TRANSITIVE LIE ALGEBROIDS AND Q-MANIFOLDS

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Mathematics
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We will start by giving an outline of the fundamentals of supergeometry and Q-manifolds. Then, we will give local description of the Atiyah sequence of a principal bundle. We will construct local charts for the manifolds involved and write the expressions explicitly in local form. The Atiyah sequence encodes the different notions in connection theory in a compact way. For example a section of the Atiyah algebroid will give a connection in the principal bundle and curvature will be the failure of this section to be a Lie algebroid morphism. We will describe the Lie brackets for the Atiyah algebroid $TP$ and the adjoint bundle $P \times G$ in local coordinates. After that, we will describe the Lie algebroid of derivations $D(E)$. We will see how the curvature of some section gives the curvature on the vector bundle $E$ and when expressed locally gives the corresponding local connection forms. On the other hand we will give an explicit expression of the morphism from $TP$ to $D(P \times V)$ where $V$ is some vector space on which $G$ acts. As a corollary we will get an isomorphism between $TF$ and $D(E)$ where $FE$ is the frame bundle of $E$ and $G$ is the general linear group of the fibre $V$. We will establish an explicit equivalence between curvature and field strength in a more general sense. We will recall constructions from the paper of Kotov and Strobl [17] that describe the construction of characteristic classes associated with a section (connection=gauge field) of a $Q$-bundle $E(M,F,\pi)$. Finally, we state and prove the non-abelian Poincaré lemma in the case when $G = \text{Diff}(F)$, the diffeomorphism group of some supermanifold $F$, which has the space of vector fields on $F$, $\mathfrak{X}(F)$, as its super Lie algebra. The diffeomorphism group is generally infinite dimensional. It is this that makes the non-abelian Poincaré lemma more interesting to applications. Then we will show how it is applied to prove that every transitive Lie algebroid is locally trivial.
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Chapter 1

Introduction

The thesis is divided into six chapters. The main tool used is the concept of $Q$-manifold. In the second chapter we outline the basics of supermanifold theory that will be needed to understand the rest of the thesis. We will introduce notions like super vector space, superalgebra, (super) modules and super Lie algebras. As their names indicate, they are the corresponding notions of the usual structures in commutative algebra. Then we introduce supermanifolds. A supermanifold is loosely speaking an ordinary manifold with its sheaf of functions enriched with odd functions. They are the main object of study in supergeometry; that is every geometric object of study lives on a supermanifold. Then we will introduce for them local coordinates. We will describe the different structures that arise in a supermanifold: vector fields, tangent and cotangent bundles, differential forms, Cartan identities, $Q$-manifolds, and super Lie groups. The definition of supermanifold uses the abstract tools of algebraic geometry and lacks intuition. By introducing the functor of points approach, some intuition can be gained about the notion of supermanifold. This will make the theory of supermanifolds look much like that of ordinary manifolds.

In the third chapter, we will describe the Atiyah sequence of a principal bundle locally. We will construct local charts for the manifolds involved and explicitly write the expressions in local form. The Atiya sequence was introduced by Michael Atiyah. It encodes the different notions in connection theory in a compact way. For example a section of the Atiyah algebroid will give a connection in the principal bundle and curvature will be the failure of this section to be a Lie algebroid morphism. We will describe the Lie brackets for the Atiyah algebroid $TP_G$ and the adjoint bundle $P \times G$ in local coordinates. On the other hand we will see that writing a section of the Atiyah algebroid $TP_G$ locally and its transformation under a coordinate change allow us to see that it encodes a connection form of the principal bundle $P$. The same thing applies for curvature. After that we will describe the Lie algebroid of derivations $\mathcal{D}(E)$. We will see how the curvature of some section gives the curvature on the vector bundle $E$ and when expressed locally gives the corresponding local connection forms. On the other hand we will give an explicit expression for the morphism from $TP_G$ to $\mathcal{D} \left( \frac{P \times V}{G} \right)$ where $V$ is some vector space on which $G$ acts. As a corollary we get an isomorphism between $\frac{TF_E}{G}$ and $\mathcal{D}(E)$ where $FE$ is the frame bundle of $E$ and $G$ is the general linear group of the fibre $V$. 
CHAPTER 1. INTRODUCTION

The fourth chapter is about the equivalence between curvature and field strength. It is known that a Lie algebroid structure on a vector bundle $E$ is equivalent to a $Q$-manifold structure on $\Pi E$. The field strength is the failure of the homological vector fields on $\Pi E$ and $\Pi TM$ to be $s$-related, where $s$ is connection in $E$. This will be made precise when we come to it. We will establish an explicit equivalence between curvature and field strength.

In the fifth chapter we recall constructions from the paper of Kotov and Strobl [17]. The main purpose is to construct characteristic classes associated with a section (connection=gauge field) of a $Q$-bundle $\mathcal{E}(\mathcal{M}, \mathcal{F}, \pi)$. This generalizes the Chern-Weil formalism of characteristic classes in principal bundles. Roughly speaking a $Q$-bundle is a fibre bundle in the category of $Q$-manifolds with some compatibility condition. A section of $Q$-bundle plays the role of a connection and the field strength the role of curvature. This approach relies on sections and field strength as much as the Chern-Weil formalism relies on connections and curvature.

In the sixth Chapter we state and prove the non-abelian Poincaré lemma in the case when $G = \text{Diff}(F)$, the diffeomorphism group of some supermanifold $F$, which has the space of vector fields on $F$, $\mathfrak{X}(F)$, as its super Lie algebra. The diffeomorphism group is generally infinite dimensional. It is this that makes the non-abelian Poincaré lemma more interesting to applications. Then we will show how it is applied to prove that every transitive Lie algebroid is locally trivial.
Chapter 2

Supergeometry

In this chapter we give an outline of the fundamentals of supergeometry that will be needed in the subsequent chapters. Supergeometry had its beginnings in particle physics in the 70s. It was devised as a theory to handle the intricacies of bosons and fermions. Supergeometry builds on the machinery and techniques of algebraic geometry and thus relies on concepts such as ringed spaces, schemes, functor of points... etc.

2.1 Supermanifolds

Definition 1. A super vector space or $\mathbb{Z}_2$-graded vector space is a vector space $V = V_0 \oplus V_1$. Elements of $V_0$ are called even and elements of $V_1$ are odd. Every element of $V$ which is either even or odd is called homogeneous. Let a nonzero element $v$ in $V_i$, $i = 0, 1$. Then the parity of $v$ is $p(v) = \overline{i} = i$. If $V_0$ is of dimension $p$ and $V_1$ is of dimension $q$, then $V$ is said to be of dimension $p|q$. The parity reversal functor $\Pi$ is defined by $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$.

Definition 2. A superalgebra is a super vector space $A$ with multiplication such that $A_i A_j \subseteq A_{i+j}$ and $A$ is an algebra. We say that $A$ is supercommutative if $a b = (-1)^{i\bar{j}} b a$ for homogeneous elements.

Suppose we have a linear map of super vector spaces $f : V \rightarrow W$. We say $f$ is even if it is parity preserving, i.e. $\overline{f(v)} = \overline{v}$ for homogeneous $v$, and $f$ is odd if it is parity reversing, i.e. $\overline{f(v)} = \overline{v} + 1$ for homogeneous $v$. If $f$ is either an even or an odd linear map, then it is said to be homogeneous. It can be seen that every linear map $f$ can be decomposed as $f = f_0 + f_1$ where $f_0$ is even and $f_1$ is odd. We denote $\text{Hom}(V, W)$ the set of all even linear maps from $V$ to $W$ and $\overline{\text{Hom}}(V, W)$ the set of all linear maps from $V$ to $W$. $\overline{\text{Hom}}(V, W)$ is a super vector space and $\overline{f(v)} = \overline{f} + \overline{v}$.

Definition 3. Let $A$ be a superalgebra and $V$ a super vector space. Then we say that $V$ is a left $A$-module (or simply $A$-module) if there is an operation $A \times V \rightarrow V$ such that $(a \cdot v) = \overline{a} + \overline{v}$ for homogeneous $a, v$ and $(ab)v = a(bv)$ and $V$ is a module in the usual sense.

We can make $V$ into a right $A$-module by defining $va = (-1)^{\overline{a}} av$ and extending by linearity. An $A$-module is said to be free if there is a basis $\{e_1, ..., e_p, v_1, ..., v_q\}$ that generate $V$ where $e_i$ are even and $v_i$ are odd.
Definition 4. A super Lie algebra is a super vector space \( L \) with a bilinear map 
\[
    [\cdot, \cdot] : L \times L \rightarrow L
\]
that satsfies:
\[
    [a, b] = \tilde{a} + \tilde{b}, \text{ i.e. } [L_i, L_j] \subseteq L_{i+j}
\]
for homogeneous elements \( a, b \).
\[
    [a, b] = -(-1)^{\tilde{a}\tilde{b}}[b, a], \text{ this is called skewsymmetry.}
\]
\[
    [a, [b, c]] + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b, [c, a]] + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}[c, [a, b]] = 0 \text{ the Jacobi identity.}
\]

Let \( A \) be a superalgebra. Then a homogeneous linear map \( D : A \rightarrow A \) is a derivation
if \( D(ab) = D(a)b + (-1)^{\tilde{a}D}aD(b) \). If \( D \) is an even (odd) linear map, then it
is called an even (odd) derivation. The set of derivations in \( A \), denoted \( \text{Der}(A) \),
is the direct sum of even and odd derivations \( \text{Der}(A) = \text{Der}(A)_0 \oplus \text{Der}(A)_1 \). Der(A)
can be made a super Lie algebra by defining
\[
    [D_1, D_2] = D_1 \circ D_2 - (-1)^{D_1D_2}D_2 \circ D_1
\]
for homogeneous derivations and extending by
linearity to inhomogeneous derivations. And this is the way we define vector fields
on supermanifolds, which will be seen later.

Now, we introduce the main objects in supergeometry, i.e. supermanifolds. Su-
permanifolds generalize the notion of smooth manifolds. In fact a smooth manifold is
no more than a special kind of supermanifold. Most objects that are associated with
a smooth manifold have their counterparts in a supermanifold; which will concisely
be described.

Definition 5. A supermanifold \( \mathcal{M} \) is a pair \( (M, \mathcal{A}) \) where \( M \) is a smooth manifold
and \( \mathcal{A} \) is a sheaf of superalgebras in \( M \) such that there is an open cover \( \{U_a\} \) of
\( M \) and \( \mathcal{A}(U_a) \cong C^\infty(U_a)[\theta^1, \ldots, \theta^q] \) for some fixed \( q \). Where \( C^\infty(U_a)[\theta^1, \ldots, \theta^q] \) is
the superalgebra with elements \( f = f_0 + f_1\theta^1 + \ldots + f_{i_1,\ldots,i_q} \theta^{i_1}\ldots\theta^{i_q} \) with \( \theta^{i_2} = 0 \) and
\( \theta^i\theta^j = -\theta^j\theta^i \) and \( f \in C^\infty(U_a), I = (i_1, \ldots, i_k) \).

Here we are using the Einstein summation convention. That is whenever an
index is repeated, then summation is implied. We can see that \( f \) is a polynomial over
the ring \( C^\infty(U_a) \) with indeterminates \( \theta^1, \ldots, \theta^q \). If \( M \) has dimension \( p \), then we say
\( \mathcal{M} = (M, \mathcal{A}) \) is a supermanifold of dimension \( (p|q) \). The manifold \( M \) is called the
support of \( \mathcal{M} \) or the underlying manifold. We call a section \( f \) of the sheaf \( (M, \mathcal{A}) \) a
function even though it is not determined by its values at its points. In some sense, a
supermanifold is an ordinary manifold with the sheaf of functions enriched with odd
ones. We write \( C^\infty(\mathcal{M}) \) for \( \mathcal{A}(M) \).

Examples of supermanifolds
The affine superspace \( \mathbb{R}^{p|q} = (\mathbb{R}^p, \mathcal{A}_q) \) where \( \mathcal{A}_q(U) = C^\infty(U)[\theta^1, \ldots, \theta^q] \) for any open
subset \( U \subseteq \mathbb{R}^p \). \( U^{p|q} = (U, \mathcal{A}_q|U) \) where \( U \subseteq \mathbb{R}^p \) is an open subset and \( \mathcal{A}_q|U \)
is the restriction of the sheaf \( \mathcal{A}_q \) to \( U \) is called a superdomain. Let \( M \) be a manifold. Then
\( \Pi\Pi\Pi M = (M, \Omega^\bullet) \) where \( \Omega^\bullet \) is the sheaf of differential forms on \( M \) is a supermanifold
called the antitangent bundle. \( \Pi\Pi\Pi^* M = (M, \mathfrak{X}^\bullet) \) where \( \mathfrak{X}^\bullet \) is the sheaf of multivector
fields on \( M \) is a supermanifold called the anticotangent bundle. We will see later why
the parity reversal functor \( \Pi \) appears here when we discuss vector bundles.
2.1.1 Local coordinates for supermanifolds

Let $\mathcal{M} = (M, \mathcal{A})$ be a supermanifold. Then, we have an open cover $\{U_\alpha\}$ of $M$ such that $\mathcal{A}(U_\alpha) \cong C^\infty(U_\alpha)[\theta^1, ..., \theta^n]$. Let $x^1, x^2, ..., x^p$ be local coordinates on $U_\alpha$. Then, we call $x^1, x^2, ..., x^p, \theta^1, \theta^2, ..., \theta^n$ local coordinates for $\mathcal{M}$. Let $U \subseteq M$ be an open subset of $M$. Then, $\mathcal{U} = (U, \mathcal{A}|_U)$ is called a submanifold of $\mathcal{M}$ where $\mathcal{A}|_U$ is the restriction sheaf.

**Definition 6.** Let $\mathcal{M} = (M, \mathcal{A})$ and $\mathcal{N} = (N, \mathcal{B})$ be two supermanifolds. Then a morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ is a pair $(f, f^*)$ such that $f : M \rightarrow N$ is a smooth map from $M$ to $N$ and $f^*$ is a morphism of sheaves, i.e. $f^* : \mathcal{B} \rightarrow f_*\mathcal{A}$ where $f_*\mathcal{A}(U) = \mathcal{A}(f^{-1}(U))$.

Let $F : \mathcal{M} \rightarrow \mathcal{N}$ and $G : \mathcal{N} \rightarrow \mathcal{P}$ where $F(f, f^*)$ and $G = (g, g^*)$ be morphisms of supermanifolds. Then, their composition is defined as follows $G \circ F = (g \circ f, f^* \circ g^*)$. The identity morphism $I = (i, i^*)$ is defined in the obvious way. $I : \mathcal{M} \rightarrow \mathcal{M}$ with $i(x) = x$ and $i^*(f) = f$. We say that $F : \mathcal{M} \rightarrow \mathcal{N}$ is a diffeomorphism if there is a morphism $G : \mathcal{N} \rightarrow \mathcal{M}$ such that $F \circ G = I_N$ and $G \circ F = I_M$. We can see that the composition of any two morphisms is still a morphism of supermanifolds and we have an identity map. Hence we can speak of the category of supermanifolds which we denote $\text{SM}$. Usually, we will write $f$ for $F$ as well.

**Theorem 1** (Chart Theorem). Let $U^{pq} \subseteq \mathbb{R}^{pq}$ be a superdomain with coordinates $y^i, \xi^j$ and $\mathcal{M}$ a supermanifold. Let $f^*$ be $p$ even functions and $\eta^q$ be $q$ odd functions on $\mathcal{M}$ such that $(v(f^1)(x), v(f^2)(x), ..., v(f^p)(x)) \in U$. Then, there is a unique morphism $f : \mathcal{M} \rightarrow U^{pq}$ such that $f^*(y^i) = f^i$ and $f^*(\xi^j) = \eta^j$.

In other words, this theorem says that a morphism of supermanifolds is determined by the pullback of the coordinate functions $y^i, \xi^j$. If $x^i, \theta^j$ are coordinates on $\mathcal{M}$ and $y^\alpha, \xi^\beta$ are coordinates on $\mathcal{N}$, then we will write a morphism $f : \mathcal{M} \rightarrow \mathcal{N}$ as $f(x^i, \theta^j) = (y^\alpha(x^i, \theta^j), \xi^\beta(x^i, \theta^j))$ to mean that $f(x) = (v(y^1)(x), v(y^2)(x), ..., v(y^p)(x))$ and $f^*(y^\alpha) = y^\alpha(x^i, \theta^j)$ and $f^*(\xi^\beta) = \xi^\beta(x^i, \theta^j)$, where $x = (x^1, x^2, ...)$. This abuse of notation will turn out to be very convenient in the supermanifold setting; it makes the theory of supermanifolds similar to that of ordinary manifolds. The theorem seems reasonable since every function can locally be approximated by a polynomial in even and odd coordinates using Taylor expansion. Once the pullback of the coordinate functions is fixed, the pullback of any polynomial is determined by the morphism of superalgebras. Actually this is the key idea in the proof of the theorem. We will give an example of how to find the pullback of a function given the pullback of the coordinate functions. Consider the map $f : \mathbb{R}^{1|2} \rightarrow \mathbb{R}^{1|2}$ with $f^*(y) = x + \theta^1\theta^2$, $f^*(\xi^1) = \theta^1$ and $f^*(\xi^2) = \theta^2$. Let $g(y) = \sin y$. Then

$$f^*(g) = \sin(x + \theta^1\theta^2)$$

$$= (x + \theta^1\theta^2) - \frac{(x + \theta^1\theta^2)^3}{3!} + \frac{(x + \theta^1\theta^2)^5}{5!} + ...$$

$$= (x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...) + \theta^1\theta^2(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...)$$

$$= \sin x + \theta^1\theta^2\cos x.$$
2.1.2 Vector fields on supermanifolds

Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold. A vector field on $\mathcal{M}$ is a derivation $X : \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$ where $U \subseteq M$ is an open subset. For a homogeneous derivation (homogeneous vector field), we have $X(fg) = X(f)g + (-1)^{\bar{s}_f \bar{s}_g} X(g)f$ for homogeneous functions $f$, $g$. To every open subset $U \subseteq M$ we have the set of derivations (vector fields) $\text{Der}_U$ which we denote $\mathfrak{X}(U)$. We have then what we call the tangent sheaf. We denote the set of vector fields on $\mathcal{M}$ (the global vector fields) $\mathfrak{X}(\mathcal{M})$. $\mathfrak{X}(\mathcal{M})$ is a $C^\infty(\mathcal{M})$-module. If $x^i, \theta^i$ are local coordinates on $\mathcal{M}$, then a vector field $X$ on $\mathcal{M}$ will be expressed locally as

$$X = a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial \theta^i}$$

with functions $a^i$, $\theta^i$, where $\frac{\partial(\theta^i f)}{\partial \theta^j} = f$ and $f$ is a function that does not contain $\theta_j$ and the definition is extended by linearity. Hence $\mathfrak{X}(\mathcal{M})$ is locally a free module. The space $\mathfrak{X}(\mathcal{M})$ is a super Lie algebra with Lie brackets defined for homogeneous vector fields by $[X, Y] = X \circ Y - (-1)^{\bar{s}_X \bar{s}_Y} Y \circ X$.

The value of a function at a point

Let $G = \mathbb{R}[\theta^1, \theta^2, ..., \theta^n]$ be the Grassmann algebra. Then an element of $G$ is of the form $s = s_0 + s_1 \theta^i + ... + s_{i_1, ..., i_n} \theta^{i_1} ... \theta^{i_n}$, where $s_0, s_1, ..., s_{i_1, ..., i_n}$ are real numbers. Then we define the value of $s$ to be $v(s) = s_0$. Now let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold and $x^i, \theta^i$ be local coordinates on it. Let $f$ be a function on $\mathcal{M}$. Then we locally have $f = f_0 + f_i \theta^i + ... + f_{i_1, ..., i_n} \theta^{i_1} ... \theta^{i_n}$. Define the value of $f$ at the point $x \in M$ as $v(f)(x) = v[f(x)] = f_0(x)$. To define $v(f)(x)$ in a coordinate-free way, we can express it in a different way. We say that it is the unique real number $s$ such that $f - s$ is not invertible in any neighborhood of $x$. From the local isomorphism given in the definition of supermanifold, we see that $s$ is unique. We have the inclusion morphism $F = (id, v)$ from $M$ to $\mathcal{M}$ where $id$ is the identity map and the valuation map $f \longrightarrow v(f) \in C^\infty(M)$ just defined. Likewise, we can define a tangent vector $w$ at a point $m \in M$. Let $\mathcal{O}_m$ be the stalk of the sheaf $\mathcal{O}$ at the point $m$. Then a homogeneous tangent vector $w$ at $m$ is a homogeneous derivation $w : \mathcal{O}_m \longrightarrow \mathcal{O}_m$ such that $w(fg) = w(f)w(g)(m) + (-1)^{\bar{s}_f \bar{s}_g} w(g)(m)w(f)(m)$. Then, the space of tangent vectors at $m$ is the direct sum of even and odd spaces of tangent vectors at $m$. It is called the tangent space at $m$ which is denoted by $T_m \mathcal{M}$. In local coordinates, $w$ is of the form

$$w = a^i \frac{\partial}{\partial x^i}|_m + b^i \frac{\partial}{\partial \theta^i}|_m$$

where $a^i$, and $b^i$ are real numbers and

$$\frac{\partial}{\partial x^i}|_m(f) = v(\frac{\partial f}{\partial x^i})(m).$$

If $X$ is a vector field on $\mathcal{M}$, then its value at a point $x \in M$ is a tangent vector $X_x$ at $x$ defined as follows $X_x(f) = v[X(f)](x)$. However the vector field $X$ is not determined by its values at the points of the underlying manifold.

**Definition 7.** Let $f : \mathcal{M} \longrightarrow \mathcal{N}$ be a map of supermanifolds. Then, the differential of $f$ at $x$ is defined as follows:
Suppose that \( D \). The proof is by noticing that \( \text{Lemma 1} \) is identically zero. Hence we have the desired result.

\[
d_x f(x) = a^A v(\frac{\partial y^B}{\partial x^A})(x) \frac{\partial}{\partial y^B} |_y.
\]

This looks exactly the same as the change of variables formula.

Example 1. Let \( x^A \) and \( y^B \) be local coordinates on \( M \) and \( N \) respectively. Then \( f(x) = (y^B(x^A)) \) and \( x = a^A \frac{\partial}{\partial x^A} |_x \). Put \( f(x) = y \) and let \( Y = d_x f(X) = b^B \frac{\partial}{\partial y^B} |_y \).

We have \( b^B = Y(y^B) = d_x f(X)(y^B) = X(y^B) = a^A v(\frac{\partial y^B}{\partial x^A})(x) \). Hence,

\[
d_x f(x) = a^A v(\frac{\partial y^B}{\partial x^A})(x) \frac{\partial}{\partial y^B} |_y.
\]

Change of variables formula

We state the following lemma without proof.

Lemma 1. Let \( f : M \to N \) be a map. Then a linear map \( D : C^\infty(N) \to C^\infty(M) \) is called a derivation over \( f \) if \( D(gh) = D(g) f^* h + (-1)^{D\bar{g}} f^* g D(h) \).

Suppose that \( D(x^A) = 0 \) for all \( A \) where \( x^A \) are local coordinates on \( N \). Then \( D \) is identically 0.

Let \( f : \mathbb{R}^{\|m} \to \mathbb{R}^{\|q} \) be a map with coordinates \( x^A \) and \( y^B \) on \( \mathbb{R}^{\|m} \) and \( \mathbb{R}^{\|q} \) respectively. Let \( g \in C^\infty(\mathbb{R}^{\|q}) \). Then

\[
\frac{\partial f^* g}{\partial x^A} = \frac{\partial y^B}{\partial x^A} f^* \frac{\partial g}{\partial y^B}.
\]

The proof is by noticing that \( D = \frac{\partial f^*}{\partial x^A} \frac{\partial}{\partial x^A} f^* \frac{\partial}{\partial y^B} \) is a derivation and \( D(y^B) = 0 \) and hence by Lemma 1 is identically zero. Hence we have the desired result.

### 2.1.3 Flow of a vector field

Let \( X \) be a vector field on a supermanifold \( M \). If \( X \) is even then the flow of \( X \) is a map \( \sigma : \mathbb{R} \times M \to M \) such that \( \frac{\partial}{\partial \tau} \circ \sigma^* = \sigma^* \circ X \) and \( \sigma_0 = \text{id} \). From the theory of differential equations, this has locally at least a solution. If \( X \) is odd and \( [X, X] = 0 \) then the flow of \( X \) is a map \( \sigma : \mathbb{R}^{0|1} \times M \to M \) such that \( \frac{\partial}{\partial \tau} \circ \sigma^* = \sigma^* \circ X \) and \( \sigma_0 = \text{id} \). In local coordinates this will be \( \frac{dx^A}{d\tau} = X^A(x(\tau)) \). Hence, \( x^A(\tau) = x_0^A + \tau X^A \) . If \( X \) is odd and \( [X, X] \neq 0 \) then the flow of \( X \) is a map \( \sigma : \mathbb{R}^{1|1} \times M \to M \) such
that \((\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial \bar{t}}) \circ \sigma^* = \sigma^* \circ X\) and \(\sigma_0 = \text{id}\), where \(\sigma_0\) is \(\sigma\) restricted to \(0 \times \mathcal{M}\). We write as well \(\sigma_t = \exp(tX)\), the exponential of the vector field \(X\).

### 2.1.4 Functor of points approach to supermanifolds

Let \(\mathcal{M} = (M, \mathcal{O})\) be a supermanifold and \(\mathcal{S}\) another supermanifold. Then an \(\mathcal{S}\)-point of \(\mathcal{M}\) is a map \(f : \mathcal{S} \rightarrow \mathcal{M}\). Then \(\mathcal{S} \rightarrow \text{Hom}(\mathcal{S}, \mathcal{M}) = \mathcal{M}(\mathcal{S})\) defines a contravariant functor from the category of supermanifolds \(\text{SM}\) to the category of sets. \(\text{Sets}\) where \(\text{Hom}(\mathcal{S}, \mathcal{M})\) is the set of morphisms between \(\mathcal{S}\) and \(\mathcal{M}\). To see this, suppose that \(\varphi : \mathcal{S}' \rightarrow \mathcal{S}\) is a map. Then we define \(\mathcal{M}_\varphi : \mathcal{M}(\mathcal{S}') \rightarrow \mathcal{M}(\mathcal{S})\) as \(\mathcal{M}_\varphi(g) = g \circ \varphi\). Suppose that \(f : \mathcal{M} \rightarrow \mathcal{N}\) is a map of supermanifolds. Then we define \(f_\mathcal{S} : \mathcal{M}(\mathcal{S}) \rightarrow \mathcal{N}(\mathcal{S})\) as \(f_\mathcal{S}(\varphi) = f \circ \varphi\) which is a natural transformation between the functors \(\mathcal{M}\) and \(\mathcal{N}\). Hence, we can think of supermanifolds as contravariant functors from the category of supermanifolds to the category of sets and the maps between them as the corresponding natural transformations between those functors. Moreover, the Yoneda Lemma says that there is a bijection between the maps and the natural transformations. Two manifolds are diffeomorphic if and only if they are naturally isomorphic as contravariant functors. Proof of the Yoneda Lemma.

**Definition 8.** Let \(F : \text{SM} \rightarrow \text{Sets}\) be a contravariant functor. Then, we say that \(F\) is representable if there is a supermanifold \(\mathcal{M}\) such that \(F\) is naturally isomorphic to \(\mathcal{M}\) as a contravariant functor.

Let \(F : \mathcal{M} \rightarrow \mathcal{N}\) be a map of supermanifolds. Then \(\varphi \in \mathcal{M}(\mathcal{S})\) is locally determined by functions \(x^i, \theta^j\) on \(\mathcal{M}\). Then \(F_\mathcal{S} : \mathcal{M}(\mathcal{S}) \rightarrow \mathcal{N}(\mathcal{S}), F_\mathcal{S}(x^i, \theta^j) = (y^\alpha, \xi^\beta)\). If \(\mathcal{S} = \mathbb{R}^{0|0}\) then \(\mathcal{M}(\mathbb{R}^{0|0}) = M\), the underlying manifold.

### 2.1.5 Tangent and cotangent bundle

Let \(x^A\) be local coordinates on a supermanifold \(\mathcal{M}\). We construct another supermanifold \(T\mathcal{M}\) which we call the tangent bundle of \(\mathcal{M}\). The supermanifold \(T\mathcal{M}\) will have coordinates \(x^A, \bar{x}^\alpha\) with \(\bar{x}^\alpha = x^A\) and \(\bar{x}^\alpha\) taking values in \(\mathbb{R}\) if it is even and \(\bar{x}^\alpha\) transforms according to the following rule

\[
\bar{x}^\alpha = x^{\prime \beta} \frac{\partial x^A}{\partial \bar{x}^\beta}.
\] (2.1)

Changing the parity of \(\bar{x}^\alpha\) we get new supermanifold \(PT\mathcal{M}\) called the antitangent bundle. Hence, on \(PT\mathcal{M}\) we have the coordinates \(x^A, dx^A\) where \(dx^A\) has the opposite parity of \(x^A\) and changes as \(\bar{x}^\alpha\). For the cotangent bundle \(T^*\mathcal{M}\) we have the coordinates \(x^A\) and \(p_A\) where \(p_A\) has the same parity as \(x^A\) and changes according to
the rule
\[ p_A = \partial x^B \partial x_A p_B. \]  
(2.2)

Changing the parity of \( p_A \) we get a new supermanifold \( \Pi T^* M \) called the **anticotangent bundle**. Hence, on \( T^* M \) we have the coordinates \( x^A, z_A \), where \( z_A \) has the opposite parity of \( x^A \) and changes as \( p_A \).

### 2.1.6 Differential forms on supermanifolds

As for ordinary manifolds, we will construct the superalgebra of differential forms on supermanifolds. We will do this by defining them locally and specifying how they transform under the change of coordinates. Let \( M = (M, \mathcal{O}) \) be a supermanifold with \( x^i, \theta^j \) local coordinates. Functions \( C^\infty(M) \) will be called zero forms and denoted \( \Omega^0(M) \). We construct one-forms \( dx^A \) with \( \tilde{dx}^A = \tilde{x}^A + 1 \) and transforms as \( dx^A = dx^B \partial x_A \partial x^B \). And we declare that \( dx^A dx^B = (-1)^{(\hat{A}+1)(\hat{B}+1)} dx^B dx^A \) where \( \tilde{x}^A = \hat{A} \).

Then a \( p \)-form is locally a polynomial of degree \( p \) on \( dx^A \) over the functions. \( \Omega^p(M) \) will stand for differential forms of degree \( p \). Then, the superalgebra of differential forms denoted by \( \Omega^*(M) \) is the direct sum of \( \Omega^p(M) \). We immediately see that \( \Omega^*(M) \) embeds naturally in \( C^\infty(\Pi T M) \). differential forms are just the functions of \( \Pi T M \) which are polynomial on the fibre coordinates. We call a function in \( C^\infty(\Pi T M) \) a pseudoform since it may not be a polynomial on the fibre coordinates.

In the case of ordinary manifolds then \( \Omega^*(M) \) is just \( C^\infty(\Pi T M) \). For a differential form \( \omega \) we define its differential as
\[ d\omega = dx^A \partial \omega \frac{\partial}{\partial x^A}. \]  
(2.3)

We can view \( d \) as a vector field on \( \Pi T M \): \( d = dx^A \partial \partial x_A \). Let \( X = X^A \partial \partial x_A \) be a homogeneous vector field on \( M \). Then, we define the **interior derivative** of \( \omega \) as
\[ i_X \omega = (-1)^{\tilde{X}} X^A \partial \omega \frac{\partial}{\partial x^A}, \]  
(2.4)

and defined for inhomogeneous vector fields by extension by linearity. The Lie derivative is defined as
\[ \mathcal{L}_X \omega = di_X \omega + (-1)^{\tilde{X}} i_X d\omega. \]  
(2.5)

Then
\[ \mathcal{L}_X = X^A \frac{\partial \omega}{\partial x_A} + (-1)^{\tilde{X}} dX^A \frac{\partial \omega}{\partial x_A}. \]  
(2.6)

and defined for an inhomogeneous vector fields by extension by linearity. The interior derivative \( i_X \) and \( \mathcal{L}_X \) can be viewed as vector fields on \( \Pi T M \) since they are derivations.
To sum up, we have the following important identities:

\[ [i_X, i_Y] = 0, \]
\[ [d, \mathcal{L}_X] = 0, \]
\[ [d, i_X] = \mathcal{L}_X, \]
\[ [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}, \]
\[ i_X df = X f, \]
\[ dd = 0. \]

Let \( f : \mathcal{M} \rightarrow \mathcal{N} \) be a map of supermanifolds. Let \( x^A \) and \( y^B \) be coordinates on \( \mathcal{M} \) and \( \mathcal{N} \) respectively. If \( \omega \in \Omega(\mathcal{N}) \) then we would like to define the pullback \( f^*(\omega) \) of \( \omega \) to \( \mathcal{M} \). First we let \( f^*(dy^B) = df^*(y^B) \). Then inductively we define \( f^*(\omega \cdot \theta) = f^*(\omega) \cdot f^*(\theta) \). This suffices to define \( f^*(\omega) \) for any \( \omega \). We can also see that this defines a map of supermanifolds \( df : \Pi T \mathcal{M} \rightarrow \Pi T \mathcal{N} \).

### 2.1.7 Fibre bundles

Let \( \mathcal{E} = (E, \mathcal{A}) \) and \( \mathcal{M} = (M, \mathcal{B}) \) and \( \mathcal{F} = (F, \mathcal{C}) \) be supermanifolds and a surjective map \( \pi : \mathcal{E} \rightarrow \mathcal{M} \) such that there is an open cover \( \{U_\alpha\} \) of \( M \) such that \( \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathcal{F} \) where \( U_\alpha = (U_\alpha, \mathcal{A}|_{U_\alpha}) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathcal{F} \\
\downarrow \pi & & \downarrow \text{pr} \\
U_\alpha & \xrightarrow{=} & U_\alpha \\
\end{array}
\]

where \( \phi_\alpha \) is the diffeomorphism between \( \pi^{-1}(U_\alpha) \) and \( U_\alpha \times \mathcal{F} \) and \( \text{pr} \) is the projection to \( U_\alpha \). Then we call \( \mathcal{E}(\mathcal{M}, \mathcal{F}, \pi) \) a fibre bundle over \( \mathcal{M} \) with fibre \( \mathcal{F} \). We have \( \pi^{-1}(U_\alpha) = (\pi^{-1}(U_\alpha), \pi^{-1}(\mathcal{B}|_{U_\alpha})) \) and \( \pi^{-1}(\mathcal{B}|_{U_\alpha}(V)) = \mathcal{A}(\pi^{-1}(V)) \). If \( \mathcal{F} = \mathbb{R}^{p|q} \) and the change of coordinates is linear then \( \mathcal{E} \) is called a (super) vector bundle. If \( \mathcal{E} = \mathcal{M} \times \mathcal{F} \) then \( \mathcal{E} \) is called a trivial fibre bundle. If \( \mathcal{E} = \mathcal{M} \times \mathbb{R}^{p|q} \) then \( \mathcal{E} \) is called a trivial vector bundle.

**Example 2.** If \( \mathcal{M} = (M, \mathcal{O}) \) is a supermanifold then \( T \mathcal{M} \) and \( T^* \mathcal{M} \) are vector bundles with fibre \( \mathbb{R}^{p|q} \) (rank \( p|q \)). As for the tangent and cotangent bundle, applying the parity reversal functor to a vector bundle \( \mathcal{E} \), i.e changing the fibre coordinates and keeping the same coordinate change between the fibres, then we get a new vector bundle \( \Pi \mathcal{E} \) with the same rank. If \( E \) is an ordinary vector bundle over \( M \), then \( \Pi E = (M, \wedge(E^*)) \) where \( \wedge(E^*) \) is the exterior bundle of \( E \).

An even section of a vector bundle \( \mathcal{E}(\mathcal{M}, \mathcal{F}, \pi) \) is map \( s : \mathcal{M} \rightarrow \mathcal{E} \) such that \( \pi \circ s = \text{id} \) and an odd section is a map \( s : \mathcal{M} \rightarrow \Pi \mathcal{E} \) such that \( \pi \circ s = \text{id} \). Then the module of sections of \( \mathcal{E} \) is \( \Gamma(\mathcal{E}) = \Gamma_0(\mathcal{E}) \oplus \Gamma_1(\mathcal{E}) \) the direct sum of even and odd sections. We denote as well \( \mathcal{C}^\infty(\mathcal{M}, \mathcal{E}) \).

### Homomorphism of fibre bundles

A fibre bundle homomorphism from the fibre bundle \( \mathcal{E}(\mathcal{M}, \mathcal{F}, \pi) \) to \( \mathcal{K}(\mathcal{N}, \mathcal{V}, \rho) \) is a pair \((f, F)\) where \( f : \mathcal{M} \rightarrow \mathcal{N} \) and \( F : \mathcal{E} \rightarrow \mathcal{K} \) such that the following diagram
commutes
\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{F} & K \\
\downarrow \pi & & \downarrow p \\
\mathcal{M} & \xrightarrow{f} & \mathcal{N}
\end{array}
\] (2.13)

If \( \mathcal{E} \) and \( \mathcal{K} \) are vector bundles, then we require \( F \) to be linear on the fibres to get a vector bundle homomorphism.

### 2.2 Q-manifolds

**Definition 9.** A \( Q \)-manifold is a supermanifold with a homological vector field \( Q \), i.e. \( [Q, Q] = 0 \) and \( \tilde{Q} = 1 \). We denote it by (\( \mathcal{M}, Q \)). A morphism of \( Q \)-manifolds from (\( \mathcal{M}, Q \)) to (\( \mathcal{N}, Q' \)) is a map \( F : \mathcal{M} \to \mathcal{N} \) such that \( Q \) and \( Q' \) are \( F \)-related, i.e. \( Q \circ F^* = F^* \circ Q' \). If \( F \) is an arbitrary map then we call \( Q \circ F^* - F^* \circ Q' \) the field strength of the map \( F \).

Let \( x^A \) be coordinates on a supermanifold \( \mathcal{M} \). Then the antitangent bundle \( \Pi \mathcal{T} \mathcal{M} \) is a \( Q \)-manifold with \( Q = d = dx^A \frac{\partial}{\partial x^A} \). If \( \mathfrak{g} \) is a Lie algebra with a basis \( \{ e_i \} \), then we have structure constants \( c_{ij}^k \) with \( [e_i, e_j] = c_{ij}^k e_k \). Then \( \Pi \mathfrak{g} \) is a \( Q \)-manifold with \( Q = -\frac{1}{2} c_{ij}^k \xi^i \xi^j \frac{\partial}{\partial \xi^k} \). If (\( \mathcal{M}, Q_M \)) and (\( \mathcal{N}, Q_N \)) are \( Q \)-manifolds, then \( \mathcal{M} \times \mathcal{N} \) is in a natural way a \( Q \)-manifold with \( Q_{\mathcal{M} \times \mathcal{N}} = Q_M \oplus Q_N \).

**Definition 10.** A graded manifold is a supermanifold whose coordinates are assigned degrees (weights) in \( \mathbb{Z} \) such that the coordinate transformations are polynomials in the coordinates with non-zero weight and preserve the weights.

Let \( x^A \) be coordinates on a supermanifold \( \mathcal{M} \). Then \( \Pi \mathcal{T} \mathcal{M} \) is a graded manifold with \( w(x^A) = 0 \) and \( w(dx^A) = 1 \). More generally, if \( \mathcal{E} \) is a vector bundle over \( \mathcal{M} \) with \( x^A \) coordinates on \( \mathcal{M} \) and \( y^B \) coordinates on the fibre, then \( \mathcal{E} \) is a graded manifold by assigning \( w(x^A) = 0 \) and \( w(y^B) = 1 \).

### 2.3 Super Lie groups

**Definition 11.** A super Lie group is a supermanifold \( \mathcal{G} = (\mathcal{G}, \mathcal{O}) \) with group structure morphisms: the multiplication map \( \mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \) satisfying the axiom of associativity \( \mu \circ (\text{id} \times \mu) = \mu \circ (\mu \times \text{id}) \), the inverse map \( i : \mathcal{G} \to \mathcal{G} \), and the identity map \( e : \mathbb{R}^{0|0} \to \mathcal{G} \) satisfying the axioms \( \mu \circ (i \times \text{id}) = \mu(\text{id} \times i) = e \) and \( \mu(\text{id} \times e) = \mu(e \times \text{id}) \), where

\[
\begin{align*}
\varepsilon : \mathcal{G} & \to \mathcal{G} \\
\varepsilon(x) &= e \\
\varepsilon^*(f) &= v(f)(e).
\end{align*}
\]

Equivalently a super Lie group can be defined using the functor of points approach which is more intuitive. If \( \mathcal{G} \) is a super Lie group then the set of \( S \)-points \( \mathcal{G}(S) \) is a group and the maps \( \mu, e \) and \( i \) will give the following natural transformations: the
multiplication map $\mu_S : \mathcal{G}(S) \times \mathcal{G}(S) \longrightarrow \mathcal{G}(S)$ in the group $\mathcal{G}(S)$, the identity map $e_S : \mathbb{R}^{0|0}(S) \longrightarrow \mathcal{G}(S)$ and the inverse map $i_S : \mathcal{G}(S) \longrightarrow \mathcal{G}(S)$. If $\phi : S \longrightarrow T$ is a morphism, then $G_\phi : \mathcal{G}(T) \longrightarrow \mathcal{G}(S)$ is a group homomorphism. Hence, $\mathcal{G}$ is a contravariant functor from the category of supermanifolds $\text{SM}$ to the category of groups $\text{Grp}$. Therefore we have the following alternative definition.

**Definition 12.** A super Lie group is a supermanifold $\mathcal{G}$ such that $\mathcal{G}(S)$ is a group for any supermanifold $S$ and $G_\phi : \mathcal{G}(T) \longrightarrow \mathcal{G}(S)$ is a group homomorphism for any morphism $\phi : S \longrightarrow T$.

### 2.3.1 The super Lie algebra of a super Lie group

If $G$ is a Lie group, then a vector field $X$ on $G$ is left-invariant if $dL_gX_h = X_{g.h}$ for all $h, g$ in $G$. $L_g$ is the left action corresponding to $g$. This can be rewritten in terms of the multiplication map $\mu$ as $(I \otimes X) \circ \mu^* = \mu^* \circ X$, where $I \otimes X$ acts on the first variable by identity and by $X$ on the second. Hence we have the following definition.

**Definition 13.** Let $\mathcal{G}$ be a super Lie group. Then $X \in \mathfrak{X}(\mathcal{G})$ is said to be left-invariant if $(I \otimes X) \circ \mu^* = \mu^* \circ X$.

**Theorem 2 ([5]).** There is a one-to-one correspondence between the left-invariant vector fields of $\mathcal{G}$ and the tangent space of $\mathcal{G}$ at the identity $T_e \mathcal{G}$.

**Example 3.** $\mathbb{R}^{1|1}$ is a super Lie group by defining $(t, \theta), (t', \theta') = (t + t' + \theta\theta', \theta + \theta')$ with identity $e = (0, 0)$ and the inverse map $(t, \theta) \longrightarrow (-t, -\theta)$. $T_e \mathbb{R}^{1|1} = \text{span}\{ \frac{\partial}{\partial t}|_e, \frac{\partial}{\partial \theta}|_e \}$. The corresponding left-invariant vector fields are $\frac{\partial}{\partial t}$ and $-\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$. Let $X = \frac{\partial}{\partial t}$ and $Y = -\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$. Then $(I \otimes X)\mu^*(s) = (I \otimes X)(t + t' + \theta\theta') = \mu^*X(s) = 1$ and $(I \otimes X)\mu^*(\xi) = (I \otimes X)(\theta + \theta') = \mu^*X(s) = 1$. The corresponding left-invariant vector field for $Y$ can be done in the same way.
Chapter 3

Local description of the Atiyah sequence

In this chapter, we describe locally the Atiyah sequence

\[ 0 \rightarrow \frac{P \times \mathfrak{g}}{G} \xrightarrow{j} \frac{TP}{G} \xrightarrow{\alpha} TM \rightarrow 0. \]

That is, we will construct local charts for the manifolds involved, write down the maps explicitly in local coordinates. We will describe the Lie brackets for the Atiyah algebroid \( \frac{TP}{G} \) and the adjoint bundle \( \frac{P \times \mathfrak{g}}{G} \). Then we define what we mean by connection in a Lie algebroid. All these will be given in local coordinates. This approach differs from the one taken by Mackenzie [19] which is mainly coordinate-free. This treatment allows us to see in a concrete way the relationship between the concepts just mentioned and the classical ones such as connection, curvature,...etc.

3.1 The Atiyah sequence

First we introduce the principal bundle which is of great importance in the theory of connections and characteristic classes.

**Definition 14.** A principal fiber bundle \( P(M, G, \pi) \) is a fiber bundle \( \pi : P \rightarrow M \) with \( G \)-action on \( P \), \( P \times G \rightarrow P \) where \( P \) and \( M \) are manifolds \( G \) a Lie group and \( \pi \) a surjective submersion such that the following conditions hold:

- The action of \( G \) is free, i.e. if \( R_g(z) = z \) for some \( z \) then \( g = 1 \).
- The quotient manifold \( P/G \) is diffeomorphic to the base manifold \( M \) via \( \pi(z) = \pi(\langle z \rangle) \), where \( \langle z \rangle \) is the equivalence class of \( z \in P \).
- There exists an open cover \( \{U_\alpha\} \) of \( M \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times G \\
\downarrow\pi & & \downarrow\text{pr} \\
U_\alpha & = & U_\alpha
\end{array}
\]
where $\phi_\alpha$ is a diffeomorphism and $\phi_\alpha(z) = (\pi(z), \varphi_\alpha(z))$ for some map $\varphi_\alpha$ and $\varphi_\alpha(z \cdot g) = \varphi_\alpha(z) \cdot g$.

If $P$ is isomorphic to $M \times G$ then $P$ is said to be a trivial principal bundle. We can have an action on the tangent bundle $TP$ in a natural way as follows. Let $(z, v) \in TP$ then $(z, v) \cdot g = (z \cdot g, dzR_g(v))$. If we let $X = (z, v)$, then $X \cdot g = dzR_gX$. Then, we can have the quotient manifold $\frac{TP}{G}$. This will be the centre of our attention in what is to come. We have $\frac{TP}{G} = \{ (z, v) : (z, v) \in TP \}$, where $(z, v)$ is the equivalence class of $(z, v)$. We will show that $\frac{TP}{G}$ is a vector bundle over $M$. We are going to construct local charts for it and a vector bundle structure. We define the projection map $\pi_2 : \frac{TP}{G} \rightarrow M$, $\pi_2(z, v) = \pi(z)$ which is well defined since $\pi(z \cdot g) = \pi(z)$. We define a vector space structure on $\frac{TP}{G}$ such that $\langle z, v \rangle$ is isomorphic to $\langle z \cdot g \rangle$. Then $\langle z \cdot g \rangle = \langle z \cdot g \cdot v \rangle$. Define $\langle z_1, v_1 \rangle + \langle z_2, v_2 \rangle = \langle z_1 + dz_1 \cdot v_2 \rangle$. This is well defined. Let $z_1 = (z_1, v_1)$ and $z_2 = (z_2, v_2)$ and let $z_1 = (u_1, w_1)$ and $z_2 = (u_2, w_1)$ be other representatives of $z_1$ and $z_2$ respectively. Then there are $h_1$ and $h_2 \in G$ such that $Y_1 = X_1 \cdot h_1$ and $Y_2 = X_2 \cdot h_2$, i.e. $(u_1, w_1) = (z_1, h_1 \cdot dz_1 \cdot R_{h_1}(v_1))$ and $(u_2, w_2) = (z_2, h_2 \cdot dz_2 \cdot R_{h_2}(v_2))$. Hence $u_1 = u_2h_2^{-1}gh_1$. Therefore

$$\langle Y_1 \rangle + \langle Y_2 \rangle = \langle u_1, w_1 + du_2R_{h_2^{-1}gh_1}(w_2) \rangle$$

$$= \langle z_1, h_1, dz_1 \cdot R_{h_1}(v_1) + du_2R_{h_2^{-1}gh_1}dz_2 \cdot R_{h_2}(v_2) \rangle$$

$$= \langle z_1 + dz_1 \cdot v_2 \rangle$$

$$= \langle z_1, v_1 + dz_2 \cdot R_{g}(v_2) \rangle$$

$$= \langle z_1, v_1 \rangle + \langle z_2, v_2 \rangle$$

$$= \langle X_1 \rangle + \langle X_2 \rangle.$$

Hence, addition is well defined. For multiplication we define $t(\langle X_1 \rangle) = \langle tX_1 \rangle$, which can easily be seen that it is well defined. Hence, $\frac{TP}{G}$ is a vector space. Now we define a map $a : \frac{TP}{G} \rightarrow TM$ by $a(\langle z, v \rangle) = (\pi(z), dz, \pi(v))$. We have $a(\langle z, g, dz \cdot R_g \rangle) = (\pi(z), dz, \pi(\pi(v)))$ since $\pi \circ R_g = \pi$. This shows that $a$ is well defined.

### 3.1.1 Local charts for $\frac{TP}{G}$

Suppose that $P = M \times G$ is the trivial principal bundle. We write the points of $TP$ as $(x, g, \bar{x}, \bar{g})$ where $x \in M$, $g \in G$, $\bar{x} \in T_xM$, $\bar{g} \in T_gG$. The elements of $\frac{TP}{G}$ become the equivalence classes as $\langle x, g, \bar{x}, \bar{g} \rangle$. Then $(x, h, \bar{x}, \bar{h})$ and $(y, u, \bar{y}, \bar{u})$ represent the same class if

\[
\begin{aligned}
y &= x \\
u &= h_1 \cdot g \\
\bar{y} &= \bar{x} \\
\bar{u} &= d_h \cdot R_g(\bar{h})
\end{aligned}
\]
for some \( g \in G \) and \( R_g \) is the right action on \( G \). And we have

\[
\pi_2 : \frac{TP}{G} \longrightarrow M
\]

\[
\pi_2(x, g, \bar{x}, \bar{g}) = x
\]

(3.2)

\[
a : \frac{TP}{G} \longrightarrow TM
\]

\[
a(x, g, \bar{x}, \bar{g}) = (x, \bar{x}).
\]

(3.3)

Let \( U_\alpha \) be some chart for \( M \), then we identify \( x \) and \( \bar{x} \) with the local coordinates \((x^1, \ldots, x^n)\) and \((\bar{x}^1, \ldots, \bar{x}^n)\) corresponding to \( U_\alpha \) to keep the notation tidy and simple. We have the bijection

\[
\phi : \frac{TP}{G} \longrightarrow TM \times \mathfrak{g}
\]

\[
\phi(x, g, \bar{x}, \bar{g}) = (x, \bar{x}, d_R h^{-1}(\bar{h}))
\]

(3.4)

\[
\phi^{-1} : TM \times \mathfrak{g} \longrightarrow \frac{TP}{G}
\]

\[
\phi^{-1}(x, \bar{x}, v) = (x, 1, \bar{x}, v).
\]

(3.5)

We define the topology of \( \frac{TP}{G} \) using these bijections and we have to check that it coincides with the quotient topology. We have the following natural projection \( i : TP \longrightarrow \frac{TP}{G} \), \( i(x, g, \bar{x}, \bar{g}) = (x, g, \bar{x}, \bar{g}) \) which is continuous. We will see later that the transition functions for \( \frac{TP}{G} \) are smooth. Therefore the two topologies coincide.

Now we consider an arbitrary principal bundle \( P \). Then we have local trivializations \( P|_{U_\alpha} \longrightarrow U_\alpha \times G = P_\alpha \) and \( \varphi_\alpha \) is equivariant, i.e. \( \varphi_\alpha(z, g) = \varphi_\alpha(z).g \). Then \( d\varphi_\alpha : TP|_{U_\alpha} \longrightarrow TP_\alpha \) is equivariant. To see this, let \((z, v) \in TP|_{U_\alpha} \) and notice that \( \varphi_\alpha(z, g) = \varphi_\alpha(z).g \) is the same as \( \varphi_\alpha R_g = R_g \varphi_\alpha \). Then

\[
d \varphi_\alpha((z, v) \cdot g) = d \varphi_\alpha d R_g(z, v)
\]

\[
= d R_g d \varphi_\alpha(z, v)
\]

\[
= d \varphi_\alpha(z, v) \cdot g.
\]

(3.6)

Hence we can define the quotient map \( d \varphi_\alpha : \frac{TP}{G}|_{U_\alpha} \longrightarrow \frac{TP_\alpha}{G} \) which is a bijection. We have as well \( d \varphi_{\beta \alpha} = d \varphi_{\beta} \varphi_\alpha : TP_{\alpha \beta} \longrightarrow TP_{\beta \alpha} \) is equivariant, where \( P_{\alpha \beta} = U_\alpha \cap U_\beta \times G \).

Hence, we have the quotient map \( d \varphi_{\beta \alpha} : \frac{TP_{\alpha \beta}}{G} \longrightarrow \frac{TP_{\beta \alpha}}{G} \) which is a bijection. Since we have local charts for \( \frac{TP_{\alpha \beta}}{G} \) we can find \( d \varphi_{\beta \alpha} \) in local coordinates. Let \( F = \varphi_{\beta \alpha} \). Then \( F : P_{\alpha \beta} \longrightarrow P_{\alpha \beta} \), \( F(x, h) = (x, g_{3 \alpha}(x).h) \). Let \( K(x, h) = g_{3 \alpha}(x).h \). Then \( K = m \circ f \) where \( f(x, h) = (g_{3 \alpha}(x), h) \) and \( m(g, h) = g.h \).

Let \( m_1(g) = m(g, h) = R_h g \) and \( m_2(h) = m(g, h) = L_g h \). Since \( K = m \circ f \), then \( dK = dm \circ df = (dm_1 + dm_2)df = (dR_h + dL_{g_{3 \alpha}(x)})df \). In matrix form we get

\[
dk = (dR_h, dL_{g_{3 \alpha}(x)})\begin{pmatrix} d g_{3 \alpha} & 0 \\ 0 & 1 \end{pmatrix}
\]

(3.7)
as block matrices. Hence
\[ dF = \begin{pmatrix} I & 0 \\ dR_h \cdot dg_{\beta\alpha} & dL_{g_{\beta\alpha}(x)} \end{pmatrix}. \] (3.8)

Hence
\[ dF(x, h, \bar{x}, \bar{h}) = \begin{pmatrix} I \\ dR_h \cdot dg_{\beta\alpha} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{h} \end{pmatrix} + \begin{pmatrix} dL_{g_{\beta\alpha}(x)} \end{pmatrix}(\bar{h}). \] (3.9)

Therefore
\[ dF(x, h, \bar{x}, \bar{h}) = (x, g_{\beta\alpha}(x) \cdot h, \bar{x}, dR_h \cdot dg_{\beta\alpha}(x) + dL_{g_{\beta\alpha}(x)}(\bar{h})). \] (3.10)

Let \( \eta_{\beta\alpha} = \phi_\beta d\bar{\phi}_\beta \phi_\alpha^{-1} \). Then
\[ \eta_{\beta\alpha}(x, \bar{x}, \bar{h}) = \phi_\beta d\bar{\phi}_\beta \phi_\alpha^{-1}(x, 1, \bar{x}, \bar{h}) = (x, g_{\beta\alpha}(x), \bar{x}, dL_{g_{\beta\alpha}}(\bar{h})). \]
\[ = (x, \bar{x}, d_{g_{\beta\alpha}}(x) R_{g_{\beta\alpha}^{-1}}(dg_{\beta\alpha}(x) + d_1 L_{g_{\beta\alpha}}(\bar{h}))). \]
\[ = (x, \bar{x}, d_{g_{\beta\alpha}}(x) R_{g_{\beta\alpha}^{-1}}dg_{\beta\alpha}(\bar{x}) + Ad(g_{\beta\alpha}(x)) (\bar{h})). \] (3.11)

That is
\[ \eta_{\beta\alpha}(X, v) = \left( X, dR_{g_{\beta\alpha}^{-1}}(x) dg_{\beta\alpha}(X) + Ad(g_{\beta\alpha}(x))(v) \right), \] (3.12)
or
\[ \eta_{\beta\alpha}(X, v) = \left( X, \Delta(g_{\beta\alpha})(X) + Ad(g_{\beta\alpha}(x))(v) \right), \] (3.13)

where \( \Delta(g) = dR_{g^{-1}(x)}dg \) is called the right Darboux derivative and \( g \) is a function taking values in \( G \). The transition functions for \( TP_G \) are smooth. Hence \( TP_G \) is a vector bundle.

### 3.1.2 Local charts for the adjoin bundle \( \frac{P \times G}{G} \)

\( \frac{P \times G}{G} \) is the associated vector bundle of \( P \). The action of \( G \) on \( P \times G \) is given by \( (z, v) \cdot g = (z \cdot g, Ad(g^{-1})(v)) \). In the case of \( P = M \times G \), i.e. it is trivial, then \( \langle x_1, h_1, v_1 \rangle = \langle x_2, h_2, v_2 \rangle \) if \( x_2 = x_1, h_2 = h_1, g, v_2 = v_1 Ad(g^{-1})(v_1) \) for some \( g \in G \).

We have the following bijection
\[
\tau : \frac{P \times G}{G} \rightarrow M \times G
\]
\[ \tau(x, h, v) = (x, Ad(h)v) \]
with inverse

\[ \tau^{-1} : M \times \mathfrak{g} \longrightarrow \frac{P \times \mathfrak{g}}{G} \]
\[ \tau_1(x,v) = (x,1,v). \]

Suppose now that we have an arbitrary principal bundle \( P \) over \( M \). We define a
projection \( \pi_1 : \frac{P \times \mathfrak{g}}{G} \longrightarrow M \), \( \pi_1(z,v) = \pi(z) \). Then we get an equivariant bijection
\( \widetilde{\varphi}_\alpha : P \times \mathfrak{g} |_{U_\alpha} \longrightarrow P_\alpha \times \mathfrak{g} \) by \( \widetilde{\varphi}_\alpha(z,v) = (\varphi_\alpha(z),v) \). Therefore we have the quotient map \( \hat{\varphi}_\alpha : \frac{P_\alpha \times \mathfrak{g}}{G} \longrightarrow \frac{P \times \mathfrak{g}}{G} \) which is a bijection as well. Using \( U_\alpha \) and \( U_\beta \) we get a
bijection \( \hat{\varphi}_{\beta\alpha} : \frac{P_\alpha \times \mathfrak{g}}{G} \longrightarrow \frac{P_\beta \times \mathfrak{g}}{G} \). We have a coordinate change
\( \chi_{\beta\alpha} = \tau_\beta \hat{\varphi}_{\beta\alpha} \tau_\alpha^{-1} : U_{\alpha\beta} \times \mathfrak{g} \longrightarrow U_{\alpha\beta} \times \mathfrak{g} \)
\[ \chi_{\beta\alpha}(x,v) = (x,\text{Ad}(g_{\beta\alpha}(x))v) . \] (3.14)

We see that the coordinate transformations are smooth and linear on the fibres.
So we define the topology of \( \frac{P \times \mathfrak{g}}{G} \) using the previous bijections. Since The natural
projection \( k : \frac{P \times \mathfrak{g}}{G} \longrightarrow P \times \mathfrak{g} \) is continuous, then it coincides with quotient topology.
Therefore we constructed a smooth vector bundle structure for \( \frac{P \times \mathfrak{g}}{G} \). Locally we have
\[ \pi_1 : \frac{P \times \mathfrak{g}}{G} \longrightarrow M \]
\[ \pi_1(x,v) = x. \] (3.15)

Define the map \( \Phi : P \times \mathfrak{g} \longrightarrow TP \) by \( \Phi(z,v) = \Phi_z(v) \) where \( \Phi_z \) is the differential
of the map \( \Phi_z : G \longrightarrow P \), \( \Phi_z(g) = z \cdot g \), i.e. \( \phi_z = d\Phi_z \). The map \( \Phi \) is equivariant, i.e.
\( \Phi((z,v) \cdot g) = \Phi(z,v) \cdot g \). Therefore we have the quotient map \( j : \frac{TP}{G} \longrightarrow \frac{P \times \mathfrak{g}}{G} \) where
\( j(z,v) = (\Phi(z,v)) \). We have the natural map
\[ i : TP \longrightarrow \frac{TP}{G} \]
\[ i(x,h,x,h) = (x,x,d_h R_{h^{-1}}(h)) . \] (3.16)

Let \( z = (x,h) \). Then we have the map \( \Phi_z : G \longrightarrow P \), \( \Phi_z(g) = (x,h \cdot g) \). Then,
\( \Phi_z(v) = (x,h,0,d_1 L_h(v)) \) because \( d\Phi_z(v) = \begin{pmatrix} 0 \\ d_1 L_h \end{pmatrix} (v) \). Hence
\[ \Phi(x,h,v) = (x,h,0,d_1 L_h(v)) , \] (3.17)

\[ j_{aa} = \phi_a \varphi_a j \hat{\varphi}_a^{-1} \tau_a^{-1} : U_\alpha \times \mathfrak{g} \longrightarrow TU_\alpha \times \mathfrak{g} \]
\[ j_{aa}(x,v) = \phi_a \varphi_a j \hat{\varphi}_a^{-1}(x,1,v) \]
\[ = \phi_a(x,1,0,d_1 L_h(v)) \]
\[ = (x,0,v) , \] (3.18)
and
\[
a : \frac{TP}{G} \longrightarrow TM
\]
\[
a(x, \bar{x}, v) = (x, \bar{x}).
\] (3.19)

We sum up everything in the following commutative diagram.
\[
\begin{array}{cccccc}
P \times G & \xrightarrow{\Phi} & TP & \xrightarrow{\pi_1} & P \times G & \xrightarrow{a} & TM & \xrightarrow{\pi_3} & 0. \\
\downarrow{k} & & \downarrow{i} & & \downarrow{\pi_2} & & \downarrow{\pi_3} & & \\
M & & M & & M & & M & & \\
\end{array}
\]
(3.20)

We want to prove that the middle row of the diagram is exact. So we have to show that \(j\) is injective and \(a\) is surjective and \(\text{Im}(j) = \ker(a)\). This can be seen at once from the local expressions of \(j\) and \(a\). Now we have to define Lie brackets for \(\frac{TP}{G}\) and \(P \times G\). Let \(E\) stand for \(\frac{TP}{G}\) and \(P \times G\) and \(I\) stand for \(i\) and \(k\). We define the set of \(G\)-invariant sections of on \(E\) as \(\Gamma E \subseteq \{X \in \Gamma E : X(z \cdot g) = X(z) \cdot g\}\).

Define \(\Psi : \Gamma \frac{E}{G} \longrightarrow \Gamma E \subseteq \{X \in \Gamma E : X(z \cdot g) = X(z) \cdot g\}\) by \((\Psi X)(z) = I_z^{-1}X(z)\pi z\) where \(I\) is the restriction of \(I\) to \(E \rightarrow \frac{E}{G}\) |$_\pi z$. Then \(\Gamma E \subseteq \{X \in \Gamma E : X(z \cdot g) = X(z) \cdot g\}\) is a \(C^\infty(M)\)-module defined by \(fX = (f \circ \pi)X\).

**Proof.** Let \(X_1 = (z, v_1), X_2 = (z, v_2) \in E_z\). Suppose that \(I_z(X_1) = I_z(X_2)\). Then \(\langle z, v_1 \rangle = \langle z, v_2 \rangle\) which implies that \(v_1 = v_2\). Hence \(X_1 = X_2\). Therefore \(I_z\) is injective.

\(I_z(\lambda X + \mu Y) = \langle \lambda X + \mu Y \rangle = \lambda \langle X \rangle + \mu \langle Y \rangle = \lambda I_z(X) + \mu I_z(Y)\). Hence \(I_z\) is an isomorphism of vector spaces.

We prove that \(\Psi\) is an isomorphism of \(C^\infty(M)\)-modules. Suppose \(X(\pi z) = \langle z, v \rangle\). Then
\[
\overline{X}(z \cdot g) = I_{z \cdot g}^{-1}X(\pi z)
= I_{z \cdot g}^{-1}(z, v)
= I_{z \cdot g}^{-1}(z \cdot g, v \cdot g)
= (z \cdot g, v \cdot g)
= (z, v) \cdot g
= I_z^{-1}X(\pi z) \cdot g
= \overline{X}(z) \cdot g.
\]

Hence, \(\overline{X} \in \Gamma E \subseteq \{X \in \Gamma E : X(z \cdot g) = X(z) \cdot g\}\) and therefore \(\Psi\) is well defined.

We now prove that \(\Psi\) is bijective. Suppose that \(\overline{X}(z) = \overline{Y}(z)\) all \(z\). Then \(I_z^{-1}X(\pi z) = I_z^{-1}Y(\pi z)\) all \(z\). Hence \(X(x) = Y(x)\) all \(x\) where \(\pi z = x\). Hence \(\Psi\) is injective.

Let \(A \in \Gamma E \subseteq \{X \in \Gamma E : X(z \cdot g) = X(z) \cdot g\}\). Then \(\Psi X = A\) where \(X(x) = \langle A(z) \rangle\) with \(\pi z = x\). Hence \(\Psi\) is
surjective. On the other hand we have
\[
\lambda X + \mu Y = I_z^{-1} (\lambda X + \mu Y)(\pi_i z)
\]
\[
= (\lambda \circ \pi_i)I_z^{-1} X(\pi_i z) + (\mu \circ \pi_i)I_z^{-1} Y(\pi_i z)
\]
\[
= (\lambda \circ \pi_i)X(z) + (\mu \circ \pi_i)Y(z)
\]
\[
= (\lambda X)(z) + (\mu Y)(z),
\]
where \(\lambda, \mu \in C^\infty(M)\). Hence \(\Psi\) is an isomorphism of \(C^\infty(M)\)-modules.

\[\square\]

### 3.1.3 Lie algebroid structure for \(\frac{TP}{G}\)

We define Lie brackets on \(\Gamma E\) by \([X,Y] = \Psi^{-1}[\Psi X,\Psi Y]\) where the second bracket is the bracket on \(\Gamma E\). In the case of \(E = TP\), they are the Lie brackets of vector fields.

**Lemma 2.** Suppose that Lie brackets \([\_]\) are given for \(\Gamma(P \times g)\) and \(\Phi[X,Y] = [\Phi X,\Phi Y]\). Then \(j[X,Y] = [jX,jY]\), where \(X = \Psi^{-1}X\).

**Proof.** We have \(jk = i\Phi\). Hence \(i\Phi[X,Y](z) = jk[X,Y](z)\). Therefore by the assumption given in the lemma \(i(\Phi X,\Phi Y)(z)) = j[X,Y](x)\). Hence \(\Phi X,\Phi Y)(x) = j[\Phi X,\Phi Y](x)\). This gives \(\Phi X,\Phi Y)(x) = j[X,Y](x)\). On the other hand \(\Phi X = jX\) since \(jk = i\Phi\). Hence \(j[X,Y] = [jX,jY]\) \(\square\).

Let \(X,Y \in \Gamma(P \times g)\) such that \(X(x,h) = (x,h,A(x,h))\) and \(Y(x,h) = (x,h,B(x,h))\). We define Lie brackets on \(\Gamma(P \times g)\) in the obvious way by setting \([X,Y](x,h) = (x,h,A(x,h),B(x,h))\). If \(X,Y \in \Gamma(P \times g)\), then \(A((x,h) \cdot g) = \Ad(g^{-1})A(x,h)\) for all \(g \in G\).

Hence \(A(x,h) = \Ad(h^{-1})A(x,1)\). We have \(\Phi X(x,h,v) = (x,h,0,d_1L_hv)\).

Hence \(\Phi X(x,h) = (x,h,0,d_1L_hA(x,h)) = (x,h,0,d_1R_hA(x,1))\) and \(\Phi Y(x,h) = (x,h,0,d_1R_hB(x,1))\). Therefore
\[
[X,Y](x,h) = (x,h,[A(x,h),B(x,h)])
\]
\[
= (x,h,\Ad(h^{-1})[A(x,1),B(x,1)]). \quad (3.21)
\]

On the other hand
\[
\Phi[X,Y](x,h) = (x,h,0,d_1L_h\Ad(h^{-1})[A(x,1),B(x,1)])
\]
\[
= (x,h,0,d_1R_h[A(x,1),B(x,1)]). \quad (3.22)
\]

Let \(\tilde{A}^x(h) = d_1R_hA(x,1)\) and \(\tilde{B}^x(h) = d_1R_hB(x,1)\). They are the right-invariant vector fields in \(G\) containing \(A(x,1)\) and \(B(x,1)\) respectively. Hence
\[
[\Phi X,\Phi Y](x,h) = \left(\begin{array}{c}
\tilde{A}^x(h)
\tilde{B}^x(h)
\end{array}\right)
\]
\[
= \left(\begin{array}{c}
x,h,0; d_1R_h\left[\begin{array}{c}
\tilde{A}^x(h)
\tilde{B}^x(h)
\end{array}\right](1)
\end{array}\right)
\]
\[
= (x,h,0,d_1R_h[A(x,1),B(x,1)]). \quad (3.23)
\]
CHAPTER 3. LOCAL DESCRIPTION OF THE ATIYAH SEQUENCE

From (3.22) and (3.23), we see that

\[ \Phi[X, Y] = [\Phi X, \Phi Y]. \] (3.24)

By Lemma (2)

\[ j[X, Y] = [jX, jY]. \] (3.25)

Write \( X(z) = (z, v(z)) \) and \( Y(z) = (z, w(z)) \). Then \( X(x) = \langle z, v(z) \rangle \) and \( Y(x) = \langle z, w(z) \rangle \) where \( \pi(z) = x \). We have \([X, Y](x) = [X, Y](x)\). Hence

\[ [X, Y](x) = \langle z, [v(z), w(z)] \rangle. \] (3.26)

If we locally have \( X(x) = (x, v(x)) \) and \( Y(x) = (x, w(x)) \), then

\[ [X, Y](x) = (x, [v(x), w(x)]), \] (3.27)

where \( X, Y \in \Gamma^{P_X \circ \pi}_{G} \).

Let \( X \in \Gamma^{TP}_{G} \) and \( X(x) = (x, \bar{x}(x), \bar{h}(x)) \). Then

\[ \bar{X}(z) = i_{\bar{x}}^{-1}(x, \bar{x}(x), \bar{h}(x)) \]
\[ = i_{\bar{x}}^{-1}(x, 1, \bar{x}(x), \bar{h}(x)) \]
\[ = i_{\bar{x}}^{-1}(x, h, \bar{x}(x), d_1 R_\eta \bar{h}(x)) \]
\[ = (x, h, \bar{x}(x), d_1 R_\eta \bar{h}(x)), \] (3.28)

where \( z = (x, h) \). Let \( \bar{x}(x) = \bar{x}^i(x) e_i(x) \) where \( e_i(x) = \frac{\partial}{\partial x_i}|_x \) and \( \bar{h}(x) = \bar{h}^j(x) u_j \)

for a basis \( \{u_j\} \) of \( g \). Since \([\bar{e}_i, \bar{e}_j] = 0\), then \([e_i, e_j] = 0\). Since \([\bar{u}_i, \bar{u}_j] = 0\), then \([u_i, u_j] = 0\). Using \( j \) we find \([u_i, u_j] = \delta^k_{ij} u_k \). Let \( X(x) = (x, \bar{x}(x), \bar{h}(x)) \) and \( Y(x) = (x, \bar{y}(x), \bar{g}(x)) \). From the local expression of \( a \), we see that \( a(e_i) = \frac{\partial}{\partial x_i} \) and \( a(u_i) = 0 \).

The Leibniz rule \([X, fY] = f[X, Y] + a(X)fY \) follows easily from the fact that

\[ \bar{X}(f \circ \pi) = a(X)(f \circ \pi). \]

\[ [X, fY] = f \circ \pi [X, Y] + \bar{X}(f \circ \pi)Y = f \circ \pi [X, Y] + a(X)(f \circ \pi)Y. \]

Hence \([X, fY] = f[X, Y] + a(X)fY \).

**Proposition 1.**

\[ [X, Y] = [\bar{x}^a e_a + \bar{h}^i u_i, \bar{y}^b e_b + \bar{g}^i u_i] \]
\[ = \left( \bar{x}^a \frac{\partial \bar{y}^b}{\partial x^a} - \bar{y}^a \frac{\partial \bar{x}^b}{\partial x^a} \right) e_b + \left( \delta^k_{ij} \bar{h}^i \bar{g}^j + \bar{x}^a \frac{\partial \bar{h}^k}{\partial x^a} - \bar{y}^a \frac{\partial \bar{h}^k}{\partial x^a} \right) u_k. \] (3.29)

If we write \( X \) as \((X, V)\) and \( Y \) as \((Y, V)\) where \( X \in \mathfrak{X}(U) \) and \( W \) a section of \( U \times g \), then

\[ [(X, V), (Y, W)] = [X, Y] + X(W) - Y(V) + [V, W]. \] (3.30)

Hence we have shown that \( T_P G \) is a Lie algebroid. This suggests the following definition of trivial Lie algebroid.

**Definition 15.** A trivial Lie algebroid is a Lie algebroid \( TM \times g \) for some manifold \( M \) with the Lie brackets defined as in (3.30). Its adjoint bundle is \( M \times g \).
3.1.4 Connection and curvature

In this section we introduce the concept of connection on a transitive Lie algebroid. We see then how to get the connection one-form in a principal bundle and we discuss curvature and see how this generalizes the usual curvature in a principal bundle and vector bundle. In doing so, we take a different approach from [19].

Definition 16. Let $A \to M$ be a transitive Lie algebroid. Then a bundle map $s : TM \to A$ such that $as = \text{id}_{TM}$ is called a Lie algebroid connection (or simply a connection). A connection reform is a bundle map $\omega : A \to L$ such that $\omega_j = \text{id}_L$.

In the case when $A = \frac{TP}{G}$, then we have

$$s : TM \to \frac{TP}{G}$$

$$s(x, \bar{x}) = (x, \bar{x}, h(x, \bar{x})). \quad (3.31)$$

For $\omega : \frac{TP}{G} \to \frac{P \times g}{G}$, we get $\omega(x, \bar{x}, v) = (x, \beta(x)(\bar{x}, v))$ where $\beta(x)$ is a linear map on the fibres ($\bar{x}, v$). Since $\omega_j(x, v) = \omega(x, 0, v) = (x, \beta(x)(0, v)) = (x, v)$, we get $\beta(x)(0, v) = v$ for all $x, v$

$$\beta(x)(\bar{x}, v) = \beta_x(\bar{x}) + v, \quad (3.32)$$

where $\beta_x(\bar{x}) = \beta(x)(\bar{x}, 0)$. We can see that $\beta_x : TU_\alpha \to g$ is a one-form.

$$\begin{array}{c|c}
P \times g & T_P \\
\downarrow k & \downarrow i \\
\frac{P \times g}{G} & \frac{TP}{G}
\end{array} \quad (3.33)$$

The connection form $\bar{\omega}$ corresponding to the connection reform $\omega$ is the bundle map $\bar{\omega} : TP \to P \times g$ such that $\bar{\omega}(z, v) = k_z^{-1}\omega(z, v)$. Let $p : M \times G \to G$ be the projection $p(x, g) = g$ and $\pi : P \to M$ the projection of $P$ on $M$ and $\omega_0 = d_hL_{h^{-1}}$ be the left Maurer-Cartan form in $G$. Then, we locally have

$$\bar{\omega}(x, h, \bar{x}, \bar{h}) = k_z^{-1}\omega(x, h, \bar{x}, \bar{h})$$

$$= k_z^{-1}\omega(x, 1, \bar{x}, d_hR_{h^{-1}}(\bar{h}))$$

$$= k_z^{-1}\omega(x, 1, \beta(x)(\bar{x}, d_hR_{h^{-1}}(\bar{h})))$$

$$= k_z^{-1}(x, h, \text{Ad}(h^{-1})\beta(x)(\bar{x}, d_hR_{h^{-1}}(\bar{h})))$$

$$= (x, h, \text{Ad}(h^{-1})\beta(x)(\bar{x}, d_hR_{h^{-1}}(\bar{h})))$$

$$= (x, h, \text{Ad}(h^{-1})\beta_x(\bar{x}) + d_hL_{h^{-1}}(\bar{h}))$$

$$= (x, h, \text{Ad}(h^{-1})\beta_x(\bar{x}) + d_hL_{h^{-1}}(\bar{h}))(\bar{x}, \bar{h})$$

$$= (x, h, \text{Ad}(h^{-1})\beta_x d\pi(\bar{x}, \bar{h}) + p^*\omega_0(\bar{x}, \bar{h})), \quad (3.34)$$

where $z = (x, h)$. Hence

$$\bar{\omega}_z = \text{Ad}(h^{-1})\beta_x d\pi + p^*\omega_0. \quad (3.35)$$
This is the expression of $\bar{\omega}$. We have to show that $\bar{\omega} \circ \Phi_z = \text{id}$.

We have

$$\bar{\omega} \circ \Phi_z(X) = k_z^{-1} \omega \langle \Phi_z(X) \rangle$$
$$= k_z^{-1} \omega \langle X \rangle$$
$$= k_z^{-1} \langle X \rangle$$
$$= X.$$  

Hence $\bar{\omega} \circ \Phi_z = \text{id}$. Let $X \in T_z P$. Then $X \cdot g \in T_{xg} P$.

We have $\text{Ad} \left( g^{-1} \right) \bar{\omega} \omega(X) = \text{Ad} \left( g^{-1} \right) k_z^{-1} \omega \langle X \rangle = \text{Ad} \left( g^{-1} \right) Y$ where $\langle Y \rangle = \omega \langle X \rangle$ and $Y \in \{ z \} \times g$. On the other hand

$$\bar{\omega} \omega(X,g) = k_z^{-1} \omega(X \cdot g) = k_z^{-1} \langle Y \rangle g = \text{Ad} \left( g^{-1} \right) Y.$$  

Hence $R^0_y \bar{\omega} = \text{Ad} \left( g^{-1} \right) \bar{\omega}$. Therefore $\bar{\omega}$ is a connection one-form.

**Proposition 2.** Consider the Atiyah Lie algebroid. Then for any connection $s$, there is a unique connection reform $\omega$ such that $j \omega + sa = \text{id}$.

**Proof.** First suppose we have $j \omega + sa = \text{id}$. Then locally we have

$$j \omega(x, \bar{x}, v) + sa(x, \bar{x}, v) = (x, \bar{x}, v).$$

Hence $j(x, \beta_x(\bar{x}) + v) + s(x, \bar{x}) = (x, \bar{x}, v)$. This implies $(x, 0, \beta_x(\bar{x}) + v) + (\bar{x}, \bar{h}(x, \bar{x})) = (x, \bar{x}, v)$.

That is $(x, \bar{x}, \beta_x(\bar{x}) + \bar{h}(x, \bar{x}) + v) = (x, \bar{x}, v)$. This says that

$$\beta_x(\bar{x}) = -\bar{h}(x, \bar{x}).$$  

(3.36)

Hence just choose $\beta_x(\bar{x}) = -\bar{h}(x, \bar{x})$. So in each local chart this is possible. But in the intersection they must coincide because $j \omega + sa = \text{id}$, and hence it is globally defined. And it is unique since $j \omega + sa = \text{id}$.

In a neighborhood $U_\alpha$, we have $s(x, \bar{x}) = (x, \bar{x}, \tilde{h}_\alpha(x, \bar{x}))$, and this transforms as

$$s(x, \bar{x}) = (x, \bar{x}, dR_{g^{-1}}(x) dg_{\beta}(\bar{x}) + \text{Ad}(g_{\beta}(x)) (\bar{h}_\alpha(x, \bar{x}))).$$

Hence we have

$$h_{\beta}(x) = \text{Ad}(g_{\beta}(x)) (\bar{h}_\alpha(x, \bar{x})) + dR_{g^{-1}}(x) dg_{\beta}(\bar{x}).$$

We will prove that

$$dR_{g^{-1}} \cdot dg_{\beta} = -dL_{g^{-1}}(x) dg_{\alpha \beta}.$$  

Let

$$m(x) = R_{g^{-1}(x_0)} g_{\beta}(x) = g_{\alpha \beta}^{-1}(x_0) g_{\alpha \beta}(x_0).$$

Then

$$d_{x_0} m = dR_{g^{-1}(x_0)} d_{x_0} g_{\beta}.$$  

And let

$$l(x) = g_{\alpha \beta}^{-1}(x_0) g_{\alpha \beta}(x) = L_{g^{-1}(x_0)} g_{\alpha \beta}(x).$$

Let $S(x) = (m(x), l(x))$ and $T(g, h) = g \cdot h$ and $T_1(g) = R_{hg}$ and $T_2(h) = L_{gh}$. We have $TS(x) = m(x) l(x) = 1$. Hence $d(1,1) T d_{x_0} S = 0$ at $x_0$. In matrix form we have
$dT = (dR_t, dL_g)$ and $dS = \begin{pmatrix} dm \\ dl \end{pmatrix}$.

Hence $dTdS = (dR_t dm + dL_m dl) = 0$. Since $m(x_0) = l(x_0) = 1$, then $dR_t(x_0) = dL_m(x_0) = \text{id}$, and therefore $dl = -dm$. We see that

$$\Delta^r(g^{-1}) = -\Delta^l(g), \quad (3.37)$$

where $\Delta^r$ and $\Delta^l$ are the right and left Darboux derivatives respectively. Hence we have the following proposition.

**Proposition 3.**

$$\bar{h}_\beta = \text{Ad} \left( g_{a\beta}^{-1} \right) \circ \bar{h}_\alpha - g_{a\beta}^* \omega_0 \quad (3.38)$$

$$\beta_\beta = \text{Ad} \left( g_{a\beta}^{-1} \right) \circ \beta_\alpha + g_{a\beta}^* \omega_0 \quad (3.39)$$

where $\omega_0$ is the standard left Maurer-Cartan form.

It could be seen that every connection $s$ defines local connection forms $(-\bar{h}_\alpha = \beta_\alpha)$.

### 3.2 The Lie algebroid of derivations $\mathfrak{D}(E)$

In this chapter we will describe the Lie algebroid of derivations $\mathfrak{D}(E)$ of a vector bundle $E$, and we will construct an explicit morphism of Lie algebroids from $TP_G$ to $\mathfrak{D}(P \times \mathcal{V}_G)$, where $V$ is a vector space on which the Lie group $G$ acts.

**Definition 17.** A Lie algebroid $A \rightarrow M$ is called transitive if the anchor $a$ is fibrewise surjective, regular if $a$ is of locally constant rank, and totally intransitive if $a = 0$.

Let $E \rightarrow M$ be a vector bundle over $M$. We shall define a Lie algebroid $\mathfrak{D}(E)$ called the Lie algebroid of derivations as follows. Let $D_x : \Gamma E \rightarrow E_x$. Then, we say that $D_x \in \mathfrak{D}(E)$ if there is a vector at $x$, $X_x \in T_x M$, such that $D_x(fs) = f(x)D_x(s) + X_x(f)s(x)$ for any function $f \in C^\infty(M)$ and any section $s \in \Gamma E$. And define $a(D_x) = X_x$. Then $\mathfrak{D}(E)$ is a vector bundle over $M$ by defining $\pi : \mathfrak{D}(E) \rightarrow M$ and $\pi(D_x) = x$. Observe that for $a(D_x) = 0$ we get the linear endomorphisms of $E$, and therefore we get the following exact sequence

$$\text{End}(E) \longrightarrow \mathfrak{D}(E) \longrightarrow TM. \quad (3.40)$$

**Definition 18.** Let $A$ be a Lie algebroid on $M$ and $E$ a vector bundle on $M$. A representation of $A$ on $E$ is a morphism of Lie algebroids $\rho : A \rightarrow \mathfrak{D}(E)$.

**Proposition 4.** $\mathfrak{D}(E)$ is a Lie algebroid.
CHAPTER 3. LOCAL DESCRIPTION OF THE ATIYAH SEQUENCE

Proof. The sections of $\mathcal{D}(E)$ are differential operators $D : \Gamma E \rightarrow \Gamma E$ such that $D(fs) = fD(s) + a(D)(f)s$ for any section $s \in \Gamma E$ and any function $f \in C^\infty(M)$. Let $D, D' \in \Gamma \mathcal{D}(E)$. Then

$$[D, D'](fs) = DD'(fs) - D'D(fs) = D(fD'(s) + a(D')(f)s) - D'(fD(s) + a(D)(f)s) = f[D, D'](s) + [a(D), a(D')](f)s,$$

where $a[D, D'] = [a(D), a(D')]$ by definition of the anchor $a$. Hence $[D, D'] \in \Gamma \mathcal{D}(E)$. For the Leibniz rule we have

$$[D, fD'] = D(fD')(s) - fD'(Ds) = DD'(s) + a(D)(f)D'(s) - fD'D(s) = f[D, D'](s) + a(D)(f)D'(s).$$

Hence $[D, fD'] = f[D, D'](s) + a(D)(f)D'$. \qed

Example 4. Let $\nabla$ be a connection on a vector bundle $E, \nabla : \mathcal{X}(M) \times \Gamma E \rightarrow \Gamma E$. We can see that $\nabla$ is a connection in the Lie algebroid $\mathcal{D}(E) \rightarrow TM$. To see this we have

$$\nabla_X fs = f\nabla_X s + X(f)s$$

with $a(\nabla_X) = (a \circ \nabla)(X) = X$, i.e. $a \circ \nabla = \text{id}$. The curvature $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the negative of the usual definition of curvature of $\nabla$.

Proposition 5. Let $s : TM \rightarrow TM \times g$ be a connection (anchor section) in the trivial Lie algebroid $TM \times g$, with $s(X) = (X, \omega(X))$. Then

$$R(X, Y) = (0, -\Omega(X, Y)). \tag{3.41}$$

Proof.

$$R(X, Y) = s[X, Y] - [sX, sY]$$
$$= ([X, Y], \omega[X, Y]) - ([X, \omega(X)), (Y, \omega(Y))]$$
$$= ([X, Y], \omega[X, Y]) - ([X, Y], X\omega(Y) - Y\omega(X) - [\omega(X), \omega(Y)]$$
$$= (0, X\omega(Y) - Y\omega(X) + \omega[X, Y] - [\omega(X), \omega(Y)])$$
$$= (0, -d\omega(X, Y) + [\omega(X), \omega(Y)])$$
$$= (0, -d\omega(X, Y) + \frac{1}{2}[\omega, \omega](X, Y))$$
$$= (0, -\Omega(X, Y))$$

where $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. \qed

We can see that to get the curvature of a connection in a vector bundle $E$ we take a section of $\mathcal{D}(E) \rightarrow TM$ and if we want to get the curvature two-form we take a
section of \( \frac{TF(E)}{G} \to TM \), where \( G \) is the general linear group of the frame bundle \( F(E) \).

### 3.2.1 Local description of \( \mathcal{D}(E) \)

Let \( \{e_i\} \) be a basis of \( E \) on some neighborhood \( U \) of \( M \). Then \( D(s) \) is determined by \( D(e_i) \) and \( d(D) \) for any section \( s = f^i e_i \) since \( D(f^i e_i) = f^i D(e_i) + a(D)(f^i)e_i \). Let \( a(D) = r^i \frac{\partial}{\partial x^i} \) and \( D(e_i) = t_i^r e_j \) for some \( r^i, t_i^r \in C^\infty(M) \). And any choice of \( r^i, t_i^r \in C^\infty(M) \) determines a differential operator \( D \in \mathcal{D}(E) \). We claim that we have an isomorphism \( \phi : \mathcal{D}(E)|_U \to TU \times \mathfrak{g} \) where \( \mathfrak{g} = \mathfrak{g}(\mathbb{R}^n) \), \( n \) is the rank of \( E \) and \( TU \times \mathfrak{g} \) is the trivial Lie algebroid. We have \( \phi(D_x) = (R(x) = r^i(x), T(x) = t_i^r(x)) \) if \( a(D_x) = r^i(x) \frac{\partial}{\partial x^i} |_x \) and \( D_x(e_i) = t_i^r(x) e_j(x) \) with \( T_{ji} = t_{ij} \). Let \( \phi D = (X, V) \) and \( \phi D' = (Y, W) \). We need to prove that

\[
\phi[D, D'] = ([X, Y], X(W) - Y(V) + [V, W]).
\]

\[
[D, D'](e_i) = DD'(e_i) - D'D(e_i)
= D(W^j_i e_j) - D'(V^j_i e_j)
= W^j_i D(e_j) + X(W^j_i) e_j - V^j_i D'(e_j) - Y(V^j_i) e_j
= V^j_i W^s_i e_j + X(W^j_i) e_j - V^j_i W^s_i e_j + Y(V^j_i) e_j
= ([V, W]^j_i + X(W^j_i) - Y(V^j_i)) e_j
\]

and \( a[D, D'] = [aD, aD'] = [X, Y] \). Hence,
\[
\phi[D, D'] = ([X, Y], X(W) - Y(V) + [V, W]).
\]

We used the opposite of the usual Lie brackets between matrices.

#### Change of coordinates

Let \( \{\bar{e}_i\} \) be another basis of \( \Gamma E|_U \) and \( \bar{e}_i \) be some other coordinates of \( U \) such that \( e_i = m_{ki} \bar{e}_k \).

We have

\[
D(e_i) = D(m_{ki} \bar{e}_k)
= m_{ki} D(\bar{e}_k) + \bar{r}^j \frac{\partial m_{ki}}{\partial \bar{x}^j} \bar{e}_k
= m_{ki} \bar{r}_k^j \bar{e}_s + \bar{r}^j \frac{\partial m_{si}}{\partial \bar{x}^j} \bar{e}_s
= (m_{ki} \bar{r}_k^s + \bar{r}^j \frac{\partial m_{si}}{\partial \bar{x}^j}) \bar{e}_s
= t_i^j m_{sj} \bar{e}_s.
\]

Hence \( t_i^j m_{sj} = m_{ki} \bar{r}_k^s + \bar{r}^j \frac{\partial m_{si}}{\partial \bar{x}^j} \).

Then \( (MT)_{si} = (TM)_{si} + \bar{r}^j \frac{\partial M}{\partial \bar{x}^j}_{si} \).

Therefore \( MT = TM + \bar{r}^j \frac{\partial M}{\partial \bar{x}^j} \).
Hence,

\[ T = M^{-1}\bar{T}M + M^{-1}r^j \frac{\partial M}{\partial x^j}. \]  

(3.42)

Or equally

**Proposition 6.**

\[ \bar{T} = MTM^{-1} + Mr^j \frac{\partial M^{-1}}{\partial x^j}. \]  

(3.43)

**Definition 19.** A transitive Lie algebroid \( A \to TM \) is locally trivial if there is an open cover \( \{ U_i \} \) of \( M \) such that there are local flat connections \( \Theta^i : TU_i \to A|_{U_i} \) and an atlas of bracket preserving trivializations \( \{ \psi_i : U_i \times g \to L|_{U_i} \} \) such that \( [\Theta^i(X), \psi_i(V)] = \psi_i(X(V)) \). The collection \( \{ U_i, \psi_i, \Theta^i \} \) is called Lie algebroid atlas.

\[
\begin{array}{cccc}
U_i \times g & \xrightarrow{j} & TU_i \times g & \xrightarrow{\alpha} & TU_i \\
\downarrow \psi_i & & \downarrow S^i & & \\
L_i & \xrightarrow{j} & A_i & \xrightarrow{\alpha} & TU_i
\end{array}
\]

where \( L_i = L_{U_i} \) and \( A_i = A_{U_i} \) and \( TU_i = TU_{U_i} \) and \( S^i \) the isomorphism defined by

\[
S^i(X, V) = \Theta^i(X) + \psi_i(V).
\]  

(3.44)

Let \( P(M, G, \pi) \) be a principal bundle and \( \rho : G \to \text{Gl}(V) \) be a representation of \( G \). Then we have the associated vector bundle \( \underbrace{P \times V}_G \) from the action \( (P \times V) \times V \to P \times V \) and \( (z \cdot g, \cdot v) = (z, g \cdot v) \) and \( \underbrace{P \times V}_G \) is a vector bundle over \( M \) with fibre \( V \). Let \( g_{\alpha \beta} \) be the transition functions of \( P \). Then the coordinate transformations of \( \underbrace{P \times V}_G \) are as \( (x, v) \to (x, \rho(g_{\beta \alpha})v) \) from \( (x, v) \to (x, h, v) \to (x, 1, h \cdot v) \to (x, h \cdot v) \) and \( (x, g_{\beta \alpha}(x) \cdot h, v) \to (x, 1, g_{\beta \alpha}(x) \cdot h, v) \to (x, 1, g_{\beta \alpha}(x) \cdot h, v) \to (x, g_{\beta \alpha}(x) h \cdot v) \). From 3.43 we can see that the change of coordinates of \( \mathcal{D}(\underbrace{P \times V}_G) \) is

\[ T = \rho(g_{\beta \alpha}) T \rho(g_{\alpha 1}^{-1}) + \rho(g_{\alpha \beta}) r^j \frac{\partial \rho(g_{\alpha \beta})}{\partial x^j}. \]  

(3.45)

Or better

\[ T_\beta = \rho(g_{\beta \alpha}) T_\alpha \rho(g_{\alpha 1}^{-1}) + \rho(g_{\beta \alpha}) X \rho(g_{\beta 1}^{-1}). \]  

(3.46)

When changing from the \( U_\alpha \) to \( U_\beta \) where \( X \) is the vector field \( r^j \frac{\partial}{\partial x^j} \). We define a morphism \( F : \underbrace{TP}_G \to \mathcal{D}(\underbrace{P \times V}_G) \) as

\[
TU_\alpha \times g \to TU_\alpha \times \text{gl}(V),
F_\alpha(X, v) = (X, d\rho(v))
\]

using local trivializations.

**Theorem 3.** The map \( F : \underbrace{TP}_G \to \mathcal{D}(\underbrace{P \times V}_G), F_\alpha(X, v) = (X, d\rho(v)) \) using local trivializations is a morphism of Lie algebroids.
Proof. We need to prove that $F$ is well defined. Actually what we have is $F = \eta_\alpha^{-1}F_\alpha\phi_\alpha$ where $\phi_\alpha$ and $\eta_\alpha$ are the trivialization maps of $TP_G$ and $\mathfrak{D}(P\times V)_G$ respectively corresponding to $U_\alpha$. If $F$ is defined using $\phi_\beta$ and $\eta_\beta$, then we should have $\eta_\alpha^{-1}F_\alpha\phi_\alpha = \eta_\beta^{-1}F_\beta\phi_\beta$, i.e. $\eta_\beta F_\alpha = F_\beta\phi_\beta\phi_\alpha^{-1}$, where $\eta_\beta = \eta_\beta\eta_\alpha^{-1}$ and $\phi_\beta = \phi_\beta\phi_\alpha^{-1}$. We have

$$\eta_\beta F_\alpha(X, v) = \eta_\beta (X, d\rho(v)) = (X, \rho(g_{\beta\alpha}) d\rho(v) \rho(g_{\beta\alpha}^{-1}) + \rho(g_{\beta\alpha}) X \rho(g_{\beta\alpha}^{-1})) . \tag{3.47}$$

On the other hand

$$F_\beta\phi_\beta\phi_\alpha(X, v) = F_\beta \left( X, \operatorname{Ad}(g_{\beta\alpha}) (v) + d \left( R_{g_{\beta\alpha}^{-1}}g_{\beta\alpha} \right) (X) \right)$$

$$= \left( X, d\rho \text{Ad}(g_{\beta\alpha}) (v) + d\rho \left( R_{g_{\beta\alpha}^{-1}}g_{\beta\alpha} \right) (X) \right)$$

$$= \left( X, \operatorname{Ad}(\rho(g_{\beta\alpha})) d\rho(v) + d \left( R_{\rho(g_{\beta\alpha}^{-1})} \rho(g_{\beta\alpha}) \right) (X) \right)$$

$$= \left( X, \rho(g_{\beta\alpha}) d\rho(v) \rho(g_{\beta\alpha}^{-1}) + \rho(g_{\beta\alpha}) X \rho(g_{\beta\alpha}^{-1}) \right) . \tag{3.48}$$

From (3.47) and (3.48) we conclude that $F$ is well defined. $F$ is a morphism of Lie algebroids. We have

$$F_\alpha [(X, v), (Y, w)] = ([X, Y], X(w) - Y(v) + [v, w])$$

$$= ([X, Y], d\rho X(w) - d\rho Y(v) + d\rho [v, w])$$

$$= ([X, Y], Xd\rho(w) - Yd\rho(v) + [d\rho(v), d\rho(w)])$$

$$= [(X, d\rho(v)), (Y, d\rho(w))]$$

$$= [F_\alpha(X, v), F_\alpha(Y, w)].$$

Hence $F$ is a morphism of Lie algebroids. 

Now let $V = \mathfrak{g}$ and $\rho$ be the adjoint representation of $G$ $\text{Ad} : G \to \text{Gl}(\mathfrak{g})$. Then $d\rho = \text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. We get the morphism $F : \frac{TP}{G} \to \mathfrak{D}(P\times \mathfrak{g})_G$ defined as $F_\alpha(X, v) = (X, \text{ad}(v))$ which is called the adjoint representation of $\frac{TP}{G}$ in $\mathfrak{D}(P\times \mathfrak{g})_G$. Consider a vector bundle $E$ with fibre $V$. Then $\frac{FE}{G}$ is isomorphic to $E$ where $G = \text{Gl}(V)$ and $FE$ is the frame bundle of $E$ and the representation of $\text{Gl}(V)$ is the identity representation, i.e. $\rho = \text{id}$. Applying the previous theorem, we get the following corollary.

**Corollary 1.** The map $F : \frac{FE}{G} \to \mathfrak{D}(E)$ defined locally as $F(X, v) = (X, v)$ is an isomorphism of Lie algebroids.
Chapter 4
Curvature and field strength

4.1 First approach

It is well known that the structure of Lie algebroid $E$ is equivalent to $Q$-manifold structure on $\Pi E$. We will prove this later in 4.2. From the Atiyah sequence

$$0 \longrightarrow \frac{P \times g}{G} \xrightarrow{j} \frac{TP}{G} \xrightarrow{a} TM \longrightarrow 0$$

we get

$$0 \longrightarrow \frac{P \times \Pi g}{G} \xrightarrow{j'\Pi} \frac{\Pi TP}{\Pi G} \xrightarrow{a\Pi} \Pi TM \longrightarrow 0.$$ 

If we have a Lie algebroid connection $\varphi : TM \longrightarrow TP$ then we have the corresponding map $\varphi^\Pi : \Pi TM \longrightarrow \Pi TP$. Let $x^i$ and $\theta^j$ be coordinates on $\Pi TM$ and $\xi^k$ be coordinates on $\frac{\Pi TP}{\Pi G}$. Then we have $\varphi^\Pi(x, \theta) = (x, \theta, \alpha^j(x) \theta^j)$ and the corresponding local one-form $\tilde{A} : TM \longrightarrow g$, $A = \alpha^j dx^j u_i$, where $\{u_i\}$ is a basis of $g$. In what follows we simply denote $\varphi^\Pi$ as $\varphi$.

Proposition 7. ([17]) Let $Q = Q_1 + Q_2$ where $Q_1 = \theta^i \frac{\partial}{\partial x^i}$ and $Q_2 = -\frac{1}{2} c_{ij}^k \xi^i \xi^j \frac{\partial}{\partial \xi^k}$. Then $(Q_1 \varphi^* - \varphi^* Q)(\omega, \alpha) = \sum_{k=1}^{k=p} (-1)^{k+1} \alpha \left( A, \ldots, \frac{k}{\partial A}, \ldots, A \right)$ where $\alpha \in \Lambda^p (\Pi TM)$ and $\omega \in C^\infty (\Pi TM)$.

We will give a proof of this proposition. First we check that $\varphi^*(\alpha \cdot \omega) = \frac{1}{p!} \alpha \cdot (A, A, \ldots, A) \cdot \omega$ and $\varphi^*(\alpha \cdot \omega) = \varphi^* \alpha \cdot \varphi^* \omega = \varphi^* \alpha \cdot \omega$. We need to prove that $\varphi^* \alpha = \alpha (A, \ldots, A)$, i.e. $\alpha \varphi = \alpha (A, \ldots, A)$.

Suppose that $\alpha = \alpha_0 \xi^1 \cdots \xi^r$, $\xi^i = \alpha^i_{jk} \theta^j$ and $\varphi^* = \alpha_0 \alpha_1 \cdots \alpha_r \theta^1 \cdots \theta^r$. Then, $\varphi^* \alpha = \varphi \alpha = \alpha_0 \alpha_1 \cdots \alpha_r \theta^1 \cdots \theta^r$. 

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We have

$$\alpha (A, \ldots, A) = \alpha_0 dx^1 \wedge \ldots \wedge dx^p \left( \alpha_{j_1}^{a_1} \theta^{j_1} u_{s_1}, \ldots, \alpha_{j_p}^{a_p} \theta^{j_p} u_{s_p} \right)$$

$$= \sum_{\sigma} \alpha_0 \sgn (\sigma) \alpha_{j_1}^{i_1(\sigma)} \ldots \alpha_{j_p}^{i_p(\sigma)} \theta^{j_1} \ldots \theta^{j_p}$$

$$= \sum_{\sigma} \alpha_0 \sgn (\sigma) \alpha_{j_1}^{i_1(\sigma)} \ldots \alpha_{j_p}^{i_p(\sigma)} \theta^{j_1(\sigma)} \ldots \theta^{j_p(\sigma)}$$

$$= \sum_{\sigma} \alpha_0 \sgn (\sigma) \alpha_{j_1}^{i_1(\sigma)} \ldots \alpha_{j_p}^{i_p(\sigma)} \sgn (\sigma) \theta^{j_1} \ldots \theta^{j_p}$$

$$= p! \alpha_0 \alpha_{j_1}^{i_1} \ldots \alpha_{j_p}^{i_p} \theta^{j_1} \ldots \theta^{j_p}. \quad (4.1)$$

Hence

$$\varphi^*(\alpha \cdot \omega) = \frac{1}{p!} \alpha (A, A, \ldots, A) \cdot \omega. \quad (4.2)$$

On the other hand

$$(Q_1 \varphi^* - \varphi^* Q) (\alpha \cdot \omega) = (Q_1 \varphi^* - \varphi^* (Q_1 + Q_2)) (\alpha \cdot \omega)$$

$$= Q_1 \varphi^* (\alpha \cdot \omega) - \varphi^* (Q_1 + Q_2) (\alpha \cdot \omega)$$

$$= Q_1 (\varphi^* (\alpha) \cdot \omega) - \varphi^* (Q_1 (\alpha \cdot \omega) + Q_2 (\alpha \cdot \omega))$$

$$= Q_1 (\varphi^* (\alpha) \cdot \omega) - \varphi^* Q_1 (\alpha \cdot \omega) - \varphi^* Q_2 (\alpha \cdot \omega)$$

$$= Q_1 (\varphi^* (\alpha) \cdot \omega) - \varphi^* (-1)^{i_1(\sigma)} \alpha Q_1 (\omega) - \varphi^* (Q_2 (\alpha) \cdot \omega))$$

$$= Q_1 \varphi^* (\alpha) \cdot \omega - (-1)^{i_1(\sigma)} \alpha Q_1 (\omega) - (-1)^a \varphi^* (\alpha) Q_1 \omega$$

$$= Q_2 (\alpha) \cdot \omega$$

$$= (Q_1 \varphi^* (\alpha) - \varphi^* Q_2 (\alpha) \cdot \omega. \quad (4.3)$$

Hence

$$(Q_1 \varphi^* - \varphi^* Q) (\alpha \cdot \omega) = (Q_1 \varphi^* (\alpha) - \varphi^* Q_2 (\alpha)) \cdot \omega. \quad (4.4)$$

We only need to prove that $Q_1 \varphi^* (\alpha) - \varphi^* Q_2 (\alpha) = \sum_{k=1}^{k=p} (-1)^{k+1} \alpha \left( A, \ldots, F, A, \ldots, A \right)$. Let $\alpha = \alpha_0 \xi^1 \ldots \xi^p$.

Then

$$Q_1 \varphi^* (\alpha) - \varphi^* Q_2 (\alpha) = \frac{\partial}{\partial x^a} \left( \alpha_0 \alpha_{j_1}^{i_1} \ldots \alpha_{j_p}^{i_p} \theta^a \theta^{j_1} \ldots \theta^{j_p} - \varphi^* \left( -\frac{1}{2} C_{ij}^k \xi^i \xi^j \frac{\partial \alpha \xi^1 \ldots \xi^p}{\partial \xi^k} \right) \right)$$

$$= \frac{\partial}{\partial x^a} \left( \alpha_0 \alpha_{j_1}^{i_1} \ldots \alpha_{j_p}^{i_p} \theta^a \theta^{j_1} \ldots \theta^{j_p} + \frac{1}{2} (-1)^{k+1} \alpha_0 C_{ij}^k \xi^i \xi^j \xi^k \ldots \xi^p \right)$$

$$= \frac{\partial}{\partial x^a} \left( \alpha_0 \alpha_{j_1}^{i_1} \ldots \alpha_{j_p}^{i_p} \theta^a \theta^{j_1} \ldots \theta^{j_p} + \frac{1}{2} (-1)^{k+1} \alpha_0 C_{ij}^k \alpha_0 \alpha_{j_1}^{i_1} \ldots \alpha_{j_k}^{i_k} \theta^a \theta^{j_1} \ldots \theta^{j_k} \ldots \theta^{j_p} \ldots \theta^{j_p} \right.$$}

Hence
\[ Q_1 \varphi^*(\alpha) - \varphi^* Q_2(\alpha) = \frac{\partial}{\partial x^a} \left( \alpha_0 \alpha_j^1 \cdots \alpha_j^p \right) \theta^a \theta^{j_1} \cdots \theta^{j_p} \]
\[ + \frac{1}{2} (-1)^{k+1} \alpha_0 \alpha_j^k \alpha_i^1 \alpha_j^1 \cdots \alpha_j^k \alpha_j^k \cdots \alpha_j^p \theta^a \theta^{j_1} \cdots \theta^{j_k} \cdots \theta^{j_p}. \]

(4.5)

On the other hand, suppose that \( A = \alpha_j^j dx^j u_i \).
Then \( F_A = dA + \frac{1}{2} [A, A] \) gives us

\[ F_A = \frac{\partial \alpha_j^j}{\partial x^a} dx^a \wedge dx^j u_i + \frac{1}{2} [\alpha_j^j dx^j u_i, \alpha_j^s dx^s u_k] \]
\[ = \frac{\partial \alpha_j^j}{\partial x^a} dx^a \wedge dx^j u_i + \frac{1}{2} \alpha_j^j \alpha_j^k dx^j \wedge dx^k c_{ls}^j u_k \]
\[ = \left( \frac{\partial \alpha_j^j}{\partial x^a} + \frac{1}{2} \alpha_j^s \alpha_j^l c_{st}^j \right) \theta^a \theta^j u_i. \]

(4.6)

\[ \alpha \left( A, \ldots, \frac{1}{k} F_A, \ldots A \right) = \alpha \left( \alpha_j^1 \theta^{j_1} u_{i_1}, \ldots, \left( \frac{\partial \alpha_j^j}{\partial x^a} + \frac{1}{2} \alpha_j^s \alpha_j^l c_{st}^j \right) \theta^{a} \theta^{j_k} u_{i_k}, \ldots, \alpha_j^p \theta^{j_p} u_{i_p} \right) \]
\[ = \alpha_j^i \cdots \left( \frac{\partial \alpha_j^j}{\partial x^a} + \frac{1}{2} \alpha_j^s \alpha_j^l c_{st}^j \right) \]
\[ \ldots \alpha_j^p \theta^{j_p} \ldots \left( \theta^a \theta^{j_k} \right) \ldots \theta^p \alpha_1^1 \ldots \xi^p \left( u_{i_1}, \ldots, u_{i_p} \right) \]
\[ = \alpha_j^i \cdots \frac{\partial \alpha_j^j}{\partial x^a} \ldots \alpha_j^p \theta^{j_p} \ldots \left( \theta^a \theta^{j_k} \right) \ldots \theta^p \alpha_1^1 \ldots \xi^p \left( u_{i_1}, \ldots, u_{i_p} \right) \]

(I)
\[ + \alpha_j^i \cdots \frac{1}{2} \alpha_j^s \alpha_j^l c_{s1}^j \ldots \alpha_j^p \theta^{j_p} \ldots \left( \theta^a \theta^{j_k} \right) \ldots \theta^p \alpha_1^1 \ldots \xi^p \left( u_{i_1}, \ldots, u_{i_p} \right). \]

(II)

We have

\[ (I) = (-1)^{k+1} \alpha_0 \alpha_j^i \cdots \frac{\partial \alpha_j^j}{\partial x^a} \ldots \alpha_j^p \theta^{j_1} \ldots \theta^{j_p} \xi^1 \ldots \xi^p \left( u_{i_1}, \ldots, u_{i_p} \right) \]
\[ = \sum_{\sigma} (-1)^{k+1} \text{sgn}(\sigma) \alpha_0 \left( \alpha_j^\sigma(1) \cdots \frac{\partial \alpha_j^\sigma(k)}{\partial x^a} \cdots \alpha_j^\sigma(p) \right) \theta^a \theta^{j_1} \ldots \theta^{j_p} \]
\[ = \sum_{\sigma} (-1)^{k+1} \text{sgn}(\sigma) \alpha_0 \left( \alpha_j^{\sigma(1)} \cdots \frac{\partial \alpha_j^{\sigma(k)}}{\partial x^a} \cdots \alpha_j^{\sigma(p)} \right) \theta^a \theta^{\sigma(1)} \ldots \theta^{\sigma(p)} \]
(II) = \sum_{e} (-1)^{k+1} \frac{1}{2} \text{sgn}(\sigma) \alpha_0 \left( \alpha_{s}^{(1)} \ldots \alpha_{s}^{(k)} \ldots \alpha_{j_p}^{(p)} \right) \alpha_{a_1}^{c} \alpha_{b_1}^{d} \theta^{a_1} \theta^{b_1} \ldots \theta^{b_p} \\
= \sum_{e} (-1)^{k+1} \frac{1}{2} \text{sgn}(\sigma) \alpha_0 \left( \alpha_{s}^{(1)} \ldots \alpha_{s}^{(k)} \ldots \alpha_{j_p}^{(p)} \right) \alpha_{a_1}^{c} \alpha_{b_1}^{d} \theta^{a_1} \theta^{b_1} \ldots \theta^{b_p} \\
= (-1)^{k+1} (p-1)! \frac{1}{2} \alpha_{0}^{c} \alpha_{j_1}^{c} \ldots \alpha_{j_k}^{c} \alpha_{j_1}^{d} \alpha_{j_1}^{d} \ldots \alpha_{j_p}^{d} \theta^{a_1} \theta^{b_1} \ldots \theta^{b_k} \ldots \theta^{b_p}.

Hence

\sum_k (-1)^{k+1} \alpha \left( A, \ldots, A \right) = \frac{1}{p!} \frac{\partial}{\partial x^a} \left( \alpha_{0}^{c} \ldots \alpha_{p}^{c} \right) \theta^{a} \theta^{b} \ldots \theta^{b_p} \\
+ \frac{1}{p!} \alpha_{0}^{c} \alpha_{1}^{d} \ldots \alpha_{p}^{d} \theta^{a_1} \theta^{b_1} \ldots \theta^{b_p} \\
= Q_{1} \varphi^*(\alpha) - \varphi^* Q_{2}(\alpha).

Therefore

\left( Q_{1} \varphi^* - \varphi^* Q \right) \left( \alpha \cdot \omega \right) = \frac{1}{p!} \sum_{k=1}^{k=p} (-1)^{k+1} \alpha \left( A, \ldots, A \right) \cdot \omega \\
= \frac{1}{p!} \sum_{k=1}^{k=p} \alpha \left( F_A, A, \ldots, A \right) \cdot \omega.

Let $\mathcal{F}_\varphi = Q_{1} \varphi^* - \varphi^* Q$ be the field strength corresponding to $\varphi$. We define a pairing $\langle F_A, \alpha \cdot \omega \rangle = p! \alpha \left( F_A, \ldots, A \right) \cdot \omega$. Then $\langle F_A, \alpha \cdot \omega \rangle = \mathcal{F}_\varphi \left( \alpha \cdot \omega \right)$

Define $I (F_A) = \mathcal{F}_\varphi$. Then $I$ is a bijection. Suppose that $I (F_A) = I (F_{A'}).$ This implies that $\mathcal{F}_\varphi (\alpha \cdot \omega) = \mathcal{F}_\varphi (\alpha \cdot \omega)$ for all $\alpha$ and $\omega$. Therefore $\alpha \left( F_A, \ldots, A \right) = \alpha \left( F_{A'}, \ldots, A' \right)$. Hence $F_A = F_{A'}$.

4.2 Lie algebroid formulation

Let $E$ and $E'$ be Lie algebroids over the same base $M$ with anchors $a$ and $a'$ respectively. Let $F: E \rightarrow E'$ be a vector bundle map such that $\alpha' F = a$. Define a map $R : \Gamma E \times \Gamma E' \rightarrow \Gamma E'$ by $R(X, Y) = F [X, Y] - [FX, FY]$. Then $R$ is bilinear. We call $R$ the curvature of the bundle map $F$. Define functions $K_{ij}^a$ such that $R(e_i, e_j) = K_{ij}^a v_a$. Then $K_{ij}^a$ determine $R$ since $R$ is bilinear over functions. Let $\{e_i\}$ and $\{v_a\}$ be the respective bases of $E$ and $E'$. Let $x^d, \xi^i$ be coordinates on $E$ and $y^d, \eta^a$ be coordinates on $E'$. Then we have the corresponding homologous vector fields on $\Pi E$ and $\Pi E'$: $Q = \xi^i a^d_{i} \frac{\partial}{\partial x^d} - \frac{1}{2} \xi^{i} \xi^{j} c_{ij}^{k} \frac{\partial}{\partial y^k}$ and $Q' = \eta^a a^d_{a} \frac{\partial}{\partial y^d} - \frac{1}{2} \eta^a \eta^b c_{ab}^{c} \frac{\partial}{\partial \eta^c}$, where $[e_i, e_j] = c_{ij}^{k} \epsilon_k^d$, $[v_a, v_b] = c_{ab}^{c} \epsilon_c^d$, $a (e_i) = a^d_{i} \frac{\partial}{\partial x^d}$ and $a (v_a) = a^d_{a} \frac{\partial}{\partial y^d}$. We have
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\[ a'F(e_i) = a(e_i) = a_i^d \frac{\partial}{\partial x^d}, \text{ and let } F(e_i) = F_i^\alpha v_\alpha. \]  

Then we get

\[
a'F(e_i) = a' (F_i^\alpha v_\alpha) = F_i^\alpha a' (v_\alpha) = F_i^\alpha a'_e \frac{\partial}{\partial y^e} = F_i^\alpha a'^e \frac{\partial x^d}{\partial y^e} \frac{\partial}{\partial x^d}.
\]

Hence

\[
a_i^d = F_i^\alpha a'^e \frac{\partial x^d}{\partial y^e}.
\]

(4.7)

If we use the same coordinates on \( M \), we get

\[
a_i^d = F_i^\alpha a'^e.
\]

(4.8)

We have

\[
[F_e^i, F_e^j] = F_i^\alpha F_j^\beta v_\alpha F_j^\beta v_\beta
\]

On the other hand, we have

\[
[F_e^i, F_e^j] = \left( F_i^\alpha F_j^\beta \frac{\partial F_i^\gamma}{\partial x^d} \right) v_\gamma
\]

(4.9)

Hence, \( F[e_i, e_j] = F[e_i, F_e^j] = c_{ij}^k F_k^\gamma v_\alpha F_j^\beta v_\beta = c_{ij}^k F_k^\gamma v_\gamma \)

Therefore

\[
K_i^\gamma = c_{ij}^k F_i^\gamma F_j^\beta e_i^\alpha = a_i^d \frac{\partial F_i^\gamma}{\partial x^d} + a_j^d \frac{\partial F_j^\gamma}{\partial x^d}.
\]

(4.10)

We now define \( \mathbf{F} = Q F^* - F^* Q' \) the field strength of \( F \). Since \( \mathbf{F} \) is a derivation over \( F \), it is determined by \( \mathbf{F}(h) \) and \( \mathbf{F}(\eta^\alpha) \) where \( h \in C^\infty(M) \).

\( \mathbf{F}(h) = Q F^* h - F^* Q' h = \xi^i a_i^d \frac{\partial h}{\partial x^d} - F_i^\alpha \xi^e a_i^e \frac{\partial h}{\partial x^d} \)

If \( h = x^d \), then \( \mathbf{F}(x^d) = 0 \) by (4.8) and therefore \( \mathbf{F}(h) = 0 \) for all \( h \) by Lemma 1.
Hence we conclude that \( F(x^d) = 0 \) is equivalent to (4.8), i.e. \( a'F = a \). We have

\[
QF^*\eta^\nu - F^*Q'\eta^\nu = Q(F^\nu_s \xi^s) - F^*(\frac{1}{2}\eta^\alpha \eta^\beta c^\nu_{\alpha\beta}) \\
= \xi^i \xi^j a^i \frac{\partial F^\nu_s}{\partial x^d} - \frac{1}{2} \xi^i \xi^j \epsilon_{ij}^k F^\nu_k + \frac{1}{2} \epsilon^\nu_{\alpha\beta} F^\alpha_i F^\beta_j \xi^i \xi^j \\
= \left( a^i \frac{\partial F^\nu_j}{\partial x^d} - \frac{1}{2} \epsilon_{ij}^k F^\nu_k + \frac{1}{2} \epsilon^\nu_{\alpha\beta} F^\alpha_i F^\beta_j \right) \xi^i \xi^j \\
= \left( \frac{1}{2} \left( a^i \frac{\partial F^\nu_j}{\partial x^d} - a^j \frac{\partial F^\nu_i}{\partial x^d} \right) - \frac{1}{2} \epsilon_{ij}^k F^\nu_k + \frac{1}{2} \epsilon^\nu_{\alpha\beta} F^\alpha_i F^\beta_j \right) \xi^i \xi^j \\
= -\frac{1}{2} \left( \epsilon_{ij}^k F^\nu_k - F^\alpha_i F^\beta_j \epsilon^\nu_{\alpha\beta} - a^i \frac{\partial F^\gamma_j}{\partial x^d} + a^j \frac{\partial F^\gamma_i}{\partial x^d} \right) \xi^i \xi^j \\
= -\frac{1}{2} K^\nu_{ij} \xi^i \xi^j. \quad (4.11)
\]

We summarize this in the following theorem.

**Theorem 4.** Let \( E \) and \( E' \) be Lie algebroids over the same base \( M \) with anchors \( a \) and \( a' \) respectively. Let \( F : E \longrightarrow E' \) be a vector bundle map. Then we have the following.

- \( F(h) = 0 \) for all \( h \in C^\infty(M) \) if and only if \( a'F = a \).
- \( F(\eta^\nu) = -\frac{1}{2} K^\nu_{ij} \xi^i \xi^j \).
- Moreover \( F = -\frac{1}{2} K^\nu_{ij} \xi^i \xi^j F^* \frac{\partial}{\partial \eta^\nu} \), and if \( a'F = a \) then \( R = \frac{1}{2} K^\alpha_{ij} e^i_\alpha \wedge e^j_\alpha \).
- \( F = 0 \) if and only if \( F \) is a morphism of Lie algebroids.

Moreover we have an explicit correspondence between the field strength \( F \) and the curvature \( R \). If \( E = TM \) and \( E' = A \) a Lie algebroid, then we have the equivalence between field strength and curvature in the usual sense. Suppose \( \{e'_i\} \) and \( \{v'_\alpha\} \) are other bases for \( E \) and \( E' \) respectively, with \( e_i = m_{ji} e'_j \) and \( v_\alpha = l^\alpha_{\beta a} v'_\beta \). Then, we have \( R(m_{is} e'_i, m_{jt} e'_j) = R(e_s, e_t) = K^\gamma_{st} v_\gamma = K^\alpha_{st} l^\alpha_\gamma v'_\gamma \).

On the other hand we have \( R(m_{is} e'_i, m_{jt} e'_j) = m_{is} m_{jt} K^\alpha_{ij} l^\alpha_\beta \).
Hence,

\[
m_{is} m_{jt} K^\alpha_{ij} = K^\beta_{st} l^\alpha_\beta. \quad (4.12)
\]

**Definition 20.** Let \( E \) and \( E' \) be Lie algebroids over bases \( M \) and \( M' \) and anchors \( a \) and \( a' \) respectively. Let \( (F, f) \) be a vector bundle morphism such that \( df \circ a = a' \circ F \).

\[
E \xrightarrow{F} E' \\
\downarrow \pi \quad \downarrow \pi' \\
M \xrightarrow{f} N'
\]

Suppose that \( s, t \) are sections of \( E \) and that there are sections \( u_i, w_j \) of \( E' \) such that \( F \circ s = r^i(u_i \circ f) \) and \( F \circ t = z^j(w_j \circ f) \). This is called \( F \)-decomposition. Then we say that \( (F, f) \) is a morphism of Lie algebroids if

\[
F \circ [s, t] = r^i z^j ([u_i, w_j] \circ f) + a(s)(z^j)(w_j \circ f) - a(t)(r^i)(u_i \circ f). \quad (4.14)
\]
The definition of morphism of Lie algebroids over arbitrary bases is due to Higgins and Mackenzie (for historical background see [19]). This definition seems to be complicated and not obvious. This is because Lie brackets are defined on sections and the bundle map \((F, f)\) do not necessarily induce a map of sections. Using the \(Q\)-manifolds approach to Lie algebroids we will see that the definition becomes very simple and natural and the same as the definition of a morphism over the same base. For more background see [19]. This definition does not depend on the \(F\)-decomposition of sections. The \(F\)-decomposition always exists. To see this suppose that \(\{e_i\}\) is a basis of \(\mathcal{E}\) and \(\{v_\alpha\}\) a basis of \(\mathcal{E}'\). Suppose that \(s = s' e_i\). Then \(F s(x) = s'(x) F'^\alpha_i(x) v_\alpha(f x)\) for some functions \(F'^\alpha_i\) on \(M\). Then \(F \circ s = s' F'^\alpha_i (v_\alpha \circ f)\) which is an \(F\)-decomposition for \(s\). Let

\[
R(s, t) = F \circ [s, t] - r^i z_i ([u_i, w_j] \circ f) - a(s)(z^j)(w_j \circ f) + a(t)(r^i)(u_i \circ f). \tag{4.15}
\]

We immediately see that \(R(s, t) = - R(t, s)\). That is \(R\) is skewsymmetric. On the other hand we have

\[
R(s, gt) = F \circ [s, gt] - r^i g z^j ([u_i, w_j] \circ f) - a(s)(g z^j)(w_j \circ f) + a(gt)(r^i)(u_i \circ f)
\]

\[
= g F \circ [s, t] + a(s)(g F \circ t - g r^i z^j ([u_i, w_j] \circ f) - a(s)(g z^j)(w_j \circ f)
\]

\[
- ga(s)(z^j)(w_j \circ f) + ga(t)(r^i)(u_i \circ f)
\]

\[
= g R(s, t).
\]

Then \(R\) is a bilinear antisymmetric map \(R : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} f^* \mathcal{E}'\). From before we get \(F \circ e_i = F'^\alpha_i (v_\alpha \circ f)\). Then

\[
R(e_i, e_j) = F \circ [e_i, e_j] - F_i^a F_j^\beta ([v_\alpha, v_\beta] \circ f) - a(e_i) \left( F_j^\beta \right) (v_\beta \circ f) + a(e_j) \left( F_i^a \right) (v_\alpha \circ f)
\]

\[
= F^i_\gamma c^\gamma_i (v_\gamma \circ f) - F_i^a F_j^\beta \left( c^\gamma_{\alpha \beta} \circ f \right) (v_\gamma \circ f) - a_i^d \frac{\partial F_j^\gamma}{\partial x^d} (v_\gamma \circ f) + a_j^d \frac{\partial F_i^\gamma}{\partial x^d} (v_\gamma \circ f)
\]

\[
= \left( F^i_\gamma c^\gamma_i - F_i^a F_j^\beta \left( c^\gamma_{\alpha \beta} \circ f \right) - a_i^d \frac{\partial F_j^\gamma}{\partial x^d} + a_j^d \frac{\partial F_i^\gamma}{\partial x^d} \right) (v_\gamma \circ f) \tag{4.16}
\]

\[
= K_{ij}^\gamma (v_\gamma \circ f) \tag{4.17}
\]

where

\[
K_{ij}^\gamma = F^i_\gamma c^\gamma_{ij} - F_i^a F_j^\beta \left( c^\gamma_{\alpha \beta} \circ f \right) - a_i^d \frac{\partial F_j^\gamma}{\partial x^d} + a_j^d \frac{\partial F_i^\gamma}{\partial x^d}. \tag{4.18}
\]

Let \(\{e_i\}\) and \(\{v_\alpha\}\) be the respective bases of \(E\) and \(E'\). Let \(x^d, \xi^i\) be coordinates on \(E\) and \(y^d, \eta^\gamma\) be coordinates on \(E'\). We have the corresponding homological vector fields on \(\Pi E\) and \(\Pi E'\), \(Q = \xi^i a^d_i \frac{\partial}{\partial x^d} - \frac{1}{2} \xi^i \xi^j c^k_{ij} \frac{\partial}{\partial \xi^k}\) and \(Q' = \eta^\gamma a^d_\gamma \frac{\partial}{\partial y^d} - \frac{1}{2} \eta^\gamma \eta^\delta c^\gamma_{\alpha \beta} \frac{\partial}{\partial \eta^\alpha}\), where \(\{e_i, e_j\} = c^k_{ij} e_k\), \(\{v_\alpha, v_\beta\} = c^\gamma_{\alpha \beta} v_\gamma\), \(a(e_i) = a_i^d \frac{\partial}{\partial x^d}\) and \(a'(v_\alpha) = a_\alpha^d \frac{\partial}{\partial y^d}\). \(a' F(e_i x) = \ldots\)
\[ df(a(e,x)) = a^d_i(x)f^q_d(x) \frac{\partial}{\partial y^q}. \] Then we have
\[
\begin{align*}
    a' F(e,x) &= a' (F^a_i(x), v_\alpha(f(x))) \\
    &= F^a_i(x) a' (v_\alpha(f(x))) \\
    &= F^a_i(x) a^q_\alpha(f(x)) \frac{\partial}{\partial y^q} f(x).
\end{align*}
\]
Hence
\[ a^d_i(x)f^q_d(x) = F^a_i(x) a^q_\alpha(f(x)). \quad (4.19) \]
In other words
\[ a^d_i f^q_d = F^a_i \left( a^q_\alpha \circ f \right). \quad (4.20) \]

We now define \( F = Q F^* - F^* Q' \) the field strength of \( F \). Since \( F \) is a derivation over \( F \), it is determined by \( F(h) \) and \( F(\eta^\nu) \) where \( h \in C^\infty(M') \).
\[ F(h) = Q F^* h - F^* Q'h = \xi^i a^d_i \frac{\partial (h \circ f)}{\partial x^i} - F^a_i \xi^j (a^q_\alpha \circ f) f^* \frac{\partial h}{\partial y^q}. \]
If \( h = y^\gamma \), then \( F(y^\nu) = 0 \) by (4.20) and hence \( F(h) = 0 \) for all \( h \) by Lemma 1. We can see that \( F(h) = 0 \) is equivalent to (4.20), i.e. \( df \circ a = a' \circ F \).

We have
\[
\begin{align*}
    QF^* \eta^\nu - F^* Q' \eta^\nu &= Q \left( F^\nu \xi^\alpha \right) - F^* \left( -\frac{1}{2} \eta^\nu \eta^\beta c^\alpha_{\beta} \right) \\
    &= \xi^i \xi^\alpha \frac{\partial F^\nu_i}{\partial x^d} - \frac{1}{2} \xi^i \xi^j c^i_{\alpha \beta} F^\nu_j + \frac{1}{2} \left( c^\nu_{\alpha \beta} \circ f \right) F^a_i F^\beta_j \xi^i \xi^j \\
    &= \left( a^d_i \frac{\partial F^\nu_i}{\partial x^d} - \frac{1}{2} c^k_{ij} F^\nu_k + \frac{1}{2} \left( c^\nu_{\alpha \beta} \circ f \right) F^a_i F^\beta_j \right) \xi^i \xi^j \\
    &= -\frac{1}{2} \left( c^k_{ij} F^\nu_k - F^a_i F^\beta_j \left( c^\nu_{\alpha \beta} \circ f \right) - a^d_i \frac{\partial F^\gamma_i}{\partial x^d} + a^d_i \frac{\partial F^\gamma_j}{\partial x^d} \right) \xi^i \xi^j \\
    &= -\frac{1}{2} K^\nu_{ij} \xi^i \xi^j. \quad (4.21)
\end{align*}
\]

Hence, \( F(\eta^\nu) = -\frac{1}{2} K^\nu_{ij} \xi^i \xi^j \) as both the left and right hand sides are derivations and coincide on all \( h \) and \( \eta^\nu \). Hence we have the following theorem.

**Theorem 5.** Let \( E \) and \( E' \) be Lie algebroids over bases \( M \) and \( M' \) and anchors \( a \) and \( a' \) respectively. Let \( (F, f) \) be a vector bundle map from \( E \) to \( E' \). Then we have the following:

- \( F(h) = 0 \) for all \( h \in C^\infty(M') \) if and only if \( df \circ a = a' \circ F \).
- \( F(\eta^\nu) = -\frac{1}{2} K^\nu_{ij} \xi^i \xi^j \).
- Moreover \( F = -\frac{1}{2} K^\alpha_{ij} \xi^i \xi^j F^* \frac{\partial}{\partial y^\alpha} \), and if \( df \circ a = a' \circ F \) then \( R = \frac{1}{2} K^\alpha_{ij} \xi^i \xi^j + \epsilon^i_j \epsilon^\nu_{\alpha} \).
- \( F = 0 \) if and only if \((F, f)\) is a morphism of Lie algebroids.
Here \( v^1_\alpha \) denotes the pullback of the section \( v_\alpha \).

Theorem 4 is a special case of Theorem 5; but nevertheless we included them both so we may see the difference between them since for the second theorem the definition of a morphism of Lie algebroid over arbitrary bases seems to be much more complicated than in the case of Lie algebroids over the same base.

**Example 5.** Let \( P = \{(l, v) \in \mathbb{C}^n \times \mathbb{C}P^{n+1}, v \in l, v \neq 0\} \) where \( \mathbb{C}P^n \) is the \( n \)-dimensional complex projective space. Then \( P \) is a principal bundle over \( \mathbb{C}P^n \) with fibre \( \mathbb{C}^* \), the nonzero complex numbers. Let \( U_a = \{[z] = [z^1, \ldots, z^{n+1}] \in \mathbb{C}P^n : z^\alpha \neq 0\} \) and \( e_\alpha [z] = \left( \frac{z^1}{z^\alpha}, \ldots, 1, \ldots, \frac{z^{n+1}}{z^\alpha} \right) \) where 1 is in the \( \alpha \)-th place. We have the trivializations \( \phi_\alpha : P_{U_a} \rightarrow U_a \times \mathbb{C}^* \), \( \phi_i ([z], k e_\alpha [z]) = ([z], k) \). Then we have \( \phi_{\alpha \beta} = \phi_\alpha \phi_\beta^{-1} : U_{\alpha \beta} \times \mathbb{C}^* \rightarrow U_{\alpha \beta} \times \mathbb{C}^* \), \( \phi_{\alpha \beta} ([z], k) = ([z], \frac{z^\alpha}{z^\beta} k) \). Hence \( g_{\alpha \beta}[z] = \frac{z^\alpha}{z^\beta} \). The action \( P \times \mathbb{C}^* \rightarrow P, (l, v) \cdot \lambda = (l, v \cdot \lambda) \). Then we get the Atiyah sequence,

\[
P \times \mathbb{C}^* \xrightarrow{\overline{\jmath}} TP \xrightarrow{\alpha} T\mathbb{C}P^n.
\]

Let \( \omega^j \) be local coordinates on \( U_a \), where \( \omega^j = x^j + iy^j \). Suppose that \( A = (r^i_j dx^j + s^j_i dy^j) \) \( e_i \) is a local one-form. Then,

\[
A = \left( r^i_j \left( \frac{d\omega^j + d\bar{\omega}^j}{2} \right) + s^j_i \left( \frac{d\omega^j - d\bar{\omega}^j}{2i} \right) \right) e_i = \left( \frac{1}{2} \left( r^i_j - i s^j_i \right) \right) d\omega^j + \frac{1}{2} \left( r^i_j + i s^j_i \right) d\bar{\omega}^j e_i = \left( \alpha^i_j d\omega^j + \bar{\alpha}^i_j d\bar{\omega}^j \right) e_i \quad (4.22)
\]

where \( \alpha^i_j = \frac{1}{2} \left( r^i_j - i s^j_i \right) \) and \( e_1 = 1 \) and \( e_2 = i \).

Let \( A[z] = (r^i_j[z] d\omega^j + \bar{r}^i_j[z] d\bar{\omega}^j) \) \( e_i \). Then

\[
dA = \left( \frac{\partial r^i_j}{\partial \omega^a} d\omega^a + \frac{\partial \bar{r}^i_j}{\partial \bar{\omega}^a} d\bar{\omega}^a \right) d\omega^j + \left( \frac{\partial \bar{r}^i_j}{\partial \omega^a} d\omega^a + \frac{\partial \bar{r}^i_j}{\partial \bar{\omega}^a} d\bar{\omega}^a \right) d\bar{\omega}^j e_i = \left( \frac{\partial r^i_j}{\partial \omega^a} d\omega^a d\omega^j + \frac{\partial \bar{r}^i_j}{\partial \omega^a} d\omega^a d\bar{\omega}^j + \frac{\partial r^i_j}{\partial \bar{\omega}^a} d\bar{\omega}^a d\omega^j + \frac{\partial \bar{r}^i_j}{\partial \bar{\omega}^a} d\bar{\omega}^a d\bar{\omega}^j \right) e_i. \quad (4.23)
\]

Since \( \mathbb{C}^* \) is abelian, then \( A \wedge A = 0 \). Hence \( \Omega = dA + \frac{1}{2} [A, A] = dA \).

Let \( \omega^j, \theta^j \) be local coordinates on \( \Pi T \mathbb{C}P^n \), where \( \theta^j = d\omega^j \) and \( \bar{\theta}^j = d\bar{\omega}^j \). Then we have coordinates \( \omega^j, \theta^j, \xi \) on \( TP \mathbb{C}^2 \), where \( \xi \) are coordinates on \( \Pi \mathbb{C} \). The homological vector field on \( \Pi T \mathbb{C}P^n \), \( Q_1 = \theta^j \frac{\bar{\omega}^j}{\bar{\omega}^a} + \bar{\theta}^j \frac{\omega^a}{\bar{\omega}^a} \) and the homological vector field on \( TP \mathbb{C}^2 \), \( Q_2 = \theta^j \frac{\bar{\omega}^j}{\bar{\omega}^a} + \bar{\theta}^j \frac{\omega^a}{\bar{\omega}^a} \) locally since the structure constants of the Lie algebra \( \mathbb{C} \) are zero.

Let \( \varphi : \Pi T \mathbb{C}P^n \rightarrow \Pi TP \mathbb{C}^2 \) be a section corresponding to the curvature one-form \( A \). Then

\[
\varphi (\omega, \bar{\omega}, \theta, \bar{\theta}) = (\omega, \bar{\omega}, \theta, \bar{\theta}, r^i_j \theta^j + \bar{r}^i_j \bar{\theta}^j). \quad (4.24)
\]
By (4.4), \((Q_1\varphi^* - \varphi^* Q_2) (\alpha, \beta) = (Q_1\varphi^* \alpha) \cdot \beta\). If \(\alpha = \xi\), then

\[
Q_1\varphi^* \alpha = \left( \frac{\partial \theta^i}{\partial \omega^j} + \frac{\partial \bar{\theta}^j}{\partial \omega^i} \right) \left( r^j_i \theta^j + \bar{r}^j_i \bar{\theta}^j \right) e_i
\]

\[
= \left( \frac{\partial r^j_i}{\partial \omega^j} \theta^j + \frac{\partial \bar{r}^j_i}{\partial \omega^i} \theta^j + \frac{\partial r^j_i}{\partial \omega^j} \bar{\theta}^j + \frac{\partial \bar{r}^j_i}{\partial \omega^i} \bar{\theta}^j \right) e_i.
\]

(4.25)

Hence \(F(\xi \beta) = K \beta\) where \(K = \left( \frac{\partial r^j_i}{\partial \omega^j} \theta^j + \frac{\partial r^j_i}{\partial \omega^i} \theta^j + \frac{\partial r^j_i}{\partial \omega^j} \bar{\theta}^j + \frac{\partial r^j_i}{\partial \omega^i} \bar{\theta}^j \right) e_i\). We have \(\varphi^* \xi = (r^j_i \theta^j + \bar{r}^j_i \bar{\theta}^j) \bar{e}_i\). Hence

\[
Q_1\varphi^* \xi = \left( \frac{\partial \theta^i}{\partial \omega^j} + \frac{\partial \bar{\theta}^j}{\partial \omega^i} \right) \left( r^j_i \theta^j + \bar{r}^j_i \bar{\theta}^j \right) e_i
\]

\[
= \left( \frac{\partial r^j_i}{\partial \omega^j} \theta^j + \frac{\partial \bar{r}^j_i}{\partial \omega^i} \theta^j + \frac{\partial r^j_i}{\partial \omega^j} \bar{\theta}^j + \frac{\partial \bar{r}^j_i}{\partial \omega^i} \bar{\theta}^j \right) \bar{e}_i.
\]

(4.26)

Hence, \(F(\xi \cdot \beta) = K \cdot \beta\). Since \(F\) is a derivation over \(\varphi\), it is completely determined.
Chapter 5

Q-bundles and characteristic classes

5.1 Q-bundles

In this chapter we will recall constructions from the paper of Kotov and Strobl [17] that describe the construction of characteristic classes associated with a section (connection=gauge field) of a Q-bundle \( E(\mathcal{M}, \mathcal{F}, \pi) \). As in the case of principal bundles, the field strength of a given section (connection) will be used to construct them. This generalizes the Chern-Weil formalism on principal bundles. We will give detailed proofs for the propositions and lemmas that were omitted in [17].

**Definition 21.** A Q-bundle is a fibre bundle \( E(\mathcal{M}, \mathcal{F}, \pi) \) where \( \pi \) is a morphism of Q-manifolds with a holonomy algebra \( g \subseteq X^0(\mathcal{F}) \) such that the local trivializations are morphisms of Q-manifolds, i.e. there is an open cover \( \{ U_\alpha \} \) of \( \mathcal{M} \) such that \( \phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathcal{F} \) and \( \phi_\alpha \) is an isomorphism of Q-manifolds with the homological vector field of \( \pi^{-1}(U_\alpha) \) is just the restriction of \( Q_\mathcal{F} \). \( \phi_{\alpha\beta} = \phi_\alpha \circ \phi^{-1}_\beta : U_{\alpha\beta} \times \mathcal{F} \rightarrow U_{\alpha\beta} \times \mathcal{F} \) \( \phi_{\alpha\beta} \in \exp(g') \) where \( g' = \text{ad}_Q(C^\infty(U_{\alpha\beta}, g)) \cap X^0(U_{\alpha\beta} \times \mathcal{F}). \)

This definition needs some explanation. We see that a Q-bundle is a fibre bundle in the category of Q-manifolds with some compatibility conditions just given in the definition. The holonomy algebra \( g \) is defined as a graded Lie subalgebra of vector fields of \( \mathcal{F} \) of negative degree and closed under the derived brackets. This means that whenever \( X \) and \( Y \) are vector fields in \( g \), then \([X,Y]_{Q_\mathcal{F}} = [[X,Q_\mathcal{F}],Y]\) is in \( g \). A gauge field (or connection) is a section of \( \mathcal{E} \), i.e. a degree preserving map \( \varphi : \mathcal{M} \rightarrow \mathcal{E} \) such that \( \pi \circ \varphi = \text{id} \). The homological vector fields \( Q_\mathcal{M} \) and \( Q_\mathcal{E} \) can be seen as odd sections \( Q_\mathcal{M} : \mathcal{M} \rightarrow \Pi T \mathcal{M} \) and \( Q_\mathcal{E} : \mathcal{E} \rightarrow \Pi T \mathcal{E} \). Suppose that \( Q_\mathcal{M} = a^A \frac{\partial}{\partial x^A} \). Then \( Q_\mathcal{M}(x) = (x^A, a^A(x)) \). Consider the following diagram:

\[
\begin{array}{ccc}
\Pi T \mathcal{M} & \xrightarrow{d\varphi} & \Pi T \mathcal{E} \\
Q_\mathcal{M} \uparrow & & \uparrow Q_\mathcal{E} \\
\mathcal{M} & \xrightarrow{\varphi} & \mathcal{E}
\end{array}
\]

where \( d\varphi : \Pi T \mathcal{M} \rightarrow \Pi T \mathcal{E}, \ d\varphi(x^A, dx^A) = \left(y^B = \varphi^B(x), \ dy^B = dx^A \frac{\partial y^B}{\partial x^A}\right) \). Now define the map \( f : \mathcal{M} \rightarrow \Pi T \mathcal{E} \), as \( f = d\varphi \circ Q_\mathcal{M} - Q_\mathcal{E} \circ \varphi \). Then we have the
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following proposition.

**Proposition 8.**

\[ f^* (dh) = F (h), \]

\[ f^* (h) = \varphi^* (h) \] where \( h \in C^\infty (\mathcal{E}) \).

**Proof.** We have

\[ f (x) = \left( y^B = \varphi^B (x), dy^B = a^A (x) \frac{\partial y^B}{\partial x^A} - \varphi^* b^B (x) \right). \] (5.1)

From the local expression we can see that \( f^* (h) = \varphi^* (h) \). Therefore

\[
\begin{align*}
  f^* (dh) &= f^* (dy^B) \frac{\partial h}{\partial y^B} \\
  &= f^* (dy^B) f^* \left( \frac{\partial h}{\partial y^B} \right) \\
  &= a^A \frac{\partial y^B}{\partial x^A} - \varphi^* b^B \frac{\partial h}{\partial y^B} \\
  &= a^A \frac{\partial \varphi^* h}{\partial x^A} - \varphi^* \left( b^B \frac{\partial h}{\partial y^B} \right). \tag{5.2}
\end{align*}
\]

On the other hand we have

\[
\begin{align*}
  F (h) &= Q_M \varphi^* (h) - \varphi^* Q_{\mathcal{E}} (h) \\
  &= a^A \frac{\partial \varphi^* h}{\partial x^A} - \varphi^* \left( b^B \frac{\partial h}{\partial y^B} \right). \tag{5.3}
\end{align*}
\]

From (5.2) and (5.3) we conclude that \( F (h) = f^* (dh) \). \( \square \)

**Proposition 9.** \( f : (\mathcal{M}, Q_M) \longrightarrow (\Pi T \mathcal{E}, Q_{\Pi T \mathcal{E}}) \) is \( Q \)-morphism, where \( Q_{\Pi T \mathcal{E}} = d + \mathcal{L}_{Q_{\mathcal{E}}} \).

Therefore we need to prove \( Q_M \) and \( Q_{\Pi T \mathcal{E}} \) are \( f \)-related.

**Proof.** We need to prove that \( Q_M \circ f^* - f^* \circ Q_{\Pi T \mathcal{E}} = 0 \).

Since \( D = Q_M \circ f^* - f^* \circ Q_{\Pi T \mathcal{E}} \) is a derivation over \( f^* \) it suffices to prove \( D = 0 \) for \( h \in C^\infty (\mathcal{E}) \) and \( dh \).

For \( h \in C^\infty (\mathcal{E}) \) we have

\[
\begin{align*}
  D (h) &= Q_M f^* h - f^* Q_{\Pi T \mathcal{E}} h \\
  &= Q_M f^* h - f^* (dh) - f^* (\mathcal{L}_{Q_{\mathcal{E}}} h) \\
  &= Q_M f^* h - F (h) - f^* (\mathcal{L}_{Q_{\mathcal{E}}} h) \\
  &= (Q_M \circ f^* - f^* \circ Q_{\mathcal{E}}) (h) - F (h) \\
  &= F (h) - F (h) \\
  &= 0.
\end{align*}
\]
For the second part we have
\[
D (dh) = Q_M f^* (dh) - f^* (d + L_{Q_\epsilon}) (dh) \\
= Q_M f^* (dh) - f^* L_{Q_\epsilon} (dh) \\
= Q_M f (h) - f^* (i_{Q_\epsilon} dh - d i_{Q_\epsilon} dh) \\
= Q_M f (h) + f^* d Q_\epsilon (h) \\
= Q_M f (h) + F Q_\epsilon (h) \\
= Q_M (Q_M \varphi^* h - \varphi^* Q_\epsilon (h)) + (Q_M \varphi^* - \varphi^* Q_\epsilon) (Q_\epsilon h) \\
= -Q_M \varphi^* Q_\epsilon h + -Q_M \varphi^* Q_\epsilon h \\
= 0.
\]

And this completes the proof. \qed

A gauge transformation is a fibrewise acting inner automorphism=vertical inner derivation. A holonomy algebra is a graded Lie subalgebra of \( \mathfrak{X}(\mathcal{F}) \) closed under derived brackets. A vector field \( X \) on \( \mathcal{E} \) is called vertical if \( X \circ \pi^* = 0 \). A vector field \( X \in \mathfrak{X}^0 (\mathcal{E}) \) is infinitesimal symmetry if \( \text{ad}_{Q_\epsilon} X = [Q_\epsilon, X] = 0 \) and an inner derivation if there is a vector field \( Y \in \mathfrak{X}(\mathcal{E}) \) such that \( X = \text{ad}_{Q_\epsilon} Y \). Let \( \sigma_t : \mathcal{E} \longrightarrow \mathcal{E} \) be the flow of a vertical vector field \( X \in \mathfrak{X}^0(\mathcal{E}) \). Then \( \sigma_t = \exp (tX) \) is locally vertical, i.e. \( \pi \circ \sigma_t = \text{id} \) in some open neighborhood. Infinitesimal symmetries are cocycles and inner derivations are coboundaries in the cochain complex \( (\mathfrak{X}, \text{ad}_{Q_\epsilon}) \).

We can see that \( g' \) consists of vertical vector fields. If \( X = [Q_\epsilon, Y] \) for some vector field \( Y \in C^\infty (\mathcal{U}_{\alpha \beta}, \mathfrak{g}) \) which is vertical, then \( X \circ \pi^* = [Q_\epsilon, Y] \circ \pi^* \). Since \( Y \) is \( \pi \)-related to 0 and \( Q_\epsilon \) is \( \pi \)-related to \( Q_M \), then \( [Q_\epsilon, Y] \) is \( \pi \)-related to \( [Q_M, 0] = 0 \). Therefore, \( X \circ \pi^* = 0 \), i.e. it is vertical. A vertical automorphism of \( \mathcal{E} \) is a fibrewise acting automorphism \( \sigma : \mathcal{E} \longrightarrow \mathcal{E} \), i.e. \( \pi \circ \sigma = \text{id} \). The set of vertical automorphisms is a subgroup of the automorphism group of \( \mathcal{E} \). Let \( \sigma_t : \mathcal{E} \longrightarrow \mathcal{E} \) be the flow of the vector field \( X \) and \( \varphi : \mathcal{M} \longrightarrow \mathcal{E} \) a section of \( \mathcal{E} \). Then, we define the action \( \sigma_t \circ \varphi = \varphi_t \) and \( \sigma_t \cdot f = f_t = d \varphi_t \circ Q_M - Q_\epsilon \circ \varphi_t \). Then the variation of \( \varphi^* \) along \( X \) is defined as
\[
(\delta_X \varphi^*) (g) = \lim_{t \to 0} \frac{\varphi_t^* (g) - \varphi^* (g)}{t} \\
\text{(5.4)}
\]
and the variation of \( f^* \) along \( X \)
\[
(\delta_X f^*) (g) = \lim_{t \to 0} \frac{f_t^* (g) - f^* (g)}{t} \text{.} \\
\text{(5.5)}
\]
We will need the following two lemmas in what follows.

**Lemma 3.** Let \( X \in \mathfrak{X}^0(\mathcal{E}) \) be a vertical vector field. Then,
\[
\delta X \varphi^* = \varphi^* \circ X \text{.} \tag{5.6}
\]

**Proof.** Let \( x^A \) be coordinates on \( \mathcal{M} \) and \( y^B \) be coordinates on the fibre \( \mathcal{F} \). Then, \( X = Y^B \frac{\partial}{\partial y^B} \) where \( Y^B \in C^\infty (\mathcal{E}) \). We have
\[ \sigma_t(x, y) = (x^A, y^B + tY^B + ...). \] Therefore, \( \varphi(x) = (x^A, s^B(x)) \) locally. Then,

\[
(\varphi_t^* g)(x) = g(x^A, s^B(x) + tY^B(x, s^B(x)) + ...) = g(x, s(x)) + \frac{\partial g}{\partial y^B} tY^B(x, s(x)) + ....
\]

Hence, \( (\delta_X \varphi^*)(g) = \frac{\partial g}{\partial y^B} Y^B(x, s^B(x)) = \varphi^* X(g). \)

**Lemma 4.** Let \( \varphi: \mathcal{M} \rightarrow \mathcal{E} \) be a morphism of \( Q \)-manifolds and \( Y \in \mathfrak{X}^{-1}(\mathcal{M}) \) such that \( X = \text{ad}_{Q\mathcal{M}}(Y) \). Let \( f: \mathcal{M} \rightarrow \Pi\mathcal{T}\mathcal{E} \) as defined previously. Then, the induced variation of \( f^* \) along \( X \) is

\[
(\delta_X f^*)(dh) = \lim_{t \to 0} \frac{f_t^*(dh) - f^*(dh)}{t} = \lim_{t \to 0} \frac{F(h) - F_1(h)}{t} = \lim_{t \to 0} \frac{Q_{\mathcal{M}}(\delta_X \varphi^*)(h) - \varphi^* Q_{\mathcal{E}}(h)}{t} = Q_{\mathcal{M}}(\delta_X \varphi^*)(h) - \varphi^* X Q_{\mathcal{E}}(h)
\]

\( = (Q_{\mathcal{M}} \varphi^*)X(h) - \varphi^* Q_{\mathcal{E}} X(h), \) since \( Q_{\mathcal{E}} \) commutes with \( X \)

\( = FX(h) = F \mathcal{L}_X(h) = f^* \mathcal{L}_X(dh), \) using Proposition 8.

For \( h \in C^\infty(\mathcal{E}) \), we have

\[
(\delta_X f^*)(h) = \lim_{t \to 0} \frac{f_t^*(h) - f^*(h)}{t} = \frac{\varphi_t^*(h) - \varphi^*(h)}{t}, \text{ by Proposition 8}
\]

\( = (\delta_X \varphi^*)(h), \) by definition

\( = \varphi^* X(h), \) by Lemma 3

\( = f^* \mathcal{L}_X(h), \) by Lemma 4.

Since \( \delta_X f^* \) is a derivation on functions, this suffices to prove the lemma.

\[ (5.7) \]

### 5.2 Characteristic classes

**Definition 22.** \( \omega \in \Omega(\mathcal{E}) \) is called generalized \( g \)-basic form if \( \mathcal{L}_X \omega = \mathcal{L}_{\text{ad}_g X} \omega = 0 \) for all \( X \in \mathfrak{g} \).
Lemma 5. Let \( \omega \in \Omega^*(F) \), where \( \mathfrak{g} \) is a subalgebra of vector fields on \( F \) closed under derived brackets. Let \( \omega' = \text{pr}^*(\omega) \) where \( \text{pr} \) is the projection from \( M \times \mathcal{F} \) to \( \mathcal{F} \). Then, \( f^*\omega' \) is invariant the action of \( \exp(\mathfrak{g}') \) where \( \mathfrak{g}' = \text{ad}_Q(C^\infty(M, \mathfrak{g})) \cap \mathfrak{x}(M \times \mathcal{F}) \) where \( Q = Q_M + Q_\mathcal{F} \).

Proof. Let \( X = \text{ad}_Q(Y) \in \mathfrak{x}(M \times \mathcal{F}) \) with \( Y \) vertical. We have \( f^*: C^\infty(N \times \mathcal{F}) \rightarrow C^\infty(N) \). Let \( Y = b^A Y_A \) where \( Y_A = \frac{\partial}{\partial y^A} \). Then,

\[
(\delta_X f*) \omega' = f^* \mathfrak{L}_{\text{ad}_Q(Y)} \omega' = f^* \left( \mathfrak{L}_{Q_M(b)} Y_A + (-1)^{\bar{b}} \mathfrak{L}_{b^A \text{ad}_Q(Y_A)} \right) \omega'.
\]

(5.8)

If we prove \( f^* \mathfrak{L}_{\mu Z} \omega' = 0 \) for all \( \mu \in C^\infty(M) \) and \( Z \in \mathfrak{x}(\mathcal{F}) \) such that \( \mathfrak{L}_Z \omega' = 0 \) then we are done.

\[
f^* \mathfrak{L}_{\mu Z} \omega' = \left( i_Z d + (-1)^{\bar{p}+\bar{Z}} d i_Z \right) \omega' = f^* \left( (-1)^{\bar{p}+\bar{Z}} d i_Z + \mu \mathfrak{L}_Z \right) \omega' = (-1)^{\bar{p}+\bar{Z}} f^* d i_Z \omega' = (-1)^{\bar{p}+\bar{Z}} f^* (d \mu)(i_Z \omega') = (-1)^{\bar{p}+\bar{Z}} F(\mu) f^*(i_Z \omega') = 0,
\]

(5.9)
since \( F = 0 \) as \( Q \) is \( \pi \)-related to \( Q_M \).

The next theorem is about the construction of the characteristic classes associated with the section \( \varphi \).

Theorem 6. Let \( \pi: E \rightarrow M \) be a \( Q \)-bundle with fibre \( F \) and \( \mathfrak{g} \) a holonomy algebra. Let \( \varphi: M \rightarrow E \) be a section. Then there is a map

\[
H^p \left( \Omega(\mathcal{F}), Q_{\text{ITF}} \right) \rightarrow H^p (C^\infty(M), Q_M) \text{ independent of homotopies of } \varphi \text{ where } Q_{\text{ITF}} = d + \mathfrak{L}_{Q_\mathcal{F}}.
\]

Proof. Let \( f_a: U_a \rightarrow \text{ITF}(U_a \times \mathcal{F}) \), where \( f_a = d\varphi_a \circ Q_{U_a} = Q_\mathcal{E} \circ \varphi_a \) and for \( \omega \in H^p \left( \Omega(\mathcal{F}), Q_{\text{ITF}} \right) \) let \( f_a(\omega_a) = \chi_a(\omega) \). Then \( \chi_a(\omega) \) is a \( Q_M \)-cocycle, i.e. \( Q_M(\chi_a(\omega)) = 0 \).

To see this we have \( Q_{\text{ITF}}(\omega) = d\omega + \mathfrak{L}_{Q_\mathcal{F}} \omega = 0 \). Therefore

\[
Q_{U_a} f_a^* \omega_a' = f^* d \omega_a' + f_a^* \left( \mathfrak{L}_{Q_{U_a}} \omega_a' + \mathfrak{L}_{Q_\mathcal{F}} \omega_a' \right), \text{ by Proposition 9}
\]

\[
= f_a^* \mathfrak{L}_{Q_{U_a}} \omega_a' + f_a^* (d \omega_a' + \mathfrak{L}_{Q_\mathcal{F}} \omega_a') = f_a^* \mathfrak{L}_{Q_{U_a}} \omega_a', \text{ since } \omega \text{ is a cocycle}
\]

\[
= 0, \text{ since } \omega_a' \text{ is vertical, i.e. in } \mathcal{F}.
\]

(5.10)

Hence \( \chi_a(\omega) \) is a \( Q_M \)-cocycle. Now we prove that the equivalence class of \( \chi_a(\omega) \) does not depend on a representative of \( \omega \). Let \( \theta \in \Omega(\mathcal{F}) \) and \( Q_{\text{ITF}} \theta = 0 \), that is a cocycle.
Therefore, \( \chi_\alpha(\omega) \) and \( \chi_\alpha(\theta) \) are in the same cohomological class.

Now let \( \varphi_t : M \longrightarrow \mathcal{E} \) be a family of sections with \( \varphi_t(x) = (x, y_t(x)) \) then we have \( \hat{\varphi} : \hat{M} \longrightarrow \hat{\mathcal{E}} \hat{\varphi}(t, dt, x) = (t, dt, x, \hat{y}(t, dt, x)) \) where \( \hat{y}(t, dt, x) = y_t(x) \) and \( \hat{M} = \Pi I \times M \) and \( \hat{\mathcal{E}} = \Pi I \times \mathcal{E} \), and \( I = [0, 1] \). Then \( \hat{\varphi} \) is a section of Q-bundle \( \hat{\pi} : \hat{\mathcal{E}} \longrightarrow \hat{\mathcal{M}} \).

If \( h \in C^\infty(\hat{\mathcal{E}}) \) then \( h = h_0(t, x, y) + dt h_1(t, x, y) \) where \( h_i \in C^\infty(I \times \mathcal{E}) \). Then we have the corresponding field strength \( \hat{F} \) and \( \hat{f} \). The supermanifolds \( \hat{M} \) and \( \hat{\mathcal{E}} \) are Q-manifolds with the holomorphic vector fields \( Q_{\hat{M}} = Q_M + Q_t \) and \( Q_{\hat{\mathcal{E}}} = Q_{\mathcal{E}} + Q_t \) respectively, where \( d_t = dt \frac{d}{dt} \).

Then we get the following:

\[
\hat{\varphi}^* h = \varphi_t^* h_0 + dt \varphi_t^* h_1,
\]
\[
\hat{F} = Q_{\hat{M}} \hat{\varphi}^* - \hat{\varphi}^* Q_{\hat{\mathcal{E}}}
\]
\[
= (Q_M + Q_t) \hat{\varphi}^* - \hat{\varphi}^* (Q_{\mathcal{E}} + Q_t).
\]

Therefore

\[
\hat{F} h = (Q_M + Q_t) \hat{\varphi}^* h - \hat{\varphi}^* (Q_{\mathcal{E}} + Q_t) h
= (Q_M + Q_t) (\varphi_t^* h_0 + dt \varphi_t^* h_1) - \hat{\varphi}^* (Q_{\mathcal{E}} + Q_t) (h_0 + dt h_1)
= (Q_M + Q_t) (\varphi_t^* h_0 + dt \varphi_t^* h_1) - \hat{\varphi}^* (Q_{\mathcal{E}} h_0 + dt Q_{\mathcal{E}} h_1 + dt h_0)
= Q_M \varphi_t^* h_0 + dt Q_M \varphi_t^* h_1 + dt \varphi_t^* h_1 - \varphi_t^* Q_{\mathcal{E}} h_0 - dt \varphi_t^* Q_{\mathcal{E}} h_1 - dt \varphi_t^* \frac{dh_0}{dt}
= F_t h_0 + dt F_t h_1 + dt \left( \frac{d \varphi_t^* h_0}{dt} - \varphi_t^* \frac{dh_0}{dt} \right).
\]

Hence we get a map \( \hat{\chi} : H^p \left( \Omega (\mathcal{F})_g ; Q_{\Pi \mathcal{F}} \right) \longrightarrow H^p \left( C^\infty \left( \hat{\mathcal{M}} \right), Q_{\hat{\mathcal{M}}} \right) \)

\( \hat{\chi}(\omega) = \hat{f}^* \omega' \).

Then

\[
\hat{\chi} \omega = \chi_\omega + dt \beta_t.
\]

Since it is a cocycle, \( \hat{\chi} \omega \) is closed with respect to \( Q_{\hat{\mathcal{M}}} \). This implies that
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\((Q_M + d_t)\hat{\chi}_t = 0\). On the other hand we have

\[
(Q_M + d_t)\hat{\chi}_t = (Q_M + d_t)(\chi_t \omega + dt \beta_t)
\]

\[
= Q_M \chi_t \omega + dt Q_M \beta_t + dt \frac{d\chi_t \omega}{dt}
\]

\[
= dt \left( Q_M \beta_t + \frac{d\chi_t \omega}{dt} \right)
\]

(5.17)

\[
= 0.
\]

From (5.17) we get \(\frac{d\chi \omega}{dt} = Q_M (\beta_t)\). Hence \(\chi_1 \omega - \chi_0 \omega = Q_M \int -\beta dt\). That is \(\chi_1 \omega\) and \(\chi_0 \omega\) represent the same cohomological class. \(\chi(\omega)\) is called the characteristic class of \(\omega\).
Chapter 6

Non-abelian Poincaré lemma when
\( G = \text{Diff}(F) \)

In his paper [35], Voronov proved the non-abelian Poincaré lemma for matrix groups and stated that it works for all groups whether finite or infinite dimensional. The non-abelian Poincaré lemma states that an odd \( g \)-valued form on a contractible manifold satisfies the Maurer-Cartan equation then it is gauge equivalent to an odd constant, i.e. to an odd element of \( g \). The meaning of this will be given later.

In this section we rewrite the non-abelian Poincaré lemma in the case when \( G = \text{Diff}(F) \), the diffeomorphism group of some supermanifold \( F \), which has the space of vector fields on \( F \), \( \mathfrak{X}(F) \), as its super Lie algebra. In this case the non-abelian Poincaré lemma becomes more interesting to applications; for instance in the theory of transitive Lie algebroids. In doing so, some definitions will have different expressions; like \( dgg^{-1} \) and \( g\omega g^{-1} \).

6.1 Proof of the non-abelian Poincaré lemma for
\( G = \text{Diff}(F) \)

Let \( M \) and \( F \) be supermanifolds and \( G = \text{Diff}(F) \) be the diffeomorphism group of \( F \). A map \( g : \Pi TM \to G \) can be seen as an even differential form taking values in \( G \). The map \( g \) will be identified with the map \( g : \Pi TM \times F \to F \) as \( g(x, dx)(y) = g(x, dx, y) \). Suppose \( F : M \to G \), with \( F(z) = g(z) \circ g^{-1}(x) = R_{g^{-1}(x)} \circ g(z) \). Then
\[
F(z, dz)(y) = g(z, dz, g^{-1}(x, dx, y)).
\]
Therefore \( dg \, g^{-1} = dxg(x, dx, g^{-1}(x, dx, y)) \).

Suppose that \( g : \Pi TM \times F \to F \), with \( g(x, dx, y) = (g^i(x, dx, y)) \).

Then
\[
d_xg(x, dx, y) = dx^a \frac{\partial g^i(x, dx, y)}{\partial x^a} \frac{\partial}{\partial y^i}.
\] (6.1)

\( d_x \) here is the differential with respect to \( x \) taking \( dx \) and \( y \) as parameters, whereas \( x^a, dx^a \) are coordinates on \( \Pi TM \) and \( y^i \) are coordinates on \( F \). Hence
\[
d_xg^{-1}(x, dx, y) = dx^a \frac{\partial g^i(x, dx, g^{-1}(x, dx, y))}{\partial x^a} \frac{\partial}{\partial y^i}.
\] (6.2)
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We see that $d_x gg^{-1}$ which is $d_x (R_{g^{-1}} \circ g)$ is an odd differential form taking values in vector fields $\mathfrak{X}(F)$.

We shall define a map

$$T : \Omega^1(M \times I, \mathfrak{X}(F)) \longrightarrow \Omega^2(M, G)$$

from the odd differential forms on $M \times I$ taking values in $\mathfrak{X}(F)$ to even differential forms on $M$ taking values in $G$, where $I$ is the unit interval $[0, 1]$.

If $\omega \in \Omega^1(M \times I, \mathfrak{X}(F))$, then it is written as

$$\omega = \omega_0(x, dx, t) + dt \omega_1(x, dx, t),$$

with

$$\omega_1(x, dx, t) = \omega^i(x, dx, t) \frac{\partial}{\partial y^i}.$$  \hfill (6.4)

Then we define $g = g_1 = T(\omega)$ where $g_t$ is the solution to the following Cauchy problem

$$\frac{dg_t(x, dx, y)}{dt} = -\omega_1(g_t(x, dx, y)).$$  \hfill (6.5)

In coordinates, it becomes

$$\frac{dg^i_t(x, dx, y)}{dt} = -\omega^i(g_t(x, dx, y)).$$ \hfill (6.6)

**Theorem 7.** Let $\omega = \omega_0(x, dx, t) + dt \omega_1(x, dx, t) \in \Omega^1(M \times I, \mathfrak{X}(F))$ and $g = g_1 \in \Omega^2(M, G)$ as above and let $\Omega = d\omega + \frac{1}{2} [\omega, \omega]$.

Then

$$-d_x gg^{-1} + \int_0^1 g g^{-1}_t \Omega g = \omega_0|_{t=1} - g(\omega_0|_{t=0}) g^{-1}.$$  \hfill (6.7)

Conjugating the formula, we get the equivalent equation

$$-g^{-1} dg + \int_0^1 g^{-1}_t \Omega g = g^{-1}(\omega_0|_{t=1}) g - \omega|_{t=0}.$$ \hfill (6.8)

First we have

$$\frac{d}{dt} \left( g^{-1}_t d_x g_t \right)(x, dx, y) = -d_y g^{-1}_t(x) \left( (d_x \omega_1)(g_t(x, dx, y)) \right).$$ \hfill (6.9)

We need the following lemma.

**Lemma 6.** Let $f_t, g_t : F \longrightarrow F$ depending on a real parameter $t$.

Then,

$$\frac{d}{dt} (f_t \circ g_t)(y) = \frac{df_t}{dt}(g_t(y)) + d_y f_t \left( \frac{dg_t}{dt}(y) \right)$$ \hfill (6.10)

and

$$\frac{dg_t^{-1}}{dt}(y) = -d_y g_t^{-1} \left( \frac{dg_t}{dt}(g_t^{-1}(y)) \right)$$ \hfill (6.11)

where $d_y f_t$ is the differential of $f_t : F \longrightarrow F$. 
Proof. We have
\[
\frac{d}{dt}(f_t \circ g_t)(y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( f_{t+\epsilon}g_{t+\epsilon}(y) - f_tg_t(y) \right)
\]
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( f_{t+\epsilon}g_{t+\epsilon}(y) - f_tg_{t+\epsilon}(y) + f_tg_{t+\epsilon}(y) - f_tg_t(y) \right)
= \frac{df}{dt}(g_t(y)) + dg_t \left( \frac{dg}{dt}(y) \right)
.
\]

The second identity follows from the first applied to \(g_t^{-1}\).

Using Lemma 6, we prove (6.9). On the other hand we have
\[
d_z \frac{d}{dt}(g_t^{-1}(x)g_t(z, y)) = \frac{d}{dt}d_z(g_t^{-1}(x)g_t(z, y)),
\]
where \(d_z\) in the right hand side means the differential of forms with respect to \(z\).

By Lemma 6, this gives us
\[
d_z \left( \frac{dg_t^{-1}}{dt}(x)(g_t(z, y)) + dg_t^{-1}(x) \left( \frac{dg_t(z, y)}{dt} \right) \right).
\]

By Lemma 6, again, this is equal to
\[
d_z \left( d_yg_t^{-1}(x) \left( \frac{dg}{dt}(x)g_t^{-1}(x)g_t(z, y)) \right) + dg_t^{-1}(x) \left( \frac{dg_t(z, y)}{dt} \right) \right).
\]

Using (6.5), putting \((z, dz) = (x, dx)\) and simplifying we get (6.9).

Equation (6.9) suggests that for a differential form \(\omega\) taking values in \(\mathfrak{X}(F)\), \(g\omega_{g^{-1}}\)

is nothing but \(d_yg(\omega)\) where \(g \in G = \text{Diff}(F)\).

To see this, let \(L : G \to G\) with \(L(h) = g \circ h\).

Then \(dL \left( \frac{dh}{dt}(y) \right) = \frac{d}{dt}(gh_t(y))\), which is equal to \(d_yg(\frac{dh_t}{dt}(y))\).

Therefore, \(dL(X) = d_yg(X)\) where \(X \in \mathfrak{X}(F)\).

And for the right multiplication \(R : G \to G\), \(R(h) = h \circ g\), we get in the same way
\(dR(X) = X\).

Since \(gXg^{-1} = d(L \circ R)(X)\), we get
\[
gXg^{-1} = d_yg(X).
\]

We now come back to the proof of (6.8).

Since \(g^{-1}dg = (g_t^{-1}dg_t)|_{t=1}\) and \(d_xg|_{t=0} = 0\) and
\[
\frac{d}{dt}(g_t^{-1}d_xg_t)(x, dx, y) = -d_yg_t^{-1}(x, dx) \left( (d_x\omega_1)(g_t(x, dx, y)) \right)
\]
we get
\[
g^{-1}dg = \int_0^1 dt \frac{d}{dt}(g_t^{-1}d_xg_t)(x, dx, y) = \int_0^1 dt \left( -d_yg_t^{-1}(x, dx) \left( (d_x\omega_1)(g_t(x, dx, y)) \right) \right)
.
\]

(6.13)

On the other hand, we have
\[
d\omega + \frac{1}{2}[\omega, \omega] = d_x\omega_0 + \frac{1}{2}[\omega_0, \omega_0] + dt \left( -d_x\omega_1 + \frac{d\omega_t}{dt} - [\omega_1, \omega_0] \right).
\]
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Hence
\[
\int_0^1 g_t^{-1} \Omega g_t = \int_0^1 dt \left( -d_y g_t^{-1}(x, dx, y) ((d_x \omega_t)(g_t(x, dx, y))) \right) + \int_0^1 dt \left( d_y g_t^{-1}(x, dx, y) \left( \frac{d\omega_t}{dt} - [\omega_t, \omega_0] \right)(g_t(x, dx, y)) \right). \tag{6.14}
\]

By (6.13) and (6.14), we get
\[
-g^{-1}dg + \int_0^1 g_t^{-1} \Omega g_t = \int_0^1 dt \left( d_y g_t^{-1}(x, dx, y) \left( \frac{d\omega_t}{dt} - [\omega_t, \omega_0] \right)(g_t(x, dx, y)) \right). \tag{6.15}
\]

On the other hand we have
\[
d dt \left( d_y g_t^{-1}(x, dx, y) \omega_0(g_t(x, dx, y)) \right) = d_y g_t^{-1}(x, dx, y) \left( \frac{d\omega_0}{dt} - [\omega_1, \omega_0] \right)(g_t(x, dx, y)), \tag{6.16}
\]
which is equivalent to
\[
d dt \left( d_y g_t^{-1}\omega_0 \right) = d_y g_t^{-1} \left( \frac{d\omega_0}{dt} - [\omega_1, \omega_0] \right). \tag{6.17}
\]

To see this, let $\omega_0 = X^i \frac{\partial}{\partial y^i}$ and $\omega_1 = Y^i \frac{\partial}{\partial y^i}$.

Then $[\omega_1, \omega_0] = (Y^k \frac{\partial X^j}{\partial y^k} - X^k \frac{\partial Y^j}{\partial y^k}) \frac{\partial}{\partial y^i}$.

Equation (6.17) is again equivalent to
\[
d dt \left( g_t^* X^j \cdot g_t \frac{\partial g_t^{-1}}{\partial y^i} \right) = g_t^* \left( \frac{dX^j}{dt} + Y^k \frac{\partial X^j}{\partial y^k} - X^k \frac{\partial Y^j}{\partial y^k} \right) \cdot g_t^* \frac{\partial g_t^{-1}}{\partial y_i}, \tag{6.18}
\]
by applying the left and right hand sides of (6.17) to $g_t^* \omega_i$. Using the Leibniz rule and the fact that $\frac{dg_t^i}{dt} = g_t^* Y^i$ and
\[
\frac{dg_t^{-1}}{dt} = -g_t^{-1} \frac{dg_t^j}{dt} \frac{\partial g_t^{-1}}{\partial y^j}
\]
and
\[
d dt \left( g_t^* X^j \right) = g_t^* \frac{dX^j}{dt} + \frac{dg_t^j}{dt} g_t^* \frac{\partial X^j}{\partial y^i}
\]
to the left hand side of (6.18), we get the right hand side. Therefore (6.15), becomes
\[
-g^{-1}dg + \int_0^1 g_t^{-1} \Omega g_t = \int_0^1 dt \frac{d}{dt} \left( d_y g_t^{-1}(x, dx, y) \omega_0(g_t(x, dx, y)) \right). \tag{6.19}
\]

And this proves the theorem.

For the following corollary and theorem, their proofs are exactly the same as in the paper [35].

**Corollary 2** (Non-abelian Algebraic Homotopy). Let $f_0, f_1 : M \to N$ be homotopic
maps and \( \omega \in \Omega^1(\mathcal{N}, \mathfrak{X}(F)) \). Then
\[
f_1^* \omega - g(f_0^* \omega) g^{-1} = -d_x gg^{-1} + \int_0^1 gg^{-1} F^* \Omega_g g	ag{6.20}
\]
where \( g_t = T(F^* \omega) \) and \( F \) is the homotopy \( F : M \times I \to \mathcal{N}, F(x,t) = f_t(x) \).
If \( \omega \) satisfies the Maurer-Cartan equation, i.e. \( d \omega + \frac{1}{2} [\omega, \omega] = 0 \), then
\[
f_1^* \omega = g(f_0^* \omega) g^{-1} - dg g^{-1}.	ag{6.21}
\]

In the case where \( M \) is a contractible manifold, the corollary will give us the following theorem.

**Theorem 8.** Let \( M \) be a contractible supermanifold and odd form \( \omega \in \Omega^1(M, \mathfrak{X}(F)) \) which satisfies the Maurer-Cartan equation. Then \( \omega \) is gauge equivalent to an odd constant (odd vector field) \( C \in \mathfrak{X}(F) \)
\[
C = g \omega g^{-1} - dg g^{-1}	ag{6.22}
\]
for some \( g \in \Omega^2(M, \text{Diff}(F)) \).

(6.22) is equivalent to
\[
C = dg(\omega) - dg g^{-1},	ag{6.23}
\]
where \( \widehat{g} : \Pi TM \times F \to \Pi TM \times F \) with \( \widehat{g}(x, dx, y) = (x, dx, g(x, dx, y)) \). The forms \( C \) and \( \omega \) are considered as vector fields on \( \Pi TM \times F \). By a direct calculation we see that
\[
d\widehat{g}(d) = dx^a \partial g^i(x, dx, y) \frac{\partial}{\partial x^a} \partial y^i = dg g^{-1}.	ag{6.24}
\]
Hence
\[
d\widehat{g}(d + \omega) = d + C.	ag{6.25}
\]

### 6.2 Applications to transitive Lie algebroids

Let \( A \to M \) be a transitive Lie algebroid over \( M \) with anchor \( a : A \to TM \) and let \( \{U_\alpha\} \) be a trivializing cover and let \( A_\alpha = A|_{U_\alpha} \). Let \( \{u_i\} \) be a basis of \( TU_\alpha \) corresponding to \( U_\alpha \) and choose \( e_i \in \Gamma A_\alpha \) such that \( a(e_i) = u_i \). This is possible since \( a \) is surjective. Let \( W_\alpha = \text{span}\{e_i\} \). Then \( \{e_i\} \) is a basis of \( W_\alpha \). What we are doing here is actually choosing a connection \( \gamma \) on \( A \) by letting \( \gamma(u_i) = e_i \) and extending linearly. At the same time we can view \( W_\alpha \) as a horizontal distribution with respect to the anchor \( a \). If \( u_i = \frac{\partial}{\partial x^i} \), then finding \( e_i \)'s such that \( [e_i, e_j] \) for all \( i, j \) amounts to finding a flat connection. Then this ensures that \( A \) is a locally trivial Lie algebroid by a theorem of Mackenzie [19].
If \( \lambda^i e_i = 0 \), then \( a(\lambda^i e_i) = 0 \). Then \( \lambda^i u_i = 0 \). That is \( \lambda^i = 0 \).
Then we have \( A_\alpha = W_\alpha \oplus L_\alpha \) where \( L_\alpha = \ker (a : A_\alpha \to TU_\alpha) \) with \( W_\alpha \) isomorphic to \( TU_\alpha \) and \( L_\alpha \) isomorphic say to \( U_\alpha \times V \) for some vector space \( V \) by the map \( \phi_\alpha \).
\( L \) is called the adjoint bundle of the Lie algebroid \( A \). Then \( A_\alpha \) is isomorphic to \( TU_\alpha \oplus (U_\alpha \times V) \) via \( \varphi_\alpha (\lambda^i e_\alpha + v) = \lambda^i u_\alpha + \phi_\alpha(v) \).
The coordinates transform as
\[
\varphi_{\beta a}(\lambda^i u_{ai} + v) = \varphi_{\beta} \varphi_a(\lambda^i u_{ai} + v)
\]
\[
= \varphi_{\beta}(\lambda^i e_{ai} + \phi_a^{-1}v)
\]
\[
= \varphi_{\beta}(\lambda^i (m_i^j e_{bj} + u_i) + \phi_a^{-1}v)
\]
\[
= \lambda^i m_i^j u_{bj} + \lambda^i \phi_{bj}(v_i) + \phi_{\beta} \phi_a^{-1}v
\]
\[
= \lambda^i m_i^j u_{bj} + \lambda^i \phi_{bj}(v_i) + \phi_{\beta a}v,
\]
(6.26)
or
\[
\varphi_{\beta a} : TU_{\alpha \beta} \times V \longrightarrow TU_{\alpha \beta} \times V
\]
\[
\varphi_{\beta a}(x + v) = X + r_x(X) + s_x(v),
\]
(6.27)
where \(a(e_{ai}) = u_{ai}\) and \(u_{ai} = m_i^j u_{bj}\). Hence \(a(e_{ai} - m_i^j e_{bj}) = 0\). Thus \(e_{ai} - m_i^j e_{bj} \in L_{\alpha \beta}\). Hence \(e_{ai} - m_i^j e_{bj} = v_i\) for some \(v_i \in L_{\beta}\). We define \(r(e_{\beta k}) = v_i\) and extend by linearity. We can rewrite (6.28) as
\[
\varphi_{\beta a} : TU_{\alpha \beta} \times V \longrightarrow TU_{\alpha \beta} \times V
\]
\[
\varphi_{\beta a}(x, \bar{x}, v) = (x, \bar{x}, r_x(\bar{x}) + s_x(v)).
\]
(6.29)
The anchor becomes \(a(x, \bar{x}, v) = (x, \bar{x})\). We can see \(A\) as a fibre bundle with base space \(TM\) and the projection being the anchor \(a\). The coordinate transformations are affine rather than linear.
Let \(\{e_a\}\) be a basis of \(W_\alpha\) that corresponds to \(\frac{\partial}{\partial x^a}\) and \(\{v_i\}\) a basis of \(L_\alpha\) and
\[
[e_a, e_b] = R_{ab}^k v_k,
\]
\[
[e_a, v_i] = Q_{ai}^k v_k,
\]
\[
[v_i, v_j] = P_{ij}^k v_k.
\]
(6.30)
Since \(a[e_a, e_b] = a[e_a, v_i] = a[v_i, v_j] = 0\).
Let \(x^a, \theta^a, \xi^i\) be coordinates on \(A\) corresponding to our charts. The homological vector field corresponding to the \(Q\)-manifold \(\Pi A\) is of the form
\[
Q = d - \frac{1}{2} \left( P_{ij}^k \xi^i \xi^j + 2Q_{a}^k \xi^a \theta^b + R_{ab}^k \theta_a \theta^b \right) \frac{\partial}{\partial \xi^k}.
\]
(6.31)
It is known [19] that the adjoint bundle \(L\) of a transitive Lie algebroid \(A\) is a Lie algebra bundle. This means that \(V\) has the structure of a Lie algebra and for local sections on \(U_{\alpha}\), we have \([x, v(x)] = (x, v(x)) = (x, v(x), w(x))\).
Then \(P_{ij}^k\) will just be constants. Let \(V = g\) some Lie algebra.
Let \(f : \Pi TM \longrightarrow \Pi A\), \(f(x, \theta) = (x, \theta, \xi^a = \omega^i_a \theta^a)\) be a connection in \(A\). The connection form taking values in \(g\) is \(\omega = \omega^a_i dx^a v_i\) with \(\{v_i\}\) a basis of \(g\).
For \(f\) to be a flat, the field strength \(F = df^* - f^*Q\) must be zero. We have
F(x^a) = F(\theta^a) = 0, by virtue of the anchor.

\[ F(\xi^i) = df^*\xi^i - f^*Q\xi^i \]

\[ = \theta^a \frac{\partial(\omega^i_k \theta^b)}{\partial \theta^a} - f^* \left( -\frac{1}{2} \left( P_{ij}^k \xi^i \xi^j + 2Q_{ia}^j \xi^i \theta^a + R_{ab}^i \theta^a \theta^b \right) \right) \]

\[ = \frac{\partial \omega^b_k}{\partial \theta^a} + \frac{\partial \omega^a_k}{\partial \theta^b} + P_{ij}^k \omega^i_a \omega^j_b + Q_{ia}^j \omega^i_a - Q_{ia}^i \omega^a_b + R_{ab}^k \theta^a \theta^b \]

Therefore, \( f \) is a flat connection iff

\[ \frac{\partial \omega^k}{\partial \theta^a} - \frac{\partial \omega^a_k}{\partial \theta^b} + P_{ij}^k \omega^i_a \omega^j_b + Q_{ia}^j \omega^i_a - Q_{ia}^i \omega^a_b + R_{ab}^k = 0 \] (6.33)

for all \( a, b, k \).

If \( M \) is a contractible manifold, then by a Mackenzie’s theorem [19] a flat connection exists and the Lie algebroid is trivial. That is there are charts such that \( R_{ab}^k = Q_{ia}^i = 0 \) in this case \( Q \) becomes

\[ Q = d - \frac{1}{2} P_{ij}^k \xi^i \xi^j \frac{\partial}{\partial \xi^k}, \] (6.34)

and (6.33) becomes

\[ \frac{\partial \omega^k}{\partial \theta^a} - \frac{\partial \omega^a_k}{\partial \theta^b} + P_{ij}^k \omega^i_a \omega^j_b = 0, \] (6.35)

which is equivalent of \( d\omega + \frac{1}{2} [\omega, \omega] = 0 \).

Let \( e_a, v_i \) coordinates on \( A_a \) and \( X_i \)

We can write \( Q \) as \( Q = d + \omega \) where \( \omega = \frac{1}{2} \left( P_{ij}^k \xi^i \xi^j + 2Q_{ia}^j \xi^i \theta^a + R_{ab}^i \theta^a \theta^b \right) \frac{\partial}{\partial \xi^k}. \)

We can consider \( \omega \) as a differential form on \( M \) taking values in \( \mathfrak{X}(\Pi \mathfrak{g}) \) which is odd in total parity.

Since \([Q, Q] = 0\), then \((d + \omega) \circ (d + \omega) = 0\) i.e. \(d \circ \omega + \omega \circ d + \omega \circ \omega = 0\) which is equivalent to \(d\omega + \frac{1}{2}[\omega, \omega] = 0\) where \(d\omega\) is the differential of \(\omega\) as a form and the Lie brackets of differential forms taking values in super Lie algebra.

Write \(\omega = \omega^i X_i\) where \(\omega^i \in \Omega(M)\) and \(X_i \in \mathfrak{X}(\Pi \mathfrak{g})\). Let \( h \in C^\infty(\Pi A) \) as \( h = S_T \xi \) locally as a product of monomials on \(\theta\) and \(\xi\). Therefore
\[(d \circ \omega + \omega \circ d)h = d(\omega^i X_i(S_0 T_\xi)) + \omega^i X_i(dS_0 T_\xi)\]
\begin{align*}
&= (-1)\bar{S}_b \bar{X}_i d(\omega^i S_0 X_i T_\xi) + (-1)\bar{S}_b \bar{X}_i \omega^i dS_0 X_i T_\xi \\
&= (-1)\bar{S}_b \bar{X}_i (d(\omega^i S_0) X_i T_\xi + (-1)\bar{X}_i \omega^i dS_0 X_i T_\xi) \\
&= (-1)\bar{S}_b \bar{X}_i (d(\omega^i S_0) + (-1)\bar{X}_i \omega^i dS_0) X_i T_\xi \\
&= (-1)\bar{S}_b \bar{X}_i (d\omega^j S_0 + (-1)\bar{\omega}^j \omega^i dS_0 + (-1)\bar{X}_i \omega^i dS_0) X_i T_\xi \\
&= -1\bar{S}_b \bar{X}_i (d\omega^j S_0) X_i T_\xi \\
&= d\omega^j X_i(S_0 T_\xi) \\
&= d\omega(h).
\end{align*}

Hence \(d \circ \omega + \omega \circ d = d\omega\) and it is easy to see that \(\omega \circ \omega = \frac{1}{2}[\omega, \omega]\).

Thus \(\omega\) is a Maurer-Cartan form. If \(M\) is contractible, then by the non-abelian Poincaré Lemma \(\omega\) is gauge equivalent to a constant \(C = d\hat{g}(\omega) - dgg^{-1}\) for some \(g : \Pi TM \rightarrow \text{Diff}(\Pi g)\). The map \(\hat{g}\) is as before in (6.23). It means there is a change of coordinates that transforms \(d + \omega\) to an odd vector field \(d + C\) on \(\Pi g\). Therefore

\[
C = -\frac{1}{2} P^k_{ij} \xi^i \xi^j \frac{\partial}{\partial \xi^k}
\]  

(6.36)

for some constants \(P^k_{ij}\). Therefore \(Q\) becomes of the form \(d - \frac{1}{2} P^k_{ij} \xi^i \xi^j \frac{\partial}{\partial \xi^k}\). Hence \(A\) is trivial. For an arbitrary transitive Lie algebroid, we can cover it by contractible open sets. Hence

**Theorem 9.** A transitive Lie algebroid \(A\) is locally trivial.

We now see how the structure constants change under coordinates change. Let \(\{\bar{e}_a\}\) and \(\{\bar{v}_i\}\) an other basis of \(A_a\), with \(\bar{e}_a = m_a^c e_c\) and \(r(\bar{e}_a) = r_a^i v_i\), \(\bar{v}_i = s_i^j v_j\). We have

\[
[\bar{e}_a, \bar{e}_b] = [m_a^c e_c + r_a^i v_i, m_b^d e_d + r_b^j v_j] \\
= m_a^c m_b^d [e_c, e_d] + m_a^c a(e_c)(m_b^d e_d - m_b^e a(e_d)(m_a^c)e_c \\
+ m_a^c r_b^i e_c)(r_b^j v_j - m_b^d a(e_d)(r_a^i v_i) - m_b^a d(a(e_d)(r_a^i v_i) \\
+ r_a^j r_b^i v_i, v_j)] \\
= \left( m_a^c m_b^d P_{cd} + m_a^c r_b^i Q_{ci} - m_b^c r_a^j Q_{jk} + m_a^c \frac{\partial r_b^k}{\partial x^e} - m_b^a \frac{\partial r_b^k}{\partial x^e} + r_a^j r_b^i P_{ij} \right) v_k \\
+ \left( m_a^d \frac{\partial m_b^k}{\partial x^d} - m_b^d \frac{\partial m_a^k}{\partial x^d} \right) e_c.
\]  

(6.37)

On the other hand

\[
[\bar{e}_a, \bar{e}_b] = \bar{R}_{ab} s_i^k v_k.
\]  

(6.38)
Hence we have
\[ \bar{R}_{ab}^k s_i^k = m_a^c m_b^d R_{cd}^{k} + m_a^c r_b^i Q_{ci}^k - m_b^c r_a^i Q_{ci}^k + m_a^c \frac{\partial r_b^k}{\partial x'} - m_b^c \frac{\partial r_a^k}{\partial x'} + r_a^i r_b^j P_{ij}^k, \] (6.39)
for all \( a, b, k \). We have
\[ m_a^d \frac{\partial m_b^c}{\partial x'} - m_b^d \frac{\partial m_a^c}{\partial x'} = 0 \] (6.40)
for all \( a, b, c \).

And
\[ [\bar{e}_a, \bar{v}_i] = [m_a^c e_c, s_i^j v_j] \]
\[ = m_a^c s_i^j [e_c, v_j] + m_a^c a(e_c)(s_i^j) v_j \]
\[ = \left( m_a^c s_i^j Q_{cj}^k + m_a^c \frac{\partial s_i^j}{\partial x'} \right) v_k \]
\[ = m_a^c \left( s_i^j Q_{cj}^k + \frac{\partial s_i^j}{\partial x'} \right) v_k. \] (6.41)

On the other hand
\[ [\bar{e}_a, \bar{v}_i] = \bar{Q}_{ai}^j s_k^j v_k. \] (6.42)

Hence we get
\[ \bar{Q}_{ai}^j s_k^j = m_a^c \left( s_i^j Q_{cj}^k + \frac{\partial s_i^j}{\partial x'} \right) \] (6.43)
for all \( a, i, k \). We have as well
\[ [\bar{v}_i, \bar{v}_j] = \left[ s_i^p v_p, s_j^q v_q \right] \]
\[ = s_i^p s_j^q P_{pq}^k v_k. \] (6.44)

On the other hand
\[ [\bar{v}_i, \bar{v}_j] = \bar{P}_{ij}^k s_k^j v_k. \] (6.45)

Hence we must have
\[ \bar{P}_{ij}^k s_k^j = s_i^p s_j^q P_{pq}^k. \] (6.46)

The construction of \( g \) ensures that the change of coordinates are affine, i.e. of the type we have constructed for the transitive Lie algebroid \( A \) and no change in the coordinates of \( M \). In that case (6.39), (6.43) and (6.46) become
\[ \bar{R}_{ab}^i s_i^k = R_{ab}^k + r_b^i Q_{ai}^k - r_a^i Q_{bi}^k + \frac{\partial r_b^k}{\partial x'} - \frac{\partial r_a^k}{\partial x'} + r_a^i r_b^j P_{ij}^k, \] (6.47)
\[ \bar{Q}_{ai}^j s_k^j = \left( s_i^j Q_{cj}^k + \frac{\partial s_i^j}{\partial x'} \right), \] (6.48)
\[ \bar{P}_{ij}^k s_k^j = s_i^p s_j^q P_{pq}^k. \] (6.49)

The non-abelian Poincaré lemma ensure that there are functions \( r_a^i \) and \( s_i^j \) such that
\[ R_{ab}^k + r_b^i Q_{ai}^k - r_a^i Q_{bi}^k + \frac{\partial r_b^k}{\partial x'} - \frac{\partial r_a^k}{\partial x'} + r_a^i r_b^j P_{ij}^k = 0 \] (6.50)
\[ \left( s^i Q^k_{aj} + \frac{\partial s^k_i}{\partial x^a} \right) = 0 \quad (6.51) \]

and \( P_{ij} \) constants.

Let \( f: E \rightarrow \Pi TM \) be a fibre bundle with \( E \) a graded manifold. Let \( Q \) be a homological vector field on \( E \) of weight 1 which is \( f \)-related to \( d \). Then we say that \( E \) is a transitive non-linear Lie algebroid. Then \( Q \) may be written locally as

\[ Q = d + r^i(x, dx, y) \frac{\partial}{\partial y^i} \quad (6.52) \]

where \( x^a, dx^a \) are coordinates on \( \Pi TM \) and \( y^i \) are coordinates on the fibre \( F \) of \( E \). Using the non-abelian Poincaré lemma as before, we can find a change of coordinates such that the \( r^i \) are only functions of \( y \).
Appendix A

A.0.1 Structure equations of Lie algebroids

In this appendix we give the structure equations of Lie algebroids and how they change under coordinate transformations. Then we use them to show the invariance of the expression of the corresponding homological vector field. Let $E \to M$ be a Lie algebroid with anchor $a : E \to TM$ and let $\{e_i\}$ be a local basis of $\Gamma(E)$. Let $[e_i, e_j] = c_{ij}^k e_k$ and $a(e_i) = a_i^j \frac{\partial}{\partial x^j}$. Then

$$
a[e_i, e_j] = a(c_{ij}^k e_k) = c_{ij}^k a(e_k) = c_{ij}^k a_s^k \frac{\partial}{\partial x^s},
$$

and

$$
[a(e_i), a(e_j)] = \left[ a_s^i \frac{\partial}{\partial x^s}, a_t^j \frac{\partial}{\partial x^t} \right]
= \left( a_s^i \frac{\partial a_t^k}{\partial x^s} - a_t^j \frac{\partial a_s^k}{\partial x^s} \right) \frac{\partial}{\partial x^k}.
$$

Since $a[e_i, e_j] = [a(e_i), a(e_j)]$ we get

$$
a_s^i \frac{\partial a_t^k}{\partial x^s} - a_t^j \frac{\partial a_s^k}{\partial x^s} = c_{ij}^k a_s^i.
$$

(A.1)

The first condition in the definition of a Lie algebroid gives us the structure equation (A.1).

From the Leibniz rule we have

$$
[e_i, fe_j] = f[e_i, e_j]
= fc_{ij}^k e_k + a_t^j \frac{\partial f}{\partial x^t} e_j.
$$

(A.2)

Hence

$$
a[e_i, fe_j] = \left( fc_{ij}^k a_s^k + a_t^j a_s^i \frac{\partial f}{\partial x^t} \right) \frac{\partial}{\partial x^s}.
$$

(A.3)
On the other hand

\[
[a(e_i), fa(e_j)] = \left[ a_i^v \frac{\partial}{\partial x^u}, f a_j^v \frac{\partial}{\partial x^v} \right] = \left( a_i^s \frac{\partial f a_j^s}{\partial x^u} - f a_j^v \frac{\partial a_i^v}{\partial x^u} \right) \frac{\partial}{\partial x^s}.
\] (A.4)

From (A.3) and (A.4) we conclude that

\[
a_i^u \frac{\partial f a_j^s}{\partial x^u} - f a_j^v \frac{\partial a_i^v}{\partial x^u} = f c_{ij}^k a_k^s + a_i^s \frac{\partial f}{\partial x^s}
\] (A.5)

for all \(i, j\).

If \(a[ei, ej] = [a(e_i), a(e_j)]\) for all \(e_i, e_j\) then \(a[X, Y] = [a(X), a(Y)]\) all \(X, Y \in \Gamma(E)\).

From the Jacobi identity, we have

\[J = [e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0.\]

Let \(\phi(i, j, k) = [e_i, [e_j, e_k]]\). Then

\[
\phi(i, j, k) = [e_i, c_{jk}^l e_l] = c_{jk}^l e_i + a(e_i)(c_{jk}^l) e_l = c_{jk}^l e_i + a_i^d \frac{\partial c_{jk}^l}{\partial x^d} e_s = \left( c_{jk}^l e_i + a_i^d \frac{\partial c_{jk}^l}{\partial x^d} \right) e_s.
\]

Therefore

\[J = \left( c_{ik}^s c_{jk}^l + c_{jk}^s c_{ki}^l + c_{ki}^l c_{ij}^l + a_i^d \frac{\partial c_{ki}^l}{\partial x^d} + a_j^d \frac{\partial c_{ij}^l}{\partial x^d} + a_k^d \frac{\partial c_{ij}^l}{\partial x^d} \right) e_s.
\]

Hence

\[c_{it}^s c_{tk}^l + c_{kt}^s c_{ti}^l + c_{ti}^l c_{kj}^l + a_i^d \frac{\partial c_{ti}^l}{\partial x^d} + a_t^d \frac{\partial c_{kj}^l}{\partial x^d} + a_k^d \frac{\partial c_{ij}^l}{\partial x^d} = 0\] (A.6)

for all \(s\). This structure equation is equivalent to the Jacobi identity.

Let \(E \rightarrow M\) be a Lie algebroid with anchor \(a : E \rightarrow TM\) and let \(\{e_i\}\) be a local basis of \(\Gamma(E)\). Let \([e_i, e_j] = c_{ij}^k e_k\) and \(a(e_i) = a_i^d \frac{\partial}{\partial x^d}\). Consider

\[Q = \xi^i a_i^d \frac{\partial}{\partial x^d} - \frac{1}{2} \xi^i c_{ij}^k \frac{\partial}{\partial \xi^k} \in \mathcal{X}(\Pi E)
\]

where \(\xi^i\) are coordinates on the fiber of \(\Pi E\) corresponding to the sections \(e_i\) and \(x^d\) are coordinates on the base \(M\). Then we have the following proposition.

**Proposition 10.** \(Q\) is a homological vector field, i.e. \([Q, Q] = 0\).

**Proof.** Let \(X = \xi^i a_i^d \frac{\partial}{\partial x^d}\) and \(Y = -\frac{1}{2} \xi^i c_{ij}^k \frac{\partial}{\partial \xi^k}\).

Then \([Q, Q] = [X, X] + [Y, Y] + 2[X, Y]\)

We have
\[ [X, X] = \left[ \xi^i a^d \frac{\partial}{\partial x^d}, \xi^s a^t \frac{\partial}{\partial x^t} \right] \]
\[ = \xi^s \xi^i a^t \frac{\partial}{\partial x^d} + \xi^i \xi^s a^t \frac{\partial}{\partial x^d} \]
\[ = \xi^s \xi^i \left( a^t \frac{\partial}{\partial x^d} - a^s \frac{\partial}{\partial x^d} \right) \frac{\partial}{\partial x^d} \]
\[ = \xi^s \xi^i c_{st} a^t \frac{\partial}{\partial x^d}, \quad (A.7) \]

by (A.1).

For the second term we get

\[ [Y, Y] = \frac{1}{4} \left[ \xi^i \xi^j c_{ij}^k \frac{\partial}{\partial \xi^k}, \xi^s \xi^t c_{st} \frac{\partial}{\partial \xi^u} \right] \]
\[ = \frac{1}{4} \left( \xi^s \xi^t c_{st} \left( \xi^i c_{uj}^k - \xi^s c_{ik}^u \right) \right) \frac{\partial}{\partial \xi^k} + \xi^i \xi^j c_{ij}^k \left( \xi^t c_{uk}^k - \xi^s c_{uk}^s \right) \frac{\partial}{\partial \xi^k} \]
\[ = \frac{1}{4} \left( \xi^i \xi^j c_{ij}^k \left( \xi^t c_{uk}^k - \xi^s c_{uk}^s \right) \right) \frac{\partial}{\partial \xi^k} + \xi^i \xi^j c_{ij}^k \left( \xi^t c_{uk}^k - \xi^s c_{uk}^s \right) \frac{\partial}{\partial \xi^k} \]
\[ = \frac{1}{2} \xi^i \xi^j c_{ij}^k \left( \xi^t c_{uk}^k - \xi^s c_{uk}^u \right) \frac{\partial}{\partial \xi^k} \]
\[ = \xi^i \xi^j \xi^k c_{ij}^k \frac{\partial}{\partial \xi^k}. \quad (A.8) \]

For the last term we have

\[ 2[X, Y] = - \left[ \xi^i a^d \frac{\partial}{\partial x^d}, \xi^s \xi^t c_{st} \frac{\partial}{\partial \xi^u} \right] \]
\[ = - \left( \xi^s \xi^t c_{st} a^d \frac{\partial}{\partial x^d} + \xi^i \xi^s a^d \frac{\partial}{\partial x^d} \right) \frac{\partial}{\partial \xi^u} \]
\[ = - \left[ X, X \right] - \xi^i \xi^j \xi^k a^t \frac{\partial}{\partial x^d} \frac{\partial}{\partial \xi^k}. \quad (A.9) \]

Hence

\[ [Q, Q] = \xi^i \xi^j \xi^k \left( c_{ij} c_{uk}^s - a^d \frac{\partial}{\partial x^d} a^t \frac{\partial}{\partial \xi^k} \right) \frac{\partial}{\partial \xi^u}. \quad (A.10) \]

Let \( \alpha_{ijk} = c_{ij} c_{uk}^s - a^d \frac{\partial}{\partial x^d} a^t \frac{\partial}{\partial \xi^k} \) and \( \xi^i \xi^j \xi^k \alpha_{ijk} = \phi_{ijk} \).
Then
\[
6[Q, Q] = \phi_{ijk} + \phi_{ikj} + \phi_{jki} + \phi_{kij} + \phi_{kji}
= (c'^{e}_{ik} c^{e}_{jk} + c'^{e}_{jki} + c'^{e}_{kk} c^{e}_{ij} + a'^{e}_{i} \frac{\partial c^{e}_{ik}}{\partial x^{d}} + a'^{e}_{j} \frac{\partial c^{e}_{ki}}{\partial x^{d}} + a'^{e}_{k} \frac{\partial c^{e}_{ij}}{\partial x^{d}}) \frac{\partial}{\partial \xi^{s}}
= 0,
\]
by (A.6). Hence \([Q, Q] = 0.\]

Let \(\phi_{\alpha} : E_{\alpha} \rightarrow U_{\alpha} \times V\) where \(E_{\alpha} = E|_{U_{\alpha}}\) be a local trivialization of \(E\). Let \(v = \phi_{\alpha}^{-1}(x, \lambda) = \phi_{\beta}^{-1}(x, m(x) \lambda).\) \(e_{i}(x) = \phi_{\alpha}^{-1}(x, u_{i})\) where \(\{u_{i}\}\) is a basis of \(V\). Hence, \(e_{i} = m_{ji} e_{j}\) and \(\bar{e}_{i} = \bar{m}_{ji} e_{j}\) where \(\bar{m}\) is the inverse matrix of \(m\) and \(\bar{e}_{i}(x) = \phi_{\beta}^{-1}(x, u_{i}).\)

We have the transformation \((x, \lambda) \rightarrow (x, \mu = m(x)\lambda).\) Therefore \(\mu^{i} = m_{ij} \lambda^{j}\) and \(\xi^{i} = m_{ij} \xi^{j}\) and \(\bar{\xi}^{i} = \bar{m}_{ij} \xi^{j}\) where \(\xi^{i}(\bar{\xi}^{i})\) are coordinates on \(V\) corresponding to \(e_{i}(\bar{e}_{i}).\)

On the other hand we have \(a(e_{i}) = a^{i}_{j} \frac{\partial}{\partial x^{j}}.\) From the linearity of the anchor we get

\[
a(m_{ji} e_{j}) = m_{ji} a(e_{j})
= m_{ji} \bar{a}_{j} \frac{\partial}{\partial x^{i}}
= m_{ji} \bar{a}_{j} \frac{\partial x^{s}}{\partial x^{i}} \frac{\partial}{\partial x^{s}}.
\]

Hence

\[
a^{i}_{j} = m_{si} \bar{a}^{i}_{s} \frac{\partial x^{j}}{\partial x^{i}}. \tag{A.11}
\]

We have as well

\[
[e_{i}, e_{j}] = [m_{si} e_{s}, m_{ij} e_{j}]
= m_{si} m_{ij} [e_{s}, e_{j}] + m_{si} a(e_{s})(m_{ij}) e_{i} - m_{ij} a(e_{i})(m_{si}) e_{s}
= m_{si} m_{ij} e_{k} + m_{si} \bar{a}^{s}_{a} \frac{\partial m_{ij}}{\partial x^{a}} \bar{e}_{i} - m_{ij} \bar{a}^{a}_{t} \frac{\partial m_{si}}{\partial x^{a}} \bar{e}_{s}
= (m_{si} m_{ij} c^{\alpha}_{st} + m_{si} \bar{a}^{s}_{a} \frac{\partial m_{uj}}{\partial x^{a}} \frac{\partial u_{j}}{\partial \xi^{\alpha}} - m_{sj} \bar{a}^{a}_{t} \frac{\partial m_{ui}}{\partial x^{a}} \frac{\partial u_{i}}{\partial \xi^{a}}) \bar{m}_{ku} e_{k}
= (m_{si} m_{ij} c^{\alpha}_{st} + m_{si} \bar{m}_{rs} a^{r}_{p} \frac{\partial x^{a}}{\partial x^{p}} \frac{\partial u_{j}}{\partial \xi^{a}} - m_{sj} \bar{m}_{rs} a^{r}_{p} \frac{\partial x^{a}}{\partial x^{p}} \frac{\partial u_{i}}{\partial \xi^{a}}) \bar{m}_{ku} e_{k}
= (m_{si} m_{ij} c^{\alpha}_{st} + a^{p}_{i} \frac{\partial m_{uj}}{\partial x^{p}} - a^{p}_{j} \frac{\partial m_{ui}}{\partial x^{p}}) \bar{m}_{ku} e_{k}.
\]

Therefore

\[
c^{k}_{ij} = (m_{si} m_{ij} c^{\alpha}_{st} + a^{p}_{i} \frac{\partial m_{uj}}{\partial x^{p}} - a^{p}_{j} \frac{\partial m_{ui}}{\partial x^{p}}) \bar{m}_{ku}. \tag{A.12}
\]
On the other hand, by using linearity and the Leibniz rule repeatedly, we get

\[ \frac{\partial}{\partial \xi^k} \xi^i \frac{\partial}{\partial \xi^k} = \tilde{m}_{ij} \tilde{m}_{jkl} \tilde{\xi}^l \tilde{\xi}^d \left( m_{si} m_{tj} \tilde{c}_s^a + a_i^t \partial m_{uj} \partial \xi^p - a_j^t \partial m_{ui} \partial \xi^p \right) \tilde{m}_{ku} m_{vk} \frac{\partial}{\partial \xi^v} \]

\[ = \tilde{m}_{ij} \tilde{m}_{jkl} \tilde{\xi}^l \tilde{\xi}^d \left( m_{si} m_{tj} \tilde{c}_s^a + a_i^t \partial m_{uj} \partial \xi^p - a_j^t \partial m_{ui} \partial \xi^p \right) \frac{\partial}{\partial \xi^u} \]

\[ = \tilde{\xi}^k \tilde{\xi}^d \left( \tilde{c}_s^a + \tilde{m}_{is} \tilde{m}_{jt} a_i^p \partial m_{uj} \partial \xi^p - \tilde{m}_{is} \tilde{m}_{jt} a_j^p \partial m_{ui} \partial \xi^p \right) \frac{\partial}{\partial \xi^u} \]  

(A.13)

Finally we get

\[ \xi^i a_i^d \frac{\partial}{\partial \xi^d} = \tilde{m}_{ij} \tilde{m}_{jkl} a_i^p \partial m_{uj} \partial \xi^p \frac{\partial}{\partial \xi^d} \]

\[ = \tilde{\xi}^d a_i^d \frac{\partial}{\partial \xi^d}. \]  

(A.14)

From (A.13) and (A.14) and the fact that \( \tilde{m}_{is} \tilde{m}_{jt} a_i^p \partial m_{uj} \partial \xi^p - \tilde{m}_{is} \tilde{m}_{jt} a_j^p \partial m_{ui} \partial \xi^p = 0 \), we can see that \( Q \) is invariant under coordinate transformations.

### A.0.2 Some other proofs

**Proposition 11.** Let \( E \to M \) be a vector bundle and anchor \( a : \Gamma(E) \to \mathfrak{X}(M) \) with \( a[X, fY] = f[X, Y] + a(X)(f)Y \). Then \( a[X, Y] = [a(X), a(Y)] \).

So if the Leibniz rule is satisfied, then the morphism condition is automatically satisfied. This means that the first condition in the definition of Lie algebroid is redundant.

**Proof.** By the Leibniz rule we have \( [[X, Y], fY] = f[[X, Y], Y] + a[X, Y](f)Y \).

On the other hand, by using linearity and the Leibniz rule repeatedly, we get

\[ [[X, Y], fY] = -[[Y, fY], X] + [[X, fY], Y] \] by the Jacobi identity

\[ = -[f[Y, Y] + a(Y)(f)Y, X] + [f[X, Y] + a(X)(f)Y, Y] \]

\[ = [X, a(Y)(f)Y] - [Y, f[X, Y] - [Y, a(X)(f)Y] \]

\[ = a(Y)(f)[X, Y] + a(X)a(Y)(f)Y - f[Y, [X, Y]] - a(Y)(f)[X, Y] - a(Y)a(X)Y \]

\[ = f[[X, Y], Y] + [a(X), a(Y)](f)Y. \]

At once we can see that \( a[X, Y] = [a(X), a(Y)] \). \( \square \)

**Lemma 7.** Suppose that \( \pi : E \to M \) be fibre bundle with fibre \( F \). If \( Q_E \in \mathfrak{X}(E) \) and \( Q_M \in \mathfrak{X}(M) \) are \( \pi \)-related, then \( Q_E = Q_M + Q' \) locally where \( Q' \) is a vector field taking values in \( \mathfrak{X}(F) \), i.e. \( Q' = v^j (x, y) \frac{\partial}{\partial y^j} \) where \( x^i \) are coordinates on \( M \) and \( y^j \) are coordinates on \( F \).

**Proof.** Let \( Q_E = u^j (x, y) \frac{\partial}{\partial x^j} + v^j (x, y) \frac{\partial}{\partial y^j} \) and \( Q_M = m^j (x) \frac{\partial}{\partial x^j} \). We have

\[ Q_E (\pi^* x^k) = Q_E (x^k) = u^k(x, y), \] and \( \pi^* Q_M (x^k) = \pi^* m^k(x) = m^k(x). \) Since \( Q_E \) and \( Q_M \) are \( \pi \)-related we have \( u^k = m^k \) and the lemma follows. \( \square \)
Bibliography


