RELIABLE CONTROLLER DESIGN
FOR A CLASS OF NONLINEAR SYSTEMS

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Engineering and Physical Sciences

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By
Zakwan Skaf
School of Electrical and Electronic Engineering
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Abstract

Control design for nonlinear systems remains an open problem in control theory despite the recent increase in research attention. This PhD work is motivated by this fact, addressing the constructive observer design approach, the output regulation problem, minimum entropy control, fault tolerant control (FTC), and iterative FTC for nonlinear systems. The main contributions of this work can be summarized as follows:

- An Iterative Learning Control (ILC)-based adaptive minimum entropy control for nonlinear systems with non-Gaussian disturbance is developed.

- A new FTC framework for nonlinear stochastic systems with non-Gaussian variables using probability density functions (PDFs) is developed.

- An ILC-based fixed structure FTC framework for output PDF shaping in general stochastic systems using Linear Matrix Inequality (LMI) techniques is established.

Moreover, the structure of the developed nonlinear controllers can easily be implemented in various practical fields of science and engineering.

Although the concept of stochastic distribution control has been used for about two decades, it remains an active area of research. To the best of our knowledge, there has been no previous research which has utilized iterative FTC for nonlinear stochastic systems subjected where non-Gaussian behaviour exists.
Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.
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Publications During PhD Study


UK, September, 2010.

Notation

\( \lambda(A) \)  
Eigenvalues of matrix \( A \)

\( \lambda_{\max}(A) \)  
Largest eigenvalue of matrix \( A \)

\( \lambda_{\min}(A) \)  
Smallest eigenvalue of matrix \( A \)

\( \| \cdot \| \)  
Euclidean norm of the real vectors and matrices

\( A^T \)  
Transpose of matrix \( A \)

\( A^{-1} \)  
Inverse of matrix \( A \)

\( I \)  
The identity matrix with appropriate dimension

\( 0 \)  
The zero matrix with appropriate dimension

In addition, symbolic function \( \text{sym}(A) \) represents the symmetric function resulting from the summation of a nominal matrix \( A \) and its transpose \( A^T \), or in other words, \( (A + A^T) \).
List of Abbreviations

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<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>DC</td>
<td>Direct Current</td>
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<tr>
<td>FDD</td>
<td>Fault Detection and Diagnosis</td>
</tr>
<tr>
<td>FDI</td>
<td>Fault Detection and Isolation</td>
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<tr>
<td>FI</td>
<td>Fault Isolation</td>
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<td>FTC</td>
<td>Fault Tolerant Control</td>
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<tr>
<td>IL</td>
<td>Iterative Learning</td>
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<td>ILC</td>
<td>Iterative Learning Control</td>
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<td>IP</td>
<td>Information Potential</td>
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<tr>
<td>LMI</td>
<td>Linear Matrix Inequality</td>
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<td>LTI</td>
<td>Linear Time Invariant</td>
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<tr>
<td>MLP</td>
<td>Multi-Layer Perception</td>
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<td>PD</td>
<td>Proportional and Differential</td>
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<td>PDE</td>
<td>Partial Differential Equation</td>
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<td>PDF</td>
<td>Probability Density Function</td>
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<tr>
<td>PI</td>
<td>Proportional and Integral</td>
</tr>
<tr>
<td>PID</td>
<td>Proportional-Integral-Differential</td>
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<tr>
<td>RBF</td>
<td>Radial Basis Function</td>
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<td>RBFNN</td>
<td>Radial Basis Function Neural Network</td>
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<td>SISO</td>
<td>Single Input Single Output</td>
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<td>SDC</td>
<td>Stochastic Distribution Control</td>
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Dedication

To My Parents,
My Wife,
And My Lovely Daughter;
I Dedicate This Thesis
Chapter 1

Introduction

1.1 Nonlinear Systems

Linear control design is a mature subject with strong theoretical foundation and a wide range of applicable design approaches. However, it has a disadvantage when it is used for practical applications such as in robotics, biomedical engineering and process control. This problem is due to the fact that almost all physical systems in science and engineering are nonlinear in nature. These nonlinearities are inherent in the system’s hardware and motion [1–3]. Therefore, nonlinear control theory is the subject of a vast and diverse range of research, worldwide.

Nonlinear systems can in some cases be simplified by the linearization method and represented as an equivalent linear system, if the nonlinearities are sufficiently smooth and the operational range of the nonlinear model is sufficiently small. Nevertheless, such approximations can be restrictive when the general case is considered, or if a larger operational range is taken into consideration [1–3]. In general, there is a considerable interest in the research and application of nonlinear control methods in various fields of engineering and science.

1.2 Output Regulation Control

The output regulation or servomechanism problem addresses the problem of a feedback controller used to achieve asymptotic tracking of reference signals and rejection of the disturbances generated by an external autonomous system, while maintaining closed-loop stability [4].

The theory of output regulation for nonlinear systems has been developed
rapidly in the last two decades. Since the regulator equations are a set of nonlinear partial differential and algebraic equations [5, 6], it is usually difficult to find an exact solution due to the nonlinearity of the system. Therefore, it is preferable to find a numerical approach to solve these regulator equations approximately. Thus, an approximate solution for the regulator equations in terms of a power series can be implemented.

1.3 Minimum Entropy Control

In stochastic systems, when the random variables or the noise are subject to non-Gaussian processes, the mean and variance are insufficient to characterize the stochastic properties, because the spread area of the non-symmetrically distributed noises cannot be described accurately by the variance of the tracking error alone [7]. Therefore, a new measure of the randomness, called the entropy of the tracking error of the closed-loop system, should be employed for the closed-loop control design of non-Gaussian stochastic systems. Entropy can represent the system uncertainties more generally because it measures the dispersion of the probability distribution. Using this concept, the main objective of minimum entropy control can be stated as formulating a deterministic control signal to minimize the entropy of the closed-loop tracking error [7].

1.4 PDF Control

Control theory has been developed rapidly recently to cope with the requirements of non-Gaussian stochastic systems such as in paper making, food processing units, combustion systems, and so on [7–9].

These developments are based on controlling the shape of the output probability density functions (PDFs) for non-Gaussian stochastic systems. This differs from traditional stochastic control where only the output mean and variance are considered [7]. With the developments in instrumentation and image processing technologies in many practical processes, the output PDFs are becoming measurable in practice where a measurement of the output PDF is required (e.g. combustion system) especially designed instruments like digital cameras with complicated built-in functions such as histogram, kernel, or other methods of density estimation and also image processing such as infrared thermographic processing are used to estimate the density of the measured output signal [10,11].
One can easily obtain the PDF from the images of the measured output. Recently, some advanced control methods, such as robust control, nonlinear control, optimal control, and adaptive control, have been used widely with non-Gaussian stochastic systems to achieve complicated control objectives [7, 12–14].

In general, the techniques of non-Gaussian, stochastic and nonlinear control design can enhance the ability of a control engineer to deal with practical non-Gaussian stochastic control problems effectively.

1.5 Fault Tolerant Control

Fault tolerant control (FTC) is used in systems that need to be able to detect faults and prevent simple faults relating to control loops from developing into production stoppages or failures at a plant level. This is achieved by combining fault diagnosis with supervisory control and re-configuring the controller to accommodate faults as shown in Fig. 1.1. The diagnosistic block uses the measured input and output and tests their consistency with the plant model. Its result is a characterisation of the fault with sufficient accuracy for the controller re-design. While, the re-design block uses the fault information and adjust the the controller to the faulty case [15]. Due to the fact that certain faults and failures cannot be avoided, these unavoidable faults should be tolerated through additional design efforts. Also, the high integrity systems must allow fault tolerance. This means that faults are compensated in such a way that they do not lead to system failures [15, 16].

Generally, there are two steps involved in making a fault-tolerant system, namely (a) fault diagnosis and (b) control re-design. In the first step, the existence of a fault should be detected and identified. In the second, the controller should be adapted to the faulty plant so that the overall performance of the faulty system can satisfy the desired performance [15, 16].

Recently, a new FTC framework was studied for stochastic dynamic systems [17]. Different from classical FTC problems, non-Gaussian stochastic variables were considered, and the output was the PDF of the system output rather than the output itself. The stochastic distribution control system is based on the fixed rational square-root B-spline approximation model.
1.6 Thesis Objectives

This section gives a summary of the thesis objectives which can be listed as follows:

1. Investigating the effectiveness of the nonlinear observer design for state estimation of continuous-time nonlinear systems, as well as for discrete-time nonlinear systems using a special Lyapunov function. The results are compared with those for a conventional nonlinear Lipschitz observer.

2. Designing a nonlinear output regulation to a well-known single-link flexible joint robot system, using a kth-order approximation solution for both continuous-time and discrete-time classes of nonlinear systems.

3. Introducing a new algorithm for an adaptive Proportional-Integrator (PI) controller for nonlinear systems subject to stochastic non-Gaussian disturbance. The minimum entropy control is applied to decrease the closed-loop tracking error on an ILC basis.

4. Introducing a new FTC system by generalizing the proposed Proportional-Integral-Differential (PID) algorithm in [18] for nonlinear continuous-time systems subject to stochastic non-Gaussian variables by using approximated PDFs via Radial Basis Function (RBF) neural networks. The main objective is to find the PID gains so that the weight dynamics can follow the desired set of weights generated from a target distribution, the closed-loop system is made stable, and the related states are constrained.
5. Since most of the nonlinear control laws are actually implemented as digital controllers, it is important to consider the development of an efficient FTC design methods for nonlinear stochastic discrete-time systems with non-Gaussian variables, under ILC basis.

1.7 Thesis Contributions and Achievements

A summary of the contributions and achievements can be stated as follows:

1. Applying the idea of an ILC-based adaptive minimum entropy control. The controller parameters are updated so that the closed-loop tracking error PDF for nonlinear systems subject to stochastic non-Gaussian disturbance is a Gaussian-like shape in the last batch.

2. The extension of the idea of tuneable RBFs to FTC, using the output PDFs of general non-Gaussian stochastic systems. This approach overcomes the problems associated with the B-spline-based functional approximation.

3. A new FTC framework, based on a PID controller, is introduced for nonlinear continuous-time systems subject to stochastic non-Gaussian variables, using approximated PDFs via RBF neural networks (RBFNNs).

4. An ILC-based fixed structure FTC framework, is studied for output PDF shaping in general stochastic systems using LMI techniques.

1.8 Thesis Organization

The thesis starts with an introduction to the research subjects and the scope and contributions of the work.

Chapter 2 contains a review of some fundamental concepts and properties of nonlinear systems, used throughout the work. We present descriptions of various nonlinear systems, and the linearization method for nonlinear systems. Finally, we summarize the stability concepts and fundamental Lyapunov stability analysis approach for unforced nonlinear systems.

A nonlinear observer design for a class of nonlinear systems is considered in Chapter 3. The nonlinear observer is designed for both continuous-time and discrete-time nonlinear systems. The nonlinear observer for state estimation is designed using a Lyapunov function. A set of sufficient conditions
for the existence of observers of a class of nonlinear systems is presented, which guarantee that the estimation error converges asymptotically to zero. Moreover, the proposed nonlinear observer is designed for a single-link flexible joint robot model and the results obtained are compared with those for the Lipschitz nonlinear observer from the estimated states point of view, as well as noise rejection, when the plant is subject to white noise.

In Chapter 4, the problem of output regulation design for a class of nonlinear systems is considered. The nonlinear output regulator is designed for continuous-time and discrete-time nonlinear systems. An approximate solution for the regulator equations using a kth-order approximation is implemented in this work. A single-link flexible joint robot model is used to demonstrate the effectiveness of the proposed method.

In Chapter 5, a PI controller for nonlinear systems subject to stochastic non-Gaussian disturbance is studied. Minimum entropy control is applied to decrease the closed-loop tracking error on an ILC basis. The control horizon is divided into a number of equal time intervals, called batches. The design procedure is divided into two main algorithms within each batch and between any two adjacent batches. A D-type ILC law is employed to tune the PI controller coefficients between two adjacent batches. However, within each batch, the PI coefficients are fixed. A sufficient condition is established to guarantee the stability of the closed-loop system. Analysis of the ILC convergence is carried out. Two examples of well-known one-link manipulators with revolute joints actuated by a DC motor and two-link robot manipulator are given, to demonstrate the use of the control algorithm.

In Chapter 6, a new FTC for general stochastic nonlinear systems is studied. Different from the formulation of classical FTC methods, it is supposed that the measured information for the FTC is the PDFs of the system output rather than its measured value. A RBFNN technique is proposed so that the output PDFs can be formulated in terms of the dynamic weights of the RBFs neural network. A Linear Matrix Inequality (LMI)-based feasible FTC method is applied such that the fault is detected and diagnosed. An illustrated example is included to demonstrate the efficiency of the proposed algorithm.

In Chapter 7, a new design for an FTC-based adaptive PI controller for nonlinear discrete-time systems, where non-Gaussian behaviour exists, is
studied. An LMI-based FTC method is presented to ensure that the fault can be estimated and compensated for. An ILC scheme is used to improve the tracking performance in the batch direction, taking advantage of the repetitive nature of batch processes.

Finally, concluding remarks are made in Chapter 8.
Chapter 2

Fundamental Concepts and Properties of Nonlinear Systems

The main goal of this chapter is to review some fundamental concepts and properties of nonlinear systems that are used in the later chapters. The background of this chapter can be found in many well-known textbooks on nonlinear control system, such as [1–3, 19, 20]. Section 2.1 presents a description of various nonlinear systems and some basic definitions. The linearization method for nonlinear systems is presented in Section 2.2. Finally, Section 2.3 summarizes the stability concepts and fundamental Lyapunov stability analysis approach for nonlinear systems.

2.1 Nonlinear Systems

Nonlinear time-variant continuous systems can be described by first-order vector differential equations of the form:

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), u(t)) \\
y(t) &= h(t, x(t), u(t))
\end{align*}
\]  

(2.1)

where \(x(t)\) is the \(m\)-dimensional state vector \((x(t) \in \mathbb{R}^m)\), \(u(t)\) represents the \(r\)-dimensional plant input vector \((u(t) \in \mathbb{R}^r)\), and \(y(t)\) is the \(n\)-dimensional output vector \((y(t) \in \mathbb{R}^n)\). In the case of discrete nonlinear systems, the system can be described by difference equations of the form:

\[
\begin{align*}
x(k+1) &= f(k, x(k), u(k)) \\
y(k) &= h(k, x(k), u(k))
\end{align*}
\]  

(2.2)
where $x(k)$ is the $m$-dimensional state vector ($x(k) \in \mathbb{R}^m$), $u(k)$ represents the $r$-dimensional plant input vector ($u(k) \in \mathbb{R}^r$), and $y(k)$ is the $n$-dimensional output vector ($y(t) \in \mathbb{R}^n$).

The existence and uniqueness of solutions cannot be guaranteed without some conditions on the function $f(t, x(t), u(t))$. The solution of differential equation (2.1) in the interval $[0, T]$ means that such $x(t)$ has everywhere derivatives when (2.1) is valid for every $t$. Therefore, the conditions on $f(t, x(t), u(t))$ requires that the nonlinear function to be locally Lipschitz [20].

**Example. 1** A classic, extensively studied nonlinear problem is the dynamics of a pendulum under the influence of gravity in Fig. 2.1. Using Newton’s second law, it can be shown that the motion of a pendulum can be described by the nonlinear equations as [20]

$$ml\ddot{\theta} + mg \sin(\theta) + kl\dot{\theta} = 0$$

(2.3)

where $g$ is the acceleration due to gravity, $k$ is a coefficient of friction, $m$ denotes the mass, and $l$ is the length of the pendulum arm. Moreover, $\theta$ is the angel between the pendulum arm and the vertical axis through the hinge. Denote $x_1 = \theta$ and $x_2 = \dot{\theta}$. This leads to the system of equations

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin(x_1) - \frac{k}{l} x_2
\end{align*}$$

(2.4)

**Definition 1** [20] The function $f$ is said to be locally Lipschitz for the variables $x(t)$ if near the point $x(0) = x_0$ it satisfies the Lipschitz criterion:

$$\|f(x(t)) - f(y(t))\| \leq U \|x(t) - y(t)\|$$

(2.5)

for all $x$ and $y$ in the vicinity of $x_0$, where $U$ is a positive constant and the norm is Euclidian. Lipschitz criterion guarantees that (2.1) has an unique solution for an initial condition $x(t_0) = x_0$.

**Definition 2** [21] A continuous system is said to be forced if an input signal is present:

$$\dot{x}(t) = f(t, x(t), u(t)), \forall t \geq 0, u(t) \neq 0$$

A continuous system is said to be unforced if there is no input signal:

$$\dot{x}(t) = f(t, x(t)), \forall t \geq 0, u(t) = 0$$
Definition 3 [1] A state $x_e$ is an equilibrium state (or equilibrium point) of the system if once $x(t)$ is equal to $x_e$ for all time. Mathematically, if the system begins from the equilibrium state, it remains in this state $x(t) = x_e, \forall t \geq t_0$ under the assumption that no external signal acts. In other words, this means that the constant vector $x_e$ satisfies

$$f(x_e) = 0 \quad (2.6)$$

The values of the equilibrium points can be determined by solving the nonlinear algebraic equation (2.6).

### 2.2 Linearization

In order to simplify nonlinear systems and make them more manageable, linearization method is quite often used. By linearizing nonlinear system about a single equilibrium state, the linear systems theory can be applied. By considering the system in (2.1), and assuming that functions $f$ and $h$ are continuously differentiable. The linearized system at $x = 0$ and $u = 0$ can be presented as [19]

$$\dot{x} = \left( \frac{\partial f}{\partial x} \right)_{(x=0,u=0)} x + \left( \frac{\partial f}{\partial u} \right)_{(x=0,u=0)} u + f_{h.o.t.}(x,u) \quad (2.7)$$

$$y = \left( \frac{\partial h}{\partial x} \right)_{(x=0,u=0)} x + \left( \frac{\partial h}{\partial u} \right)_{(x=0,u=0)} u + h_{h.o.t.}(x,u)$$

where $f_{h.o.t.}(x,u)$ and $h_{h.o.t.}(x,u)$ represent the higher-order terms in $x$ and $u$. 

Figure 2.1: Illustration of a pendulum.
Denoting
\[ A = \left( \frac{\partial f}{\partial x} \right)_{(x=0,u=0)}, \quad B = \left( \frac{\partial f}{\partial u} \right)_{(x=0,u=0)} \]
\[ C = \left( \frac{\partial h}{\partial x} \right)_{(x=0,u=0)}, \quad D = \left( \frac{\partial h}{\partial u} \right)_{(x=0,u=0)} \]
(2.8)
the linearized system can be written as
\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]
(2.9)

2.3 Stability Concepts

Stability is the most significant qualitative property of automatic control systems. The concept of stability is so important since every control system must be primarily stable, then other properties can be studied. The most powerful approach for studying the stability of nonlinear control system was introduced in the late 19\textsuperscript{th} century by the Russian mathematician, A. Lyapunov. Lyapunov introduced two approaches for the stability of the nonlinear system [19]. The first approach is the linearization method, where the stability of the nonlinear system can be found by analyzing the stability of a linear model that is obtained by linearization of the nonlinear system in the vicinity of the equilibrium state [19]. However, this method is limited just to local stability. The second approach is known as the direct method or Lyapunov second method. The direct method is a generalization of the energy concepts associated with a mechanical system [19].

Definition 4 [20] The equilibrium point \( x = 0 \) of \( \dot{x} = f(x) \) is said to be stable if, for each \( \varepsilon > 0 \), there is \( \delta = \delta(\varepsilon) \) such that
\[
\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq 0
\]

Definition 5 [20] An equilibrium point \( x = 0 \) of \( \dot{x} = f(x) \) is asymptotically stable if it is stable and \( \delta \) can be chosen such that
\[
\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0
\]

Theorem 1. [20] Let \( x = 0 \) be an equilibrium point for \( \dot{x} = f(x) \) and \( D \subset \mathbb{R}^n \) be a domain containing \( x = 0 \). Let \( V : D \to \mathbb{R} \) be a continuously differentiable function such that
\[
V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}\]
\[ \dot{V}(x) \leq 0 \text{ in } D \]

Then, \( x(0) = 0 \) is stable. Moreover, if

\[ \dot{V}(x) < 0 \text{ in } D - \{0\} \]

then \( x(0) = 0 \) is asymptotically stable.

**Definition 6** [21] A necessary and sufficient condition for a real \((P)\) and symmetric \((P = P^T)\) matrix to be positive definite is one of the following:

1) \( x^T P x > 0, \forall x \neq 0, \)

2) All eigenvalues of the matrix \( P \) satisfy \( \lambda_i(P) > 0, \)

3) A matrix \( W \) exists, such that \( P = W^T W \) is valid.

4) All upper left minors (submatrices) \( P_k \) have positive determinants,

### 2.4 Summary

In fact, all the systems in the real world are inherently nonlinear. Therefore, linear control theory cannot be applied without modification. A nonlinear system can be linearized around an equilibrium point if the nonlinearities in the system are smooth and the operating range of the nonlinear system is small. The nonlinearities should be taken into account when looking at the general case.

Stability is a fundamental issue in control system analysis and design. Lyapunov’s stability concept is defined in order to characterize the stability behaviour of nonlinear systems. Two methods are introduced by Lyapunov to determine this stability. The linearization method is restricted to the small motion of nonlinear systems around an equilibrium point, as well as the weakness of nonlinearities in the system. Only local stability can be justified using this method, because it depends on the linear model of the system. Lyapunov’s second method, based on the Lyapunov functions, is a general method, and is applicable to all dynamic systems.
Chapter 3

Nonlinear Observer Design

3.1 Introduction

In this chapter, the problem of state observer design for a class of nonlinear systems is considered. The nonlinear observer will be designed for continuous-time and discrete-time nonlinear systems. The nonlinear observer for state estimation is designed using a Lyapunov function. A set of sufficient conditions for the existence of an observer of the nonlinear systems is presented, which guarantees that the estimation error converges asymptotically to zero. This observer is compared with the Lipschitz nonlinear observer from the estimated states point of view; also noise rejection is considered, when the plant is subject to white noise. An example of a single-link flexible joint robot model is given to demonstrate the effectiveness of the proposed observer.

3.2 Literature Review

Research into observer design has been regarded as an important research area over the past few decades. In many engineering applications, it is desirable to measure all the state variables directly, which may be extremely expensive, or even impossible. Therefore, the problem of estimating the state of a dynamic system from its outputs and inputs plays a vital role in the theory of systems. The concept of a system observer is an auxiliary system that models the real system in order to provide an estimate of the target system of interest such that the error dynamics between the real and the auxiliary systems go to zero as time approaches infinity. This concept was first introduced by Kalman in the early 1960s [22, 23], and then by Luenberger [24–26]. The Luenberger observer was first proposed and developed in [24], and further developed in [25]. In general,
the Luenberger observer has a relatively simple design that makes it an attractive general design technique [16,26]. Later on, the Luenberger observer was extended to discrete-time linear systems in [27]. Although the Kalman filter has been in use for more than three decades and has been described in many papers and books, its design is still an area of concern for many researchers and studies. It could be argued that the Kalman filter is one of prominent observers used against a wide range of disturbances [16,28]. For linear systems, it has been extensively studied, and proven extremely useful for both continuous-time and discrete-time linear systems [27,29,30]. However, for nonlinear systems, the theory of observers is neither as complete or as successful as it is for linear systems. Therefore, in spite of the many approaches that exist, observer design for nonlinear systems remains an open problem in control theory as there is no universal nonlinear observer design methodology that can be applied to all nonlinear systems. Various approaches of nonlinear observer design have been investigated for different types of nonlinear systems.

In early work on observer design for nonlinear systems, the theory of the observer mainly concentrated on linear systems where the nominal observer gains are determined from a linearized model [31]. One of the widely used alternative methods for estimating the states of a nonlinear system is the extended Kalman filter. It is obtained by linearizing the dynamics and the observation along the trajectory of the estimate. However, this is only a convenient method, in the sense that the estimate converges to the true state if the initial error is not too large and the linearization does not present any singularity. Moreover, linear approaches may not work efficiently when they are used in strongly nonlinear systems. The state observation of nonlinear dynamic systems with bounded nonlinearities/uncertainties was discussed in [32]. The authors presented an observer design method using Lyapunov and min-max methods.

Nonlinear observer design is generally a difficult problem because there is no general theory for the design of observers for nonlinear systems. As a result, the existing nonlinear observers reported in the literatures tend to group into two major categories. In the first category, the observer design is based on a nonlinear state transformation such that the error dynamic of the state is linear. Necessary and sufficient conditions for the existence of the state transformation have been established in [31,33–35]. On the other hand, the second category is based on designing observers for the original nonlinear systems without the need for state
The first attempts to construct observers for the original nonlinear systems were reported by Thau in \[36\]. The high-gain observer is quite popular in system theory and has been widely dealt with in the literature, as can be seen, for example, in \[31, 33, 34, 42\]. Other nonlinear observer design techniques include the pole placement approach, and riccati equations approach which were studied in \[43\] and \[37\], respectively. Also, in \[44\] and \[45\], a new method for nonlinear observers design was studied, where a general set of necessary and sufficient conditions were derived using Lyapunov's auxiliary theorem. However, the application of the previous direct nonlinear observer design technique is quite limited due to the strict condition that the system must be in a semi-triangular structure. Moreover, an observer for Lipschitz nonlinear systems was studied in \[36, 38, 39, 46, 47\]. The proposed observer design techniques were based on quadratic Lyapunov functions and thus depend on the existence of a positive definite solution to an algebraic ricatti equation \[38, 39\]. The existence of a stable observer for Lipschitz nonlinear systems was addressed, and a sufficient condition was given on the Lipschitz constant. However, the approach suffers from the limitation that the Lipschitz constant has to be small. Recently, in \[48\], the one-sided Lipschitz condition was introduced instead of the Lipschitz condition. Based on this, a set of sufficient conditions for the existence of an observer of the class of nonlinear systems were established, which were less conservative than the results based on the Lipschitz condition \[48\].

The above results were all obtained for the continuous case. On the other hand, observer design for nonlinear discrete-time systems has also attracted much interest from many researchers since digital technology is increasingly being used in industrial applications, and the controller is implemented by digital computers, see for example \[49–54\]. Later on, in \[55\] and \[56\], the observer construction was studied in the context of solving simultaneous nonlinear equations using Newton's algorithm. However, this method requires the iterative solution of a set of nonlinear algebraic equations for each time interval, and the convergence conditions check for each time interval is quite complex in practice. In addition, in \[57\], results were established for the class of Lipschitz systems. In \[58\], the problem of observer design for a class of Lipschitz nonlinear discrete-time systems was solved via simple and useful transformations. Both full and reduced order state observers were established. From the Lyapunov functions, a set of sufficient conditions for asymptotic stability was obtained in terms of LMIs. In addition, a simple and useful reduced-order observer for a large class of nonlinear discrete-time systems was presented in the recent work \[59\], where under general conditions a set of
asymptotic convergence conditions for a proposed state observer was established that appears to be nonrestrictive in the presence of mild nonlinearities. A recent notable piece of research appeared in [60], where the observer design problem was translated into the problem of solving a system of first-order, linear, inhomogeneous functional equations. A set of necessary and sufficient conditions for solvability was derived using results from functional equations theory. Finally, immersion techniques were investigated for observer design for nonlinear discrete-time systems, in [61]; this can be considered as an extension of the immersion and dynamic observer error linearization techniques to nonlinear discrete-time systems. The new technique was motivated by the fact that there exists a class of nonlinear systems that cannot be transformed into observer form via diffeomorphism. A set of necessary and sufficient conditions for observer design was obtained, and a constructive algorithm was also provided for dynamic observer error linearization.

3.3 Basics of Nonlinear Observers

Consider the nonlinear system

\[ \dot{x}(t) = f(x(t), u(t)) \]
\[ y(t) = h(x(t)) \]

(3.1)

where \( x(t) \in \mathbb{R}^n \) is the unmeasured state vector, \( u(t) \in \mathbb{R}^r \) is the measurable input vector, and \( y(t) \in \mathbb{R}^p \) is the measurable output vector. Suppose the system has an equilibrium point at \((x^*, u^*)\), i.e. \( f(x^*, u^*) = 0 \). A local observer for (3.1) is another an auxiliary dynamical system, driven by inputs \( y(t) \) and \( u(t) \), that produces an estimate \( \hat{x}(t) \) of \( x(t) \) such that the error \( e(t) = x(t) - \hat{x}(t) \) converges to zero as time tends to infinity:

\[ \|x(t) - \hat{x}(t)\| \to 0, \text{ when } t \to \infty \]

provided \( x(t) \) remains sufficiently close to \( x^* \).

The measurement error or residual is used to drive a replica of the system so that the observer equations are

\[ \dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + L(y(t) - h(\hat{x}(t))) \]

(3.2)
where $L$ is the observer gain to be determined. The error dynamics can be computed as

$$
\dot{e}(t) = \dot{x}(t) - \hat{\dot{x}}(t) = f(x(t), u(t)) - f(x(t) - e(t), u(t)) + L(x(t) - e(t), h(x(t)) - h(x(t) - e(t)))
$$

(3.3)

Let us consider (3.3) as a differential equation that defines $e(t)$, the error response to an exogenous input $x(t)$. The system (3.2) is an observer, if $L$ can be chosen so that this equilibrium point is asymptotically stable.

### 3.4 Nonlinear Observability

A crucial issue when attempting to design an observer for a system is of course whether it is possible to estimate the state. Observability of a system is the property that the state can be uniquely determined from the input and the output signals. In this section, a more precise definition of observability will be given. The observability condition is required on the system for possible solutions to solve the observer problem. Roughly speaking, the results from the system are observable, if for any pair of initial states produce identical output under any input, i.e. is distinguishable. System (3.1) is said to be observable at $x_0$ or alternatively at time $t_0$ if the state vector $x(t_0)$ can be determined from the observation of $y(t)$ over a finite time interval, $t_0 < t < t_1$. For Single-Input-Single-Output (SISO) time-invariant linear systems with $m$ states in the state space representation, if the rank of the following matrix

$$
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{m-1}
\end{bmatrix}
$$

(3.4)

is equal to $m$, then the system is observable. This condition called the observability rank condition. Therefore, the pair of $(A, C)$ is an observable pair, if the observability rank condition is satisfied. For nonlinear observability, consider the
following affine system

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\
y(t) &= h(x(t))
\end{align*}
\]

(3.5)

where \( x(t) \in \mathbb{R}^m \), \( u(t) \in \mathbb{R}^r \), \( y(t) \in \mathbb{R}^n \), \( f(\cdot) \) and \( g(\cdot) \) are smooth vector fields defined on \( D \subset \mathbb{R}^n \). Moreover, \( h(\cdot) \) is a bounded smooth function on \( D \). The Lie derivative of a smooth function is defined as follows.

**Theorem 2.** [62] Let \( h : D \rightarrow \mathbb{R} \) be a smooth function, and \( f : D \rightarrow \mathbb{R}^n \) be a smooth vector field. The Lie derivative of \( h \) with respect of \( f \), written as \( L_fh \), is defined by

\[
L_fh(x) = \frac{\partial h}{\partial x}f(x)
\]

(3.6)

The system is locally observable at \( x_0 \) if there exists a neighbourhood of \( x_0 \) such that every \( x \) in that neighbourhood other than \( x_0 \) is distinguishable from \( x_0 \). The system satisfies the observability rank condition if any of the observability matrices are of rank \( m \). The observability matrices are given by

\[
\begin{bmatrix}
dL^0_fh_j \\
dL^1_fh_j \\
dL^2_fh_j \\
\vdots \\
dL^{m-1}_fh_j
\end{bmatrix}, \text{for } 1 \leq j \leq n
\]

(3.7)

### 3.5 Observer Design for Continuous-time Systems

This section presents the problem of nonlinear observer design for the following continuous-time nonlinear systems:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Gg(x(t)) \\
y(t) &= Cx(t)
\end{align*}
\]

(3.8)

where \( x(t) \in \mathbb{R}^m \) is the unmeasured state vector, \( u(t) \in \mathbb{R}^r \) is the measurable input vector, and \( y(t) \in \mathbb{R}^n \) is the measurable output vector. Moreover, \( A, B, G \) and \( C \) represent the identified coefficient matrices of the nonlinear system with suitable dimensions. \( g(x(t)) \) is a nonlinear vector function of the nonlinear
system and is supposed to satisfy \( g(0) = 0 \), and the following norm condition

\[
\| g(x_1(t)) - g(x_2(t)) \| \leq \| U(x_1(t) - x_2(t)) \| \tag{3.9}
\]

for any \( x_1(t) \) and \( x_2(t) \), where \( U \) is a known matrix.

The following nonlinear observer for the continuous-time nonlinear system (3.8) is constructed

\[
\dot{x}(t) = Ax(t) + Bu(t) + Gg(x(t)) + Le(t) \\
\dot{y}(t) = Cx(t) \\
e(t) = y(t) - \hat{y}(t) = C(x(t) - \hat{x}(t)) \tag{3.10}
\]

where \( \hat{x}(t) \) is the estimated state vector, \( L \in \mathbb{R}^{m \times r} \) is the observer gain matrix to be determined, and \( e(t) \) is the residual signal.

By defining \( e(t) = x(t) - \hat{x}(t) \), the estimation error system can be formulated from (3.8) and (3.10) to give

\[
\dot{e}(t) = (A - LC)e(t) + G(g(x(t)) - g(\hat{x}(t))) \tag{3.11}
\]

The objective is to find \( L \) such that the estimation error system (3.11) is asymptotically stable.

**Theorem 3.** If for the parameter \( \lambda > 0 \), there exist matrices \( P = P^T > 0 \), and \( R \) of appropriate dimensions satisfying

\[
\begin{bmatrix}
\Pi_0 & \Pi_1 \\
\Pi_1^T & -I
\end{bmatrix} < 0 \tag{3.12}
\]

where

\[
\Pi_0 = (PA - RC) + (PA - RC)^T + \frac{1}{\lambda^2}U^TU \
\Pi_1 = \lambda PG
\]

and \( \Pi_1 = \lambda PG \), then estimation error system (3.11) with gain \( L = P^{-1}R \) is asymptotically stable.

**Proof.** Denote the Lyapunov function candidate as follows:

\[
\Phi(e(t), x(t), \hat{x}(t), t) = e^T(t)Pe(t) + \frac{1}{\lambda^2} \int_0^t \| Ue(\tau) \|^2 - \| g(x(\tau)) - g(\hat{x}(\tau)) \|^2 d\tau
\]

(3.14)
Following (3.9), it is noted that \( \Phi (e(t), x(t), \dot{x}(t), t) \geq 0 \) holds for all arguments. The time derivative of the above Lyapunov candidate will be along the trajectories of (3.11) and by using the completion-of-square method, it can be shown that

\[
\dot{\Phi} (e(t), x(t), \dot{x}(t), t) = e^T(t) \left[ P(A - LC) + (A - LC)^T P \right] e(t) \\
+ 2e^T(t)P\left[ Gg(x(t)) - Gg(\dot{x}(t)) \right] \\
+ \frac{1}{\lambda^2} \left[ \| Ue(t) \|^2 - \| g(x(t)) - g(\dot{x}(t)) \|^2 \right] \\
\leq e^T(t) \left[ P(A - LC) + (A - LC)^T P \right] e(t) \\
+ e^T(t)\lambda^2 PGG^T P e(t) \\
+ e^T(t) \left[ \frac{1}{\lambda^2} U^TU \right] e(t) \\
\leq e^T(t) \Psi_0 e(t) \tag{3.15}
\]

By denoting \( R = PL \) and using Schur complement formula with respect to (3.12), \( \Psi_0 < -\eta I \), where

\[
\Psi_0 = P(A - LC) + (A - LC)^T P + \lambda^2 PGG^T P + \frac{1}{\lambda^2} U^TU
\]

Thus, under (3.12), it can be seen that \( \dot{\Phi} (e(t), x(t), \dot{x}(t), t) < -\eta \| e(t) \|^2 \)

which means that \( \Phi (e(t), x(t), \dot{x}(t), t) < 0 \), and implies that the estimation error system is asymptotically stable.

The observer design presented in Theorem 3 is a special case for later use in this work.

### 3.6 Observer Design for Discrete-time Systems

This section considers the problem of nonlinear observer design for the following discrete-time nonlinear systems:

\[
x(k+1) = \bar{A}x(k) + \bar{B}u(k) + \bar{G}g(x(k)) \\
y(k) = \bar{C}x(k) \tag{3.17}
\]

where \( x(k) \in \mathbb{R}^m \) is the unmeasured state vector, \( u(k) \in \mathbb{R}^r \) is the measurable input vector, and \( y(k) \in \mathbb{R}^n \) is the measurable output vector. Moreover, \( \bar{A}, \bar{B}, \bar{G} \) and \( \bar{C} \) represent the identified coefficient matrices of the nonlinear system with suitable dimensions. \( g(x(k)) \) is a nonlinear vector function of the nonlinear system and is supposed to satisfy \( g(0) = 0 \), and the following norm condition

\[
\| g(x_1(k)) - g(x_2(k)) \| \leq \| U_1(x_1(k) - x_2(k)) \| \tag{3.18}
\]
for any \( x_1(k) \) and \( x_2(k) \), where \( U_1 \) is a known matrix.

The following nonlinear observer for the discrete-time nonlinear system (3.17) is constructed

\[
\hat{x}(k+1) = \bar{A}\hat{x}(k) + \bar{B}u(k) + \bar{G}g(\hat{x}(k)) + L_1\epsilon_1(k)
\]

\[
\hat{y}(k) = \bar{C}\hat{x}(k)
\]

\[
\epsilon_1(k) = y(k) - \hat{y}(k) = \bar{C}(x(k) - \hat{x}(k)) \tag{3.19}
\]

where \( \hat{x}(k) \) is the estimated state vector, \( L_1 \in \mathbb{R}^{m \times r} \) is the observer gain matrix to be determined, and \( \epsilon_1(k) \) is the residual signal.

Be defining \( e(k) = x(k) - \hat{x}(k) \), the estimation error system can be formulated from (3.17) and (3.19) to give

\[
e(k+1) = (\bar{A} - L_1\bar{C})e(k) + \bar{G}(g(x(k)) - g(\hat{x}(k))) \tag{3.20}
\]

The objective is to find \( L_1 \) such that the estimation error system (3.20) is asymptotically stable.

**Theorem 4.** If for the parameter \( \lambda_1 > 0 \), there exist matrices \( P_1 = P_1^T > 0 \), and \( R_1 \) satisfying

\[
\begin{bmatrix}
    -P + \frac{1}{\lambda_1}U_1^TU_1 & 0 & \bar{A}^TP - \bar{C}^TR_1^T \\
    0 & -\frac{1}{\lambda_1} & \bar{G}^TP \\
    P\bar{A} - R\bar{C} & PG & -P
\end{bmatrix} < 0 \tag{3.21}
\]

then estimation error system (3.20) with gain \( L_1 = P_1^{-1}R_1 \) is asymptotically stable.

**Proof.** Denote the Lyapunov function candidate as follows:

\[
\Phi_1(e(k), x(k), \hat{x}(k), k) = e^T(k)P e(k) + \frac{1}{\lambda_1^2} \sum_{i=1}^{k-1} \left[ ||U_1e(i)||^2 - ||g(x(i)) - g(\hat{x}(i))||^2 \right] \tag{3.22}
\]
Differentiating (3.22) over the time gives

\[ \Delta \Phi_1 = \Phi_1(k+1) - \Phi_1(k) \]
\[ = e(k)^T \left( (\bar{A} - L_1 \bar{C})^T P_1 (\bar{A} - L_1 \bar{C}) - P_1 \right) e(k) \]
\[ + (g(x(k)) - g(\hat{x}(k)))^T \hat{G}^T P_1 \hat{G} \left( g(x(k)) - g(\hat{x}(k)) \right) \]
\[ + 2e(k)^T (\bar{A} - L_1 \bar{C})^T P_1 (g(x(k)) - g(\hat{x}(k))) \]
\[ + \frac{1}{\lambda_1^2} \left[ \| U_1 e(k) \|^2 - \| g(x(k)) - g(\hat{x}(k)) \|^2 \right] \]
\[ = e(k)^T \left( (\bar{A} - L_1 \bar{C})^T P_1 (\bar{A} - L_1 \bar{C}) - P_1 + \frac{1}{\lambda_1^2} U_1^T U_1 \right) e(k) \]
\[ + (g(x(k)) - g(\hat{x}(k)))^T \left( \hat{G}^T P_1 \hat{G} - \frac{1}{\lambda_1^2} \right) \left( g(x(k)) - g(\hat{x}(k)) \right) \]
\[ + 2e(k)^T (\bar{A} - L_1 \bar{C})^T P_1 \hat{G} \left( g(x(k)) - g(\hat{x}(k)) \right) \]  

(3.23)

According to the Lyapunov stability theory [63], this Lyapunov function guarantees the asymptotic stability of the estimation error in (3.20) if

1. \( \Phi_1(e(k), x(k), \hat{x}(k), k) > 0 \) for all \( e(k) \neq 0 \) and

2. \( \Delta \Phi_1 < 0 \) for all trajectories of the error (3.20)

Condition (1) is satisfied if the matrix \( P_1 \) is positive definite. Condition (2) can be rewritten as

\[ \Delta \Phi_1 = \begin{bmatrix} e(k)^T (g(x(k)) - g(\hat{x}(k))) \end{bmatrix} \Psi_1 \begin{bmatrix} e(k) \\ (g(x(k)) - g(\hat{x}(k)) \end{bmatrix} < 0 \]  

(3.24)

where

\[ \Psi_1 = \begin{bmatrix} (\bar{A} - L_1 \bar{C})^T P_1 (\bar{A} - L_1 \bar{C}) - P_1 + \lambda_1^{-2} U_1^T U_1 & (\bar{A} - L_1 \bar{C})^T P_1 \hat{G} \\ \hat{G}^T P_1 (\bar{A} - L_1 \bar{C}) & \hat{G}^T P_1 \hat{G} - \lambda_1^{-2} I \end{bmatrix} \]  

(3.25)

Then it can be deduce that (3.20) is asymptotically stable if inequality (3.24) is satisfied. By denoting \( R_1 = P_1 L_1 \) and using Schur complement formula, it is clear that (3.25) is equivalent to (3.21). When (3.21) is feasible, a positive definite matrix \( P_1 \) can be determined, and the observer gain can be calculate by the following equation

\[ L_1 = P_1^{-1} R_1 \]  

(3.26)
3.7 Simulation Results

In order to show the effectiveness of the proposed method, the following one-link manipulator with revolute joints actuated by a DC motor is considered, as shown in Fig. 3.1. The elasticity of the joint can be well-modelled by a linear torsional spring [64]. The elastic coupling of the motor shaft to the link introduces an additional degree of freedom. The nominal model states of the presented one-link flexible joint robot in Fig. 3.1 are: motor position and velocity, and the link position and velocity. The output of system is the motor position, and the input is the motor velocity [64].

The single-link flexible joint robot is described by the following equations:

\[
\begin{align*}
\dot{\theta}_m &= \omega_m \\
\dot{\omega}_m &= \frac{k}{J_m} (\theta - \theta_m) - \frac{B}{J_m} \omega_m + \frac{K_\tau}{J_m} u \\
\dot{\theta}_1 &= \omega_1 \\
\dot{\omega}_1 &= -\frac{k}{J_1} (\theta - \theta_m) - \frac{mgh}{J_1} \sin(\theta_1)
\end{align*}
\]

(3.27)

where all the variables and parameters are defined as follows:

- $J_m$ is the inertia of the motor ($Kgm^2$)
- $J_1$ stands for the inertia of the link ($Kgm^2$)
- $k$ represents the tensional constant of the spring ($Nmrad^{-1}$)
- $K_\tau$ is the Amplifier gain ($NmV^{-1}$)
- $B$ stands for the Viscous friction coefficient ($NmV^{-1}$)
- $m$ stands for pointer mass ($Kg$)
- $g$ is the acceleration of gravity ($ms^{-2}$)
- $h$ is the link length ($m$)
- $\theta_m$ is the angular rotation of the motor ($rad$)
- $\theta_1$ is the angular rotation of the link ($rad$)
- $\omega_m$ represents the angular velocity of the motor ($rads^{-1}$)
- $\omega_1$ stands for the angular velocity of the link ($rads^{-1}$)

Thus, the system dynamics is nonlinear and of the form

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Gg(x(t)) \\
y(t) &= Cx(t)
\end{align*}
\]

(3.28)
with

\[
x = \begin{bmatrix} \theta_m & \omega_m & \theta_1 & \omega_1 \end{bmatrix}^T
\]

(3.29)

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-48.6 & -1.25 & 48.6 & 0 \\
0 & 0 & 0 & 1 \\
19.5 & 0 & -19.5 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
21.6 \\
0 \\
0
\end{bmatrix},
G = \begin{bmatrix}
0 \\
0 \\
0 \\
-3.33
\end{bmatrix},
g(x) = \begin{bmatrix}
0 \\
0 \\
0 \\
\sin(x_3)
\end{bmatrix}
\]

and

\[
C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
\]

The values of constants used for observer design and simulation in this work are summarized in Table. 3.1.

<table>
<thead>
<tr>
<th>The parameter values of robot</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motor inertia, $J_m$ (Kgm(^2))</td>
<td>$3.7 \times 10^{-3}$</td>
</tr>
<tr>
<td>Link inertia, $J_1$ (Kgm(^2))</td>
<td>$9.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>Pointer mass, $m$ (Kg)</td>
<td>$2.1 \times 10^{-1}$</td>
</tr>
<tr>
<td>Link length, $l$ (m)</td>
<td>$3.1 \times 10^{-1}$</td>
</tr>
<tr>
<td>Torsional spring constant, $k$ (Nmrad(^{-1}))</td>
<td>$1.8 \times 10^{-1}$</td>
</tr>
<tr>
<td>Viscous friction coefficient, $B$ (NmV(^{-1}))</td>
<td>$4.6 \times 10^{-2}$</td>
</tr>
<tr>
<td>Amplifier gain, $K_\tau$ (NmV(^{-1}))</td>
<td>$8.0 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

The Lipschitz constant of $g(x)$ with respect to $x$ is $U = 3.33$. The observer for the flexible link robot (3.28) is in the form of (3.10). Using Theorem 2 with $\lambda = 2$ it can be calculated that
Chapter 3. Nonlinear Observer Design

To simulate the system (3.27), the initial value of $x$, $x(0)$, is chosen to be $[1, 1, 1, 1]^T$; the initial value of $\dot{x}$, $\dot{x}(0)$, is chosen to be $[0, 0, 0, 0]^T$. The system is assumed to be unforced i.e. $(u = 0)$. The simulation results of the observer are shown in Fig. 3.2 and Fig. 3.3. Fig. 3.2 shows the motor angular position, motor angular velocity, and their estimates. Fig. 3.3 shows the link angular position, the link angular velocity and their estimates. From both figures, it is obvious that the estimates converge to the true states. The estimation error is driven to zero after a short time, ensure adequate state estimation in Fig. 3.4.

To show the effectiveness of our observer design, the continuous-time nonlinear observer simulation results is compared with the simulation results for the Lipschitz nonlinear observer approach which has been presented in [65] by applying both of them for the single-link flexible joint robot model under the same initial value of $x$, $\dot{x}$, $T$, and $u$. The problem of nonlinear observer design for nonlinear systems using the Lipschitz approach has already been reported in the literature and details can be found in [65]. Based on [65], the observer for the flexible link robot (3.28) is in the form of (3.10) with

$$P = \begin{bmatrix} 2.5478 & -0.0780 & 0.0067 & 0.0020 \\ -0.0780 & 0.0213 & -0.0820 & 0.0224 \\ 0.0067 & -0.0820 & 0.5560 & -0.0699 \\ 0.0020 & 0.0224 & -0.0699 & 0.0547 \end{bmatrix}, \quad R = \begin{bmatrix} 93.7079 \\ -0.2276 \\ -1.2026 \\ 0.0347 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 57.5711 \\ 681.0302 \\ 74.3309 \\ -184.9217 \end{bmatrix}$$

![Figure 3.1: Robotic system with flexible joint](image)
Throughout this section our observer approach in this chapter is referred as Lyapunov observer design or new observer design and the proposed algorithm for the Lipschitz nonlinear observer approach which has been presented in [65] is as Lipschitz observer design. Figs. 3.5 and 3.6 show the motor angular position, motor angular velocity and their estimates, the link angular position, the link angular velocity and their estimates in both observer approaches, respectively. From both figures, it is clear that the estimates converge to the true states, but the estimated states in the new observer design converge faster than the estimated states in the Lipschitz observer method.
Chapter 3. Nonlinear Observer Design

Figure 3.4: Estimation error

Figure 3.5: Comparison between motor angular position, motor angular velocity, and their estimates

Figure 3.6: Comparison between link angular position, link angular velocity, and their estimates
Figure 3.7: Comparision between motor angular position and its estimates with white noise

Figure 3.8: Comparision between motor angular position and its estimates with white noise (nonlinear Lipschitz observer)

Figure 3.9: Estimation error with white noise.
Figure 3.10: Estimation error with white noise (nonlinear Lipschitz observer).

Figure 3.11: Estimated PDF of estimation error with white noise

Figure 3.12: Estimated PDF of estimation error with white noise (nonlinear Lipschitz observer).
Moreover, when the system is subject to white noise, the effectiveness of each observer design is judged by comparing the error estimation, Kernel Density amplitude and its distribution type. Figs. 3.7 and 3.8 show the motor angular position and its estimate when the system is subject to white noise in both observer approaches, respectively.

The estimation errors of the observers in Figs. 3.9 and 3.10 show that the new observer has less amplitude variation than the Lipschitz observer. Moreover, the Kernel density distributions of the estimation error as shown in Figs. 3.11 and 3.12 show that both observers distributions are non-symmetric. In addition, the new observer’s deviation is better than that of the Lipschitz observer. As a result, the proposed observer has better performance and noise rejection than the Lipschitz observer.

3.8 Summary

Motivated by the fact that observer design for nonlinear systems remains an open problem in control theory, as there is no universal nonlinear observer design methodology that can be applied to all nonlinear systems, a new nonlinear observer design technique, based on a Lyapunov function for a class of nonlinear systems, is introduced in this chapter. This technique deals with both continuous-time and discrete-time nonlinear systems. A set of sufficient conditions for the existence of observers of the class of nonlinear systems is presented. The observer gains can be obtained by solving LMIs.

The simulation results have demonstrated the capabilities of the nonlinear observer in accurately estimating the state variables of the system, which represent the single-link flexible joint robot model. Subsequently, the estimation error is driven to zero after a short time ensuring adequate state estimation.

For the purposes of comparison with the proposed observer, a nonlinear Lipschitz observer is considered. The simulation results revealed that the proposed observer is superior in terms of disturbance rejection when the observers are subject to white noise, compared with the Lipschitz observer.
Chapter 4

Output Regulation for Nonlinear Systems: Preliminary Study

4.1 Introduction

In this chapter, the problem of output regulation design for a class of nonlinear systems is considered. The nonlinear output regulator will be designed for continuous-time nonlinear systems as well as for discrete-time nonlinear systems. It is known that in both nonlinear classes the feedback controller that solves the nonlinear servomechanism problem depends on the solution of a set of nonlinear functional equations called the nonlinear regulator equations. The exact solution of the regulator equations is usually difficult due to the nonlinearity of the system. Therefore, an approximate solution using a kth-order approximation approach is implemented in this work. The tracking performance of the system is compared with other tracking performances from the point of view of the maximal steady-state tracking errors of the closed-loop system when the plant is subject to different reference inputs.

Although the single-link flexible joint robot system is well-known, the application of the nonlinear output regulation design methodology on this system is unique, and has not been previously carried out to be the best of our knowledge. The simulation shows the effectiveness of the proposed controller approach.

4.2 Literature Review

This section introduces a brief history of the output regulation problem, or the servomechanism problem, which includes tracking reference signals and rejecting
disturbances generated using an external autonomous system called the exosystem. A number of industrial cases can be considered as practical examples of output regulation control. For example, the cruise control of automobiles, the manipulation of robot arms, the orbiting of satellites, and motor speed regulation [4]. For linear systems, this problem was discussed in the 1970s by [5,66,67]. For nonlinear systems, extensive research has been carried out, starting with the work in [68], which presented solutions to the local output regulation problem for general nonlinear continuous-time systems. The necessary and sufficient conditions for the solution of the output regulation problem when the exosystem is neutrally stable were given in [68]. In [69], the authors studied a servomechanism problem for nonlinear plants subject to constant or slowly varying exogenous signals via stabilization on the manifold of constant operating points corresponding to zero error. The extended linearization methodology was applied by Huang and Rugh in [70] to the nonlinear servomechanism problem for the case of time-varying exogenous signals. These works have been followed by an extensive amount of research dealing with different aspects of the output regulation problem for nonlinear systems [71–73].

The necessary and sufficient conditions for the solution of the nonlinear, discrete-time output regulation problem, were generalized in [74, 75]. Also, various aspects of the output regulation problem were discussed in [6, 76, 77]. The solvability of the problem depends on the solvability of the regulator equations. For continuous-time nonlinear systems, these are a set of partial differential and algebraic equations [68], while for discrete-time nonlinear systems they are a set of nonlinear functional equations [78]. Due to their nonlinear nature, it is impossible to solve these equations analytically for general nonlinear systems. Thus, various approximation methods have been presented to obtain approximate controllers in studies such as [76, 78, 79].

### 4.3 Nonlinear Output Regulation

Similar to [4], the typical problem form of nonlinear output regulation is described in Fig. 4.1, where the nonlinear system in the following form:

\[
\begin{align*}
\dot{x}(t) &= F(x(t), u(t), d(t)), x(0) = x_0 \\
y(t) &= H(x(t), u(t), d(t))
\end{align*}
\] (4.1)
where \( x(t) \) is the plant state, \( u(t) \) stands for the plant input, \( y(t) \) presents the plant output, and \( d(t) \) is the disturbance signal generated by an exogenous system. Moreover, there is a reference input generated by an exogenous system. The tracking error can be defined by

\[
e(t) = y(t) - r(t) \tag{4.2}
\]

where \( r(t) \) is the reference input. A classical nonlinear feedback control law will be in the following form:

\[
\begin{align*}
    u(t) &= k(z(t)) \\
    \dot{z} &= g(z(t), e(t))
\end{align*} \tag{4.3}
\]

where \( k \) and \( g \) are some nonlinear functions. The objective of the control law is that the closed-loop system be stable and the output be able to track the reference input asymptotically, which means

\[
\lim_{t \to \infty} (y(t) - r(t)) = 0 \tag{4.4}
\]

By combining the reference input \( r(t) \) and disturbance \( d(t) \) into a single exogenous signal vector \( \nu \), the exosystem can be presented as follow [4]

\[
\begin{align*}
    \dot{\nu} &= a(\nu(t)), \nu(0) = \nu_0 \tag{4.5}
\end{align*}
\]

As a result, we can put (4.1), (4.2), and (4.5) together as follows:

\[
\begin{align*}
    \dot{x}(t) &= f(x(t), u(t), \nu(t)) \\
    \dot{\nu} &= a(\nu(t)) \\
    e(t) &= h(x(t), u(t), \nu(t)) \tag{4.6}
\end{align*}
\]
4.4 Output Regulation for Continuous-time Nonlinear Systems

Similar to [68], the continuous-time nonlinear system is considered in (4.6), where \( x(t) \) is the \( n \)-dimensional state, \( u(t) \) presents the \( m \)-dimensional plant input, \( y(t) \) is the \( p \)-dimensional plant output, and \( \nu(t) \) stands for a \( q \)-dimensional exogenous signal. It is assumed that all the functions \( f(\ldots) \), \( h(\ldots) \) and \( a(.) \) are sufficiently smooth and satisfy \( f(0,0,0) = 0, h(0,0,0) = 0 \). The error signal can be measured as

\[
e(t) = H(x(t), u(t), \nu(t)) - r(t)
\]  

(4.7)

The control design to the plant in (4.6) can be provided by state feedback strategy. The approach of design is proposed in [68, 80]. A state feedback controller law considered here is described by

\[
u(t) = k(x(t), \nu(t))
\]  

(4.8)

By substituting (4.8) in (4.6), the closed-loop system can be put into the following form:

\[
\dot{x}(t) = f_c(x(t), \nu(t))
\]

\[
e(t) = h_c(x(t), \nu(t))
\]  

(4.9)

where

\[
f_c(x(t), \nu(t)) = f(x(t), k(x(t), \nu(t)), \nu(t)),
\]

\[
h_c(x(t), \nu(t)) = h(x(t), k(x(t), \nu(t)), \nu(t))
\]

The theory of the nonlinear regulator problem is based on the following three basic assumptions [68]:

a) The equilibrium of exosystem at \( \nu = 0 \) is Lyapunov stable, and all the eigenvalues of \( \frac{\partial a}{\partial \nu}(0) \) have zero real parts.

b) The pair \( \left( \frac{\partial f}{\partial x}(0,0,0), \frac{\partial f}{\partial u}(0,0,0) \right) \) is stabilizable.

c) The pair \( \left( \frac{\partial h}{\partial x}(0,0,0), \frac{\partial h}{\partial \nu}(0,0,0) \right), \left[ \begin{bmatrix} \frac{\partial f}{\partial x}(0,0,0) & \frac{\partial f}{\partial u}(0,0,0) \\ 0 & \frac{\partial a}{\partial \nu}(0,0,0) \end{bmatrix} \right] \) is detectable.
Similar to [68], the conditions for the solution of this problem depends on the existence of a zero-error manifold for that plant, which means an existence of sufficiently smooth functions \(x(\nu)\) and \(u(\nu)\) defined on an open neighbourhood \(\Gamma\) of the origin such that \(x(0) = 0\) and \(u(0) = 0\), and

\[
\frac{\partial x(\nu)}{\partial \nu} a(\nu) = f(x(\nu), u(\nu(t)), \nu) = 0 = h(x(\nu), u(\nu(t)), \nu)
\]  

(4.10)

When such functions exit, the solution of the problem can be found [68]. Due to the difficulty in solving the partial differential equation (PDE)(4.10) for \(x(\nu)\) and \(u(\nu)\), a kth-order approximation solution has been presented in [68], and then modified in [70] by adding a sufficient condition for the existence of a kth-order approximation solution. Suppose \(x_k(\nu)\) and \(u_k(\nu)\) is a kth-order approximate solution of (4.10), then a control law can be given by

\[
u(t) = u_k(\nu(t)) + K(\nu(t))[(x(t) - x_k(\nu(t))]
\]  

(4.11)

where \(K(\nu(t))\) can be chosen such that

\[
\frac{\partial f}{\partial x}(x_k(\nu(t)), u_k(\nu(t)), \nu) + \frac{\partial f}{\partial u}(x_k(\nu(t)), u_k(\nu(t)))K(\nu(t))
\]  

(4.12)

has desired eigenvalue location for each \(\nu \in \Gamma\)

4.5 Output Regulation for Discrete-time Nonlinear Systems

Similar to [74,75], the discrete-time nonlinear system is considered as follows:

\[
x(k + 1) = f(x(k), u(k), \nu(k)), x(0) = x_0, k \geq 0
\]

\[
\nu(k + 1) = a(\nu(k)), \nu(0) = \nu_0
\]

\[
e(k) = h(x(k), u(k), \nu(k))
\]  

(4.13)

where \(x(k)\) is the state, \(u(k)\) represents the plant input, \(e(k)\) is the error output, and \(\nu(k)\) stands for the exogenous signal. It is assumed that all the functions \(f(\ldots, \ldots), h(\ldots, \ldots)\) and \(a(\cdot)\) are sufficiently smooth and satisfy \(f(0,0,0) = 0, h(0,0,0) = 0.\)
The main objective is to find a state feedback controller law of the form

\[ u(k) = k(x(k), \nu(k)) \]  \hspace{1cm} (4.14)

such that the equilibrium of the closed-loop system is locally asymptotically stable, and the error output the plant go to zero asymptotically for all sufficiently small initial conditions.

The conditions for the solution of this problem has been studied in [76]. Under some similar standard assumptions to the previous section, the problem is solvable if and only if the following algebraic functional equations are solvable

\[ x(a(\nu(k)) = f(x(\nu), u(\nu(k)), \nu) \]
\[ 0 = h(x(\nu), u(\nu(k)), \nu) \]  \hspace{1cm} (4.15)

If the solutions of the discrete regulator equations in(4.15) are possible, the control law can be given by

\[ u(k) = u(\nu(k)) + K(\nu(t))[x(k) - x(\nu(k))] \]  \hspace{1cm} (4.16)

where \( K \) can be chosen such that all the eigenvalues of the following matrix

\[ \frac{\partial f}{\partial x}(0, 0, 0) + \frac{\partial f}{\partial u}(0, 0, 0)K(\nu(t)) \]  \hspace{1cm} (4.17)

are inside the unit circle.

Similar to the continuous-time case, due to the difficulty in solving the equation (4.15) and obtaining the exact solution, a kth-order output regulation problem for the continuous-time systems has been extended to the discrete-time systems. Suppose \( x_k(\nu) \) and \( u_k(\nu) \) are a kth-order approximate solution of (4.15), then a control law can be given by

\[ u(k) = u_k(\nu(k)) + K(\nu(k))[x(k) - x_k(\nu(k))] \]  \hspace{1cm} (4.18)

### 4.6 Simulation Results of Output Regulation for Continuous-time Nonlinear Systems

This section considers the approximate asymptotic tracking problem for the one-link manipulator described in chapter 3. The single-link flexible joint robot is
described by the following equations:

\begin{align*}
\dot{\theta}_m(t) &= \omega_m(t) \\
\dot{\omega}_m(t) &= \frac{k}{J_m}(\theta_1(t) - \theta_m(t)) - \frac{B}{J_m}\omega_m(t) + \frac{K_T}{J_m}u(t) \\
\dot{\theta}_1(t) &= \omega_1(t) \\
\dot{\omega}_1(t) &= -\frac{k}{J_1}(\theta_1(t) - \theta_m(t)) - \frac{mgh}{J_1}\sin(\theta_1(t)) \\
y(t) &= \theta_m(t) \\
\end{align*}

Denote \(x_1(t) = \theta_m(t), x_2(t) = \omega_m(t), x_3(t) = \theta_1(t), x_4(t) = \omega_1(t)\). Equation (4.19) can be represented as follows.

\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{k}{J_m}x_1(t) - \frac{B}{J_m}x_2(t) + \frac{k}{J_m}x_3(t) + \frac{K_T}{J_m}u(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= \frac{k}{J_1}x_1(t) - \frac{k}{J_1}x_3(t) - \frac{mgh}{J_1}\sin(x_3(t)) \\
y(t) &= x_1(t) \\
\end{align*}

The objective is to design a state-feedback controller such that the angular of the motor asymptotically follows a sinusoidal reference input \(A_m\sin(\omega t)\), where \(\omega\) is fixed.

The exosystem in this example will be defined as follows:

\begin{align*}
\dot{\nu}_1(t) &= \nu_2(t) \\
\dot{\nu}_2(t) &= -\omega_1(t) \\
\dot{\nu}_1(0) &= 0 \\
\dot{\nu}_2(0) &= A_m \\
\end{align*}

which is the same as \(\nu(t) = A_m\sin(\omega t)\). The error equation can be represented by the following equation:

\[e(t) = x_1(t) - \nu_1(t)\]

As the system satisfying the conditions in [70] for the existence of a kth-order approximation solution for the plant, a third order approximation solution will be considered in this example.

The scalar form of the regulator equations with the system in (4.20) can be represented by the following equation:
\[
\frac{\partial x_1(\nu)}{\partial \nu} A_1 \nu = x_2(\nu) \tag{4.23}
\]
\[
\frac{\partial x_2(\nu)}{\partial \nu} A_1 \nu = -a x_1(\nu) - b x_2(\nu) + a x_3(\nu) + c u(\nu)
\]
\[
\frac{\partial x_3(\nu)}{\partial \nu} A_1 \nu = x_4(\nu)
\]
\[
\frac{\partial x_4(\nu)}{\partial \nu} A_1 \nu = d x_1(\nu) - d x_3(\nu) - n \sin x_3(\nu)
\]
\[
x_1(\nu) = \nu_1
\]

where \( a = \frac{k}{J_m} \), \( b = \frac{B}{J_m} \), \( c = \frac{K}{J_m} \), \( d = \frac{k}{J_1} \), and the value of \( n = \frac{m g h}{J_1} \).

It can be noticed that
\[
x_1(\nu) = \nu_1
\]
\[
x_2(\nu) = \omega \nu_2
\]
\[
u(\nu) = \frac{1}{c} \left[ -\omega^2 \nu_1 + a \nu_1 + b \omega \nu_2 - a x_3(\nu) \right]
\]
with two functions \( x_3(\nu) \) and \( x_4(\nu) \) satisfying
\[
\frac{\partial x_3(\nu)}{\partial \nu} A_1 \nu = x_4(\nu) \tag{4.25}
\]
\[
\dot{x}_3(\nu) = \frac{\partial x_3(\nu)}{\partial \nu_1} \nu_1 + \frac{\partial x_3(\nu)}{\partial \nu_2} \nu_2 \tag{4.26}
\]

Similar to [4], it can be assumed the Taylor series solution of (4.25) and (4.26) as follows:
\[
x_3^{(3)}(\nu) = a_{10} \nu_1 + a_{01} \nu_2 + a_{30} \nu_1^3 + a_{21} \nu_1^2 \nu_2 + a_{12} \nu_1 \nu_2^2 + a_{03} \nu_2^3
\]
\[
x_4^{(3)}(\nu) = b_{10} \nu_1 + b_{01} \nu_2 + b_{30} \nu_1^3 + b_{21} \nu_1^2 \nu_2 + b_{12} \nu_1 \nu_2^2 + b_{03} \nu_2^3
\]

By substituting (4.27) in (4.25) and (4.26), and determining the coefficients, the solution of the third-order approximation of \( x_3(\nu) \) and \( x_4(\nu) \) will in the following form:
\[
x_3^{(3)}(\nu) = a_{10} \nu_1 + a_{30} \nu_1^3 + a_{12} \nu_1 \nu_2^2
\]
\[
x_4^{(3)}(\nu) = b_{01} \nu_2 + b_{21} \nu_1^2 \nu_2 + b_{03} \nu_2^3
\]
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where
\[
a_{10} = \frac{d}{n + d - \omega^2}
\]
\[
a_{12} = \frac{-6b_2\omega^2}{n + d + (6b_1 - 7)\omega^2}
\]
\[
a_{30} = b_2(1 - \frac{-6b_1\omega^2}{n + d + (6b_1 - 7)\omega^2})
\]
\[
b_{01} = \frac{d\omega}{n + d - \omega^2}
\]
\[
b_{21} = 3\omega b_2(1 - \frac{-6b_2\omega^2}{n + d + (6b_1 - 7)\omega^2}) - \frac{126b_2\omega^3}{n + d + (6b_1 - 7)\omega^2}
\]
\[
b_{03} = \omega(\frac{-6b_2\omega^2}{n + d + (6b_1 - 7)\omega^2})
\]
\[
b_1 = -\frac{2\omega^2}{n + d - 3\omega^3}
\]
\[
b_2 = -\frac{0.166n}{n + d - 3\omega^2a_{10}^3}
\]

By using \(u(\nu)\) in (4.24), the third-order approximation of \(u(\nu)\) will be as follows:
\[
u^{(3)}(\nu) = \frac{1}{c} \left[ -\omega^2\nu_1 + a\nu_1 + b\omega\nu_2 - ax^{(3)}_3(\nu) \right]
\]

The Jacobian linearization of the plant in (4.20) is given by
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{k}{J_m} & -\frac{B}{J_m} & \frac{k}{J_m} & 0 \\
0 & 0 & 0 & 1 \\
\frac{k}{J_1} & 0 & -\frac{k}{J_1} - \frac{mgh}{J_1} & 0
\end{bmatrix}, B = \begin{bmatrix}
0 \\
\frac{K_c}{J_m} \\
0 \\
0
\end{bmatrix}, C = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}, D = 0
\]

Thus a feedback gain \(K\) can be chosen such that the matrix \(A + BK\) is Hurwitz. By choosing \(K = [12.8508, 1.3782, -8.4631, 2.3098]\), the eigenvalues of \(A + BK\) will place at \(-8.1423 \pm 2.4399i, -5.9157 \pm 7.4718i\).

Figure 4.2 depicts the reference signal \(y_r(t) = A_m \sin \omega(t)\) and the controlled output \(y(t)\) for the closed loop system when \(A_m = 2\) and \(\omega = 0.4\pi\). While, Fig.4.3 depicts the error of the system.

A Matlab simulation is run for different values of \(A_m\) and \(\omega\) to check the tracking performance of the closed-loop system. Figs. 4.4 and 4.5 show the tracking performance of the closed-loop system resulting from the third-order controller with \(A_m = 3, 4\) and \(\omega = 0.4\pi\). Moreover, Figs. 4.6, 4.7 and 4.8 show
the tracking performance of the closed-loop system resulting from the third-order controller with $A_m = 2, 3, 4$ and $\omega = 0.2\pi$.

The system is investigated for different values of $A_m$ and $\omega$. It is clear from the graphs that the maximal steady-state tracking error of the closed-loop system under the third-order controller for $A_m = 4$ and $\omega = 0.4\pi$ are far larger than those when $A_m = 2, 3$ and $\omega = 0.4\pi$. Also, the error when $A_m = 2$ and $\omega = 0.4\pi$ are considerably better than those when $A_m = 3$ and $\omega = 0.4\pi$. However, the performance is good for all values of $A_m$. The same results can be obtained to occur for different values of $A_m$ and $\omega = 0.2\pi$. 

Figure 4.2: Tracking performance: $A_m = 2$ and $\omega = 0.4\pi$.

Figure 4.3: System error: $A_m = 2$ and $\omega = 0.4\pi$.
Figure 4.4: System error: $A_m = 3$ and $\omega = 0.4\pi$.

Figure 4.5: System error: $A_m = 4$ and $\omega = 0.4\pi$.

Figure 4.6: System error: $A_m = 2$ and $\omega = 0.2\pi$. 
Figure 4.7: System error: $A_m = 3$ and $\omega = 0.2\pi$.

Figure 4.8: System error: $A_m = 4$ and $\omega = 0.2\pi$.

### 4.7 Simulation Results of Output Regulation for Discrete-time Nonlinear Systems

This section considers the approximate asymptotic tracking problem for the discretized model of the one-link manipulator with revolute joints actuated by a DC motor system. By using Euler’s method with $T$ as the sampling time to discretize
Chapter 4. Nonlinear Output Regulation

the continuous model in (4.20), the discrete-time model will be as follows:

\[
\begin{align*}
x_1(k+1) &= x_1(k) + Tx_2(k) \\
x_2(k+1) &= x_2(k) + T \left( -\frac{k}{J_m}x_1(k) - \frac{B}{J_m}x_2(k) + \frac{k}{J_m}x_3(k) + \frac{K}{J_m}u(k) \right) \\
x_3(k+1) &= x_3(k) + Tx_4(k) \\
x_4(k+1) &= x_4(k) + T \left( \frac{k}{J_1}x_1(k) - \frac{k}{J_1}x_3(k) - \frac{mgh}{J_1} \sin x_3(k) \right) \\
y(k) &= x_1(k)
\end{align*}
\] (4.31)

The exosystem in this example will be defined as follows:

\[
\nu(k+1) = A_1 \nu(k)
\] (4.32)

where

\[
A_1 = \begin{bmatrix}
\cos \omega & \sin \omega \\
-\sin \omega & \cos \omega
\end{bmatrix}, \nu(k) = \begin{bmatrix} \nu_1(k) \\ \nu_2(k) \end{bmatrix}, \nu(0) = \begin{bmatrix} 0 \\ A_m \end{bmatrix}
\]

The error equation can be represented by the following equation:

\[
e(k) = x_1(k) - \nu_1(k)
\] (4.33)

The discrete regulator equations with the discrete system can be presented by the following equation:

\[
\begin{align*}
x_1(A_1 \nu) &= x_1(\nu) + Tx_2(\nu) \\
x_2(A_1 \nu) &= x_2(\nu) + T (-ax_1(\nu) - bx_2(\nu) + ax_3(\nu) + cu(\nu)) \\
x_3(A_1 \nu) &= x_3(\nu) + Tx_4(\nu) \\
x_4(A_1 \nu) &= x_4(\nu) + T (dx_1(\nu) - dx_3(\nu) - 3.33 \sin x_3(\nu))
\end{align*}
\] (4.34)

Equations (4.34) can be partially solved as follows:

\[
\begin{align*}
x_1(\nu) &= \nu_1 \\
x_2(\nu) &= \frac{1}{T}[\cos \omega - 1, \sin \omega] \nu \\
u(\nu) &= \frac{1}{cT}x_2(\nu)(A_1 - I) + a\nu_1 + bx_2(\nu) - ax_3(\nu)
\end{align*}
\] (4.35)

with \(x_3(\nu)\) and \(x_4(\nu)\) satisfying the following equations:

\[
\begin{align*}
x_3(A_1 \nu) &= x_3(\nu) + Tx_4(\nu) \\
x_4(A_1 \nu) &= x_4(\nu) + T (dx_1(\nu) - dx_3(\nu) - 3.33 \sin x_3(\nu))
\end{align*}
\] (4.36)
The equations in (4.36) can be viewed as centre manifold equations with the following nonlinear difference equations:

\[ x_3(k+1) = x_3(k) + Tx_4(k) \]
\[ x_4(k+1) = x_4(k) + T (dv_1 - dx_3(\nu) - 3.33 \sin x_3(\nu)) \]

By eliminating \( x_4(\nu) \) from (4.36) gives

\[ x_3(A_1^2\nu) = 2x_3(A_1\nu) - (1 + T^2d)x_3(\nu) - 3.33T^2_x(\nu) + T^2dv_1 \] (4.38)

As long as we can obtain the function \( x_3(\nu) \) by solving (4.38), we can obtain \( x_1(\nu), x_2(\nu) \), and \( u(\nu) \) from (4.34), and \( x_4(\nu) \) from

\[ x_4(A_1\nu) = \frac{x_3(A_1\nu) - x_3(\nu)}{T} \] (4.39)

The approach of a third-order state-feedback controller for a plant can be seen in more details in [4]. A third-order polynomial approximation for \( x_3(\nu) \) denoted by \( x_3^{(3)}(\nu) \) can be obtained by solving (4.38), where \( x_3^{(3)}(\nu) \) is given by the following equation:

\[ x_3^{(3)}(\nu) = a_{10}\nu_1 + a_{01}\nu_2 + a_{30}\nu_1^3 + a_{21}\nu_1^2\nu_2 + a_{12}\nu_1\nu_2^2 + a_{03}\nu_2^3 \] (4.40)

where

\[ a_{10} = s_1a_{01} \]
\[ a_{01} = \frac{T^2d}{s_1 \cos 2\omega - 2s_1 \cos \omega - \sin 2\omega + 2 \sin \omega + s_1s_2} \]
\[ a_{30} = \frac{s_5}{-s_3a_{21} - s_4a_{12} + s_6} \]
\[ a_{03} = \frac{s_3a_{12} - s_4a_{21} + s_6}{s_7} \]
\[ a_{12} = \frac{s_1s_{12} - s_9s_{13}}{s_8s_{11} - s_9s_{10}} \]
\[ a_{21} = \frac{-s_{10}s_{12} - s_8s_{13}}{s_8s_{11} - s_9s_{10}} \] (4.41)
and

\[ s_1 = \frac{2 \cos \omega - \cos 2\omega + s_2}{\sin 2\omega - 2 \sin \omega} \]
\[ s_2 = -(1 + T^2d + nT^2) \]
\[ s_3 = -3 \sin 2\omega + 2 \sin \omega \]
\[ s_4 = 2 \sin^2 \omega \]
\[ s_5 = 3 \cos^2 \omega - 6 \sin 2\omega - 6 \cos \omega - s_1 \]
\[ s_6 = \frac{nT^2a_{10}^3}{6} \]
\[ s_7 = \frac{nT^2a_{01}^3}{6} \]
\[ s_8 = s_{14} - s_{15} \frac{s_4}{s_5} + 3s_4 \frac{s_3}{s_5} - s_{14} \]
\[ s_9 = -s_{15} \frac{s_3}{s_5} + 3s_4 \frac{s_4}{s_5} \]
\[ s_{10} = s_{16} + s_{15} \frac{s_3}{s_5} - 3s_4 \frac{s_4}{s_5} - s_{16} \]
\[ s_{11} = -s_{15} \frac{s_4}{s_5} - 3s_4 \frac{s_3}{s_5} \]
\[ s_{12} = s_{15}a_{30} + 3s_4a_{03} - 3s_4a_{03} - 3s_{15}a_{30} \]
\[ s_{13} = s_{15}a_{03} + 3s_4a_{30} - 3s_4a_{30} - 3s_{15}a_{03} \]
\[ s_{14} = 9 \cos^2 \omega - 6 \sin 2\omega + 4 \sin \omega \]
\[ s_{15} = 9 \sin 2\omega - 6 \sin \omega \]
\[ s_{16} = 9 \cos^2 \omega - 7 \sin^2 \omega - 6 \cos \omega - s_2 \]

After determining \( x_3^{(3)}(\nu) \), we can obtain the third-order approximation of \( x(\nu) \) and \( u(\nu) \). Thus the third-order state feedback controller is given by the following equation:

\[ u(k) = u^{(3)}(\nu(k)) + K(x(k) - x^{(3)}(\nu(k))) \quad (4.42) \]

where the feedback gain \( K \) is selected such that the eigenvalues of the matrix \( A + BK \) are \(-3.2865 \pm 4.1510i, -4.5235 \pm 1.3555i\)

Fig. 4.9 shows the tracking performance of the closed-loop system resulting from the third-order controller with \( \omega = 0.4\pi \), \( A_m = 2 \), and \( T = 0.001s \). Fig. 4.10 depicts the error of the system. Moreover, Figs. 4.11, and 4.12 show the tracking performance of the closed-loop system resulting from the third-order controller with \( A_m = 3, 4 \) and \( \omega = 0.4\pi \).
With similar discussion to the previous section, it can be concluded that the maximal steady-state tracking errors of the closed-loop system under the third-order controller for $A_m = 4$ and $\omega = 0.4\pi$ is far larger than that of the system.
Figure 4.12: Tracking performance: $A_m = 4$ and $\omega = 0.4\pi$.

with $A_m = 3, 4$ and $\omega = 0.4\pi$. Also, the above mentioned maximal steady-state tracking errors measures as of the system with $A_m = 2$ and $\omega = 0.4\pi$ are considerably better than that of the system with $A_m = 3$ and $\omega = 0.4\pi$.

4.8 Summary

Although output regulation has been in use for more than three decades, and has been described in several papers and books, its design is still an area of concern, as mentioned in many studies. Therefore, an output regulation design for both continuous-time and discrete-time classes of nonlinear systems has been proposed here. A kth-order approximation has been applied to solve the nonlinear regulator equations, due to their complex nature.

The simulated example of a single-link flexible joint robot system has further confirmed the expected results of the proposed algorithm. The simulations demonstrate that the feedback controller law, based on the kth-order approach, can be designed. A comparison between the different resulting tracking performances was performed in the previous section. The maximal steady-state tracking error is used as the point of view when the plant is subject to different exosystems, with various values of the amplitudes and frequencies (i.e. $A_m$ and $\omega$). It is clear from the previous sections that the maximal steady-state tracking errors of the closed-loop system with high value of $A_m$, and under the same fixed value of $\omega$, are significantly larger for high values of $A_m$ than for low values, holding $\omega$ fixed. However, the performance is acceptable for both continuous-time and discrete-time controllers, for all selected values of $A_m$. 
Chapter 5

An Adaptive PI Controller for non-Gaussian Stochastic Systems

5.1 Introduction

In this chapter, a new algorithm for an adaptive PI controller for nonlinear systems subject to stochastic non-Gaussian disturbance is studied. The minimum entropy control is applied to decrease the closed-loop tracking error on an ILC basis. The key issue here is to divide the control horizon into a number of equal time intervals called batches. Within each interval, there are a fixed number of sample points. The design procedure is divided into two main algorithms, within each batch and between any two adjacent batches. A D-type ILC law is employed to tune the PI controller coefficients between two adjacent batches. However, within each batch, the PI coefficients are fixed. A sufficient condition is established to guarantee the stability of the closed-loop system. An analysis of the ILC convergence is carried out. Two illustrated examples of one-link manipulator with revolute joints, actuated by a DC motor and two-link robot manipulator, are included to demonstrate the use of the control algorithm, and satisfactory results are obtained.

5.2 Literature Review

Research into controller design for stochastic systems has been regarded as an important aspect of this field of research over the past few decades. This is mainly due to the fact that a large class of physical systems has random inputs, time delays, uncertainties and noise. Therefore, an ideal control design should be such that the system attenuates the stochastic behaviour, or sufficiently minimizes the
effect caused by the randomness. In most stochastic or PDF control system designs, the output tracking error signal (the difference between the set point and the system output) of the closed-loop control system is a central index that represents the control performance of the closed-loop systems. Under the assumption that the random variables or the noise in the stochastic system are subject to Gaussian processes, the tracking error is used to represent the closed-loop performance. The statistics of the tracking error thus characterize the performance of the controller. The first approach to controller design focused on mean and variance control [81], minimizing the uncertainties of the closed-loop stochastic systems. Later on, linear optimal control [82] and linear quadratic martingale control [83] were developed. Further research has been performed into optimal stochastic control, optimal adaptive predictive control for nonlinear stochastic systems, and stochastic adaptive control strategies presented in [84–86], respectively. Also, the sliding mode control problem for a class of linear continuous-time systems with Markovian jump parameters was solved in [87], and robust fuzzy control for uncertain Markovian stochastic systems was developed in [88]. More recently, robust $H_\infty$ control of uncertain stochastic, nonlinear systems [89], and an adaptive output-feedback controller for a stochastic nonlinear system [90] were introduced.

On the other hand, when the random variables or the noise in the stochastic system are subject to non-Gaussian processes (i.e., systems with non-symmetrically distributed noises), the mean and variance are insufficient to characterize the stochastic properties, since the spread of the non-symmetrically distributed noises cannot be accurately described by the variance of the tracking error measures alone [7]. This is because the shape of the tracking error PDF will be related to both the control parameters and the statistics of the noise. Therefore, a new measure of randomness, called the entropy of the tracking error of the closed-loop system, should be employed for the closed-loop control design of non-Gaussian stochastic systems. Thus, the main objective of minimum entropy control would be formulating a deterministic control signal to minimize the entropy of the closed-loop tracking error. Entropy has a significant advantage in dealing with non-Gaussian systems. This is because entropy provides a general description of the uncertainties of stochastic systems, without constraints the of using certain distributions. Moreover, it has been proved that the minimum entropy algorithm is similar to the minimum variance algorithm, when the noise of the stochastic system follows a Gaussian behaviour [91,92]. For an arbitrary continuous random
variable $x \in [a, b]$. The entropy of a $x$ is defined as

$$H(x) = -\int_a^b \gamma(x) \ln \gamma(x) \, dx$$  \hfill (5.1)

where $\gamma(x)$ is the PDF of $x$. From (5.1) it is clear that the entropy is a measure of the amount of uncertainty represented by the probability distribution and provides a more general character of the stochastic system, which is subject to arbitrary noises with any PDF shape. Suppose $x$ is a Gaussian variable, $a = -\infty$, and $b = +\infty$. Also, consider that the PDF of $x$ is Gaussian distribution defined as follows.

$$\gamma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$  \hfill (5.2)

where $m$ and $\sigma^2$ are the mean and the variance of the distribution, respectively. By substituting (5.2) in (5.1), it can be seen that the entropy $H(x)$ is as follows.

$$H(x) = 0.5(1 + \ln(2\pi\sigma^2))$$  \hfill (5.3)

From (5.3), it can be seen that minimizing entropy is equivalent to the minimization of the variance of $x$, and in this way, minimum variance control may be regarded as a special case of minimum entropy control. The entropy was used in different definitions. Initially, the entropy was presented as a measure of the uncertainty about its actual structure in both continuous and discrete time [93]. Later on, a new definition was presented in [94] by considering the exponential nature of the information gain function. Also, the entropy was represented as an average information content in a given probability density function [10]. In fact, entropy concept was applied widely in the design of control algorithms for stochastic systems. Among various existing approaches is the work of [91], where the entropy was applied to characterize and minimize the closed loop randomness for general stochastic systems subject to arbitrary bounded random inputs. The control input was formulated which minimizes the output entropy and guarantee the local stability of the closed-loop system. Later on, in [92], the authors developed a control algorithm for the control of the output mean values and the minimization of the closed-loop entropy for nonlinear affine and non-Gaussian stochastic systems. Moreover, a novel controller design for a linear time-invariant stochastic system subject to a bounded random input by using Youla parameterization was established in [95]. Also, a minimum entropy control problem was solved for nonlinear ARMA systems over a communication network with a stochastic delay in the communication channels, where the probability density
function of the tracking error was estimated by using a neural network [96].

In recent work, an adaptive control for general nonlinear, non-Gaussian, unknown stochastic systems was established [97]. For this purpose, the control horizon was divided into a number of equal time intervals called batches. Within each interval, there were a fixed number of sample points. The design procedure was divided into two main algorithms, within each batch and between any two adjacent batches. Minimum entropy control was applied to decrease the closed-loop randomness of the output on an ILC basis. Neural network dynamics were constructed to perform both the modelling and control of the plant.

It should be noted that the minimum entropy control for stochastic system discussed in this chapter is different from that of the minimum entropy based-$H_\infty$ control introduced in [98]. According to [92], the definition of the entropy used $H_\infty$ control for LTI systems in [98] at $s = \infty$ is defined in terms of the closed-loop transfer function as follows [98].

$$I_d(G; l; \infty) = -\frac{l^2}{2\pi} \int_{-\infty}^{+\infty} \ln |\det(I - l^2 G^*(j\omega) G(j\omega))| \, d\omega \quad (5.4)$$

where the transfer function $G$ is supposed to be strictly satisfies $\|G\|_\infty < l$, where $l$ is a constant, $G^*$ denotes the adjoint of $G$. $I_d$ is finite if $G$ is strictly proper. From (5.4) it can be seen that the term entropy in the $H_\infty$ control is a measure of the mutual information between input and output of the system which is different from the entropy definition in this chapter where the mutual information maximisation idea is presented as a probabilistic point of view.

The main objective of this chapter is to investigate a new algorithm for an adaptive PI controller for robotic manipulators with non-Gaussian noise by combining the D-type ILC law with a minimum tracking error entropy control strategy.

### 5.3 Problem Formulation

In this chapter, the following nonlinear stochastic model between the output and input will be considered

$$x(i + 1) = Ax(i) + Bu(i) + g(x(i)) + d(i)$$
$$y(i) = Cx(i) \quad (5.5)$$
where \( x(i) \in \mathbb{R}^n \) is the state vector, \( u(i) \in \mathbb{R}^r \) is the measurable input vector and \( y(i) \in \mathbb{R}^p \) is the measurable output vector. Moreover, \( A, B \) and \( C \) represent the known parametric matrices of the dynamic part of the system with suitable dimensions. \( d(i) \) is the bounded non-Gaussian random noise. \( g(x(i)) \) is a nonlinear vector function that stands for the nonlinear dynamics of the model and is supposed to satisfy \( g(0) = 0 \), and the following Lipschitz condition, similar to [99]

\[
\|g(x_1(i)) - g(x_2(i))\| \leq \|U(x_1(i) - x_2(i))\| \tag{5.6}
\]

for any \( x_1(i) \) and \( x_2(i) \), where \( U \) is a known matrix. Denoting the desired system output as \( r(i) \), the tracking error can be expressed as

\[
e(i) = r(i) - y(i) \tag{5.7}
\]

which is a non-Gaussian random process due to the affect of the non-Gaussian disturbance on the closed-loop performance. Theoretically, it is expected that such a randomness or uncertainty is minimized by the controller function. In other words, the purpose of the PI controller design in this chapter is to establish a control signal so that the randomness or uncertainty in the system output and closed-loop tracking error is minimized. The controller coefficients will be trained by an ILC tuning mechanism to minimize the entropy of the closed-loop tracking error. For this purpose, the well-known Renyi\'s entropy measure for a random error \( e \) can be expressed as follows

\[
H(e) = \frac{1}{1 - \alpha} \log \left( \int \gamma^\alpha(e) de \right) \tag{5.8}
\]

where \( \gamma \) presents the probability density function of the random error. The following Kernel density estimation method is used to estimate the tracking error probability distribution function within each batch using the sampled tracking error data [10].

\[
\gamma(e) \approx \hat{\gamma}(e) = \frac{1}{N} \sum_{i=1}^{N} K_\sigma(e - e_i) \tag{5.9}
\]

where \( K_\sigma(.) \) is a real, symmetrical Kernel function with the specifications stated in [10], and \( \sigma > 0 \) is the order of Renyi\'s quadratic entropy. The chosen Kernel function in this work is expressed as follows:

\[
K_\sigma(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \tag{5.10}
\]
The choice of the Kernel function is actually dependent on the level of smoothness the designer expects from the pdf estimation. The ILC-based controller tuning objective function can be written as follows:

\[ H(e) = \frac{1}{1-\alpha} \log(V_{Ra}(e)) \]  

(5.11)

where \( V_{Ra}(e) \) is usually called the Information Potential (IP), and can be further expressed as

\[ V_{Ra}(e) = \frac{1}{N^\alpha} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} K_\sigma(e_i - e_j) \right)^\alpha \]  

(5.12)

### 5.4 Controller Algorithm Design

PI controllers are widely used in both theoretical studies and practical applications of industrial control over the last five decades [100]. The popularity of PI controller can be attributed to its robust performance in a wide range of operating conditions and to its very simple structure.

As discussed before, the aim of the PI controller design is to minimize the randomness of the tracking error, so that the entropy of the tracking error can be minimized. For this purpose, the control horizon will be divided into a number of equal time-domain intervals called batches as \( \{T_1, T_2, T_3, \ldots, T_k, \ldots\} \). Within each interval, there are a fixed number of sample points \( \{1, 2, \ldots, N\} \), where \( N \) is sampling numbers in a batch. In this case, the batch length \( N \) should be selected large enough so that the system almost reaches the steady state with each batch. In each batch, a number of sampled tracking error can be collected. Between any two adjacent batches, the PI coefficients are tuned by ILC-based method to minimize the tracking error entropy (see Fig. (5.1)). The ILC tuning mechanism of PI coefficients needs to go on at the beginning of the \( k \) batches until the PI coefficients cannot be further tuned.

A generalized PI controller with tunable coefficients is considered as adaptive controller in this work as follows

\[ \xi(i) = \xi(i - 1) + T_s e(i - 1) \]

\[ u(i) = K_pe(i) + K_i \xi(i) \]  

(5.13)

where \( e(i) = r - y(i) \) and \( T_s \) is the sampling time.

Assume that the current instant is at the beginning of the interval \( T^k \), the
system tracking error within this interval should have the following sample values \( \{e_k^1, e_k^2, e_k^3, \ldots, e_k^N\} \). Within this interval, the sampled tracking error can be used to measure the entropy and construct the PDF. Therefore, the entropy of sample \( \{e_{k-1}^1, e_{k-1}^2, e_{k-1}^3, \ldots, e_{k-1}^N\} \) in the interval \( T_{k-1} \) can be expressed as follows

\[
H_{k-1}(e) = \frac{1}{1 - \alpha} \log \left\{ \frac{1}{N\alpha} \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} K_{\sigma}(e_{i}^{k-1} - e_{j}^{k-1}) \right]^{\alpha-1} \right\} \tag{5.14}
\]

and the information potential for \( \{e_{k-1}^1, e_{k-1}^2, e_{k-1}^3, \ldots, e_{k-1}^N\} \) in the interval \( T_{k-1} \) can be expressed according to the following equation

\[
V_{Ra}^{k-1}(e) = \frac{1}{N\alpha} \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} K_{\sigma}(e_{i}^{k-1} - e_{j}^{k-1}) \right]^{\alpha-1} \tag{5.15}
\]

By substituting (5.15) into (5.14), the entropy can be obtained as follows:

\[
H_{k-1}(e) = \frac{1}{1 - \alpha_{Ra}} (e) \tag{5.16}
\]

The task of tuning the PI controller coefficients between adjacent batches can be solved by the following D-type ILC law which can be only considered as a local optimal solution

\[
K_P(k) = K_P(k - 1) - \lambda_P \frac{\partial H_{k-1}(e)}{\partial K_P^{k-1}} \\
K_I(k) = K_I(k - 1) - \lambda_I \frac{\partial H_{k-1}(e)}{\partial K_I^{k-1}} \tag{5.17}
\]

where \( \lambda_P \) and \( \lambda_I \) are the ILC learning rates chosen so that the iterative control law is convergent \([101]\). \( K_P(k) \) and \( K_I(k) \) are the PI controller coefficients within
the $k^{th}$ batch. In ILC, the PI coefficients in the $k^{th}$ batch is based on the PI coefficients in the $(k - 1)^{th}$ batch and a correcting function, which is basically dependent on the gradient of closed-loop entropy tracking error of PI coefficients in the $(k - 1)^{th}$ batch.

The $k^{th}$ component of $\frac{\partial H(e)}{\partial K_P}$ and $\frac{\partial H(e)}{\partial K_I}$ can be approximated as follows:

$$\frac{\partial H(e)}{\partial K_P} = \frac{H^{k-1}(e) - H^{k-2}(e)}{\Delta K_P}$$

$$\frac{\partial H(e)}{\partial K_I} = \frac{H^{k-1}(e) - H^{k-2}(e)}{\Delta K_I}$$

(5.18)

where

$$\Delta K_P = K_P(k - 1) - K_P(k - 2)$$

$$\Delta K_I = K_I(k - 1) - K_I(k - 2)$$

(5.19)

The statement of the convergence in this section may raise some issues about the learning parameters, type of convergence that need to be addressed more clearly. Therefore, it should be noted that the proposed convergence analysis is in introductory stage of development and future efforts need to address constraints on learning rates precisely. As it will be seen, the so called convergence laws in this work determine the constraints based on which the ILC-based cost function is decreasing along with the batches. For simplicity, we only discuss the sufficient convergence conditions to guarantee the convergence of the above proposed ILC algorithm. The key issue is that the learning rate of PI coefficients should be decreased batch-by-batch. In other words, the entropy of closed-loop tracking error should be decreased batch-by-batch. This would be equivalent to

$$H^k(e(i)) < H^{k-1}(e(i))$$

(5.20)

Since $\alpha > 0$, then inequality (5.20) can be re-written as

$$\log \left( V_{Ra}^{k-1}(e(i)) \right) < \log \left( V_{Ra}^k(e(i)) \right)$$

(5.21)

which means that

$$\log \left( \frac{V_{Ra}^{k-1}(e(i))}{V_{Ra}^k(e(i))} \right) < 0$$

(5.22)
which is equivalent to the following

\[ \frac{V_{Ra}^{k-1}(e(i))}{V_{Ra}^k(e(i))} < 1 \]

Since the IP is non-negative, inequality (5.21) would mean

\[ \Delta V_{Ra,k} = V_{Ra}^k(e(i)) - V_{Ra}^{k-1}(e(i)) > 0 \]  \hspace{1cm} (5.24)

By using (5.12), the following approximation can be made.

\[
\Delta V_{Ra,k} \approx \Delta \left[ \frac{1}{N^\alpha} \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} K_\sigma(e_i - e_j) \right]^\alpha \right] \\
= \frac{1}{N^\alpha} \sum_{i=1}^{N} \left\{ \sum_{j=1}^{N} K_\sigma(e_k(i) - e_k(j)) \right\}^{\alpha-1} \times \\
\left\{ \sum_{j=1}^{N} \dot{K}_\sigma(e_k(i) - e_k(j)) \Delta(e_k(i) - e_k(j)) \right\} \right] \]  \hspace{1cm} (5.25)

As such, when PI controller is running, the learning parameters \( \lambda_P \) and \( \lambda_I \) must be chosen so that the (5.20) to (5.25) together with (5.19) are satisfied. The algorithm can be summarized as follows:

Step 1: Collect error sample from interval \( T^{k-1} \)

Step 2: At the beginning of interval \( T^k \), calculate the rate of changes of the closed-loop tracking error entropy as shown in (5.18).

Step 3: Update the PI coefficients using (5.17)

Step 4: Calculate the control signal using (5.13), and apply it to the system, and return to step 1.

This process will repeat until the end of control horizon is reached.

\section*{5.5 Closed Loop Stability}

In this section, the stability of the closed-loop system given by (5.5) and (5.13) will be considered. Corresponding to (5.5) and (5.13), the closed-loop system
equation within the \( k^{th} \) batch can be written as

\[
M_k(i + 1) = \bar{A}_k M_k(i) + \bar{B}_k r + \bar{F}g(M_k(i))
\] (5.26)

where

\[
M_k(i) = \begin{bmatrix} e_k(i) \\ \xi(i) \end{bmatrix}, \quad \bar{A}_k = \begin{bmatrix} A - BK_P & -BK_I \\ T_s I & I \end{bmatrix}
\] (5.27)

and

\[
\bar{B}_k = \begin{bmatrix} I - A \\ 0 \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad g(M_k(i)) = \begin{bmatrix} g(x(i)) \\ 0 \end{bmatrix}
\] (5.28)

To prove the stability of the closed loop system, let us consider the following Lyapunov candidate

\[
V(M_k(i), i) = M_k^T(i)P_k^{-1}M_k(i) + \lambda^2 \sum_{i=1}^{k-1} [||\bar{U}M_k(i)||^2 - ||g(M_k(i))||^2]
\] (5.29)

Differentiating equation (5.29) over the time gives

\[
\Delta V(i) = V(M_k(i + 1), i + 1) - V(M_k(i), i)
= \bar{S}_k^T(i)N_k \bar{S}_k(i) + 2\bar{S}_k(i)\bar{A}_k^T P_k^{-1} \bar{B}_k r
+ 2g^T(M_k(i))\bar{F}^T P_k^{-1} \bar{B}_k r
+ r^T \bar{B}_k P_k^{-1} \bar{B}_k r < 0
\] (5.30)

where

\[
N_k = \begin{bmatrix} \bar{A}_k^T P_k^{-1} \bar{A}_k - P_k^{-1} + \lambda^2 \bar{U}^T \bar{U} & \bar{A}_k^T P_k^{-1} \bar{F} \\ \bar{F}^T P_k^{-1} \bar{A}_k & \bar{F}^T P_k^{-1} \bar{F} - \lambda^2 I \end{bmatrix}
\] (5.31)

Using the well known Schur complement formula, (5.31) can be further reduced to

\[
N_{1,k} = \begin{bmatrix} -P_k^{-1} + \lambda^2 \bar{U}^T \bar{U} & 0 & \bar{A}_k^T \\ 0 & -\lambda^2 I & \bar{F}^T \\ \bar{A}_k & \bar{F} & -P_k \end{bmatrix} < 0
\] (5.32)

By pre-multiplying \( N_{1,k} \) by \( \text{diag}(P_k^T, I, I) \) and post multiplying it by \( \text{diag}(P_k, I, I) \) and also applying Schur complement formula, the necessary condition for stability
will be as follows

\[
N_{2,k} = \begin{bmatrix}
-P_k & 0 & P_k^T \bar{A}_k^T & \lambda P_k \bar{U}_k^T \\
0 & -\lambda^2 I & \bar{F}^T & 0 \\
\bar{A}_k P_k & \bar{F} & -P_k & 0 \\
\lambda \bar{U}_k P_k & 0 & 0 & -I
\end{bmatrix} < 0
\] (5.33)

If (5.33) holds, a positive scalar \( \delta \) exists so that \( N_k \leq -\delta I \). Along with (5.26) it can be verified that

\[
\Delta V(i) \leq -\delta \|\bar{S}_k\|^2 + 2\|\bar{A}_k^T P_k^{-1} \bar{B}_k r\| \|\bar{S}_k\|^2 \\
+ 2 g^T(M_k(i)) \bar{F}_k^T P_k^{-1} \bar{B}_k r \\
+ r^T \bar{B}_k^T P_k^{-1} \bar{B}_k r
\] (5.34)

It is obvious that the right-hand side of inequality (5.34) is a second degree polynomial with respect to \( \bar{S}_k(i) \). It can be shown that \( \Delta V(i) \leq 0 \) holds if

\[
\bar{S}_k(i) \geq \delta^{-1}(\|\bar{A}_k^T P_k^{-1} \bar{B}_k r\| \\
+ \sqrt{\|\bar{A}_k^T P_k^{-1} \bar{B}_k r\|^2 + \delta (c_1 + c_2)})
\] (5.35)

where

\[
c_1 = 2 g^T(M_k(i)) \bar{F}_k^T P_k^{-1} \bar{B}_k r \\nc_2 = r^T \bar{B}_k^T P_k^{-1} \bar{B}_k r
\]

This means that the stability of the closed-loop system can be checked.

**5.6 Simulation Results**

**5.6.1 One-link Manipulator**

Similar to the previous chapter, in this section the one-link manipulator with revolute joints actuated by a DC motor is considered.

The system dynamics are nonlinear, and of the form

\[
x(i + 1) = Ax(i) + Bu(i) + g(x(i)) \\
y(i) = Cx(i)
\] (5.36)

with

\[
x = \begin{bmatrix}
\theta_m & \omega_m & \theta_1 & \omega_1
\end{bmatrix}^T
\] (5.37)
By Adding a non-Gaussian bounded noise such as \( d(x) = 1 - \sin(x^2) \) to the system, where \( x \in [-3, +3] \), the probability density estimate of the sample in the vector \( d \) can be shown in Fig. 5.2

![Figure 5.2: Probability density estimate of noise](image)

By assuming the following initial value \([1; 1; 0; 0]\), considering the total number of batches as 250, and choosing the ILC learning rates \( \lambda_P(1) = 15 \), \( \lambda_I(1) = 120 \). In addition, the initial values of the PI controller as chosen as \( K_P(1) = 200 \), \( K_I(1) = 100 \). The following results have been obtained to show the effectiveness of the proposed method.

Minimizing the closed loop tracking error entropy should make the error PDF a Gaussian-like shape, which means that the randomness of the closed loop tracking error entropy loop system is minimized batch by batch. In other words, the PDF of the closed loop tracking error should change along with the batches and move towards a Gaussian-like PDF shape. Fig. 5.3 shows that the proposed algorithm works as expected by comparing the shape of error PDF in the first batch and the last iteration. Moreover, the 3-D mesh of the PDF of the closed loop tracking error further confirms that the closed loop tracking error tends to be a Gaussian-like shape. The result is illustrated in Fig. 5.4.

The ILC-based calculated control signal should result in the tracking error PDF to tend to a narrowly distributed Gaussian shape along with the ILC closed loop operation. The corresponding trend is shown in Fig. 5.5. As depicted in Figs. 5.6 and 5.7, the ILC learning rates converge throughout the batches. Finally, the ILC-based tuning algorithm sets PI controller in the last batch to \( K_P = 145.5 \) and \( K_I = 10.5 \).
Chapter 5. An Adaptive PI Controller for non-Gaussian Stochastic Systems

Figure 5.3: The error PDF for the first and last batch

Figure 5.4: The 3D mesh of the tracking error through ILC
Figure 5.5: The error entropy along with the batches

Figure 5.6: $\lambda_P$ converge
5.6.2 Two-link Robot Manipulator

The dynamic equations of the robot manipulator are a set of highly nonlinear coupled differential equations. Using the Lagrange Euler formulation, the dynamic equation of a n-joint of an ideal rigid robot arm can be expressed with the equation of motion given by [102],

\[
M(q)\ddot{q} + c(q, \dot{q})\dot{q} + g(q) = u \tag{5.38}
\]

where \( q \in \mathbb{R}^n \) is the joint angular position vector of the robot manipulator; \( u \in \mathbb{R}^n \) is the applied joint torques; \( M(q) \in \mathbb{R}^{m \times n} \) is the inertia matrix; \( c(q, \dot{q}) \in \mathbb{R}^{m \times n} \) is the effect of Coriolis and centrifugal forces; \( g(q) \in \mathbb{R}^n \) is the gravitational torques.

In this work, a two-link robot manipulator as shown in Fig. 5.8 is considered. The parameter matrices are as follows [103]:

\[
M(q) = \begin{bmatrix}
\theta_1 + \theta_2 + 2\theta_3 \cos(q_2) & \theta_2 + 2\theta_3 \cos(q_2) \\
\theta_2 + 2\theta_3 \cos(q_2) & \theta_2 \\
\end{bmatrix}
\tag{5.39}
\]

\[
c(q, \dot{q}) = \begin{bmatrix}
-\theta_3 \sin(q_2)\dot{q}_2 & -\theta_3 \sin(q_2)(\dot{q}_1 + \dot{q}_2) \\
\theta_3 \sin(q_2)\dot{q}_1 & 0 \\
\end{bmatrix}, \tag{5.40}
\]
Figure 5.8: Two-link robot manipulator.

\[ g(q) = \begin{bmatrix} g(\theta_4 + \theta_5) \cos(q_1) + g\theta_6 \cos(q_1 + q_2) \\ g\theta_6 \cos(q_1 + q_2) \end{bmatrix}, \quad (5.41) \]

where \( g \) is the gravitational acceleration and

\[ \theta_1 = m_1 l_{c1}^2 + m_2 l_{1}^2 + I_1 \quad (5.42) \]
\[ \theta_2 = m_2 l_{c2}^2 + I_2 \quad (5.43) \]
\[ \theta_3 = m_2 l_1 l_{c2} \quad (5.44) \]
\[ \theta_4 = m_1 l_{c1} \quad (5.45) \]
\[ \theta_5 = m_1 l_1 \quad (5.46) \]
\[ \theta_6 = m_2 l_{c2} \quad (5.47) \]

The numerical values for the two-link robot are presented in Table 5.1. The desired trajectories are:

\[ q_d = \begin{bmatrix} q_{d1} \\ q_{d2} \end{bmatrix} = \begin{bmatrix} 3.14(1 - \exp(-0.5t)) \\ 3.14(1 - \exp(-0.5t)) \end{bmatrix} \quad (5.48) \]

The robotic control systems under communication environment are stochastic in nature because of the random time delays caused by the communications on the employed networks. However, for the simulation in this work, it is assumed that the system (5.38) is stochastic in the sense that it is corrupted with the
Table 5.1: The parameter values of two-link robot

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 = m_2 )</td>
<td>( 10Kg )</td>
</tr>
<tr>
<td>( l_{c1} = l_{c2} )</td>
<td>( 0.5m )</td>
</tr>
<tr>
<td>( l_1 = l_2 )</td>
<td>( 1m )</td>
</tr>
<tr>
<td>( I_1 = I_2 )</td>
<td>( 1Kg.m^2 )</td>
</tr>
<tr>
<td>( g )</td>
<td>( 9.81Kg/s^2 )</td>
</tr>
</tbody>
</table>

Figure 5.9: The error PDF for the first and last batch

process noise \( d \) which follows non-Gaussian distribution. Equation (5.38) can be written as follows:

\[
M(q)\ddot{q} + c(q, \dot{q})\dot{q} + g(q) + d = u
\]  

(5.49)

Assuming the following initial value \([1; 1; 0; 0]\). Fig. 5.9 shows that the proposed algorithm works as expected by comparing the shape of error PDF in the first batch and the last iteration. Moreover, the 3-D mesh of the PDF of the closed loop tracking error further confirms that the closed loop tracking error tends to be a Gaussian-like shape, as shown in Fig. 5.10. Although some small fluctuations can be seen in the entropy variations, the overall trend of the closed-loop error entropy suggests a minimum has been achieved. Fig. 5.11 illustrates how the closed-loop error entropy is minimised along with the batches.

By choosing the ILC learning rates \( \lambda_P(1) = 2, \lambda_I(1) = 5 \). Figs. 5.12 and 5.13 show the ILC learning rates converge throughout the batches. Also, the ILC-based tuning algorithm sets PI controller to \( K_P = 169.3I \) and \( K_I = 79.5I \) in the last batch as shown in Figs 5.14 and 5.15. From simulated results which are shown in Figs. 5.16 through 5.19, it can be seen that both \( q_1 \) and \( q_2 \) converge to the desired trajectories in Figs. 5.16 and 5.17; From Figs. 5.18 and 5.19, it is obvious that chattering of the control input is eliminated by the application of the proposed method.
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Figure 5.10: The 3D mesh of the tracking error through ILC

Figure 5.11: The error entropy along with the batches

Figure 5.12: \( \lambda_P \) converge of two-link robot
Figure 5.13: $\lambda_I$ converge of two-link robot

Figure 5.14: $K_P$ variation of two-link robot

Figure 5.15: $K_I$ variation of two-link robot
Figure 5.16: The response of actual and desired path of link 1

Figure 5.17: The response of actual and desired path of link 2

Figure 5.18: The control input of link 1
5.7 Summary

As an application of stochastic distribution control, the idea of ILC-based adaptive minimum entropy control was applied to nonlinear stochastic systems subject to non-Gaussian disturbance in this chapter. A modified PI controller was developed for the purpose of minimum entropy. In this algorithm, the whole control horizon is divided into a number of equal time intervals, called batches. Within each interval, there are a fixed number of sample points. Within each batch, the minimum entropy control is realized, whilst between any two adjacent batches, the coefficients of the PI controller are tuned. Since the closed-loop tracking error signal is a non-Gaussian stochastic process, the minimization of the mean square of the error is not an adequate method of characterizing the randomness in the closed-loop tracking error. Therefore, the objective of the optimization technique is to update the controller parameters in such a way that the entropy of the closed-loop tracking error is minimized as the batches progress.

A simple ILC law, namely the D-Type ILC, was applied for the tuning between any two adjacent batches, and a condition for the ILC convergence was established. The closed-loop stability of the system was discussed.

Two illustrated examples of one-link manipulator and two-link robot manipulator were included to demonstrate the effectiveness of the proposed control algorithm. The simulation results demonstrate that the randomness of the closed-loop tracking error is minimized batch by batch, and the ILC learning rates converge throughout the batches. Moreover, the error PDF is a Gaussian-like shape in the
last batch.
Chapter 6

Fault Tolerant Control Based on Stochastic Distribution

6.1 Introduction

In this chapter, a new fault tolerant control (FTC) via a controller reconfiguration approach for general stochastic, nonlinear systems is studied. Unlike the formulation of classical FTC methods, it is supposed that the measured information for the FTC is the probability density functions (PDFs) of the system output rather than its measured value. A radial basis function neural network (RBFNN) technique is proposed so that the output PDFs can be formulated in terms of the dynamic weighting of the RBFs neural network. As a result, the nonlinear FTC problem subject to dynamic relation between the input and the output PDFs, can be transformed into a nonlinear FTC problem subject to the dynamical relationship between the control input and the weights of the RBFNN. The main objective of FTC requires detecting the occurrence of faults and maintaining the performance of the system in the presence of faults, at a satisfactory level. The FTC design consists of two steps. The first step is fault detection and diagnosis (FDD), which can provide an alarm when there is a fault in the system and also locate which component has a fault. The second step is adapting the controller to the faulty case, so that the system can achieve its target. A linear matrix inequality (LMI)-based feasible FTC method is applied such that the fault can be detected and diagnosed. A simulated example is included to demonstrate the efficiency of the proposed algorithm, and satisfactory results are obtained.
6.2 Literature Review

Under the assumption that the random variables or the noise in the stochastic system are subject to Gaussian processes, the following approaches have been widely applied in theoretical studies: minimum variance control [81], whose purpose is minimizing the variations in the controlled system outputs or tracking errors, linear optimal control [82], linear quadratic martingale control [83], and stochastic control for systems with Markovian jump parameters [87]. In all these methods, the targets are the mean and variance of the output. However, this assumption may not hold in some applications. For example, many variables in the paper-making systems do not obey Gaussian distributions [7, 12]. Therefore, a new measure of randomness, called the PDF control, should be employed for general stochastic systems with non-Gaussian variables [7]. In PDF control problems, the control objective is to design a control signal so that the PDF shape of the output variable follows a desired distribution.

There are many stochastic systems in practice whose outputs are the PDF of the system output [7], rather than the actual output values. For such cases, the measured output PDFs can be used as an output for the feedback control. Such types of stochastic systems are called Stochastic Distribution Control (SDC) systems [7]. Practical examples of SDC systems in industrial applications include: Molecular weight distribution control [104, 105], combustion flame distribution processes [8, 9], particle size distribution control in polymerisation and powder processing industries [106, 107], and the wet-end of paper-making [7]. A well-known example of an SDC system in practice is the 2D paper web solid distribution in paper-making processes [108]. For instance, in Fig. 6.1, the PDF distribution of the grammage of the finished sheet can be measured online via digital cameras. Such images from these digital cameras can be processed and used as a feedback signal to the control system [108].

For the system represented in Fig. 6.2 of a food processing unit, wheat particles pass through two engraved rollers, which rotate in an opposite direction and driven by two motors. The gap between the two rollers and the speed of the motors, can be regarded as inputs that control the wheat particle size distribution after they have passed through the two rollers, and can be adjusted, so as to make sure that the broken wheat particles have a desired PDF.
Another example is the combustion system shown in Fig. 6.3, which is comprised of the fuel input together under some operating conditions and produces a flame (or a temperature) distribution inside the combustion chamber. Several digital cameras can be used to measure the flame distribution which can be further transferred into the temperature distribution. An efficient combustion means that the distribution shape of temperature needs to be controlled. Since the distribution of flames is directly related to the distribution of temperature, this can also be formulated as to control the fuel flow rate so that the flames
distribution can be made to follow a target distribution. The advantage of using flames distribution control in a practical situation is that a fast closed loop control can be established. Also, this overcomes the difficulties caused by long time delays for existing combustion control systems seen in different industrial applications.

![Flame distribution in combustion process.](image)

SDC was originally developed by Professor Hong Wang in 1996, when he considered a number of challenging paper machine modelling and control problems [7]. The process and the control were presented in a PDF form. As such, the purpose of the controller design was to obtain the PDF of the controller so that the closed-loop PDF would follow the pre-specified PDF. Since then, rapid developments have been made and introduced in different control applications [14].

The most exciting PDF control approaches are based on the B-spline model. Four types of model have been used in PDF control strategies. The linear B-spline model is easy to set up and widely used to model the dynamic output PDFs, but the output PDF of the model may be negative, and the integral of the PDF in its definition interval must be equal to 1, which has to be considered in the controller design [109]. The square-root B-spline model achieves more robustness, since it guarantees the positiveness of the PDF, but the constraint still needs to be taken into account in the controller design [110]. Although the rational B-spline model can meet the constraint, it cannot guarantee that the output PDF of the model is positive [111]. Only the rational square-root B-spline model can satisfy the constraint and guarantee the positiveness of the PDF at the same
time [112]. Moreover, multi-layer perception (MLP) neural network models have been applied to the shape control for the output PDFs [113]. Recently, a RBFNN has been used to approximate the output PDF of the system [114]. In this work, we have used RBFs instead of B-Splines which help generalize the output PDF expression and overcome the problems with B-spline-based functional approximations.

Many effective fault detection and diagnosis (FDD) strategies have been developed by researchers in the last several decades to cover various types of faulty systems [115, 116]. For stochastic systems, many significant schemes have been introduced and applied to practical process successfully. In general, the following approaches have been widely applied and developed for this problem: filter-or observer-based approaches [116, 117], identification-based approaches [118], and static approaches based likelihood approaches [119]. For dynamic stochastic systems, the filter based FDD approaches, have been presented as an effective way for Gaussian variables in stochastic systems. However, in many practical process, non-Gaussian variables exist in many stochastic systems. In this case, the filter-based FDD for non-Gaussian stochastic systems may be incapable. Therefore, for non-Gaussian stochastic systems, a new FDD approach has been established by using output distribution function for general stochastic systems in [116], where the dynamical system was supposed to be a precise linear model and the design algorithm required some technical conditions that were hard to verify. That work was motivated by the retention system of the paper making process, where the system output is replaced by the measured output PDFs to generate the residual of the filter [116]. The residual signal is calculated via the use of either the weighted integration or the integration of the square of the difference between the measured and the estimated PDFs. This method was the first attempt focussing on the application of the PDF model. However, there was a criticism that the used linear B-spline model cannot guarantee the output PDF of the model is positive [99, 117]. Subsequently, an improved design approach has been applied for the general stochastic system by using a square root B-spline model and non-linear filter design [99].

Due to the high demand for reliability and safe operation, many FTC methods were developed in the past four decades, which have the capability of detecting the occurrence of faults and maintaining the performance of the system in the presence of faults at a prescribed level [15, 16]. In most cases, the literatures on the FTC algorithms for stochastic systems have been presented under the
assumption that the random variables or the noise in the stochastic system are subject to Gaussian distribution [120, 121]. In [17], a nonlinear adaptive observer-based fault diagnosis algorithm has been presented for the SDC systems that are based on the rational square-root B-spline approximation model. When faults occur in the system, the controller was redesigned. This method is suffering from the complexity of modelling of stochastic distribution control. As such, there is a need to develop FTC methods that can be applied to general stochastic systems subject to arbitrary variables distribution, and reduced the complexity of SDC modelling.

6.3 Introduction to Fault Tolerant Control

Fault-tolerant control is used in systems that need to be able to detect faults and prevent simple faults related to control loops from developing into production stoppages or failures at a plant level. This is obtained by combining fault detection with supervisory control and re-configuration to accommodate faults.

6.3.1 Faults

A fault in a dynamic system is a deviation of the system structure or system parameters from the nominal situation, such as the loss of sensor, the blocking of an actuator, and leakage in pipes [122]. This fault changes the performance of the closed-loop system which further results in a loss of the system function. The faults are often classified according to the location as follows (Fig. 6.4)

a) Plant faults.

b) Sensor faults.

c) Actuator faults.

The time dependency of faults can be distinguished as shown in Fig. 6.5
a) Abrupt Fault: Represents bias in the monitored signal and can be modeled as stepwise function.

b) Incipient Fault: Represents drift in the monitored signal and can be modeled as ramp function.

c) Intermittent Fault: It can be modeled as impulse.

![Figure 6.5: Typical fault classes.](image)

The faults can be further classified with regard to the process models into two types as follows.

a) Additive faults: Influence a variable $X$ by an addition of the fault $f$ (e.g., as offsets of sensors)

b) Multiplicative faults: Represented by the product of a variable $Y$ with the fault $f$ (e.g., as parameter changes within a process).

### 6.3.2 Fault Diagnosis

Once the fault occurred, the location and the value of the fault have to be determined. Generally, there are three steps in diagnostic process [15]

a) **Fault detection**: This step determines the time at which the system is subject to some fault

b) **Fault isolation**: This step determines the location of the fault.

c) **Fault identification and fault estimation**: This step determines the kind of fault and its value.

### 6.3.3 Controller re-Design

Controller re-design is the problem of modifying the control structure and control law to satisfy the requirements on the closed loop system after a fault has occurred in the system. Two main ways of controller re-design have to be suggested [15].
1) **Fault accommodation.** If it is possible to adapt the controllers parameters, such that the faulty system can satisfy the control requirements, then using this modified control law is called fault accommodation. In this case, the input and output of the plant remain the same for the fault system.

2) **Fault reconfiguration.** When fault accommodation is impossible, then the faulty system requires the selection of alternative input and output signals. Then, a new control has to be designed on-line.

---

### 6.4 Problem Formulation

#### 6.4.1 Model Representation

Similar to [7, 99], denote $y(t) \in [a, b]$ as a uniformly bounded random process variable defined on $t \in [0, +\infty)$ and assume that $y(t)$ represents the output of the stochastic system in Fig. 6.6. Let $u(t) \in R^r$ be the control signal and is supposed to control the shape of the PDF of $y(t)$. At any time $t$, $y(t)$ can be described by its PDF $\gamma(y, u(t))$, which is defined by

$$P(a \leq y(t) < \xi) = \int_{a}^{\xi} \gamma(y, u(t))dy$$

where $P(a \leq y(t) < \xi, u(t))$ denotes the probability of output variable $y$ lying between $a$ and $\xi$ when the control is applied to the system. Similar to [7], it is assumed that the interval $[a, b]$ is known and the probability density function is continuous, measurable, and bounded. The well-known RBF neural networks can be used to approximate the square root of the output PDF as

$$\sqrt{\gamma(y, u(t))} = R(y)V(t) + r_n(y)h(V(t)) + \omega(y, u(t), F(t))$$

where $R(y) = [r_1(y), \ldots, r_{n-1}(y)]$ and $V(t) = [v_1(t), \ldots, v_{n-1}(t)]^T\quad (6.1)$

where the output PDF is denoted by $\gamma(y, u(t))$. $F(t)$ is supposed to be an actuator fault to be diagnosed and compensated. The term $\omega(y, u(t), F(t))$ represents the model uncertainties or the error term on the approximation of PDFs. In addition, $\omega(y, u(t), F(t))$ must satisfy the following condition [7]

$$|\omega(y, u(t), F(t))| \leq \delta$$

where $\delta > 0$ is a known positive constant. In (6.1), $r_l(y)$ and $v_l(t)\; (l = 1, 2, \ldots n)$ are the activation function and weight element corresponding to RBF neural network used for PDF modelling, respectively.
Similar to [13, 123], the RBF activation functions are chosen as of Gaussian shapes and expressed as

\[ r_l(y) = \exp\left(-\frac{(y_j - \mu_l)^2}{2\sigma_l^2}\right) \]  \hspace{1cm} (6.2)

where \(\mu_l, \sigma_l\) are the centers and widths of the RBF basis functions, respectively.

Similar to [7], the following constraint over the states should be satisfied

\[ V^T(t)\Lambda_4V(t) - \Lambda_3 \geq 0 \]  \hspace{1cm} (6.3)

which is equivalent to

\[ V^T(t)\Lambda_5V(t) \leq 1 \]  \hspace{1cm} (6.4)

where

\[ \Lambda_5 = \Lambda_1 - \Lambda_3^{-1}\Lambda_2^T\Lambda_2 \]
\[ \Lambda_4 = \Lambda_1\Lambda_3 - \Lambda_2^T\Lambda_2 \]
\[ \Lambda_1 = \int_a^b R^T(y)R(y)dy \]  \hspace{1cm} (6.5)
\[ \Lambda_2 = \int_a^b R^T(y)r_n(y)dy \]
\[ \Lambda_3 = \int_a^b r_n(y)^2dy \]

With the previous definition, it can be verified that

\[ h(V(t)) = \frac{1}{\Lambda_3}(-\Lambda_2V(t) + \sqrt{\Lambda_3 - V^T(t)\Lambda_4V(t)}) \]  \hspace{1cm} (6.6)

In (6.6), the nonlinear function \(h(V(t))\) should satisfy the following Lipschitz condition within its operating region for any \(V_1(t), V_2(t)\) and a known matrix \(U_1\).

\[ \|h(V_1(t)) - h(V_2(t))\| \leq \|U_1(V_1(t) - V_2(t))\| \]  \hspace{1cm} (6.7)
### 6.4.2 Nonlinear Dynamic Weight Model

In many cases, the dynamic relation between the input and the output PDFs can be transformed into dynamic relation between the control input and the weights of the RBFs neural network approximation to the output PDFs. In this section, the following nonlinear weighting model will be established, which is different from the model considered in [99]

\[
\dot{x}(t) = Ax(t) + Bu(t) + Gg(x(t)) + F(t)
\]

\[
V(t) = Ex(t)
\]  

(6.8)

where \(x(t) \in R^n\) is the state vector, and \(u(t) \in R^r\) is the measurable input vector. Moreover, \(A, B, G\) and \(E\) represent the identified coefficient matrices of the weight system with suitable dimensions. \(g(x(t))\) is a nonlinear vector function that stands for the nonlinear dynamics of the model and is supposed to satisfy \(g(0) = 0\), and the following Lipschitz condition, similar to [99]

\[
\|g(x_1(t)) - g(x_2(t))\| \leq \|U_2(x_1(t) - x_2(t))\|
\]  

(6.9)

for any \(x_1(t)\) and \(x_2(t)\), where \(U_2\) is a known matrix. \(F(t)\) is an actuator fault to be estimated and rejected, denoted as

\[
F(t) = \begin{cases} 
F, & t \geq T > 0 \\
0, & t \leq T 
\end{cases}
\]  

(6.10)

where \(T > 0\) is an appropriate time parameter.

### 6.5 Fault Detection

In order to detect the fault based on the changes of PDFs, the following nonlinear observer is considered [100]:

\[
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + Gg(\hat{x}(t)) + Le(t)
\]

\[
\epsilon(t) = \int_a^b \mu(y)(\sqrt{\gamma(y, u(t), F)} - \sqrt{\hat{\gamma}(y, u(t))})dy
\]

\[
\sqrt{\hat{\gamma}(y, u(t))} = R(y)E\hat{x}(t) + h(E\hat{x}(t))r_n(y)
\]  

(6.11)

where \(\hat{x}(t) \in R^n\) is the estimated state, \(L \in R^{n \times p}\) is the filter gain to be determined. Residual \(\epsilon(t)\) is formulated as an integral of the difference between the
measured PDFs and the estimated ones, where, $\mu(y) \in R^{p \times 1}$ is a pre-specified weighting vector.

Denoting the estimation error as $e(t) = x(t) - \hat{x}(t)$, the dynamic of the estimation error will be expressed as

$$
\dot{e}(t) = (A - L\Gamma_1)e(t) + [Gg(x(t)) - Gg(\hat{x}(t))] - L\Gamma_2[h(Ex(t)) - h(E\hat{x}(t))] - L\Delta(t) + F(t)
$$

(6.12)

where

$$
\Gamma_1 = \int_a^b \mu(y)R(y)E\,dy
$$

$$
\Gamma_2 = \int_a^b \mu(y)r_n(y)dy
$$

$$
\Delta(t) = \int_a^b \mu(y)\omega(y, u(t))dy
$$

It can be seen that

$$
e(t) = \Gamma_1 e(t) + \Gamma_2 [h(Ex(t)) - h(E\hat{x}(t))] + \Delta(t)
$$

From $|\omega(y, u(t), F)| \leq \delta$, it can be verified that

$$
\|\Delta(t)\| = \|\int_a^b \mu(y)\omega(y, u(t))dy\| \leq \tilde{\delta}
$$

(6.13)

where $\tilde{\delta} = \delta\|\int_a^b \mu(y)dy\|$.

In this section, the objective is to find $L$ such that the system (6.12) is stable if $F = 0$, which can be formulated in the following theorem

**Theorem 5.** [100] For the parameter $\lambda_i > 0 (i = 1, 2)$, if there exist matrices $P > 0$, and $R$ satisfying

$$
\Psi_0 = \begin{bmatrix}
\Pi_0 & \Pi_1 \\
\Pi_1^T & -I
\end{bmatrix} < 0
$$

(6.14)

where

$$
\Pi_0 = P(A - L\Gamma_1) + (A - L\Gamma_1)^T P + \frac{1}{\lambda_1} E^T U_1^T U_1 E + \frac{1}{\lambda_2} U_2^T U_2
$$

and

$$
\Pi_1 = \begin{bmatrix}
\lambda_1 R \Gamma_2 & \lambda_2 PG
\end{bmatrix}
$$
Then in the absence of fault, the error dynamic system with gain $L = P^{-1}R$ is stable and the error satisfies
\[
\|e(t)\| \leq \vartheta_0 = \max \left\{ \|e(0)\|, 2\varrho^{-1}\tilde{\delta} \|R\| \right\}
\] (6.15)

Theorem 5 presents a necessary condition for fault detection. In order to detect $F$, we select $\epsilon(t)$ as the residual signal and propose the following result to determine the threshold. The fault value that indicates the working status of the system has one of three possible values: fault, non-fault, or unknown. A fault value indicates that the system has a fault and might work incorrectly; a non-fault value indicates that the system works correctly; and an unknown value indicates that the working status of the system is uncertain.

**Theorem 6.** [100] For the parameter $\lambda_i > 0 (i = 1, 2)$, if there exit matrices $P > 0$, and $R$ satisfying (6.14), then fault $F$ can be detected by the following criterion
\[
\|e(t)\| > \vartheta = \vartheta_0 (\|\Gamma_1\| + \|\Gamma_2\| \|\|U\|\|) + \tilde{\delta}
\] (6.16)

which means that (7.26) implies $F \neq 0$

Once fault is detected, it needs to be estimated, which follows an adaptive fault diagnosis algorithm in the next section.

### 6.6 Fault Diagnosis

Once the fault is detected based on Theorem 5 and 10, the fault value must be estimated. For this purpose, the following adaptive observer is considered [100]:

\[
\dot{x}(t) = A\hat{x}(t) + Bu(t) + Gg(\hat{x}(t)) + Le(t) + \hat{F}(t)
\] (6.17)

\[
\sqrt{\gamma(y, u(t))} = R(y)E\hat{x}(t) + r_n(y)h(E\hat{x}(t))
\]

\[
\dot{\hat{F}}(t) = -\Upsilon_1 \hat{F}(t) + \Upsilon_2 \epsilon(t)
\]

where $\hat{F}(t)$ is the estimated fault. $\Upsilon_1 > 0$ and $\Upsilon_2$ are the learning operators to be determined by the diagnosis algorithm in (6.17). Denoting $\hat{F}(t) = F(t) - \hat{F}(t)$ and $\epsilon(t) = x(t) - \hat{x}(t)$. The dynamic of the estimation error will be expressed as

\[
\dot{\epsilon}(t) = (A - L\Gamma_1)\epsilon(t) + [Gg(x(t)) - Gg(\hat{x}(t))] - L\Gamma_2[h(Ex(t)) - h(E\hat{x}(t))]
\]

\[
- L\Delta(t) + \hat{F}(t)
\] (6.18)
Similar to previous FDD algorithms in [116], in this section it is assumed that 
\[ \dot{F}(t) = 0, \|F(t)\| \leq M/2, \|\dot{F}(t)\| \leq M/2, \text{and consequently } \|\dot{F}(t)\| \leq M. \] Then the size of the fault can be diagnosed by the third equation of 6.17.

**Theorem 7.** [100] For the parameter \( \lambda_i > 0 (i = 1, 2) \), if there exit matrices \( P > 0, R, \Lambda_i > 0 (i = 1, 2, \Lambda_1 > 0) \), and constants \( \theta_i > 0 (i = 1, 2, 3) \) satisfying

\[
\Psi_i = \begin{bmatrix}
    \Pi_0 & P - \Gamma_1^T \Gamma_2^T & \Pi_2 & 0 & E^T U_1^T \\
    P - \Upsilon_2 \Gamma_1 & -2 \Upsilon_2^T & 0 & \Pi_3 & 0 \\
    \Pi_2^T & 0 & -I & 0 & 0 \\
    0 & \Pi_3^T & 0 & -I & 0 \\
    U_1 E & 0 & 0 & 0 & \theta_3 I
\end{bmatrix} < 0 \quad (6.19)
\]

where

\[
\Pi_0 = P(A - L \Gamma_1) + (A - L \Gamma_1)^T P + \frac{1}{\lambda_1} E^T U_1^T U_1 E + \frac{1}{\lambda_2} U_2^T U_2
\]

\[
\Pi_2 = \begin{bmatrix}
    \lambda_1 R \Gamma_2 & \lambda_2 P G & \theta_1 R
\end{bmatrix}
\]

and

\[
\Pi_3 = \begin{bmatrix}
    \theta_2 R \Upsilon_2 & \theta_3 \Upsilon_2 \Gamma_2
\end{bmatrix}
\]

Then under the diagnosis filter (6.17) with gain \( L = P^{-1} R \), error system (6.18) is stable and satisfies

\[
\|e(t)\|^2 \leq \kappa^{-1} \left( (\theta_1^{-2} + \theta_2^{-2}) \delta^2 + \|\Upsilon_1\| M^2 \right) \quad (6.20)
\]

### 6.7 Controller Design

The purpose of control algorithm design of stochastic systems is to choose a control input such that the actual output PDF of the system is made as close as possible to a desired PDF. The desired PDF can be expressed as

\[
\sqrt{g(y)} = R(y)V_g + r_n(y)h(V_g),
\]

where \( V_g \) represents the desired weights. For simplicity, \( \omega(y, u(t), F(t)) \) is considered to be negligible and thus omitted hereafter. The tracking error can be defined as

\[
e(y, t) = \sqrt{g(y)} - \sqrt{\gamma(y, u(t))} = R(y)\omega(t) + [h(V_g) - h(V(t))] r_n(y) \quad (6.21)
\]
which is a function of both \( y \) and time instant \( t \), where the weight error vector is defined as

\[
\omega(t) = V_g - V(t)
\]

Since the \( h(V(t)) \) is a continuous function, it can be noted that \( e(y, t) \to 0 \), can be satisfied if and only if \( \omega(t) \to 0 \) \[100\]. Therefore, the PDF tracking problem can be reduced to weight vectors tracking problem. A FTC design for general non-Gaussian stochastic systems using a classical proportional-integral-derivative (PID) controller will be used for the dynamic weight system. PID controllers are widely used in both theoretical studies and practical application \[124–126\]. For output PDF control, the classical PID controller is not realized in stochastic systems since \( y \) is a variable defined on \([a, b]\), the control input is a twin variable function of \( y \) and \( t \), and the stochastic system can only accept one input value at a time instant \[109\]. Therefore, a pseudo propositional-integral-derivative control law will be used. The pseudo-PID control law can be given by

\[
u(t) = K_P \omega(t) + K_I \int_0^t \omega(\tau) d\tau + K_D \dot{\omega}(t), \quad (6.22)
\]

where \( K_P, K_I \) and \( K_D \) are control gains to be determined.

Substituting (6.22) in (6.8) yields the following closed-loop system:

\[
\begin{bmatrix}
\dot{\omega}(t) \\
\dot{s}(t)
\end{bmatrix} =
\begin{bmatrix}
\Delta^{-1} \Pi & -\Delta^{-1} BK_I \\
I & 0
\end{bmatrix}
\begin{bmatrix}
\omega(t) \\
s(t)
\end{bmatrix} +
\begin{bmatrix}
-\Delta^{-1} A \\
0
\end{bmatrix}
V_g +
\begin{bmatrix}
-\Delta^{-1} G \\
0
\end{bmatrix}
F(t) +
\begin{bmatrix}
0 \\
0
\end{bmatrix}
g(x(t))
\]

(6.23)

where \( s(t) = \int_0^t \omega(\tau) d\tau \), \( \dot{s}(t) = \omega(t) \), \( \Pi = (A - BK_P) \), and \( \Delta = I + BK_D \). \( \Delta \) is assumed invertible to avoid the singularity. The objective of controller is finding \( K_P, K_I \) and \( K_D \) such that (6.23) is stable and \( \omega(t) \) is convergent in the presence of the actuator fault.

**Theorem 8.** The closed-loop system (6.23) combined with the weight dynamical system (6.8) and the pseudo-PID control law (6.22) is asymptotically stable, \( \lim_{t \to \infty} \omega(t) = 0 \), and the controller parameters can be calculated using \( K_P = Q_T^T Q_1^{-1} T \), \( K_I = P_1 P_2 \), and \( K_D = P_3^T P_1^{-T} \) if the following two algebraic matrix inequalities:

\[
P = \begin{bmatrix}
P_1 & P_2 \\
P_2^T & P_3
\end{bmatrix} > 0
\]

(6.24)
Chapter 6. Fault tolerant control

\[
\tilde{\Psi} = \begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \tilde{U} & 0 \\
\Phi_{12}^T & \Phi_{22} & \Phi_{23} & 0 & 0 \\
\Phi_{13}^T & \Phi_{23}^T & -\lambda^{-2}I & 0 & 0 \\
\tilde{U}^T & 0 & 0 & -\lambda^{-2}I & 0 \\
0^T & 0 & 0 & 0 & -2\Gamma_1 I
\end{bmatrix} < 0 \quad (6.25)
\]

with the following constraint

\[\bar{\alpha}^{-1}\tilde{\sigma}_3 \|\Lambda_5\| \leq 1 \quad (6.26)\]

are solvable for \( P = P^T > 0 \) and matrices \( K_P, K_I, \) and \( K_D, \) where

\[
\begin{align*}
\Phi_{11} &= (P_1 \Delta^{-1} \Pi + P_2) \\
\Phi_{12} &= -P_1 \Delta^{-1} BK_I + \Pi^T \Delta^{-T} P_2 + P_3 \\
\Phi_{13} &= -P_1 \Delta^{-1} G \\
\Phi_{22} &= (-P_2^T \Delta^{-1} BK_I) \\
\Phi_{23} &= -P_2^T \Delta^{-1} G
\end{align*}
\]

In this case, the output PDF converges to the desired PDF in the presence of the actuator fault.

**Proof:** For this purpose, the following Lyapunov function is considered.

\[
V_1(\omega, s, t) = \begin{bmatrix} \omega^T(t) & s^T(t) \end{bmatrix} P \begin{bmatrix} \omega(t) \\ s(t) \end{bmatrix} + \tilde{F}(t)\tilde{F}(t) + \int_0^t (\|\lambda U w(\tau)\|^2 - \|\lambda g(x(\tau))\|^2) d\tau
\]

where \( \tilde{F}(t) = F(t) - \hat{F}(t) \). Following (6.7) and (6.9), it is noted that \( V(\omega, s, t) \geq 0 \). Furthermore, it can be seen that the time derivative of the above Lyapunov candidate will lead to the following equation:

\[
\dot{V}_1(\omega, s, t) = \begin{bmatrix} \omega^T(t) & s^T(t) \end{bmatrix} P \begin{bmatrix} \dot{\omega}(t) \\ \dot{s}(t) \end{bmatrix} + \begin{bmatrix} \dot{\omega}^T(t) & \dot{s}^T(t) \end{bmatrix} P \begin{bmatrix} \omega(t) \\ s(t) \end{bmatrix} + 2\tilde{F}^T(t)\dot{F}(t) + \|\lambda U w(t)\|^2 - \|\lambda g(x(t))\|^2 \quad (6.27)
\]

According to fault diagnosis algorithm in previous section, it can be constructed the filter as follows.

\[
\dot{\hat{F}}(t) = -\Gamma_1 \hat{F}(t) + \Gamma_2 \epsilon(t) \quad (6.28)
\]
Then, it can be seen that
\[
\frac{\partial}{\partial t} (\tilde{F}^T(t)\tilde{F}(t)) = 2\tilde{F}^T(t)\dot{\tilde{F}}(t) = 2\tilde{F}^T(t)(\Gamma_1\dot{F}(t) - \Gamma_2\epsilon(t)) = 2\tilde{F}^T(t)(\Gamma_1(F(t) - \tilde{F}(t)) - \Gamma_2\epsilon(t)) = 2\tilde{F}^T(t)\Gamma_1F(t) - 2\tilde{F}^T(t)\Gamma_1\tilde{F}(t) - 2\tilde{F}^T(t)\Gamma_2\epsilon(t)
\]

By substituting (6.23) and (6.29) in (6.27) it can be verified that

\[
\begin{align*}
\dot{\psi}_1(\omega, s, t) &= \begin{bmatrix} \omega^T(t) & s^T(t) \end{bmatrix} \Psi_1 \begin{bmatrix} \omega(t) \\ s(t) \end{bmatrix} + 2 \begin{bmatrix} \omega^T(t) & s^T(t) \end{bmatrix} \begin{bmatrix} -P_1\Delta^{-1}A \\ -P_2^T\Delta^{-1}A \end{bmatrix} V_g \\
&+ 2 \begin{bmatrix} \omega^T(t) & s^T(t) \end{bmatrix} \begin{bmatrix} -P_1\Delta^{-1}A \\ -P_2^T\Delta^{-1}A \end{bmatrix} F(t) + 2 \begin{bmatrix} \omega^T(t) & s^T(t) \end{bmatrix} \begin{bmatrix} -P_1\Delta^{-1}G \\ -P_2^T\Delta^{-1}G \end{bmatrix} g(x(t)) \\
&+ \|\lambda\tilde{U}w(t)\|^2 - \|\lambda g(x(t))\|^2 + 2\tilde{F}^T(t)\Gamma_1F(t) - 2\tilde{F}^T(t)\Gamma_1\tilde{F}(t) - 2\tilde{F}^T(t)\Gamma_2\epsilon(t) \\
&\leq \bar{q}^T\Psi_2\bar{q} + 2\bar{q}^T \begin{bmatrix} -P_1\Delta^{-1}A \\ -P_2^T\Delta^{-1}A \end{bmatrix} V_g + 2\bar{q}^T \begin{bmatrix} -P_1\Delta^{-1}G \\ -P_2^T\Delta^{-1}G \end{bmatrix} F(t) - 2\tilde{F}^T(t)\Gamma_2\epsilon(t)
\end{align*}
\]

(6.30)

where

\[
\Psi_1 = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix}
\]

\[
\bar{q} = \begin{bmatrix} \omega(t) \\ s(t) \\ g(x(t)) \\ \tilde{F}(t) \end{bmatrix}^T
\]

and

\[
\Psi_2 = \begin{bmatrix} \Phi_{11} + \lambda^2\tilde{U}^T\tilde{U} & \Phi_{12} & -P_1\Delta^{-1}G & 0 \\ * & \Phi_{22} & -P_2\Delta^{-1}G & 0 \\ * & * & -\lambda^{-2} & 0 \\ * & * & * & -2\Gamma_1 \end{bmatrix}
\]

(6.31)

By applying Schur complement formula to (6.31), (6.25) can be obtained. If (6.25) holds, a positive scalar \( \alpha \) exists so that \( \bar{\Psi} \leq -\alpha I \). Along with (6.23) it can
be verified that
\[
\dot{V}(\omega, s, t) \leq -\bar{\alpha}\|\bar{q}\|^2 + 2\|\bar{q}\| \left[ \begin{array}{c}
-P_1\Delta^{-1}A \\
-P_2\Delta^{-1}A \\
0 \\
0
\end{array} \right] \|V_g\| + 2\|\bar{q}\| \left[ \begin{array}{c}
-P_1\Delta^{-1} \\
-P_2^{T}\Delta^{-1} \\
0 \\
2\Xi_1
\end{array} \right] \|F(t)\| - 2\|\bar{F}(t)\|\|\Gamma_2\|\|\epsilon(t)\| \tag{6.32}
\]

It is obvious that the right-hand side of inequality is a second order degree polynomial with respect to \[\omega(t) \quad s(t) \quad g(x(t)) \quad \bar{F}(t)\] T. Denote
\[
\bar{\sigma}_0 = \left\| \begin{array}{c}
-P_1\Delta^{-1}A \\
-P_2\Delta^{-1}A \\
0 \\
0
\end{array} \right\|V_g\|, \quad \bar{\sigma} = \left\| \begin{array}{c}
-P_1\Delta^{-1} \\
-P_2^{T}\Delta^{-1} \\
0 \\
2\Xi_1
\end{array} \right\|F(t)\|
\]
\[
\bar{\sigma}_1 = -2\|\bar{F}(t)\|\|\Gamma_2\|\|\epsilon(t)\| = -2\|M\|\|\Gamma_2\|\bar{\sigma}_2
\]

where
\[
\|\epsilon\| \leq \|\Gamma_1\|\|e\| + \|\Gamma_2\|\|h(E(x(t))) - h(E\hat{x}(t))\| + \|\Delta(t)\| \\
\leq \|\Gamma_1\|\|e\| + \|\Gamma_2\|\|U_1\|\|E\|\|e\| + \|\Delta(t)\| \\
\leq \bar{\vartheta}_0(\|\Gamma_1\| + \|\Gamma_2\|\|U_1\|\|E\| + \hat{\delta} = \bar{\sigma}_2 \tag{6.33}
\]

Thus, it can be shown that \(\dot{V}(\omega, s, t) \leq 0\) holds if
\[
\|\bar{q}\| \geq \bar{\alpha}^{-1}(\|\bar{\sigma}_0 + \bar{\sigma}\| + \sqrt{\|\bar{\sigma}_0 + \bar{\sigma}\|^2 + \bar{\alpha}\|\bar{\sigma}_1\|}) = \bar{\alpha}^{-1}\bar{\sigma}_3 = \bar{\sigma}_4 \tag{6.34}
\]

which implies
\[
\|\bar{q}\| \leq \max\{\|\bar{q}(0)\|, \bar{\sigma}_4\} \tag{6.35}
\]

This confirm that the closed-loop system as a bounded and internally stable system.

Using (6.34), the state constraint (6.4) can be written as
\[
V^T(t)A_5V(t) \leq \|q\|^2\|A_5\| \leq \bar{\sigma}_4\|A_5\| \leq 1 \tag{6.36}
\]

which means that (6.26) holds true.

To discuss the system tracking performance, suppose that \(\bar{\varphi}\) is a trajectory of the nonlinear closed-loop system (6.23), and
\[
\lim_{t \to \infty} \tilde{\varphi}(t) = \tilde{\varphi}_0
\]

where \( \tilde{\varphi}_0 \) is a constant vector, it follows that

\[
\lim_{t \to \infty} \dot{\tilde{\varphi}}(t) = 0
\]

From system equation (6.23) it can be see that \( \dot{s}(t) = \omega(t) \). Thefore, it can be concluded that \( \lim_{t \to \infty} \omega(t) = 0 \)

In order to prove the internal stability, \( P, K_P, K_I, \) and \( K_D \) should be found so that (6.25) is feasible.

After exploring the solvability conditions, the feasible design procedure must be developed. From Theorem 8, a necessary condition for (6.25) is given by

\[
\Phi_{22} = (-P_2^T \Delta^{-1} B K_I) < 0 \quad (6.37)
\]

which includes a strong nonlinearity with respect to \( P_2, K_D, \) and \( K_I \).

**Lemma 2:** If there exist a vector \( P_I \) and a matrix \( P_D \) so that the following LMI holds,

\[
(-BP_I - BP_D B^T) < 0 \quad (6.38)
\]

then (6.37) holds for some \( P_2, \) and \( K_I \). In this case,

\[
K_I = P_I P_2 \quad (6.39)
\]

and \( K_D \) can be solved via

\[
P_I K_D^T = P_D \quad (6.40)
\]

**Proof:** Without any loss of generality, we choose \( P_2 = (1/2) \beta I > 0 \) for an appropriate \( \beta > 0 \). (6.37) is equivalent to (6.38) by pre-multiplying \( \Delta P_2^{-T} \) and post-multiplying \( P_2^{-1} \Delta^T \), and denoting \( P_I = K_I P_2^{-1} \) and \( P_D = K_I P_2^{-1} K_D^T \).

Using the results from Theorem 8, the previous LMI in (6.37) can be used in order to achieve the proportional coefficient of pseudo-PID controller.

**Lemma 3:** For the \( \beta, K_I, \) and \( K_D \) obtained from Lemma 2, if there exist \( Q_1 > 0, Q_P, \) and \( \pi > 0 \) such that the following LMI is satisfied,
where

\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} + \pi^2 I & -\Delta^{-1}G & Q_1 & Q_1\bar{U} & 0 \\
* & \Psi_{22} & -Q_1P_2^T\Delta^{-1}G & 0 & 0 & 0 \\
* & * & -\lambda^{-2} & 0 & 0 & 0 \\
* & * & * & -\beta^{-1} & 0 & 0 \\
* & * & * & * & -\lambda^{-2} & 0 \\
* & * & * & * & * & -2\Gamma_1 \\
\end{bmatrix} < 0 \tag{6.41}
\]

then closed-loop system (6.23) combined with the weight dynamical system (6.8) and the pseudo-PID control law (6.22) is asymptotically stable and the output PDF \( \gamma(y, u) \) follows the desired PDF \( g(y) \). In this case, the \( K_p \) coefficient can be obtained by

\[
K_p = Q_p^TQ_1^{-T}
\]

**Proof:** Substituting the obtained \( P_2, K_I \) and \( K_D \) into (6.25), denoting \( Q_1 = P_1^{-1} \), and pre-multiplying by \( \text{diag}(P_1^{-1}, I, I, I, I) \) and pos-multiplying it by \( \text{diag}(P_1^{-1}, I, I, I, I)^T \) yields

\[
\begin{bmatrix}
\dot{\Psi}_{11} & \dot{\Psi}_{12} & \dot{\Psi}_{13} & 0 \\
\dot{\Psi}_{12}^T & \dot{\Psi}_{22} & \dot{\Psi}_{23} & 0 \\
\dot{\Psi}_{13}^T & \dot{\Psi}_{23}^T & -\lambda^{-2} & 0 \\
0 & 0 & 0 & -2\Gamma_1 \\
\end{bmatrix} \tag{6.43}
\]

where

\[
\begin{align*}
\dot{\Psi}_{11} &= (\Delta^{-1}AQ_1^T - \Delta^{-1}BQ_p^T) + Q_1(P_2 + P_2^T)Q_1^T + Q_1\lambda^2\bar{U}^T\bar{U}Q_1^T \\
\dot{\Psi}_{12} &= -\Delta^{-1}BK_I + (Q_1A^T - Q_pB^T)\Delta^{-T}P_2 + Q_1P_3 \\
\dot{\Psi}_{13} &= -\Delta^{-1}G \\
\dot{\Psi}_{22} &= (-P_2^T\Delta^{-1}BK_I) \\
\dot{\Psi}_{23} &= -Q_1P_2^T\Delta^{-1}G
\end{align*}
\]
(6.43) is equivalent to

\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} + Q_1 P_3 & -\Delta^{-1}G & Q_1 & Q_1 \bar{U} & 0 \\

\Psi_{12}^T + P_3^T Q_1^T & \Psi_{22} & -Q_1 P_2^T \Delta^{-1}G & 0 & 0 & 0 \\

-G^T \Delta^{-T} & -G^T \Delta^{-T} P_2 Q_1^T & -\lambda^{-2} & 0 & 0 & 0 \\

Q_1^T & 0 & 0 & -\beta^{-1} & 0 & 0 \\

\bar{U}^T Q_1^T & 0 & 0 & 0 & -\lambda^{-2} & 0 \\

0 & 0 & 0 & 0 & 0 & -2\Gamma_1
\end{bmatrix} < 0
\]  

(6.44)

by using the well-known Schur complement formula. Denote \( Q_3 = Q_1, P_3 = \pi^2 I \). It is clear that (6.44) is equivalent to (6.41) when \( P_3 = \pi^2 Q_3^{-1} \). By using Schur complement formula, (6.24) hold if

\[ P_3 - 0.25\beta^2 P_1^{-1} > 0 \]

Using the LMI-toolbox, a feasible PID controller design algorithm can be summarized as follows.

**Step 1:** Solve the LMI (6.38) with respect to \( P_I \) and \( P_D \).

**Step 2:** Apply (6.39) to calculate \( K_I \) with respect to \( \beta \).

**Step 3:** Solve \( K_D \) via (6.40).

**Step 4:** Solve the LMI (6.41) for \( Q_1 > 0 \) and \( Q_P \).

**Step 5:** Compute \( K_P \) through \( K_P = Q_P^T Q_1^{-T} \).

**Step 6:** Calculate the control input based on (6.22) and apply it to the stochastic system.

### 6.8 An Illustrated Example

In this section, a simulation study of the proposed method will be described. First, the system model and RBF neural network components are introduced, and then the performance of the pseudo-PID control law will be investigated. For a stochastic system with non-Gaussian process, it is supposed that the output PDF can be formulated by using three-layer neural network with three radial basis activation functions with the following initial conditions over its definition interval \([a, b]\).

\[
y \in [0, 1],
\]

\[
\mu_1 = 0.25, \mu_2 = 0.5, \mu_3 = 0.75
\]

\[
\sigma_1 = \sigma_2 = \sigma_3 = 0.06
\]
This would mean that the output PDF of the stochastic system will be described as follows.

\[
\sqrt{\gamma(y, u(t))} dy = R(y)V(t) + r_3(y)h(V(t))
\]  

(6.45)

where

\[
R(y) = [r_1(y), r_2(y)]
\]

and

\[
V(t) = [v_1(t), v_2]^T
\]

The weight vector behaves dynamically as described in (6.8) with the following parameters:

\[
A = \begin{bmatrix} 2 & -1 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 0.8 & 0.3 \\ 0.4 & 1 \end{bmatrix}, G = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.4 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

The nonlinear function was chosen as follows.

\[
g(V(t)) = \begin{bmatrix} 0 \\ \sqrt{v_1^2 + v_2^2} \end{bmatrix}
\]

The initial value of the weight vector is set as \(V(0) = [0.001, 0.001]^T\). In addition, the matrices \(U_1\) and \(U_2\) were chosen as follows.

\[
U_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Furthermore, the following results can be easily obtained from (6.5)

\[
\Lambda_1 = \begin{bmatrix} 0.1153 & 0.0013 \\ 0.0013 & 0.1153 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} 0 & 0.014 \\ 0.014 & 0 \end{bmatrix}, \Lambda_3 = 0.1060
\]

It can be tested that \(\Gamma_1 = [1.84, 2.21], \Gamma_2 = 0.15, \hat{\delta} = 0.0007\) for \(\mu(y) = 2.3\) and \(\delta = 0.0003\).

For simulation purposes, 100 uniformly distributed samples of the output and 50 for the time samples were used. In addition, it is desired that the measured output PDF follows a distribution as described by RBF basis functions parameters desired as follows:

\[
\mu_{g1} = 0.25, \mu_{g2} = 0.5, \mu_{g3} = 0.75
\]

\[
\sigma_{g1} = \sigma_{g2} = \sigma_{g3} = 0.06
\]
Moreover, the desired dynamical weights are set as $V_g = [1; 1.5]$. With the above parameters, the 2-D plot of the desired PDF shape is shown in Fig. 6.7. Fig. 6.8 shows the initial distribution of the three RBF basis functions over the definition interval of the system output. The desired PDF shape with 100 samples defined.
for the output variable and 50 for the time sample is shown in Fig. 6.9.

To demonstrate the effectiveness of the proposed algorithm, the fault is chosen to be a time-variant signal as $F(t) = 3 + 0.1 \sin(0.05t)$, and it is supposed to
commence at $T = 10s$.

Firstly, the fault detection problem is considered. Using Theorem 5 with $\lambda_1 = \lambda_2 = 1$ it can be calculated that

$$P = \begin{bmatrix} 5.7884 & 9.7090 \\ 9.7090 & 17.9827 \end{bmatrix}, \quad R = \begin{bmatrix} 32.2681 \\ 44.2770 \end{bmatrix}, \quad L = \begin{bmatrix} 115.3022 \\ -15.7995 \end{bmatrix}$$

Next, the fault diagnosis problem is considered for the above system and fault. Using Theorem 7 with $\theta_1 = 3, \theta_2 = \theta_3 = 1.2$, the following results can be obtained:

$$P = \begin{bmatrix} 11.1862 & -1.9324 \\ -1.5797 & 9.7355 \end{bmatrix}, \quad R = \begin{bmatrix} 11.1862 \\ -1.5797 \end{bmatrix}, \quad L = \begin{bmatrix} 5.7443 \\ -1.18 \end{bmatrix}$$

$\Upsilon_1 = 0.15, \Upsilon_2 = 0.001$

By applying the nonlinear fault isolation filter, the estimated fault (6.28) should track the real fault profile as close as possible. Fig. 6.10 shows the response of the residual signal by using the diagnosis nonlinear filter. Fig. 6.11 shows that such a filter can effectively diagnose the actuator fault. After the fault occurred, it can be computed via (6.38) that

$$P_I = \begin{bmatrix} 2.171 & -0.623 \\ -0.841 & 1.754 \end{bmatrix}, \quad P_D = \begin{bmatrix} 1.485 & 0.032 \\ 0.032 & 1.751 \end{bmatrix}$$

with which it can be obtained that

$$K_I = \begin{bmatrix} 2.1667 & -0.6239 \\ -0.8414 & 1.7481 \end{bmatrix}$$

and

$$K_D = \begin{bmatrix} 0.8000 & 0.4000 \\ 0.3000 & 1.0000 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \beta = 2$$

Solving (6.41) produces $\pi^2 = 5.3318$ and

$$Q_P = \begin{bmatrix} 12.244 & -1.923 \\ -1.571 & 10.624 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1.097 & 0.017 \\ 0.0167 & 1.094 \end{bmatrix}$$

Thus, it can be obtained that

$$K_P = \begin{bmatrix} 20.8008 & -3.0891 \\ -2.2932 & 18.9790 \end{bmatrix}$$
Fig. 6.12 clearly demonstrates the effect of the fault-tolerant action on maintaining the correct value for the controlled weights.

\[ P = \begin{bmatrix} 0.911 & -0.014 & 1 & 0 \\ -0.014 & 0.914 & 0 & 1 \\ 1 & 0 & 25.911 & -0.397 \\ 0 & 1 & -0.397 & 25.992 \end{bmatrix} \]

6.9 Summary

A new FTC algorithm against the actuator fault is presented for general non-Gaussian stochastic nonlinear systems. Unlike existing FTC methods, the measured information is the PDF of the system output rather than its value, where RBFNN technique is proposed so that the output PDFs can be formulated in terms of the dynamic weightings, and the problem can be transformed into a nonlinear FTC problem subject to the weight dynamical systems. An LMI-based feasible FTC method is applied such that the fault can be detected and diagnosed.

After a fault happens in the nonlinear system, the fault estimation information is introduced into the controller reconfiguration, to keep the stability and tracking convergence of the post-fault system, leading to the fault tolerant control of the whole stochastic nonlinear system. The solvability conditions are obtained in terms of a group of matrix inequalities for the pseudo-PID tracking control problem. Feasible controller design procedures are presented to guarantee closed-loop stability and tracking convergence. The illustrated example demonstrates the effectiveness of the control algorithm, and shows that satisfactory estimation and control performance can be achieved.

Finally, the major difference between the proposed FTC problem in this chapter and that in [17] is that, here, the FTC problem is first investigated by using the measured output PDFs using RBFNNs instead of a fixed B-spline expansion. Also, we provide the stability and tracking performance analysis for the weight dynamical system, and the constraint of the states is guaranteed. Moreover, compared with the controller algorithm in [18], there are two main advantages of the PID control design proposed here. First, to meet the requirement of practical modelling and application, we introduce a general PID control design for nonlinear stochastic systems subject to non-Gaussian parameters. Second, the controller parameters have been adapted such that the faulty system can satisfy the control requirements.
Chapter 7

Iterative Fault Tolerant Control Based on Stochastic Distribution

7.1 Introduction

In the previous chapter, a new FTC design to general nonlinear continuous-time stochastic systems was developed. However, most of the nonlinear control laws are implemented as digital controllers in reality. Motivated by this fact, a new design of a FTC-based an adaptive, fixed-structure PI controller, with constraints on the state vector for nonlinear discrete-time system subject to stochastic non-Gaussian disturbance is studied. The objective of the reliable control algorithm scheme is to design a control signal such that the actual PDF of the system is made as close as possible to a desired PDF, and make the tracking performance converge to zero, not only when all components are functional but also in case of admissible faults. An LMI-based FTC method is presented to ensure that the fault can be estimated and compensated for. An ILC scheme is used to improve the tracking performance in the batch direction, taking advantage of the repetitive nature of batch processes.

Similarly to in the previous chapter, a RBFNN is used to approximate the output PDF of the system. Thus, the aim of the output PDF control will be a RBF weight control with an adaptive tuning of the basis function parameters. The key issue here is to divide the control horizon into a number of equal time intervals called batches. Within each interval, there are a fixed number of sample points. The design procedure is divided into two main algorithms, within each batch, and between any two adjacent batches. A P-type ILC law is employed to tune the parameters of the RBFNN so that the PDF tracking error decreases along with the batches.
Sufficient conditions for the proposed fault tolerance are expressed as LMIs. An analysis of the ILC convergence is carried out. Finally, the effectiveness of the proposed method is demonstrated with an illustrated example.

7.2 ILC-based PDF Control

To begin this section, a brief history of some significant developments in the field of ILC presented. The term Iterative Learning Control was first presented in [127] and discussed in more detail in [128,129]. Traditional iterative learning controllers have been developed over the past decade for nonlinear systems with nonlinearities satisfying the global Lipschitz continuous condition. The application of ILC to nonlinear systems proved to be a good way of improving the performance of these systems, which has led the development of different methods dealing with nonlinear dynamics [130–132]. Examples of systems that operate in a repetitive manner include robot arm manipulators and chemical batch processes. In each of these tasks, the system is required to perform the same action over and over again.

By using information from previous repetitions, a suitable control action can be found iteratively. A classical control law is of the form:

$$u_k(i) = u_{k-1}(i) + \lambda J_{k-1}(i)$$  \hspace{1cm} (7.1)

where $i$ represents the time instant which satisfies $0 \leq i \leq m$. $m$ is the total number of time samples within a batch. $J(i)$ stands for a function that is related to the tracking performance index, and $\lambda$ is a learning rate gain, which is chosen so that the iterative control law is convergent.

In the stochastic control area, the ILC has been applied successfully to nonlinear stochastic systems [133]. Some effort has been made to apply ILC in the design of the output PDF in order to control its shape [13,114]. As shown in Fig (7.1), the control horizon has been divided into a number of equal time-domain intervals called batches indexed by $k = (1, 2, \ldots)$, and these batches are specified by $[(k-1)(N + \Delta N), (k)(N + \Delta N)]$. Within each batch there are a fixed number of sample points $N$ which is considered as the batch length and should be large enough so that the system reaches the steady state within each batch; $\Delta N$ is the time period between adjacent batches. The design procedure is divided into two main algorithms, within each batch and between any two adjacent batches. Within batches, fixed basis functions are used to generate the required control.
Then, the control law is implemented in the stochastic system in such a way that the closed-loop system is stable. Meanwhile, between adjacent batches the RBF parameters are updated to ensure the measured output PDF is closer to the desired PDF within the next batch.

Figure 7.1: The ILC-based output PDF control scheme

7.3 Model Representation

In this section, this approach is different from the results in [99], a discrete-time RBF neural networks square root model is introduced to approximate the output PDFs, and then formulate a discrete-time nonlinear model for the weighting vectors.

Consider $u(k) \in R^{r}$ as the input of a discrete-time dynamic stochastic system, $y(k) \in [a, b]$ as the output, and $F(k)$ as the fault. At sample time $k$, $y(k)$ can be described by its PDF $\gamma(y, u(k))$, which is defined by

$$P(a \leq y(k) < \xi) = \int_{a}^{\xi} \gamma(y, u(k), F(k))dy$$

where $P(a \leq y(k) < \xi, u(k))$ denotes the probability of output variable $y$ lying between $a$ and $\xi$ when the control is applied to the system.

It is assumed that the PDF is measurable and defined on a known interval $[a, b]$. The well-known RBF neural networks can be used to approximate the square root of the output PDF as follows [7].

$$\sqrt{\gamma(y, u(k), F(k))} = R(y)V(k) + r_{n}(y)h(V(k)) + \omega(y, u(k), F(k))$$  (7.2)
where $\gamma(y, u(k), F(k))$ is the output measured PDF.

$$R(y) = [r_1(y), r_2(y), \ldots, r_{n-1}(y)]$$

$$V(k) = [v_1(k), v_2(k), \ldots, v_{n-1}(k)]^T$$

$$\Lambda_4 = \Lambda_2^T \Lambda_2 - \Lambda_3 \Lambda_1$$

$$\Lambda_1 = \int_a^b R^T(y)R(y)dy$$

$$\Lambda_2 = \int_a^b R^T(y)r_n(y)dy$$

$$\Lambda_3 = \int_a^b r_n^2(y)dy$$

and

$$h(V(k)) = \frac{1}{\Lambda_3} (-\Lambda_2 V(k) + \sqrt{V^T(k)\Lambda_4 V(k)})$$

$F(k)$ is supposed to be an actuator fault, to be diagnosed and compensated. Term $\omega(y, u(k), F(k))$ represents the model uncertainties or the error term on the approximation of PDFs. In addition, $\omega(y, u(k), F(k))$ must satisfy the following condition [7]:

$$|\omega(y, u(k), F(k))| \leq \delta$$

where $\delta > 0$ is a known positive constant. In (7.2), $R(y)$ and $V(k)$ are the activation function and weight element corresponding to RBF neural network used for PDF modelling, respectively. Similar to [13, 123], the RBF activation functions are chosen as of Gaussian shapes and expressed as follows.

$$r_i(y) = \exp \left( -\frac{(y_j - \mu_i)^2}{2\sigma_i^2} \right)$$

In (7.4), the nonlinear function $h(V(k))$ should satisfy the following Lipschitz condition for any $V_1(k), V_2(k)$ and a known matrix $U_1$.

$$\|h(V_1(k)) - h(V_2(k))\| \leq \|U_1(V_1(k) - V_2(k))\|$$

### 7.3.1 Nonlinear Dynamic Weight Model

In many cases, the dynamic relation between the input and the output PDFs can be transformed into dynamic relation between the control input and the weights of the RBFs neural network approximation to the output PDFs. In this section,
the following discrete-time nonlinear weighting model will be used
\[
x(k+1) = Ax(k) + Bu(k) + Gg(x(k)) + DF(k) \\
V(k) = Ex(k)
\] (7.7)

where \(x(k) \in \mathbb{R}^n\) is the state vector, and \(u(k) \in \mathbb{R}^r\) is the measurable input vector. Moreover, \(A, B, G, D\) and \(E\) represent the identified coefficient matrices of the weight system with suitable dimensions. \(g(x(k))\) is a nonlinear vector function that stands for the nonlinear dynamics of the model, and is supposed to satisfy \(g(0) = 0\), and the following Lipschitz condition.
\[
\|g(x_1(k)) - g(x_2(k))\| \leq \|U_2(x_1(k) - x_2(k))\| \tag{7.8}
\]
for any \(x_1(k)\) and \(x_2(k)\), where \(U_2\) is a known matrix. \(F(k)\) is an actuator fault to be estimated and rejected with \(\|F(k)\| \leq b (b > 0\) is a constant number).

With model (7.7), equation (7.2) can be written as a nonlinear function of \(x(k)\) as follows:
\[
\sqrt{\gamma(y,u(k),F(k))} = R(y)Ex(k) + r_n(y)h(Ex(k)) + \omega(y,u(k),F(k)) \tag{7.9}
\]

Different from the models considered in [99, 116], the proposed discrete-time square root RBFNN model is more practical and better suited to digital control.

### 7.4 Fault Detection

In order to detect the fault based on the changes of PDFs, the following nonlinear observer is considered:

\[
\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + Gg(\hat{x}(k)) + L\epsilon(k) \\
\epsilon(t) = \int_a^b \mu(y)(\sqrt{\gamma(y,u(k),F)}) - \sqrt{\hat{\gamma}(y,u(k)))} dy \\
\sqrt{\hat{\gamma}(y,u(k))} = R(y)E\hat{x}(k) + h(Ex(k))r_n(y)
\] (7.10)

where \(\hat{x}(k) \in \mathbb{R}^n\) is the estimated state, \(L \in \mathbb{R}^{n \times p}\) is the filter gain to be determined. Residual \(\epsilon(k)\) is formulated as an integral of the difference between the measured PDFs and the estimated ones, where, \(\mu(y) \in \mathbb{R}^{p \times 1}\) is a pre-specified weighting vector.
Denoting the estimation error as $e(k) = x(k) - \hat{x}(k)$, the dynamic of the estimation error will be expressed as

$$e(k + 1) = (A - L\Gamma_1)e(k) + [Gg(x(k)) - Gg(\hat{x}(k))]$$

$$- L\Gamma_2[h(Ex(k)) - h(E\hat{x}(k))] - L\Delta(k) + DF(k)$$

(7.11)

where

$$\Gamma_1 = \int_a^b \mu(y)R(y)Edy$$
$$\Gamma_2 = \int_a^b \mu(y)r_n(y)dy$$
$$\Delta(k) = \int_a^b \mu(y)\omega(y, u(k))dy$$

(7.12)

It can be seen that

$$\epsilon(k) = \Gamma_1 e(k) + \Gamma_2[h(Ex(k)) - h(E\hat{x}(k))] + \Delta(k)$$

From $|\omega(y, u(k), F)| \leq \delta$, it can be verified that

$$\|\Delta(k)\| = \|\int_a^b \mu(y)\omega(y, u(k))dy\| \leq \tilde{\delta}$$

(7.13)

where $\tilde{\delta} = \delta \|\int_a^b \mu(y)dy\|$.

**Theorem 9.** For the parameter $\lambda_i > 0 (i = 1, 2)$, if there exist matrices $P > 0$, and $R$ satisfying

$$\Psi = \begin{bmatrix}
-P + \lambda_2^2 U_2^T U_2 + \lambda_2^2 E^T U_1 U_1^T E & 0 & 0 & A^T P - \Gamma_1^T R^T \\
0 & -\lambda_2^2 I & 0 & G^T P \\
0 & 0 & -\lambda_2^2 I & \Gamma_2^T R^T \\
PA - R\Gamma_1 & PG & -R\Gamma_2 & -P
\end{bmatrix} < 0$$

(7.14)

then in the absence of fault, the error dynamic system with gain $L = P^{-1}R$ is stable and the error satisfies $\lim_{k \to \infty} e(k) = 0$.


**Proof:** For this purpose, the following Lyapunov function is considered.

\[
\Phi(k) = e^T(k)Pe(k) + \lambda_2 \sum_{i=1}^{k-1} \left[ ||U_1Ee(i)||^2 - ||h(Ex(i)) - h(E\hat{x}(i)||^2 \right] \\
+ \lambda_2 \sum_{i=1}^{k-1} \left[ ||U_2e(i)||^2 - \|g(x(i)) - g(\hat{x}(i)||^2 \right]
\]

In the absence of \(F(k)\), along with (7.11) it can be verified that

\[
\Delta \Phi = \Phi(k+1) - \Phi(k) \\
= e^T(k+1)Pe(k+1) - e^T(k)Pe(k) + \lambda_2 \left[ ||U_1Ee(k)||^2 - ||h(Ex(k)) - h(E\hat{x}(k)||^2 \right] \\
+ \lambda_2 \left[ ||U_2e(k)||^2 - \|g(x(k)) - g(\hat{x}(k)||^2 \right] \\
= S_k^T \Psi_1 S_k + 2S_k^T \left[ -(A - L\Gamma_1)^T PL \right. \\
- G^TPL \\
\left. \Gamma_2^T L^TPL \right] \Delta(k) + \Delta^T(k)L^TPL\Delta(k) < 0
\]

(7.16)

where

\[
\Psi_1 = \begin{bmatrix}
\Psi_2 & (A - L\Gamma_1)^T PG & -(A - L\Gamma_1)^T PL \Gamma_2 \\
* & -\lambda_2 I + G^T PG & -G^TPL \Gamma_2 \\
* & * & -\lambda_2 I + \Gamma_2^T L^TPL \Gamma_2
\end{bmatrix}
\]

(7.17)

\[
S_k^T = \left[ e^T(k), (g(x(k)) - g(\hat{x}(k))^T, (h(Ex(k)) - h(E\hat{x}(k))^T \right]
\]

\[
\Psi_2 = \left[ (A - L\Gamma_1)^T P(A - L\Gamma_1) - P + \lambda_2^2 U_2^T U_2 + \lambda_2^2 E^TU_1^TU_1 \right]
\]

Denote \(R = PL\), it can be seen that

\[
\Delta \Phi = S_k^T \Psi_1 S_k + 2S_k^T \left[ -(A - L\Gamma_1)^T R \right. \\
- G^TR \\
\left. \Gamma_2^T R^TP^{-1} R \right] \Delta(k) + \Delta^T(k)R^TP^{-1} R\Delta(k) < 0
\]

(7.18)

By using the Schur complement formula, (7.17) can be further reduced to

\[
\Psi_5 = \begin{bmatrix}
\Psi_3 & \Psi_4^T \\
\Psi_4 & -P
\end{bmatrix} < 0
\]

(7.19)

where

\[
\Psi_3 = \begin{bmatrix}
-P + \lambda_2^2 U_2^T U_2 + \lambda_2^2 E^TU_1^TU_1 \ E & 0 & 0 \\
* & -\lambda_2^2 I & 0 \\
* & * & -\lambda_2^2 I
\end{bmatrix}
\]

(7.20)
and

\[ \Psi_4 = \begin{bmatrix} PA - R\Gamma_1 & PG - R\Gamma_2 \end{bmatrix} \]  

(7.21)

which is equivalent to (7.14). If (7.14) holds, a positive scalar exists \( \varrho \) so that

\[ \Psi \leq -\varrho I. \]

Thus, it can be seen that

\[ \Delta \Phi \leq -\varrho \| S_k \|^2 - 2\| \tilde{A}^T R \| \| \Delta(k) \| \| S_k \| + \| \Delta(k) \|^2 \| R^T P^{-1} R \| \]

(7.22)

where

\[ \tilde{A} = \begin{bmatrix} A - L\Gamma_1 & PG - R\Gamma_2 \end{bmatrix} \]  

(7.23)

It can be shown that

\[ \| S_k \| \geq \tilde{\varrho}^{-1} \left( \| \tilde{A}^T R \| + \sqrt{\| \tilde{A}^T R \|^2 + \varrho \| R^T P^{-1} R \|} \right) \]

(7.24)

which implies

\[ \| S_k \| = \tilde{\varrho} = \max \left\{ \| S_k(1) \|, \tilde{\varrho}^{-1} \left( \| \tilde{A}^T R \| + \sqrt{\| \tilde{A}^T R \|^2 + \varrho \| R^T P^{-1} R \|} \right) \right\} \]

(7.25)

This means that the error system in (7.11) is asymptotically stable under \( \Psi < 0 \). Because \( \lim_{k \to \infty} \epsilon(k) = 0 \), equations (7.6) and (7.8) guarantee that \( \lim_{k \to \infty} (h(Ex(k)) - h(\hat{E}\hat{x}(k))) = 0 \) and \( \lim_{k \to \infty} (g(x(k)) - g(\hat{x}(k))) = 0 \), which implies that \( \lim_{k \to \infty} \Delta \Phi(k) = 0 \).

Theorem 9 provides a sufficient criterion for the stability of the error system in the absence of the fault, which presents a necessary condition for fault detection. In order to detect \( F \), the following theorem should be considered.

**Theorem 10.** For the parameter \( \lambda_i > 0 (i = 1, 2) \), if there exit matrices \( P > 0 \), and \( R \) satisfying satisfying (6.14), then fault \( F \) can be detected by the following criterion

\[ \| \epsilon(k) \| > \tilde{\varrho} = \tilde{\varrho}_0 (\| \Gamma_1 \| + \| \Gamma_2 \| \| U_1 E \|) + \tilde{\delta} \]

(7.26)

which means that (7.26) implies \( F \neq 0 \)

Once fault is detected, it needs to be estimated, which follows an adaptive fault diagnosis algorithm in the next section.
Chapter 7. Iterative Fault tolerant control

7.5 Fault Diagnosis

Once the fault is detected, the fault value must be estimated. For this purpose, the following observer is considered:

\[ \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + Gg(\hat{x}(k)) + L\epsilon(k) + D\hat{F}(k) \]  

(7.27)

\[ \sqrt{\gamma}(y,u(k)) = R(y)E\hat{x}(k) + r_n(y)h(E\hat{x}(k)) \]

\[ \hat{F}(k+1) = \Upsilon_1\hat{F}(k) + \Upsilon_2\epsilon(k) \]

where \( \hat{F}(k) \) is the estimation of \( F(k) \). \( \Upsilon_1 (-I < \Upsilon_1 < I) \) and \( \Upsilon_2 \) are the learning operators to be determined together with \( L \) by the diagnosis algorithm in (7.27).

Denoting \( \bar{F}(k) = F(k) - \hat{F}(k) \) and \( e(k) = x(k) - \hat{x}(k) \). The dynamic of the estimation error will be expressed as

\[ e(k+1) = (A - L\Gamma_1)e(k) + [Gg(x(k)) - Gg(\hat{x}(k))] - L\Gamma_2[h(Ex(k)) - h(E\hat{x}(k))] \]

\[ - L\Delta(k) + D\bar{F}(k) \]  

(7.28)

and

\[ \bar{F}(k+1) = F(k+1) - \hat{F}(k-1) \]

\[ = F(k+1) + \Upsilon_1\hat{F}(k) - \Upsilon_2\epsilon(k) \]

\[ = F(k+1) - \Upsilon_1F(k) + \Upsilon_1\hat{F}(k) - \Upsilon_2\epsilon(k) \]

\[ = \Delta F(k) + \Upsilon_1\hat{F}(k) - \Upsilon_1\Gamma_1e(k) - \Upsilon_2\Gamma_2[h(Ex(k)) - h(E\hat{x}(k))] - \Upsilon_2\Delta(k) \]  

(7.29)

where

\[ \Delta F(k) = F(k+1) - \Upsilon_1F(k) \]

**Theorem 11.** For the parameter \( \lambda_i > 0 (i = 1, 2) \), if there exist matrices \( P > 0 \), \( R \), and \( \Upsilon_i > 0 (i = 1, 2) \) satisfying

\[ \Psi = \begin{bmatrix} \Psi_1 & 0 & \Gamma_1^T \Upsilon_2 \Gamma_2 & \Gamma_1^T \Upsilon_2 \Upsilon_1 & (A - L\Gamma_1)^TP \\ * & -\lambda_2^2I & 0 & 0 & G^TP \\ * & * & -\lambda_2^2I + \Gamma_1^T \Upsilon_2 \Gamma_2 & -\Gamma_1^T \Upsilon_2 \Upsilon_1 & \Gamma_1^T \Upsilon_2 R^T \\ * & * & * & -I + \Upsilon_1^T \Upsilon_1 & D^TP \\ * & * & * & * & -P \end{bmatrix} < 0 \]  

(7.30)

where
\[ \bar{\Psi}_1 = -P + \lambda_2^2 U_2^T U_2 + \lambda_1^2 E^T U_1^T U_1 E + \Gamma_1^T \Upsilon_2^T \Upsilon_2 \Gamma_1 \]

then the filtering gain \( L = P^{-1} R \), the error dynamic system is stable and the error satisfies

\[
\| \tilde{S}_k \| \leq \max \left\{ \| \tilde{S}_k(1) \|, \phi_0^{-1} \left( \| \Xi_1 \| + \sqrt{\| \Xi_1 \|^2 + \theta_1 \| \Xi_2 \|} \right) \right\} \quad (7.31)
\]

**Proof:** For this purpose, the following Lyapunov function is considered.

\[
\Phi_1(k) = e^T(k) P e(k) + \lambda_1^2 \sum_{i=1}^{k-1} \left[ \| U_1 E e(i) \|^2 - \| h(x(e(i))) - h(\hat{x}(i)) \|^2 \right]
+ \lambda_2^2 \sum_{i=1}^{k-1} \left[ \| U_2 e(i) \|^2 - \| g(x(i)) - g(\hat{x}(i)) \|^2 \right] + \bar{F}^T(k) \bar{F}(k) \quad (7.32)
\]

It can be verified that

\[
\Delta \Phi_1 = \Phi_1(k+1) - \Phi_1(k)
= e^T(k+1) P e(k+1) + \bar{F}^T(k+1) \bar{F}(k+1) - e^T(k) P e(k) - \bar{F}^T(k) \bar{F}(k)
+ \lambda_1^2 \left[ \| U_1 E e(k) \|^2 - \| h(x(k)) - h(\hat{x}(k)) \|^2 \right] + \lambda_2^2 \left[ \| U_2 e(k) \|^2 - \| g(x(k)) - g(\hat{x}(k)) \|^2 \right]
= \tilde{S}_k^T \bar{\Psi}_2 \tilde{S}_k + \Delta F^T(k) \Delta F(k) - 2 \Delta F^T(k) \Upsilon_2 \Delta(k) + \Delta^T(k) \left( L^T P L + \Upsilon_2^T \Upsilon_2 \right) \Delta(k)
- 2 \tilde{S}_k^T \begin{bmatrix}
(A - L \Gamma_1)^T P L & -\Gamma_1^T \Upsilon_2^T \\
G^T P L & 0 \\
-\Gamma_2^T L^T P L & -\Gamma_2^T \Upsilon_2^T \\
D^T P L & \Upsilon_2^T
\end{bmatrix} \Delta(k) + 2 \tilde{S}_k^T \begin{bmatrix}
-\Gamma_1^T \Upsilon_2^T \\
0 \\
-\Gamma_2^T \Upsilon_2^T \\
\Upsilon_2^T
\end{bmatrix} \Delta F(k) < 0
\quad (7.33)
\]

where

\[
\bar{\Psi}_2 = \begin{bmatrix}
\bar{\Psi}_3 & (A - L \Gamma_1)^T P G & \bar{\Psi}_4 & \bar{\Psi}_5 \\
* & -\lambda_2^2 I + G^T P G & -G^T P L \Gamma_2 & G^T P D \\
* & * & \bar{\Psi}_6 & -\Gamma_2^T L^T P D - \Gamma_2^T \Upsilon_2^T \Upsilon_1 \\
* & * & * & D^T P D - I + \Upsilon_2^T \Upsilon_1
\end{bmatrix}
\quad (7.34)
\]

\[
\tilde{S}_k^T = \begin{bmatrix}
e^T(k), (g(x(k)) - g(\hat{x}(k)))^T, (h(x(k)) - h(\hat{x}(k)))^T, \bar{F}^T(k)
\end{bmatrix}
\]

\[
\bar{\Psi}_3 = (A - L \Gamma_1)^T P (A - L \Gamma_1) - P + \lambda_2^2 U_2^T U_2 + \lambda_1^2 E^T U_1^T U_1 E + \Gamma_1^T \Upsilon_2^T \Upsilon_2 \Gamma_1
\]

\[
\bar{\Psi}_4 = -(A - L \Gamma_1)^T P L \Gamma_2 + \Gamma_1^T \Upsilon_2^T \Upsilon_2 \Gamma_2
\]

\[
\bar{\Psi}_5 = (A - L \Gamma_1)^T P D - \Gamma_1^T \Upsilon_2^T \Upsilon_1
\]
\[
\bar{\Psi}_6 = -\lambda_2^2 I + \Gamma_2^T L^T P L \Gamma_2 + \Gamma_2^T \Upsilon_2 \Upsilon_2 \Gamma_2
\]

Denote \( R = P L \), it can be seen that

\[
\Delta \Phi_1 = \bar{S}_k^T \bar{\Psi}_2 \bar{S}_k + \Delta F^T(k) \Delta F(k) - 2 \Delta F^T(k) \Upsilon_2 \Delta(k) + \Delta^T(k) (R^T P^{-1} R + \Upsilon_2^T \Upsilon_2) \Delta(k)
\]

\[
-2\bar{S}_k^T \left\{ \begin{bmatrix} (A - L \Gamma_1)^T R \\ G^T R \\ -\Gamma_2^T L^T R \\ D^T R \end{bmatrix} + \begin{bmatrix} -\Gamma_1^T \Upsilon_2 \\ 0 \\ -\Gamma_2^T \Upsilon_2 \\ \Upsilon_1^T \end{bmatrix} \right\} \Delta(k) + 2\bar{S}_k^T \begin{bmatrix} -\Gamma_1^T \Upsilon_2 \\ 0 \\ -\Gamma_2^T \Upsilon_2 \\ \Upsilon_1^T \end{bmatrix} \Delta F(k) < 0
\]

By using the Schur complement formula, (7.34) can be further reduced to

\[
\bar{\Psi}_9 = \begin{bmatrix} \bar{\Psi}_7 & \bar{\Psi}_7^T \\ \bar{\Psi}_8 & -P \end{bmatrix} < 0
\]

where

\[
\bar{\Psi}_7 = \begin{bmatrix} \bar{\Psi}_1 & 0 & \Gamma_1^T \Upsilon_2 \Upsilon_2 \Gamma_2 & \Gamma_1^T \Upsilon_2 \Upsilon_1 \\ * & -\lambda_2^2 I & 0 & 0 \\ * & * & -\lambda_2^2 I + \Gamma_2^T \Upsilon_2 \Upsilon_2 \Gamma_2 & -\Gamma_2^T \Upsilon_2 \Upsilon_1 \\ * & * & * & -I + \Upsilon_1^T \Upsilon_1 \end{bmatrix}
\]

and

\[
\bar{\Psi}_8 = \begin{bmatrix} PA - R \Gamma_1 & PG & -R \Gamma_2 & PD \end{bmatrix}
\]

which is equilevent to (7.30). If (7.30) holds, a positive scalar exists \( \varrho_1 \) so that \( \bar{\Psi} \leq -\varrho_1 I \). Thus, it can be seen that

\[
\Delta \Phi_1 \leq -\varrho_1 \| S_k \|^2 - 2 \left\{ \| \tilde{A}^T R \| + \| \tilde{T}^T \| \| \Upsilon_2 \| \right\} \| \Delta(k) \| \| S_k \| + 2 \| \tilde{T}^T \| \| \Delta F(k) \| \| S_k \| \\
+ \| \Delta F(k) \|^2 - 2 \| \Delta F(k) \| \| \Upsilon_2 \| \| \Delta(k) \| + \| \Delta(k) \|^2 \left\{ \| R^T P^{-1} R \| + \| \Upsilon_2 \|^2 \right\}
\]

\[
\leq -\varrho \| S_k \|^2 - 2 \| \Xi_1 \| \| S_k \| + \Xi_2
\]

where

\[
\tilde{A} = \begin{bmatrix} A - L \Gamma_1 & PG & -R \Gamma_2 & PD \end{bmatrix}
\]

\[
\tilde{T} = \begin{bmatrix} -\Upsilon_2 \Gamma_1 & 0 & -\Upsilon_2 \Gamma_2 & \Upsilon_2 \end{bmatrix}
\]

\[
\Xi_1 = \delta \left\{ \| \tilde{A}^T R \| + \| \tilde{T}^T \| \| \Upsilon_2 \| \right\} - \| \tilde{T}^T \| \| \Delta F(k) \|
\]

and
\[
\Xi_2 = \|\Delta F(k)\|^2 - 2\|\Delta F(k)\|\|\Upsilon_2\|\delta + \tilde{s}^2 \left\{\|R^TP^{-1}R\| + \|\Upsilon_2\|^2\right\}
\]

It can be shown that
\[
\|\tilde{S}_k\| \geq \varrho_1^{-1} \left( \|\Xi_1\| + \sqrt{\|\Xi_1\|^2 + \varrho_1 \|\Xi_2\|} \right)
\]

which implies
\[
\|\tilde{S}_k\| \leq \max \left\{ \|\tilde{S}_k(1)\|, \varrho_1^{-1} \left( \|\Xi_1\| + \sqrt{\|\Xi_1\|^2 + \varrho_1 \|\Xi_2\|} \right) \right\}
\]

This means that the error system in (7.28) is asymptotically stable under \(\bar{\Psi} < 0\).

## 7.6 Fault Tolerant Control

### 7.6.1 Problem Formulation

Similar to [123], consider \(u_k(i) \in \mathbb{R}^r\) is the input of a discrete-time dynamic stochastic system at the \(i^{th}\) time instant within the \(k^{th}\) batch, \(y \in [a, b]\) is the output. At sample time \(k\), \(y\) can be described by its PDF \(\gamma_k(y, u_k(i))\). Assuming that \([a, b]\) is known and the probability density function is continuous and bounded within each iteration. The well-known RBF neural networks can be used to approximate the square root of the output PDF as follows.

\[
\sqrt{\gamma(y, u_k(i))} = \sum_{l=1}^{n} \nu_{l,k}(u_k(i)) r_{l,k}(y) \tag{7.42}
\]

where \(\gamma(y, u_k(i))\) is the output PDF measured at the \(i^{th}\) time instant within the \(k^{th}\) batch. Also \(\nu_{l,k}(i)\) is the \(l^{th}\) weight element of the RBFNN in the \(i^{th}\) (\(i = 1, 2, \ldots, m\)) sample time within the \(k^{th}\) batch, and \(r_{l,k}(y)\) denotes the \(l^{th}\) \(i^{th}\) (\(l = 1, 2, \ldots, n\)) RBF activation function within \(k^{th}\) batch. Assume \(n\) and \(k\) represent the number of RBFs and the batch length, respectively. The RBF activation functions are expressed as follows [114].

\[
r_{l,k}(y) = e^{\lambda_1} \left( -\frac{(y_j - \mu_{l,k})^2}{2\sigma_{l,k}^2} \right) \tag{7.43}
\]

where \(\mu_{l,k}, \sigma_{l,k}\) are the centres and widths of the RBF basis functions within \(k^{th}\) batch, respectively.

Different from [99,116], the output PDF described in (7.42) can be re-written
as the following vector form.

\[
\sqrt{\gamma(y, u_k(i)))} = \begin{bmatrix} R_k(y) \\ r_{n,k}(y) \end{bmatrix} \begin{bmatrix} V_k(i) \\ \nu_{n,k}(i) \end{bmatrix} \tag{7.44}
\]

where

\[
R_k(y) = [r_{1,k}(y), r_{2,k}(y), \ldots, r_{n-1,k}(y)]
\]

\[
V_k(i) = [\nu_{1,k}(i), \nu_{2,k}(i), \ldots, \nu_{n-1,k}(i)]^T.
\]

Since, \(\gamma(y, u_k(i)))\) is a probability density function, it must satisfy the following integral constraint [7].

\[
\int_a^b (\sqrt{\gamma(y, u_k(i)))})^2 dy = 1 \tag{7.45}
\]

By substituting \(\gamma(y, u_k(i)))\) with (7.44) and solving the equation for \(\nu_{n,k}(i)\) similar to [7, 100], it can be shown that the following state constraint shall be satisfied within each batch to guarantee that the measured \(\gamma(y, u_k(i)))\) is a probability density function [7].

\[
V_k^T(i)Q_{ab,k}V_k(i) \leq 1 \tag{7.46}
\]

where

\[
Q_{ab,k} = b_{1,k} - b_{a,k}^{-1}b_{2,k}^Tb_{2,k}^T
\]

\[
b_{1,k} = \int_a^b R_k^T(y)R_k(y)dy
\]

\[
b_{2,k} = \int_a^b r_{n,k}(y)R_k(y)dy
\]

\[
b_{3,k} = \int_a^b r_{n,k}^2(y)dy
\]

It has been proven in [7] that \(Q_{ab,k}\) is always positive definite. When (7.46) holds, it can be seen that \(\nu_{n,k}\) can be represented as a known nonlinear function of \(V_k\) called \(h(V_k)\). Thus the output PDF described in (7.42) can be re-written as follows.

\[
\sqrt{\gamma(y, u_k(i)))} = R_k(y)V_k(i) + r_{n,k}(y)h(V_k) \tag{7.47}
\]

In (7.47), the nonlinear function \(h(V_k)\) should satisfy the following Lipschitz condition

\[
\|h(V_{1,k}(i)) - h(V_{2,k}(i))\| \leq \|\bar{U}_1(V_{1,k}(i) - V_{2,k}(i))\| \tag{7.48}
\]

where \(\bar{U}_1\) is a known matrix.
Thus the dynamic model between the output PDF and the RBF neural network weight vectors in the presence of the actuator fault will be established as follows.

\[
V_k(i + 1) = A_k V_k(i) + B_k u_k(i) + G g(V_k(i)) + DF_k(i)
\]

\[
\sqrt{\gamma(y, u_k(i))} = R_k(y) V_k(i) + r_{n,k}(y) h(V_k(i))
\]  

(7.49)

Similar to [99], the nonlinear dynamics of the model in (7.49) is supposed to satisfy the following Lipschitz condition.

\[
\|g(V_{1,k}(i)) - g(V_{2,k}(i))\| \leq \|U_2(V_{1,k}(i) - V_{2,k}(i))\|
\]

(7.50)

where \(U_2\) is a known matrix.

### 7.6.2 Controller Design

A generalized PI controller with tuneable coefficients is considered as adaptive controller in this work as follows

\[
\xi_k(i) = \xi_k(i - 1) + T_s e_k(i - 1)
\]

\[
u_k(i) = K_{P,k} e_k(i) + K_{I,k} \xi_k(i)
\]

(7.51)

where \(e_k(i) = V_g - V_k(i)\) represents the dynamical weight tracking error, and \(T_s\) is the sampling time. Substituting (7.51) in (7.49) yields the following closed-loop system for the weight control loop with the \(k^{th}\) batch:

\[
M_k(i + 1) = A_k M_k(i) + B_k V_g + G g(M_k(i)) + DF_k(i)
\]

(7.52)

where

\[
M_k(i) = \begin{bmatrix} V_k(i) \\ \xi_k(i) \end{bmatrix}, \quad A_k = \begin{bmatrix} A_k - B_k K_{P,k} & B_k K_{I,k} \\ -T_s I & I \end{bmatrix}
\]

\[
\tilde{B}_k = \begin{bmatrix} B_k K_{P,k} \\ T_s I \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 0 \\ D \end{bmatrix}, \quad g(M_k(i)) = \begin{bmatrix} g(V_k(i)) \\ 0 \end{bmatrix}
\]

\[
\tilde{A}_k = \begin{bmatrix} A_k & 0 \\ -T_s I & I \end{bmatrix}, \quad \tilde{B}_k = \begin{bmatrix} B_k \\ 0 \end{bmatrix}
\]

Denote
The following theorem represents the solvability conditions of the general PI controller

**Theorem 12.** Within the $k^{th}$ batch, for the parameter $\lambda$, if there exist matrices $P > 0$, and $R$ satisfying the following LMIs for any initial condition $M(0)$ satisfying constraint (7.46)

$$\tilde{\Psi}_k = \begin{bmatrix}
-P_k & 0 & 0 & 0 & 0 & P_k^T \tilde{A}_k^T + R^T \tilde{B}_k^T & \lambda P_k \tilde{U}_2^T \\
0 & -\lambda^2 I & 0 & 0 & 0 & \tilde{G}^T & 0 \\
0 & 0 & -I + \Upsilon_1^T \Upsilon_1 & \Upsilon_1^T & -\Upsilon_1^T \Upsilon_2 & \tilde{D}^T & 0 \\
0 & 0 & \Upsilon_1 & I & -\Upsilon_2^T & 0 & 0 \\
0 & 0 & -\Upsilon_2^T \Upsilon_1 & -\Upsilon_2 & \Upsilon_2^T \Upsilon_2 & 0 & 0 \\
\tilde{A}_k P_k + \tilde{B}_k R & \tilde{G} & \tilde{D} & 0 & 0 & -P_k & 0 \\
\lambda \tilde{U}_2 P_k & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0$$

and

$$2\tilde{\alpha} \|Q_{ab}\| \|\tilde{B}_k V_g\|^2 \leq \lambda_{\min}(P_k)$$

then, the closed loop system is stable with $\lim_{k \to \infty} e(i) = 0$, and the controller parameters can be calculated by using

$$\begin{bmatrix}
K_P & K_I
\end{bmatrix} = RP^{-1}$$

**Proof:** For this purpose, the following Lyapunov function is considered.

$$\Phi_3(i) = M_k^T(i) P^{-1} M_k(i) + \lambda^2 \sum_{j=1}^{i-1} \left[ \|\tilde{U}_2 M_k(i)\|^2 - \|g(M_k(i))\|^2 \right] + \tilde{F}_k^T(i) \tilde{F}_k(i)$$

(7.55)

It can be verified that

$$\Delta \Phi_3 = \Phi_3(M_k(i + 1), i + 1) - \Phi_3(M_k(i), i)$$

$$= \tilde{M}_k^T(i) \tilde{\Psi}_{1,k} \tilde{M}_k(i) + 2\tilde{M}_k(i) \tilde{N}_k P_k^{-1} \tilde{B}_k V_g + V_g^T \tilde{B}^T P^{-1} \tilde{B} V_g$$

(7.56)

where

$$\tilde{\Psi}_{1,k} = \begin{bmatrix}
\tilde{N}_k & \tilde{A}_k P_k^{-1} \tilde{G} & \tilde{A}_k P_k^{-1} \tilde{D}_k & 0 & 0 \\
\tilde{G}^T P_k T \tilde{A}_k & \tilde{G}^T P_k T \tilde{G} - \lambda^2 I & \tilde{G}^T P_k T \tilde{D}_k & 0 & 0 \\
\tilde{D}_k P_k T \tilde{A}_k & \tilde{D}_k P_k T \tilde{G} & \tilde{N}_k & \Upsilon_1 & -\Upsilon_1^T \Upsilon_2 \\
0 & 0 & \Upsilon_1 & I & -\Upsilon_2^T \\
0 & 0 & -\Upsilon_2^T \Upsilon_1 & -\Upsilon_2 & \Upsilon_2^T \Upsilon_2
\end{bmatrix}$$
\[ \tilde{N}_k = [\tilde{A}_k^T, \tilde{G}_k^T, \tilde{D}_k^T, 0, 0] \]
\[ \tilde{N}_{k,1} = \tilde{A}_k^T P_k^{-1} \tilde{A}_k - P_k^{-1} + \lambda^2 \tilde{U}_2^T \tilde{U}_2 \]
\[ \tilde{N}_{k,2} = \tilde{D}_k^T P_k^{-1} \tilde{D}_k - I + \Upsilon_1^T \Upsilon_1 \]

And

\[ \tilde{M}_k^T(i) = [M_k^T(i), g^T(M_k(i)), \tilde{F}_k^T, \Delta F^T(k), \epsilon(k)] \]

By using the well-know Schur complement formula, (7.56) can be as follows.

\[ \tilde{\Psi}_{2,k} = \begin{bmatrix} -P_k^{-1} + \lambda^2 \tilde{U}_2^T \tilde{U}_2 & 0 & 0 & 0 & 0 & A_k^T \\ 0 & -\lambda^2 I & 0 & 0 & 0 & \tilde{G}^T \\ 0 & 0 & -I + \Upsilon_1^T \Upsilon_1 & \Upsilon_1^T & -\Upsilon_1^T \Upsilon_2 & \tilde{D}^T \\ 0 & 0 & \Upsilon_1 & I & -\Upsilon_2^T & 0 \\ 0 & 0 & -\Upsilon_2^T \Upsilon_1 & -\Upsilon_2 & \Upsilon_2^T \Upsilon_2 & 0 \\ \tilde{A}_k & \tilde{G} & \tilde{D} & 0 & 0 & -P_k \end{bmatrix} < 0 \] (7.57)

By pre-multiplying \( \tilde{\Psi}_{2,k} \) by \( \text{diag} \left( P_k^T, I, I, I, I, I \right) \) and post multiplying it by \( \text{diag} \left( P_k, I, I, I, I, I \right) \) and applying the well-know Schur complement formula, the condition for stability will be as follows.

\[ \tilde{\Psi}_{3,k} = \begin{bmatrix} -P_k & 0 & 0 & 0 & 0 & P_k^T A_k^T & \lambda P_k \tilde{U}_2^T \\ 0 & -\lambda^2 I & 0 & 0 & 0 & \tilde{G}^T & 0 \\ 0 & 0 & -I + \Upsilon_1^T \Upsilon_1 & \Upsilon_1^T & -\Upsilon_1^T \Upsilon_2 & \tilde{D}^T & 0 \\ 0 & 0 & \Upsilon_1 & I & -\Upsilon_2^T & 0 & 0 \\ 0 & 0 & -\Upsilon_2^T \Upsilon_1 & -\Upsilon_2 & \Upsilon_2^T \Upsilon_2 & 0 & 0 \\ \tilde{A}_k P_k & \tilde{G} & \tilde{D} & 0 & 0 & -P_k & 0 \\ \lambda \tilde{U}_2 P_k & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \] (7.58)

By substituting matrices \( \tilde{A} \) and \( \tilde{B} \) into \( \tilde{\Psi}_3, \tilde{\Psi} \) can be obtained.

If (7.53) holds, a positive scalar \( \tilde{\alpha} \) exists so that \( \tilde{\Psi} \leq -\tilde{\alpha} I \). Along with (7.52) it can be verified that

\[ \Delta \Phi_3 \leq -\tilde{\alpha} \| \tilde{M}_k \|^2 + 2 \| \tilde{M}_k \| \| \tilde{N}^T_k P^{-1} BV_g \| + \| V_g \|^2 \| \tilde{B}^T P^{-1} \tilde{B} \| \] (7.59)

It is obvious that the right-hand side of inequality is a second order degree polynomial with respect to \( \| \tilde{M}_k \| \). Denote

\[ \tilde{\sigma} = V_g^T \tilde{B}^T P^{-1} \tilde{B} V_g \] (7.60)
Thus, it can be shown that $\Delta \Phi_3 \leq 0$ holds if
\[
|\dot{M}_k| \geq \alpha^{-1} \left( \| \tilde{N}_k^T P^{-1} \tilde{B} V_g \| + \sqrt{\| \tilde{N}_k^T P^{-1} \tilde{B} V_g \|^2 + \alpha \bar{\sigma}} \right)
\] (7.61)
which implies
\[
\| \dot{M}_k \| \leq \max \left\{ \| M_k(1) \|, \alpha^{-1} \left( \| \tilde{N}_k^T P^{-1} \tilde{B} V_g \| + \sqrt{\| \tilde{N}_k^T P^{-1} \tilde{B} V_g \|^2 + \alpha \bar{\sigma}} \right) \right\}
\] (7.62)
This confirms that the closed-loop system is bounded and internally stable.

Based on (7.52) and (7.55) it can be shown that
\[
\Delta \Phi_3 = \Phi_3(M_k(i + 1), i + 1) - \Phi_3(M_k(i), i)
= \dot{M}_k(i) (\tilde{A}_k^T P_k^{-1} \tilde{S}_k - P_k^{-1}) \dot{M}_k(i) + 2 \dot{M}_k(i) \tilde{N}_k^T P_k^{-1} \tilde{B}_k V_g + V_g^T \tilde{B}^T P^{-1} \tilde{B} V_g
\] (7.63)

It can be verified that $\tilde{A}_k^T P_k^{-1} \tilde{S}_k - P_k^{-1} < -\alpha^{-1} I$ as long as $\tilde{A}_k^T P_k^{-1} \tilde{S}_k - P_k^{-1} < 0$ is guaranteed by (7.53) Thus it can be seen that
\[
\Delta \Phi_3 \leq -\alpha^{-1} \| \dot{M}_k(i) \|^2 + 2 \dot{M}_k(i) \tilde{N}_k^T P_k^{-1} \tilde{B}_k V_g + V_g^T \tilde{B}^T P^{-1} \tilde{B} V_g
\leq -\alpha^{-1} \| \dot{M}_k(i) \|^2 - (P_k^{-1/2} \tilde{N}_k^T \dot{M}_k(i) - P_k^{-1/2} \tilde{B}_k V_g) (\tilde{N}_k \dot{M}_k(i) - P_k^{-1/2} \tilde{B}_k V_g) + 2 V_g^T \tilde{N}_k^T P_k^{-1} \tilde{B}_k V_g
\leq -\alpha^{-1} \| \dot{M}_k(i) \|^2 + 2 \lambda_{\max}(P^{-1}) \| \tilde{B}_k V_g \|^2
\] (7.64)

Denote
\[
\hat{\beta} = 2 \lambda_{\max}(P^{-1}) \| \tilde{B}_k V_g \|^2
\]
Then it can be shown that $\Delta \Phi_3 < 0$ holds if
\[
\| \dot{M}_k(i) \|^2 > \alpha \hat{\beta}
\]
From the constraint (7.46) it can be seen that
\[
V_k^T(i) Q_{ab} V_k(i) < \| V_k(i) \|^2 \| Q_{ab} \|
\leq \| \dot{M}_k(i) \|^2 \| Q_{ab} \|
\leq \alpha \hat{\beta} \| Q_{ab} \| \leq 1
\] (7.65)

Thus, the constraint in (7.54) can be guaranteed by the obtained results from (7.64) and (7.65)
To discuss the system tracking performance, suppose that $\tilde{\varphi}_1(i)$ and $\tilde{\varphi}_2(i)$ are two trajectories of the nonlinear closed-loop system (7.52) corresponding to fixed initial conditions and fault, and $V_g$ is the input. Denote the error between the two trajectories

$$\chi(i) = \tilde{\varphi}_1(i) - \tilde{\varphi}_2(i)$$

with

$$\chi(1) = 0$$

Then, the dynamic of $\chi(i+1)$ can be presented as follows.

$$\chi(i+1) = \tilde{A}_k \chi(i) + \tilde{G} [g(\tilde{\varphi}_1) - g(\tilde{\varphi}_2)]$$  (7.66)

The following Lyapunov function will be considered

$$\Phi_4 (\chi(i), \tilde{\varphi}_1(i), \tilde{\varphi}_2(i), i) = \chi^T(i) P^{-1} \chi(i) + \lambda^2 \sum_{j=1}^{i-1} \left( \|U \chi(i)\|^2 - \|g(\tilde{\varphi}_1) - g(\tilde{\varphi}_2)\|^2 \right)$$  (7.67)

It can be verified that

$$\Delta \Phi_4 = \dot{\chi}^T(i) \dot{\Psi}_{4,k} \dot{\chi}(i) < -\tilde{\alpha} \| \dot{\chi}(i) \|$$  (7.68)

where

$$\dot{\chi}(i) = [\chi^T(i), g(\tilde{\varphi}_1) - g(\tilde{\varphi}_2)]$$

$$\dot{\Psi}_{4,k} = \begin{bmatrix} \tilde{A}_k^T P_k^{-1} \tilde{A}_k - P_k^{-1} + \lambda^2 \tilde{U}_2^T \tilde{U}_2 & \tilde{A}_k^T P_k^{-1} \tilde{G}_k \\ \tilde{G}_k^T P_k^{-1} \tilde{A}_k & \tilde{G}_k^T P_k^{-1} \tilde{G}_k - \lambda^2 \lambda \end{bmatrix}$$

This means that the closed loop system is exponentially stable around $\chi = 0$ neighborhood. Thus, the tracking performance of the system has been satisfied.

### 7.7 Tuning of Radial Basis Function

Similar to [123], the following P-type ILC law will be used to tune the basis function parameters (RBF centres and widths) between any two batches

$$\mu_{l,k} = \mu_{l,k-1} + \Lambda_\mu E_{k-1}$$

$$\sigma_{l,k} = \sigma_{l,k-1} + \Lambda_\sigma E_{k-1}$$  (7.69)

where the performance indices of the $(k-1)^{th}$ batch will be as follows.
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\[ E_{k-1} = [J_{k-1}(1), J_{k-1}(2), ..., J_{k-1}(m)]^T \]

where \( m \) represents the total number of time instants within a batch. \( J_{k-1}(i) \) is the performance at the \( i^{th} \) sampling instant of the \( (k-1)^{th} \) batch, it can be expressed as follows.

\[ J_{k-1}(i) = \int_a^b \left( \sqrt{\gamma(y,u_{k-1}(i))} - \sqrt{g(y)} \right)^2 dy \]

In addition, the learning parameters in (7.69) are defined as

\[ \Lambda_\mu = \alpha_\mu [\lambda_1, \lambda_2, ..., \lambda_m] \]
\[ \Lambda_\sigma = \alpha_\sigma [\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_m] \]  

(7.70)

where \( \lambda, \bar{\lambda} \) are the learning elements, and \( \alpha_\mu, \alpha_\sigma \) are the learning rates to be determined.

### 7.8 Convergence Analysis

Similar to [123], the learning vectors in (7.70) should be selected carefully to ensure the convergence of the ILC-based tuning algorithm between batches. Therefore, the closed loop performance should satisfy the following condition between batches

\[ \frac{F_k}{F_{k-1}} = \frac{\sum_{i=1}^m J_k(i)}{\sum_{i=1}^m J_{k-1}(i)} \leq 1 \]  

(7.71)

where

\[ F_k = \sum_{i=1}^m J_k(i) \]  

(7.72)

where \( F_k \) is the measure of the overall closed loop performance within the \( k^{th} \) batch. Since \( J_k(i) \) is non-negative, it can be verified that

\[ \Delta F_k = F_k - F_{k-1} \leq 0 \]

(7.73)

The conditions of convergence have been discussed in [114], which can be summarized as follows:

\[ \sum_{i=1}^m \int_a^b \left[ \left( \sqrt{\gamma_{k-1,i}(y)} - \sqrt{g(y)} \right) \Delta \sqrt{\gamma_{k-1,i}(y)} \right] dy \leq 0 \]  

(7.74)

together with

\[ \Delta R_{l,k-1}(y) = \frac{y_j - \mu_{k-1}}{\sigma_{k-1}^2} R_{l,k-1}(y) \Lambda_\mu E_{k-1} + \frac{(y_j - \mu_{k-1})^2}{\sigma_{k-1}^3} R_{l,k-1}(y) \Lambda_\mu E_{k-1} \]  

(7.75)
Figure 7.2: Three basis functions for approximating the output PDF

\[
\Delta \gamma_{k-1,i}(y) = \sqrt{\gamma_{k-1,i}(y)} - \sqrt{\gamma_{k-2,i}(y)} = \sum_{l=1}^{n} V_l(i) \Delta R_{l,k-1}(y) \quad (7.76)
\]

where

\[
\Delta R_{l,k-1}(y) = R_{l,k-1}(y) - R_{l,k-2}(y)
\]

\[
\approx \frac{\partial}{\partial \mu_{k-1}} R_{l,k-1}(y) \Delta \mu_k + \frac{\partial}{\partial \sigma_{k-1}} R_{l,k-1}(y) \Delta \sigma_k \quad (7.77)
\]

and

\[
\Delta \mu_k = \mu_k - \mu_{k-1}
\]

\[
\Delta \sigma_k = \sigma_k - \sigma_{k-1} \quad (7.78)
\]

### 7.9 An Illustrated Example

In this section, a simulation study of the proposed method will be described. First, the system model and RBF neural network components are introduced, and then the performance of the FTC control law will be investigated.

For a stochastic system with non-Gaussian process, it is supposed that the output PDF can be formulated by using three-layer neural network with three radial basis activation functions as shown in Fig. 7.2 with the following initial conditions over its definition interval \([a, b]\).

\[
y \in [0, 2],
\]
Chapter 7. Iterative Fault tolerant control

Figure 7.3: Desired output PDF

Figure 7.4: Fault and its estimation under the filter

Figure 7.5: 2D plot of measured and desired PDF, $k=2$
This would mean that the output PDF of the stochastic system is described as:

\[
\sqrt{\gamma(y,u(k),F(k))} = R(y)V(k) + r_n(y)h(V(k)) + \omega(y,u(k),F(k)) \tag{7.79}
\]

where

\[
R(y) = [r_1(y), r_2(y)]
\]

and

\[
V(t) = [v_1(t), v_2]^T
\]

The weight vector behaves dynamically as described in (7.7) with the following parameters:

\[
A = \begin{bmatrix} -0.45 & 0.03 \\ 0.1 & -0.28 \end{bmatrix}, \quad B = \begin{bmatrix} 0.45 & 0.01 \\ 0.01 & -0.86 \end{bmatrix}, \quad G = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad D = E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

The nonlinear function was chosen as follows.

\[
g(V(t)) = \begin{bmatrix} 0 \\ \sqrt{v_1^2 + v_2^2} \end{bmatrix}
\]

The initial value of the weight vector is set as \(V_1(0) = [0.001, 0.001]^T\), and the ILC learning rates are defined as

\[
\Lambda_\mu = \begin{bmatrix} -7.8 & 0 & 0 \\ 0 & -13.1 & 0 \\ 0 & 0 & -16.9 \end{bmatrix} \begin{bmatrix} 0.0030 \times j \\ 0.0015 \times j \\ 0.0015 \times j \end{bmatrix}
\]

along with

\[
\Lambda_\sigma = \begin{bmatrix} -0.71 & 0 & 0 \\ 0 & -0.71 & 0 \\ 0 & 0 & -0.71 \end{bmatrix} \begin{bmatrix} 0.005 \times j \\ 0.005 \times j \\ 0.005 \times j \end{bmatrix}
\]

where \(j = 0, 1, \ldots, 20\). In addition, the matrices \(U_1\) and \(U_2\) were chosen as follows.

\[
U_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
For simulation purposes, 200 uniformly distributed samples of the output and 20 for the time samples were used. Also, the RBF basis functions parameters of the desired output PDF are as follows:

\[
\mu_{g1} = 0.2, \mu_{g2} = 0.8, \mu_{g3} = 1.3
\]

\[
\sigma_{g1} = \sigma_{g2} = \sigma_{g3} = 0.1
\]

Moreover, the desired dynamical weights are set as

\[
V_g = [0.09 : 0.06].
\]

With the above parameters, the 3-D plot of the desired PDF shape is shown in Fig. 7.3. Assume that the modelling error satisfies

\[
|\omega(y, u(k), F)| \leq 0.002.
\]

The bound of modelling error satisfies \(\tilde{\delta} = 0.0008\) for \(\mu(y) = 1\). From (7.3), it can be compute that

\[
\Lambda_1 = \begin{bmatrix}
0.3010 & 0.0347 \\
0.0347 & 0.3010
\end{bmatrix}, \Lambda_2 = \begin{bmatrix}
0.0001 & 0.0350 \\
0.0350 & 0.3010
\end{bmatrix}, \Lambda_3 = 0.3413
\]

Also, from (7.12), it can be seen that

\[
\Gamma_1 = \begin{bmatrix}
0.0389 & 0.0348
\end{bmatrix}, \Gamma_2 = 0.4225
\]

To demonstrate the effectiveness of the proposed algorithm, the fault is chosen to be a constant signal as \(F = 0.8\), and it is supposed to commence at \(T = 2s\). Firstly, we consider the fault detection problem. Using Theorem 9 with \(\lambda_1 = \lambda_2 = 1\) it can be calculated that

\[
P = \begin{bmatrix}
1.5941 & 0.1479 \\
0.1479 & 95.5499
\end{bmatrix}, R = \begin{bmatrix}
14.9186 \\
50.1447
\end{bmatrix}, L = \begin{bmatrix}
9.3113 \\
0.5104
\end{bmatrix}
\]

Next, the fault diagnosis problem is considered for the above system and fault. Using Theorem 11, the following results can be obtained:

\[
P = \begin{bmatrix}
2.2720 & -0.0226 \\
-0.0226 & 2.3052
\end{bmatrix}, R = \begin{bmatrix}
-3.9556 \\
0.9197
\end{bmatrix}, L = \begin{bmatrix}
-1.7372 \\
0.3819
\end{bmatrix}
\]

\[
\Upsilon_1 = 0.97, \Upsilon_2 = 1.3
\]

By applying the nonlinear fault isolation filter, the estimated fault (7.27) should track the real fault profile as close as possible. Fig. 7.4 shows that such a filter can effectively diagnose the actuator fault.

The difference of the PDF tracking within the second batch between the measured and desired PDF is shown in Fig. 7.5. The parameters of controller in this batch are
After the final batch ($k = 10$), the coefficients of PI controller are as follows.

$$K_P = 1.0e+003 \times \begin{bmatrix} -5.5224 & -3.5340 \\ 7.4317 & 4.4533 \end{bmatrix}$$

and

$$K_I = 1.0e-002 \times \begin{bmatrix} 0.0394 & 0.0394 \\ 0.4860 & 0.4860 \end{bmatrix}$$

As a result, the corresponding of the weight control loop with the last batch of operation can be described in Fig. 7.6. Fig. 7.6 clearly demonstrates the effect of the fault-tolerant action on maintaining the correct value for the controlled weights. Moreover, it reflects the effectiveness of LMI feasibility results.

The 3D mesh plot of the output PDF in the last batch of operation is shown in Fig. 7.7. In addition, the PDF tracking performance within the last batch of operation is shown in Fig. 7.8.

Finally, the trend of the ILC performance function along the batches implies the effectiveness of the proposed algorithm is shown in Fig. 7.9.
Figure 7.7: 3-D mesh plot of the measured output PDF after the fault happens

Figure 7.8: 2D plot of measured and desired PDF, k=10

Figure 7.9: performance function of the ILC
7.10 Summary

An ILC-based FTC method is presented, for the shape control of the output PDFs for general stochastic systems with non-Gaussian variables, using a generalized fixed-structure PI controller with constraints on the state vector that results from the application of square-root PDF modelling. The whole control horizon is divided into a number of batches. Within each batch, the state-constrained generalized PI controller is used to shape the output PDF using an LMI approach. Between any two adjacent batches, the parameters of the RBF basis functions are tuned. A P-Type ILC is applied to achieve such tuning between batches, where a sufficient condition for the ILC convergence has been established.

After a fault happens, the fault estimation information is introduced into the controller reconfiguration to retain the stability and tracking convergence of the post-fault system, leading to the FTC of the whole stochastic nonlinear system. The simulation of the illustrated example demonstrates the use of the control algorithm, and shows that a satisfactory estimation and control performance is established.

A major difference between the FTC framework in this chapter and that in [17] is the ILC-based SDC with the application of RBFNNs instead of a fixed rational square-root B-spline expansion, which has the advantage of more general output PDF approximation in terms of tuneable RBF parameters. The application of ILC decreases the complexity of the stochastic model through the use of tuneable basis functions. Moreover, a fixed-structure PI controller is easy to implement in practical process control problems.
Chapter 8

Final Conclusion

Each chapter has been summarized individually and detailed conclusions have been drawn at the end of each chapter. This chapter provides some general overall comments, as well as suggestions for future work.

The chapter is divided into the following sections: (1) summaries and remarks on the main subject of this thesis; (2) suggestions and proposal for future work.

8.1 Summary and Remarks

State estimation of nonlinear systems satisfying a Lipschitz condition has been an important topic of research in nonlinear system theory for over three decades. In this thesis, a new observer design technique is developed for a class of nonlinear systems based on Lyapunov functions. This technique is developed to deal with both continuous-time and discrete-time nonlinear systems. A set of sufficient conditions for the existence of observers was presented. The observer gain is obtained by solving LMIs. The proposed observer was compared with a nonlinear Lipschitz observer from the point of view of convergence speed, as well as noise rejection, the performance of the new observer was clearly superior than that of the Lipschitz observer.

An output regulation design for both continuous-time and discrete-time classes of nonlinear systems was addressed in Chapter 4. Due to the complexities involved in solving nonlinear regulator equations, a kth-order approximation was applied to solve them here. A comparison between the output responses of the closed-loop system, when the plant was subject to different exosystems, with various values of amplitudes and frequencies, was performed. The maximal steady-state
tracking error was used as the point of view. The application of nonlinear output regulation design to a single-link flexible joint robot system using a kth-order approximation is unique and has not been previously carried out. As a result, the successful utilization of an output regulator on this type of system can be considered as a significant contribution to the output regulator research.

An adaptive PI controller was developed for the shaping of the closed-loop tracking error PDF for nonlinear systems subjected to non-Gaussian disturbance. The whole control horizon was divided into a number of batches. Within each batch, there are a fixed number of sample points. Within each batch, the minimum entropy control is realized, whilst between any two adjacent batches, the coefficients of the controller are tuned. A D-type ILC law, was applied for the tuning between batches, and a condition for the ILC convergence was established. The closed-loop stability of the system was discussed. The proposed method was implemented on two illustrated examples of one-link manipulator and two-link robot manipulator to demonstrate the effectiveness of the proposed control algorithm. The randomness of the closed-loop tracking error was minimized batch by batch, and the error PDF was a Gaussian-like shape in the last batch.

Motivated by the many stochastic systems in practice, where the output concerned is the PDF of the system output ([7, 14, 108, 112]), we studied a new FTC framework via a controller re-design approach for continuous-time stochastic nonlinear systems using PDFs. Different from the classical FTC problems, non-Gaussian stochastic variables were of interest and the output was the PDF of the system output rather than the output itself. Using the RBFNN approximation, the output PDFs control problem was reduced to the control of a weighting dynamics system, and consequently the dynamical relationship between the control input and the weights of the RBF neural network was set up through a nonlinear model. Thus, the proposed FTC problem was transformed into a nonlinear FTC problem subject to the continuous-time nonlinear weighting system. A generalized PID control strategy is proposed in the continuous-time context, and LMI-based algorithms are provided to formulate the gains of PID controller for the PDF tracking problem. In addition, LMI techniques were used to design a fault diagnostic filter, based on convex algorithms, to estimate the size of the fault and compensate for the fault. Finally, a simulated example was given to demonstrate the effectiveness of the proposed approach.
In addition to developing a FTC design for general nonlinear stochastic systems in Chapter 6, a new FTC framework via a controller re-design approach for general, stochastic distribution control in the discrete-time case, using PDFs, based on ILC theory was proposed. Similarly to Chapter 5, the control horizon was divided into batches. Also, a RBFNN was used to approximate the output PDF of the system. In Chapter 7, the ILC-based stochastic distribution control with application of RBFNNs was used instead of fixed B-spline expansions, which has the advantage of more general output PDF approximation in terms of tuneable RBF parameters. The design procedure was divided into two main algorithms, within each batch and between any two adjacent batches. The objective of the control design within batches was to find a fixed-structured PI controller with state constraints using an LMI approach such that the closed-loop weight control system was stable and the weights tracked the desired weight. Between any two adjacent batches, a P-type ILC law was employed to tune the parameters of the RBFNN so that the PDF tracking error decreased as the batches progressed, where a sufficient condition for the ILC convergence had been made. An LMI-based feasible FTC method was applied such that the fault could be detected and diagnosed. An illustrated example was included to demonstrate the efficiency of the proposed algorithm, and satisfactory results were obtained.

8.2 Future Work

The following future work is proposed:

1. The nonlinear observer design under the minimum error entropy frame could be generalized to deal with delays, and uncertainty systems.

2. An investigation could be carried out into the FTC performance-based PID/PI when the stochastic discrete-time system is subject to a time delay and parametric uncertainty. The performance could be compared to other FTC framework-based conventional controllers.

3. Similarly, an investigation of the ILC FTC performance-based PI when the non-Gaussian singular stochastic discrete-time system is subject to a time delay and parametric uncertainty could be carried out. Once again, the performance could be compared with other ILC FTC framework-based conventional controllers.

4. The FTC using PDFs could be applied to practical industrial and modern systems.
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