessity part of the result. There are two cases to consider. First, suppose all sellers tie at some price $P \notin [\hat{P}(N), P^*(N)]$. If $P < \hat{P}(N)$ then Lemma 1.4 implies $\pi(P, N) < 0$ and some seller would have a profitable deviation by setting price $\bar{P}$. If $P \in (P^*(N), P_{mon}]$ then Lemma 1.5 implies some seller could profitably undercut. Finally, if $P \in (P_{mon}, \bar{P}]$ then A4 implies that some seller could profitably deviate by setting price equal to $P_{mon}$. Second, suppose a set of sellers, $A \subset N, A \neq N$, tie at the minimum price. There are some sellers, $N \setminus A$, which are undercut, receive zero demand and earn zero profit. We now show that this cannot be an equilibrium. Let $x = 1/|A|$. The profit of the sellers tied at the minimum price is:

$$xD(P)P - C(xD(P)) \geq 0.$$  

The share of the market demand which a seller not tied at the minimum price could obtain by tying at the minimum price, which we shall denote by $y$, is:

$$y = 1/(|A| + 1).$$

The demand shares are such that:

$$0 < y < x.$$  

Therefore there exists a $\gamma \in (0, 1)$ such that:

$$\gamma x = y.$$
By the convexity of the cost function:

\[ \gamma C(xD(P)) + (1 - \gamma)C(0) > C(yD(P)). \]

As \( C_i(0) = 0 \) this simplifies to:

\[ \gamma C(xD(P)) > C(yD(P)). \]

As \( \gamma > 0 \) and \( (xD(P))P - C(xD(P)) \geq 0 \) we have:

\[ \gamma (xD(P))P - C(xD(P)) \geq 0. \]

\[ \gamma xD(P)P - \gamma C(xD(P)) \geq 0. \]

Then \( \gamma x = y \) yields:

\[ yD(P)P - \gamma C(xD(P)) \geq 0. \]

Then \( \gamma C(xD(P)) > C(yD(P)) \) gives:

\[ yD(P)P - C(yD(P)) > 0. \]

Which means a seller posting a price above the minimum price has a profitable deviation by joining the minimum price tie. ■

**Proposition 1.2** If \( P^C \) is a price-taking equilibrium, all sellers have strictly convex costs, there is equal sharing at price ties, and A1-A4 are satisfied, then all sellers quoting the price-taking equilibrium is a pure strategy Bertrand equilibrium. From the necessity part of Proposition 1.1, this implies \( P^C \in [\hat{P}(N), P^*(N)]. \)

**Proof.** If \( P^C \in (0, \bar{P}) \) is a price-taking equilibrium then we know that \( \sum_{i \in N} h_i(P^C) = D(P^C) \). As the sellers in the market are symmetric they have the same competitive supplies. Therefore we also have \( nh(P^C) = D(P^C) \) and \( h(P^C) = \frac{1}{n} D(P^C) \). Now suppose all sellers quote the price-taking equilibrium. As the market demand is shared equally each seller obtains profit of \( P^C h(P^C) - C(h(P^C)) = \pi^*(P^C) \). Suppose a seller were to deviate from this price. If a seller deviated to a \( P' < P^C \) then the maximum profit they could obtain
is $\pi^*(P')$. From Lemma 1.2 and $P' < P^C$ we have that $\pi^*(P') < \pi^*(P^C)$ and this is not a profitable deviation. Finally suppose a seller were to increase their price. Then they lose all the demand and earn zero profit. However, as $P^C > 0$ Lemma 1.2 implies that $\pi^*(P^C) > 0$ and this is not a profitable deviation. ■

It it worth making a number of comments about these results. First, the standard intuition that price competition between sellers producing a perfectly homogeneous good results in sellers making zero profit does not hold. For any price $P \in (\hat{P}(N), P^*(N)]$ the sellers earn strictly positive profits. The reasoning behind the result is that when sellers’ costs are convex then undercutting the market and serving all the market demand is not profitable because costs rise by a greater multiple than the extra revenue does. Second, as $P^C \in [\hat{P}(N), P^*(N)]$ the result gives a strategic foundation for price-taking behaviour as the competitive outcome can be supported as a Bertrand equilibrium. Finally, it is well-known that repeated interaction between sellers with constant marginal costs may permit collusive outcomes providing sellers do not discount future profits too much. In the model with strictly convex costs it is possible that $P'(N) \in [\hat{P}(N), P^*(N)]$. That is, the collusive outcome may be achievable in a one-shot price-setting game.\footnote{We shall not pursue this matter further here but the reader can find a more detailed account of this result in Dastidar (2001).} Having established that a pure strategy Bertrand equilibrium always exists with convex costs and equal sharing at price ties we now turn to the mixed extension of the price-setting game. The next two results were discovered by Hoernig (2002).

**Proposition 1.3** In a market with symmetric sellers with strictly convex costs and equal sharing at price ties, if A1-A4 are satisfied, then there is a continuum of symmetric mixed strategy Bertrand equilibria with $\text{supp}(\sigma) = [P', P^*(N)]$ and $\bar{P} < P' < P^*(N)$. In the mixed strategy equilibrium the measure $\sigma$ is atomic.

**Proof.** We shall show that one can construct a probability measure with the required properties. Let $F(P)$ be the cumulative probability distribution played in equilibrium. This distribution will be atomless on $[P', P^*(N)]$ and have an atom at $P^*(N)$. The expected
payoff in the equilibrium is equal to \( \pi(P') > 0 \) because any seller could always obtain this profit with certainty by pricing at \( P' \). Therefore if a seller is indifferent between playing any price in the interval \([P', P^*(N)]\) the distribution function \( F(P) \) must satisfy:

\[
(1 - F(P))^{n-1} \pi(P) = \pi(P').
\]

Rearranging this expression we obtain:

\[
F(P) = 1 - (\pi(P')/\pi(P))^{1/(n-1)}.
\]

The distribution function then satisfies \( F(P') = 0 \) and \( F'(P) > 0 \) for all \( P \in [P', P^*(N)] \). The size of the atom placed at \( P^*(N) \) is equal to \((\pi(P')/\pi(P^*(N)))^{1/(n-1)}\). As positive probability is placed at \( P^*(N) \) there must be no benefit from undercutting this price. That is, the expected profit from tieing with other sellers at this price must equal the monopoly profit at this price. This is satisfied as by definition \( P^*(N) \) is the price at which monopoly profit equals tied profit. ■

**Proposition 1.4** In a market with symmetric sellers with strictly convex costs and equal sharing at price ties, if A1-A4 are satisfied, then for any finite set of prices belonging to \((\hat{P}(N), P^*(N)]\) there is a symmetric mixed strategy Bertrand equilibrium in which sellers place probability mass on the finite number of prices.

**Proof.** Suppose we pick \( l \) prices belonging to the interval \((\hat{P}(N), P^*(N)]\) with \( l \geq 1 \). We shall show by induction that there is a mixed strategy equilibrium with the sellers mixing over these price. If \( l = 1 \) then this is a (degenerate) mixed strategy equilibrium. Suppose that \((P_1, ..., P_l)\) is a mixed strategy equilibrium over the \( l \geq 1 \) prices in which the expected payoff is \( \pi_l \). Consider another price \( P_{l+1} \) with \( \hat{P}(N) < P_{l+1} < P_1 \). The sellers are indifferent between playing \( P_{l+1} \) and playing the mixed strategy equilibrium \((P_1, ..., P_l)\) with probabilities \( p \) and \( 1 - p \) respectively if:

\[
\sum_{m=1}^{n} \left( \frac{n-1}{m-1} \right) p^{m-1}(1-p)^{n-m} \pi(P_{l+1}, W(m)) = (1-p)^{n-1} \pi_l \quad \text{with} \quad |W(m)| = m. \tag{1.7}
\]

The expression on the left-hand side is the expected profit from tieing at price \( P_{l+1} \) with the other sellers given that each seller plays \( P_{l+1} \) with probability \( p \). The right-hand side
is the expected payoff from playing the $l$-price mixed distribution $(P_1, \ldots, P_l)$. Evaluating the equation at $p = 0$ the left-hand side is equal to $\pi(P_{l+1})$ which is strictly less than the right-hand side which is equal to $\pi_l$ as $\pi(P_{l+1}) < \pi(P_l)$. Evaluating the equation at $p = 1$ the left-hand side is $p^{a-1}\pi(P_{l+1}, N)$ which is strictly greater than the right-hand side which is equal to zero. By the continuity of eq.(1.7) there exists a $p^* \in (0, 1)$ which makes the two sides equal. Then this gives a non-degenerate probability distribution $((1 - p^*)(P_1, \ldots, P_l), p^*)$ and the expected profit in this distribution is $\pi_{l+1} = (1 - p^*)^{n-1}\pi_l$. A routine check of possible deviations shows that sellers do not possess any profitable deviations from playing the non-degenerate mixed strategy equilibrium. ■

As noted at the beginning, the most striking result about Bertrand price competition is that the perfectly competitive outcome may be achieved by a small number of sellers acting as price-makers. We now present the standard ‘Bertrand paradox’ result in our general model.

**Proposition 1.5** In a market with symmetric sellers with linear costs and equal sharing at price ties, if A1-A4 are satisfied, then all sellers quoting the price $P' = \hat{P}(N) = P^*(N)$ is the unique symmetric pure strategy Bertrand equilibrium.

**Proof.** If costs are linear then $C(Q) = cQ$ with $c \in [0, \bar{P})$. Then $P' = c$ and if all sellers quote this price they make zero profit. If a seller undercuts this price by posting a price $P'' < c$ then the profit which they obtain is $(P'' - c)D(P'') < 0$ and this is not a profitable deviation. If a seller increases their price then they lose all demand and obtain zero profit which is not an improvement upon setting price equal to $c$. Therefore all sellers quoting marginal cost is a symmetric Bertrand equilibrium. To see that there are no other symmetric equilibria note that if all sellers quoted a price $P' < c$ then the profit they would obtain is $(P' - c)\frac{1}{n}D(P') < 0$ and one seller could profitably deviate by setting price equal to $c$. If all sellers quoted a price $P' \in (c, \bar{P})$ then as $\pi(P) > \pi(P, N)$ for all $P \in (c, \bar{P})$ one seller could profitably deviate to a price $P' - \epsilon$. Finally, if all sellers posted $\bar{P}$ then one seller could profitably deviate by posting $P^{mon}$ in the market. Therefore $P' = c$ is the unique symmetric pure strategy Bertrand equilibrium. ■
It worth noting that the Bertrand paradox outcome is such that each seller plays a weakly dominated strategy (pricing at marginal cost is weakly dominated by any price above marginal cost but below $\bar{P}$). Therefore, the equilibrium is not trembling-hand perfect and is not a best response to a rival playing a completely mixed strategy. This property of the equilibrium is important when we consider the existence of a Bayesian Bertrand equilibrium. The mixed extension of the Bertrand game with linear costs was first analyzed by Harrington (1989) who showed that all equilibria of the game must have at least two sellers placing all probability mass at marginal cost. We now present this result.

**Proposition 1.6** In a duopoly market, $n = 2$, with symmetric sellers with linear costs and equal sharing at price ties, if $A1$-$A4$ are satisfied, then the unique mixed strategy Bertrand equilibrium is for both sellers to place all probability mass on $P' = \hat{P}(N) = P^*(N)$.

**Proof.** Suppose each seller plays a mixed strategy given by probability measure $\sigma_i$. Let $P'_i = \inf(supp(\sigma_i))$ and $P''_i = \sup(supp(\sigma_i))$. Suppose also that each seller has a marginal cost of production given by $c \in (0, \bar{P})$. First, we must have $\min\{P'_1, P'_2\} \geq c$ or one would make negative expected profits in equilibrium and has a profitable deviation by shifting all probability mass to $c$. Now suppose that $\max\{P''_1, P''_2\} = P_{max} > c$ and $P_{max} \leq \bar{P}$. At most only one seller has an atom on $P_{max}$ because if both sellers had an atom at this point one seller could profitably deviate by shifting probability mass below $P_{max}$. The seller placing an atom at $P_{max}$ is then undercut with certainty and earns zero profit in equilibrium. If no seller has an atom at $P_{max}$ then at least one seller has probability mass arbitrarily close to $P_{max}$ as this is the supremum of the supports. This seller is then undercut with certainty and earns zero profit in equilibrium. Therefore at least one seller earns zero profit in any equilibrium.

Suppose the seller earning zero profit is seller 1. If $P''_2 > c$ then seller 1 could earn positive expected profit by shifting all probability mass to a price $P \in (c, P''_2)$. Therefore we must have $P''_2 = c$. However if, $P''_2 = c$ and $P''_1 > c$ then seller 2 has a profitable deviation by shifting all probability mass to a price $P \in (c, P''_1)$. Therefore we must have $\max\{P''_1, P''_2\} = c$ and $\min\{P'_1, P'_2\} \geq c$ which together imply that both sellers place all probability mass on $c$. 

$\blacksquare$
Surprisingly, the Bertrand game does not possess an equilibrium in either pure or mixed strategies when sellers have strictly concave costs and the market demand is shared equally at price ties. Shapiro (1989) was the first to note that adding an avoidable fixed cost to the model with constant variable cost results in the non-existence of pure strategy equilibrium. The non-existence of a mixed strategy equilibrium has, for a long time, been an open question. In a forthcoming paper Dastidar (2010) proves that there is no mixed strategy equilibrium. Here we provide an original proof, similar to a proof presented by Baye and Kovenock (2008), to establish the non-existence of mixed strategy equilibrium.

Lemma 1.9 In a market with symmetric sellers with strictly concave costs and equal sharing at price ties if $\pi(P,W) \geq 0$ with $2 \leq W \leq n$ then $\pi(P) > \pi(P,W)$.

Proof. See the Appendix.

Proposition 1.7 In a market with symmetric sellers with strictly concave costs and equal sharing at price ties, if A1-A4 are satisfied, then there does not exist any pure strategy Bertrand equilibrium.

Proof. There are two possible cases to consider. The first is when the sellers play a symmetric pure strategy equilibrium. The second is when the sellers play an asymmetric pure strategy equilibrium. We consider the cases separately.

Case 1. Suppose the sellers quote the same price $P_i = P'$ for all $i \in N$. From Lemma 1.7 we know that $\hat{P}(W) > \tilde{P} > P^*(W)$. If $P' < \hat{P}(N)$ then $\pi(P',N) < 0$ and one seller could profitably deviate to any price greater than $P'$ and earn zero profit. If $P' \geq \hat{P}(N)$ then $\pi(P', N) \geq 0$. From Lemma 1.9 we know that $\pi(P') > \pi(P', N) \geq 0$. Then one seller could deviate to a price $P' - \epsilon$ and earn strictly higher profits provided that $\epsilon > 0$ is sufficiently small. Therefore there does not exist any symmetric pure strategy Bertrand equilibrium.

Case 2. Suppose the sellers play an asymmetric pure strategy equilibrium. Let $P'$ denote the lowest price quoted by the sellers and let $P''$ denote the highest price quoted by the sellers. Suppose that the set of sellers posting the minimum price is $W$ with $2 \leq |W| \leq n - 1$. From Lemma 1.7 we know $\hat{P}(W) > \tilde{P} > P^*(W)$. Exactly the same argument in case one shows
that there is no price at which more than one seller can tie and no seller has a profitable deviation. Therefore the seller posting $P'$ must be unique. If $P' < \tilde{P}$ then this seller would make negative profits and would have a profitable deviation by posting price $\tilde{P}$ which guarantees zero profit. If $P' = \tilde{P}$ then the seller makes zero profit and could profitably deviate to a price $P' + \epsilon$ which is still strictly less than the other sellers’ prices and earn $\pi(P' + \epsilon) > 0$. Finally, suppose that $P' > \tilde{P}$. Then the seller posting $P''$ could profitably deviate to a price $P \in (\tilde{P}, P')$ and earn strictly positive profit. Therefore there does not exist any pure strategy Bertrand equilibrium. ■

**Proposition 1.8** In a duopoly market, $n = 2$, with symmetric sellers with strictly concave costs and equal sharing at price ties, if A1-A4 are satisfied, then there does not exist any mixed strategy Bertrand equilibrium.

*Proof.* Suppose each seller plays a mixed strategy given by probability measure $\sigma_i$, $i = 1, 2$. Let $P_i' = \inf(\text{supp}(\sigma_i))$ and $P_i'' = \sup(\text{supp}(\sigma_i))$. If $\min\{P_1', P_2'\} < \tilde{P}$ then one seller would make negative expected profit in equilibrium and would have a profitable deviation by shifting all probability mass to $\tilde{P}$. Therefore we must have $\min\{P_1', P_2'\} \geq \tilde{P}$. Let $P_{\max} = \max\{P_1'', P_2''\}$. We know from Proposition 1.7 that there does not exist any pure strategy Bertrand equilibrium therefore it must be that $P_{\max} > \tilde{P}$ in any mixed strategy equilibrium. Otherwise $\min\{P_1', P_2'\} \geq \tilde{P}$ and $P_{\max} \leq \tilde{P}$ imply that both sellers place all probability mass at $\tilde{P}$ which is not an equilibrium point. Now at most one seller places an atom at $P_{\max}$. If both sellers placed an atom at $P_{\max}$ then one seller could profitably deviate by shifting probability mass to a lower price. The seller placing an atom at $P_{\max}$ is then undercut almost surely and must earn zero expected profit in any equilibrium. If no seller places an atom at $P_{\max}$ then as this is the maximum of the supremums of the supports then at least one seller has probability mass in any open interval of $P_{\max}$ and must be undercut almost surely. Therefore at least one seller earns zero expected profit in any mixed strategy equilibrium. Suppose seller 1 earns zero expected profit. If $P_2'' > \tilde{P}$ then as $\sigma_2$ can have at most countably many atoms there must exist a price $P \in (\tilde{P}, P'')$ which has measure zero under $\sigma_2$, seller 1 could shift all probability mass to this point and earn
positive expected profit. If $P''_2 = \tilde{P}$ then seller 2 would earn zero profit. As $P_{\text{max}} > c$ seller 2 could deviate to a price $P \in (\tilde{P}, P''_1)$ which has measure zero under $\sigma_1$ and earn positive expected profit. Therefore if no seller has a profitable deviation then $P_{\text{max}} \leq \tilde{P}$. However, $P_{\text{max}} \leq \tilde{P}$ and $\min\{P'_1, P'_2\} \geq \tilde{P}$ imply that both sellers must place all probability mass on $\tilde{P}$ which is not an equilibrium point. We can conclude that there does not exist any mixed strategy Bertrand equilibrium. ■

Up until now we have assumed that all sellers in the market are symmetric. This is clearly unrealistic but is a convenient theoretical assumption. However, even in a market with asymmetric convex costs a pure strategy Bertrand equilibrium exists provided that the market demand is split according to capacity sharing at price ties. The case where sellers have constant marginal costs is often cited as a problem in textbooks to show that no well-defined equilibrium exists. In fact, this is not the case if one permits mixed strategies as was first shown by Deneckere and Kovenock (1992) and later rediscovered by Blume (2003). We now present the results.

**Proposition 1.9** In a market with sellers with strictly convex costs and capacity sharing at price ties, if A1-A4 are satisfied, then there exists a pure strategy Bertrand equilibrium.

**Proof.** We proceed in two steps. First, we show that under if the assumptions are satisfied then the market possesses a price-taking equilibrium. Second, we show that no seller can profitably deviate from quoting the price-taking equilibrium.

**Step 1.** Define the excess demand function of the market to be $f(P) = D(P) - \sum_{i \in N} h_i(P)$. As this is the sum of continuous functions $f(P)$ is continuous in $P$. Now $f(0) = D(0) > 0$ and $f(\bar{P}) = -\sum_{i \in N} h_i(\bar{P}) < 0$ as from Lemma 1.1 $h_i(0) = 0$ and $h'_i(\bar{P}) > 0$. As $f(P)$ is continuous the intermediate value theorem guarantees that there exists a $P^C \in (0, \bar{P})$ such that $f(P^C) = 0$.

**Step 2.** Suppose all sellers quote price $P^C$. The profit which they obtain at this price, given that demand is shared according to capacity sharing, is:

$$\frac{h_i(P^C)}{\sum_{i \in N} h_i(P^C)} P^C D(P^C) - C_i\left(\frac{h_i(P^C)}{\sum_{i \in N} h_i(P^C)} D(P^C)\right).$$
Now as $\sum_{i \in N} h_i(P^C) = D(P^C)$ the profit simplifies to give:

$$P^C h_i(P^C) - C_i(h_i(P^C)) = \pi^*_i(P^C).$$

Now suppose that a seller deviates from quoting the price-taking equilibrium. If a seller posts a price $P' < P^C$ the maximum profit they could obtain is $\pi^*_i(P')$ and from Lemma 1.2 we know that $\pi^*_i(P') < \pi^*_i(P^C)$ and this is not a profitable deviation. If a seller raises their price then they obtain zero profit which cannot be an improvement as $\pi^*_i(P^C) > 0$. Therefore all sellers quoting the price-taking equilibrium is a pure strategy Bertrand equilibrium. ■

**Proposition 1.10** In a duopoly market, $n = 2$, with asymmetric linear costs and equal sharing at price ties, if A1-A4 are satisfied, then there exists a continuum of mixed strategy Bertrand equilibria.

**Proof.** Suppose the marginal costs of sellers 1 and 2 are given by $c_2 > c_1 > 0$. Let $P_{\text{mon}}(c_1) = \arg \max_{P \in [0, \bar{P}]} \pi_1(P)$ so $P_{\text{mon}}(c_1)$ is the monopoly price for seller 1. Also we shall retain the assumption that each seller’s monopoly profit function is strictly concave in price. This then ensures that $P_{\text{mon}}(c_1)$ is unique. If $c_2 > P_{\text{mon}}(c_1)$ then it is clear that seller 1 can post price $P_{\text{mon}}(c_1)$ and obtain its monopoly profit. seller 2 can do no better than posting any price strictly above $P_{\text{mon}}(c_1)$. However, if $c_2 \leq P_{\text{mon}}(c_1)$ then if both sellers post prices equal to $c_2$ they split the demand equally and seller 1 could profitably deviate by posting a price $c_2 - \epsilon$, $\epsilon > 0$. This is the well-known “open-set” problem of the Bertrand equilibrium with asymmetric costs. We shall show that there is a mixed strategy equilibrium with seller 1 posting price equal to $c_2$ and seller 2 mixing on an interval $(c_2, c_2 + \eta]$ with $\eta > 0$ being arbitrarily small. Let $F(P, \eta)$ denote the cumulative distribution function of the mixing distribution seller 2 plays. For simplicity suppose $F(P, \eta)$ is uniform. If $f(P, \eta)$ is the density function then as $\eta \to 0$, $f(P, \eta) \to \infty$. The expected profit which seller 1 could obtain from posting a price $P \in (c_2, c_2 + \eta]$ is given by:

$$(1 - F(P, \eta))(P - c_1)D(P).$$

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If this is not a profitable deviation we require that the derivative of this expected profit function is negative everywhere on \((c_2, c_2 + \eta]\). The derivative is given by:

\[
-f(P, \eta)D(P)(P - c_1) + (1 - F(P, \eta))(D'(P)(P - c_1) + D(P)).
\]

As \(D(P)\) is \(C^2\) all the terms are bounded. Therefore as \(f(P, \eta) \to \infty\) as \(\eta \to 0\) if \(\eta\) is sufficiently small then seller 1 can do no better than to post price \(c_2\). The open-set problem is solved because seller 2 posts a price higher than \(c_2\) almost surely. Note that there are different types of mixed strategies which seller 2 could play. One alternative is for seller 2 to mix according to:

\[
F(P) = 1 - \frac{\pi_1(c_2)}{\pi_1(P)}
\]

on the interval \((c_2, P^{\text{mon}}(c_1))\) and place an atom equal to \((\pi_1(c_2)/\pi_1(P^{\text{mon}}(c_1)))\) at \(P^{\text{mon}}(c_1)\).

Finally, the same argument can be made for seller 1 playing any price in the interval \(P \in (c_1, c_2)\) with seller 2 mixing on an interval above. ■

We now turn to the final sharing rule which we introduced at the beginning: winner-takes-all sharing. It turns out that equilibrium existence under this rule is guaranteed under weak conditions on the primitives.

**Proposition 1.11** In a market with symmetric sellers and winner-takes-all sharing at price ties, if A1-A4 are satisfied, then there exists a pure strategy Bertrand equilibrium in which all sellers quote \(\tilde{P}\).

**Proof.** If all sellers quote \(\tilde{P}\) then the expected payoff to each seller is \(\frac{1}{n}\pi(\tilde{P}) = 0\). If a seller were to deviate to a \(P' < \tilde{P}\) then their profit would be \(\pi(P') < 0\) which is not a profitable deviation. If a seller were to increase their price they would lose any probability of serving the market and earn zero profit. Therefore all sellers quoting \(\tilde{P}\) is a pure strategy Bertrand equilibrium. ■

Note that this result does not depend upon the convexity/concavity of the cost function. Moreover, the result is a type of Bertrand paradox result in that all sellers earn zero expected profits at the equilibrium. Therefore it tends to be suggested that the winner-takes-all sharing rule is competitive. Nevertheless, we know from the previous results that if costs
are strictly convex then $\hat{P}(N) < \bar{P}$ so there always exists a lower equilibrium price under the equal sharing rule compared with winner-takes-all sharing. Also, it is possible that the equilibrium price under winner-takes-all sharing is higher than under Cournot competition.\footnote{The reader can check that in a market with four symmetric sellers, $n = 4$, with costs given by $C(Q) = Q^2$ and market demand given by $D(P) = \max\{0, 10 - P\}$ the unique symmetric Cournot equilibrium results in a lower price than under winner-takes-all competition.}

One can go further by permitting discontinuities in the monopoly profit function and still guarantee that an equilibrium exists and that sellers earn zero profit in any equilibrium points. We present this result below.

**Definition 1.4** A market with symmetric sellers and monopoly profit function $\pi(P)$ possesses an initial breakeven price if there exists a $c \in \mathbb{R}_+$ such that $\pi(c) = 0$ and $P < c$ implies $\pi(P) < \pi(c)$.

**Definition 1.5** A real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ is left lower semicontinuous if for all $x' \in \mathbb{R}$ we have $\liminf_{x \uparrow x'} f(x) \geq f(x')$.

**Proposition 1.12** In a market with symmetric sellers and winner-takes-all sharing at price ties if (i) there exists an initial breakeven price (ii) $\pi(P)$ is left lower semicontinuous and (iii) $\pi(P)$ is bounded above then all sellers earn zero expected profits in any Bertrand equilibrium and such an equilibrium always exists.

**Proof.** We refer the reader to Baye and Morgan (2002). ■

**Example 1.1**

We now illustrate some of the results presented in this section by a couple of examples. Consider a duopoly market $N = \{1, 2\}$ in which the market demand is piecewise-affine and $D(P) = \max\{0, 10 - P\}$. Each seller has a symmetric cost function equal to $C(Q) = Q^2$. Therefore the sellers’ costs are strictly convex. The monopoly profit and shared profit functions are:

$$\pi(P) = P(10 - P) - (10 - P)^2$$
\[ \pi(P, N) = \frac{1}{2} P(10 - P) - \frac{1}{4}(10 - P)^2. \]

Routine calculations then reveal that \( \tilde{P} = 5 \), \( \hat{P}(N) = \frac{10}{3} \) and \( P^*(N) = 6 \). Therefore both sellers quoting any price in the interval \([\frac{10}{3}, 6]\) is a pure strategy Bertrand equilibrium. It is also worth noting that the competitive supply of each seller is \( h(P) = \frac{P}{2} \) and the price-taking equilibrium is \( P^C = 5 \). Now to see that there is a mixed strategy equilibrium suppose each seller plays a mixed strategy given by the following distribution function:

\[
F(P) = \begin{cases} 
1 - \frac{9}{(20-2P)(2P-10)} & \text{if } P \in \left[\frac{11}{2}, 6\right); \\
1 & \text{if } P = 6.
\end{cases}
\] (1.8)

The distribution function \( F(P) \) is atomic at \( P = 6 \). A routine check shows that no seller can profitably deviate if both mix according to \( F(P) \). In Figure 1 we illustrate the monopoly and shared profit functions and the set of pure strategy Bertrand equilibria. Now suppose we again have a duopoly market we the market demand as above but each seller’s cost function is given by \( C(Q) = 5Q - \frac{1}{6}Q^2 \) then \( C'(Q) = 5 - \frac{1}{3}Q > 0 \) for all \( Q \in [0, 10] \) and \( C''(Q) = -\frac{1}{3} \).

Therefore the cost function is strictly concave. Routine calculations reveal that \( \tilde{P} = 4 \), \( \hat{P}(N) = \frac{50}{11} \) and \( P^*(N) = \frac{10}{3} \). In this case there does not exist any pure or mixed strategy Bertrand equilibrium. In Figure 2 we illustrate the monopoly and shared profit function so the reader can contrast them with the case of strictly convex costs.

### 1.3.2 Unbounded profits

In deriving the results above we assumed that the market demand possesses a finite choke-off price which we denoted by \( \bar{P} \). Relaxing this assumption might not seem to make much difference to the results. However, under certain conditions the non-existence of a finite choke-off price can result in equilibria which are quite different from the Bertrand paradox outcome. We now state the next result which was originally discovered by Baye and Morgan (1999).

**Proposition 1.13** In a market with symmetric sellers with linear costs and equal sharing at price ties if \( \pi(P) \) is continuous and \( \pi'(P) > 0 \) for all \( P \in (0, \infty) \) then there is a continuum of
Figure 1.1: The set of Bertrand equilibria with strictly convex costs

Figure 1.2: Non-existence of pure and mixed strategy Bertrand equilibria
atomless mixed strategy Bertrand equilibria in which all sellers earn positive expected profits.

Proof. Suppose each sellers has a marginal cost of $c > 0$ then pick any $P' > c$. We shall now show that there is a mixed strategy equilibrium in which each seller earns expected profits of $\pi(P')$. Consider the case where each sellers mixes according to an atomless cumulative distribution function $F(P)$ on $[P', \infty)$. If each seller is indifferent between playing any price in the support, which must be the case in any mixed strategy equilibrium then we require $F(P)$ to satisfy:

$$(1 - F(P))^{n-1} \pi(P) = \pi(P').$$

Rearranging this we obtain a closed form expression for the mixing distribution:

$$F(P) = 1 - \left(\frac{\pi(P')}{\pi(P)}\right)^{\frac{1}{n-1}}.$$  

As $\pi(P)$ is continuous, $F(P)$ is continuous and as $\pi'(P) > 0$ we know $\lim_{P \to \infty} \pi(P) = \infty$. This implies $\lim_{P \to \infty} F(P) = 1$. Therefore $F(P)$ has all the required properties of an atomless distribution function. Given that each seller is indifferent between playing any price in the interval $[P', \infty)$ the only possible deviation for a seller is to play some price $P < P'$. However, as any seller can obtain $\pi(P')$ because all sellers price above $P'$ almost surely it is not profitable to post any price less than $P'$.

1.3.3 Superadditivity and subadditivity

The result on equilibrium existence have tended to be stated in terms of the convexity/concavity of the cost function. A number of recent papers, including Saporiti and Coloma (2010) and Dastidar (2010) have found that there is a weaker concept, that of superadditivity/subadditivity, which can be used to characterize equilibrium existence in Bertrand games. We now introduce this notion.

Definition 1.6 A cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly subadditive (superadditive) on $\mathbb{R}_+$ if and only if $C(x + y) < (>) C(x) + C(y)$ for all $x, y \in (0, \infty)$.

The intuition behind strict subadditivity is that it costs less to produce any given quantity than to split the quantity up and produce it separately.
Lemma 1.10 If $C(\cdot)$ is strictly concave on $\mathbb{R}_+$ then it is strictly subadditive on $\mathbb{R}_+$.

Proof. See the Appendix.

Lemma 1.11 If $C(\cdot)$ is strictly convex on $\mathbb{R}_+$ then it is strictly superadditive on $\mathbb{R}_+$.

Proof. See the Appendix.

The proofs of the following two results can be found in Dastidar (2010).

Proposition 1.14 In a market with symmetric sellers with strictly superadditive costs and equal sharing at price ties, if A1-A4 are satisfied, then there is a continuum of pure strategy Bertrand equilibria.

Proposition 1.15 In a market with symmetric sellers with strictly subadditive costs and equal sharing at price ties, if A1-A4 are satisfied, then there does not exist any Bertrand equilibrium in either pure or mixed strategies.

The results in Propositions 1.14 and 1.15 together with Lemmas 1.10 and 1.11 then yield Propositions 1.1, 1.7 and 1.8 as special cases.

1.3.4 Comparison with Bertrand-Edgeworth games

At this point it is interesting to compare the results we have established with the other canonical model of price competition: Bertrand-Edgeworth competition. As noted in the introduction, most of the early research on partial equilibrium price-setting games analyzed models where sellers had capacity constraints or were free to ration the market demand. In contrast, the model of Bertrand competition assumes that a seller must meet all the demand forthcoming- an assumption which has been justified by there being large costs of turning consumers away in certain markets. In this section we shall focus on a duopoly market, $n = 2$, and analyze the existence of equilibrium when sellers are free to ration the market demand. Moreover, we shall assume A1-A2 are satisfied and we shall stick with the
assumption that if sellers tie at the same price they split the market demand equally. If the sellers tie at the same price then the demand forthcoming to seller $i$ is:

$$d_i(P) = \frac{1}{2}D_i(P) \quad \text{if} \quad P_i = P_j.$$  

At any given price a seller would never serve more than their competitive supply $h_i(P)$. Therefore a seller may raise its price above its rival’s price and still receive market demand because its rival does not serve all the demand. We shall assume that the market demand is rationed according to the surplus-maximizing rule which means the consumers who value the good most highly buy from the lowest priced seller.\footnote{For a comprehensive survey of the alternative rationing rule, that of proportional rationing, the reader is referred to Vives (1999, Ch.5).} Formally, if $P_i > P_j$ then the demand which seller $i$ receives is:

$$d_i(P) = \max\{0, D_j(P) - h_j(P_j)\}.$$  

The profits of the sellers are a mapping $\Pi_i : S \rightarrow \mathbb{R}$ with $S = [0, \bar{P}]^2$ with $\Pi_i(P_i, P_j) = P_i x_i - C_i(x_i)$ and $x_i = \min\{h_i(P_i), d_i(P_i)\}$. We now state a result contained in Dasgupta and Maskin (1986) regarding the existence of equilibrium is discontinuous games.

\textbf{Theorem 1.1} In a game with a finite number of players with non-empty and compact strategy spaces, $S_i$, if the payoffs $u_i$ are continuous except on the set (of measure zero) $\hat{S} = \{s \in \times_{i \in N} S_i : s_i = s_j\}$, $\sum_{i \in N} u_i$ is upper semicontinuous and $u_i$ is bounded and left lower semicontinuous in own strategy then the game possesses a mixed strategy equilibrium.

We now present the following result, established by Dixon (1984) which shows that if sellers have symmetric, weakly convex costs, then the Bertrand-Edgeworth duopoly possesses a mixed strategy equilibrium.

\textbf{Proposition 1.16} In the Bertrand-Edgeworth duopoly with payoffs given by $\Pi_i : S \rightarrow \mathbb{R}$ and symmetric weakly convex costs there exists a mixed strategy equilibrium.

\textit{Proof.} We can note that the strategy spaces $S_i = [0, \bar{P}]$ are non-empty and compact. The profits of the sellers $\Pi_i$ are discontinuous on the set of prices $\hat{S} = \{(P_i, P_j) \in [0, \bar{P}]^2 : P_i = P_j\}$
which has measure zero in the joint strategy space. As the market demand is bounded the profits are also bounded.

All that remains to be shown, to apply Theorem 1, is that \( \Pi_i(P_i, P_j) \) is left lower semicontinuous in \( P_i \) and \( \sum_{k \in \{i,j\}} \Pi_k(P_i, P_j) \) is upper semicontinuous. If \( \Pi_i(P_i, P_j) \) is left lower semicontinuous in \( P_i \) then as seller \( i \) increases its price \( \Pi_i(P_i, P_j) \) should “jump down” at any points of discontinuity approached from the left. Formally:

\[
\lim \inf_{P \uparrow P_i} \Pi_i(P, P_j) \geq \Pi_i(P_i, P_j).
\]

As the only points of discontinuity are \( \dot{S} = \{(P_i, P_j) \in [0, \bar{P}]^2 : P_i = P_j\} \) this condition is trivially satisfied: if a seller raises its price up to the point it ties with its rival’s price then, given the equal sharing at price ties, its demand and profits suffer a discontinuous drop. To see that the sum of the profits is upper semicontinuous note that:

\[
\sum_{k \in \{i,j\}} \Pi_k(P_i, P_j) = \sum_{k \in \{i,j\}} P_kx_k - \sum_{k \in \{i,j\}} C(x_k).
\]

The revenue part of the expression, \( \sum_{k \in \{i,j\}} P_kx_k \), is continuous in prices, and therefore upper semicontinuous. Then \( \sum_{k \in \{i,j\}} \Pi_k(P_i, P_j) \) is upper semicontinuous if \( -\sum_{k \in \{i,j\}} C(x_k) \) is upper semicontinuous. We now show that this is the case. Consider a sequence of prices \( (P_{in}, P_{jn}) \to (P^*_i, P^*_j) \). This sequence of prices we generate a sequence of corresponding outputs, \( (x_{in}, x_{jn}) \), for each of the sellers. Now although individual seller’s outputs may change discontinuously as the prices change the total output changes continuously in price. To see this let \( P^\text{max}_n = \max\{P_{in}, P_{jn}\} \) then the total demand is:

\[
x_n = x_{in} + x_{jn} = \min\{D(P_n^\text{max}), h(P_{in}) + h(P_{jn})\}.
\]

It is clear that the expression on the right-hand side is continuous in prices. It is now a simple matter to show that \( \sum_{k \in \{i,j\}} \Pi_k(P_i, P_j) \) is upper semicontinuous in prices. First, suppose \( (P^*_i, P^*_j) \in S \setminus \dot{S} \) and we have \( P^*_i \neq P^*_j \) and \( x_{kn} \to x^*_k \) then:

\[
\lim_{(P_{in}, P_{jn}) \to (P^*_i, P^*_j)} \sum_{k \in \{i,j\}} C(x_{kn}) = \sum_{k \in \{i,j\}} C(x^*_k).
\]
Second, suppose \((P^*_i, P^*_j) \in \hat{S}\) then by the convexity of the cost function we have:

\[
\lim_{(P^i_n,P^j_n) \to (P^*_i,P^*_j)} \sum_{k \in \{i,j\}} C(x_{kn}) \geq 2C\left(\frac{x^*_i + x^*_j}{2}\right).
\]

These then imply that:

\[
\lim_{(P^i_n,P^j_n) \to (P^*_i,P^*_j)} \sum_{k \in \{i,j\}} \Pi_k(P^i_n, P^j_n) \leq \sum_{k \in \{i,j\}} \Pi_k(P^*_i, P^*_j).
\]

Therefore the conditions of Theorem 1.1 are satisfied and we can conclude that there exists a mixed strategy equilibrium. ■

**Proposition 1.17** In the Bertrand-Edgeworth duopoly with payoffs given by \(\Pi_i : S \to \mathbb{R}\) and symmetric concave costs there does not exist any pure or mixed strategy equilibrium.

**Proof.** To establish the result it is sufficient to note that a seller with concave costs would be willing to supply all the demand forthcoming provided the market price is \(P \geq \tilde{P}\). Therefore if a Bertrand-Edgeworth equilibrium were to exist then it would also be a Bertrand equilibrium and we know from Propositions 1.7 and 1.8 that there does not exists a Bertrand equilibrium in pure or mixed strategies when sellers have concave costs. ■

### 1.4 Uncertainty and Bayesian equilibrium

In the previous sections it has been assumed that the characteristics of the market, the number of sellers, each seller’s cost type etc., were known by each seller in the market. This is not particularly realistic but it makes the models tractable. In this section we consider whether it is possible to relax the assumption that the characteristics of the market are known with certainty. Generally speaking, equilibrium existence is not easily established in price-setting games with incomplete information. However, we shall consider three particular aspects of market competition which have been successfully modeled. First, we consider what equilibrium prices are posted by sellers when the set of sellers active in the market is uncertain. Second, we consider what properties the equilibrium has when sellers have
incomplete information regarding the cost types of other sellers. Finally, we study a market model where the quality of the good traded by sellers is not known with certainty so that prices may play an additional role of signaling the quality of the good traded in the market. Despite the problems analyzed here being different to the previous sections, the reader will find that we still stick with the general framework of Bertrand competition which we have analyzed so far.

1.4.1 Pricing under uncertainty

Consider a market in which there is a fixed set of sellers but each seller has a probability \( \alpha \in (0, 1) \) of being inactive in the market. If a seller is inactive then it does not post a price in the market. The following result regarding equilibrium existence was established by Janssen and Rasmusen (2002).

**Proposition 1.18** In a market with symmetric sellers, linear costs, equal sharing at price ties and probability \( \alpha \in (0, 1) \) that each seller is inactive in the market then, if A1-A4 are satisfied, there does not exist a pure strategy Bertrand equilibrium but there does exist a mixed strategy equilibrium.

**Proof.** Suppose each seller has a marginal cost of \( c > 0 \). To see that there does not exist a pure strategy Bertrand equilibrium suppose each seller posted a price \( P_i \) and assume, without loss of generality that \( c \leq P_1 \leq P_2 \ldots \leq P_n \leq \bar{P} \). First, note that \( P_1 \) must be strictly greater than \( c \) because seller 1 could deviate to a price \( P \in (c, \bar{P}) \) and earn expected profit of \( \alpha^{n-1}\pi(P) > 0 \). Now as \( \pi(P) > \pi(P, W) \) whenever \(|W| \geq 2\) no seller would want to tie with any other seller at the same price. Therefore the best response of seller 1 is to post price \( P_1 = P_2 - \epsilon, \epsilon > 0 \). However, this means seller 2 would have a profitable deviation by posting \( P_2 - \epsilon \). This possibility of undercutting, and no sellers wanting to tie at the same price mean there is no figuration of prices with \( c < P_1 \leq P_2 \ldots \leq P_n \leq \bar{P} \) such that no seller has a profitable deviation.

To see that there exists a mixed strategy equilibrium first note that there exists a unique \( P' \in (c, P^{mon}) \) such that \( \pi(P') = \alpha^{n-1}\pi(P^{mon}) \). This is the case because \( \pi(P) \) is continuous
in price, \( \pi(c) = 0 \) and \( 0 < \alpha^{n-1}\pi(P_{mon}) < \pi(P_{mon}) \) so the intermediate value theorem guarantees that there exists such a \( P' \in (c, P_{mon}) \). The uniqueness follows from the strict concavity of \( \pi(P) \) which implies \( \pi'(P) > 0 \) for all \( P \in (0, P_{mon}) \). Now suppose each seller plays an atomless mixed strategy on the interval \([P', P_{mon}]\). We shall denote the mixed strategy by the cumulative distribution function \( F(P) \). If the expected payoff in the equilibrium is given by \( U \geq 0 \) then if each seller is indifferent between playing each price in the interval \([P', P_{mon}]\) we require that \( F(P) \) satisfy:

\[
\sum_{r=0}^{n-1} \binom{n-1}{r} (\alpha)^{n-1-r}(1-\alpha)^{n-1}(1-F(P))^r \pi(P) = U \quad \text{for all} \quad P \in [P', P_{mon}].
\]

Setting \( F(P_{mon}) = 1 \) in the left-hand side of the equation gives \( U = \alpha^{n-1}\pi(P_{mon}) \). We can rewrite the expected payoff equation as:

\[
\sum_{r=0}^{n-1} \binom{n-1}{r} (\alpha)^{n-1-r}(1-\alpha)^{n-1}(1-F(P))^r = \frac{\alpha^{n-1}\pi(P_{mon})}{\pi(P)}.
\]

Then setting \( P = P' \) on the right-hand side, and using the fact that \( \pi(P') = \alpha^{n-1}\pi(P_{mon}) \), implies \( F(P') = 0 \). As the expected payoff equation is a continuous function it can be checked that there is an implied value of \( F(P) \) which has all the required properties of an atomless distribution function on \([P', P_{mon}]\).

Now suppose each seller mixes according to \( F(P) \) and earn an expected payoff of \( \alpha^{n-1}\pi(P_{mon}) \). Consider what happens if a seller deviates from this. If a seller deviated to, or shifting probability mass onto, a price \( P \geq P_{mon} \) then the expected profit from playing this price is \( \alpha^{n-1}\pi(P) < \alpha^{n-1}\pi(P_{mon}) \). Therefore this is not a profitable deviation. If a seller deviates to a price \( P < P' \) then they earn \( \pi(P) \) with certainty. However, they could always earn \( \pi(P') \) with certainty as \( F(P) \) is atomless. We can conclude that there is not profitable deviations for the sellers and \( F(P) \) is a mixed strategy equilibrium.

\[ \blacksquare \]

### 1.4.2 Existence of Bayesian equilibrium

In contrast to the previous section suppose that the set of sellers is fixed but that each seller has incomplete information regarding the cost types of their rivals. Specifically, suppose that
each seller has a constant marginal cost but that this cost is uncertain. There are two types of marginal cost, high and low, with \( c_L < c_H \), and each seller has probability \( \alpha \in (0, 1) \) of having marginal cost \( c_L \) and probability \( 1 - \alpha \) of having marginal cost \( c_H \). We shall assume that \( \pi_i(P) \) is strictly concave in price and \( P^\text{mon}_L > c_H \) so the costs of the sellers are not too far apart. As we showed earlier the existence of equilibrium with asymmetric marginal costs has been a problematic issue. The case of uncertain marginal cost is a more demanding issue as we have to establish the existence of a Bayesian equilibrium for each type. The following result was established by Routledge (2010).

**Proposition 1.19** In a market with sellers which have linear costs and uncertain marginal cost which is \( c_L \) with probability \( \alpha \in (0, 1) \) and \( c_H \) with probability \( 1 - \alpha \), with \( c_L < c_H \), then if there is equal sharing at price ties, and A1-A4 are satisfied, then there exists a mixed strategy Bayesian Bertrand equilibrium.

**Proof.** Let \( \pi_i(P) = (P - c_i)D(P) \) denote the monopoly profit function of a seller with cost type \( i \) and let \( P^\text{mon}_i \) be the monopoly price of a seller with cost type \( i \). A routine check of possibly pure strategies for low-cost sellers, similar to that in the previous Proposition, shows that there is no pure strategy Bayesian equilibrium as there are no equilibrium prices for low-cost types. We shall show that there exists a mixed strategy equilibrium in which high-cost sellers price at marginal cost with probability one and low-cost sellers play an atomless mixed strategy on the interval \( [P', c_H] \) with \( P' \) satisfying \( \pi_L(P') = (1 - \alpha)^{n-1}\pi_L(c_H) \). Note that as \( \pi_L(P) \) is continuous in price, strictly concave and \( P^\text{mon}_L > c_H \) there exists a unique \( P' \in [c_L, c_H] \) such that \( \pi_L(P') = (1 - \alpha)^{n-1}\pi_L(c_H) \). Each low-cost seller mixes on the interval \( [P', c_H] \) according to a mixed strategy given by cumulative distribution function \( F(P) \). If \( U \) is the expected payoff which low-cost sellers earn in equilibrium then the payoff indifference property of the Nash equilibrium requires that \( F(P) \) satisfies:

\[
\sum_{r=0}^{n-1} \binom{n-1}{r} (1 - \alpha)^{n-1-r}(\alpha)^{n-1}(1 - F(P))^r \pi_L(P) = U \quad \text{for all } P \in [c_L, c_H].
\]

Evaluating the left-hand side at \( F(c_H) = 1 \) then gives \( U = (1 - \alpha)^{n-1}\pi_L(c_H) \). Now the reader can follow the same steps as in the previous Proposition to convince himself that there exists
a distribution function which gives low-cost sellers an expected payoff of \((1 - \alpha)^{n-1}\pi_L(c_H)\).

Now suppose a seller were to deviate from playing these strategies. It is evident that no high-cost seller can do any better than pricing at marginal cost. If a low-cost seller deviates to, or places probability mass on, a \(P > c_H\) they earn zero profit which is strictly worse than \((1 - \alpha)^{n-1}\pi_L(c_H) > 0\). If a low-cost seller were to deviate to a price \(P < P'\) then the strict concavity of the profit function and \(P_{L\text{mon}} > c_H\) imply \(\pi_L(P) < \pi_L(P')\). Therefore these strategies constitute a Bayesian Bertrand equilibrium.

It is worth noting that Proposition 1.19 gives another reason why the Bertrand paradox outcome may fail: with probability \(\alpha^n\) all sellers in the market are of the same cost type yet they price above marginal cost and earn positive expected profits because of the uncertainty regarding other sellers’ cost types. At a technical level, the result occurs because the Bertrand paradox is an equilibrium in weakly dominated strategies and if there is any probability, no matter how small, that a rival charges a price above marginal cost then pricing at marginal cost is no longer a best response.

1.4.3 Existence of fully-revealing Bayesian equilibrium

We have assumed that the good which sellers produce in the market is perfectly homogeneous so that consumers always prefer to buy from the lowest priced seller. In this section we relax this assumption and consider the case where the quality of the good traded may be different but that consumers cannot distinguish between goods of different quality when they make their purchase. In this case, there is a possibility that prices play an additional role of signaling the quality of the good which sellers are offering in the market. To model this, we shall assume that there are two types of good traded: high and low quality. As in the previous section, the marginal cost of production depends upon the quality of the good produced and \(c_H > c_L \geq 0\). The market demand for the different quality goods is simpler than we have used previously in that there is a unit mass of consumers which demand a unit of the good inelastically provided that the price of the good is less than or equal to the valuation of the good. The valuations are \(V_H > V_L\) with \(V_H > c_H\) and \(V_L > c_L\). The quality which each seller
produces is determined by a probability distribution and each seller has probability $\alpha \in (0, 1)$ that the quality of their good is high and probability $1 - \alpha$ that the quality is low. We now have a three-stage game: in the first stage the random draw determines the quality of each seller’s good and this is private information although the probability distribution is common knowledge; in the second stage sellers simultaneously and independently post prices in the market; in the third stage consumers decide which sellers to buy from. The solution concept will be that of perfect Bayesian equilibrium. An equilibrium will be said to be fully-revealing if the prices which high and low quality sellers post in the market are almost surely different.

We now present the following result which was established by Janssen and Roy (2010).

**Proposition 1.20** In the market described above, where the quality of sellers goods is uncertain, there exists a fully-revealing Bayesian Bertrand equilibrium provided the number of sellers in the market is sufficiently large.

**Proof.** A routine examination of pricing strategies shows that there cannot exist any fully-revealing equilibrium in pure strategies. However, we shall show how to construct a fully-revealing mixed strategy equilibrium. The high-quality sellers will post the same price $P_H \in [c_H, V_H]$ whenever they are active in the market. The low-quality sellers will play an atomless mixed strategy, denoted by cumulative distribution function $F(P)$, on the interval $[P_L, P_H]$. The upper bound of the support satisfies:

$$\bar{P}_L = P_H - (V_H - V_L).$$

This means consumers are indifferent between purchasing off a low-quality type which post price $\bar{P}_L$ and a high-quality seller which posts price $P_H$. As the distribution function $F(P)$ will be atomless low-quality sellers will post prices below $\bar{P}_L$ almost surely. Therefore high-quality sellers will only make sales when all other sellers in the market are of high-quality. The expected payoff to low-quality sellers is given by:

$$U_L = (\bar{P}_L - c_L)\alpha^{n-1} = (P_H - (V_H - V_L) - c_L)\alpha^{n-1}.$$
Then the lower bound of the support must be such that low-quality sellers can guarantee a payoff of $U_L$ by pricing at $P_L$:

$$P_L - c_L = U_L.$$  

Note that the prices satisfy $c_L < P_L < \bar{P}_L < P_H$. Now if each seller mixes according to $F(P)$ on the interval $[P_L, \bar{P}_L]$ then it must be that $F(P)$ satisfies:

$$(\alpha + (1 - \alpha)(1 - F(P)))^{n-1}(P - c_L) = U_L \quad \text{for all } P \in [P_L, \bar{P}_L].$$

This is simply the payoff indifference property of the Nash equilibrium. Now we can solve for $F(P)$:

$$F(P) = \frac{1}{1 - \alpha} \left[ 1 - \left( \frac{U_L}{P - c_L} \right)^{\frac{1}{n-1}} \right].$$

It can be verified that this function is a probability distribution function on $[P_L, \bar{P}_L]$. Now we need to check that no seller can deviate from playing these strategies. If a low-quality seller were to mimic a high-quality seller and post price $P_H$ then the expected profit they would obtain is:

$$\frac{1}{n}(P_H - c_L)\alpha^{n-1}.$$  

This is not a profitable deviation if:

$$\frac{1}{n}(P_H - c_L)\alpha^{n-1} \leq U_L.$$  

Rearranging this last inequality and substituting the expression for $U_L$ we get:

$$P_H \geq c_L + \frac{V_H - V_L}{1 - (1/n)}.$$  

Now the payoff which a high-quality seller obtains from playing $P_H$ is $U_H = \frac{1}{n}(P_H - c_H)\alpha^{n-1}$. If a high-quality seller imitates a low-quality seller and posts a price $P \in [P_L, \bar{P}_L]$, then if $q(P)$ denotes the expected quantity sold, the payoff which the high-quality seller obtains is:

$$(P - c_H)q(P) = (P - c_L)q(P) - (c_H - c_L)q(P) = U_L - (c_H - c_L)q(P) \quad (1.9)$$

$$\leq U_L - (c_H - c_L)q(\bar{P}_L) \quad (1.10)$$

$$= (\bar{P}_L - c_H)\alpha^{n-1}. \quad (1.11)$$
Therefore this is not a profitable deviation for the high-quality seller if:

\[ \frac{1}{n}(P_H - c_H)\alpha^{n-1} \geq (P_L - c_H)\alpha^{n-1}. \]

Rearranging this inequality we obtain:

\[ P_H \leq c_H + \frac{V_H - V_L}{1 - (1/n)}. \]

The possible deviations give an upper and lower bound upon the value of \( P_H \):

\[ \max\{c_H, c_L + \frac{V_H - V_L}{1 - (1/n)}\} \leq P_H \leq \min\{c_H + \frac{V_H - V_L}{1 - (1/n)}, V_H\}. \]

A necessary and sufficient conditions for this inequality to be satisfied is:

\[ \frac{1}{n} \leq \frac{V_L - c_L}{V_H - c_L}. \]

This final inequality is satisfied provided \( n \) is sufficiently large. Also we must specify appropriate out-of-equilibrium beliefs of consumers which sustain the equilibrium. If consumers assign probability one that a sellers quality is low whenever there price, \( P \), is such that \( P \neq P_H \) and \( P \in [P_L, P_L] \) then there is no profitable deviations for either quality seller outside of the played strategy sets, and as no seller can mimic another type, the strategies form a fully-revealing Bayesian Bertrand equilibrium. ■

### 1.5 Sharing rules

In analyzing Bertrand games up to this point we have considered a number of sharing rules: equal sharing, capacity sharing and winner-takes-all. We found that the existence of equilibrium often depends upon the sharing rules. For example, in a market with symmetric concave costs there does not exist a pure strategy Bertrand equilibrium under the equal sharing rule but this is not the case when the sharing rule is winner-takes-all. In this section we take a different tack by assuming the sharing rule may be arbitrary and try to uncover general properties of the sharing rules which guarantee the existence of a Bertrand equilibrium. We now introduce some properties of sharing rules.
Definition 1.7 A sharing rule is weakly (strictly) tie-decreasing for seller \( i \in N \) at \( P \in \mathbb{R}^+ \) if \( \pi_i(P) \geq (>) \pi_i(P,W) \), for all \( W \in \chi^W \), \(|W| \geq 2\), and \( \pi_i(P,W) > 0 \). The sharing rule is weakly tie-decreasing if it is weakly tie-decreasing for all \( i \in N \) and \( P \in \mathbb{R}^+ \).

Definition 1.8 A sharing rule is coalition monotone if \( \sum_{i \in V} \pi_i(P,V) \leq \sum_{i \in W} \pi_i(P,W) \) for all \( P \in \mathbb{R}^+ \) and \( V \subset W \in \chi^W \).

Definition 1.9 A sharing rule is sum upper semicontinuous if \( \limsup_{P' \to P} \sum_{i \in W} \pi_i(P',W) \leq \sum_{i \in W} \pi_i(P,W) \) for \( P \in \mathbb{R}^+ \) and \( W \in \chi^W \).

If a sharing rule is tie-decreasing then the profit a seller obtains from serving all the market demand is above its tied profit whenever tied profit is positive. Coalition monotonicity means that adding more sellers to a price tie does not reduce the sum of the sellers’ profits. Finally, sum upper semicontinuous means that the sum of sellers’ tied profits “jump up” at discontinuities. We now present the following result contained in Hoernig (2006).

Proposition 1.21 In a Bertrand game in which A1-A4 are satisfied and the sharing rule is weakly tie-decreasing, coalition monotone and sum upper semicontinuous there exists a mixed strategy Bertrand equilibrium.

Proof. The result will be established by showing that the game satisfies all the conditions of Theorem 5 in Dasgupta and Maskin (1986).

First, the action space of each seller is \( S_i = [0, \bar{P}] \) which is compact and in any equilibrium a player must earn profit at least as great as zero. If this were not the case, and some seller earned less than zero, then they could deviate to \( \bar{P} \) and earn zero profit.

Second, we show that the payoffs \( u_i(P_i, P_{-i}) \) defined by eq.(1.2) are weak lower semicontinuous. For any vector of prices, if \( i \in N \setminus W \) then \( u_i(P_i, P_{-i}) = 0 \) as the seller is undercut and \( u_i(\cdot, P_{-i}) \) is continuous at \( P_i \). If \( i \in W \) then \( u_i(P_i, P_{-i}) = \pi_i(P_i, W) \). Then if \( P_i' < P_i \) we have \( u_i(P_i', P_{-i}) = \pi_i(P_i') \) and if \( P_i' > P_i \) then \( u_i(P_i', P_{-i}) = 0 \). The payoffs are weak lower semicontinuous if there exists a \( \lambda \in [0,1] \) such that \( \lambda \liminf_{P_i' \uparrow P} \pi_i(P_i') \geq \pi_i(P,W) \). As \( \pi_i(P) \) is continuous in price \( \lambda \liminf_{P_i' \uparrow P} \pi_i(P_i') = \pi_i(P) \) and as the sharing rule is weakly tie-decreasing the condition is satisfied with \( \lambda = 1 \) if \( \pi_i(P) \geq 0 \) and \( \lambda = 0 \) if \( \pi_i(P) < 0 \).
Third, we show that the sum of the payoffs $\sum_{i \in N} u_i(P_i, P_{-i})$ is upper semicontinuous. For $i \in N \setminus W$ the payoff is zero and the sum of such payoffs is zero so we only have to consider the sum of the payoffs of the winners. If $P$ is the minimum price in the market let $P'$ denote a price in a small neighbourhood of $P$ and consider a $V \subset W$. The sum of the payoffs is upper semicontinuous if and only if:

$$\sum_{i \in W} u_i(P, P_{-i}) \geq \max \left\{ \sum_{i \in V} u_i(P, P_{-i}), \limsup_{P' \to P} \sum_{i \in V} u_i(P', P_{-i}) \right\} \quad \forall \quad V \subset W.$$

This condition implies that $\sum_{i \in W} u_i(P_i, P_{-i})$ is upper semicontinuous at $P$, and as $\sum_{i \in V} u_i(P_i, P_{-i})$ must also satisfy the equality, a sufficient condition for this criterion to be satisfied is that $\sum_{i \in V} u_i(P_i, P_{-i}) \leq \sum_{i \in W} u_i(P_i, P_{-i})$. That is, the sharing rule be coalition monotone.

Then as payoffs are bounded are discontinuous on a set of measure zero in the joint action spaces all the conditions of Dasgupta and Maskin (1986) Theorem 5 are satisfied and we can conclude that there exists a mixed strategy Bertrand equilibrium. ■

## 1.6 Coalitional deviations

In this section we consider what prices might emerge in the Bertrand game when sellers may group together and collectively change their prices. Two different possibilities will be considered. First, what prices might emerge when we apply the refinement of coalition proofness to the Bertrand equilibrium set. Second, we introduce a new core concept, which we term the Bertrand core, where a group of traders, buyers and sellers, may form a market and trade. The notion of coalition proofness was introduced by Bernheim et al. (1987) and applied to Bertrand games by Chowdhury and Sengupta (2004). The concept of the Bertrand core is new and a fuller account of this core concept is explored in Chapter 2.

### 1.6.1 Existence of coalition-proof Bertrand equilibrium

The concept of coalition proofness requires that no group of sellers has a profitable deviation from a price vector and that this deviation be self-enforcing. Using the payoffs defined by eq.(1.2) we define what is meant by a profitable deviation.
**Definition 1.10** At a price vector \((P_1, ..., P_n)\) a coalition \(T \subset N\) has a profitable deviation if there exists a \(P'(T) = \{P'_i\}_{i \in T}\) such that:

\[
u_i(P'_i, P'(T \setminus i), P(N \setminus T)) > u_i(P_i, P_{-i}) \quad \forall \ i \in T.
\]

In words a coalition \(T\) has a profitable deviation at \((P_1, ..., P_n)\) if the members of \(T\) can deviate to a new set of prices \(P'(T)\) which give them a higher payoff providing the sellers not in \(T\) continue to play their prices in \((P_1, ..., P_n)\) which we denote by \(P(N \setminus T)\). The relationship with the Nash equilibrium is given below.

**Definition 1.11** A price vector \((P_1, ..., P_n)\) is a Nash equilibrium if no coalition \(T\), with \(|T| = 1\), has a profitable deviation at \((P_1, ..., P_n)\).

A coalition-proof Nash equilibrium is a Nash equilibrium where no coalition of sellers has a self-enforcing profitable deviation from the Nash equilibrium. This is defined inductively as follows. A coalition \(T\) with \(|T| = 1\) has a self-enforcing profitable deviation from \((P_1, ..., P_n)\) if \(T\) has a profitable deviation \((P_1, ..., P_n)\). A coalition \(T\), with \(|T| = 2\) has a self-enforcing profitable deviation from \((P_1, ..., P_n)\) if \(T\) has a profitable deviation \(P'(T)\) and there does not exist \(S \subset T, S \neq T\), which has a self-enforcing profitable deviation from \(P'(T)\). We could then repeat this induction to define a self-enforcing profitable deviation for any coalition \(T \subseteq N\). We can now define the notion of coalition-proof Bertrand equilibrium.

**Definition 1.12** A vector of prices \((P_1, ..., P_n)\) is a coalition-proof Bertrand equilibrium if \((P_1, ..., P_n)\) is a pure strategy Bertrand equilibrium and no coalition \(T \subset N\) has a self-enforcing profitable deviation from \((P_1, ..., P_n)\).

**Lemma 1.12** In a market with strictly convex costs, if A1-A2 are satisfied, then there exists a unique price-taking equilibrium.

**Proof.** The reader is referred to the proof of Proposition 1.9.

**Lemma 1.13** In a market with convex costs and capacity sharing at price ties we have:

(i) \(\pi_i(P, N)\) is continuous in \(P\) for all \(P \in [P^C, \bar{P}]\) where \(P^C\) is the unique price-taking
equilibrium.

(ii) \( \pi_i(P, N) > 0 \) for \( P \geq P^C \).

(iii) For all \( P < P^C \) and \( T \subset N \), \( \pi_i(P, T) \leq \pi_i(P, N) \) and the inequality holds strictly if \( P > 0 \).

Proof. See the Appendix.

For any \( P \in [0, \bar{P}] \) let \( \Gamma(P) = \{ T \subset N, T \neq N : T \) has a profitable deviation at \( (P, N) \} \) and \( \Delta = \{ P \in [0, \bar{P}] : \Gamma(P) = \emptyset \} \). The proofs of the following three lemmas can be found in the Appendix.

Lemma 1.14 \( P^C \in \Delta \).

Lemma 1.15 If \( P^S = \sup \Delta \) then \( P^S \in \Delta \) and \( P^S < \bar{P} \).

Lemma 1.16 If all sellers charge a price \( P \) with \( P > P^S \) then there exists a \( T \neq N \) which has a self-enforcing profitable deviation at \( (P, N) \).

The next result shows that in a Bertrand game the set of coalition-proof Bertrand equilibria is non-empty. This is an interesting result because Berheim et al. (1987) showed that many games, including simple Cournot games, may fail to possess a coalition-proof Nash equilibrium.

Proposition 1.22 In a market with sellers with strictly convex costs and capacity sharing at price ties, if A1-A4 are satisfied, then there exists a coalition-proof Bertrand equilibrium in which all sellers charge the same price.

Proof. Consider the case where all sellers charge \( P^S \). From Lemma 1.15 no proper subset of \( N \) has a profitable deviation from this price. If the grand coalition of all \( N \) sellers does not have a self-enforcing profitable deviation from this price then \( P^S \) is a coalition-proof Bertrand equilibrium. Suppose the grand coalition does has a self-enforcing profitable deviation. As all sellers must be strictly better off from the deviation they must charge a common price.
say $P'$. If this is self-enforcing Lemma 1.16 tells us that $P' < P^S$. From part $(iii)$ of Lemma 1.13 we can restrict attention to $P' \in [P^C, P^S]$. Define the set:

$$R = \{ P \in [P^C, P^S] : \pi_i(P, N) \geq \pi_i(P^S, N) \quad \forall \quad i \}.$$ 

Let $P(1) = \arg \max_{P \in R} \pi_1(P, N)$. Note that $R$ is non-empty as $P^S \in R$ and by part $(i)$ of Lemma 1.13 $\pi_1(P, N)$ is continuous on $[P^C, P^S]$ so $R$ is a closed set. Therefore $P(1)$ is well-defined.

We now claim that all sellers quoting $P(1)$ is a coalition-proof Bertrand equilibrium. Now the grand coalition cannot have a self-enforcing profitable deviation from $P(1)$ because the price charged cannot be more than $P^S$ by Lemma 1.16 and as a result seller 1 cannot be made better off than at $(P(1), N)$. If there is a profitable deviation it must come from a subset $T$ which is a strict subset of $N$. As $P(1) \geq P^C$ part $(ii)$ of Lemma 1.13 tells us that the sellers make non-negative profit charging $P(1)$. The coalition $T$ must deviate to a common price $P'' < P(1)$ such that $\pi_i(P'', T) > \pi_i(P(1), N)$ for all $i \in T$. As $\pi_i(P(1), N) \geq \pi_i(P^S, N)$ because $P(1) \in R$, we then have $\pi_i(P'', T) > \pi_i(P^S, N)$ for all $i \in T$. This means $T$ has a profitable deviation from $P^S$ and contradicts $P^S \in \Delta$. ■

This result demonstrates that with possibly asymmetric sellers the set of coalition-proof Bertrand equilibria is non-empty. However, when sellers are symmetric the set of coalition-proof Bertrand equilibria is a singleton. Therefore if we admit coalitional deviations from a market with symmetric sellers with strictly convex costs then we have a unique equilibrium. We now present this result.

**Proposition 1.23** In a market with symmetric sellers with strictly convex costs and capacity sharing at price ties, if $A1-A4$ are satisfied, then there exists a unique coalition-proof Bertrand equilibrium in which all sellers quote $P^{CP} = \min\{P^S, P^J(N)\}$.

**Proof.** We refer the reader to Chowdhury and Sengupta (2004, p.316).
1.6.2 Sequential implementation of the coalition-proof equilibrium

In their paper introducing the notion of coalition-proofness, Bernheim et al. (1987) presented an interesting interpretation of coalition-proofness. If a Nash equilibrium is coalition-proof then if all players of the game are in a room each player could leave the room, one by one, and write down their strategy choice on a piece of paper. The players remaining in the room could decide whether or not to form a coalition. If an equilibrium is coalition-proof then the equilibrium can be implemented by each player writing down their strategy and leaving the room, one by one, and this is true no matter what order the players leave the room. We now consider whether the symmetric coalition-proof Bertrand equilibrium can be implemented, as a subgame-perfect Nash equilibrium, in a sequential-move game where players announce their prices, one by one, and after all prices have been announced, the profit accrue to the sellers. Now it it straightforward to show that if the price space is the real line then the “open-set” problem we noted earlier may preclude the existence of a well-defined best response after some seller has announced their price. To overcome this difficulty we shall assume that the price space is discrete \( F = \{0, \epsilon, 2\epsilon, \ldots\} \), \( \epsilon > 0 \). Let \( P^J_{\epsilon}(N) = \arg \max_{P \in F} \pi(P, N) \), \( P^S_{\epsilon} = \min\{P \in F : P \geq P^S\} \) and \( P^{CP}_{\epsilon} = \min\{P^J_{\epsilon}(N), P^S_{\epsilon}\} \).

We now present the following result which shows the the unique coalition-proof Bertrand equilibrium can be implemented in the sequential move game.

**Proposition 1.24** In a market with symmetric sellers with strictly convex costs and capacity sharing at price ties, if A1-A4 are satisfied and the price space is \( F \), the unique-coalition proof Bertrand equilibrium is for all sellers to quote \( P^{CP}_{\epsilon} \). If \( \epsilon \) is sufficiently small, then the unique coalition-proof Bertrand equilibrium can be implemented in a sequential move game where sellers announce their prices.

**Proof.** We refer the reader to Chowdhury and Sengupta (2004, p.318).
1.6.3 The Bertrand core and market contracts

The topic of coalitional deviations in Bertrand games has only recently been considered but the possibility that a group of traders may deviate from a larger group of traders and improve their outcomes by doing so has a long tradition in economic theory. Edgeworth’s seminal core concept in exchange theory looks for trading allocations which are immune to groups of traders being able to improve their outcomes trading by themselves. However, in a homogeneous good market with buyers and sellers the possibility of coalitional deviations by traders on both sides of the market has not been generally considered. One of the reasons why this is the case is that two assumptions are often made regarding the type of contract which price-setting sellers make in the market. First, the Bertrand assumption, is that sellers commit to supplying all the demand forthcoming from buyers. Second, the Bertrand-Edgeworth assumption is that sellers post prices with no commitment to supply any particular quantity. Suppose that sellers make contracts with some subset of buyers to supply all the demand forthcoming from these buyers, but not from buyers with whom they do not make a contract. This seems to be a more realistic way of modeling market contracts and is somewhere between the extremes of committing to supply all the market demand and not committing to any particular quantity. If contracts can be formed in this way then a trading price for the whole market must be immune to any group of buyers and sellers forming new contracts and trading by themselves. In Chapter 2 we pursue this issue in more detail and introduce a new core notion for an oligopoly market which we term the Bertrand core. We find that in large markets the only price which remains in the Bertrand core is the price-taking equilibrium and this is true even when the limit market contains uncountably many Bertrand equilibria. Therefore this type of contract provides a new foundation for price-taking behaviour in large markets.

1.6.4 Appendix

Proof of Lemma 1.1 As $h_i(P) = \arg \max_{Q \in \mathbb{R}^+} \pi_i(Q)$ if $P = 0$ then the profit of the seller is $\pi_i(Q) = -C_i(Q)$. Therefore the profit maximizing output is $Q = 0$. To establish the second
part of the lemma note that \( h_i(P) \) must satisfy the first-order condition for maximization:

\[
P - C_i'(h_i(P)) = 0.
\]

Differentiating w.r.t. \( P \) we obtain:

\[
1 - C_i''(h_i(P)) h_i'(P) = 0.
\]

Rearranging:

\[
h_i'(P) = 1/C_i''(h_i(P)).
\]

As sellers have strictly convex cost functions \( C_i''(\cdot) > 0 \) and therefore \( h_i'(P) > 0 \). ■

**Proof of Lemma 1.2** From the definition \( \pi_i^*(P) = Ph_i(P) - C_i(h_i(P)) \) and therefore:

\[
\pi_i^*(P) = h_i(P) + Ph_i'(P) - C_i'(h_i(P)) h_i'(P).
\]

Factorizing:

\[
\pi_i^*(P) = h_i(P) + h_i'(P)[P - C_i'(h_i(P))].
\]

From the first-order condition \( P - C_i'(h_i(P)) = 0 \) therefore:

\[
\pi_i^*(P) = h_i(P).
\]

From Lemma 1.1 we know that \( h_i(P) > 0 \) for all \( P > 0 \) which establishes the result. ■

**Proof of Lemma 1.3** From the definition of the monopoly profit function we have \( \pi(0) = 0 - C(Q) < 0 \) and as \( \hat{P} > 0 \) and \( C'(0) = 0 \) these imply \( \pi(P_{mon}) > 0 \). As the profit function is continuous, the intermediate value theorem guarantees that \( \exists \hat{P} \in [0, P_{mon}) \) such that \( \pi(\hat{P}) = 0 \). The uniqueness follows from the strict concavity of the monopoly profit function. ■

**Proof of Lemma 1.4** From the definition of the shared profit function \( \pi(0, W) = 0 - C(Q/W) < 0 \) and from A5 we have \( \pi(P_j(W), W) > 0 \). Therefore the continuity of the shared profit function guarantees that \( \exists \hat{P}(W) \in [0, P_j(W)) \) such that \( \pi(\hat{P}, W) = 0 \). Again the uniqueness follows from the strict concavity of the shared profit function. ■
Before proving the next lemmas we note the following property.\footnote{The proofs of the next three lemmas are shortened versions of those presented by Dastidar (2006). The interested reader is referred there for a fuller account.} If the cost function $C(\cdot)$ is strictly convex then for $2 \leq |W| \leq n$ we have:

$$
\frac{C(D(P)) - C(D(P)\frac{P}{|W|})}{D(P) - D(P)\frac{P}{|W|}} > \frac{C(D(P)\frac{P}{|W|}) - C(0)}{D(P)\frac{P}{|W|}}.
$$

Rearranging this inequality, and noting that $C(0) = 0$, we obtain:

$$
\pi(P) < |W|\pi(P,W).
$$

We can repeat these steps to show that if $C(\cdot)$ is strictly concave then:

$$
\pi(P) > |W|\pi(P,W).
$$

*Proof of Lemma 1.5* To see that $P^*(W)$ exists define the function:

$$
g(P,W) = \pi(P) - \pi(P,W)
$$

Then $g(0,W) < 0$ and $g(P^{\text{mon}},W) > 0$ by A4. Therefore as $g(P,W)$ is continuous in price $\exists P^*(W) \in [0,P^{\text{mon}}]$ such that $g(P^*(W),W) = 0$. To see that $P^*(W)$ is unique note that the mean value theorem guarantees that $\exists y \in (D(P)\frac{P}{|W|},D(P))$ such that:

$$
C(D(P)) - C(D(P)\frac{P}{|W|}) = (D(P) - D(P)\frac{P}{|W|})C'(y).
$$

Using this result the function $g(P,W)$ can be rewritten as:

$$
g(P,W) = \frac{|W| - 1}{|W|}D(P)(P - C'(y))
$$

Now suppose that $C(\cdot)$ is convex. The $C'(\cdot)$ is increasing over $(\frac{D(P)}{|W|},D(P))$. We have:

$$
\frac{|W| - 1}{|W|}D(P)(P - C'(D(P))) < g(P,W)
$$

and

$$
g(P,W) < \frac{|W| - 1}{|W|}D(P)(P - C'(\frac{D(P)}{|W|})).
$$
Therefore \( g(P, W) = 0 \) means:

\[
\frac{|W| - 1}{|W|} D(P)(P - C'(D(P))) < 0 < \frac{|W| - 1}{|W|} D(P)(P - \frac{D(P)}{|W|}).
\]

The left-hand side of the last inequality implies that when \( g(P, W) = 0 \), \( P - C'(D(P)) < 0 \) and \( \pi'(P) > 0 \) which implies that \( P^* < P^{\text{mon}} \). The right-hand side of the last inequality also implies that \( P - \frac{D(P)}{|W|} > 0 \). Together these inequalities imply \( \pi'(P) > \pi'_P(P, W) \).

Therefore we have that \( g'_P(P^*(W), W) > 0 \). This last fact implies that \( P^*(W) \) is unique.

To establish the same result for the case of strictly concave costs one can simply reverse the relevant inequalities.

Note that we have established the uniqueness of \( P^* \) on the domain \((0, P^{\text{mon}})\) rather than on \((0, \bar{P})\). This does not restrict the results because the assumption that monopoly profit is higher than tied profits at the collusive price (A4) means that the highest price in any equilibrium must be lower than \( P^{\text{mon}} \). Therefore limiting attention to \((0, P^{\text{mon}})\) does not affect the results. ■

Proof of Lemma 1.6 We noted earlier that if \( C(\cdot) \) is strictly convex then \( \pi(P) < |W|\pi(P, W) \).

We now have \( \pi(\bar{P}) = 0 < |W|\pi(\bar{P}, W) \) and \( \pi(\bar{P}, W) > 0 > \pi(\bar{P}(W), W) = 0 \). This implies \( \bar{P}(W) < \bar{P} \).

By definition \( \pi(P^*(W)) = \pi(P^*(W), W) \). We have \( \pi(P^*(W)) < |W|\pi(P^*(W), W) \).

Together these imply that \( \pi(P^*(W)) < |W|\pi(P^*(W)) \). This then implies that \( \pi(P^*(W)) > 0 \) which gives \( P^*(W) > \bar{P} \).

Proofs of Lemma 1.7 and 1.8 The proof of Lemma 1.7 can be established by repeating the steps in the proof of Lemma 1.6 and by noting that \( \pi(P) > |W|\pi(P, W) \). The proof of Lemma 1.8 can be established by inspection of the relevant functions. ■

Proof of Lemma 1.9 At the beginning of the Appendix we proved that if costs are strictly concave then \( \pi(P) > |W|\pi(P, W) \). As \( |W| > 1 \) the result is immediately clear. ■

The proofs of the following two lemmas are contained in Dastidar (2010).

Proof of Lemma 1.10 Suppose \( C(\cdot) \) is strictly concave and satisfies \( C(0) = 0 \). Then we can let \( \alpha = \frac{a}{b} \) with \( 0 < a < b \) so \( \alpha \in (0, 1) \). By the strict concavity of the cost function we have:

\[
C(a) = C((1 - \alpha)(0) + \alpha b) > (1 - \alpha)C(0) + \alpha C(b) = \frac{a}{b} C(b).
\]

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Rearranging gives:
\[
\frac{C(a)}{a} > \frac{C(b)}{b}.
\]
We then have:
\[
C(a + b) = \frac{a}{a+b} C(a + b) + \frac{b}{a+b} C(a + b)
\leq \frac{aC(a)}{a} + \frac{bC(b)}{b} = C(a) + C(b).
\]
\[
\text{Proof of Lemma 1.11} \quad \text{Suppose } C(\cdot) \text{ is strictly convex. Choose an } x, y \in (0, \infty) \text{ and assume, w.l.o.g., that } x \geq y. \text{ We then have:
\[
C(x) = \int_0^x C'(t)dt, \quad C(y) = \int_0^y C'(t)dt \quad \text{and} \quad C(x + y) = \int_0^{x+y} C'(t)dt.
\]
Then:
\[
C(x + y) = \int_0^{x+y} C'(t)dt = C(x) = \int_0^x C'(t)dt + C(x) = \int_x^{x+y} C'(t)dt.
\]
We then have that:
\[
C(x + y) - C(x) - C(y) = \int_x^{x+y} C'(t)dt - \int_0^y C'(t)dt.
\]
As \(C(\cdot)\) is strictly convex and \(x \geq y\) we know that:
\[
\int_x^{x+y} C'(t)dt > yC'(x) \geq yC'(y) \quad \text{and} \quad \int_0^y C'(t)dt < yC'(y).
\]
Therefore
\[
\int_x^{x+y} C'(t)dt - \int_0^y C'(t)dt > 0.
\]
Which implies:
\[C(x + y) > C(x) + C(y).\]

The proofs of the following four lemmas can be found in Chowdhury and Sengupta (2004).
\textbf{Proof of Lemma 1.13} Part (i). As $\pi_i(P,N) = PD(P)\beta(P) - C_i(\beta(P)D(P))$ with $\beta(P) = h_i(P)/\sum_{j\in N} h_j(P)$. Therefore $\pi_i(P,N)$ is the sum of continuous functions and is continuous.

Part (ii). Note that at $P^C$ each seller obtains $\pi_i^*(P^C) > 0$. For any $P > P^C$ each seller obtains demand of $\beta(P)D(P)$ and $\beta(P)D(P) < h_i(P)$ whenever $P > P^C$. We have:

\[\pi_i(P,N) = \beta(P)D(P)P - C_i(\beta(P)D(P)).\]

By the convexity of the cost function:

\begin{align*}
\beta(P)D(P)V(P,\beta(P)D(P)) &= \beta(P)D(P)[P - C_i(\beta(P)D(P))]/\beta(P)D(P) \quad (1.15) \\
&\geq \beta(P)D(P)[P - C_i(h_i(P))/h_i(P)] \quad (1.16) \\
&\geq \beta(P)D(P)[P - C'_i(h_i(P))] = 0. \quad (1.17)
\end{align*}

Part (iii). If $P < P^C$ then $\beta(P)D(P) > h_i(P)$ and if $T \subset N$, $T \neq N$, then $\beta(P)D(P) > \beta(P)D(P)$. As the the function $\pi_i(Q) = PQ - C_i(Q)$ is strictly concave in $Q$ and $h_i(P)$ is the profit maximizing quantity this implies $\pi_i(P,T) < \pi_i(P,N)$ if $P < P^C$. ■

\textbf{Proof of Lemma 1.14} From part (ii) of Lemma 1.13 we know $\pi_i(P^C,N) > 0$. Now if $P^C \notin \Delta$ there exists a $T \in \Gamma(P^C)$, $T \neq N$. To enact the profitable deviation the coalition must set a common price $P' < P^C$. Then part (iii) tells us that the deviation is not profitable for the coalition $T$. ■

\textbf{Proof of Lemma 1.15} From Lemma 1.14 $P^C \in \Delta$ so $P^S \geq P^C$ From part (ii) of Lemma 1.13 we know that $\pi_i(P^S,N) > 0$ for all $i \in N$. If $P^S \notin \Delta$ then there exists $T \neq N$ and $P' < P^S$ such that $\pi_i(P',T) > \pi_i(P^S,N)$ for all $i \in T$. As $\pi_i(P,N)$ is a continuous function of $P$ on $[P^C, \bar{P}]$ there exists $\delta > 0$ such that $\pi_i(P',T) > \pi_i(P,N)$ if $P \in [P^S - \delta, P^S]$. This contradicts that $P^S = \sup \Delta$. ■

\textbf{Proof of Lemma 1.16} If we let $\Gamma^*$ denote the set of coalitions that has a profitable deviation at $(P,N)$ and select a $T^* \in \Gamma^*$ with the property that no proper subset of $T^*$ has a profitable deviation from $(P,N)$. As $P > P^S$ the set $\Gamma^*$ is non-empty and as $N$ is finite $T^*$ exists. As $P > P^S \geq P^C$ we know from part (ii) of Lemma 1.13 that $\pi_i(P,N) > 0$ for all $i \in N$. For $T^* \neq N$ to have a profitable deviation there must exist a $P' < P$ such that $\pi_i(P',T^*) >$
\( \pi_i(P, N) \) for all \( i \in T^* \). If this deviation is not self enforcing then there exists \( S \subset T^* \), \( S \neq T^* \), that has a profitable deviation from \( ((P', T^*), (P, N \setminus T^*)) \). As the coalition \( S \) must set a common price \( P'' < P' \) this implies that \( S \) has a profitable deviation from \( (P, N) \). Since \( S \) is a proper subset of \( T^* \) this contradicts the definition of \( T^* \). \( \blacksquare \)
Chapter 2

On the Bertrand core and equilibrium of a market

2.1 Introduction

A problematic issue in economic theory is the study of price-making behaviour and the formation of price-making contracts. The original insight by Joseph Bertrand (1883), and the later formalization of his insight, showed that subject to certain technical conditions, such as smoothness of market demand and constant returns to scale costs, price competition between two or more sellers is sufficient to obtain the competitive equilibrium of a market. However, as was shown in Chapter 1, this outcome is well-known to fail under different market conditions such as when sellers have limited capacities or decreasing returns to scale costs. In this chapter we reconsider the problem of establishing what price a homogeneous good might be traded at in a market where sellers have strictly convex costs and act as strategic price-makers. The difference in this chapter is that we introduce the possibility that traders may choose to form coalitions and trade by themselves. To study which prices may result in the market we introduce a new core concept which we term the Bertrand core. A trading price is said to be in the Bertrand core if all sellers serve the market and no subset of buyers and sellers can improve their outcomes trading by themselves.
The Bertrand core is an original combination of the classical ideas of Bertrand and Edgeworth. It is well-known that Edgeworth criticized Bertrand’s insight regarding price competition which resulted in the study of markets with capacity constraints and decreasing returns to scale costs. However, Edgeworth’s other seminal insight, that of the core of an economy, introduced in Edgeworth (1881), has tended to be studied solely in the context of general equilibrium exchange or cooperative game theory. This chapter combines Edgeworth’s insight regarding the core with Bertrand price competition. Mas-Colell et al. (1995, p.655) note that there is a close relationship between Bertrand price competition and the market competition in the Edgeworth core. The seminal result of Debreu and Scarf (1963) showed that as an economy is replicated the only allocations which remain in the core are Walrasian allocations. In this chapter we find that there are some deep similarities between the Edgeworth core and the Bertrand core. Whereas Walrasian allocations always belong to the Edgeworth core we show that price-taking equilibria always belong to the Bertrand core. Moreover, we establish a partial equilibrium analogue of the Debreu-Scarf result: as the number of traders in the market is replicated the only price which remains in the Bertrand core in the limit is the competitive equilibrium. Remarkably, this result remains valid even when the limit market possesses uncountably many pure strategy Bertrand equilibria. Therefore, we are able to provide a strategic price-making foundation for price-taking behaviour in large markets which does not depend upon rationing rules or sellers playing mixed strategies.

This chapter provides a descriptive theory of price formation which improves upon existing theories of price formation in two respects. First, a number of papers have shown the existence of a price-setting equilibrium but there tends to be multiplicity of equilibrium, even in large markets. Second, when an equilibrium, in pure strategies, fails to exist, a

\[1\] At a technical level the models display a number of similarities. Walrasian allocations belong to the Edgeworth core and competitive equilibria belong to the set of Bertrand equilibria (subject to the sharing rule). Moreover, generically the Edgeworth core has uncountably many allocations and there are generically uncountably many Bertrand equilibrium prices.

\[2\] This result still holds even if traders increase arbitrarily provided that all traders do not vanish as a fraction of the limit economy (Hildenbrand and Kirman, 1988, pp.190-9).
mixed strategy usually exists and this equilibrium exhibits some form of convergence to the competitive equilibrium in large markets. Here we overcome these problems by providing a framework where a pure strategy equilibrium always exists and this remains the only price equilibrium in large markets.

This chapter is related to a number of papers which have considered strategic price-making foundations of competitive equilibrium. Dixon (1992) analyzed a model where sellers had symmetric, strictly convex costs and showed that if sellers post prices and can commit to supplying a quantity greater than their competitive supplies, subject to a no-bankruptcy condition, then the price-taking equilibrium can be sustained as a pure strategy Nash equilibrium. A sufficient condition for this was found to be that all but one seller could supply the market demand at the competitive price without incurring a loss. In an influential paper, Dastidar (1995) considered price competition, with a commitment to supply all demand forthcoming, between sellers with strictly convex costs. In a market with symmetric sellers and equal sharing at prices ties it was shown that there are uncountably many pure strategy Bertrand equilibria and the competitive equilibrium belongs to the set (Vives, 1999, p.122). Chowdhury and Sengupta (2004) considered when the refinement of coalition-proofness reduces the equilibrium set in standard Bertrand games. It was established that if sellers have symmetric costs then the game admits a unique coalition-proof Bertrand equilibrium. They showed that if one considers sequences of economies then as the number of sellers in the market becomes large the set of coalition-proof equilibria coincides with the competitive equilibrium of the market provided all sellers are active in the limit. Yano (2006a) analyzed a market model with free entry where sellers had u-shaped average costs. Sellers posted prices and a set of quantities they were willing to sell at the posted prices. It was shown that under certain conditions the competitive outcome is a Nash equilibrium of the game despite only a small number of sellers being active in the market. In a related paper, Yano (2006b) showed that the Bertrand paradox and Edgeworth criticism could be obtained as special cases of the game where sellers post prices and quantities.

We follow the tradition of these papers by analyzing price competition between sellers
producing a single perfectly homogeneous good. However, unlike most of the previous literature, we model the demand side of the market in an explicit manner by assuming that there is a finite number of buyers. This framework then permits a rich set of trading possibilities as any subset of buyers and sellers could trade by themselves. We also allow for asymmetries between buyers and sellers so the model imposes few restrictions upon buyers’ market demands and sellers’ cost functions.

The notion of the Bertrand core introduced here brings new insights to the types of market contracts which price-setting sellers make with buyers. Traditionally, two different approaches have been considered in the literature. First, Bertrand competition assumes that sellers post a price in the market with a commitment to supply all the demand forthcoming from buyers.\(^3\) Second, Bertrand-Edgeworth competition assumes that sellers post prices but do not give any commitment to supply any quantity demanded so that sellers would never produce more than their competitive supply at any given price.\(^4\) The model presented here assumes that the market contracts may be somewhere between these two extremes in that sellers may make contracts with specific buyers to supply all demand forthcoming from these buyers, which may be more than their competitive supply, but that sellers make no commitment to buyers in the market with whom they do not trade. Therefore the market contracts may have elements of both Bertrand competition and Edgeworth competition. A market contact which is in the Bertrand core is immune to a group of traders leaving the market and forming contracts in this way.

The concept of the Bertrand core also considers some subtle issues regarding communication and cooperation between sellers in the market. A set of traders may communicate with each other to form a market within the grand coalition but once the market is agreed upon the sellers act non-cooperatively in offering price contacts to buyers in the market. This is a limited form of cooperation which is at the heart of the Bertrand core. However, one could go further and consider what happens when sellers communicate to form markets and are

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\(^3\)This assumption is sometimes justified on the basis that there may be large costs involved in turning customers away (Dixon, 1990).

\(^4\)Papers in this tradition include Allen and Hellwig (1986a,b) and Vives (1986).
willing to cooperate in setting prices. In this case, the appropriate solution concept is that of the coalition-proof Bertrand equilibrium introduced in Chowdhury and Sengupta (2004). To study this possibility, we introduce the concept of the coalition-proof Bertrand core which combines the possibility of coalitional improvements with the concept of coalition-proofness. Throughout the chapter it is assumed that sellers cannot write binding contracts so any solution must at least be a Nash equilibrium of the game being considered.

In the following section we present the market model, define the Bertrand core and present the results. The final section presents some suggestions for future research.

2.2 The trading game

Consider the market for a perfectly homogeneous good. In the market there is a finite set of buyers \( B = \{1, \ldots, b\}, \ b \geq 2, \) and a finite set of sellers \( S = \{1, \ldots, s\}, \ s \geq 2. \) Each seller in the market has a cost function \( C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) which is \( C^2, \) strictly convex and satisfies \( C_i(0) = 0 \) and \( C_i'(0) = 0. \) Each buyer in the market has a demand function \( D_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and for each \( j \in B \) there exist strictly positive finite real numbers \( \bar{p}_j, \bar{q}_j \) such that \( D_j(\bar{p}_j) = 0 \) and \( D_j(0) = \bar{q}_j. \) The market demand of each buyer is \( C^2 \) on \((0, \bar{p}_j)\) and \( D_j'(p) < 0 \) for all \( p \in (0, \bar{p}_j). \) In what follows we shall make frequent use of sellers’ competitive supplies. The profit of each seller, as a function of quantity, is \( \pi_i(q) = pq - C_i(q). \) The competitive supply of the seller, as a function of price, is \( h_i(p) = \arg \max_{q \in \mathbb{R}_+} \pi_i(q). \) As each seller’s cost function is strictly convex the function \( \pi_i(q) \) is strictly concave in \( q \) and \( h_i(p) \) is well-defined and single-valued. Also let \( \pi_i^*(p) = ph_i(p) - C_i(h_i(p)) \) so \( \pi_i^*(p) \) is the value function. We shall want to consider a Bertrand price competition game between possible subsets of buyers and sellers so let \( \chi^B = \{ M : M \in 2^B \setminus \{\emptyset\} \} \) and let \( \chi^S = \{ M : M \in 2^S \setminus \{\emptyset\} \}. \) The set \( \chi^B \) is all the non-empty subsets of buyers and \( \chi^S \) is all the non-empty subsets of sellers. For any set of traders \( T' = (B', S') \in \chi^B \times \chi^S \) consider a classical Bertrand price game between these buyers and sellers. Each seller simultaneously and independently chooses a \( p_i \in \mathbb{R}_+ \) with a commitment to supply all the demand forthcoming from the buyers \( B'. \) If a seller posts the unique minimum price in the market then it serves all the demand forthcoming at
that price. If a seller is undercut then it obtains no demand and its profit is zero. If a seller ties with other sellers at the minimum price then a sharing rule describes how the market demand is shared. Throughout we shall assume the market demand is shared according to capacity sharing.\footnote{This sharing rule has been used by, amongst others, Dastidar (1997) and Chowdhury and Sengupta (2004).} Let \( \beta_i(p) = h_i(p)/\sum_{j \in A} h_j(p) \) be the share of the market demand which seller \( i \) obtains when it ties with \( A \setminus \{i\} \) other sellers at minimum price \( p \). Fix a vector of prices \( p \in \mathbb{R}^{\mid S'\mid} \). Let \( D(B', p) = \sum_{j \in B'} D_j(p) \) and \( \pi_i(p, T') \) denote the profit of seller \( i \) at price vector \( p \) in the market with \( T' \) traders. We can summarize this profit as:

\[
\pi_i(p, T') = \begin{cases} 
 p_i D(B', p_i) - C_i(D(B', p_i)) & \text{if } p_i < p_k \ \forall k \neq i; \\
 p_i \beta_i(p_i) D(B', p_i) - C_i(\beta_i(p_i) D(B', p_i)) & \text{if } i \text{ ties with } A \setminus \{i\} \text{ at min price;}
\end{cases}
\]

\[
0 & \text{if } p_i > p_k \text{ for some } k.
\]

(2.1)

We shall let \( G = (B, S) \in \chi^B \times \chi^S \) denote the grand coalition of all buyers and sellers. Now in a market with a given set of traders some sellers may be able to improve their outcomes by affecting a coalitional change in their prices.

\textbf{Definition 2.1} Fix a coalition of traders \( T' = (B', S') \in \chi^B \times \chi^S \) and a price vector \( p \in \mathbb{R}^{\mid S'\mid} \). A coalition of sellers \( A \subset S' \) has an improvement upon a price vector \( p \) if there is a vector \( p'(A) = \{p'_i\}_{i \in A} \) such that \( \pi_i(p'(A), p(S' \setminus A), T') > \pi_i(p, T') \) for all \( i \in A \).

A coalition of sellers, \( A \subset S' \), has an improvement upon a price vector \( p \in \mathbb{R}^{\mid S'\mid} \) if the coalition has another vector of prices which they can post in the market, \( p'(A) = \{p'_i\}_{i \in A} \), which will result in higher profit provided all sellers not in the coalition \( S' \setminus A \) continue to post their prices in the price vector \( p \) which we denote by \( p(S' \setminus A) \). Given this concept of an improvement upon a price vector we can now define the concept of a pure strategy Bertrand equilibrium for a market.

\textbf{Definition 2.2} For any coalition of traders \( T' = (B', S') \in \chi^B \times \chi^S \) a price vector \( p \in \mathbb{R}^{\mid S'\mid} \) is a pure strategy Bertrand equilibrium if no coalition of sellers \( A \subset S' \), \( |A| = 1 \), has an improvement upon price vector \( p \).
A price vector is a pure strategy Bertrand equilibrium of a market if no seller can unilaterally change their price and obtain higher profit. For any set of traders $T' = (B', S') \in \chi^B \times \chi^S$ we shall let $\mathcal{E}(T') \subset \mathbb{R}^{\mid S'\mid} \cup \{\emptyset\}$ denote the set of pure strategy Bertrand equilibria of the market formed by the traders. In defining the notion of an improvement upon a price vector we restricted the deviating coalition to the set of sellers. However, when we introduce the Bertrand core we shall want to consider the possibility that a coalition of traders, buyers and sellers, can enact an improvement upon a price vector. We now introduce this concept.

**Definition 2.3** Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$ and a price vector $p \in \mathbb{R}^{\mid S'\mid}$. A coalition of traders $A = (B'', S'') \subset T'$ has an improvement upon price vector $p$ if there exists a price vector $p' \in \mathcal{E}(A)$ such that $\min p' < \min p$ and $\pi_i(p', A) > \pi_i(p, T')$ for all $i \in S''$.

A coalition of traders, sellers and buyers, has an improvement upon a price vector if the coalition can form a market and there is an equilibrium trading vector which is an improvement for both the buyers and sellers in the coalition.\(^6\) The new trading vector must be an improvement for buyers in that they obtain the good at a lower price which is represented by $\min p' < \min p$. The new trading vector must be an improvement for sellers in that they obtain higher profits at the new equilibrium which is represented by $\pi_i(p', A) > \pi_i(p, T')$. Note that the difference between Definition 2.1 and Definition 2.3 is that in Definition 2.1 the improvement upon a price vector is enacted by a subset of the sellers whereas in Definition 2.3 the improvement is enacted by some subset of sellers and buyers. The concept of an improvement enacted solely by sellers was considered by Chowdhury and Sengupta (2004) in the context of coalition-proof equilibria. The concern here is about the stronger concept of an improvement enacted by both sides of the market. Later in the section we shall return to the relationship between coalition-proof equilibria and the Bertrand core. We are now ready to introduce the Bertrand core.

**Definition 2.4** A price vector $p \in \mathbb{R}^*_+$ is in the Bertrand core if all sellers quote the same

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\(^6\)For the special case where a coalition containing only one seller forms it will be assumed that the seller posts price $\min\{p^m, \min p - \epsilon\}$, $\epsilon > 0$, where $p^m$ denotes the monopoly price of the coalition market.
price, earn non-negative profits, and no coalition of traders, \( A \subset G \), has an improvement upon \( p \).

A price vector is in the Bertrand core if all sellers serve market demand and no coalition contained within, or including, the grand coalition, has an improvement upon the price vector. We shall let \( \mathcal{C}(G) \subset \mathbb{R}_+^s \cup \{\emptyset\} \) denote the price vectors in the Bertrand core.

**Remark 1** \( \mathcal{C}(G) \subset \mathcal{E}(G) \). The Bertrand core is a subset of the set of pure strategy Bertrand equilibria of the grand coalition. To see this, note that if there were a price vector in which all sellers quote the same price, earn non-negative profit and were not a pure strategy Bertrand equilibrium then there would be an improvement upon this price vector. However, a price vector which is a pure strategy Bertrand equilibrium may not be in the Bertrand core. See the example below.

**Remark 2** It is worth noting that the Bertrand core is neither wholly cooperative nor non-cooperative. It is cooperative in the sense that a coalition of traders, buyers and sellers, may recognize that they can improve their outcomes by forming a market within the grand coalition. However, once the market is formed the sellers act non-cooperatively in offering price-making contracts to the buyers in the market. Later we shall consider which prices remain in the Bertrand core when sellers are willing to communicate to form a market and are willing to cooperate in setting prices.

**Remark 3** A price vector which is not in the Bertrand core is a vector which is not robust to traders forming contracts which may be somewhere between the extremes of standard price-making contracts. Price-setting games have traditionally assumed that either sellers commit to supplying all demand from the grand coalition (Bertrand competition) or make no commitment to supply any particular demand (Bertrand-Edgeworth). The notion of an improvement by a coalition in Definition 2.3 admits sellers to form contracts which are between these two extremes in that sellers may supply more than their competitive supply but less than the demand from the grand coalition.

**Lemma 2.1** \( h_i(0) = 0 \) and \( h'_i(p) > 0 \) for all \( i \in S \).
Proof. See the Appendix.

**Lemma 2.2** \( \pi_i^*(p) > 0 \) for all \( p > 0 \).

*Proof.* See the Appendix.

### 2.2.1 Price-taking equilibrium and the Bertrand core

Having defined the Bertrand core and related notions of improvements upon price vectors we now show that, under the assumptions made here, the Bertrand core is non-empty. This is established by showing that the competitive equilibrium of the market belongs to the Bertrand core. In a market where all sellers take prices as given a price-taking, or competitive, equilibrium is a price such that the quantities the sellers are willing to supply to the market is exactly equal to the quantity demanded by the buyers.

**Definition 2.5** For any coalition of traders \( T' = (B', S') \in \chi^B \times \chi^S \) a price-taking equilibrium is a \( p' \in \mathbb{R}_+ \) such that \( \sum_{i \in S'} h_i(p') = D(B', p') \).

For any coalition of traders \( T' = (B', S') \in \chi^B \times \chi^S \) we shall let \( \mathcal{P}(T') \subset \mathbb{R}_+ \cup \{\emptyset\} \) denote the price-taking equilibria of the market composed of the \( T' \) traders.

**Proposition 2.1** For any \( T' = (B', S') \in \chi^B \times \chi^S \), \( \mathcal{P}(T') \neq \emptyset \) and \( |\mathcal{P}(T')| = 1 \).

*Proof.* Define the function \( f(p) = D(B', p) - \sum_{i \in S'} h_i(p) \). The function \( f(p) \) is the excess demand function. From the first-order condition \( h_i(p) = C_i'^{-1}(p) \) and as the cost function \( C_i(\cdot) \) is \( C^2 \) the first derivative is continuous and the inverse of the first derivative is continuous. Therefore \( f(p) \) is a is a continuous function of price. Note that \( f(0) = \sum_{j \in B'} \bar{q}_j > 0 \) and letting \( \bar{p} = \max\{\bar{p}_j : j \in B'\} \) we have \( f(\bar{p}) = -\sum_{i \in S'} h_i(\bar{p}) < 0 \). As \( f(p) \) is continuous, the intermediate value theorem guarantees that there exists a \( p' \in (0, \bar{p}) \) such that \( f(p') = 0 \) which implies \( D(B', p') = \sum_{i \in S'} h_i(p') \). To see that the price-taking equilibrium is unique note that \( f'(p) < 0 \). ■

The next result was first established by Dastidar (1997).
Proposition 2.2 For any \( T' = (B', S') \in \chi^B \times \chi^S \) if \( p' \in \mathcal{P}(T') \) then \((p', \ldots, p') \in \mathcal{E}(T') \) provided \(|S'| \geq 2\).

Proof. Suppose we have a market with \( T' = (B', S') \in \chi^B \times \chi^S \) traders. If each seller quotes price \( p' \) to the buyers, with \( p' \in \mathcal{P}(T') \), the profit which each seller obtains at this price is:

\[
p'\beta_i(p')D(B', p') - C_i(\beta_i(p')D(B', p')).
\]

As \( \beta_i(p') = h_i(p')/\sum_{j \in S'} h_j(p') \) and \( \sum_{j \in S'} h_j(p') = D(B', p') \) the profit of each seller simplifies to:

\[
p'h_i(p') - C_i(h_i(p')) = \pi_i^*(p').
\]

Now consider whether any seller could profitably deviate from quoting this price. If a seller were to quote a price \( p'' < p' \) then the maximum profit they could obtain is \( \pi_i^*(p'') \). Lemma 2.2 then implies \( \pi_i^*(p'') < \pi_i^*(p') \) and this is not a profitable deviation. If a seller increases their price then as \(|S'| \geq 2\) they lose all demand and earn zero profit which is not a profitable deviation. Therefore \((p', \ldots, p') \in \mathcal{E}(T')\).

Proposition 2.3 \( \mathcal{C}(G) \neq \emptyset \).

Proof. We shall show that if \( p^C \in \mathcal{P}(G) \) then \((p^C, \ldots, p^C) \in \mathcal{C}(G)\). That is, the price-taking equilibrium for the whole market belongs to the Bertrand core. Suppose a coalition of traders \( T' = (B', S') \in \chi^B \times \chi^S \) deviate from the grand coalition. The profit which a seller \( i \in S' \) earned at the price-taking equilibrium was \( \pi_i^*(p^C) \). Suppose that \( p' \in \mathcal{E}(T') \) is the equilibrium price at which trade takes place amongst \( T' \) traders. Let \( p'_j = \min p' \). If \( p'_j < p^C \) then the maximum profit seller \( j \) obtains from deviating is \( \pi_j^*(p'_j) < \pi_j^*(p^C) \) and deviating is not profitable for seller \( j \). If \( p'_j \geq p^C \) then the deviating coalition is not a strict improvement for buyers. Therefore \((p^C, \ldots, p^C) \in \mathcal{C}(G)\).

The results show that the Bertrand core is non-empty as the price-taking equilibrium for the grand coalition belongs to the core. We illustrate these results in an example.

Example 2.1 Consider a market with two buyers, \( B = \{1, 2\} \), and three sellers, \( S = \{1, 2, 3\} \). The market demand of each buyer is given by the piecewise-affine function \( D(p) = \ldots \)
Each seller’s cost function is given by \( C(q) = q^2 \). Standard calculations\(^7\) reveal that the Bertrand equilibrium set for the grand coalition is \( \mathcal{E}(G) = \{ p \in \mathbb{R}^3_+ : p_i = p_j, \forall j \neq i, p_i \in [2\frac{1}{2}, 5\frac{5}{7}] \} \). There are a number of different coalitions which could deviate from the market. One possibility is that a single seller leaves the market and trades with a subset of buyers. Routine calculations show that prices in the interval \((4\frac{6}{11}, 5\frac{5}{7})\) can be improved upon by this coalition. Second, a coalition with two sellers and one buyer, \( B' = \{1\} \) and \( S' = \{1, 2\} \), could form. Routine calculations show that \( \mathcal{E}(B', S') = \{ p \in \mathbb{R}^2_+ : p_i = p_j, \forall j \neq i, p_i \in [2, 4\frac{2}{7}] \} \). Of the possible coalition prices it is straightforward to check that all prices in the interval \([2, 3\frac{13}{19})\) represent profitable deviations from the whole market. The final possible coalition is that of two buyers and two sellers, \( B'' = \{1, 2\} \) and \( S'' = \{1, 2\} \). The set of equilibria of this market is \( \mathcal{E}(B'', S'') = \{ p \in \mathbb{R}^2_+ : p_i = p_j, \forall j \neq i, p_i \in [3\frac{1}{3}, 6] \} \). Of the possible coalition prices the prices in the interval \((4\frac{6}{11}, 5\frac{5}{7})\) represent profitable deviations from the whole market. Therefore the Bertrand core is \( \mathcal{C}(G) = \{ p \in \mathbb{R}^3_+ : p_i = p_j, \forall j \neq i, p_i \in [3\frac{13}{19}, 4\frac{6}{11}] \} \subset \mathcal{E}(G) \). Note that the competitive supply of each seller is \( h(p) = \frac{p}{2} \) and the price-taking equilibrium is \( p^C = 4 \).

### 2.2.2 A limit result on the Bertrand core

The results in Propositions 2.1 and 2.3 show that the price-taking equilibrium of the grand coalition belongs to the Bertrand core. This provides a strategic foundation for price-taking behaviour. However, this strategic foundation is weak in that the Bertrand core will typically contain prices which are different from the price-taking equilibrium, as illustrated in Example 2.1. This raises the question of whether a stronger foundation for price-taking behaviour can be established. In this section we show that as the set of traders in the market becomes large the only price which remains in the Bertrand core is the price-taking equilibrium even when the pure strategy Bertrand equilibria of the limit market remains unchanged. To understand how contracts may be formed in large markets we introduce the notion of a replicated market. Formally, the \( r \in \mathbb{N} \) replication of the market with \( T' = (B', S') \in \chi^B \times \chi^S \) is the market

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\(^7\)See Vives (1999, pp.120-2) or Dastidar (1995).

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in which there are \( r \) number of each type of buyer and seller. Following the notation used above we shall let \( \mathcal{P}_r(T') \subset \mathbb{R}_+ \cup \{0\} \) denote the price-taking equilibria of the \( r \)-replicated market, \( \mathcal{E}_r(T') \subset \mathbb{R}_+^{r|S'|} \cup \{0\} \) will denote the set of pure strategy Bertrand equilibria of the \( r \)-replicated market, and \( \mathcal{C}_r(G) \subset \mathbb{R}_+^{rs} \cup \{0\} \) will denote the set of Bertrand core prices of the \( r \)-replicated grand coalition.

Proposition 2.4 For any \( T' = (B',S') \in \chi^B \times \chi^S \), \( \mathcal{P}_r(T') = \mathcal{P}(T') \) for all \( r \in \mathbb{N} \).

Proof. Define the excess demand of the replicated market as \( f(p,r) = rD(B',p) - \sum_{i \in S'} rh_i(p) \). Factorizing gives \( f(p,r) = r(D(B',p) - \sum_{i \in S'} h_i(p)) \). As \( r \in \mathbb{N} \), \( f(p',r) = 0 \) if and only if \( f(p) = 0 \). ■

Proposition 2.5 \( \mathcal{C}_r(G) \neq \emptyset \) for all \( r \in \mathbb{N} \).

Proof. As \( \mathcal{P}_r(G) = \mathcal{P}(G) \) for all \( r \in \mathbb{N} \) the same steps used in the proof of Proposition 2.3 establish that if \( p^C \in \mathcal{P}(G) \) the price vector \( (p^C,\ldots,p^C) \in \mathcal{C}_r(G) \) for all \( r \in \mathbb{N} \). ■

The result in Proposition 2.5 shows that the Bertrand core is non-empty for any \( r \)-replication of the grand coalition. However, it does not give any insight as to which other prices, if any, remain in the Bertrand core and the set of pure strategy Bertrand equilibria. We now show that as the market becomes large the only trading price which remains in the Bertrand core is the price-taking equilibrium. Let \( p^*(r) = \sup \{ \max \mathbf{p} : \mathbf{p} \in \mathcal{C}_r(G) \} \) and \( p_*(r) = \inf \{ \min \mathbf{p} : \mathbf{p} \in \mathcal{C}_r(G) \} \). The price \( p^*(r) \) is the supremum of prices quoted by any seller in the \( r \)-replicated Bertrand core and \( p_*(r) \) is the infimum of prices quoted by any seller in the \( r \)-replicated Bertrand core.

Lemma 2.3 The sequences \( \{p_*(r)\}_{r \in \mathbb{N}} \) and \( \{p^*(r)\}_{r \in \mathbb{N}} \) satisfy \( p_*(r + 1) \geq p_*(r) \) and \( p^*(r + 1) \leq p^*(r) \). That is, the sequences \( \{p_*(r)\}_{r \in \mathbb{N}} \) and \( \{p^*(r)\}_{r \in \mathbb{N}} \) are monotone.

Proof. See the Appendix.

We now present the limit result on the Bertrand core.

Proposition 2.6 As \( r \to \infty \), \( p^*(r) \to p^C \) and \( p_*(r) \to p^C \) with \( p^C \in \mathcal{P}(G) \).
Proof. The result will be established by showing that if \( \lim_{r \to \infty} p_*(r) < p^C \) then there exists \( \bar{r} \in \mathbb{N} \) such that the sellers quoting \( p_*(r) \) can form a coalition and improve upon quoting \( p_*(r) \) for all \( r \geq \bar{r} \). Similarly if \( \lim_{r \to \infty} p^*(r) > p^C \) then sellers quoting \( p^*(r) \) can form a coalition and improve upon quoting \( p^*(r) \) for all \( r \geq \bar{r} \).

**Step 1.** Consider the sequence of prices \( \{p_*(r)\}_{r \in \mathbb{N}} \). This sequence is bounded above by \( p^C \). From Lemma 2.4 we know that the sequence is monotone. Therefore the sequence \( \{p_*(r)\}_{r \in \mathbb{N}} \) must converge (Rudin, 1976, p.55). Suppose \( \lim_{r \to \infty} p_*(r) = p_* < p^C \).

**Step 2.** Consider the profit which each seller earns when all sellers post price \( p_* \). As \( p_* < p^C \) a straightforward check of the excess demand function shows that, for any \( r \)-replication, sellers posting \( p_* \) serve market demand greater than their competitive supply and earn strictly less than \( \pi^*_i(p_*) \). Therefore, by the continuity of the profit function, there exists \( \epsilon > 0 \) such that \( \pi^*_i(p_* - \epsilon) \) is greater than the profit any seller earns from the sequence of prices \( p_*(r) \in \mathcal{N}_i(p_*) \).

**Step 3.** As the sequence \( \{p_*(r)\}_{r \in \mathbb{N}} \) is convergent fix an \( r' \) such that \( p_*(r) \in \mathcal{N}_i(p_*) \) for all \( r \geq r' \). Let \( \hat{p} = p_* - \epsilon \). If the profit from \( \pi^*_i(\hat{p}) \) is strictly higher than posting the prices \( p_*(r) \), \( r \geq r' \) then the continuity of \( \pi^*_i(\cdot) \) means there is a \( \delta > 0 \) such that \( \pi^*_i(p), p \in [\hat{p} - \delta, \hat{p}] \), is also strictly greater than the profit from posting prices \( p_*(r), r \geq r' \).

**Step 4.** Now fix an \( i \in S \) and fix a \( j \in B \) such that \( D_j(p^C) > 0 \). Then consider the mapping \( g(p) = h_i(p)/D_j(p) \). This is a continuous mapping provided \( D_j(p) \neq 0 \). As \( g(p) \) is continuous, the image of \([\hat{p} - \delta, \hat{p}]\) under \( g(\cdot) \) is a compact connected interval which we shall denote by \( g([\hat{p} - \delta, \hat{p}]) \).

**Step 5.** By the everywhere denseness of the rationals in the real line there must exist a \( z \in \text{int}(g([\hat{p} - \delta, \hat{p}]))) \) with \( z \in \mathbb{Q}^+ \). As \( z \in \mathbb{Q}^+ \) we can write \( z \) as \( z = x/y \) with \( x, y \in \mathbb{N} \) and \( y \geq 2 \). Now consider the sequence of replicated markets with \( \bar{r} \geq \max \{x, y, r'\} \). Let \( p(z) \) denote the pre-image of \( z \) under \( g(\cdot) \).

**Step 6.** Consider the market formed by \( x \) buyers of type \( j \) and \( y \) sellers of type \( i \). From the mapping we know that \( z = g(p(z)) \) which means \( xD_j(p(z)) = yh_i(p(z)) \). That is, \( p(z) \) is a price-taking equilibrium for the market \( x \) buyers of type \( j \) and \( y \) sellers of type \( i \).
From Proposition 2.2 we know that all sellers quoting price \( p(z) \) is a pure strategy Bertrand equilibrium for the coalition and as \( p(z) \in (\hat{p} - \delta, \hat{p}) \) the buyers obtain the good at a price lower than any price in the sequence \( p_*(r), r \geq \bar{r} \) and each seller obtains strictly higher profit than in the grand coalition. Therefore this coalition improves upon the sequence \( p_*(r), r \geq \bar{r} \) and it must be the that \( \lim_{r \to \infty} p_*(r) = p^C \).

**Step 7.** Consider the sequence of prices \( \{p^*(r)\}_{r \in \mathbb{N}} \). As the sequence is bounded below by \( p^C \) and is monotone the sequence converges. Suppose \( \lim_{r \to \infty} p^*(r) = p^* > p^C \). One can repeat Steps 2-6 with \( p_* = p^* \) to show that a coalition has an improvement provided the market is replicated sufficiently many times. ■

**Example 2.2** Consider the market in Example 2.1. We found that the Bertrand core was \( \mathcal{C}(G) = \{ p \in \mathbb{R}^3_+ : p_i = p_j, \forall j \neq i, p_i \in [\frac{13}{19}, \frac{4}{11}] \} \). The set of Bertrand equilibria was \( \mathcal{E}(G) = \{ p \in \mathbb{R}^3_+ : p_i = p_j, \forall j \neq i, p_i \in [\frac{2}{5}, \frac{7}{5}] \} \) and the price-taking equilibrium was \( p^C = 4 \). Now consider what happens as the market is replicated. Routine calculation reveal that \( \mathcal{E}_r(G) = \{ p \in \mathbb{R}^3_+ : p_i = p_j, \forall j \neq i, p_i \in [\frac{2}{5}, \frac{7}{5}] \} \) for all \( r \in \mathbb{N} \). The set of pure strategy Bertrand equilibria is not reduced as the market is replicated. However, we know from Proposition 2.6 that as \( r \to \infty \), \( p^*(r) \to 4 \) and \( p_*(r) \to 4 \). Price-taking behaviour prevails in the Bertrand core as the market becomes large.

### 2.2.3 On the relationship between coalition-proof contracts and the Bertrand core

In defining the Bertrand core we assumed that the sellers act non-cooperatively once any market of traders was formed. However, if the sellers could communicate to improve their outcomes, and were willing to cooperate in setting prices, but could not form binding agreements, then the analysis may be quite different. Chowdhury and Sengupta (2004) considered which equilibrium prices survive the Nash equilibrium refinement of coalition-proofness introduced by Bernheim et al. (1987). In this section we review the differences and similarities between the types of price-setting contracts which are coalition-proof and the price-setting contracts in the Bertrand core. As we shall see, the two concepts are independent but can be
combined to give a new core notion which we term the coalition-proof Bertrand core. Recall the earlier definition of an improvement upon a price vector by a coalition of sellers.

**Definition 2.6** Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$ and a price vector $p \in \mathbb{R}^{|S'|}$. A coalition of sellers $A \subset S'$ has an improvement upon price vector $p$ if there is a vector $p'(A) = \{p'_i\}_{i \in A}$ such that $\pi_i(p'(A), p(S' \setminus A), T') > \pi_i(p, T')$ for all $i \in A$.

The coalition-proof Bertrand equilibrium is defined inductively. Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$. A coalition of sellers $A \subset S'$, $|A| = 1$, has a self-enforcing improvement upon price vector $p$ if $A$ has an improvement upon price vector $p$. A coalition of sellers $A \subset S'$, $|A| = 2$, has a self-enforcing improvement upon $p$ if $A$ has a price vector $p'(A)$ which is an improvement upon $p$ and no strict subset of $A$, which would be a coalition of cardinality one, has an improvement upon $p'(A)$. We could continue this process to define a self-enforcing improvement for $|A| = 3, 4, ...$. Then any coalition of sellers $A \subset S'$ has a self-enforcing improvement upon $p$ if $A$ has a price vector $p'(A)$ which is an improvement upon $p$ and no strict subset $A' \subset A$, $A' \neq A$, has a self-enforcing improvement upon $p'(A)$.

**Definition 2.7** Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$. A price vector $p \in \mathbb{R}^{|S'|}$ is a coalition-proof Bertrand equilibrium of the market with $T'$ traders if no coalition of sellers has a self-enforcing improvement upon $p$.

We shall let $\mathcal{E}_{CP}(T') \subset \mathbb{R}^{|S'|} \cup \{\emptyset\}$ denote the set of coalition-proof Bertrand equilibria of the market with $T'$ traders.

**Proposition 2.7** For any coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$, $\mathcal{E}_{CP}(T') \neq \emptyset$.

**Proof.** See Chowdhury and Sengupta (2004, Prop. 1). ■

The result in Proposition 2.7 is interesting because many games fail to possess a coalition-proof Nash equilibrium. We now can consider what the relationship is between the set of coalition-proof Bertrand equilibria and the Bertrand core. The next example shows that the a coalition-proof Bertrand equilibrium may, or may not, belong to the Bertrand core.
Example 2.3 Consider a market with a grand coalition of two sellers and two buyers $S = \{1, 2\}$ and $B = \{1, 2\}$. Each seller has a cost function given by $C(q) = q^2$. Each buyer has a market demand given by $D(p) = \max\{0, 1 - \frac{1}{2}p\}$. Routine calculations show that the unique coalition-proof Bertrand equilibrium is for each seller to quote price $p_i = \frac{4}{3}$ (see Chowdhury and Sengupta (2004, Example 2)). Moreover, if one considers each of the possible coalitions of traders which could form it is straightforward to show that no coalition can improve upon each seller quoting this price. Therefore in this example the intersection of the set of coalition-proof Bertrand equilibria and the Bertrand core is nonempty $C(G) \cap E^{CP}(G) \neq \emptyset$. However, consider a market with four sellers, $S = \{1, \ldots, 4\}$, each with costs given by $C(q) = q^2$ and eight buyers $B = \{1, \ldots, 8\}$ each of whom has demand given by $D(p) = \max\{0, \frac{1}{2} - \frac{1}{4}p\}$. In this market the unique coalition-proof equilibrium has each seller quoting price $p = \frac{14}{13}$. The sellers earn a profit of $\frac{48}{169}$. Now consider a coalition of traders $T' = (B', S')$ composed of five buyers and two sellers $B' = \{1, \ldots, 5\}$ and $S' = \{1, 2\}$. Suppose each seller quotes the price $\frac{14}{13} - \epsilon$, $\epsilon > 0$, to the buyers. A routine check shows that $(\frac{14}{13} - \epsilon, \frac{14}{13} - \epsilon) \in E(T')$. The profit which each seller earns is strictly higher than $\frac{48}{169}$. Therefore this coalition improves upon the coalition-proof Bertrand equilibrium and $C(G) \cap E^{CP}(G) = \emptyset$.

Remark 4 The results in Example 2.3 illustrate that it is not easy to compare the set of coalition-proof Bertrand equilibria and the Bertrand core. This is because the Bertrand core is in a sense stronger and weaker than the requirement of coalition-proofness. It is stronger in that it permits a wider range of coalitions to form and improve upon price vectors. The Bertrand core permits buyers and sellers to form a coalition whereas the coalition-proof Bertrand equilibrium only permits sellers to form a coalition. However, the Bertrand core is weaker in that it permits a coalition to trade at a price vector which may not be a coalition-proof Bertrand equilibrium.

The refinement of coalition-proofness is interesting because it applies to situations where traders communicate and cooperate in setting prices but cannot write binding contracts. In defining the Bertrand core it was implicitly assumed that once a coalition forms and an equilibrium trading vector is agreed upon then this is how trade takes place. However,
coalition-proofness raises the possibility that if the contracts are not binding then some subset of sellers in a coalition market may be able to change their trading prices at the expense of other traders in the coalition. To rule out these possibilities we now introduce the concept of a coalition-proof improvement upon a price vector.

**Definition 2.8** Fix a coalition of traders $T' = (B', S') \in \chi^B \times \chi^S$ and a price vector $p \in \mathbb{R}_+^{S'}$. A coalition of traders $A = (B'', S'') \subset T'$ has a coalition-proof improvement upon price vector $p$ if there exists a price vector $p' \in \mathcal{E}_{CP}(A)$ such that $\min p' < \min p$ and $\pi_i(p', A) > \pi_i(p, T')$ for all $i \in S''$.

We can now define a coalition-proof analogue of the Bertrand core in which coalitions may form but sellers exhibit limited cooperation in setting prices.

**Definition 2.9** A price vector $p \in \mathbb{R}_+^s$ is in the coalition-proof Bertrand core if $p$ is a coalition-proof Bertrand equilibrium, all sellers quote the same price, and no coalition of traders, $A \subset G$, has a coalition-proof improvement upon $p$.

We shall let $C_{CP}(G) \subset \mathbb{R}_+^s \cup \{\emptyset\}$ denote the price vectors in the coalition-proof Bertrand core. The next example shows that the coalition-proof Bertrand core may be empty.

**Example 2.4** Consider a market with two sellers, $S = \{1, 2\}$, and two buyers, $B = \{1, 2\}$. The sellers have the same cost function given by $C(q) = q^2$. The demand of buyers 1 is given by $D_1(p) = \max\{0, 4 - \frac{1}{2}p\}$ and the demand of buyer two is $D_2(p) = \max\{0, 6 - \frac{1}{3}p\}$. Routine calculations show that the unique coalition-proof Bertrand equilibrium for the grand coalition is for both sellers to quote a price of 6 to the buyers $\mathcal{E}_{CP}(G) = \{6, 6\}$. At this equilibrium each seller earns a profit of 8. Now suppose one of the sellers and buyer 1 were to form a coalition. If the seller quotes price $p = 6 - \epsilon$, $\epsilon > 0$, to buyer one then as $\epsilon \rightarrow 0$, the profit of the seller tends towards 9.\(^8\) Therefore this market is a coalition-proof improvement upon the grand coalition and the coalition-proof Bertrand core is empty $C_{CP}(G) = \{\emptyset\}$.

\(^8\)The best response of the seller is not well-defined, but we could have assumed that the price space is finite and consider the limit as the price space becomes finer.
Unfortunately, it is not easy to identify conditions which guarantee the non-emptiness of the coalition-proof Bertrand core but it seems to be the case that the coalition-proof Bertrand core is more likely to be non-empty when the demand side of the market contains few buyers compared with the supply side of the market as this limits the types of coalition markets which could form. Investigating the properties of the coalition-proof Bertrand core is left as a topic for future research.

2.2.4 Market contracts, coalitions and the Bertrand core: a summary

In this section we outline the relationship between the standard solution concepts in price-setting games and the coalitional concepts introduced here. As noted at the beginning, we have assumed throughout that traders cannot write binding contracts so any solution must be at least a Nash equilibrium of the game under consideration. In Table 1 we show the different solution concepts depending upon the degree of cooperation between sellers and the types of contracts they can form. The standard contract type, that of Bertrand competition, assumes that sellers enter into the market with a commitment to supply demand forthcoming from all buyers. In this case, even if sellers communicate, the inability to form coalitions and the assumption that sellers act non-cooperatively in setting prices, means that the solution concept is the standard Bertrand equilibrium. If sellers offer Bertrand contracts but are willing to communicate and cooperate in setting prices then the solution concept is the coalition-proof Bertrand equilibrium introduced in Chowdhury and Sengupta (2004). However, if traders can form coalitions, as we considered here, but sellers act non-cooperatively in setting prices, then the solution concept is the Bertrand core. If traders can form coalitions and sellers are willing to cooperate in setting prices then the solution concept is the coalition-proof Bertrand core.
### Table 2.1: Market contracts: a summary of solution concepts.

<table>
<thead>
<tr>
<th></th>
<th>Non-cooperative price-setting</th>
<th>Cooperative price-setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bertrand contracts</td>
<td>Bertrand equilibrium</td>
<td>Coalition-proof Bertrand equilibrium</td>
</tr>
<tr>
<td>Coalition contracts</td>
<td>Bertrand core</td>
<td>Coalition-proof Bertrand core</td>
</tr>
</tbody>
</table>

2.3 Conclusion

The concept of the Bertrand core is an original combination of the early insights of Bertrand and Edgeworth regarding the formation of contracts and exchange. The exchange game presented here provides an elegant and tractable model for studying the formation of price-making contracts. The results show that even if we permit coalitions of traders to form within the grand coalition the Bertrand core is non-empty and as market become large we should expect price-making contracts to be close to the competitive equilibrium. However, if traders communicate, cooperate, but cannot commit to binding contracts, then the coalition-proof Bertrand core is the appropriate core concept.

Given that the Bertrand core is a new concept for analyzing market exchange there are several possibilities for future research. First, Chowdhury and Sengupta (2004) presented a limit result regarding the set of coalition-proof Bertrand equilibria which showed that under certain conditions the limit equilibrium set is the competitive equilibrium as the number of sellers becomes large. It may be possible to use this result to show that the coalition-proof Bertrand core is non-empty in markets with large numbers of sellers. This would then provide a foundation for price-taking behaviour even when sellers act cooperatively in setting prices. Aumann (1964) showed that in an economy with an atomless measure space of traders the Edgeworth core is equal to the set of Walrasian allocations. A similar result is likely to hold regarding the Bertrand core in markets with demand generated by an atomless measure space of buyers.

Second, the exchange game which we studied assumed that the set of traders was known with certainty and there was complete information regarding traders’ types. This is clearly a
restrictive assumption. Janssen and Rasmussen (2002) analyzed a price game where the set of traders was inactive with some exogenous probability. This possibility of inactivity could be studied in the context of the Bertrand core. Given a probability that some traders are inactive coalitions may prefer not to form because sellers could end up supplying the whole of the demand from the coalition and contracts may be quite different from those considered here. Alternatively, the exchange game could be extended to admit the possibility that traders have incomplete information regarding each others’ cost types.

Third, we assumed that there were no non-convexities in the market. The analysis could be extended to cover the cases where each of the sellers incurs a sunk/avoidable fixed cost upon trading. This then produces non-convexities in the cost function. Saporiti and Coloma (2010) considered a market with this characteristic and showed that the existence of a competitive equilibrium guarantees the non-emptyness of the set of Bertrand equilibria. However, a Bertrand equilibrium may exist but a market may fail to possess a competitive equilibrium.

Finally, we remarked that the Bertrand core is neither wholly cooperative nor non-cooperative. It should be possible to analyze refinements of the core which admit greater cooperation between traders within any coalition. In the coalition-proof Bertrand core there is limited cooperation between traders, but the inability to form binding contracts means that the solution must be a Nash equilibrium of the game. If traders could write binding contracts and were cooperative it would of great interest to study the core and bargaining set of the game and compare it with the Bertrand core and the coalition-proof core.

2.4 Appendix to Chapter 2

Proof of Lemma 2.1. As \( h_i(p) = \arg \max_{q \in \mathbb{R}^+} \pi_i(q) \) if \( p = 0 \) then the profit of the seller is \( \pi_i(q) = -C_i(q) \). Therefore the profit maximizing output is \( q = 0 \). To establish the second

---

9Kaneko (1977) studied a price-setting game, similar to the exchange game presented here, and characterized both the core and the bargaining set.
part of the lemma note that \( h_i(p) \) must satisfy the first-order condition for maximization:

\[
p - C_i'(h_i(p)) = 0.
\]

Differentiating w.r.t. \( p \) we obtain:

\[
1 - C_i''(h_i(p))h_i'(p) = 0.
\]

Rearranging:

\[
h_i'(p) = \frac{1}{C_i''(h_i(p))}.
\]

As sellers have strictly convex cost functions \( C_i''(\cdot) > 0 \) and \( h_i'(p) > 0 \).

**Proof of Lemma 2.2.** From the definition \( \pi_i^*(p) = ph_i(p) - C_i(h_i(p)) \) and:

\[
\pi_i''(p) = h_i(p) + ph_i'(p) - C_i'(h_i(p))h_i'(p).
\]

Factorizing:

\[
\pi_i''(p) = h_i(p) + \frac{h_i'(p)}{h_i'(p) - p - C_i'(h_i(p))}.
\]

From the first-order condition \( p - C_i'(h_i(p)) = 0 \) therefore:

\[
\pi_i''(p) = h_i(p).
\]

From Lemma 2.1 we know that \( h_i(p) > 0 \) for all \( p > 0 \) which establishes the result.

**Proof of Lemma 2.3.** Suppose a contradiction that \( p_*(r) > p_*(r+1) \). Then there exists a \( p' \) such that \( p_*(r) > p' \geq p_*(r+1) \) and \( p' \in C_{r+1}(G) \). As \( p_*(r) > p' \) all sellers quoting \( p' \) does not belong to the \( r \)-replicated Bertrand core. Therefore there is a subset of the \( r \)-replicated traders which have an improvement upon \( p' \). The profit which sellers would obtain from posting \( p' \) in the \( r \)-replicated market is:

\[
p'rD(B, p')\beta_i(p') - C_i(rD(B, p')\beta_i(p')).
\]

This simplifies to:

\[
\frac{p'D(B, p')h_i(p')}{\sum_{j \in S} h_j(p')} - C_i(\frac{p'D(B, p')h_i(p')}{\sum_{j \in S} h_j(p')}).
\]
Note that profit does not depend on \( r \). Therefore sellers would obtain the same profit from posting price \( p' \) in the \( r + 1 \)-replicated market. However, the same subset of traders which had an improvement upon \( p' \) would also have an improvement upon \( p' \) in the \( r + 1 \)-replicated market. This contradicts \( p' \in \mathcal{C}_{r+1}(G) \). The same proof can be used to establish that \( p^*(r) \geq p^*(r + 1) \). ■
Chapter 3

On revealed preferences in oligopoly games

3.1 Introduction

Suppose we make a finite set of observations $T = \{1, \ldots, m\}, m \geq 1$, of a perfectly homogeneous-good oligopoly market. There is a finite number of sellers $N = \{1, \ldots, n\}, n \geq 2$, which compete in the market. In each observation we observe each seller’s price, their output, and possibly their cost information. Given this information how should we go about checking whether these observations are consistent/inconsistent with sellers playing a Nash equilibrium in prices? That is to say, if each seller simultaneously and independently chooses a price, with a commitment to supply all the demand forthcoming at that price, a classical Bertrand game, what observable restrictions does the Nash equilibrium in prices impose upon the outcomes? In this chapter we solve this theoretical problem by providing a complete characterization of the observations which can be ‘rationalized’ by the classical Bertrand model. We identify two main conditions which are economically meaningful and take the form of linear inequalities. The first of these conditions is what we term the *monopoly deviation condition* which requires that we do not observe a situation where one seller could profitably deviate by serving the entire market demand. The second condition is what we term the
tie deviation condition which requires that we do not observe a situation where one seller could profitably deviate by joining a price tie. Together with a weak monotonicity condition on observed costs we find that these conditions provide a complete characterization of the observations which can be Bertrand rationalized with standard primitives.

This chapter continues the tradition in economic theory of analyzing what structure, if any, various equilibrium concepts impose upon observable outcomes. Although the preference based approach to economic theory, where we state specific primitives and analyze equilibrium existence and comparative statics etc., is still the way economic theory is mainly done, the establishment of observable restrictions provides a useful complement to primitive-based theory for a number of reasons. First, it allows us to establish whether economic equilibrium concepts impose any structure upon the observable outcomes and therefore whether it is possible, even at a theoretical level, to refute equilibrium concepts. Second, by asking whether a set of observations can be revealed consistent/inconsistent with a given equilibrium concept we often permit greater variety of economic primitives than when we directly state primitives and analyze equilibrium properties. For example, when analyzing classical Bertrand games it is typical to assume that cost functions of sellers are convex. Here we impose no such restrictions. Instead, we simply require that observations are consistent with an increasing cost function. Finally, the structure of equilibrium sets is a useful addition to the stock of theoretical knowledge regarding canonical economic models.

There is a substantial literature on revealed preferences in economic models so we shall only mention the key touchstones in the literature here. The most frequent recourse to revealed preference is in the context of consumer theory where Samuelson’s weak axiom of revealed preference is well-known to constitute a necessary condition for the existence of a ‘rationalizing’ preference relation. That is to say, for those observations which violate the weak axiom it is not possible to find a utility function which would generate the observations from utility maximization. An early counter example, by John R. Hicks, showed that in a three-commodity economy consumer’s Walrasian demands may satisfy the weak axiom and still exhibit intransitive choices (and therefore be revealed inconsistent with utility maxi-
Afriat (1967) was a seminal paper in that it looked for necessary and sufficient conditions upon price/quantity observations for the existence of a rationalizing utility function. The key condition was a strengthening of the weak axiom, a form of cyclical consistency, which is equivalent to Hendrik Houthakker’s strong axiom of revealed preference.

This early literature on revealed preferences in consumption theory clearly illustrates the method of characterizing observable restrictions. The method starts from the basis that there is something unobservable (preference) which is fixed across observations, and something which is observable (budget sets/choices) which change across observations. The literature then asks what restrictions must be placed on the observables for them to be consistent which the existence of a well-behaved unobservable. Following in this tradition we shall say that a set of oligopoly observations is Bertrand rationalizable if there exists a market demand for each observation and a cost function for each seller which is able to account for the observed market outcomes. Hence, the analogue with consumer theory is that the cost function represents the type of each of the sellers which is fixed across observations whereas the changes in the market demand represent the observable which accounts for the changes in observed outcomes.

The canonical general equilibrium model which brings consumption choices together with market clearing conditions was thought to have few observable restrictions because of the well-known result that generically economies have a finite, and cardinally odd, set of equilibria but no other general restrictions are imposed upon the equilibrium price set. The results of Debreu-Mantel-Sonnenschein went further by showing that for any bounded continuous function defined on a compact subset of the price space (not including the origin) which satisfies Walras law and is homogeneous of degree zero in prices there exists an economy which generates the function as the excess demand. For a long time these results were

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1 See, for example, Mas-Colell et al. (1995, p.35). Although observations which satisfy the weak axiom of revealed preference also satisfy the weak weak axiom of revealed preference and can be rationalized by a complete, but not necessarily transitive, preference relation (Jerison and Quah, 2008).

2 Forges and Minelli (2009) have extended Afriat’s theorem to budget sets which may have nonlinear frontiers such as in strategic market games.

3 This remains true even if one restricts attention to distribution economies where individual endowments
thought to seriously undermine the usefulness of the general equilibrium model. However, in an important paper Brown and Matzkin (1996) analyzed what restrictions observations of prices and individual endowments must satisfy to belong to the equilibrium manifold. That is to say, what restrictions must prices and endowments satisfy so that there exist quasiconcave utility functions for each trader such that the observed prices and endowments are points on the equilibrium manifold generated by those preferences. They showed that there exist non-trivial restrictions and exhibited simple exchange economies which could not be rationalized as Walrasian equilibria.

There is however a fundamental difference between this chapter and the seminal works of Afriat (1967) and Brown and Matzkin (1996) in that the latter were characterizing the observable outcomes of non-strategic equilibrium concepts whereas this chapter is concerned with the strategic Nash equilibrium. It is only in the last ten years or so that research has turned to applying the revealed preference method to the Nash equilibrium.

However, as the Nash equilibrium is a full rationality concept it is perhaps not surprising that it should be amenable to characterization in terms of observables. This more recent literature is what is most closely related to this chapter. Sprumont (2000) considered the case where we observe players choices from all possible subsets of strategy choices, all action spaces were finite, and identified two conditions, expansion and contraction consistency, which are necessary and sufficient to be able to rationalize the observed choices as Nash equilibria. Zhou (2005) considered two-player games where players' strategy sets were the unit interval and assumed that we observe a finite subset of choices. He then asked what conditions the observed choices must satisfy to be able to find payoff functions for the players which are quasiconcave in own are collinear and the distribution of income is independent of prices (Kirman and Koch, 1986).

4The equilibrium manifold being the graph of the Walras correspondence.

5Although as Rizvi (2006, p.239) notes the Brown and Matzkin results do not overturn Kenneth Arrow's statement that "in the aggregate, the hypothesis of rational behaviour has in general no implications" because their results rely on us being able to observe individual endowments.

6A survey of the recent developments of testable restrictions in both equilibrium theory and game theory is provided by Carvajal et al. (2004).

7Anticipating this work, Sprumont (2000, p.221) noted at the end of his paper that it would be interesting to characterize observable restrictions in games with more "economic flesh" such as oligopoly games.
strategy and continuous in all strategies which would explain the observed choices as Nash equilibria. By exploiting the path connectedness of the best response correspondence she provided a no-improper-crossing condition which if satisfied means the observations admit rationalizing payoff functions. Most closely related to this chapter is work by Carvajal et al. (2010) which considered the Cournot oligopoly model. They assumed we make a finite number of observations of a homogeneous-good oligopoly market and in each observation we observe a single market price, sellers’ outputs and possibly also cost information. They show that if the observations satisfy a marginal condition, which roughly means that we do not observe instances where sellers can profitably deviate by reducing their outputs, then providing observed costs are co-monotone with outputs, the observations can be rationalized as Cournot equilibria. If cost information is unavailable, then any set of observations can be Cournot rationalized if one permits a general increasing cost function. However, they introduce a stronger criterion of a ‘convincing rationalization’, which means that the cost function must be constructed so that the marginal cost lies between the observed marginal costs, and show that this imposes restrictions upon price/output observations.

This chapter aims to add to this literature by examining the other benchmark oligopoly model. We start from the same point as Carvajal et al. (2010) by assuming that we observe sellers’ prices, outputs and possibly cost information and look for the restrictions which the Bertrand equilibrium imposes upon the observables. The case when sellers have the same price in a given observation could clearly also be consistent with sellers choosing outputs in equilibrium. This special case is one of the most interesting aspects of the results provided here as we shall be able to perform revealed preference analysis on example observations to establish their consistency with either the Bertrand or Cournot equilibrium concepts. However, given the idealized nature of the Bertrand model it is unlikely that the conditions provided here could be used to test real-world market competition. Nevertheless, given the prominence of the model in the literature we consider the characterizations to be of considerable theoretical interest. In the next section of the chapter we set out the general theoretical problem and provide the major results. The mathematical requirements are
limited as we only use basic set theory to organize the observations. In the next section we present a couple of simple examples and use them to apply the theoretical results. The final section draws some conclusion regarding future directions for research in this area.

### 3.2 Revealed Nash equilibria in oligopoly games

Before addressing the main problem of observable restrictions which the Nash equilibrium imposes upon oligopoly outcomes we shall first define the standard notions of Bertrand and Cournot equilibrium as well as when a set of observations admit a rationalization by either of the models. Although there is a vast literature regarding the two models, the aim here is to present the simplest form of the equilibrium concept. First consider a perfectly homogeneous-good market with \( N = \{1, ..., n\}, \ n \geq 2 \), sellers. Each seller has a cost function \( C_i : \mathbb{R}_+ \to \mathbb{R}_+ \) which is strictly increasing, continuous and satisfies \( C_i(0) = 0 \). The market demand \( D : \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous and strictly decreasing whenever \( D > 0 \). We shall let the function \( F : \mathbb{R}_+ \to \mathbb{R}_+ \) denote the inverse market demand in what follows. Now consider first the notion of a Bertrand equilibrium. Each seller simultaneously and independently chooses a \( P_i \in \mathbb{R}_+ \), sellers in the market commit to supplying all the demand forthcoming at any price.\(^8\) If a seller posts the unique minimum price in the market then it obtains all the market demand and its payoff is \( P_iD(P_i) - C_i(D(P_i)) \). If a seller ties with \( m - 1 \) sellers at the minimum price then they share the demand equally between themselves and the payoff of each seller is given by \( \frac{1}{m}P_iD(P_i) - C_i(\frac{1}{m}D(P_i)) \). If a seller is undercut by any other seller in the market then it obtains zero demand, and given the assumption that the cost function passes through the origin, its payoff is zero. These payoffs are summarized below.

\[
\pi_i(P_i, P_{-i}) = \begin{cases} 
P_iD(P_i) - C_i(D(P_i)) & \text{if } P_i < P_j \ \forall j \neq i; \\
\frac{1}{m}P_iD(P_i) - C_i(\frac{1}{m}D(P_i)) & \text{i ties with } m - 1 \text{ sellers;} \\
0 & \text{if } P_i > P_j \text{ for some } j.
\end{cases}
\]  

\[(3.1)\]

\(^8\)This is what distinguishes Bertrand competition from Bertrand-Edgeworth competition.
Given this specification of the payoffs we can state the Bertrand equilibrium as a Nash equilibrium of this price-setting game.

**Definition 3.1** A pure strategy Bertrand equilibrium is a Nash equilibrium of the game with payoffs defined by eq. (1). That is, a vector of prices \((P^B_i, P^B_{-i})\) such that \(\pi_i(P^B_i, P^B_{-i}) \geq \pi_i(P_i, P^B_{-i})\) for all \(P_i \in \mathbb{R}_+\) and \(i \in N\).

Now consider the alternative case when sellers choose outputs which are sent to the market. For a more detailed exposition of the Cournot model the reader is referred to Vives (1999. Ch.4). Each seller simultaneously and independently chooses a \(Q_i \in \mathbb{R}_+\). Given the total output chosen by the sellers the market demand clears this output and sends back a single market-clearing price. The payoff which any seller receives, given the vector of chosen outputs is \((Q_i, Q_{-i})\), is given below.

\[
\pi_i(Q_i, Q_{-i}) = F(\sum_{j=1}^{n} Q_j)Q_i - C_i(Q_i) \tag{3.2}
\]

The Cournot equilibrium is then a Nash equilibrium of this output-setting game.

**Definition 3.2** A pure strategy Cournot equilibrium is a Nash equilibrium of the game with payoffs defined by eq. (2). That is, a vector of outputs \((Q^C_i, Q^C_{-i})\) such that \(\pi_i(Q^C_i, Q^C_{-i}) \geq \pi_i(Q_i, Q^C_{-i})\) for all \(Q_i \in \mathbb{R}_+\) and \(i \in N\).

These are the the most commonly used equilibrium concepts in oligopoly theory and yet little has been known, until recently, about what structure they impose upon the observable outcomes in the market. We now turn to this problem. Suppose we make a finite number of observations, \(T = \{1, ..., m\}, \ m \geq 1\), of a homogenous-good oligopoly market. In each period we observe the price of each seller, \(P_{it}\), their output, \(Q_{it}\), and their costs incurred, \(C_{it}\). The total set of observations can then be summarized as \((P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}\). To organize the observations let \(P^*_t = \min_{i \in N} P_{it}\). That is, \(P^*_t\) is the minimum price posted in the market in observation \(t\). Let \(Q^*_t = \sum_{i \in N} Q_{it}\) denote the aggregate output produced in observation \(t\). The set of sellers which tie at the minimum price, what we shall informally call the set of ‘active sellers’, is given by \(A_t = \{i \in N : P_{it} = P^*_t\}\).
We shall impose certain restrictions upon the type of observations which we make to ensure that they are not immediately inconsistent with the models. For example, if we were to observe an observation where sellers have different prices then such a set of observations is clearly never going to be consistent with sellers choosing outputs and the market sending back a single market-clearing price. Similarly, if we make observations where two sellers produce output at different prices, so the law of one price fails, then such a set of observations is always going to be inconsistent with the Bertrand model which postulates that all trade takes place at the minimum price. Therefore we shall introduce the notion of a generic homogeneous-good set of oligopoly observations which has the basic features of a perfectly homogeneous-good market.

**Definition 3.3** A set of oligopoly observations is a generic homogeneous-good market data set if it satisfies the following conditions:

i) \( P_{it} > 0, \quad Q_{it} \geq 0, \quad C_{it} \geq 0, \quad P_{it}Q_{it} \geq C_{it} \) and \( Q_{it} \neq Q_{it'} \) whenever \( t \neq t' \).

ii) If \( P_{it} > P_{jt} \) then \( Q_{it} = 0 \).

iii) If \( P_{it} = P_{jt} = P^*_t \) then \( Q_{it} = Q_{jt} \).

iv) \( |A_t| \geq 2 \).

The first part imposes mild conditions that we observe positive prices, non-negative outputs, revenues are greater than costs and sellers’ outputs change across observations. The requirement that outputs change across observations means we observe some variation in the data and it stops the data contradicting itself. The second part means that the market is consistent with the law of one price: if any sellers raises their price above that of another seller then they receive zero demand and produce zero output. The third part states that sellers tieing at the minimum price split the demand equally. In the literature on price games, alternative sharing rules at minimum price ties have been used, but as this is the most common rule used in the literature we shall consider this case here. The final part states that we do not observe a monopolist in any observation. A special type of data set is what we shall term a single-price data set: this is a generic homogeneous-good data set with the additional property that \( P_{it} = P^*_t \) for all \( i \in N \) and \( t \in T \). This special case...
where we observe a single price by all sellers in each observation is particularly interesting from a theoretical perspective because we shall be able to analyze whether observations are consistent/inconsistent with either the Bertrand or Cournot equilibrium concepts.

When a set of observations is consistent with sellers playing a Nash equilibrium in prices or outputs we shall term the observations as rationalizable by the Bertrand or Cournot models respectively. As noted at the beginning, we shall assume that variations in observed outcomes are due to changes in the market demand across observations with sellers’ cost functions fixed across observations. The method of revealed preference starts from the basis that there is something observable which changes across observations and something unobservable which remains fixed across observations. The method then asks what restrictions must be placed upon the observables such that they are consistent with the existence of a well-behaved unobservable. Formally, we define the notion of Bertrand rationalizability below.

**Definition 3.4** A set of generic homogeneous-good observations, \((P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}\), is Bertrand rationalizable if there exist \(C^2\) functions, \(\bar{C}_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) for each \(i \in N\), \(\bar{D}_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) for each \(t \in T\), such that:

i) \(\bar{C}_i(0) = 0\) and \(\bar{C}_i'(x) > 0\) for all \(x > 0\).

ii) \(\bar{D}_t(x) \geq 0\) and \(\bar{D}_t'(x) \leq 0\) with the latter inequality holding strictly whenever \(\bar{D}_t(x) > 0\).

iii) \(\bar{C}_i(Q_{it}) = C_{it}\) and \(\bar{D}_t(P^*_{it}) = Q^*_{it}\).

iv) The set of observed prices \((P_{1t}, ..., P_{nt})\) is a Bertrand equilibrium in pure strategies for each \(t \in T\).

The first three parts state that we can find standard demands for each observation and cost functions for each seller such that the cost and demand functions explain the observed demands and costs. The final part states that these cost and demand functions must also be such that the observed set of prices in each observation constitute a pure strategy Bertrand equilibrium. The alternative concept of Cournot rationalizability requires that we can find inverse demands for each observation and cost functions for each seller such that these functions explain the observed outputs and costs. Moreover, the cost and inverse demand functions must be such that the observed outputs in each observation constitute a pure
strategy Cournot equilibrium. For completeness we state this formally below.

**Definition 3.5** A single-price data set, \((P^*_t, Q_{it}, C_{it})_{i \in N, t \in T}\), is Cournot rationalizable if there exist \(C^2\) functions, \(\bar{C}_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) for each \(i \in N\), \(\bar{F}_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) for each \(t \in T\), such that:

i) \(\bar{C}_i(0) = 0\) and \(\bar{C}_i'(x) > 0\) for all \(x > 0\).

ii) \(\bar{F}_t(x) \geq 0\) and \(\bar{F}_t'(x) \leq 0\) with the latter inequality holding strictly whenever \(\bar{F}_t(x) > 0\).

iii) \(\bar{C}_i(Q_{it}) = C_{it}\) and \(\bar{F}_t(Q^*_t) = P^*_t\).

iv) The set of observed outputs \((Q_{1t}, ..., Q_{nt})\) is a Cournot equilibrium in pure strategies for each \(t \in T\).

Before stating conditions which characterize the rationalizable sets we shall introduce some notation which helps organize the observations. Let \(R_i(t) = \{t' \in T : Q_{it'} \geq Q^*_i\}\). That is, \(R_i(t)\) is the set of observations when the output of seller \(i\) is greater than the aggregate output in observation \(t\). Let \(S_i(t) = \{t' \in T : Q_{it'} < Q_{it}\}\). The set \(S_i(t)\) is those observations when the output of seller \(i\) is less than its own output in observation \(t\). We shall also want to compare sellers’ outputs with regards to the following quantity \(\hat{Q}_t = Q_t^*/(|A_t| + 1)\). We introduce \(M_i(t) = \{t' \in T : Q_{it'} \geq \hat{Q}_t\}\) which is the set of observations when the output of seller \(i\) is greater than or equal to \(\hat{Q}_t\).

**Definition 3.6** A set of generic homogeneous-good oligopoly observations, \((P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}\), satisfy the increasing cost condition (ICC) if, whenever \(S_i(t) \neq \emptyset\), then \(C_{it} - C_{it'} > 0\) for all \(t' \in S_i(t)\).

The interpretation of the increasing cost condition is straightforward: it states that whenever we observe a seller producing a higher output then its observed costs should increase. It should be clear that this is a necessary condition for rationalization by either oligopoly model. If violated then we would not be able to construct an increasing cost function which explains the observed costs.

**Definition 3.7** A set of generic homogeneous-good oligopoly observations, \((P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}\), satisfy the monopoly deviation condition (MDC) if, whenever \(R_i(t) \neq \emptyset\), then \(P^*_t Q_{it} - C_{it} \geq P^*_t Q^*_t - C_{it'}\) for all \(t' \in R_i(t)\) with the inequality holding strictly whenever \(Q_{it'} > Q^*_i\).
The monopoly deviation condition requires that the observed profit of any sellers is at least as great as the profit they could obtain from supplying the entire market demand and incurring a cost at least as large as that required to meet the demand. Note that if the relevant inequalities are not defined then this does not violate the condition. The key point is that we do not observe a violation of the condition. The intuition behind the condition is that when it is satisfied we shall be able to find cost and demand function which are consistent with sellers not wanting to post a price which undercuts the observed minimum price.

**Definition 3.8** A set of generic homogeneous-good oligopoly observations, \((P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}\), satisfy the tie deviation condition (TDC) if, whenever \(M_i(t) \neq \emptyset\), then \(P_i^* \hat{Q}_t \leq C_{it}'\) for all \(t' \in M_i(t)\) and \(i \in N \setminus A_t\) with the inequality holding strictly whenever \(Q_{it'} > \hat{Q}_t\).

The tie deviation condition states that if a seller is not active in a given observation then the revenue it could obtain from tying at the minimum price, note that \(\hat{Q}_t\) is the share of the demand which the seller could obtain from tying at the minimum price, is less than the cost which the seller would obtain from supplying a quantity at least as high as \(\hat{Q}_t\). As with the monopoly deviation condition, if the relevant inequalities are not defined then this does not violate the condition. All that matters is that we do not observe a violation when the relevant quantities are defined. The intuition behind the condition is that if satisfied we shall be able to construct cost and demand functions such that inactive sellers cannot profitably deviate by joining a minimum price tie.

**Definition 3.9** A single-price data set, \((P_i^*, Q_{it}, C_{it})_{i \in N, t \in T}\), satisfies the marginal condition (MC) if, whenever \(S_i(t) \neq \emptyset\), then \(P_i^*Q_{it'} - C_{it'} < P_i^*Q_{it} - C_{it} for all t' \in S_i(t)\).

The marginal condition applies to single-price data sets and says we do not observe a period where one seller could reduce its output and increase its profits provided that the market price remains unchanged. As with the previous conditions, if the inequality is not defined this does not contradict the condition. We now present the first main result which is a complete characterization of which generic homogeneous-good data sets can be Bertrand rationalized.
Theorem 3.1 A set of generic homogeneous-good oligopoly observations, \((P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}\), is Bertrand rationalizable if and only if it satisfies ICC, MDC and TDC.

Proof. First we prove the necessity part of the result and second the sufficiency part. It should be clear from the definition of Bertrand rationalizability that if ICC is violated we cannot construct a cost function which satisfies part \((i)\) and \((iii)\) of Definition 4. Suppose instead that MDC is violated. Then there is an \(i, t\) and \(t' \in R_i(t)\) such that \(P^*_t Q_{it} - C_{it} \leq P^*_t Q^*_t - C_{it'}\) and \(Q_{it'} > Q^*_t\). If the observations are rationalizable we have that \(P^*_t Q_{it} - \bar{C}_i(Q_{it}) \leq P^*_t \bar{D}_t(P^*_t) - \bar{C}_i(Q_{it'})\). Then seller \(i\) could set a price \(P^*_t - \epsilon\), and by choosing \(\epsilon > 0\) to be sufficiently small, we have \(\bar{D}_t(P^*_t - \epsilon) < Q_{it'}\) and \(\bar{C}_i(\bar{D}_t(P^*_t - \epsilon)) < \bar{C}_i(Q_{it'})\). This means seller \(i\) could obtain a strictly higher profit by deviating to price \(P^*_t - \epsilon\) and contradicts the observed prices being a Bertrand equilibrium.\(^9\) Suppose that TDC is violated. Then there is an \(i, t\) and \(t' \in M_i(t)\) such that \(i \in N \setminus A_t, P^*_t \hat{Q}_t - C_{it'} \geq 0\) and \(Q_{it'} > \hat{Q}_t\). If the observations are rationalizable we have \(P^*_t \hat{Q}_t - \bar{C}_i(Q_{it'}) \geq 0\). As \(i \in N \setminus A_t\) the seller obtains zero observed profit in period \(t\), but this means seller \(i\) could deviate and join the minimum price tie at \(P^*_t\) and obtain \(P^*_t \hat{Q}_t - \bar{C}_i(\hat{Q}_t) > 0\) as \(\bar{C}_i(\hat{Q}_t) < \bar{C}_i(Q_{it'})\).\(^10\)

To see that the conditions in the theorem are also sufficient we first show how to construct sellers’ cost functions given the observed costs. Second we show how to construct the market demands. Finally we show that the constructed functions satisfy all the requirements of Definition 4. In constructing the relevant functions it is helpful to introduce the following notation. We shall let \(r_i(t) = \{t' \in R_i(t) : Q_{it'} < Q_{it} \forall t \in R_i(t)\}\) so that \(r_i(t)\) is that observation in \(R_i(t)\) which minimizes the output of seller \(i\) across the observations in \(R_i(t)\). We shall let \(s_i(t) = \{t' \in S_i(t) : Q_{it'} > Q_{it} \forall t \in S_i(t)\}\). That is, \(s_i(t)\) is that observation belonging to \(S_i(t)\) which maximizes the output of seller \(i\) across the observations in \(S_i(t)\). Let \(m_i(t) = \{t' \in M_i(t) : Q_{it'} < Q_{it} \forall t \in M_i(t)\}\). That is, \(m_i(t)\) is that observation belonging to \(M_i(t)\) which minimizes the output of seller \(i\) across the observations in \(M_i(t)\).

First, as the observed costs satisfy ICC we can construct a smooth, strictly increasing cost function for each seller with the properties that \(\bar{C}_i(Q_{it}) = C_{it}\) and \(\bar{C}_i(0) = 0\). Now we

\(^9\)The same argument applies when \(P^*_t Q_{it} - C_{it} < P^*_t Q^*_t - C_{it'}\) and \(Q_{it'} = Q^*_t\).

\(^10\)The same argument applies when \(P^*_t Q_{it} - C_{it'} > 0\) and \(Q_{it'} = \hat{Q}_t\).
impose the following three restrictions upon these cost functions:

(1) For all $t \in T$ such that $R_i(t)$ is non-empty and $Q_{ir_i(t)} > Q_{i1}^*$ choose the cost function so that $C_i(Q_{i1}^*) > \max\{P_t^*(Q_{i1}^* - Q_{it}) + C_{it}, C_{is_i(t)}\}$.

(2) Define $V_i = \{t' \in T : R_i(t') = \emptyset\}$, $v_i = \{t' \in T : Q_{i1}^* \leq Q_{i1}^* \forall t \in V_i\}$ and $w_i = \max_{t \in V_i} P_t^*(Q_{i1}^* - Q_{it}) + C_{it}$. Then the cost functions are constructed so that $C_i(Q_{i1}^*) > v_i$.

(3) For all $i \in N \setminus A_t$ and $Q_{im_i(t)} > \hat{Q}_t$ construct the cost function so that $C_i(\hat{Q}_t) > P_t^*\hat{Q}_t$.

The restriction in (1) can always be satisfied because as MDC is satisfied we have that $P_t^*Q_{it} - C_{it} > P_t^*Q_{i1}^* - C_{ir_i(t)}$ and the cost function can be chosen so that $C_i(Q_{i1}^*) = C_{ir_i(t)} - \epsilon$ provided $\epsilon > 0$ is sufficiently small. The restriction in (2) can always be satisfied because it states that for those observations when we do not observe the seller $i$ producing an output at least as large as the aggregate market output we can choose the cost function so that MDC is satisfied. The restriction in (3) can be satisfied because as TDC is not violated we have that whenever $i \in N \setminus A_t$ then $P_t^*\hat{Q}_t - C_{im_i(t)} < 0$ whenever $Q_{im_i(t)} > \hat{Q}_t$ and the cost function of seller $i$ can be constructed so that $C_i(\hat{Q}_t) = C_{im_i(t)} - \epsilon$ provided $\epsilon > 0$ is sufficiently small.

To see how to construct the market demands for each observation consider the following piecewise-affine market demand $\bar{D}_t(P_t) = \max\{0, 2Q_{i1}^* - Q_{i1}^*P_t/P_t^*\}$. This is strictly decreasing on the interior of the output space and satisfies $\bar{D}(P_t^*) = Q_{i1}^*$. Note that the revenue as a function of price is $R(P_t) = 2Q_{i1}^*P_t - Q_{i1}^*P_t^2/P_t^*$. The marginal revenue is $R'(P_t) = 2Q_{i1}^* - 2Q_{i1}^*P_t/P_t^*$. Therefore $R'(P_t^*) = 0$ and revenue is maximized at $P_t^*$. This means $P_t\bar{D}_t(P_t) < P_t^*\bar{D}_t(P_t^*)$ whenever $P_t \neq P_t^*$. This property will be useful in showing that these functions are sufficient for Bertrand rationalizability.

To see that these functions are sufficient for Bertrand rationalizability note that we have satisfied parts (i) – (iii) of Definition 3.4 and that all that remains to be shown is that given the constructed functions the set of observed market prices constitutes a pure strategy Bertrand equilibrium. Consider any seller $i \in A_t$ or $i \in N \setminus A_t$ increasing their price. As $|A_t| \geq 2$ they lose any demand and make zero profit which cannot be an improvement upon the observed profit. Second, consider a seller $i \in N \setminus A_t$ which lowers their price to tie at $P_t^*$.

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11 The kink which occurs when demand becomes zero could be smoothed out on a small interval to give a $C^2$ function.
As the constructed cost function satisfies $P_t^*Q_t - \bar{C}_i(\hat{Q}_t) \leq 0$ this cannot be an improvement upon zero profit. Finally, suppose a seller undercut the market and posts a price $P_t^* - \epsilon$, $\epsilon > 0$. The profit the seller would obtain is $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^*))$. As the market demand achieves maximum revenue at $P_t^*$ we have $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) < P_t^*\bar{D}_t(P_t^*) = P_t^*Q_t^*$. As the market demand is strictly decreasing and cost functions strictly increasing we have $\bar{C}_i(\bar{D}_t(P_t^* - \epsilon)) > \bar{C}_i(Q_t^*)$. Combining these two inequalities gives $(P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^*)) < P_t^*Q_t^* - \bar{C}_i(Q_t^*)$. As the constructed cost functions satisfy $P_t^*Q_t - \bar{C}_i(Q_t) \geq P_t^*Q_t^* - \bar{C}_i(Q_t^*)$ we have $P_t^*Q_t - \bar{C}_i(Q_t) > (P_t^* - \epsilon)\bar{D}_t(P_t^* - \epsilon) - \bar{C}_i(\bar{D}_t(P_t^*))$. We can conclude that no seller can profitably undercut the market and the set of observed prices constitutes a pure strategy Bertrand equilibrium. ■

When we have the special case of a single-price data set this could be consistent with sellers choosing outputs and letting the market determine a single-market clearing price. It turns out that the structure imposed by the Cournot equilibrium upon the observable market outcomes can be characterized by the increasing cost condition and the marginal condition.

**Theorem 3.2** A single-price data set, $(P_t^*, Q_t, C_t)_{i \in N, t \in T}$, is Cournot rationalizable if and only if it satisfies ICC and MC.

**Proof.** See Carvjal et al. (2010, p.25, Thereom 5).12 ■

The result in Theorem 3.2 actually applies to a broader class of data sets than that considered here in that one could allow sellers to produce different quantities and the result still holds. It is worth noting that the characterizations in Theorem 3.1 and 3.2 are quite different in terms of what they impose upon the observables. The key condition for a set of observations to be Bertrand rationalizable is that we do not observe a situation where a seller can benefit from increasing its output and serving the entire market demand (monopoly deviation) whereas the condition for Cournot rationalizability is that we do not observe a situation where a seller can benefit from decreasing its output (marginal condition). Despite it often being assumed that price competition will result in lower prices than quantity competition

12They assume that the data satisfies the increasing cost condition and therefore state the result in solely in terms of the marginal condition which they term the ‘discrete marginal condition’.
it is well-known that when the set of Bertrand equilibria of a market is non-empty then it is possible for the Cournot equilibrium price to also constitute a Bertrand equilibrium (Vives, 1999, p.122). Therefore it is interesting to establish which market outcomes are consistent with both of the oligopoly models. We provide the following characterization.

**Theorem 3.3** A single-price data set, \((P^*_t, Q_t, C_t)_{i \in N, t \in T}\), is Bertrand and Cournot rationalizable if and only if it satisfies ICC, MC and MDC.

**Proof.** The result follows from combining the conditions in Theorems 3.1 and 3.2 and noting that a single-price data set trivially satisfies TDC.

The previous results started from the assumption that we are able to observe prices, outputs and cost information.\(^{13}\) What happens to these results if we drop cost information from the data set? That is, can any set of generic homogeneous-good market observations be Bertrand or Cournot rationalized? If any set of generic observations of prices and outputs could be explained by the models then they can not be refuted even at a theoretical level. It turns out that there are non-trivial restrictions imposed by the Bertrand equilibrium upon generic data sets. To see this consider the following example of a symmetric duopoly, \(n = 2\), with two observations, \(m = 2\).

<table>
<thead>
<tr>
<th>t</th>
<th>((P, Q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 4)</td>
</tr>
<tr>
<td>2</td>
<td>(2, 2)</td>
</tr>
</tbody>
</table>

If these observations are Bertrand rationalizable then we should be able to find a cost function for the first observation which satisfies \(0 < \bar{C}(4) \leq 4\) as sellers do not make losses. In the second observation the monopoly deviation condition requires that we can choose the cost function so that \(4 - \bar{C}(2) \geq 8 - \bar{C}(4)\) with \(0 < \bar{C}(2) < \bar{C}(4)\). Inspection of these inequalities reveals that they cannot be simultaneously satisfied and the table is an example

\(^{13}\)We could have started instead by assuming that we observe prices, outputs and profits and then inferred cost information as the difference between revenue and costs.
of a data set of prices and outputs which cannot be Bertrand rationalized. We now introduce two conditions for generic price and output data sets.

**Definition 3.10** A generic homogeneous-good set of prices and outputs, \((P_{it}, Q_{it})\), satisfies the revenue condition (RC) if, whenever \(R_i(t) \neq \emptyset\), then \(P^*_t Q_{it} > P^*_t (Q_t^* - Q_{it})\) for all \(t' \in R_i(t)\) and \(i \in A_t\).

The revenue condition states that the observed revenue which a seller obtains when it supplies an output at least as large as the observed aggregate output must be strictly greater than the increase in the revenue which a seller would obtain from supplying the entire market at the existing price.

**Definition 3.11** A generic homogeneous-good set of prices and outputs, \((P_{it}, Q_{it})\), satisfies the tied revenue condition (TRC) if \(i \in A_t\) then we do not observe a \(t' \in T\) such that \(i \in N \setminus A_{t'}\) and \(P^*_t Q_{t'} \geq P^*_t Q_{it}\) and \(Q_{t'} \leq Q_{it}\) with at least one of the inequalities holding strictly.

The tied revenue condition requires that if we observe a seller supplying the market in a given observation then we should not observe the same seller not supplying the market whenever it can obtain at least as high a revenue and produce a strictly smaller output or produce the same output and obtain a strictly higher revenue. We now present the following result which shows that both the revenue condition and tied revenue condition are necessary conditions for Bertrand rationalizability.

**Proposition 3.1** A generic homogeneous-good set of prices and outputs, \((P_{it}, Q_{it})\), is Bertrand rationalizable only if it satisfies RC and TRC.

**Proof.** Suppose we have a generic set of observations which violate RC. Then there is an \(i, t\) and \(t' \in R_i(t)\) such that \(P^*_t Q_{it'} \leq P^*_t (Q_t^* - Q_{it})\). If the observations are rationalizable we should be able to find a cost function such that \(C_i(Q_{it'}) \leq P^*_t Q_{it'}\) and \(C_i(Q_{it}) \leq P^*_t Q_{it}\) as sellers do not make losses. MDC also requires that \(C_i(Q_{it'}) - C_i(Q_{it}) > P^*_t (Q_t^* - Q_{it})\).

However, if \(C_i(Q_{it'}) \leq P^*_t Q_{it'}\) we must have that \(C_i(Q_{it'}) - C_i(Q_{it}) < P^*_t (Q_t^* - Q_{it})\) and the observations are not Bertrand rationalizable.
Suppose that we have a set of generic observations which violate TRC. Then there is an
\( i, t \) and \( t' \) such that \( i \in A_t, i \in N \setminus A_t' \) and \( P_t^* \hat{Q}_t > P_t^* Q_{it} \) and \( \hat{Q}_t \leq Q_{it} \). If the observations are rationalizable we should be able to find a cost function so that \( \hat{C}_i(Q_{it}) \leq P_t^* Q_{it} \). However, as \( \hat{Q}_t \leq Q_{it} \) and \( P_t^* \hat{Q}_t > P_t^* Q_{it} \) this implies that \( P_t^* \hat{Q}_t - \hat{C}_i(Q_{it}) > 0 \) and that seller \( i \) has a profitable deviation by posting the minimum price in observation \( t' \).\(^1_{14}\)

If one returns to the example data set it is clear that the reason that set of observations
could not be Bertrand rationalized is because it violates RC: in observation one each seller
produces a output equal to the aggregate market output in the second observation and
obtains a revenue of 4. The revenue condition then requires that \( 4 > 2(4 - 2) \) which is
violated. We shall introduce a strengthening of the revenue condition which can be applied
to single-price observations of prices and outputs.

**Definition 3.12** A single-price data set of prices and outputs, \( (P_t^*, Q_{it})_{i \in N, t \in T} \), satisfies the
strengthened revenue condition (SRC) if, whenever \( R_i(t) \neq \emptyset \), then \( P_t^* Q_{it} > P_t^* Q_{i(t)} \) for all \( t' \in R_i(t) \).

The following result shows that this condition alone is sufficient for a single-price set of prices
and outputs to be Bertrand rationalizable.

**Proposition 3.2** A single-price data set of prices and outputs, \( (P_t^*, Q_{it})_{i \in N, t \in T} \), is Bertrand
rationalizable if it satisfies SRC.

**Proof.** As SRC is satisfied we have \( P_t^* Q_{ir_i(t)} > P_t^* Q_{it} \) whenever this is defined.\(^1_{15}\) Subtracting \( P_t^* Q_{it} \) from each side of the inequality gives \( P_t^* Q_{ir_i(t)} - P_t^* Q_{it} > P_t^* Q_{i'} - P_t^* Q_{it} \). The rationalizing cost function can then be chosen so that \( \hat{C}_i(Q_{ir_i(t)}) = P_t^* Q_{ir_i(t)}, \hat{C}_i(Q_{it}) = P_t^* Q_{it} \) and the constructed costs satisfy ICC. We then also have \( \hat{C}_i(Q_{ir_i(t)}) - \hat{C}_i(Q_{it}) > P_t^*(Q_{i'} - Q_{it}) \) and MDC is satisfied. As the data set is a single-price data set TDC is trivially satisfied and
the rationalizability of the data set follows from Theorem 1. \( \blacksquare \)

All the characterizations of price-quantity observations considered up to now have been
partial characterizations. However, as found the conditions which completely characterize

\(^1_{14}\)The same argument applies when \( P_t^* \hat{Q}_t \geq P_t^* Q_{it} \) and \( \hat{Q}_t < Q_{it} \).

\(^1_{15}\)Recall that \( r_i(t) = \{ t' \in R_i(t) : Q_{it'} < Q_{it} \ \forall t \in R_i(t) \} \) was introduced in the proof of Theorem 1.
observations where cost information is available we can use that result to provide a complete characterization of which price-quantity observations can be Bertrand rationalized. We now present the result.

**Proposition 3.3** A generic homogeneous-good set of prices and outputs, \((P_{it}, Q_{it})_{i \in N, t \in T}\), is Bertrand rationalizable iff there exists a set of costs \((C_{it})_{i \in N, t \in T}\) such that the set \((P_{it}, Q_{it}, C_{it})_{i \in N, t \in T}\) satisfies ICC, MDC and TDC.

Up until now we have not considered what structure the Cournot equilibrium imposes upon single-price data sets of prices and outputs and what the relationship is with the Bertrand equilibrium conditions. It turns out that the Cournot equilibrium imposes no restrictions upon prices and outputs. The following result states that any single-price data set of prices and outputs can be Cournot rationalized.

**Proposition 3.4** Any single-price data set of prices and outputs, \((P^*_t, Q_{it})_{i \in N, t \in T}\), is Cournot rationalizable.


This represents a significant difference between the two models. If one permits a general increasing cost function then any single-price data set of prices and outputs can be Cournot rationalized whereas the Bertrand equilibrium imposes non-trivial restrictions even upon single-price data sets. Once cost information is unavailable the Cournot model cannot be refuted. However, if one restricts the rationalizing cost function beyond requiring that it just explain the observed costs and be strictly increasing then it is possible that some single-price data sets of prices and outputs cannot be Cournot rationalized.\(^{16}\)

\(^{16}\)Carvajal et al. (2010) introduce the notion of a ‘convincing rationalization’ where the cost function must be chosen so that the marginal cost lies between the observed marginal costs and show that this restriction rules out certain types of observations being Cournot rationalizable.
3.3 Examples of revealed Nash equilibria in oligopoly games

One of the advantages of the revealed preference method is that we can write down possible observations and test their compatibility with different equilibrium concepts. We now proceed in this fashion and apply the conditions derived in the previous section to two simple data sets to demonstrate their theoretical usefulness.

3.3.1 Example 1

<table>
<thead>
<tr>
<th></th>
<th>((P, Q, C))</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=1</td>
<td>(3, 1, 1)</td>
</tr>
<tr>
<td>t=2</td>
<td>(4, 2, 7)</td>
</tr>
</tbody>
</table>

Consider the example observations given in the table of a symmetric duopoly, \(n = 2\), with two observations, \(m = 2\). This is a single-price data set and could potentially be consistent with either the Bertrand or Cournot equilibrium solutions. First, we can note that \(S_i(2) = \{1\}\). The increasing cost condition requires \(C_{i2} - C_{i1} = 7 - 1 > 0\) which is satisfied. Second, note that \(R_i(1) = \{2\}\) as in observation two each seller produces an output equal to the aggregate market output in observation one. The monopoly deviation condition requires that we do not observe a seller able to increase its observed profits by supplying the entire market demand. This means we require \(P_i^*Q_{i1} - C_{i1} = (3)(1) - (1) = 2 \geq (3)(2) - 7 = -1 = P_1^*Q_1^* - C_{i2}\) which is satisfied. As this is a single-price data set the tie deviation condition is trivially satisfied and we can conclude that the observations can be Bertrand rationalized. Now consider the marginal condition. We require that we do not observe a situation where a seller could benefit from reducing its output. This means we require \(P_2^*Q_{i2} - C_{i2} = (4)(2) - 7 = 1 > (4)(1) - 1 = 3 = P_2^*Q_{i1} - C_{i1}\) which is violated. We can conclude that this set of observations cannot be Cournot rationalized.
3.3.2 Example 2

<table>
<thead>
<tr>
<th>t</th>
<th>((P, Q, C))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(3, 1, 2)</td>
</tr>
<tr>
<td>2</td>
<td>(4, 2, 4)</td>
</tr>
</tbody>
</table>

Consider the example observations given in the table above of a symmetric duopoly, \(n = 2\), and two observations, \(m = 2\). Again as this is a single-price data set the observations could potentially be consistent with sellers playing either the Bertrand or Cournot equilibrium. First, note that \(S_i(2) = \{1\}\). The increasing cost condition requires \(C_{i2} - C_{i1} = 4 - 2 > 0\) which is satisfied. Second, note that \(R_i(1) = \{2\}\) as in observation two each seller produces an output equal to the aggregate output in the first observation. The monopoly deviation condition requires that \(P^*_iQ_{i1} - C_{i1} = (3)(1) - 2 = 1 \geq (3)(2) - 4 = 2 = P^*_1Q^*_1 - C_{i2}\) which is violated. Therefore this set of observations cannot be Bertrand rationalized. The marginal condition requires that no seller can benefit by reducing their outputs. The condition requires that \(P^*_2Q_{i2} - C_{i2} = (4)(2) - 4 = 4 > (4)(1) - 2 = 2 = P^*_2Q_{i1} - C_{i1}\). Therefore this set of observations can be Cournot rationalized.

3.4 Conclusion to Chapter 3

Following on from the works of Sprumont (2000), Zhou (2005) and Carvajal et al. (2010) which characterized Nash equilibrium sets in abstract games, the aim of this chapter has been to show that the method of revealed preferences can be applied to oligopoly games. To this end, we have provided characterizations of observations which can be rationalized by the Bertrand and Cournot equilibrium solutions. The results show that both equilibrium concepts impose non-trivial restrictions upon observables and it is possible for a set of market observations to be revealed inconsistent with either of the equilibrium concepts. Moreover, the conditions which characterize the sets, monopoly deviation, tie deviation and marginal condition, are economically intuitive and take the form of linear inequalities. However, if we
only observe prices and outputs then the Cournot equilibrium imposes no restrictions upon the observables whereas the Bertrand equilibrium does.

One drawback of the model presented here is that it imposed strong, and unrealistic, assumptions on the types of observations we make. This is most evident in the assumption of equal sharing at price ties, which is not observed in the real world. It seems likely that some of the concepts presented here could be extended to consider more general tie-breaking rules at the minimum price. For example, if we defined the Bertrand game to be one with capacity sharing at price then a larger set of observations would be capable to being rationalized. However, the Bertrand game still has a number of features which tend not to be observed in the real-world, most notably, the assumption that sellers which do not tie at the minimum price lose all their demand. If we are keen to use the revealed preference method to test real-world data, then it seems likely that an alternative model of price-setting competition should be taken as the benchmark. For example, in Bertrand-Edgeworth competition where sellers are free to ration the demand which they receive, then a homogeneous-good may be traded at different prices, which is true of real-world markets, and the market outcomes may be quite different from Bertrand or Cournot competition. The difficulty in proceeding in this direction is that the equilibrium solution is often in mixed strategies and it is not straightforward how revealed preference theory should be extended to test stochastic choice behaviour.\footnote{In an elegant paper, Bandyopadhyay et al. (1999) initiated research in this direction by providing a stochastic version of the weak axiom of revealed preference.}
Chapter 4

On the existence of Bayesian Bertrand equilibrium

4.1 Introduction

In this chapter we consider a partial equilibrium model of price competition between a finite set of sellers producing a perfectly homogeneous good. Each seller has constant returns to scale production technology and there is a finite number of cost types in the market. Unlike previous chapters we introduce incomplete information. Sellers know their own cost type but do not know the cost types of their rivals. The sellers engage in Bertrand price competition, with a commitment to supply all demand forthcoming to them, with the difference that cost information is private information. We find that in equilibrium sellers mix over a set of connected intervals. However, as the market becomes large then marginal cost pricing holds approximately. The existence if mixed strategy Bertrand equilibria has not received much attention until recently. Harrington (1989) was the first paper to examine the mixed extension of the Bertrand game and found that there exist no other mixed equilibria. Moreover, he showed that if one requires that equilibria of the Bertrand game survive the stronger criterion of admissibility, which eliminates weakly dominated strategies, then the Bertrand game fails to possess an admissable equilibrium. The assumption that the market
demand possesses a finite choke-off price plays an important role in the Bertrand paradox outcome. Baye and Morgan (1999) showed that if the market demand fails to possess a finite choke-off price, and monopoly profit is unbounded, then the Bertrand game also possesses a continuum of atomless mixed strategy equilibria in which sellers earn positive expected profits. This result was illustrated in Chapter 1.

There are, however, only a small number of papers which have studied the impact of uncertainty and incomplete information in Bertrand games. Spulber (1995) considered the case where there is a continuum of cost types and sellers had weakly convex costs. It was shown that the game possesses a pure strategy equilibrium which converges to marginal cost pricing as the market became large. As was shown in Chapter 1, Jannsen and Rasmusen (2002) analyzed a price game where there was a fixed set of sellers, with constant marginal cost and an exogenous probability that a seller is inactive. They established that the game has a mixed strategy equilibrium and showed that under certain specifications the expected Bertrand price may be higher than the Cournot price- a reversal of the usual outcome. Routledge (2010) considered the classical Bertrand game, with a fixed set of sellers with constant returns to scale costs, where the cost type of each seller was drawn from a binary distribution and showed that there exists a mixed strategy Bayesian equilibrium.

The work presented here is related to recent existence results established for games with incomplete information where best responses are monotone in players’ types. Athey (2001) considered games with continuous payoffs, and games with discontinuous payoffs such as auctions and Bertrand competition, where players’ types were drawn from atomless distributions and found conditions for the existence of pure strategy Nash equilibrium. It is possible that there exist some relations between the results presented there and the model we analyze as the Bertrand game we consider has an equilibrium in which sellers’ strategies are monotone in type (higher cost sellers post higher prices). However, we consider a model with a finite number of types so the distribution of types is not atomless.

This chapter continues this recent literature of analyzing Bertrand games under incomplete information. We introduce a general model, similar to that in Routledge (2010), but
with finitely many cost types. The sellers in the market possess a common prior over the type space. It is shown that under standard assumptions a pure strategy Bayesian Bertrand equilibrium fails to exist but that a mixed strategy equilibrium always exists. We provide a complete characterization of this mixed strategy equilibrium and are able to obtain closed form expressions for the mixing distributions. In this equilibrium the Bertrand paradox fails. Nevertheless, as the market becomes large we show the the mixing equilibrium converges to the Bertrand paradox in that the probability distributions exhibit almost everywhere pointwise convergence to marginal cost pricing. In the next section of the chapter we set out the model and present a simple example to illustrate the equilibrium result. In the final section we present some suggestions for future research.

4.2 The Bertrand game

Consider a perfectly homogeneous good market in which the set of sellers is \( N = \{1, ..., n\} \), \( n \geq 2 \). Each seller competes in the market by simultaneously and independently choosing a price \( P_i \in \mathbb{R}_+ \) with a commitment to supply all market demand forthcoming at that price. The market demand \( D : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is such that there exist positive finite real numbers \( \bar{P}, \bar{Q} \) with \( D(\bar{P}) = 0 \) and \( D(0) = \bar{Q} \). The market demand is \( C^2 \) on \((0, \bar{P})\) and decreasing on the interior of the output space so \( D'(P) < 0 \) for all \( P \in (0, \bar{P}) \). The set of possible types of the sellers is given by \( M = \{1, ..., m\} \), \( m \geq 2 \). Given the set of types, \( M \), there is a finitely-additive probability measure \( \mu(\cdot) \) with support, \( supp(\mu) \subset M \), from which each seller’s type is an independent draw. All sellers have constant returns to scale costs and the type determines the marginal cost of the seller. It will be assumed, without loss of generality, that \( 0 < c_1 < ... < c_m < \bar{P} \). Therefore the cost function of seller \( i \) with type \( j \in M \) is given by \( C_i(Q) = c_jQ \). In what follows it will be useful to denote \( \mu_j^+ = \mu(M \setminus \{j, j-1, ..., 1\}) \) with \( \mu(M) = 1 \). That is, \( \mu_j^+ \) is the probability that a seller has marginal cost higher than \( c_j \). The monopoly profit of each seller is a mapping \( \pi : \mathbb{R}_+ \times M \rightarrow \mathbb{R} \). The specification of the cost
and demand functions means monopoly profit is:

\[ \pi(P, j) = (P - c_j)D(P). \]  \hfill (4.1)

If a seller ties with \( s - 1 \) sellers at the minimum price it shall be assumed that they split
the demand equally and we shall denote the shared profit of a seller with type \( j \) when it ties
with \( s - 1 \) sellers at the minimum price by:

\[ \hat{\pi}(P, j, s) = \frac{1}{s}(P - c_j)D(P). \]  \hfill (4.2)

Throughout it will be assumed that \( D''(P) < 0 \) which implies \( \pi(P, j) \) and \( \hat{\pi}(P, j, s) \) are
strictly concave in price. The monopoly price of type \( j \) is \( P^{mon}(c_j) = \arg \max_{P \in \mathbb{R}^+} \pi(P, j) \).
Finally it shall be assumed that \( c_m \leq P^{mon}(c_1) \) which ensures that the cost types of the
sellers are not too spaced out. As is standard in Bertrand games if a seller posts the unique
minimum price then it serves all the demand and obtains its monopoly profit at that price.
If a seller ties with \( s - 1 \) other sellers at the minimum price then it obtains \( \hat{\pi}(P, j, s) \). If a
seller is undercut then it serves zero demand and its profit is zero. Given these assumptions
the profit of seller \( i \) given that it has type \( j \in M \) and the vector of prices posted in the
market is \((P_1, ..., P_n)\) is summarized below.

\[ E_i(P_1, ..., P_n, j) = \begin{cases} 
\pi(P_i, j) & \text{if } P_i < P_k \forall k \neq i; \\
\hat{\pi}(P_i, j, s) & \text{if } i \text{ ties with } s - 1 \text{ sellers at min price}; \\
0 & \text{if } P_i > P_k \text{ for some } k. 
\end{cases} \]  \hfill (4.3)

The timing if the game is as follows: each seller’s cost type is determined by an independent
draw from probability measure \( \mu(\cdot) \). Each seller only knows their own cost type and the
probability measure is common knowledge to all sellers. sellers simultaneously and indepen-
dently choose a \( P_i \in \mathbb{R}_+ \). The payoffs of the sellers are given by eq.(4.3). Therefore this
is a classic one-shot Bertrand game, with a fixed set of sellers, with the sole difference that
each seller has incomplete information about other sellers’ types. We shall let \( A_i = \mathbb{R}_+ \)
denote the strategy space of seller \( i \) and we shall let \( \Delta A_i \) denote the set of Borel probability
measures on \( A_i \). A mixed strategy for seller \( i \) in this game is a mapping \( f_i : M \rightarrow \Delta A_i \). We
shall let \( \mathcal{G} = [N, M, \mu(\cdot), \{\Delta A_i, E_i\}_{i \in N}] \) denote the one-shot incomplete information game. A *Bayesian Bertrand equilibrium* is then a set of strategies \((f_i^*, f_{-i}^*)\) with the Nash equilibrium property in the one-shot incomplete information game.\(^1\) In analyzing equilibrium existence it will be helpful to consider the following function \( g(P, j, j + q) = \pi(P, j)/\pi(P, j + q) \). The function \( g(P, j, j + q) \) is just the ratio of the monopoly profit of seller with type \( j \) to that with type \( j + q \). Before proceeding to the analysis on equilibrium existence we state four lemmas which are useful in understanding the results. The proofs of the following lemmas are contained in the Appendix.

**Lemma 4.1** The function \( g(P, j, j + q) \) satisfies \( g'_P(P, j, j + q) < 0 \) whenever defined.

**Lemma 4.2** \( P_{\text{mon}}'(c_j) > 0. \)

**Lemma 4.3** If \( \text{supp}(\mu) = M \) there exists a unique \( \hat{P}_{m-1} \in (c_{m-1}, c_m) \) such that \( (\mu^+_{m-2})^{n-1}\pi(\hat{P}_{m-1}, m-1) = (\mu^+_{m-1})^{n-1}\pi(c_m, m-1). \)

**Lemma 4.4** If \( \text{supp}(\mu) = M \) then for each \( m - s \in M, s \geq 2 \), there exists a unique \( \hat{P}_{m-s} \in (c_{m-s}, \hat{P}_{m-s+1}) \) such that \( (\mu^+_{m-s-1})^{n-1}\pi(\hat{P}_{m-s}, m-s) = (\mu^+_{m-s})^{n-1}\pi(\hat{P}_{m-s+1}, m-s). \)

### 4.2.1 Equilibrium existence

Before stating the equilibrium properties of the game it will be helpful to develop some intuition about the results. Consider a duopoly market, \( n = 2 \), with a market demand as described above and each seller has the same constant marginal cost \( c \in [0, \bar{P}] \). In this price-setting game the result is that the unique Nash equilibrium, pure or mixed, is for both sellers to price at marginal cost and earn zero profits: the competitive equilibrium outcome prevails.

There are two points worth making. First, there does not exist any configuration of prices above marginal cost such that no seller has a profitable deviation. Second, the equilibrium

\(^1\)To be precise each seller, given their type, and the common knowledge of the probability measure, chooses their strategy to maximize their expected payoff taking the other sellers' strategies as fixed. This must hold for almost every type and strategy. A Nash equilibrium is then a vector of strategies such that each seller is simultaneously maximizing their payoff.
outcome is for each seller to play a weakly dominated strategy. Now it is well-known that a weakly dominated strategy is never a best response to a rival playing a completely mixed strategy (one which assigns positive probability to every strategy). In a Bertrand game with constant returns to scale costs it is the case that marginal cost pricing is never a best response to a rival playing a strategy which places positive probability above marginal cost because a seller could deviate to a price above marginal cost but below the upper bound of the rival’s mixed strategy and earn positive expected profits. However, as there does not exist any configuration of prices above marginal cost such that no sellers have profitable deviations, if a seller always prices above their rival’s marginal cost (which is the case when sellers have different marginal costs) a pure strategy equilibrium typically fails to exists. In the Bertrand game with incomplete information this is exactly what happens. The possibility of earning higher profits by pricing above marginal cost leads to the non-existence of a pure strategy equilibrium. Although not immediately clear, we shall show that the game always possesses a mixed strategy Bayesian equilibrium.

**Proposition 4.1** The game $G$ possesses a pure strategy Bayesian Bertrand equilibrium if and only if $\mu(\cdot)$ is degenerate. That is, $G$ possesses a pure strategy Bayesian Bertrand equilibrium if and only if there exists a $j \in M$ such that $\mu(j) = 1$.

**Proof.** First we prove the necessity part of the result and second the sufficiency part.

**Necessity.** Suppose $\mu(\cdot)$ is not degenerate. We shall show that there does not exist any pure strategy Bayesian Bertrand equilibrium. If $\mu(\cdot)$ is not degenerate then there exist $j, k \in \text{supp}(\mu)$ such that $c_j < c_l$ for all $l \in \text{supp}(\mu)$ and $c_k < c_l$ for all $l \in \text{supp}(\mu) \setminus \{j\}$.

First, consider a symmetric pure strategy with $f_i(j) = f_j$ for all $i \in N$ with $f_j \in [c_j, \bar{P})$ and $f_i(k) = f_k$ for all $i \in N$ with $f_k \in [c_k, \bar{P})$. The strategy spaces are restricted because no type $l \in \text{supp}(\mu)$ would post a price in the interval $[0, c_l)$ as they would make negative expected profit and could always deviate to $c_l$. Now suppose, without loss of generality, that $f_j < f_k$ (the same argument will apply when $f_j \geq f_k$). The expected profit of type $j$ then includes $\mu(j)^{n-1}\hat{\pi}(f_j, j, n)$. That is, the expected profit of type $j$ includes the probability that all other sellers are of the same type, $\mu(j)^{n-1}$, multiplied by the shared payoff $\hat{\pi}(f_j, j, n)$. 

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However, as \( \hat{\pi}(f_j, j, n) < \pi(f_j, j) \) whenever \( f_j \in (c_j, \bar{P}) \) if \( f_j > c_j \) one seller could deviate to \( f_j - \epsilon, \epsilon > 0 \), and increase their expected profit. If \( f_j = c_j \) then a seller of type \( j \) earns zero expected profit. Then one seller could deviate to \( f_k - \epsilon, \epsilon > 0 \), and earn expected profit arbitrarily close to \( \mu(k)^{n-1}\hat{\pi}(f_k - \epsilon, j) > 0 \). Therefore there exists no symmetric pure strategy equilibrium.

Second, consider the possibility that sellers play an asymmetric pure strategy equilibrium. As \( \mu(\cdot) \) is not degenerate there exist \( j, k \in \text{supp}(\mu) \) such that \( c_j > c_l \) for all \( l \in \text{supp}(\mu) \) and \( c_k > c_l \) for all \( l \in \text{supp}(\mu) \setminus \{j\} \). Let \( f_i(j) = f_{ij} \) denote the strategy of seller \( i \) when it has type \( j \) and \( f_i(k) = f_{ik} \) denote the strategy of seller \( i \) when it has type \( k \). In any pure strategy equilibrium we must have \( \min\{f_{ij}\}_{i \in N} \geq c_j \) otherwise at least one seller would make negative expected profits when it is type \( j \) and could profitably deviate to marginal cost pricing. As a result, a seller of type \( k \) can always secure an expected payoff arbitrarily close to \( \mu(j)^{n-1}\pi(c_j, k) > 0 \) by setting \( f_{ik} = c_j - \epsilon, \epsilon > 0 \). Now consider the strategies of the sellers when they are of type \( k \), \( \{f_{ik}\}_{i \in N} \). Let \( f_{1k} \in \min\{f_{ik}\}_{i \in N} \) and \( f_{2k} \in \min\{f_{ik}\}_{i \in N \setminus 1} \). As each seller of type \( k \) can obtain profit arbitrarily close to \( \mu(j)^{n-1}\pi(c_j, k) > 0 \) it must be the case that \( f_{1k} > c_k \). Then the same reasoning as in the previous paragraph shows that no sellers of type \( k \) would ever want to tie at the same price so all sellers of type \( k \) post different prices. Therefore \( f_{1k} < f_{2k} \). If seller 1 earns higher expected profit than seller 2 when it is of type \( k \) this would mean seller 2 has a profitable deviation by posting price \( f_{1k} - \epsilon \) when it is of type \( k \). If seller 2 earns higher profit than seller 1 when it is of type \( k \) then seller 1 would have a profitable deviation by posting price \( f_{2k} - \epsilon \) when it is of type \( k \). Finally, if sellers 1 and 2 earn the same expected profit when they are of type \( k \) then seller 2 would have a profitable deviation by posting \( f_{1k} - \epsilon \) when it is of type \( k \) because it earns additional profit from not being undercut when seller 1 is of type \( k \). Therefore there exists no pure strategy Bayesian Bertrand equilibrium.

\(^2\text{Note that as the price space is the real line the best response is not well-defined. This is the standard “open-set” problem which often occurs in Bertrand games. We shall stick with the convention that when the best response is not well-defined sellers price arbitrarily close to the limit best response.}\)
Sufficiency. If \( \mu(\cdot) \) is degenerate then the game possesses a pure strategy Bayesian Bertrand equilibrium as the market demand \( D(P) \) is continuous, bounded from above and below, sellers have constant returns to scale costs, so the game meets all the conditions for the Bertrand paradox outcome and the strategy \( f_i(l) = c_i \) for all \( i \in N \) and \( l \in M \) is a pure strategy Bayesian Bertrand equilibrium. ■

Proposition 4.2 The game \( \mathcal{G} \) possesses a mixed strategy Bayesian Bertrand equilibrium.

Proof. To show that \( \mathcal{G} \) possesses a mixed strategy equilibrium we shall proceed in two steps. First, we characterize the mixing distributions used by the sellers. Second, we show that no seller can profitably deviate from playing these strategies.

Step 1. If \( \mu(\cdot) \) is degenerate then we know the game possesses a pure strategy equilibrium from Proposition 4.1. Suppose that \( \mu(\cdot) \) is not degenerate. We shall assume from now on that \( \text{supp}(\mu) = M \). If \( \text{supp}(\mu) \neq M \) then eliminate the types in \( M \) which have measure zero and redefine the highest cost type with positive probability \( c_m \), the next highest type with positive probability \( c_{m-1} \) and so on so that the notation employed here matches the types with positive probability. We shall return to those types with measure zero at the end of the proof. If \( \mu(\cdot) \) is not degenerate we must have \( m \geq 2 \). We now show that there exists a mixed strategy Bayesian Bertrand equilibrium with \( f_i(m) = c_m \) for all \( i \in N \). Type \( m-1 \) plays an atomless mixed strategy which we denote by the cumulative distribution function \( F_{m-1}(P) \) with support \( \text{supp}(F_{m-1}(P)) = [\tilde{P}_{m-1}, c_m] \) with \( \tilde{P}_{m-1} \) defined by Lemma 4.3. Any type \( m-s \in M, s \geq 2 \), plays an atomless mixed strategy denoted by cumulative distribution function \( F_{m-s}(P) \) with support \( \text{supp}(F_{m-s}(P)) = [\tilde{P}_{m-s}, \tilde{P}_{m-s+1}] \) with the bounds of the support defined as in Lemma 4.4. To see that the relevant mixed strategies exist consider type \( m-1 \). Let \( \alpha_{m-1} \) denote the expected payoff to a seller of type \( m-1 \) in the mixed strategy equilibrium. If each seller of type \( m-1 \) is willing to mix over \( \text{supp}(F_{m-1}(P)) \) then \( F_{m-1}(P) \) must satisfy:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu_{m-1}^+)^{n-1-r} \mu(m-1)^{n-1} (1 - F_{m-1}(P))^r \pi(P, m-1) = \alpha_{m-1}. \tag{4.4}
\]
Eq.(4.4) is the standard indifference property of the Nash equilibrium. We can now show that for any \( P \in [\tilde{P}_{m-1}, c_m] \) there is an implied value of \( F_{m-1}(P) \) which has the required properties of a distribution function. Setting \( F_{m-1}(c_m) = 1 \) on the left-hand side of eq.(4.4) then implies that \( \alpha_{m-1} = (\mu^+_{m-1})^{n-1}\pi(c_m, m-1) \). We can then rewrite eq.(4.4) as:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu^+_{m-1})^{n-1-r}\mu(m-1)^{n-1}(1 - F_{m-1}(P))^r = \frac{(\mu^+_{m-1})^{n-1}\pi(c_m, m-1)}{\pi(P, m-1)}.
\] (4.5)

Evaluating the right-hand side of eq.(4.5) at \( \tilde{P}_{m-1} \) and noting from Lemma 4.3 that \((\mu^+_{m-2})^{n-1}\pi(\tilde{P}_{m-1}, m-1) = (\mu^+_{m-1})^{n-1}\pi(c_m, m-1)\) gives:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu^+_{m-1})^{n-1-r}\mu(m-1)^{n-1}(1 - F_{m-1}(\tilde{P}_{m-1}))^r = (\mu^+_{m-2})^{n-1}.
\] (4.6)

Eq.(4.6) then implies that \( F_{m-1}(\tilde{P}_{m-1}) = 0 \). As the right-hand side of eq.(4.5) is strictly decreasing for all \( P \in [\tilde{P}_{m-1}, c_m] \) this implies that \( F_{m-1}(P) \) is smoothly increasing. Hence, the implied function \( F_{m-1}(P) \) has all the required properties of an atomless cumulative distribution function.

Now consider any type \( m - s \in M, s \geq 2 \). Let \( \alpha_{m-s} \) denote the expected payoff to type \( m - s \) in the mixed strategy equilibrium. If any seller of type \( m - s \) is willing to mix over \( \text{supp}(F_{m-s}(P)) = [\tilde{P}_{m-s}, \tilde{P}_{m-s+1}] \) then \( F_{m-s}(P) \) must satisfy:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu^+_{m-s})^{n-1-r}\mu(m-s)^{n-1}(1 - F_{m-s}(P))^r\pi(P, m-s) = \alpha_{m-s}. \tag{4.7}
\]

Then setting \( F_{m-s}(\tilde{P}_{m-s+1}) = 1 \) on the left-hand side of eq.(4.7) implies

\[
\alpha_{m-s} = (\mu^+_{m-s})^{n-1}\pi(\tilde{P}_{m-s+1}, m-s).
\]

We can then rewrite eq.(4.7) as:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu^+_{m-s})^{n-1-r}\mu(m-s)^{n-1}(1 - F_{m-s}(P))^r = \frac{(\mu^+_{m-s})^{n-1}\pi(\tilde{P}_{m-s+1}, m-s)}{\pi(P, m-s)}.
\] (4.8)

Evaluating the right-hand side of eq.(4.8) at \( \tilde{P}_{m-s} \) and using the result in Lemma 4.4 that

\[
(\mu^+_{m-s-1})^{n-1}\pi(\tilde{P}_{m-s}, m-s) = (\mu^+_{m-s})^{n-1}\pi(\tilde{P}_{m-s+1}, m-s) \]

gives:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu^+_{m-s})^{n-1-r}\mu(m-s)^{n-1}(1 - F_{m-s}(\tilde{P}_{m-s}))^r = (\mu^+_{m-s-1})^{n-1}. \tag{4.9}
\]
Eq. (4.9) then implies that \( F_{m-s}(\tilde{P}_{m-s}) = 0 \). As the right-hand side of eq. (4.8) is strictly decreasing for all \( P \in [\tilde{P}_{m-s}, \tilde{P}_{m-s+1}] \) this means there is a smooth strictly increasing implied value of \( F_{m-s}(P) \) which satisfies eq. (4.7). This implied function constitutes a mixed strategy for type \( m - s \). We can go further by noting that for any type \( j \in M \setminus \{m\} \) the mixed strategy defined above must also satisfy:

\[
(\mu_j^+ + \mu(j)(1 - F_j(P)))^{n-1}\pi(P, j) = \alpha_j.
\] (4.10)

Re-arranging eq. (4.10) gives:

\[
\mu_j^+ + \mu(j)(1 - F_j(P)) = \left( \frac{\alpha_j}{\pi(P, j)} \right)^{1/(n-1)}.
\]

Which gives:

\[
\mu(j)(F_j(P) - 1) = \mu_j^+ - \left( \frac{\alpha_j}{\pi(P, j)} \right)^{1/(n-1)}.
\]

Therefore:

\[
F_j(P) = 1 + \frac{1}{\mu(j)} \left[ \mu_j^+ - \left( \frac{\alpha_j}{\pi(P, j)} \right)^{1/(n-1)} \right].
\] (4.11)

Eq. (4.11) gives a closed form expression for the mixing distribution used by each type \( j \in M \setminus \{m\} \).

**Step 2.** Now consider whether any seller can benefit from deviating from the mixed strategies described in Step 1. It is clear that no seller of type \( m \) can benefit from increasing their price above \( c_m \), and no seller of type \( M \setminus \{m\} \) could improve their payoff by deviating to a price above \( c_m \). Consider an arbitrary type \( j \in M \setminus \{m\} \) and suppose a seller of type \( j - 1 \) posts a price \( P' \in \text{supp}(F_j(P)) \). Their expected profit is given by:

\[
\sum_{r=0}^{n-1} \binom{n-1}{r} (\mu_j^+)^{n-1-r} \mu(j)^{n-1} (1 - F_j(P'))^r \pi(P', j - 1).
\]

If this is not a profitable deviation we require:

\[
\sum_{r=0}^{n-1} \binom{n-1}{r} (\mu_j^+)^{n-1-r} \mu(j)^{n-1} (1 - F_j(P'))^r \pi(P', j - 1) \leq (\mu_{j-1}^+)^{n-1} \pi(\tilde{P}_j, j - 1). \tag{4.12}
\]

The left-hand side of eq. (4.12) can be rewritten, using the characterization in eq. (4.7), as:

\[
\left[ \frac{(\mu_{j-1}^+)^{n-1} \pi(\tilde{P}_j, j)}{\pi(P', j)} \right] \pi(P', j - 1) \leq (\mu_{j-1}^+)^{n-1} \pi(\tilde{P}_j, j - 1). \tag{4.13}
\]
Which simplifies to give:
\[
\frac{\pi(P', j - 1)}{\pi(P', j)} \leq \frac{\pi(\hat{P}_j, j - 1)}{\pi(\hat{P}_j, j)}. \tag{4.14}
\]

Using the result in Lemma 4.1 and noting that \( P' \geq \hat{P}_j \) we can conclude that eq.(4.14) is satisfied and a seller of type \( j - 1 \) cannot profitably deviate by posting a price in the support of type \( j \). Now consider type \( j - 2 \). If this type posts a price \( P' \) in the support of type \( j \) then following the previous steps we can establish its expected profit is given by:
\[
\left[ \frac{\left(\mu_{j-1}^+\right)^{n-1}\pi(\hat{P}_j, j)}{\pi(P', j)} \right] \pi(P', j - 2).
\]

If this is not a profitable deviation we require:
\[
\left[ \frac{\left(\mu_{j-1}^+\right)^{n-1}\pi(\hat{P}_j, j)}{\pi(P', j)} \right] \pi(P', j - 2) \leq \left(\mu_{j-2}^+\right)^{n-1}\pi(\hat{P}_{j-1}, j - 2). \tag{4.15}
\]

Eq.(4.13) can be rewritten as:
\[
\left[ \frac{\left(\mu_{j-1}^+\right)^{n-1}\pi(\hat{P}_j, j)}{\pi(P', j)} \right] \leq \frac{\left(\mu_{j-1}^+\right)^{n-1}\pi(\hat{P}_j, j - 1)}{\pi(P', j - 1)}. \tag{4.17}
\]

Using the equality \( \left(\mu_{j-1}^+\right)^{n-1}\pi(\hat{P}_j, j - 1) = \left(\mu_{j-2}^+\right)^{n-1}\pi(\hat{P}_{j-1}, j - 1) \) eq.(4.17) can be rewritten as:
\[
\left[ \frac{\left(\mu_{j-1}^+\right)^{n-1}\pi(\hat{P}_j, j)}{\pi(P', j)} \right] \leq \frac{\left(\mu_{j-2}^+\right)^{n-1}\pi(\hat{P}_{j-1}, j - 1)}{\pi(P', j - 1)}. \tag{4.18}
\]

Substituting the right-hand side of eq.(4.18) in place of the expression in brackets in the left-hand side of eq.(4.16) we have:
\[
\frac{\left(\mu_{j-2}^+\right)^{n-1}\pi(\hat{P}_{j-1}, j - 1)\pi(P', j - 2)}{\pi(P', j - 1)} \leq \left(\mu_{j-2}^+\right)^{n-1}\pi(\hat{P}_{j-1}, j - 2). \tag{4.19}
\]

Which reduces to:
\[
\frac{\pi(P', j - 2)}{\pi(P', j - 1)} \leq \frac{\pi(P_{j-1}, j - 2)}{\pi(P_{j-1}, j - 1)}.
\]

Using the result in Lemma 4.1 and noting that \( P' > \hat{P}_{j-1} \) we can conclude that this inequality is satisfied which implies eq.(4.16) is satisfied. Hence a seller of type \( j - 2 \) cannot profitably deviate by posting a price \( P' \in supp(F_j(P)) \). We could then repeat these steps to show that no type \( j - s \in M, s \geq 2 \), could profitably deviate from posting a price \( P' \in supp(F_j(P)) \).
As type \( j \in M \setminus \{m\} \) was chosen arbitrarily, we can conclude that no seller can profitably deviate by increasing their price.

Now consider whether any seller can deviate by reducing their price. Clearly no seller of type \( m \) would want to price below \( c_m \). However, suppose a seller of type \( j + 1 \) deviates to a price \( P' \in \text{supp}(F_j(P)) \) with \( j \in M \setminus \{m,m-1\} \). The expected profit of the seller is given by:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu_j^+)^{n-1-r} \mu(j)^{n-1}(1 - F_j(P'))^r \pi(P', j + 1).
\]

If this is not a profitable deviation then:

\[
\sum_{r=0}^{n-1} \binom{r}{n-1} (\mu_j^+)^{n-1-r} \mu(j)^{n-1}(1 - F_j(P'))^r \pi(P', j + 1) \leq (\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j + 1). \tag{4.20}
\]

The left-hand side of eq.(4.20) can be rewritten, again using the characterization in eq.(4.7), as:

\[
\left[ \frac{(\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j)}{\pi(P', j)} \right] \pi(P', j + 1) \leq (\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j + 1). \tag{4.21}
\]

Which simplifies to give:

\[
\frac{\pi(\tilde{P}_{j+1}, j)}{\pi(\tilde{P}_{j+1}, j + 1)} \leq \frac{\pi(P', j)}{\pi(P', j + 1)}. \tag{4.22}
\]

Using the result in Lemma 4.1 and noting that \( P' \leq \tilde{P}_{j+1} \) we can conclude that eq.(4.22) is satisfied. Now suppose a seller of type \( j + 2 \) deviates to a price \( P' \in \text{supp}(F_j(P)) \). Following the steps above, we can establish that this is not a profitable deviation if:

\[
\left[ \frac{(\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j)}{\pi(P', j)} \right] \pi(P', j + 2) \leq (\mu_{j+2}^+)^{n-1} \pi(\tilde{P}_{j+2}, j + 2). \tag{4.23}
\]

Eq.(4.21) can be rewritten as:

\[
\left[ \frac{(\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j)}{\pi(P', j)} \right] \leq \frac{(\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j + 1)}{\pi(P', j + 1)}. \tag{4.24}
\]

Then using the equality \( (\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j + 1) = (\mu_{j+2}^+)^{n-1} \pi(\tilde{P}_{j+2}, j + 1) \) eq.(4.24) can be rewritten as:

\[
\left[ \frac{(\mu_{j+1}^+)^{n-1} \pi(\tilde{P}_{j+1}, j)}{\pi(P', j)} \right] \leq \frac{(\mu_{j+2}^+)^{n-1} \pi(\tilde{P}_{j+2}, j + 1)}{\pi(P', j + 1)}. \tag{4.25}
\]
Substituting the right-hand side of eq.(4.25) in place of the expression in brackets in the left-hand side of eq.(4.23):

\[ \left[ \frac{(\mu_{j+2}^+)\mu_{j+2}}{\mu_{j+2}} \right] \pi(P', j + 1) \leq (\mu_{j+2}^+)\mu_{j+2} \pi(P_{j+2}, j + 1). \] (4.26)

Which simplifies to give:

\[ \frac{\pi(P_{j+2}, j + 1)}{\pi(P', j + 1)} \leq \frac{\pi(P_{j+2}, j + 1)}{\pi(P_{j+2}, j + 1)}. \] (4.27)

From Lemma 4.1 and \( P' < \tilde{P}_{j+2} \) we can conclude that eq.(4.27) is satisfied which implies eq.(4.23) is satisfied. Having established that type \( j + 2 \) cannot profitably deviate by pricing in the support of type \( j \) we could continue to repeat the steps to show no type \( j + s, s \geq 2 \), could profitably deviate by doing so, and as type \( j \) was chosen arbitrarily, we can conclude that no type can profitably deviate by reducing their price. Therefore the set of strategies \( f_i(j) = F_j(P) \) for \( j \in M \setminus \{m\} \) and \( f_i(m) = c_m \) for all \( i \in N \) constitute a symmetric Bayesian Bertrand equilibrium.\(^3\) ■

The properties of the mixed strategy equilibrium in Proposition 4.2 are that all sellers except those with the highest marginal cost earn positive expected profits and price above marginal cost with probability one. This is a more realistic outcome than the Bertrand paradox where just two sellers results in the competitive equilibrium outcome. However, an interesting question is what happens to the mixed strategy equilibrium in Proposition 4.2 as the market becomes large. We now turn to this problem and shall denote the mixed strategies which sellers play in equilibrium by \( F_j(P, n) \) to show that the distribution function depends on prices and the number of sellers in the market. Also, the lower bound on the mixed strategy equilibrium shall be denoted by \( \tilde{P}_j(n) \). We shall let \( \delta_j(P) \) denote the distribution function which places all mass on \( c_j \). That is, \( \delta_j(P) = 0 \) for all \( P \in [0, c_j) \) and \( \delta_j(P) = 1 \) for all \( P \in [c_j, \infty) \). It turns out that the equilibrium distributions converge, in a specific way, to marginal cost pricing. Therefore, the Bertrand paradox should still be expected to hold approximately in large markets. Before showing this we introduce a lemma.

\(^3\)At the beginning we eliminated types with measure zero. One could simply assign the strategy of marginal cost pricing to these types and the Nash equilibrium property then holds for almost every type.
Lemma 4.5 If $\text{supp}(\mu) = M$, then as $n \to \infty$, $\tilde{P}_j(n) \to c_j$ for all $j \in M$.

Proof. From the mixed strategy equilibrium in Proposition 4.2 $f_i(m) = c_m$ so for the highest cost types the convergence is trivial. Consider type $m - 1$. From Lemma 4.3 the price $\tilde{P}_{m-1}(n)$ satisfies $(\mu_{m-2}^+)_{n-1} \pi(\tilde{P}_{m-1}(n), m-1) = (\mu_{m-1}^+)_{n-1} \pi(c_m, m-1)$. As $n \to \infty$ then $(\mu_{m-1}^+/\mu_{m-2}^+)_{n-1} \to 0$ which from Lemma 4.3 implies $\tilde{P}_{m-1}(n) \to c_{m-1}$. Repeating the steps here and using Lemma 4.4 we can establish the result for any $m - s \in M$, $s \geq 2$. ■

Proposition 4.3 As $n \to \infty$, $F_j(P, n) \to^p \delta_j(P)$ for a.e. $P \in \mathbb{R}_+$. That is, as $n \to \infty$ the function $F_j(P, n)$ converges pointwise to $\delta_j(P)$ for almost every $P \in \mathbb{R}_+$.

Proof. First eliminate those types in $M$ which have measure zero according to $\mu(\cdot)$ so that the remaining types satisfy $\text{supp}(\mu) = M$.\textsuperscript{4} From the definition of pointwise convergence\textsuperscript{5} we need to show that for any $\epsilon > 0$ and each $P \in \mathbb{R}_+$ there exists $\bar{n} \in \mathbb{N}$ such that $|F_j(P, n) - \delta_j(P)| < \epsilon$ for all $n > \bar{n}$. First note that $\tilde{P}_j(n) > c_j$ for all $n \in \mathbb{N}$. Therefore for any $P \in [0, c_j)$ we have $|F_j(P, n) - \delta_j(P)| = 0$. Similarly for any $P > c_{j+1}$ Lemma 4.5 implies that for $n$ large enough $\tilde{P}_{j+1}(n) < P$ and then $|F_j(P, n) - \delta_j(P)| = 0$. Finally, consider any $P' \in (c_j, c_{j+1}]$ we then have $\pi(P', j) > 0$. From Lemma 4.5 we have that as $n \to \infty$, $\pi(\tilde{P}_{j+1}(n), j) \to \pi(c_{j+1}, j)$ and $\pi(c_{j+1}, j) > 0$. If we define the function $h(P', n)$ as:

$$h(P', n) = \left(\frac{\pi(\tilde{P}_{j+1}(n), j)}{\pi(P', j)}\right)^{\frac{1}{n-1}}.$$  

As $n \to \infty$ we have $h(P', n) \downarrow 1$. We can then rewrite $F_j(P', n)$ using eq.(4.11) as:

$$F_j(P', n) = 1 + \frac{1}{\mu(j)} [\mu_j^+ - \mu_j^+ h(P', n)]. \quad (4.28)$$

Therefore:

$$|F_j(P', n) - \delta_j(P')| = \left|\frac{\mu_j^+}{\mu(j)}(1 - h(P', n))\right|.$$  

Given that $h(P', n) \downarrow 1$ as $n \to \infty$ there exists $\bar{n} \in \mathbb{N}$ such that $|F_j(P', n) - \delta_j(P')| < \epsilon$ for all $n > \bar{n}$. To see that the convergence holds only almost everywhere note that $\tilde{P}_j(n) > c_j$

\textsuperscript{4} Note that at the end of Proposition 4.2 the strategy of marginal cost pricing was assigned to those types with measure zero so pointwise convergence is trivially satisfied for these types.

\textsuperscript{5} See, for example, Morgan (2005, p.75).
for all \( n \in \mathbb{N} \) so \(|F_j(c_j, n) - \delta_j(c_j)| = 1\) for all \( n \in \mathbb{N} \). Pointwise convergence fails at \( c_j \in \mathbb{R}_+ \) which has Lebesgue measure zero. ■

### 4.2.2 Uniqueness of equilibrium

As we have constructed an equilibrium in the price-setting game it would be desirable to know whether or not this equilibrium is unique. A familiarity with the model suggests that this equilibrium is the unique symmetric equilibrium. However, a proof of this result would be difficult to provide. If one considers asymmetric equilibria it is not clear whether there could be other equilibria, especially when \( n = 2 \). A open question for future research is to clarify these questions regarding the uniqueness of equilibrium. Moreover, if there exist other equilibria we should also like to note whether there is convergence to marginal cost pricing in large markets amongst these other equilibrium points.

### 4.2.3 Example

We now illustrate the equilibrium existence results in an example. Consider a duopoly market, \( N = \{1, 2\} \), with piecewise-affine market demand given by \( D(P) = \max\{0, 10 - P\} \). The set of types is \( M = \{1, 2, 3\} \) with the common prior being \( \mu(i) = \frac{1}{3} \) for all \( i \in M \). The corresponding marginal costs are \( c_1 = 1 \), \( c_2 = 2 \) and \( c_3 = 3 \). This market satisfies all the conditions given above. Routine calculations reveal that \( \tilde{P}_2 \) is the relevant solution to

\[
\tilde{P}_2^2 - 12\tilde{P}_2 + 23\frac{1}{2} = 0.
\]

Solving gives \( \tilde{P}_2 = 6 - \frac{1}{2}\sqrt{50} \approx 2.464 \). Then \( \tilde{P}_1 \) is the relevant solution to

\[
\tilde{P}_1^2 - 11\tilde{P}_1 + 15 + \frac{1}{3}\sqrt{50} = 0.
\]

Solving we obtain \( \tilde{P}_1 = \frac{1}{2}(11 - \sqrt{61 - \frac{4}{3}\sqrt{50}}) \approx 1.909 \). Therefore in equilibrium sellers of type one mix over the interval \( [\frac{1}{2}(11 - \sqrt{61 + \frac{4}{3}\sqrt{50}}), 6 - \frac{1}{2}\sqrt{50}] \), sellers of type two mix over the interval \( [6 - \frac{1}{2}\sqrt{50}, 3] \) and sellers of type three price at marginal cost. The mixing distributions which types one and two use in equilibrium are given by:

\[
F_1(P) = 3 - \frac{22.071}{(P - 1)(10 - P)},
\]

\[
F_2(P) = 2 - \frac{21}{3(P - 2)(10 - P)}.
\]

Figures one and two present graphs of these distribution functions.
Figure 4.1: Graph of Distribution Function $F_1(P)$

Figure 4.2: Graph of Distribution Function $F_2(P)$
4.3 Conclusion to Chapter 4

The classical model of Bertrand competition assumes that sellers possess complete information about their rivals’ costs. This chapter has shown that introducing incomplete information results in a pure strategy Bayesian Bertrand equilibrium failing to exist. However, a mixed strategy equilibrium always exists and has been characterized in this chapter. Generally, only a small amount of recent research has examined equilibrium existence in price-setting games with incomplete information. There are a number of possible extensions for future research. First, the model presented here assumed that sellers always serve the demand forthcoming. However, in the related model of Bertrand-Edgeworth competition the market demand may be rationed. As far as the author is aware, there are no papers which have analyzed Bertrand-Edgeworth competition between sellers which have incomplete information. Deneckere and Kovenock (1996) considered Bertrand-Edgeworth competition when sellers have asymmetric marginal costs. It would be interesting to extend their analysis to the case with uncertain marginal costs. Second, Janssen and Roy (2010) considered Bertrand competition between sellers whose goods may be of different quality and showed that there exists a fully-revealing Bayesian equilibrium. The results established in this chapter could be used to provide an extension of their model to the case where there are potentially finitely many types of quality traded in the market.

4.4 Appendix to Chapter 4

Proof of Lemma 4.1. From the definition of \( g(p,j,j+q) \) we have:

\[
g'_p(P,j,j+q) = \frac{\pi(P,j+q)\pi'_p(P,j) - \pi(P,j)\pi'_p(P,j+q)}{\pi(P,j+q)^2}. \tag{4.29}
\]
Proof of Lemma 4.4. First suppose \( \tilde{\pi} \) implies that \( \tilde{\pi} \) and \( 0 \) exists a \( \tilde{\pi} \) monopoly profit function is continuous, the intermediate value theorem guarantees there exists a \( \tilde{\pi} \) monopoly profit function and \( 0, \) the intermediate value theorem ensures there exists a \( \tilde{\pi} \) monopoly profit function implies \( \tilde{\pi} = \pi \) for all \( P \). From Lemma 4.2, \( P \) monopoly profit function and \( \tilde{P} \) we have \( \pi(c_m, m-1) > 0 \). Therefore \( 0 < (\mu_{m-1}^{+})^{n-1}/(\mu_{m-2}^{+})^{n-1} \) \( \pi(c_m, m-1) ) = \pi(c_m, m-1) \). From Lemma 4.2 we have \( P_{\text{mon}}(c_{m-1}) \) and as \( P_{\text{mon}}(c_1) \geq c_m \) by assumption, we have \( P_{\text{mon}}(c_{m-1}) \geq c_m \). The strict concavity of the monopoly profit function implies \( \pi'_{p}(P, m-1) > 0 \) for all \( P \in (0, c_m) \) which implies that \( \tilde{P}_{m-1} \) is unique. ■

Proof of Lemma 4.3. If \( \text{supp}(\mu) = M \) then \( (\mu_{m-1}^{+})^{n-1}/(\mu_{m-2}^{+})^{n-1} \in (0, 1) \). From the monopoly profit function and \( c_{m-1} < c_m < \tilde{P} \) we have \( \pi(c_m, m-1) > 0 \). Therefore \( 0 < (\mu_{m-1}^{+})^{n-1}/(\mu_{m-2}^{+})^{n-1} \) \( \pi(c_m, m-1) < \pi(c_m, m-1) \). As the monopoly profit function is continuous, and \( \pi(c_{m-1}, m-1) = 0 \), the intermediate value theorem ensures there exists a \( \tilde{P}_{m-1} \in (c_{m-1}, c_m) \) such that \( (\mu_{m-2}^{+})^{n-1} \pi(\tilde{P}_{m-1}, m-1) = (\mu_{m-1}^{+})^{n-1} \pi(c_m, m-1) \). From Lemma 4.2 we have \( P_{\text{mon}}(c_{m-1}) \geq P_{\text{mon}}(c_1) \) and as \( P_{\text{mon}}(c_1) \geq c_m \) by assumption, we have \( P_{\text{mon}}(c_{m-1}) \geq c_m \). The strict concavity of the monopoly profit function implies \( \pi'_{p}(P, m-1) > 0 \) for all \( P \in (0, c_m) \) which implies that \( \tilde{P}_{m-1} \) is unique. ■

Proof of Lemma 4.4. First suppose \( m-2 \in M \). If \( \text{supp}(\mu) = M \) then \( (\mu_{m-2}^{+}/\mu_{m-3}^{+}) \in (0, 1) \). From Lemma 4.3 we have \( \tilde{P}_{m-1} > c_{m-1} \). Therefore \( \tilde{P}_{m-1} > c_m \) and \( \pi(\tilde{P}_{m-1}, m-2) > 0 \) and \( 0 < (\mu_{m-2}^{+}/\mu_{m-3}^{+}) \pi(\tilde{P}_{m-1}, m-2) < \pi(\tilde{P}_{m-1}, m-2) \). As \( \pi(c_{m-2}, m-2) = 0 \) and the monopoly profit function is continuous, the intermediate value theorem guarantees there exists a \( \tilde{P}_{m-2} \in (c_{m-2}, \tilde{P}_{m-1}) \) such that \( (\mu_{m-3}^{+})^{n-1} \pi(\tilde{P}_{m-2}, m-2) = (\mu_{m-2}^{+})^{n-1} \pi(\tilde{P}_{m-1}, m-2) \). From Lemma 4.2, \( P_{\text{mon}}(c_{m-2}) \geq P_{\text{mon}}(c_1) \) and \( P_{\text{mon}}(c_1) \geq c_m \) by assumption. Therefore \( P_{\text{mon}}(c_{m-2}) \geq c_m \) and the strict concavity of the profit function implies \( \pi'_{p}(P, m-2) > 0 \) for all \( P \in (c_{m-2}, \tilde{P}_{m-1}) \) which implies \( \tilde{P}_{m-2} \) is unique. Having established that \( \tilde{P}_{m-2} \) exists we can then repeat the steps exactly to establish \( \tilde{P}_{m-3} \) exists if \( m-3 \in M \). As \( M \) is finite repeated application of the steps proves the result. ■
Chapter 5

Conclusion

In this book the aim has been to provide a unified treatment of the model of Bertrand competition. A state-of-the-art survey of results regarding price-making behaviour by strategic agents has been provided and a number of open questions, regarding the structure of the equilibrium set and the existence of Bayesian equilibrium have been analyzed. Given the importance of prices in economic theory, and the fact that the study of price-making behaviour has tended to be a non-trivial problem, it seems likely that the topic will remain of some interest to economic theorists in the immediate future. It seems fitting, then, to close this monograph with some tentative comments regarding possible extensions for future research and continuing open problems in the literature.

First, all the models presented in this book have been Marshallian. That is to say, it has been assumed that the market for the good under consideration could be studied without considering the impacts elsewhere in the economy. Although this is appropriate in specific economies, such as when agents have quasilinear utilities, it is not likely to be valid in general. A more general treatment of price-making behaviour would take into account changes which the price of one good has upon demands for other goods. In other words, it is desirable for us to formulate a Walrasian approach to price-making behaviour. However, this is notoriously difficult. When sellers quote prices which are non-Walrasian some agent in the economy must find their economic choices rationed and the primitives of an economy must be augmented to
include rationing rules which describe how exchange takes place when there is rationing. In a remarkable, but little cited paper, Funk (1995) provided a Walrasian perspective upon price-making behaviour and showed that every Walrasian equilibrium could be approximated by a Bertrand equilibrium. It seems possible that some of the ideas analyzed in this monograph could be restated in a general equilibrium context using the model of Funk (1995) as a benchmark.

Second, the notion of the Bertrand core, introduced in chapter two, raises a number of subtle issues regarding contract formation and cooperation amongst market traders. In the Bertrand core it was assumed that sellers act non-cooperatively in setting prices. In the coalition-proof Bertrand core sellers cooperative in setting prices but cannot write binding contracts. A natural question is what outcomes emerge when sellers fully cooperate in setting prices. Kaneko (1977) defined the core of a price-setting game and showed that when sellers had identical and constant marginal costs the core was empty. In the more general trading game analyzed in chapter two it is possible that the core, as defined by Kaneko (1977), is non-empty. Further research is required to establish the precise relationship between these different core concepts.

Finally, in chapter four we analyzed the classical Bertrand game in which sellers had incomplete information. However, there is recent research which has analyzed oligopoly games when traders have asymmetric information (Einy et al., 2002). As the Bertrand core shares a number of similarities with the Edgeworth core analyzed in general equilibrium exchange it would be interesting to see whether it is possible to analyze the Bertrand core with differential information (Glycopantis and Yannelis, 2005). For example, if sellers have asymmetric information about each others’ costs and the set of buyers in the market is the Bertrand core non-empty? What happens as the set of traders in the market becomes large? As the Bertrand core is a new concept these types of questions have previously not been possible to analyze in the context of price-making contracts. When sellers have asymmetric information regarding each others’ costs and the market demand proving that a Bayesian equilibrium exists is a non-trivial question.
Bibliography


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