A Rationale for Tying Merchants’ Membership of Platforms Serving Independent Markets

A thesis submitted to the University of Manchester for the degree of Doctor of Social Sciences in the Faculty of Humanities

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# Contents

I Buyers, Sellers and Payment-Card Platforms \hfill 13

1 Introduction \hfill 15
  1.1 The Model \hfill 15
  1.2 The Research Question \hfill 20
  1.3 Outcome without Tying \hfill 20
  1.4 Outcome with Tying \hfill 21
    1.4.1 Case (i): \( \lambda \geq \rho \) \hfill 22
    1.4.2 Case (ii): \( \lambda > \rho \) \hfill 22
    1.4.3 Consumer-Surplus when Platforms are Tied \hfill 24
  1.5 Pass-on Test \hfill 24

2 Literature Review \hfill 28
  2.1 Pricing in Two Side-Markets \hfill 29
    2.1.1 Definition of a two-sided market \hfill 29
    2.1.2 Origin and lineage of recent work on two-sided markets \hfill 31
    2.1.3 Seminal work on two sided markets \hfill 32
  2.2 Payment-Card Platforms \hfill 34
  2.3 Tying and Foreclosure \hfill 36
  2.4 Tying in Two-Sided Markets \hfill 39

3 My Contribution \hfill 42
  3.1 The Pass-On Test and the Legality of Tying \hfill 42
  3.2 Further Formalization \hfill 43
  3.3 The Outcome without Tying \hfill 44
  3.4 The Outcome with Tying \hfill 44

4 The Framework \hfill 46
  4.1 Assumptions \hfill 46
    4.1.1 The Model Developed by Rochet and Tirole \hfill 46
    4.1.2 Further Assumptions and Refinements \hfill 47
  4.2 Players and Markets \hfill 48
    4.2.1 Buyers, Sellers and Markets \hfill 48
    4.2.2 Platforms and Networks \hfill 49
    4.2.3 Sequence of Events and Information \hfill 49
  4.3 Transactions via a Platform \hfill 50
    4.3.1 Transaction Costs \hfill 50
    4.3.2 Transaction Fees \hfill 50
    4.3.3 Buyers’ Membership Decision \hfill 51
  4.4 Membership and Usage of Platforms \hfill 52
    4.4.1 Sellers’ Membership Decision \hfill 52
    4.4.2 Tying Platforms \hfill 53
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.4</td>
<td>Buyers’ Usage Decision</td>
<td>53</td>
</tr>
<tr>
<td>4.5</td>
<td>Net-Benefit from Platforms</td>
<td>54</td>
</tr>
<tr>
<td>4.5.1</td>
<td>Quality of Service</td>
<td>54</td>
</tr>
<tr>
<td>4.5.2</td>
<td>Partial-Demand for a Seller</td>
<td>55</td>
</tr>
<tr>
<td>4.5.3</td>
<td>Membership and a Seller’s Marginal Cost</td>
<td>56</td>
</tr>
<tr>
<td>4.5.4</td>
<td>The End-User Benefit</td>
<td>57</td>
</tr>
<tr>
<td>4.5.5</td>
<td>The Extra-Surplus</td>
<td>60</td>
</tr>
<tr>
<td>4.6</td>
<td>Payoffs and Consumer Surplus</td>
<td>60</td>
</tr>
<tr>
<td>4.6.1</td>
<td>Sellers’ Profit</td>
<td>60</td>
</tr>
<tr>
<td>4.6.2</td>
<td>Networks’ Profits</td>
<td>61</td>
</tr>
<tr>
<td>4.6.3</td>
<td>Consumer Surplus and Social Welfare</td>
<td>63</td>
</tr>
<tr>
<td>4.7</td>
<td>Switching Off Platform $C_B$</td>
<td>63</td>
</tr>
</tbody>
</table>

II Networks are prohibited from Tying the Membership of Platforms  65

5  
5.1   | Game I: Tying is prohibited                | 68   |
5.2   | Players                                    | 68   |
5.3   | Parameters                                 | 68   |
5.4   | Sequence of Actions                        | 69   |
5.5   | Extra-Surplus                              | 69   |
5.6   | Payoffs                                    | 70   |
5.6   | Decision Rule                              | 72   |

6  
6.1   | Sellers Set Prices                         | 74   |
6.2   | Payoffs and Best Response                  | 74   |
6.3   | The SPNE                                   | 75   |
6.4   | Demands, Markups and Payoffs               | 76   |
6.5   | Properties of the Profit Function          | 77   |

7  
7.1   | Sellers’ Membership Decisions              | 80   |
7.2   | Defining a Seller’s Best-Response          | 80   |
7.3   | Sellers’ Best-Response in Market $\zeta$   | 81   |
7.3.1 | Subgame [1]                                 | 82   |
7.3.2 | Subgame [2]                                 | 84   |
7.3.3 | Subgame [3]                                 | 85   |

8  
8.1   | Network A’s Best-Response                  | 89   |
8.2   | The Monopoly Fee in Market $\zeta$         | 89   |
8.2   | Best-Response in Market $\varphi$          | 90   |

9  
9.1   | Equilibrium Outcomes                       | 92   |
9.2   | Equilibrium Fees                           | 92   |
9.3   | End-User Benefit                           | 93   |
9.4   | Membership and Prices                      | 94   |
9.4   | Consumer-Surplus                           | 95   |
10 **Game II: Tying is Enforced**

10.1 Players .......................... 100
10.2 Parameters .......................... 100
10.3 Sequence of Actions .................. 101
10.4 Extra-Surplus ......................... 101
10.5 Payoffs ........................... 102
10.6 Sellers’ Decision Rule ............... 104
10.7 Sellers’ Price-Setting Decision ....... 104

11 **The Sellers’ Dichotomy** ............. 107

11.1 Defining a Seller’s Best Response .... 107
11.2 Dominance, Equivalence and Feasibility ... 107
11.3 Criteria for a Restricted SPNE ........ 108
11.4 Independence of "Irrelevant" Alternatives ... 109
11.5 No More than Two Relevant Options ... 110
  11.5.1 Outside Options versus Joining Network B ..... 110
  11.5.2 Multihoming versus Joining Network A .... 111
  11.5.3 The Relevant Set ............... 111
11.6 Existence and Uniqueness ............ 111

12 **Eight Membership Configurations** .... 113

12.1 Being "on" a Platform .................. 113
12.2 Classifying Equilibria ................ 114
12.3 Adding Up Rules .................... 115
12.4 Consistency of Choices ............... 115
12.5 Constraints on $J$ and $K$, given $I$ ... 116
12.6 Possible Membership Configurations .... 119

13 **Three Classes of Subgame** ......... 121

13.1 The Tie is either "Slack" or "Binding" .... 121
13.2 Classifying Subgames ............... 123
13.3 Configurations in Subgame [1] ........ 124
13.4 Configurations in Subgame [2] ........ 124
13.5 Configurations in Subgame [3] ....... 125
13.6 Summary .......................... 126

14 **Elliptic Curves** .................... 128

14.1 The Action-Space of Network A ........ 128
14.2 Preconditions for the Stability of (303) .... 129
14.3 Preconditions for the Stability of (003) .... 130

15 **Membership in Subgame [1a]** ...... 133

15.1 Four Possible Configurations .......... 133
  15.1.1 The Stability of (330) ............ 133
  15.1.2 The Stability of (221) ............ 134
  15.1.3 The Stability of (112) ............ 136
  15.1.4 The Stability of (003) ............ 137
15.2 Regions of Stability ............... 138
15.3 Mutually Exclusive Areas .......... 139
20 Equilibrium Outcomes 184
20.1 Equilibrium Fees ........................................... 184
20.2 End-User Benefit ........................................... 186
20.3 Membership and Prices .................................... 187
20.4 Consumer-Surplus ........................................... 190

IV Comparing Outcomes, with and without Tying 193
21 The Pass on Test 194
21.1 Consumer-Surplus Generated by Platforms .............. 194
21.2 Levels of Inter-Network Competition .................... 194
21.3 "Strong" Intra-Network Competition .................... 195
21.4 "Moderate" Intra-Network Competition .................. 196
21.5 "Weak" Intra-Network Competition ...................... 196
21.6 The Main Theorem ........................................... 200

22 Conclusion 202
22.1 Summary ....................................................... 202
22.2 Further work .................................................. 204

V Appendix 208
A Existence and Uniqueness 209
A.1 Set Up: Two Membership Options ....................... 209
A.2 Payoffs ......................................................... 210
A.3 Unilateral Deviation ......................................... 211
A.4 Criteria for RSPNE’s ......................................... 212
A.5 Existence and "Uniqueness" of RSPNE’s ................... 213

B Configurations and Relevant Options 214
B.1 Membership when \( M(F) \subseteq \{ a, a \} \) ..................... 214
B.2 Membership when \( M(F) \subseteq \{ a, b \} \) ..................... 215
B.3 Membership when \( M(F) \subseteq \{ b, a \} \) ..................... 216
B.4 Membership when \( M(F) \subseteq \{ b, h \} \) ..................... 217

C Properties of the Inner Oval 219
C.1 An Upper Bound for \( f^2 \) ..................................... 219
C.2 The Slope: \( \frac{dy}{dx} \) ............................................ 220
C.3 Convexity and Concavity ................................... 221
C.4 Polar Coordinates ............................................ 221
C.5 Upper Bound on the Radius, \( a \) ............................ 222
C.6 Lower Bound on the Radius, \( a \) ............................ 222
C.7 Stationary Points ............................................. 223
C.8 Maximum and Minimum Radius .......................... 225
C.9 Summary (Inner Oval) ....................................... 227
List of Figures

Figures in Part II.

Figure 1  End-User Benefit in Subgame [1]  pp. 83
Figure 2  End-User Benefit in Subgame [2]  pp. 84
Figure 3  End-User Benefit in Subgame [3]  pp. 86
Figure 4  Extra Consumer Surplus without Tying  pp. 96

Figures in Part III.

Figure 5  Slackness when \( z < 0 \)  pp. 122
Figure 6  Slackness when \( z > 0 \)  pp. 122
Figure 7  Classifying Subgames  pp. 123
Figure 8  Inner Oval  pp. 130
Figure 9  Outer Oval  pp. 131
Figure 10 Regions of Stability in Subgame [1a]  pp. 139
Figure 11 Regions of Stability in Subgame [1b]  pp. 149
Figure 12 Regions Defined by the Number of Sellers on \( C_A \)  pp. 155
Figure 13 Regions Defined by the Number of Sellers on \( D_A \)  pp. 156
Figure 14 \( \theta(\rho) \) and \( \omega(\rho) \) versus \( \rho \)  pp. 178
Figure 15 \( \theta(\rho) \) and \( \phi(\rho) \) versus \( \rho \)  pp. 181
Figure 16 \( \theta(\rho) \), \( \phi(\rho) \) and \( \omega(\rho) \) versus \( \rho \)  pp. 182
Figure 17 Extra Consumer Surplus with Tying  pp. 191
Figure 18 Unique Point of Intersection  pp. 200
Figure 19 Effect of Tying on the Consumer Surplus  pp. 201
# List of Tables

**Tables in Part III.**

- **Table 1**: Eight Configurations pp. 120
- **Table 2**: Impossible Configurations within a Subgame pp. 126
- **Table 3**: Areas of the Action-Space in Subgame [1a] pp. 139
- **Table 4**: Areas of the Action-Space in Subgame [1b] pp. 150
- **Table 5**: Configurations in Subgame [1] pp. 154
- **Table 6**: Areas in the 1st Quadrant pp. 165

**Tables in the Appendix.**

- **Table 7**: Configurations in Subgame [2] pp. 249
- **Table 8**: Configurations in Subgame [3] pp. 252
Abstract

A Rationale for Tying Merchants’ Membership of Platforms Serving Independent Markets. This thesis was submitted to the University of Manchester for the degree of Doctor of Social Sciences in the Faculty of Humanities by Michael King during August 2010.

This study analyses the effect of tying sellers’ membership of a monopoly platform to membership of another platform, which operates in an otherwise competitive market. Visa’s contentious use of the honour-all-cards rule to tie their debit and credit cards is an example of such a tie-in.

There has been a move to judge tying cases under "rule of reason", which permits dubious practices when they are indispensable to creating economic benefit. However, a proportion of the extra-surplus must be passed on to consumers ("pass on test").

Rochet and Tirole (2008) claimed that tying payment cards raised Visa’s profit without harming end-users. However, this doesn’t fully address the concerns of regulators. Hence, my thesis investigates whether tying satisfied the "pass on test".

Part I: In Rochet-Tirole (2008) sellers operate in two independent markets (ç and ð). Network A runs platforms in both markets; and Network B only operates in market ð. The price-level (buyer-fee plus seller-fee) on a network’s platforms is exogenously determined but they can choose the price-structure. This study extends this framework by explicitly modelling competition in the product market.

Part II: Platform competition leads to a price-structure that maximizes the net-benefit received by buyers and sellers. In contrast, a monopoly platform extracts most of the surplus by encouraging excessive use of payment-cards. Therefore, if tying is prohibited, then competition for sellers in market ð leads to an optimal price-structure. However, Network A extracts most of the surplus created by its monopoly platform. Finally, if the average transaction-cost, ρ, exceeds the price-level, ρ, then the net-benefit generated by a monopoly platform remains strictly positive.

Part III: By tying its platforms, Network A can exclude Network B. However, Network A is unable to exclude Network B just by matching the net-benefit it generates; rather, it must "compensate" sellers for the extra competition they face from being on the same network. Therefore, if tying is permitted, then the total net-benefit on Network A exceeds the maximum benefit that can be generated by a single platform.

Part IV: It was found that if transaction-fees, ρ, are high relative to transaction-costs, τ, then tying always increases the consumer surplus. However, if transaction-fees, ρ, are low relative to transaction-costs, τ, then tying doesn’t benefit consumers; and will reduce the consumer-surplus if their transaction-costs are sufficiently high.
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Part I

Buyers, Sellers and Payment-Card Platforms
Outline of Part I

Chapter 1 provides a summary of the thesis.

Chapter 2 summarizes the relevant literature.

Chapter 3 explains the contribution of this thesis.

Chapter 4 sets up the framework and derives key equations.
Chapter 1

Introduction

Chapter 1 provides a summary of the thesis. Section (1) provides a sketch of the model. Section (2) explains the research question and the motivation for the study. Section (3) describes the outcome without tying. Section (4) describes the outcome with tying. Section (5) explains the pass on test and summarizes my main results.

This thesis analyses the effect on consumers of a network tying sellers’ membership of a monopoly platform to membership of a separate platform, which operates in an otherwise, competitive market. The study is motivated by Visa’s contentious use of the honour-all-cards rule to tie their debit and credit cards.

In recent years, there has been a move in the European Community and the US to judge tying cases under what has become known as the "rule of reason". For example, the Treaty of the European Communities (TEC), under Article 81(3), permits practices that might otherwise be seen as harmful (such as abuse of a dominant position) when they are indispensable to creating economic benefit through improved market efficiency. However, consumers must receive a fair share of the resulting benefit. That is, some of the extra-surplus must be passed on to consumers; this requirement will be referred to as the "pass on test".

Rochet and Tirole (2008) found that allowing Visa to tie its credit and debit card platforms would raise "social welfare", which was defined as the sum of Visa’s profit and the surplus received by merchants and cardholders. That is, the positive effect on Visa’s profit outweighs any negative effect it might have on end-users. However, it can reasonably be argued that this finding does not fully address the concerns of regulators because it does not show that consumers benefit from the imposition of a tie. That is, Rochet and Tirole didn’t show that Visa’s tie satisfied the "pass on test". Hence, my thesis investigates whether tying platforms that operate in separate markets benefits consumers.

1.1 The Model

Buyers can be distinguished by whether or not they need store-credit to make purchases, which can be modelled by imagining that there exist two independent markets, $k \in \{c, d\}$. Suppose there is a continuum of buyers $j \in [0, 1]$.
who are divided between the market for credit-goods \( (k = \zeta) \) and the market for debit-goods \( (k = \phi) \); where these markets are served by competing sellers \( i = \{1, 2, 3\} \) who operate in both markets and have the same marginal cost \( \delta \).

A buyer incurs a transaction-cost \( T(j) \in [0, 2\tau] \) when payment is made by conventional means (cash or payment-on-account) but payment-platforms give users the chance to avoid these transaction costs. The marginal cost of processing transactions is \( \gamma \) and platforms charge transaction-fees for their services (buyer-fees and a seller-fees). Furthermore, a given platform operates in just one market. That is, debit-cards facilitate transactions in market \( \phi \); and credit-cards facilitate transactions in market \( \zeta \).

Suppose that there are two competing networks \( n \in \{A, B\} \). Network A runs a platform in both markets; whereas, Network B only operates in market \( \phi \). Hence, Network A has a monopoly in market \( \zeta \) but faces competition in market \( \phi \). To soften this competition, Network A could impose a tie-in on sellers that want access to its monopoly platform. That is, Network A might not allow sellers to join Platform \( C_A \) unless they also join Platform \( D_A \).

In the case of payment-card associations, inter-bank competition determines the price-level (seller-fee plus buyer-fee). However, affiliated banks effectively choose the price-structure (buyer-fee : seller-fee); which is done by agreeing the level of an "interchange-fee" paid by a merchant’s bank to a card-holder’s bank. Hence, the price-level, \( \rho \), is exogenously fixed but networks can still compete on price-structure. The assumption that the price-level is fixed means that seller-fees are determined once buyer-fees have been set. That is, if \( f_n^k \) denotes the seller-fee set by network \( n \) in market \( k \), then its buyer-fee automatically becomes \( \rho - f_n^k \). This framework was used in Rochet and Tirole (2008) to investigate regulatory issues raised by an antitrust case involving Visa and MasterCard.

My study builds on the framework of Rochet and Tirole by adding the following elements:

- Competition occurs between sellers offering horizontally differentiated goods (Hotelling model); and the "travel" cost parameter is \( \sigma \).
- The retail market contains a small number of large firms (e.g. supermarkets).
- Sellers set prices after making their membership decisions.
- Buyers’ transaction costs are uniformly distributed over \( [0, 2\tau] \), which implies that buyers have linear demand for platform-services.
- Sellers have no transaction costs and so their motivation for joining platforms derives from the opportunity to increase market-share by raising the quality-of-service (QoS).
- The buyer-fees set by the platforms are non-negative. However, seller-fees can be negative.

My analysis also requires the following parameter assumption:

- The degree of horizontal product differentiation, \( \sigma \), is large relative to the average payment-related transaction cost, \( \tau \).
**End-User Benefit.** These assumptions made it possible to investigate the effect of tying on the net-benefit received by end-users. If a seller joins a platform where the seller-fee is \( f \), then the decrease in their customers’ transaction-costs is

\[
\hat{U}(f) = \mathbb{E}[T(j) - (\rho - f) | T(j) \geq \rho - f] \Pr(T(j) \geq \rho - f)
\]

but the increase in their own marginal cost (due to the seller-fee) is

\[
\hat{S}(f) = f \cdot \Pr(T(j) \geq \rho - f)
\]

Hence, the extra-surplus available to a seller and their customers (end-user benefit) becomes

\[
\hat{W}(f) = \hat{U}(f) - \hat{S}(f)
\]

The concept of end-user benefit is central to much of the analysis.

Since the price-level (buyer-fee plus seller-fee) is fixed, it follows that the end-user benefit generated by a platform is determined by its price-structure (buyer-fee : seller-fee). Hence, the maximum benefit occurs when buyers pay the entire transaction-fee and the platform is made free to sellers:

\[
\max_f \hat{W}(f) = \hat{W}(0)
\]

Charging buyers ensures that the platform is only used when their transaction-cost, \( T(j) \), exceeds the total transaction-fee, \( \rho \). Whereas, if the seller-fee is positive, then the buyer-fee is less than \( \rho \); which encourages buyers with low transaction costs to use the platform. Hence, high seller-fees tend to decrease the average net-benefit generated by a platform. For example, if \( f = \rho \), then all customers use the platform (because it’s free for buyers) and the net-benefit becomes \( \hat{W}(\rho) = \tau - \rho \). Note that this is negative when \( \rho > \tau \).

This analysis implies that when \( \rho > \tau \), there exists a threshold, \( \omega(\rho) \in (0, \rho) \), such that: if \( f > \omega(\rho) \), then \( \hat{W}(f) < 0 \); whereas, if \( f \leq \omega(\rho) \), then \( \hat{W}(f) \geq 0 \). Since transaction costs are uniform on \([0, 2\tau]\), the average payment-related transaction cost incurred by card-users is

\[
\mathbb{E}[T(j) | T(j) \geq \rho - f] = \frac{1}{2} (\rho - f + 2\tau)
\]

This implies that the threshold for a non-negative end-user benefit is defined by

\[
\frac{1}{2} (\rho - \omega(\rho) + 2\tau) = \rho
\]

Rearrangement gives

\[
\omega(\rho) = 2\tau - \rho
\]

A seller might want to deter the excessive use of payment-cards by surcharging card-users. However, this is assumed not to occur because of the no-surcharge rule (which operates in some countries) or because of the cost of administering a system of two-prices. Hence, the seller-fee is passed on to buyers in the form of higher prices, which reduces the consumer-surplus.

**Sellers’ Membership Decisions.** Let \( m_i^k \in \{o, a, b, h\} \) denote the membership of seller \( i \) in market \( k \), where the possibilities are: outside-options, "o";
join Network A, "a"; join Network B, "b"; or multihome, "h". Note that if tying is prohibited, then sellers can make different membership decisions in each market.

Because sellers can multihome they can join a platform without necessarily being "on" the platform. This is because when given a choice of platforms, sellers will never use the platform with the higher buyer-fee. Hence, it’s useful to introduce the following "usage" functions:

\[ A(f_k, m^k_i) = 1(m^k_i = a) + 1(m^k_i = b), f^k_A \geq f^k_B \]
\[ B(f_k, m^k_i) = 1(m^k_i = b) + 1(m^k_i = h), f^k_A < f^k_B \]

where \( 1(\cdot) \) is an indicator function, which takes the value 1 if a statement is true and is 0 otherwise. The first function takes the value 1 if seller \( i \) is "on" Network A, and is 0 otherwise. Similarly, the second function takes the value 1 if seller \( i \) is "on" Network B, and is 0 otherwise.

**Seller’s Profit.** A given seller operates in both markets and can set different prices for debit-goods (purchased with cash) and credit-goods (payment on-account). In a particular market, a seller competes against a rival who offers a horizontally differentiated version of the good. Furthermore, competition for market-share occurs with respect to price and quality-of-service (customers’ average transaction-cost), which is determined by a seller’s membership decision. If they don’t join a platform, then the average transaction costs is \( \tau \); whereas, if they join network \( n \) in market \( k \), then their customers’ average transaction-cost becomes \( \tau - \hat{U}(f^k_n) \), where \( f^k_n \) is the seller-fee. (Note that if a seller multihomes, then they’re on the platform with the lower buyer-fee; because buyers always use the card that’s cheapest for them.)

This combination of Bertrand competition and quality competition can be analyzed by introducing the idea of an "effective-price", which is defined as the ordinary price plus the payment-related transaction costs incurred by customers. The effective price can be written as follows:

\[ p^k_i = P(f_k, m^k_i, p^k_i) + \tau \quad \text{if } m^k_i = o \]
\[ p^k_i = P(f_k, m^k_i, p^k_i) + \tau - \hat{U}(f^k_B) \quad \text{if } m^k_i = b \]
\[ p^k_i = P(f_k, m^k_i, p^k_i) + \tau - \hat{U}(f^k_A) \quad \text{if } m^k_i = a \]
\[ p^k_i = P(f_k, m^k_i, p^k_i) + \tau - \hat{U}(\max\{f^k_A, f^k_B\}) \quad \text{if } m^k_i = h \]

where \( P(f_k, m^k_i, p^k_i) \) is the ordinary price. Because buyers take expected transaction-costs into account before making their purchasing decision, the market-share of seller \( i \) is determined by the differential in effective prices: \( p^k_{-i} - p^k_i \)\(^1\). However, platform membership raises a seller’s marginal cost because they pay the seller-fee, \( f \), on a proportion of their sales. Hence, if they join a platform, then their markup becomes

\[ \text{markup} = \text{ordinary price} - \text{marginal cost} \]

\(^{1}\)Wright (2003) and Wright (2004) introduced the idea of effective prices. This concept was also made use of by Rochet and Tirole.
Using the definitions of $p_i^k$ and $\hat{W}(f)$, this can be re-expressed as

$$\text{markup} = p_i^k - \tau - \delta + \hat{W}(f)$$

Therefore, a seller’s marginal cost becomes

$$\text{marginal cost} = \tau + \delta - \hat{W}(f)$$

This analysis implies that sellers can increase their competitiveness by joining a platform, providing it offers a positive end-user benefit. In particular, sellers are reluctant to join platforms that encourage excessive use of payment-cards because the high seller-fee raises their costs without greatly improving their quality-of-service. Furthermore, when offered a choice of platforms, sellers prefer to be on the platform that offers the maximum end-user benefit. That is, sellers prefer to join networks where buyers pay most of the transaction fee, which means that networks compete on price-structure. This suggests that if networks compete for sellers in market $d$, then the equilibrium seller-fee maximizes a platform’s end-user benefit:

$$\hat{f}_i^d = \arg \max_f \hat{W}(f) = 0$$

**Platform’s Profit.** Since the price-level is exogenously fixed, so is a platform’s markup, $\mu = \rho - \gamma$. However, to make a profit a platform must be active, which requires at least one of the sellers to have joined. Once a seller is on the platform the proportion of sales made via the platform depends on buyers’ demand; which is the fraction of buyers with transaction costs higher than the buyer-fee:

$$Q(f) = \Pr(T(j) \geq \rho - f)$$

$$= \frac{1}{2\tau} [2\tau - \rho + f]$$

$$= \frac{1}{2\tau} [\omega(\rho) + f]$$

Therefore, networks choose a price-structure that maximizes the volume of transactions. In general, this involves making platform(s) as cheap for card-users as possible, while still attracting sellers.

A monopoly platform would set the highest possible seller-fee, while still offering a non-negative end-user benefit. Also, buyer-fees are constrained to be non-negative and so the seller-fee can’t exceed $\rho$. Finally, when tying is prohibited, Network $A$ can’t use Platform $C_A$ to help them attract sellers in market $d$. Therefore, if tying is prohibited, then the seller-fee chosen by Network $A$ in market $c$ becomes:

$$\hat{f}_A^c = \max \left\{ f : \hat{W}(f) \geq 0, f \leq \rho \right\} = \min \{ \rho, \omega(\rho) \}$$

If there were no platforms, then in equilibrium each seller would receive one-third of the market. Furthermore, since the degree of product differentiation, $\sigma$, is large relative to the average transaction-cost, $\tau$, the presence of platforms can only perturb this outcome. Hence, if one seller joins, then the volume of transactions is approximately $\frac{1}{3} Q(f_i^k)$. Since the proportion of sellers on Network $A$ in market $k$ is given by $\sum_i A(f_k, m_i^k)$, it follows that
sellers’ demand for Network A becomes \( \frac{1}{3} \sum_i A(f_k, m_k^i) \). Similarly, sellers’ demand for Network B becomes \( \frac{1}{3} \sum_i B(f_k, m_k^i) \). Finally, the profit generated by a platform is the markup \( \mu \) multiplied by the volume of transactions, which implies that the profit made by each network in market \( k \) becomes

\[
\Upsilon_A(f_k, m_k) = \frac{1}{3} \mu Q(f_k^A) \sum_i A(f_k, m_k^i)
\]

\[
\Upsilon_B(f_k, m_k) = \frac{1}{3} \mu Q(f_k^B) \sum_i B(f_k, m_k^i)
\]

A network’s payoff is the profit received from its (active) platforms:

\[
\Pi_A = \Upsilon_A(f_{\psi}, m_{\psi}) + \Upsilon_A(f_{d\cdot}, m_{d\cdot})
\]

\[
\Pi_B = \Upsilon_B(f_{d\cdot}, m_{d\cdot})
\]

### 1.2 The Research Question

The analysis suggests that networks and sellers prefer opposite price-structures: networks want low buyer-fees to increase the volume of transactions; whereas, sellers want higher buyer-fees to deter the excessive use of payment-cards. Furthermore, it can be seen that platform competition leads to a price-structure that maximizes end-user benefit; whereas, a monopoly platform extracts most of the surplus by encouraging excessive use of payment-cards.

It is assumed that Network A competes for sellers in market \( d\cdot \) (debit-cards) but has a monopoly in market \( \psi \) (credit-cards). Is the distortion caused by Network A’s monopoly credit-card platform reduced by tying? And, if so, does a tie also satisfy the pass-on test? An answer can be found by comparing the outcome that occurs without tying to that which occurs if tying is permitted.

Let \( \Delta\Phi_I \) denote the extra consumer-surplus generated by platforms when tying is prohibited; and let \( \Delta\Phi_{II} \) denote the extra consumer-surplus generated by platforms when tying is permitted. The pass-on test involves comparing \( \Delta\Phi_I \) and \( \Delta\Phi_{II} \).

### 1.3 Outcome without Tying

When tying is prohibited, it’s convenient to analyze each market separately:

**Market \( \psi \).** Since Network A has a monopoly in market \( \psi \), they can choose a low buyer-fee to encourage a high volume of transactions. Providing Platform \( C_A \) offers a non-negative end-user benefit, sellers prefer to join Platform \( C_A \) rather than use outside-options (that is, cash). Hence, Network A extracts most of the surplus generated by Platform \( C_A \). Whether or not Platform \( C_A \) offers any benefit at all depends on how the average transaction cost, \( \tau \), compares to the price-level, \( \rho \). There are two cases: \( \rho \geq \tau \); and \( \rho < \tau \).

- If transaction-fees are high relative to the transaction-costs (\( \rho \geq \tau \)), then Network A sets the highest seller-fee possible, subject to offering a non-negative end-user benefit. Hence, the seller-fee on Platform \( C_A \) becomes \( \hat{f}_A^A = \omega(\rho) \). Since \( \hat{W}(\omega(\rho)) = 0 \), this implies that there is no end-user benefit on Platform \( C_A \).
• If transaction-fees are low relative to the transaction-costs ($\rho < \tau$), then Network $A$ makes Platform $C_A$ free for buyers and expensive for sellers, $\hat{f}_A^\rho = \rho$. Hence, in market $\varsigma$, all payments are made by card, which implies that the net-change in the surplus available to buyers and sellers becomes $\hat{W}(\rho) = \tau - \rho$. By assumption, $\tau > \rho$, which implies that Platform $C_A$ still generates a strictly positive end-user benefit.

Therefore, the optimal seller-fee in market $\varsigma$ is

$$\hat{f}_A^\varsigma = \min \{\omega(\rho), \rho\}$$

It follows that the end-user benefit offered by Network $A$, in market $\varsigma$, becomes:

$$\hat{W}(\hat{f}_A^\varsigma) = \hat{W}(\min \{\omega(\rho), \rho\}) \geq 0$$

**Market $\varphi$.** If tying is not permitted, then sellers just choose the best platform in each market, namely, the platform with the highest end-user benefit. Hence, a form of Bertrand competition forces the networks to make their platforms free for sellers:

$$\hat{f}_A^\varphi = \hat{f}_B^\varphi = 0$$

This implies that buyers pay the entire transaction-fee. Since the networks set the same buyer-fees, consumers are (formally) indifferent about which network they use. However, inertia leads buyers to remain with the incumbent, namely, Network $A$. It follows that the end-user benefit offered by Network $A$, in market $\varphi$, becomes:

$$\hat{W}(\hat{f}_A^\varphi) = \hat{W}(0) = \max_f \hat{W}(f)$$

That is, sellers and their customers receive the maximum possible net-benefit from platforms.

**Consumer-Surplus.** Because sellers make identical membership decisions, the sellers are equally competitive in both markets. It follows that competition forces sellers to pass the extra-surplus on to their customers. This suggests that:

**Claim 1.1 (consumer-surplus without tying.)** If tying is prohibited, then the increase in the consumer-surplus due to platforms becomes

$$\Delta \Phi_I = \hat{W}(0) + \hat{W}(\min \{\omega(\rho), \rho\})$$

### 1.4 Outcome with Tying

As in the previous case, sellers wish to be on the platform that offers the higher end-user benefit. Tying forces sellers to choose between networks rather than platforms and so sellers have to consider the combined end-user benefit on a network’s platforms. Furthermore, Network $A$ can always offer a higher combined end-user benefit than Network $B$.

There exists a threshold, $\lambda \in (0, \tau)$, such that the nature of the outcome depends on whether or not $\rho$ exceeds this threshold. Hence, there are two cases to consider: (i) $\rho \leq \lambda$; and (ii) $\rho > \lambda$. 


1.4.1 Case (i): $\rho \leq \lambda$

If $\rho \leq \lambda < \tau$, then platforms are very cheap relative to the benefit that comes from avoiding transaction costs. It follows that access to Platform $C_A$ is valuable to sellers, because it significantly increases their quality-of-service.

It can be shown that if tying is enforced and $\rho \leq \lambda$, then Network $A$ is able to exclude Network $B$, despite setting the highest possible seller-fees on both its platforms. That is, Network $A$ is able to attract all the sellers even when $f^i_A = f^i_B = \rho$. Making its platforms free to buyers encourages an excessive use of payment-cards. However, since $\rho < \tau$, it follows that $\rho < 2\tau - \rho \equiv \omega(\rho)$, which implies that the end-user benefit on Network $A$’s platforms is still strictly positive.

Claim 1.2 (Exclusion of Network B when $\rho \leq \lambda$) If $\rho \leq \lambda$, then Network $A$ excludes Network $B$. Furthermore, Network $A$ is able to attract all the sellers despite making both its platforms free to buyers (and expensive for sellers). That is, the fees on Network $A$ are $f^i_A = f^i_B = \rho$.

1.4.2 Case (ii): $\rho > \lambda$

If $\rho > \lambda$, then platforms aren’t particularly cheap relative to the benefit that comes from avoiding transaction costs. This means that access to Platform $C_A$ is, generally, desirable but sellers are prepared to forego it when Network $A$ encourages an excessive use of payment-cards by setting low buyer-fees (and high seller-fees). It follows that Network $A$ must compete with Network $B$ for sellers.

Consider what happens when the fees on Network $A$ are:

\[
\begin{align*}
    f^i_A &= \min\{\omega(\rho), \rho\} \\
    f^i_B &= \min\{\omega(\rho), f^i_B\}
\end{align*}
\]

Let these fees be referred to as Network $A$’s default fees, because it corresponds to Network $A$’s best-response when tying is prohibited. It can be shown that if Network $A$ sets its default fees, then sellers’ membership is as follows:

- Network $A$ offers a non-negative end-user benefit in market $\zeta$. Hence, sellers (slightly) prefer Platform $C_A$ to outside-options.

- If Network $B$ offers a positive end-user benefit, then this benefit is matched by Network $A$; whereas, if Network $B$ offers a negative end-user benefit, then Network $A$ offers a non-negative end-user benefit. Hence, the sellers prefer Platform $D_A$ to either Platform $D_B$ or outside-options.

The payoff received by Network $A$ from this default-option is

\[
\Pi_A = \mu.Q(\min\{\omega(\rho), \rho\}) + \mu.Q(\min\{\omega(\rho), f^i_B\})
\]

Network $A$ can (and will) veto any outcome that generates a lower payoff than the default-payoff. The default-payoff can be compared with the payoff
received from accommodating Network B. It was found that it is never in the interest of Network A to share the market with Network B. This is why it was assumed that there’s a small number of large retailers. This assumption ensures that the extra market share that comes from attracting the final seller always outweighs the benefit from slightly reducing the end-user benefit and increasing buyers’ demand.

Claim 1.3 (Exclusion of Network B when $\rho > \lambda$.) If $\rho > \lambda$, then Network A never accommodates Network B and always offers a sufficiently high combined end-user benefit to attract all the sellers.

The analysis shows that Network A will exclude Network B by offering a sufficiently high combined end-user benefit. However, because sellers choose between networks rather than platforms, tying gives Network A flexibility over how it apportions the combined end-user benefit across its platforms. In particular, Network A can increase the seller-fee on Platform $D_A$ above its default level, $f_A^{d_A} > \min\{\omega(\rho), f_B^{d_B}\}$, providing they compensate by reducing the seller-fee on Platform $C_A$ below its default level, $f_A^{c_A} < \min\{\omega(\rho), \rho\}$. This is equivalent to decreasing the end-user benefit on Platform $D_A$ and increasing it on Platform $C_A$.

Furthermore, since buyers have linear demand and $\hat{W}(f)$ is a decreasing concave function of $|f|$, the volume of transactions on a platform can be re-expressed as a decreasing concave function of its end-user benefit. This implies that reducing the benefit on Platform $D_A$ while increasing it on Platform $C_A$, tends to increase the overall profit. (Consider a mean preserving spread.) Therefore, the extra flexibility afforded by tying increases Network A’s profit.

This suggests that Network B will offer the higher end-user benefit in market $d$, while Network A offers a strictly positive end-user benefit in market $c$. In this situation, sellers may join opposite networks. For example, Seller 1 could join Network B while Seller 2 could join Network A. Should such an outcome occur, Seller 1 receives an advantage in market $d$, while Seller 2 receives an advantage in market $c$.

Moreover, this type of outcome tends to increase sellers’ equilibrium profits because it enables them to further differentiate themselves by specializing in different markets. That is, each seller has a strictly higher quality-of-service than their rival in one market. Whereas, if sellers join the same network, then they are equally competitive and so competition forces them to pass the extrasurplus on to customers. That is, in a symmetric equilibrium, the sellers’ only source of rents is horizontal product differentiation. Hence, if platforms are tied and Network B offers the higher end-user benefit in market $d$, then sellers have two "strategies":

1. Join the network that offers the highest combined end-user benefit, namely Network A.

2. Put further distance between themselves and their rival by foregoing access to Platform $C_A$ but gain an advantage in market $d$ by being on Platform $D_B$ rather than Platform $D_A$. 

23
This suggests that Network A is unable to exclude Network B just by matching its benefit; rather, it must "compensate" sellers for the extra competition they face from being on the same network. In other words, tying makes it possible for sellers to specialize in a particular market; and to close-off this possibility, Network A must offer a strictly higher total benefit than that offered by Network B.

Claim 1.4 ("Compensation" is needed to overcome sellers’ rivalry.) If \( \rho > \lambda \), then to attract all the sellers (and exclude Network B), the combined end-user benefit on Network A’s platforms must exceed the maximum benefit that can be generated by a single platform:

\[
\hat{W}(\tilde{f}_A^e) + \hat{W}(\tilde{f}_A^d) > \hat{W}(0)
\]

It can be shown that there exists an equilibrium in which Network B tries to attract sellers by making its platform free to sellers, \( \tilde{f}_B^e = 0 \), but is, ultimately, excluded because Network A offers a higher combined end-user benefit. Furthermore, it can be shown that this is the only possible equilibrium. Finally, because sellers make identical membership decisions, they are equally competitive. It follows that competition forces sellers to pass the extra-surplus on to their customers.

1.4.3 Consumer-Surplus when Platforms are Tied

The outcome when platforms are tied can be summarized as follows. If \( \rho \leq \lambda \), then Network A is able to exclude Network B even when it encourages an excessive volume of transactions by making both its platforms free to buyers: \( \tilde{f}_A^e = \tilde{f}_A^d = \rho \). Whereas, if \( \rho > \lambda \), then Network A is in competition with Network B for the membership of sellers.

Claim 1.5 (Consumer-Surplus with tying.) There are two cases: (i) \( \rho \leq \lambda \); and (ii) \( \rho > \lambda \).

(i) If \( \rho \leq \lambda \), then the increase in the consumer-surplus due to platforms becomes:

\[
\Delta \Phi_{II} = 2\hat{W}(\rho)
\]

(ii) If \( \rho > \lambda \), then the increase in the consumer-surplus due to platforms becomes:

\[
\Delta \Phi_{II} = \hat{W}(\tilde{f}_A^e) + \hat{W}(\tilde{f}_A^d) > \hat{W}(0)
\]

1.5 Pass-on Test

The legality of tying depends on whether part of the surplus created by improved market efficiency is passed on to consumers. Hence, the consumer surplus created when tying is prohibited, \( \Delta \Phi_I \), should be compared to that
generated when platforms are tied, $\Delta \tilde{\Phi}_{II}$. There are two cases to consider: (i) $\rho \leq \lambda(\rho)$; and (ii) $\rho > \lambda(\rho)$.

**Case (i).** Firstly, if tying is prohibited, then the consumer-surplus is $\Delta \tilde{\Phi}_{I} = \tilde{W}(0) + \tilde{W}(\min\{\rho, \omega(\rho)\})$, which becomes $\Delta \tilde{\Phi}_{I} = \tilde{W}(0) + \tilde{W}(\rho)$ when $\rho \leq \lambda < \tau$.\(^2\) Secondly, if platforms are tied and $\rho \leq \lambda$, then Network A is able to exclude Network B despite charging sellers the full fee. Hence, when tying is permitted, the consumer-surplus becomes $\Delta \tilde{\Phi}_{II} = 2\tilde{W}(\rho)$. Finally, since $\tilde{W}(0) > \tilde{W}(\rho)$, it follows that $\Delta \tilde{\Phi}_{II} - \Delta \tilde{\Phi}_{I} = \tilde{W}(\rho) - \tilde{W}(0) < 0$, which implies that tying lowers the consumer surplus.

**Claim 1.6** (*The effect of tying when $\rho \leq \lambda$.*) If $\rho \leq \lambda$, then $\Delta \tilde{\Phi}_{II} < \Delta \tilde{\Phi}_{I}$. That is, if transaction-fees, $\rho$, are very low relative to transaction-costs, $\tau$, then tying harms consumers and fails the pass-on test.

**Case (ii).** When tying is prohibited, the consumer-surplus is $\Delta \tilde{\Phi}_{I} = \tilde{W}(0) + \tilde{W}(\min\{\rho, \omega(\rho)\})$, which can be re-expressed as\(^3\)

$$\Delta \tilde{\Phi}_{I} = \begin{cases} 
\tilde{W}(0) + \tilde{W}(\rho) & \text{if } \rho \leq \tau \\
\tilde{W}(0) & \text{if } \rho > \tau 
\end{cases}$$

However, if platforms are tied and $\rho > \lambda$, then Network A still competes with Network B for the membership of sellers. That is, to attract all the sellers, Network A must offer a strictly higher combined end-user benefit than Network B:

$$\Delta \tilde{\Phi}_{II} = \tilde{W}(\tilde{f}_{A}) + \tilde{W}(\tilde{f}_{B}) > \tilde{W}(0)$$

It can be seen that if $\rho > \tau$, then tying benefits consumers but the situation is less clear when $\lambda < \rho \leq \tau$. However, more detailed analysis shows that there exists a threshold, $\kappa \in (\lambda, \tau)$, such that if $\rho > \kappa$, then tying benefits consumers, otherwise, tying harms consumers.

**Claim 1.7** (*The effect of tying when $\rho > \lambda$.*) If $\rho > \lambda$, then tying may benefit consumers. There exists a threshold, $\kappa \in (\lambda, \tau)$, such that $\Delta \tilde{\Phi}_{II} > \Delta \tilde{\Phi}_{I}$ iff $\rho > \kappa$. That is, if transaction-fees, $\rho$, are high relative to transaction-costs, $\tau$, then tying benefits consumers and satisfies the pass-on test.

The main result of this thesis is that tying benefits consumers when fees are high and harms consumers when fees are low (Theorem 21.1). The intuition for this result can be summarized as follows.

**Cheap Platforms.** Consider the case where platforms are relatively cheap and tying is prohibited. Platforms are most profitable when relatively high seller-fees are used to subsidize the participation of buyers. However, competition for sellers’ membership forces both networks to offer the maximum end-user benefit in market $d$ and to make their platforms free for sellers.

\(^2\)If $\rho \leq \lambda(\rho) < \tau$, then $\rho < \omega(\rho)$.

\(^3\)Firstly, $\rho > \omega(\rho)$ iff $\rho > \tau$. Secondly, $\tilde{W}(\omega(\rho)) = 0$. 
\(f_A^I = f_B^I = 0\). In contrast, Network A has a monopoly in market \(\varsigma\), which makes it possible to attract all the sellers in this market, providing it offers a non-negative end-user benefit. Moreover, if \(\rho < \tau\), then \(\hat{W}(\rho) > 0\), which implies that sellers will join Platform \(C_A\) even when they are required to pay the entire fee \((f_A^I = \rho)\).\footnote{It can be shown that if \(\rho < \tau\), then \(\rho < \omega\). Since the seller-fee must be below \(\rho\), Platform \(C_A\) necessarily offers a positive end-user benefit and, hence, Network A never meets merchants’ resistance in market \(\varsigma\).} This analysis suggests that the extra-surplus generated by platforms when tying is prohibited becomes \(\Delta \hat{\Phi}_I = \hat{W}(0) + \hat{W}(\rho)\).

Now consider the case where tying is enforced. Network B will still try to resist exclusion by making its platform free to sellers \((f_B^I = 0)\). However, Network A now has much greater flexibility over its choice of fees. Since the end-user benefit generated by a platform decreases as the seller-fee increases and the seller-fee can’t exceed \(\rho\), it follows that \(\hat{W}(f_A^I) + \hat{W}(f_B^I) > 2\hat{W}(\rho)\). Hence, the surplus offered by Network B is \(\hat{W}(0)\), whereas, the combined surplus offered by Network A necessarily exceeds \(2\hat{W}(\rho)\). Moreover, if fees are sufficiently low, then \(\hat{W}(\rho)\) will become comparable to \(\hat{W}(0)\), which implies that \(2\hat{W}(\rho)\) can become almost double the size of \(\hat{W}(0)\). In this situation, Network A will necessarily offer the highest combined surplus even when sellers pay the full fee on both its platforms. That is, if platforms are relatively cheap, then access to a platform in market \(\varsigma\) is important to sellers because it significantly affects their competitiveness even when sellers pay entire fee. Furthermore, as \(\rho\) decreases, any loss suffered from being on Platform \(D_A\) rather than Platform \(D_B\) becomes ever more minimal. It follows that if platforms are relatively cheap, then tying gives Network A substantial leverage over sellers. Moreover, this leverage can be strong enough to exclude Network B despite setting the maximum possible seller fees on both its platforms. This analysis suggests that for sufficiently low values of \(\rho\), the combined surplus generated by platforms when tying is enforced becomes \(\Delta \hat{\Phi}_{II} = 2\hat{W}(\rho)\).

It can be seen that if platforms are cheap, then \(\Delta \hat{\Phi}_{II} - \Delta \hat{\Phi}_I = \hat{W}(\rho) - \hat{W}(0)\). Finally, since \(\hat{W}(0) > \hat{W}(\rho)\), it follows that tying reduces the surplus and fails the pass-on-test.

**Expensive Platforms.** Consider the case where platforms are relatively expensive and tying is prohibited. Platforms are most profitable when high seller-fees are used to subsidize the participation of buyers. However, in market \(d\) competition for sellers forces both networks to offer the maximum possible end-user benefit on their platforms \((f_A^d = f_B^d = 0)\). But Network A has a monopoly in market \(\varsigma\) and so can attract all the sellers in this market, providing it offers a non-negative end-user benefit on Platform \(C_A\). It can be shown that if \(\rho > \tau\), then \(\rho > \omega\), which implies that Network A is able to increase the seller-fee until it meets merchants’ resistance and Platform \(C_A\) generates no end-user benefit \((f_A^e = \omega)\). This analysis suggests that the combined surplus generated by platforms when tying is prohibited becomes \(\hat{W}(0) + \hat{W}(\omega)\). Finally, since \(\hat{W}(\omega) = 0\), it follows that \(\Delta \hat{\Phi}_I = \hat{W}(0)\).

Now consider the case where platforms are relatively expensive and tying is enforced. Network B will still try to resist exclusion by making its platform free to sellers \((f_B^d = 0)\). However, by tying its platforms Network A can exclude Network B despite setting a positive seller-fee in market \(d\). That is, Network A can increase \(f_A^d\) above zero providing they compensate by reducing \(f_A^e\) below
Hence, tying leads to more balanced fees on Network A’s platforms. Since Network B makes its platform free to sellers, it follows that neither network has the best platform in both markets. That is, in market $d$, $\hat{W}(f_A^d) < \hat{W}(0)$, whereas, in market $\varsigma$, $\hat{W}(f_A^\varsigma) > 0$. This gives sellers the option to specialize by joining different networks (asymmetric membership). If this were to happen, then the sellers on Network A would be efficient in market $\varsigma$ but inefficient in market $d$, whereas, the opposite would be true for sellers on Network B. Furthermore, such specialization would enable sellers to retain part of the surplus created by platforms, which gives sellers an incentive to specialize and creates a kind of repulsion between them. It follows that in order to attract all the sellers, and exclude Network B, Network A must overcome this repulsion. Hence, Network A must offer a strictly higher total benefit than that offered by Network B. That is, $\hat{W}(f_A^\varsigma) + \hat{W}(f_A^d) > \hat{W}(0)$. Therefore, the combined surplus generated by platforms when tying is enforced becomes $\Delta \tilde{\Phi}_{II} > \hat{W}(0)$.

It can be seen that if platforms are expensive, then $\Delta \tilde{\Phi}_{II} - \Delta \tilde{\Phi}_I > 0$, which implies that tying increases the surplus and satisfies the pass-on-test.

\(^5\)In other words $f_A^d$ approaches $f_A^\varsigma$. 

27
Chapter 2

Literature Review

Chapter 2 summarizes the relevant literature. Section (1) looks at pricing in two-sided markets. Section (2) looks at payment card associations and the interchange fee. Section (3) looks at tying and foreclosure.

Payment card platforms charge cardholders and merchants but tend to adopt a fee structure that favours cardholders over merchants to maximize the volume of transactions. This choice of price structure has been a source of contention for over twenty years and has been the subject of investigations by courts and regulators in a number of countries. The antitrust case brought against Visa in 1998 can be seen as part of this long-standing dispute.

Visa had a dominant position in the credit-card market but faced competition in the debit-card market. Rival debit-cards used low merchant fees to steer merchants away from Visa's debit-card. Visa responded by demanding that merchants who wished to join its credit-card also join its debit-card, but an antitrust investigation in 2003-2004 forced Visa to abandon this tie-in and pay substantial damages to WalMart and other merchants. Was this a socially desirable outcome? An answer to this question should draw on three areas of the IO literature:

**Pricing in Two-Sided Markets**: Platforms can charge fees to both sides for their services. Hence, platforms have a price-level (sum of fees) and a price-structure (ratio of fees). The literature on two-sided markets suggests that the price structure chosen by a social planner is similar to the monopoly price structure; although, not surprisingly, the monopoly price-level is higher. Furthermore, it has been shown that competition between platforms distorts the price-structure because sellers choose the cheapest platform from their point of view (assuming buyers multihome).

**Payment-Card Associations**: An association, such as Visa, is composed of banks that agree to pass on payments from a cardholder's account to a merchant's account. It is agreed that the merchant's bank will pay a small fee to the seller's bank for passing on the payment; this is referred to as the "interchange-fee". Because the level of the interchange-fee is agreed collectively by the banks, some regulators have accused them of collusion. However, an established body of literature on payment-cards argues that payment-card associations are unable to affect the price-level because the banks in the association still compete. Hence, the price-level is determined by the number of banks that belong to the association and the degree to which they are able to
differentiate themselves in the eyes of consumers rather than the interchange-fee\(^1\). Therefore, the collective setting of the interchange-fee enables the association to cooperatively determine the price-structure but not the price-level and so does not (necessarily) amount to price-fixing.

_Tying and Foreclosure:_ A firm with a monopoly in one market can monopolize the market for complementary goods by imposing a tie. Consider a multi-product system such as a razor with replaceable blades: a tie may be sufficient to exclude rival blade manufacturers and monopolize the adjacent market. Consumers may be harmed if the integrated monopolist is able to raise the price of blades. However, tying enables the monopolist to meter usage with a two-part tariff. If consumers are heterogenous, then tying may increase efficiency by enabling a monopolist to meter demand for the system\(^2\).

### 2.1 Pricing in Two Side-Markets

What do credit card associations, games consoles, operating systems, internet backbones, and price comparison sites have in common? At first sight they are engaged in very different activities but they are all examples of platform businesses, which facilitate interactions between two distinct groups of customers. The two-sided nature of these businesses means that they need to take account of similar issues when devising their business models.

ICT technology has greatly reduced the cost of acquiring, processing and transmitting information. The ICT revolution and the rise of the internet has led to the emergence of some significant two-sided industries. This trend is likely to continue and intensify over the coming years. Hence, it is important to understand the economics behind this growing class of industries.\(^3\)

Finally, regulatory authorities need to take into account the two-sided nature of these industries when framing legislation. The antitrust principles that apply in one-sided markets may not be applicable in two-sided markets and so research is needed to ensure that regulation is appropriate.\(^4\)

#### 2.1.1 Definition of a two-sided market

What distinguishes a two-sided business from an ordinary one-sided business? Firstly, platforms act as intermediaries that facilitate direct interactions between two distinct groups of users. At first sight this is a characteristic that is shared by any business that exists within a value chain. For example, a car producer might be said to operate a kind of platform, connecting consumers to tyre makers. However, there is no direct transaction between tyre makers and consumers and so this is not a true platform.\(^5\)

\(^1\)The acquiring side of the business is assumed to be competitive.

\(^2\)There is an analogous complementarity between a seller’s membership and a buyer’s usage: a seller’s membership is similar to purchasing the razor and a buyer’s use of the platform is like purchasing blades. Clearly this analogy isn’t perfect because unlike multi-product systems, multi-sided markets involve cross-group externalities.

\(^3\)Roson (2005)

\(^4\)Evans (2005)

\(^5\)Car producers substitute themselves for consumers in their dealings with tyre makers; and then bundle the tyres with the car.
The second key characteristic of two-sided businesses is that they can charge both sides of the market for transacting via the platform. Hence, platforms have a price structure as well as a price level, which determines how the overall charge for using the platform is allocated between the two groups. It may be that the benefit from interaction is skewed so that members of group-A value interactions with members of group-B far more than the reverse. This will be taken in to account when the platform sets the price structure. The platform will maximize the volume of transactions by choosing a price structure that enables it to cross subsidize between different groups of users.

Indirect externalities are a third defining feature of a two-sided market. A user belonging to one group benefits from interacting with a member of the other group. Members on one side of the market benefit when an additional member of the opposite group joins the platform. Furthermore, they receive this additional benefit without being charged any extra. Hence, platforms are characterized by indirect network externalities. The existence of indirect network externalities is important because two-sided businesses generate profit by internalizing these externalities via the imposition of an appropriate price structure.

Rochet-Tirole (2003) argued that the presence of network externalities is not sufficient for a business to be two-sided. They argue that the defining feature of a two-sided business is that the price structure affects the volume of transactions on the platform.\(^6\) Transaction costs need to be sufficiently high to prevent customers from internalizing the indirect network externalities.\(^7\) If the two sides can negotiate and financial transactions can occur between them, then they can undermine the platform’s preferred price structure. For example, members of the side that are ‘penalized’ by being charged a higher price may threaten not to use the platform unless it is agreed that they can pass some of the cost on to members of the other side. Such agreements neutralize the impact of the platform’s price structure.

To some extent businesses can choose whether to adopt a one sided or a two sided business model. Consider a company that produces razors and razor blades. Such a company has adopted a one-sided business model because it only serves consumers. However, the company could decide to stop manufacturing razor blades and allow other firms to produce razor blades that are compatible with its razor. The blade producers would be required to pay royalties on the blades sold to consumers. In this case the company has adopted a two-sided business model in which its razor functions as a platform.

A two-sided business may need to impose special rules on its customers for it to be viable. For example, credit card associations impose the no surcharge rule and the honour-all-cards rule on merchants. Such rules are imposed to prevent the platform’s price structure being undermined. The imposition of such rules is often contentious and attracts the attention of regulatory authorities. However, the viability of the platform may be compromised if such rules are not permitted.

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\(^6\)It’s possible for the fee structure to be neutral with regard to volume (very much like the classic VAT example of neutrality).

\(^7\)Of course, the transaction cost must be sufficiently high for most potential users in order for the platform business to be viable. If the two groups can interact for free by some other means, then the platform has little to offer.
2.1.2 Origin and lineage of recent work on two-sided markets

The literature on the economics of two-sided markets is related to the work on network economics and multi-product pricing (system goods). In many ways the study of two-sided markets is a synthesis of these two mature areas of the literature on Industrial Organization. Work on the role of (intra-side) network externalities in the development of markets was initiated by Arthur (1989), David (1985) and Farrell-Saloner (1985). Many markets with network externalities are two sided but this was not recognized in the early studies.

Multi-product pricing concerns situations where a consumer has to coordinate their purchases of various products to generate utility (e.g. razors and razor blades). In such markets the price of one product will affect demand for the other. It may be profitable to charge below cost on the durable component of a system in order to establish a large customer base. The loss can then be recouped by charging above marginal cost on the nondurable component of the system: giving away razors to sell razor blades.

Unlike two-sided markets, multi-product markets are not affected by non-internalized externalities. When a consumer makes a decision to buy a razor he takes into account the fact that he will need to buy razor blades. If demand for razors were to increase, then this would result in an increase in demand for razor blades; no third party is affected. In two-sided markets end users do not fully internalize the welfare impact of using the platform.

Given the prevalence of two-sided industries it may seem surprising that economists have paid little attention to them. In fact the issue of pricing and business models in two-sided markets has been studied since the 1980s. In particular, the cooperative setting of the interchange fee by banks belonging to a credit card association has been extensively studied. Credit card associations provide a payment platform connecting merchants and consumers. An interchange fee is paid by the merchant’s bank to the consumer’s bank when it transmits the payment. VISA has been accused of colluding in order to raise the price charged to merchants. However, studies have effectively shown that the interchange fee determines the price structure and that cooperative setting of this fee is socially efficient. Similar considerations have led to suggestions that internet backbones should introduce bilateral fees for passing on traffic.

Internet sites connecting buyers and sellers have also received some attention in the literature. These platforms may be auction sites such as eBay or price comparison sites such as Shopper.com. Most recently there has been some work on software platforms, though this important area has yet to receive adequate attention. Hence, there has been work on two sided markets but it is only recently that the similarities between seemingly rather different businesses have been highlighted.

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8 These studies investigated the role of network externalities in competition between de facto standards such as Betamax and VHS.


10 This strategy enables the firm to discriminate between heavy users of the system and light users of the system, which is known as second degree price discrimination.


2.1.3 Seminal work on two sided markets

Three seminal papers on competition in two-sided markets are:

- Armstrong (2006)
- Rochet-Tirole (2003)

Caillaud-Jullien (2003) analyzes competition between intermediaries who help agents with particular requirements find agents on the other side of the market with matching characteristics. They regard their model as particularly relevant to the intermediation services offered on the internet, who they refer to as Cybermediaries.\(^{15}\)

The intermediaries operate in a two-sided market. Hence, there exist strong indirect inter-group network externalities. This is because the probability of a successful match depends on the number of individuals on the other side of the market that have joined the site; and once membership fees have been paid this additional benefit isn’t charged for. An intermediary charges fees to both sides for the service (which can be negative). In order to create a viable platform the intermediary must solve the ‘chicken and egg’ problem: in order to attract buyers, a platform must have a large number of registered sellers, however, the sellers only join if they anticipate a large number of buyers on the platform. This problem is solved by charging fees that take into account the inter-group network externalities.

Caillaud-Jullien set up a model in which there is assumed to be a unique match between individuals on each side of the market. That is, sellers offer very specific products and buyers have very specific needs. An intermediary owns a matching technology which makes it possible to identify matching agents, providing they have registered.\(^{16}\) Furthermore, the matching technology is imperfect in that it isn’t guaranteed to identify matching agents even when they have both registered. This creates an incentive for the agents to multihome (join more than one platform) in the hope of increasing their chances of finding a partner. Although in some cases agents are unable to multihome.\(^{17}\)

Caillaud-Jullien consider the case where an incumbent platform competes against an entrant. They assumed that the incumbent has a slight advantage in that if an individual is indifferent then they remain with the incumbent platform. That is, there exists a degree of inertia that favours the established intermediary, which can make it difficult for the entrant to solve the chicken and egg problem. They found that there were two distinct types of outcome, depending on whether or not multihoming was allowed.

Firstly, it was found that the incumbent can use negative prices to subsidize the participation of one group; once individuals from this group have registered the other side has to follow. This is referred to as a "divide and conquer" strategy. However, the threat of entry means that the incumbent makes little

\(^{15}\)These include: online estate agents; B2B services; price comparison sites; auction sites; and recruitment sites.

\(^{16}\)This is a database detailing the characteristics of registered agents.

\(^{17}\)For example, a farmer can only attend one market; and most people rarely have time to read more than one paper.
profit. Hence, the outcome is efficient: the price level is quite competitive; and since all the users are on a single platform, network externalities are maximized.

Secondly, the introduction of transaction fees and the option of multihoming makes it easier for the entrant to create a viable platform. Hence, multihoming and transaction fees tend to decrease the profits made by the incumbent. Indeed, rents can be exhausted by the need to deter entry. Finally, if one side multihomes while the other side singlehomes, then there is an equilibrium in which the intermediaries share the market.

Armstrong (2006) investigates the case of "competitive bottlenecks", which is where one side of the market singlehomes and the other side multihomes. A bottleneck occurs because an intermediary restricts access to the singlehoming side by charging high fees to the multihoming side. Finally, when there is more than one platform, the "bottlenecks" are in competition.

Armstrong's argument is as follows. It may not be possible for one side to join more than one platform (e.g. exclusive contracts or time constraints). Hence, platforms are differentiated by their membership on the singlehoming side of the market. Because a platform has exclusive rights over access to a particular member of the singlehoming side, they are able to extract rents from members of the multihoming side. When the multihoming side is heterogeneous with respect to the gross benefit received from access to the single homingside, there will be socially too few of the multihoming side on the platforms.

Armstrong found that when the platforms can charge transaction fees, as well as membership fees, competition is reduced. He argues that this is because transaction fees are only paid once users have successfully interacted (e.g. a sale has been made). This reduces the inter-group network externalities because, on average, members of group 1 are charged extra for every addition member of group 2 that joins. The analysis demonstrates that this makes it easier to persuade members of the multihoming side to join.

Finally, Armstrong claims that there are few (if any) examples of markets where both groups multihome. Hence, he doesn't see much value in discussing this configuration. The implication is that although sellers (or advertisers) multihome, buyers (or viewers) singlehome. However, there seems to be an important difference between multihoming-in-membership and multihoming-in-usage. Whereas, it's rare for buyers to multihome-in-usage, they often multihome-in-membership. For example, it's relatively cost-less to carry more than one payment-card in my pocket or download more than one search engine on my desktop. So perhaps "global" multihoming is more relevant than Armstrong imagines.

Rochet-Tirole (2003) investigates the factors which determine the price structure chosen by firms that adopt a two sided business model and explores the impact of these choices on social welfare.

Rochet-Tirole (2003) found a formula for the price structure chosen by a profit maximizing monopolist. The price structure is given by the ratio of the price elasticity of demand on either side of the market. Rochet-Tirole also found the prices set by a social planner whose objective is to maximize social welfare subject to a balanced budget (Ramsey prices). In this model social welfare is defined as the sum of buyers' and sellers' surplus. It was found that if the benefit distributions on each side of the market are sufficiently
similar, then the Ramsey price structure is identical to that chosen by a private monopolist. However, the price level chosen by the private monopolist obeys the Lerner formula whereas a Ramsey planner would set the price level equal to the marginal cost of a transaction.

Rochet-Tirole (2003) next considered the case of two competing platforms that are horizontally differentiated with respect to buyers. Platform owners demand that sellers who sign up to their platform agree to let buyers choose the platform on which the transaction occurs. This is the honour-all-cards rule.

In the case of two competing platforms buyers and sellers have the option to ‘multihome’. An agent is said to multihome when they belong to both platforms. A seller will always prefer to transact on the platform with the lowest seller price but they may decide to multihome rather than forfeit the opportunity to transact with those buyers who refuse to use their preferred platform. However, even sellers who receive a high benefit from transacting via the platforms may decide to only sign up to their preferred platform if they expect most buyers to multihome. By refusing to sign up to the platform with the higher seller price, sellers can force buyers to use their preferred platform. This has the effect of intensifying price competition for sellers. That is, platform owners have an incentive to try to undercut the seller price of their rival in the hope of ‘steering’ sellers towards their platform; and once sellers have migrated to their platform buyers will be forced to follow.

Rochet-Tirole used this framework to set up a simultaneous move game between the platforms in which the prices charged to each side are strategic variables. The price structure that characterizes the symmetric Nash equilibrium was found to differ fundamentally from the price structure chosen by the Ramsey planner because the possibility of ‘steering’ forces platforms to make concessions to sellers. ‘Steering’ has the effect of tilting the price structure in favour of sellers. Hence, platform competition does not necessarily lead to an efficient price structure and can even lead to a price structure that is less efficient than that chosen by a private monopolist. In effect, Rochet-Tirole (2003) found that there may be a trade off between a less efficient price structure and a more efficient price level under platform competition.

This counter intuitive result suggests that competition authorities should tread cautiously when attempting to regulate firms that operate in two-sided markets. Well established results that hold for one-sided markets may not hold for two-sided markets. Evans (2003) and Roson (2005) argue that a failure to account for the special features of two-sided markets has led to a number of confused and erroneous applications of antitrust law and the imposition of regulations that undermine platform businesses.

2.2 Payment-Card Platforms

A payment-card association such as Visa is composed of affiliated high-street banks. When a buyer pays by card their bank (issuer) debits their account, \( \text{price} + \text{buyer-fee} \), and passes on the payment to the seller’s bank (acquirer) who then credits the seller’s account, \( \text{price} - \text{seller-fee} \). Finally, an interchange-fee is paid by the acquirer to the issuer for passing on the payment. The level of the interchange fee is determined cooperatively by the banks. A higher interchange-fee raises the cost of serving sellers and lowers the cost of serv-
ing buyers. Hence, an increase in the interchange fee is generally associated with an increase in the seller-fee and a decrease in the buyer-fee. Therefore, the price-level (sum of fees) is fixed by inter-bank competition. However, a payment-card association (or rather its affiliated banks) can determine the price-structure (ratio of fees) by choosing an appropriate interchange-fee.

The cooperative nature of the way the interchange fee is agreed led to allegations of collusion on the part of the banks affiliated to Visa. It was argued that the interchange fee was, basically, a tax on the consumers and retailers; which was then redistributed to the association’s members. However, the collective determination of interchange fees was found to be legal in the 1984 NaBanco decision. Baxter (1983) showed that if issuers and acquirers are perfectly competitive, then the efficient level of the interchange fee is strictly positive. In particular, Baxter claimed that the collective determination of the interchange fee was not a straightforward case of price fixing. The NaBanco decision rested partly on Baxter’s analysis, in which he appeared as an expert witness.

However, in recent years the collective determination of the interchange fee has come under attack in the EU and Australia. The basis of the new charges is that the association is acting unreasonably because it sets the interchange fee at a level that maximizes profit rather than social welfare. That is, high interchange fees are being used to subsidize cardholders; which promotes excessive use of payment cards. Furthermore, it was claimed by regulators that Baxter overestimates merchants’ resistance. Baxter assumed that cardholders are not informed about the sellers’ membership policies before they visit. Hence, there was not attempt to account for temptation to increase market share by raising quality of service. (That is, the decision to accept cards is much like installing more checkouts.) Renewed interest in the role of the interchange fee led to a further work on payment-card associations. Two notable studies are Schmalensee (2002) and Rochet-Tirole (2002).

Schmalensee (2002) shows how the interchange fee balances charges between cardholders and merchants under imperfect competition. He initially considers a situation where there is a single issuer and a single acquirer. A two stage game occurs between issuers and acquirers. In the first stage the interchange fee is set so as to maximize (a weighted sum) of the banks’ profits. In the second stage of the game the issuer sets the cardholder-fee and the acquirer sets the merchant-fee. Using this framework a pure strategy Nash equilibrium is found. Schmalensee then generalizes the set up to allow for an oligopoly on both sides of the market. The partial demand of an individual bank depends on the extent to which it deviates from the mean fee set by its rivals. Hence, the banks on each side engage in price competition and Schmalensee focusses on the symmetric equilibrium of this game.

It was found that the privately optimal interchange fee depends on the differences between the cost of serving consumers (borne by the issuer) and the cost of serving merchants (borne by the acquirer). This assumes that

\[18\text{ Reserve Bank of Austrailia (2002)\quad 19\text{ Schmalensee does note that the structure of the model leads to double marginalization, which raises prices, lowers demand and reduces total profits. He note that in this situation issuers and acquirers might want to merge in order to internalize the effects of double marginalization.}\]
certain important functions are more efficiently carried out on one side rather than the other. It was found that if issuers and acquirers incurred the same costs, then a zero interchange fee would be optimal. Moreover, it was found that in non-extreme situations the profit maximizing level of the interchange fee also maximizes the welfare of merchants and consumers.

Rochet-Tirole (2002) also investigates the role of the interchange fee in payment card associations. Unlike Schmalensee, Rochet and Tirole provide a micro-foundation from which they derive the demand for payment-card services. Merchants are assumed to be homogeneous in that they receive equal benefits from processing card payments (rather than cash payments). Whereas, consumers are assumed to be heterogeneous, in that they vary with respect to how much they value the option of paying by card. Merchants are assumed to have sufficient market power that they can sustain a positive markup. Rochet and Tirole set up a four stage game:

1. The association sets the interchange fee.
2. Issuers set cardholder fees and acquirers set merchant fees.
3. Merchants make their membership decision.
4. Merchants set prices.

Issuers and acquirers are treated as distinct businesses. The acquirers are competitive and essentially just pass the interchange fee on to merchants, whereas, issuers are able to differentiate themselves and hence can sustain a positive margin. Rochet and Tirole improved on other studies by endogenizing the sellers’ membership decision. In particular, their analysis took account of the strategic benefits that an individual merchant gains from accepting cards.

Once again, the study concludes that there is nothing intrinsically anti-competitive about the way the interchange fee is agreed. In particular, a positive interchange fee is needed to maximize social welfare. However, they found that the merchant fee does tend to exceed the technical benefit (from reduced transaction costs) that a merchant receives from payment cards. Moreover, the interchange fee can be too high because the threshold at which merchants refuse to join is determined by the average benefit of a cardholder rather than the marginal benefit of a cardholder.\(^{20}\) Hence, the privately optimal interchange fee can lead to an over provision of card services.

### 2.3 Tying and Foreclosure

It is helpful to begin with a brief review of tying, in general. A firm is said to engage in tying when they make the purchasing of one good (tying good) conditional on buying another good (tied good). Mixed bundling occurs when a firm sells two goods together as a single product while at the same time making the products available separately. In contrast, pure bundling refers to the practice of only selling the bundle. The distinction between pure bundling and tying is that the tied good is available on its own under tying but not under pure bundling.\(^{21}\)

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\(^{20}\)The marginal benefit is the benefit of the last cardholder to take a card.

\(^{21}\)Tirole (2005)
According to the "leverage theory", tying is a mechanism whereby a firm with a dominant position in one market (tying good) can use this market power to foreclose sale in a second market (tied good).\textsuperscript{22} Through this foreclosure, the dominant firm is able to monopolize the second market. For this reason tying has been viewed with great suspicion and has been treated harshly by the courts.

Historically, tying and exclusive contrast were seen as restrictions on trade and illegal \textit{per se}.\textsuperscript{23} For example, in 1914 the Clayton Antitrust Act was enacted in the US.\textsuperscript{24} This act extends the Sherman Antitrust Act of 1890 so as to specifically prohibit exclusive contracts and tying. However, from its inception, the prohibitions contained in the Clayton Act conflicted with what has become known as the "Rule of Reason"; which was a doctrine developed by the Supreme Court in its interpretation of the Sherman Antitrust Act.\textsuperscript{25} This ruling of the Supreme Court declared that only contracts that \textit{unreasonably} restrained trade were subject to the antitrust laws and that monopoly power was not \textit{per se} illegal. Moreover, according to the rule of reason, practices that restrict trade are classified as reasonable when they improve market efficiency and do not harm consumers.

Restrictions on tying have been criticized on the grounds that tying can’t be used to increase the profit made by a monopolist.\textsuperscript{26} The Chicago School defence of tying runs as follows:

Suppose that there are two markets, Market 1 and Market 2, and imagine that a firm has a monopoly in Market 1 but that Market 2 is competitive. The marginal costs in Market 1 and Market 2 are $c_1$ and $c_2$, respectively. The gross value of the goods in Market 1 and Market 2 are $u_1$ and $u_2$, respectively. Finally, suppose that the firm imposes a tie-in; and let the price of the bundle be $p$. A consumer’s net benefit from the bundle is $u_1 + u_2 - p$ and the net benefit from just purchasing the second good is $u_2 - c_2$. It follows that the bundle will only be purchased if $p \leq u_1 + c_2$. That is, $p^* = u_1 + c_2$. The profit made from selling the bundle is $\pi(p) = p - c_1 - c_2$, where $p \leq u_1 + c_2$. Therefore, the maximum profit from tying is $\pi(p^*) = u_1 - c_1$; which is identical to the profit from just selling the first good. The argument can be summed up by saying that the firm is in possession of a single rent and tying will not help the firm extract more profit than this rent already gives them. If tying can’t be used to profitably monopolize adjacent markets, then there must be other, more innocuous, reasons for tying such as:

- reduced transaction costs and lower distribution costs;
- guaranteeing that the non-durable component of a systems good is compatible with the durable component;
- protection of intellectual property and "good will".

\textsuperscript{22}The original (static) form of leverage theory was developed by the courts rather than by academic economists.
\textsuperscript{23}When a practice is \textit{per se} illegal it is sufficient to show that a company used the particular strategy. The prosecution does not need to show that any harm to consumers was caused.
\textsuperscript{25}This was first applied in the case of Standard Oil Co. of New Jersey v. United States, 221 U.S. 1 (1911).
\textsuperscript{26}Posner (1976) and Bork (1978)
• price discrimination through metering.

This analysis suggests that tying may only exclude competitors coincidentally. That is, if the increase in efficiency is large, then tying is a profit maximizing strategy whether or not this causes the foreclosure of potential rivals.\textsuperscript{27} Arguments of this type may have led to a sightly more sympathetic view of tying in the Courts. For example, in the 1984 case involving Jefferson Parish Hospital, the Supreme Court recognized the welfare enhancing effects of tying in its decision. (Although, the Supreme Court ruled that the tie was illegal because, on balance, it was harmful.)\textsuperscript{28}

During the 1980s there was a particular focus on how tying could be used to support price discrimination (Schmalensee (1982)). The argument can be summarized as follows:

Consider a multi-product system such as a razor with replaceable blades and suppose that a firm with a popular razor is able to ensure that only its blades are compatible with the razor. Tying enables the monopolist to meter usage with a two-part tariff. If consumers are heterogenous, then tying may increase efficiency by enabling a monopolist to meter demand for the system. Without the tie, heavy users of the system are indirectly subsidized by light users of the system because both sides have to purchase the durable component at the same price. But then heavy users get a significant usage out of the system, whereas, light users have to pay a high fixed cost but then don’t use it all that much. Indeed, the price of the durable component may deter some users from purchasing the system at all, in which case the market isn’t covered. By using a tie to create an integrated system, the monopolist is able to price discriminate between heavy users and light users of the system. This may lead to lower prices for light users and hence, increased coverage.

However, at the start of the 1990s a revised form of the leverage theory emerged (Whinstone (1990)). Suppose there are two markets, Market 1 and Market 2, and that a firm has a monopoly in Market 1. Furthermore, imagine that there is a fixed cost associated with operating in Market 2. (For example, an entry cost.) It’s argued that by tying the two goods, the Monopolist can commit himself to an aggressive pricing strategy because every sale made by a rival firm results in a lost sale of the monopoly good. Hence, the Monopolist will try and undercut a rival operating in the second market. In particular, the optimal price of the bundle may be sufficiently low that the rival firm is unable to cover the fixed costs of its operation. Providing the tie is technologically irreversible this will be perceived as a credible threat by a potential rival; and the threat of a price-war may be enough to deter a rival from entering Market 2 (or encourage exit). Therefore, tying enables a dominant firm to extend its monopoly into adjacent markets. For this reason, tying can be both anti-competitive and profitable.

Since the impact of tying is not clear cut, Evans, Padilla and Ahlborn (2004) argue that it should be considered under the "rule of reason": if the increase in efficiency outweighs the harm done by a decrease in competition, then the tie should be permitted. They argue that tying is only harmful if it

\textsuperscript{27}That is, "no cost predation" or "blockaded entry" can not be ruled out.

\textsuperscript{28}Jefferson Parish Hospital District No. 2 v. Hyde, 466 U.S. 2 (1985)
is part of a predatory strategy designed to exclude competitors. In its 2001 decision in the Microsoft case the D.C. Circuit Court of Appeals explicitly took into account the efficiency effects of tying. Evans, Padilla and Ahlborn (2004) argue that this sets a precedent for adopting a rule of reason approach to tying.

2.4 Tying in Two-Sided Markets

There are two prominent competition cases in which the owners of a monopoly platform have been accused of anti-competitive tying:

- The first was *United States v Microsoft*. This was an antitrust case in which Microsoft was accused of tying its dominant operating system to Internet Explorer, which allegedly foreclosed the market for internet browsers.

- The second was *US Merchants v Visa & MasterCard*. This was a class-action antitrust lawsuit, initiated by WalMart, in which US merchants accused Visa and MasterCard of using the honour-all-cards rule to tie debit and credit cards. These cases are discussed in turn:

**United States v Microsoft.** The prosecution argued that Microsoft tied together the Windows 98 operating system and the Internet Explorer (IE) browser. Furthermore, it was alleged that they manipulated the application programming interfaces (APIs) of Windows so as to favour Internet Explorer over third party browsers such as Netscape Navigator. The D.C. Circuit Court of Appeals rejected the claim that a browser was just an inextricable part of an operating system but held that the tie was not *per se* illegal. That is, Microsoft did tie products that were independent but that didn’t necessarily mean that the practice was illegal. Hence, the case was to be judged under the rule of reason, with both sides bringing in economists to argue their case.

The debate surrounding the Microsoft case took into consideration the fact that operating systems are a platform that link PC users and applications developers. The prosecution alleged that Microsoft wanted to maintain an "applications barrier" to entry faced by a firm that wanted to offer a rival operating system (Carlton-Waldman (2002)). In the absence of compatible applications, a company that wanted to launch a rival operating system would need to develop its own suite of applications; which, indirectly, raises the cost of entering the market for operating systems. Whereas, if Microsoft allowed applications developers to produce applications that were easily portable across operating systems, then producing a commercially viable rival to Windows would be significantly cheaper.

The settlement of 2004 required that Microsoft share its APIs with third-party applications developers, in the hope that over time the applications

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29 This is the voluntary loss of profit motivated by the hope that his will force a rival to exit the market; after which, the predatory firm will be able to (more than) recover its losses.

30 Richard Schmalensee (Dean of MITs Sloan School of Management) acted as Microsoft's expert witness. David S. Evans (vice president at National Economic Research Associates, Inc) also worked with Microsoft on the antitrust case.
barrier to entry would decrease. However, the settlement did not prevent Microsoft from tying other software with Windows.

Furthermore, recent work suggests that tying can improve welfare in two-sided markets when consumers are able to multihome (Choi (2007), Whinston (2001)). It was argued that bundling an essential product (operating system) with an application (media player) encourages consumers to multihome. This makes it possible for content providers to supply platform-specific content because they know that consumers are on both platforms. This means that content providers don’t need to incur the cost of duplicating content for a range of different platforms. Finally, because the cost of supplying content is lower (as porting costs have been removed), more content is made available to viewers. Choi’s analysis was particularly relevant to the case brought against Microsoft by the European Union in which they were criticized for bundling Windows Media player with their operating system.

WalMart v Visa & MasterCard. WalMart initiated a class action against Visa and MasterCard over a tie-in imposed on merchants whereby they were obliged to accept the association’s debit card if they accepted the association’s credit card. It was claimed that the debit and credit markets were separate, and the credit card associations were using their dominant position in the credit card market to deter the entry of on-line debit cards.

In 2003 Visa and MasterCard signed a settlement in which they agreed to compensate merchants and ceased to tie their debit and credit cards. That is, a US merchant is now able to accept Visa’s (or MasterCard’s) credit-card without having to also accept Visa’s (or MasterCard’s) debit-card. However, the legality of the honour-all-cards rule within a specific market has not been challenged; nor has the legality of the no-surcharge rule.

Rochet-Tirole (2008) found tying makes the association less sensitive to competitive pressure in the debit card market. This allows the platform to achieve a better price structure, which increases the volume of transactions and raises social welfare. Rocher-Tirole (2008) claims that the welfare implications of tying in two-sided markets are significantly different from those in one-sided markets, and that traditional antitrust methodology does not necessarily apply. The argument of Rochet and Tirole can be summarized as follows:

Firstly, payment-card associations are not-for-profit organizations, which help banks provide payment services for their customers. This means that the association is not able to set the price-level; which is the merchant-fee plus the cardholder-fee. It is argued that the price-level depends on the degree of competition between affiliated banks and the cost of processing transactions. Hence, because Visa is a not-for-profit association, the price-level must track the opportunity cost of providing the service.

Secondly, the role of the association is to set the interchange-fee, which is paid by the acquirer to the issuer for passing on the payment. This interchange-fee determines the price-structure chosen by the banks, which is the ratio of the merchant-fee to the cardholder-fee. Preexisting economic analysis strongly suggests that the interchange-fee can’t affect the price-level (Baxter (1983)). Moreover, the US courts seem to have accepted this claim.\footnote{This observation might be particularly relevant to games consoles; where some games developers incur the cost of developing the same game for different platforms.}

\footnote{In the NaBanco case affiliated banks were accused of using the interchange-fee as a mechanism for price-fixing. Expert evidence provided by William F Baxter (antitrust professor at...}
Finally, Rochet and Tirole argue that competition from rival networks distorts the price-structure in the debit-card market. That is, because cardholders multihome, merchants have an incentive to accept only the debit-card with the lowest merchant-fee. By single-homing, merchants can force cardholders to use the card with the lowest merchant-fee. A debit-card platform is only active if it can encourage merchants to "get on board". Hence, competition for merchants in the debit-card market forces the networks to set low merchant-fees and, consequently, high cardholder-fees. (Rochet and Tirole refer to this phenomenon as "steering"). Therefore, buyers are penalized; which results in a low volume of debit card transactions.

This analysis leads Rochet and Tirole to claim that unrestrained competition forces networks to focus on the system’s own elasticities, rather than, the more socially relevant elasticities of end-users. By tying debit and credit cards on the merchant side of the market, Visa was able to rebalance its price-structure. That is, Visa could increase the buyer-fee on its credit-card by slightly reducing the buyer-fee on its debit-card. The tie-in meant that a merchant’s acceptance decision binds over both cards, which gave Visa much greater flexibility.

Moreover, providing the entry costs are low, Whinston’s foreclosure argument isn’t relevant. That is, the incumbent association can’t extinguish the threat of entry and hence must remain reasonably competitive.\textsuperscript{33} Alternatively, on-line debit cards are already up and running (e.g. PayPal) and naturally complement auction sites, such as eBay. This provides a niche-market in which on-line debit-cards can survive.

\textsuperscript{33}In their model there are no fixed costs and so the entrant is always a potential threat with or without the tie.

Stanford) was taken to shown that the price-level is neutral with respect to the interchange-fee. In 1986 the court ruled that an interchange-fee determined by the payment-card associations, rather than through individual bargaining between banks, did not constitute price fixing.
Chapter 3

My Contribution

Chapter 3 explains the contribution of this thesis. Section (1) explains the importance of the pass on test to the legality of tying. Section (2) describes how my study formalizes the analysis of Rochet and Tirole. Section (3) describes the outcome without tying and how this differs from the version given by Rochet and Tirole. Section (4) describes the outcome with tying and the effect of sellers’ rivalry on the surplus offered by Network A.

3.1 The Pass-On Test and the Legality of Tying

Rochet-Tirole (2008) has shown that allowing Visa to tie its credit and debit card platforms would raise "social welfare" because it leads to a more efficient price-structure without affecting the price-level. In their analysis "social welfare" is defined as the sum of Visa’s profit and the surplus received by merchants and cardholders\(^1\). That is, the positive effect on Visa’s profit outweighs any negative effect it might have on end-users. Indeed, they claim that even though end-users might not benefit from the tie, they will never be harmed.

However, it can reasonably be argued that their analysis does not fully address the concerns of regulators. In the past authorities in the US took a strong line on tying: it was deemed anti-competitive and \emph{per se} illegal. In recent years there has been a move in the US and elsewhere, to judge tying cases under what has become known as the "rule of reason". This essentially says that the onus is on a firm to justify a tie: it must be socially beneficial and increased efficiency must outweigh any reduction in competition. Hence, an issue arises as to how one should weigh profits and consumer surplus.

Antitrust enforcers focus on consumer welfare and so increased profitability is not legally sufficient to justify a tie that excludes rivals. In short, some of the efficiency gain must be passed on to consumers and this benefit must exceed the loss from reduced competition. For example, the Treaty of the European Communities (TEC), under Article 81(3), permits practices that might otherwise be seen as harmful when they are indispensable to creating economic benefit through improved market efficiency. However, a fair proportion of the extra-surplus must be passed on to consumers; this requirement will be referred to as the "pass on test".

Rochet and Tirole proved that Visa’s attempt to tie debit and credit cards raised "social welfare" but they didn’t demonstrate that it benefits consumers.

\(^1\)In their model the entrant is excluded with or without tying and so can never make a profit.
This suggests that further work is needed to fully address the concerns of regulators. My study is focussed on the question of whether tying platforms that operate in different markets benefits consumers.

My study found that whether or not tying satisfies the pass-on test depends on how the transaction-fee, $\rho$, compares to the average transaction-cost, $\tau$. If $\rho < \tau$, then tying tends to harm consumers; whereas, if $\rho > \tau$, then tying increases the surplus passed on to consumers.

### 3.2 Further Formalization

A principal aim of my thesis was to formalize the analysis given in Rochet-Tirole (2008). The article has a relatively informal style, which makes it highly readable and brings out the economic intuition behind their defence of tying. However, the downside of this is that it blurs the distinction between the model and the analysis. Some of the difficulties I incurred when attempting to replicate the results were as follows:

- The sellers are not provided with membership strategies. It’s clear from the discussion what their membership options are but no symbols (or action sets) are introduced to represent them.

- The networks’ payoff functions are not formally introduced until near the end of the article. This makes it difficult to investigate what happens off the equilibrium path.

- It’s clear that the volume of transactions on a platform is the product of buyers’ demand and the proportion of sellers on the platform. A function was introduced for buyers’ demand but there is no corresponding function for sellers’ demand.

- There is an implicit assumption that Network A’s problem always has an interior solution. Hence, the possibility of corner solutions isn’t explored.

In order to build on their work it’s necessary to unravel the model from the analysis. Firstly, Rochet and Tirole seem to regard the relationship between the effective price, ordinary price and extra-surplus as part of the analysis, whereas, I include this when setting up the model. Secondly, I have formally introduced the sellers’ membership strategies so as to specify their payoff function. Furthermore, sellers make their pricing decision after making their membership decision. Thirdly, "usage functions" were introduced so that an expression for sellers’ demand can be constructed. Finally, Rochet and Tirole worked with buyer-fees rather than seller-fees. The two fees are, essentially, interchangeable but it turns out to be much easier working with seller-fees rather than buyer-fees (particularly, when analyzing a seller’s membership decision).

There are two key assumptions in my model that are absent for that of Rochet and Tirole. Firstly, it is assumed that the degree of product differentiation, $\sigma$, is large relative to transaction costs, $\tau$. This assumption was made so that the presence of platforms can only perturb the outcome in the product market. That is, membership of platforms can effect the sellers’ competitiveness but not to an extent that greatly alters their market shares in equilibrium.
That is, this assumption was made to ensure that the market share of each seller remains roughly one-third and is necessary to justify an approximation that was made while deriving a formula for the volume of transactions on a platform. Furthermore, without this assumption networks might have an incentive to lower their seller-fees, because it would make their sellers more competitive, boosting market share, and raising the volume of transactions on their network.

Secondly, it’s assumed that there is a very small number of large sellers (supermarkets) and that the market is covered. This assumption is needed to ensure that Network A will never choose to accommodate Network B. If there were a very large number of sellers, each with a very small share of the market, then a network can still have a high volume of transactions even if some of the sellers join the other network. In this kind of situation Network A might be prepared to see a small number of sellers leave its network if it allows them to raise the seller-fee on its platforms. However, Network A has no such incentive when there is a small number of sellers.

### 3.3 The Outcome without Tying

My study and that of Rochet and Tirole both investigate the behavior of sellers and networks when tying is prohibited. The main difference between my analysis and theirs is that under certain conditions Network A’s monopoly platform still generates a positive end-user benefit.

Rochet and Tirole assume that Network A can always increase buyers’ demand by lowering the buyer-fee (and increasing the seller-fee). They claim that Network A will continue to reduce the buyer-fee on Platform C_A until they encounter merchants’ resistance, which occurs when the end-user benefit becomes negative.

However, if \( \rho \) is small relative to \( \tau \), then the buyer-fee becomes negative before Network A encounters merchants’ resistance. Once the platform has been made free for buyers, all the buyers prefer to pay by card rather than use cash. (This is because they avoid payment-related transaction costs and the service is free.) Since Network A can’t increase the overall price-level, it has no incentive to raise the seller-fee beyond the point where its service is made free to buyers. Therefore, for sufficiently low values of \( \rho \), the binding constraint is \( f_A^* \leq \rho \) (rather than \( \bar{W}(f_A^*) \geq 0 \)). Hence, if \( \rho < \tau \), then the end-user benefit generated by a monopoly platform is \( \tau - \rho > 0 \).

### 3.4 The Outcome with Tying

My study and that of Rochet and Tirole both investigate the behavior of sellers and networks when tying is permitted. The main difference between my analysis and theirs is that under certain conditions Network A can exclude Network B despite making both its platforms free to buyers. That is, there exist circumstances under which Network A can exclude Network B even when it encourages the excessive use of both its platforms.

I found that there exists a threshold, \( \lambda(\rho) \in (0, \tau) \), such that the nature of the outcome depends on whether or not \( \rho \) exceeds this threshold. Hence, there are two cases to consider: (i) \( \rho \leq \lambda(\rho) \); and (ii) \( \rho > \lambda(\rho) \).
Low Transaction-Fees. If $\rho \leq \lambda(\rho)$, then platforms are very cheap relative to the benefit that comes from avoiding transaction costs. By tying its platforms, Network A threatens to bar sellers from accessing its monopoly platform, namely, Platform $C_A$. Hence, when $\rho < \tau$, Network A has significant "leverage" over sellers. It follows that tying makes it easier for Network A to exclude Network B. Therefore, when $\rho$ is low relative to $\tau$, tying reduces the benefit received by end-users; and more of the surplus is extracted by Network A. This suggests that in some circumstances the leverage argument against tying seems valid in the case of payment-card associations. This possibility is not taken into account in Rochet and Tirole’s study.

High Transaction-Fees. If $\rho > \lambda(\rho)$, then platforms aren’t particularly cheap relative to the benefit that comes from avoiding transaction costs. This means that access to Platform $C_A$ is, generally, desirable but sellers are prepared to forego the platform if Network A encourages an excessive use of payment-cards by setting low buyer-fees (and high seller-fees). It follows that Network A must compete with Network B for sellers.

Furthermore, it was found that tying gives sellers the opportunity to specialize by joining different networks. If sellers were to join different networks, then those on Network A would be more efficient in market $\mathfrak{c}$; whereas, those on Network B would be more efficient in market $\mathfrak{d}$. Hence, asymmetric outcomes enable the sellers to further differentiate themselves (and increase the rents they command). That is, specializing is potentially profitable for the sellers because it allows them to capture part of the surplus generated from reducing transaction costs. (Note that when all the sellers are on the same network they are equally efficient and so competition forces them to pass the end-user benefit on to their customers.)

This suggests that Network A is unable to exclude Network B just by matching its benefit; rather, it must "compensate" sellers for the extra competition they face from being on the same network. In other words, Network A must overcome the inter-seller repulsion created by opportunity to specialize. Therefore, the combined end-user benefit on Network A must exceed the maximum end-user benefit that a single platform can generate. This important mechanism was not discussed by Rochet and Tirole.
Chapter 4

The Framework

Chapter 4 sets up the framework and derives key equations. The chapter contains the following sections. Section (1) outlines the assumptions. Section (2) introduces the players and the markets. Section (3) examines transactions on a platform. Section (4) summarizes a seller’s membership options and explains how it determines which platforms can be used by their customers. Section (5) examines the net-benefit from platforms. Section (6) derives the players’ payoffs and the consumer surplus. Section (7) explains that Network A has a monopoly in market $\mathcal{C}$.

4.1 Assumptions

4.1.1 The Model Developed by Rochet and Tirole

My study follows Rochet and Tirole (2008) by assuming the following:

Buyers are divided between two independent markets: $\mathcal{C}$ and $\mathcal{D}$. In the context of the Visa’s antitrust case consumers divide into those that need store-credit in order to make their purchases (payment on account) and those that have sufficient money in their current-account to make purchases using cash: $\mathcal{C}$ is the market for "credit-goods" and $\mathcal{D}$ is the market for "debit-goods". Finally, each seller operates in both markets; it is convenient to think of a seller running one store in market $\mathcal{D}$ and another in market $\mathcal{C}$.

Buyers incur transaction costs associated with making their payment: firstly, in market $\mathcal{C}$ there is hassle associated with arranging store-credit (e.g. forms and credit checks); secondly, in market $\mathcal{D}$ buyers have to find a cash machine and queue up$^1$.

A credit-card platform serves market $\mathcal{C}$ by offering an alternative to store credit; and a debit-card platform serves market $\mathcal{D}$ by offering an alternative to cash. Following Rochet and Tirole, I make the simplifying assumption that buyers in market $\mathcal{C}$ never switch to market $\mathcal{D}$ and vice versa. This can be motivated as follows: firstly, buyers in the credit market can’t use cash or debit-cards because of liquidity problems; secondly, buyers in the debit market have money in their pockets (or accounts) and don’t want the hassle of remembering to payoff their credit-card.

$^1$There is an risk associated with carrying large amounts of cash.
There are two competing payment-card networks. Network \( A \) offers a credit-card (\( C_A \)) as well as a debit-card (\( D_A \)), whereas Network \( B \) only offers a debit-card (\( D_B \)). In the context of the antitrust case Network \( A \) represents Visa and Network \( B \) represents an entrant. It is assumed that nothing differentiates the rival debit-card platforms. Hence, buyers are only concerned about fees. This creates strong competition on the buyer-side of the market.

Membership of platforms is free to both buyers and sellers but networks charge transaction fees to both sides.\(^2\) Rochet and Tirole assumed that the price-level on a platform is fixed by competition between banks in the payment-card association:

\[
\text{buyer-fee} + \text{seller-fee} = \text{constant}
\]

Hence, the price-level is exogenously determined by factors like the number of banks in the association and the degree to which the banks are able to differentiate themselves. Since the price-level is fixed, an increase in a platform's buyer-fee corresponds to an equal decrease in the seller-fee and vice versa. This is a major simplification but it seems to necessary to make the analysis tractable, moreover, it also brings the role of the price-structure (buyer-fee : seller-fee) into sharp focus.

The honour-all-cards (HAC) rule means that if a seller joins a network, then they have to give buyers are free choice of whether or not to use the platform. Furthermore, when a seller is on both \( D_A \) and \( D_B \) the buyer chooses which (if any) platform to use. Sellers that join the platform are not allowed to surcharge customers that choose to use the platform. This prevents them passing on the seller-fee. This can be motivated as follows. In some countries payment-card associations make the no-surcharge rule a condition of membership. Alternatively, the cost of administering a system of two prices, could out weigh the benefit if the seller-fee was sufficiently small.

Without the tie, buyers and sellers are free to join any combination of platforms. In particular, they can join both debit-cards. With the tie, *sellers can only join \( C_A \) if they also agree to join \( D_A \).* However, buyers face no such restrictions.

Visa attempted to use the HAC rule to force sellers, on their dominant credit-card platform, to allow customers to use Visa’s debit-card. In effect, Visa was using the HAC rule to tie its debit and credit cards on the seller-side of the market. However, if tying is not permitted, then sellers are free to join Visa’s credit-card without being forced to join Visa’s debit-card.

Finally, I follow Rochet and Tirole by assuming that buyers and sellers have a very slight preference for Network \( A \). That is, if they are otherwise indifferent between using \( D_A \) or \( D_B \), then they use \( D_A \). This could be justified by assuming that users have a certain degree of inertia and so only switch to the entrant iff something breaks their indifference.

### 4.1.2 Further Assumptions and Refinements

My study adds to the framework of Rochet and Tirole (2008) as follows:

\(^2\)It is also assumed that there is no cost associated with joining (e.g. the card just arrives in the post and can be easily carried in a pocket).
**Oligopolistic Sellers:** A Hotelling model was added to analyze competition between sellers in the product markets. Specifically, each product market is served by three competing sellers offering horizontally differentiated goods. This is intended to represent the situation where a small number of dominant supermarkets (Tesco, Sainsbury’s and Asda) serve a community that is distributed over a certain geographical area.\(^3\)

The number of sellers in the model is constrained above and below. If there were a very large number of sellers, then Network \(A\) might choose to accommodate Network \(B\) because the extra market share that comes from gaining the final seller would be small relative to the increase in sellers’ demand that comes from decreasing the buyer-fee.\(^4\) On the other hand, my analysis assumes that there is a reasonable degree of competition in the product market.

Due to these constraints I decided to have three sellers. It’s not crucial to have three sellers, because I can generate very similar results with two sellers or four sellers. However, the number of sellers can’t be arbitrarily large, otherwise, it becomes prohibitively costly for Network \(A\) to overcome the inter-seller repulsion.

**Only Buyers incur Transaction Costs:** Sellers do not incur transaction costs but can increase their quality-of-service by joining platforms. That is, a seller is motivated to join a platform if it lowers the average transaction-cost incurred by their customers, which helps them to increase their market share. Finally, this assumption (and the fixed price level) has the effect of ensuring the extra-surplus received by a seller (and their customers) decreases as the buyer-fee increases because the total transfer to the platform is fixed but buyers with ever lower transaction costs are encouraged to use the platform.

**Uniformly Distributed Transaction Costs:** I have assumed that buyers’ transaction costs are uniformly distributed, which results in linear demand for platforms on the buyer-side of the market. This assumption (and the Hotelling model) makes it possible to derive closed-form parametric expressions for the increase in the surplus. Therefore, specifying the distribution of the transaction costs enables more detailed welfare analysis.

### 4.2 Players and Markets

#### 4.2.1 Buyers, Sellers and Markets

There exist two distinct markets (credit-goods and debit-goods) and these markets are indexed by \(k \in \{c,d\}\).

\(^3\)Note that it is necessary not to assume that there isn’t a large number of sellers because in this case the incumbent may be tempted to accommodate the entrant. That is, the increased market share from attracting the last seller may not exceed the increased volume of transaction that comes form a slightly lower buyer-fee (and a slightly higher seller-fee).

\(^4\)Firstly, the sellers are attracted to the network with low seller-fees and high buyer-fees. Secondly, if both networks offer a similar level of extra surplus, then a seller prefers the network with fewer sellers on. (That is, there is a degree of repulsion between the sellers.) Suppose there were 100 sellers and Network \(A\) sets its fees at a level which is sufficient to attract the seller 99 but not seller 100. Network \(A\) could decrease its seller-fees further so as to attract the final seller but they might do better to forego the last seller because to overcome the inter-seller repulsion they would have to substantially decrease their fees.
Buyers are evenly divided between these markets. In each market there is a continuum of buyers and each buyer has inelastic unit demand for their designated good. The buyers in market \( k \in \{c, d\} \) are indexed by \( j_k \in [0, 1] \). Buyers have a constant marginal utility of money, \( \psi = 1 \). Finally, the gross benefit received from consuming one unit of the good is \( \nu \) (money equivalent). However, further units generate no benefit.

Buyers are served by three competing sellers (supermarkets), who are active in both markets. The sellers are indexed by \( i \in \{1, 2, 3\} \) (Sainsbury’s, Tesco, Asda) and the marginal cost of supplying a unit of the good is \( \delta \) (wholesale price).

The buyers live in a community that is based around three streets (of unit length) which intersect to form an equilateral triangle. Suppose that the buyers are uniformly distributed along each street with the sellers at either end. That is, there is on seller at each corner of the triangle. The buyers incur travel-costs when travelling to the sellers to purchase goods. The travel cost parameter is \( \sigma \).

Finally, the cost of manufacturing a unit of the good is small relative to the gross benefit generated from consuming it, \( \delta < \frac{1}{10} \nu \). Travel costs are also small relative to the gross benefit generated by a good, \( \sigma < \frac{1}{10} \nu \). Hence, markets are always covered.

### 4.2.2 Platforms and Networks

When purchases are made by conventional means such as cash (outside options) buyers incur transaction costs. (These are distinct from the travel-costs.) However, sellers can help their customers to avoid transaction costs by joining platforms, which are run by two competing networks. The networks are indexed by \( n \in \{A, B\} \) and the marginal cost of processing transactions on a platform is \( \gamma \). Suppose that membership of the platforms is free but networks charge transaction-fees to both sides: buyer-fees and seller-fees. Finally, it’s assumed that a platform can only operate in one specific market and so a network runs two platforms, one in each market:

- Network \( A \) runs platform \( C_A \) in market \( c \) and platform \( D_A \) in market \( d \).
- Network \( B \) runs platform \( C_B \) in market \( c \) and platform \( D_B \) in market \( d \).

### 4.2.3 Sequence of Events and Information

The sequence of actions is as follows:

1. Networks set transaction fees.
2. Buyers make their membership decision.
3. Sellers make their membership decision.
4. Sellers set prices.
5. Buyers decide which seller to purchase from.
6. Nature allocates transaction costs to buyers and their type is revealed.
7. Buyers decide whether to make a transaction via any of the platforms.

It is assumed that everything that has occurred in earlier stages is common knowledge to all agents. In particular, buyers know which platforms a seller has joined. However, buyers don’t know their transaction cost until they have selected a seller.

4.3 Transactions via a Platform

4.3.1 Transaction Costs

Purchases can be made without using the platforms (e.g. using cash) but in such cases buyers incur transaction costs. The average transaction cost is $\tau$. The buyers’ transaction costs are uniformly distributed on the interval $[0, 2\tau]$, and buyers be indexed in increasing order of their transaction costs. Hence, the transaction cost of buyer $j_k \in [0, 1]$ is

$$T(j_k) \equiv 2\tau j_k$$ (4.3.1)

It is assumed that sellers don’t incur transaction costs, but they are motivated to join networks because it helps them increase their market share by offering a better quality-of-service to their customers.

The level of horizontal differentiation (due to travel costs) is large relative the average transaction cost, $\tau < \frac{1}{10}\sigma$. This implies that the presence of platforms can only perturb the outcome in the product market. In particular, sellers who decide not to join a network are still able to win a positive market share.

Finally, without loss of generality, one of the parameters in the model can be set to unity. The remaining parameters are then measured in units of this quantity. Throughout the analysis it’s convenient to notionally set $\tau = 1$ and measure the other parameters in units of the average transaction cost. (Although, $\tau$ will continue to appear throughout the analysis.)

4.3.2 Transaction Fees

Membership is free to both sides but networks charge buyers and sellers transaction fees. My study follows Rochet and Tirole by assuming that the price-level on a platform is exogenously fixed. Hence,

$$\text{buyer-fee} + \text{seller-fee} = \rho,$$

where $\rho \in (0, 2\tau)$ is the price-level on a platform. That is, an increase in the seller-fee creates an equivalent decreases in the buyer-fee and vice-versa. The price-level, $\rho$, is determined by the degree of inter-network competition between affiliated banks and the extent to which banks are able to differentiate themselves on the buyer-side of the market. Hence, in the short term, $\rho$ is beyond the control of the network.\(^5\) The consequences of this assumption are now explored.

\(^5\)In the long term, the association could reduce $\rho$ (and $\mu$) by expanding the network. That is, by increasing the number of affiliated banks the network can increase inter-bank competition.
A payment-card association such as Visa is composed of affiliated high-street banks. When a buyer pays by card their bank (issuer) debits their account, \( \text{price} + \text{buyer-fee} \), and passes on the payment to the seller’s bank (acquirer) who then credits the seller’s account, \( \text{price} - \text{seller-fee} \). Finally, an interchange-fee is paid by the acquirer to the issuer for passing on the payment. The level of the interchange fee is determined cooperatively by the banks. The interchange-fee set by network \( n \in \{A,B\} \) in market \( k \in \{c,d\} \) is denoted by \( f_n^k \).

A higher interchange-fee raises the cost of serving sellers and lowers the cost of serving buyers. Hence, an increase in the interchange fee is, generally, associated with an increase in the seller-fee and a decrease in the buyer-fee. Indeed, it’s convenient to assume that the seller-fee is entirely determined by the interchange-fee:

\[
\text{seller-fee} = f_n^k \\
\text{buyer-fee} = \rho - f_n^k
\]

Therefore, the price-level (sum of fees) is fixed by inter-bank competition but a network (or rather its affiliated banks) can determine the price-structure (ratio of fees) by choosing an appropriate interchange-fee.

Buyers become members of a platform if its buyer-fee is lower than their transaction cost. It is reasonable to assume that networks never charge a buyer-fee greater than \( 2\tau \) because otherwise no buyers will join; and the platform will be inactive. Also, there is no advantage to be gained by setting a buyer-fee lower than zero because once a platform is made free, any further "reduction" in the buyer-fee has no effect on buyer’s demand. Hence, the buyer-fee on a platform is (weakly) less than the maximum transaction cost, \( 0 \leq \rho - f_n^k \leq 2\tau \), which implies that

\[-(2\tau - \rho) \leq f_n^k \leq \rho \]

The quantity on the LHS in () brackets has intuitive meaning. Since \( 2\tau \) is the maximum possible transaction-cost and \( \rho \) is the price-level, it follows that the maximum extra-surplus received by buyers and sellers is

\[
\omega(\rho) \equiv 2\tau - \rho \tag{4.3.2}
\]

Let \( \omega(\rho) \) be referred to as the maximum surplus per transaction (MSPT). Therefore, the set of possible seller-fees can be expressed as follows:

\[
f_n^k \in [-\omega(\rho), \rho]
\]

This shows that the seller-fee can be negative. A platform is said to "subsidize" sellers if \( f_n^k \leq 0 \) and not "subsidize" sellers if \( f_n^k > 0 \).

It is useful to introduce notation for the vector of fees in each market. The vector of transaction fees in market \( k \) is denoted by

\[
f_k \equiv (f_A^k, f_B^k) \in [-\omega(\rho), \rho]^2
\]

### 4.3.3 Buyers’ Membership Decision

Buyers can mutihome with regard to membership. That is, they can carry more than one card in their pockets.
Buyers prefer to use a platform if its buyer-fee is less than their transaction cost. Since membership is free, all buyers with a transaction cost that exceeds a platform’s buyer-fee will join the platform. Hence, if the buyer-fee is \( \rho - f_n(k) \), then the proportion of buyers that join the platform is

\[
Q(f_n^k) \equiv \frac{2\tau - (\rho - f_n^k)}{2\tau},
\]

which can be re-expressed as

\[
Q(f_n^k) = \frac{\omega(\rho) + f_n^k}{2\tau}
\]  

(4.3.3)

This shows that buyers’ demand is an increasing linear function of the seller-fee (in other words, the interchange fee). Note that \( Q(-\omega) = 0 \) and \( Q(\rho) = 1 \).

Finally, note that the average transaction cost, \( \tau \), can’t be directly observed. However, it can be inferred using observed seller-fees and the volume of transactions. In particular, it will be found that when tying is prohibited, there is an equilibrium in which platforms are made free to sellers in market \( \mathcal{d} \). From this we get

\[
\tau = \frac{\rho}{2[1 - Q(0)]}
\]

Hence, \( \tau \) can be estimated using the volume of transactions on a platform where buyers pay the entire transaction fee.

### 4.4 Membership and Usage of Platforms

#### 4.4.1 Sellers’ Membership Decision

Buyers prefer to use a platform if its transaction cost exceeds its buyer-fee but they can only use the platform if their chosen seller is a member. Membership is free for both sides but networks impose the following conditions on sellers who choose to join: (1) surcharging buyers for using the platform is not permitted; and (2) members must allow customers to use the platform if they wish.

In each market, a seller decides which platform(s) to join. Furthermore, sellers can make different membership decisions in each market. The membership decision of seller \( i \in \{1, 2, 3\} \) in market \( k \in \{c, d\} \) is denoted by

\[
m_i^k \in \mathbb{M} \equiv \{\sigma, a, b, h\},
\]

where the membership options are defined as follows:

- "\( \sigma \)" denotes outside-options (neither network).
- "a" denotes joining Network \( A \).
- "b" denotes joining Network \( B \).
- "h" denotes multi-homing (joining both networks).
In market $k$, the vector of sellers’ membership is

$$m_k = (m_{k1}, m_{k2}, m_{k3}) \in M^3$$

Finally, it is useful to introduce a decision-rule that comes into play when sellers are indifferent between two (or more) options. Firstly, suppose that sellers have slight preference for the option that involves joining the maximum number of platforms. Secondly, suppose that a seller has a slight preference for Network $A$ (incumbent) over Network $B$ (entrant). Hence, the options can be put in increasing order of preference as follows: $a, b, a, h$.

The decision rule can be operationalized as follows. Let the elements of $\{0, 1, 2, 3\}$ be put in one-to-one correspondence with the elements of $M$:

$$a = 0$$
$$b = 1$$
$$a = 2$$
$$h = 3$$

The decision-rule can now be characterized as follows: if a seller is indifferent between the elements of $\bar{M}(\subseteq M)$, then they choose $\max \bar{M}$.

### 4.4.2 Tying Platforms

Visa was accused to using the Honour-All-Cards (HAC) rule to tie its debit-card and credit-card on the seller side of the market. This can be modelled by imposing the restriction that a seller can only join a network if it is a member of both its platforms:

$$m_i^c = m_i^d$$

This greatly reduces their choice of membership options (from 16 to 4).

### 4.4.3 Buyers’ Usage Decision

A buyer prefers to use a platform rather than outside-options if $\rho - f_k^b < T(j_k)$. However, they can only use a given platform if their chosen seller is a member. Furthermore, if their chosen seller is a member of both networks, then the buyer chooses which platform to use. Given the choice, a buyer naturally prefers the network with the lower buyer-fee; and when the buyer-fees are equal, they are assumed to have a slight preference for $A$ over $B$. This makes sense if Network $A$ is seen as an incumbent network and Network $B$ is seen as an entrant because buyers will have got into the habit of using Network $A$ and so inertia means that they continue to do so unless Network $B$ sets lower fees.

A seller is said to be "on" the platform if some of its customers use it. Because sellers can multihome they can join a platform without necessarily being "on" the platform. This is because when given a choice of platforms, buyers will never use the platform with the higher buyer-fee. Hence, it’s useful to introduce the following "usage" functions:

$$A(f_k, m_i^c) \equiv 1(m_i^c = a) + 1(m_i^c = h).1(\rho - f_A^k \leq \rho - f_B^k)$$

(4.4.1)

$$B(f_k, m_i^c) \equiv 1(m_i^c = b) + 1(m_i^c = h).1(\rho - f_A^k > \rho - f_B^k)$$

(4.4.2)
where $1(s)$ is a function that takes the value 1 if $s$ is true and 0 if $s$ is false. The first function takes the value 1 if seller $i$ is on Network $A$, and is 0 otherwise. Similarly, the second function takes the value 1 if seller $i$ is "on" Network $B$, and is 0 otherwise. Note that it is not possible for a seller to be "on" more than one network, in a given market, even though they can be members of both networks.

It is sometimes convenient to re-express these usage functions as tables. The usage function of Network $A$ becomes

\[
A(f_k, m_k^i) = \begin{cases} 
0 & f^k_A \geq f^k_B, \quad f^k_A < f^k_B \\
1 & m_k^i = o \\
1 & m_k^i = a \\
0 & m_k^i = b \\
0 & m_k^i = h
\end{cases}
\]

and the usage function of Network $B$ becomes

\[
B(f_k, m_k^i) = \begin{cases} 
0 & f^k_A \geq f^k_B, \quad f^k_A < f^k_B \\
0 & m_k^i = o \\
0 & m_k^i = a \\
1 & m_k^i = b \\
1 & m_k^i = h
\end{cases}
\]

### 4.5 Net-Benefit from Platforms

#### 4.5.1 Quality of Service

The average reduction in the transaction costs incurred by a seller's customers is referred to as the seller's "quality-of-service". This will depend on which platforms the seller joined and their buyer-fees. There are essentially four situations:

- If the seller is on neither network, then customers incur their transaction cost.
- If the seller joins Network $A$, then customers use the platform when $T(j_k) > \rho - f_A^k$.
- If the seller joins Network $B$, then customers use the platform when $T(j_k) > \rho - f_B^k$.
- If the seller is a member of both networks, then buyers only ever use the platform with the lower buyer-fee

The quality-of-service received by customers of seller $i$ in market $k$ is

\[
U(f_k, m_k^i) = \begin{cases} 
0 & m_k^i = o \\
\hat{U}(f_A^k) & m_k^i = a \\
\hat{U}(f_B^k) & m_k^i = b \\
\hat{U}(\max\{f_A^k, f_B^k\}) & m_k^i = h
\end{cases}
\]  

(4.5.1)
where

\[ \hat{U}(f) = \mathbb{E}[T(j_k) - (\rho - f)|T(j_k) \geq \rho - f]. \Pr(T(j_k) \geq \rho - f) \quad (4.5.2) \]

is the average reduction in transaction costs that comes from using a platform when its buyer-fee is \( \rho - f \).

### 4.5.2 Partial-Demand for a Seller

Buyers have inelastic unit demand for the good and choose which of the sellers to purchase it from. Furthermore, a buyer knows which platforms the sellers joined before they set out on their shopping trip. Hence, when making their purchasing decision, the buyers take into consideration prices and whether or not a given seller offers them the chance to reduce transaction costs by using a platform.

Membership of platforms increases the quality of service received by customers. Hence, it can be seen as a way of reducing the "effective" price, where the "effective" price is defined as the "real" price plus the average cost incurred by customers while making their payment (transaction-cost or buyer-fee). This implies that the "effective" price set by seller \( i \) in market \( k \) is

\[ p_k^i = P(f_k, m_k^i, p_k^i) + \tau - U(f_k, m_k^i), \quad (4.5.3) \]

where \( P(f_k, m_k^i, p_k^i) \) is the "real" price set by seller \( i \) in market \( k \). It is convenient to let the "effective" price be a seller’s decision variable (rather than the "real" price). The vector of prices in market \( k \) is

\[ p_k = (p_k^1, p_k^2, p_k^3) \in \mathbb{R}^3 \]

(Note that I will refer to the effective price as simply the "price".)

Buyers belong to a community that is based around three streets that connect so as to form a triangle. The sellers are located at the corners of this triangle. (One seller at each corner.) Each street contains one-third of the sellers and the streets are of equal length.

A given buyer is located somewhere on one of these streets and must choose between the sellers at either end. That is, there is competition between the two nearest sellers for their custom; and they never visit the remaining seller whatever the prices. When choosing which seller to visit the buyer takes into account their prices and the distance they need to travel from their location on the street. Travel costs are linear and the cost per unit distance is \( \sigma \).

The partial demand of seller \( 1 \) can be found as follows. Seller \( 1 \) attracts buyers from the two streets either side of it (adjacent streets). Their partial-demand can be found by combining their market share in each street and multiplying the result by 1/3. The market shares (m.s) of Seller \( 1 \) in each adjacent street are:

\[ \text{m.s. in } 1^{st} \text{ street } = \frac{\sigma + p_k^2 - p_k^1}{2\sigma} \]

\[ \text{m.s. in } 2^{nd} \text{ street } = \frac{\sigma + p_k^3 - p_k^1}{2\sigma} \]
Hence, the partial demand of Seller 1 becomes
\[
\text{partial-demand} = \frac{2\sigma + p_2^k + p_3^k - 2p_1^k}{6\sigma}
\]
The partial demand of the other sellers can be found by interchanging their labels.

It has been shown that a seller’s demand depends on its own price and the average price set by its rivals. Hence, it’s useful to create (1 x 2) vectors from the prices set by a seller’s rivals:

\[
\begin{align*}
  p_{-1}^k &\equiv (p_2^k, p_3^k) \\
p_{-2}^k &\equiv (p_1^k, p_3^k) \\
p_{-3}^k &\equiv (p_1^k, p_2^k)
\end{align*}
\]

It’s also useful to introduce the following notation for the average price set by a seller’s rivals:

\[
\begin{align*}
  \langle p_{-1}^k \rangle &\equiv \frac{p_2^k + p_3^k}{2} \\
  \langle p_{-2}^k \rangle &\equiv \frac{p_1^k + p_3^k}{2} \\
  \langle p_{-3}^k \rangle &\equiv \frac{p_1^k + p_2^k}{2}
\end{align*}
\]

In terms of this notation, the partial demand for seller \(i\) in market \(k\) becomes

\[
D(p_i^k, \langle p_{-i}^k \rangle) = \frac{\sigma + \langle p_{-i}^k \rangle - p_i^k}{3\sigma}
\]

where "\(-i\)" denotes the rivals of seller \(i\).

### 4.5.3 Membership and a Seller’s Marginal Cost

The marginal cost of supplying a unit of the good is denoted by \(\delta\). However, joining platforms tends to raise a seller’s marginal cost because they have to pay the seller-fee on a fraction of their transactions. There are four situations to consider:

- If the seller is on neither network, then they never have to pay the fee.
- If the seller joins Network A, then customers use the platform when \(T(j_k) \geq \rho - f_A^k\). Hence, the average increase in the seller’s marginal cost is \(f_A^k \Pr(T(j_k) \geq \rho - f_A^k)\).
- If the seller joins Network B, then customers use the platform when \(T(j_k) \geq \rho - f_B^k\). Hence, the average increase in the seller’s marginal cost is \(f_B^k \Pr(T(j_k) \geq \rho - f_B^k)\).
- If the seller is a member of both networks, then buyers only ever use the platform with the lowest buyer-fee. Hence, the average increase in the seller’s marginal cost is \(\max\{f_A^k, f_B^k\} \Pr(T(j_k) \geq \min\{\rho - f_A^k, \rho - f_B^k\})\).
Therefore, the increase in the marginal cost of seller \(i\) in market \(k\) is

\[
S(f_k, m_k^i) = \begin{cases} 
0 & \text{if } m_k^i = 0 \\
\hat{S}(f_A^k) & \text{if } m_k^i = a \\
\hat{S}(f_B^k) & \text{if } m_k^i = b \\
\hat{S}(\max\{f_A^k, f_B^k\}) & \text{if } m_k^i = h
\end{cases}
\] (4.5.6)

where

\[
\hat{S}(f) = f \cdot \Pr(T(j_k) \geq \rho - f),
\] (4.5.7)

is the increase in a seller’s marginal cost due to joining a platform when its buyer-fee is \(\rho - f\).

### 4.5.4 The End-User Benefit

Joining the platforms increases a seller’s average marginal cost but it also increases its quality-of-service by giving customers the chance to reduce their transaction costs. Platforms serve buyers and sellers and collectively are known as "end-users". The net-benefit received by a seller and their customers from a platform will be referred to as the platform’s end-user benefit. This concept (and terminology) was introduced in Rochet and Tirole (2008).

When the buyer-fee is \(\rho - f\), the end-user benefit generated by a platform is

\[
\hat{W}(f) \equiv \mathbb{E}[T(j_k) - \rho | T(j_k) > \rho - f] \cdot \Pr(T(j_k) > \rho - f)
\]

This can be rewritten as

\[
\hat{W}(f) \equiv \mathbb{E}[T(j_k) - (\rho - f) - f | T(j_k) > \rho - f] \cdot \Pr(T(j_k) > \rho - f)
\]

Since the expectations operator is linear, this gives

\[
\hat{W}(f) \equiv \mathbb{E}[T(j_k) - (\rho - f)] \cdot \Pr(T(j_k) > \rho - f) - \mathbb{E}[f | T(j_k) > \rho - f] \cdot \Pr(T(j_k) > \rho - f)
\]

Because \(f\) isn’t stochastic, this gives

\[
\hat{W}(f) \equiv \mathbb{E}[T(j_k) - (\rho - f)] \cdot \Pr(T(j_k) > \rho - f) - f \cdot \Pr(T(j_k) > \rho - f)
\]

The first term is the quality-of-service, \(\hat{U}(f)\). And the second term is the extra-cost incurred by sellers because of the seller-fee, \(\hat{S}(f)\). Therefore, the end-user benefit becomes

\[
\hat{W}(f) = \hat{U}(f) - \hat{S}(f)
\] (4.5.8)

Since the end-user benefit is central to much of the analysis, it’s useful to have a closed-form expression. This expression is derived as follows. If the buyer-fee is \(f - \rho\), then the end-user benefit generated by it is

\[
\hat{W}(f) = \mathbb{E}[T(j_k) - \rho | T(j_k) > \rho - f] \cdot \Pr(T(j_k) > \rho - f)
\]
Since $T(j_k)$ is uniformly distributed over $[0, 2\tau]$, this can be re-expressed as

$$\hat{W}(f) = \frac{1}{2\tau} \int_{0-f}^{2\tau} (t - \rho) \, dt = \frac{1}{4\tau} \left[ t^2 - 2\rho t \right]_{0-f}^{2\tau}$$

Hence, the end-user benefit becomes

$$\hat{W}(f) = \frac{1}{4\tau} \left[ 4\tau^2 - (\rho - f)^2 + 2\rho(\rho - f) - 4\rho \tau \right]$$

It can be seen that $\hat{W}(f)$ is a negative quadratic. Completing the square gives

$$\hat{W}(f) = \frac{1}{4\tau} (2\tau - \rho)^2 - \frac{1}{4\tau} f^2$$

The maximum surplus per transaction (MSPT) was defined as $\omega(\rho) = 2\tau - \rho$. Therefore, the end-user benefit generated by a platform becomes

$$\hat{W}(f) = \frac{1}{4\tau} (\omega(\rho))^2 - f^2 \quad (4.5.9)$$

It is useful to discuss some of the properties of the end-user benefit function. Firstly, it can be seen that the roots of $\hat{W}(f)$ are:

$$f = -\omega(\rho)$$
$$f = \omega(\rho)$$

(Remember that $f \in [-\omega(\rho), \rho]$, which implies that the upper root could lie the domain.) Since $\hat{W}(f)$ is a negative quadratic, it follows that:

$$\hat{W}(f) \geq 0 \iff f \in [-\omega(\rho), \omega(\rho)]$$

Therefore, providing there is an intermediate interchange-fee (between $-\omega(\rho)$ and $\omega(\rho)$), a platform increases the surplus available to buyers and sellers.

Secondly, it can be seen that the maximum occurs at $f = 0$. That is, a platform generates its highest level of end-user benefit when it’s made free to sellers (no interchange-fee). This implies that for all $f \in \mathbb{R}$,

$$\hat{W}(f) \leq \hat{W}(0) = \frac{1}{4\tau} (\omega(\rho))^2$$

Thirdly, $\hat{W}(f)$ is a negative quadratic function of $f$, where $f$ is bounded above and below: $f \in [-\omega(\rho), \rho]$. It follows that, there are minima at $f = -\omega(\rho)$ and $f = \rho$. Since the function is symmetric about its maximum ($f = 0$), the global minimum occurs at the point furthest from the origin. Hence,

$$\min_{f \in [-\omega(\rho), \rho]} \hat{W}(f) = \begin{cases} 
-\omega(\rho) & \text{if } \omega(\rho) \geq \rho \\
\rho & \text{if } \omega(\rho) < \rho
\end{cases}$$

Therefore, for all $f \in [-\omega(\rho), \rho]$,

$$\hat{W}(f) \geq \min \{0, \frac{1}{4\tau} (\omega(\rho))^2 - \rho^2 \}$$
By definition $\omega(\rho) = 2\tau - \rho$, which implies that:

$$\omega(\rho)^2 = 4\tau^2 + \rho^2 - 4\tau\rho$$

Using this identity, the result can be re-expressed as

$$\hat{W}(f) \geq \min\{0, \tau - \rho\}$$

Finally, since the function is quadratic, a given level of benefit can be generated by two different fees:

$$\hat{W}(f) = \hat{W}(-f)$$

In other words, $f$ and $-f$ correspond to the same level of end-user benefit. The first is a fee and the second is a subsidy.

**Lemma 4.1 (End-User Benefit and Seller-Fees)** If the seller-fee on a platform is $f$, then the end-user benefit generated is $\hat{W}(f) = \frac{1}{4\tau}(\omega(\rho)^2 - f^2)$, where $\omega(\rho) = 2\tau - \rho$. This function has the following properties:

1. There are roots at $-\omega(\rho), \omega(\rho)$, which implies that $\hat{W}(f) \geq 0$ iff $f \leq \omega(\rho)$. Furthermore, since $f \geq -\omega(\rho)$, it follows that $\hat{W}(f) < 0$ iff $f > \omega(\rho)$.

2. A platform generates its highest level of end-user benefit when it’s made free to sellers. This implies that for all $f \in \mathbb{R}$,

$$\max_f \left\{ \hat{W}(f) : f \in [-\omega(\rho), \rho] \right\} = \hat{W}(0) = \frac{1}{4\tau}\omega(\rho)^2$$

3. A platform generates its lowest level of end-user benefit (which could even be negative) if it sets a very high seller-fee ($f = \rho$) or very low seller-fee ($f = -\omega(\rho)$). This implies that:

$$\min_f \left\{ \hat{W}(f) : f \in [-\omega(\rho), \rho] \right\} = \min\{0, \tau - \rho\}$$

4. The function is symmetric about its maximum: for all $f \in [-\omega(\rho), \rho]$, $\hat{W}(f) = \hat{W}(-f)$.

As explained the price-level, $\rho$, is exogenously fixed by inter-bank competition. Hence, in the short term $\rho$ is beyond the control of the networks. However, the price-level probably changes more over time than other parameters. In particular, the price-level will tend to vary more than the average transaction cost, $\tau$. Therefore, towards the end of the analysis, it’s useful to allow $\rho$ to vary within the interval $(0, 2\tau)$ and to investigate how this effects end-user benefit:

If $\rho \to 0$, then $\omega(\rho) \to 2\tau$. Hence, if $f \in (-\omega(\rho), \rho)$, then as $\rho \to 0$, $0 \leq \hat{W}(f) < \tau$. Whereas, if $\rho \to 2\tau$, then $\omega(\rho) \to 0$. Hence, if $f \in (-\omega(\rho), \rho)$, then as $\rho \to 2\tau$, $-\tau < \hat{W}(f) \leq 0$. Therefore, for all $\rho \in (0, 2\tau)$, if $f \in (-\omega(\rho), \rho)$, then

$$-\tau < \hat{W}(f) < \tau$$
4.5.5 The Extra-Surplus

It has been shown that the end-user benefit received by a seller and their customers from a platform depends on its seller-fee. If the seller-fee is \( f \), then the end-user benefit is \( W(f) = U(f) - S(f) \). The value of \( f \) depends their membership decision, \( m_i^k \), and the fees set by the networks, \( f_k \). In particular, if they use outside options, then there is no change in the surplus.

The extra-surplus is the change in the surplus available to a seller and their customers due to reduced transaction costs. The extra-surplus available to seller \( i \) and their customers in market \( k \) is

\[
W(f_k, m_i^k) = U(f_k, m_i^k) - S(f_k, m_i^k)
\]

where \( U(f_k, m_i^k) \) is the quality-of-service and \( S(f_k, m_i^k) \) is the increase in the seller’s marginal cost due to the seller-fee. There are four possible cases:

- If the seller isn’t on either platform, \( m_i^k = o \), then it receives no benefit: \( U(f_k, m_i^k) = S(f_k, m_i^k) = 0 \).
- If the seller joins Network A, \( m_i^k = a \), then \( U(f_k, m_i^k) = \hat{U}(f_A^k) \) and \( S(f_k, m_i^k) = \hat{S}(f_A^k) \).
- If the seller joins Network B, \( m_i^k = b \), then \( U(f_k, m_i^k) = \hat{U}(f_B^k) \) and \( S(f_k, m_i^k) = \hat{S}(f_B^k) \).
- If the seller multihomes, \( m_i^k = h \), then buyers use the platform with the lower buyer-fee (and higher seller-fee). Hence, \( U(f_k, m_i^k) = \hat{U}(\max \{ f_A^k, f_B^k \}) \) and \( S(f_k, m_i^k) = \hat{S}(\max \{ f_A^k, f_B^k \}) \).

Since \( W(f) = \hat{U}(f) - \hat{S}(f) \), it follows that the extra-surplus received by seller \( i \) in market \( k \) becomes:

\[
W(f_k, m_i^k) = \begin{cases} 
0 & \text{if } m_i^k = o \\
\hat{W}(f_A^k) & \text{if } m_i^k = a \\
\hat{W}(f_B^k) & \text{if } m_i^k = b \\
\hat{W}(\max \{ f_A^k, f_B^k \}) & \text{if } m_i^k = h 
\end{cases}
\]

4.6 Payoffs and Consumer Surplus

4.6.1 Sellers’ Profit

A seller’s marginal cost of supplying the good in market \( k \) is

\[
\text{marginal-cost} = \delta + S(f_k, m_i^k),
\]

where \( \delta \) is the cost of manufacturing the good (wholesale price). Marginal cost, price and markup are related as follows:

\[
\text{markup} = \text{real price} - \text{marginal-cost},
\]

Hence, the markup of seller \( i \) is

\[
\text{markup} = P(f_k, m_i^k, p_i^k) - \delta - S(f_k, m_i^k),
\]
where $P(f, m^k_i, p^k_i)$ is the "real" price. The markup can be re-expressed in terms of the "effective" price:

\[
\text{markup} = P(f, m^k_i, p^k_i) - \delta - S(f, m^k_i)
\]

\[
= \{ P(f, m^k_i, p^k_i) + \tau - U(f, m^k_i) \} + U(f, m^k_i) - \tau - \delta - S(f, m^k_i)
\]

\[
= p^k_i - \delta - \tau + U(f, m^k_i) - S(f, m^k_i)
\]

This becomes

\[
\text{markup} = p^k_i - \delta - \tau + W(f, m^k_i),
\]

where $W(f, m^k_i)$ is the extra-surplus. By definition,

\[
\text{marginal-cost} = \text{price} - \text{markup}
\]

Therefore, the "effective" marginal cost of seller $i$ in market $k$ becomes

\[
C(f, m^k_i) \equiv \delta + \tau - W(f, m^k_i) \quad (4.6.1)
\]

A seller's profit in a particular market is its markup multiplied by its partial demand. Hence, the profit of seller $i$ in market $k$ is

\[
\Gamma_i(f, m^k_i, p_k) \equiv \{ p^k_i - C(f, m^k_i) \} . D(p^k_i, \langle p_{-i}^k \rangle) \quad (4.6.2)
\]

Since a seller operates in both markets, their payoff becomes

\[
\Pi_i = \sum_{k \in \{1,4\}} \Gamma_i(f, m^k_i, p_k) \quad (4.6.3)
\]

Finally, note that the degree of horizontal product differentiation isn't directly observable. However, it can be inferred from the profit made by retailers. In particular, it will be shown that in equilibrium all sellers make the same membership choice: $m^k_i = \hat{m}_k$. It follows that they have the same marginal costs; which means that the sellers to set identical prices and each receive one-third of the market. Hence, in equilibrium, $\sigma = 3\Gamma_i$. Therefore, $\sigma$ can be estimated from the profit made by retailers.

### 4.6.2 Networks' Profits

Buyers can only use a particular platform if the seller they visit is a member. Hence, the number of transactions that occur on a given platform via a particular seller is the seller’s market share multiplied by the platform’s usage function (see the section on a buyer’s usage decision). The proportion of total sales made by sellers who are "on" the platform will be referred to as sellers’ demand for the platform. Hence, in market $k$, sellers’ demand for each network
Sellers’ demand for \( A \) is\(^6\)
\[
\frac{1}{3} \sum_{i \in \{1,2,3\}} A(f_k, m^k_i)
\]
Sellers’ demand for \( B \) is\(^6\)
\[
\frac{1}{3} \sum_{i \in \{1,2,3\}} B(f_k, m^k_i)
\]
The volume of transactions on a platform is the proportion of buyers on the platform (buyers’ demand) multiplied by sellers’ demand:
\[
\text{volume on } A = \frac{1}{3} Q(f^k_A) \sum_{i \in \{1,2,3\}} A(f_k, m^k_i)
\]
\[
\text{volume on } B = \frac{1}{3} Q(f^k_B) \sum_{i \in \{1,2,3\}} B(f_k, m^k_i)
\]
And the markup on a platform is
\[
\mu = \rho - \gamma, \quad (4.6.4)
\]
where \( \gamma \) is the marginal cost of processing transactions. The markup on a platform is fixed because the price-level, \( \rho \), on a network is exogenously determined by inter-bank competition.

A network’s profit in market \( k \) is the markup, \( \mu \), multiplied by the volume of transactions. Hence, in market \( k \), the the profit of Network \( A \) is
\[
\Pi_A(f_k, m_k) \equiv \frac{1}{3} \mu.Q(f^k_A) \sum_i A(f_k, m^k_i) \quad (4.6.5)
\]
\(^6\)This is an approximation because the number of transactions will depend on the number of sellers that join and the market share of those sellers. Hence, in market \( k \), sellers’ demand for Network \( A \) becomes \( \sum_i D(p^k_i, \langle p^k_i \rangle)A(f_k, m^k_i) \) and sellers’ demand for Network \( B \) becomes \( \sum_i D(p^k_i, \langle p^k_i \rangle)B(f_k, m^k_i) \). However, providing \( \tau \) is small relative to \( \sigma \) the existence of platforms will only perturb the outcome. The argument is as follows.

Sellers set price once fees and membership have been determined. It can be shown that given fees, \( f_k = (f^k_n)_n \), and membership, \( m_k = (m^k_i)_i \), a seller’s market share becomes
\[
\tilde{D}_i(f_k, m_k) = \frac{1}{15\sigma} \left( 5\sigma + 2W(f_k, m^k_i) - \sum_{j \neq i} W(f_k, m^j_i) \right)
\]
It can be shown that \(-\tau \leq W(f_k, m^k_i) \leq \tau\), which implies that
\[
\frac{1}{15\sigma} \left( 5\sigma - 4\tau \right) \leq \tilde{D}_i(f_k, m_k) \leq \frac{1}{15\sigma} \left( 5\sigma + 4\tau \right)
\]
This can be re-expressed as
\[
\frac{1}{3} - \frac{4}{15} \left( \frac{\tau}{\sigma} \right) \leq \tilde{D}_i(f_k, m_k) \leq \frac{1}{3} + \frac{4}{15} \left( \frac{\tau}{\sigma} \right)
\]
Finally, it can be seen that as \( \tau/\sigma \to 0 \), \( \tilde{D}_i(f_k, m_k) \to 1/3 \), regardless of \( f_k \) and \( m_k \).

Therefore, the approximation is reasonable providing \( \tau \) is small relative to \( \sigma \). Essentially, my approximation ignores the possibility that by choosing a fee structure that increases the competitiveness of affiliated sellers a network could increase the volume of transactions on its platforms.
and the profit of Network $B$ is

$$\Upsilon_B(f_k, m_k) \equiv \frac{1}{3} \mu Q(f_B^k) \sum_i B(f_k, m_k^i)$$

(4.6.6)

Since platforms are active in both markets, the payoff of network $n \in \{A, B\}$ is

$$\Pi_n = \sum_{k \in \{c, d\}} \Upsilon_n(f_k, m_k)$$

(4.6.7)

### 4.6.3 Consumer Surplus and Social Welfare

Consumers have a constant marginal utility of money, $\psi = 1$. The gross benefit from consuming one unit of a good is $\nu$ (money value) so the average net-benefit received by customers of seller $i$ in market $k$ is$^7$

$$\text{average net-benefit} = \frac{\nu}{\text{gross benefit}} - \frac{p_k^i}{\text{transaction}} - \frac{1}{2} \sigma D(p_k^i, \langle p_{-i}^k \rangle)$$

Hence, the net-benefit received by customers of seller $i$ in market $k$ is the product of the average net-benefit and the seller’s partial demand:

$$\Theta(p_k^i, \langle p_{-i}^k \rangle) \equiv \left\{ \nu - p_k^i - \frac{1}{2} \sigma D(p_k^i, \langle p_{-i}^k \rangle) \right\} \cdot D(p_k^i, \langle p_{-i}^k \rangle)$$

(4.6.8)

The total surplus received by buyers, in a given market, is found by adding the net-benefit received from each of the sellers. Finally, the total consumer surplus is found by summing across markets. Therefore, the consumer surplus is

$$\Phi(P) \equiv \sum_{k \in \{c, d\}} \sum_{i \in \{1, 2, 3\}} \Theta(p_k^i, \langle p_{-i}^k \rangle)$$

(4.6.9)

Finally, social welfare is taken to be the sum of profits and the consumer surplus:

$$\Psi(F, M, P) \equiv \sum_i \Pi_i(F, (m_k^i)_k, P) + \sum_n \Pi_n(F, M) + \Phi(P)$$

(4.6.10)

### 4.7 Switching Off Platform $C_B$

So far we have discussed the case where the networks are symmetric; that is, each network is active in both markets. However, following Rochet and Tirole, I wish to investigate the situation where an incumbent network (Visa) has a monopoly in one market and competes against a potential entrant in the other market. In particular, suppose that Network $A$ has a monopoly in market $c$ (credit-cards) but competes against Network $B$ in market $d$ (debit-cards). That is, suppose that Platform $C_B$ has been "switched off". It can be show that a platform can be "switched off" be setting the maximum possible buyer-fee (and minimum possible seller-fee). The argument is as follows:

$^7$This expression takes account of transaction costs because it’s in terms of "effective price" rather than "real price".
If $f_B^c = -\omega(\rho)$, then the buyer-fee becomes $\rho - f_B^c = 2\tau$. Since the maximum transaction cost is $2\tau$, no buyers will use a platform with this price-structure. Hence, if $f_B^k = -\omega(\rho)$, then $Q(f_B^k) = 0$. Since buyers’ demand goes to zero, Platform $C_A$ offers no end-user benefit and makes no profit. Therefore, if $f_B^k = -\omega(\rho)$, then Platform $C_B$ is switched off. From this point on, this constraint is imposed on the fees of Network $B$.

It is convenient to introduce the following notation. The fees set by Network $A$ are $f_A^c = x$, $f_A^d = y$, and the fees set by Network $B$ are $f_B^c = -\omega$, $f_B^d = z$. Hence, fees in market $c$ are $f_c = (x, -\omega(\rho))$ and fees in market $d$ are $f_d = (y, z)$. The matrix of fees becomes

$$F \equiv (f_c^T, f_d^T) = \begin{pmatrix} x & y \\ -\omega & z \end{pmatrix}$$

Finally, the set of possible fees is

$$\mathbb{F} \equiv \{(f_c^T, f_d^T) : f_c = (x, -\omega(\rho)), f_d = (y, z)\} \subseteq [-\omega(\rho), 2\tau]^4$$
Part II

Networks are prohibited from Tying the Membership of Platforms
Outline of Part II

The Game

Chapter 5 sets up the game played by sellers and networks when tying is prohibited.

Stage 3: Sellers’ Prices

Chapter 6 analyzes the prices set by sellers in the final stage of the game.

Stage 2: Sellers’ Membership Decisions

Chapter 7 analyzes a seller’s optimal choice of membership in each market.

Stage 1: Networks’ Seller-Fees

Chapter 8 characterizes Network A’s best response.

Chapter 9 determines the seller-fees that will be set by networks and equilibrium outcomes.
The Game
Chapter 5

*Game I: Tying is prohibited*

Chapter 5 sets up the game played by sellers and networks when tying is prohibited. Section (1) introduces the payers. Section (2) summarizes the parameters. Section (3) describes the timing and actions available to the players. Section (4) summarizes the concept of end-user benefit. Section (5) defines the payoffs. Section (6) describes the decision rule, which applies when a seller is indifferent between two or more options.

5.1 Players

This part of the study investigates what happens when networks are prohibited from tying a seller’s membership. The framework can be used to construct a model, *Game I*, in which sellers are not restricted in their choice of membership. (That is, a seller is free to choose their membership options so that $m_i^c \neq m_i^d$.) The players in the game are:

- Network $A$ ($n = A$)
- Network $B$ ($n = B$)
- Seller 1 ($i = 1$)
- Seller 2 ($i = 2$)
- Seller 3 ($i = 3$)

5.2 Parameters

The parameters of the model can be summarized as follows:

- The gross benefit (money value) from consuming one unit of the good is $\nu$. This is assumed to be at least an order of magnitude larger than the other parameters.
- The wholesale price of the good is $\delta$.
- The travel cost parameter is $\sigma$.
- Payment related transaction costs are uniformly distributed over $[0, 2\tau]$, where the average transaction cost is $\tau$, where $\tau < \frac{1}{10} \sigma$.  

68
• The marginal cost of processing a transaction on a platform (payment-card) is $\gamma$.

• The price-level on a network’s platforms is $\rho \in (0, 2\tau)$. The level is exogenously determined by the degree of competition between affiliated banks.

From these parameters the following quantities can be derived:

• A platform’s markup is $\mu = \rho - \gamma$.

• The maximum surplus-per-transaction (MSPT) received by end-users (buyers and sellers) is $\omega = 2\tau - \rho$.

### 5.3 Sequence of Actions

The game is sequential with players choosing their actions as follows:

**Stage 1.** Networks set seller-fees on their platforms: Networks $A$ is active in both markets, $\varsigma$ and $\varrho$. Whereas, Network $B$ is only active in market $\varrho$ (because Platform $C_B$ is switched off). The seller-fees set by Network $A$ in markets $\varsigma$, $\varrho$ are $x, y \in [-\omega, \rho]$, respectively. The seller-fee set by Network $B$ in market $\varrho$ is $z \in [-\omega, \rho]$. The fee vectors in markets $\varsigma$, $\varrho$ are $f_\varsigma = (x, -\omega)$, $f_\varrho = (y, z)$.

**Stage 2.** The set of membership options is $M = \{o, b, a, h\}$, where the elements are as follows: $o$ denotes outside-options (cash); $b$ denotes membership of Network $B$; $a$ denotes membership of Network $A$; and $h$ denotes multi-homing (multiple membership). Seller $i \in \{1, 2, 3\}$ decides which platforms (if any) to join in each market: $m^i_\varsigma \in M$ and $m^i_\varrho \in M$. The membership vectors in markets $\varsigma$, $\varrho$ are $m_\varsigma = (m^i_1, m^i_2, m^i_3)$, $m_\varrho = (m^i_1, m^i_2, m^i_3)$.

**Stage 3.** Seller $i \in \{1, 2, 3\}$ chooses effective prices in each market: $p^i_\varsigma \in \mathbb{R}$ and $p^i_\varrho \in \mathbb{R}$. The price vectors in markets $\varsigma$, $\varrho$ are $p_\varsigma = (p^i_1, p^i_2, p^i_3)$, $p_\varrho = (p^i_1, p^i_2, p^i_3)$.

It is assumed that everything that has occurred in earlier stages is common knowledge.

### 5.4 Extra-Surplus

The end-user benefit generated by a platform with seller-fee $f$ is

$$\hat{W}(f) = \frac{1}{4}(\omega^2 - f^2)$$

This is the increased surplus available to buyers and sellers due to a decrease in transaction costs.

The extra-surplus is the change in the surplus available to a seller and their customers due to reduced transaction costs. This depends on seller-fees and
which platforms a seller joined. The extra-surplus available to seller $i$ and their customers in market $\varsigma$ is$^1$

$$W(\mathbf{f}_\varsigma, m^\varsigma_i) = \begin{cases} 
0 & \text{if } m^\varsigma_i \in \{a, b\} \\
\hat{W}(x) & \text{if } m^\varsigma_i \in \{a, h\} 
\end{cases} \quad (5.4.1)$$

The extra-surplus available to seller $i$ and their customers in market $\theta$ is

$$W(\mathbf{f}_\theta, m^\theta_i) = \begin{cases} 
0 & \text{if } m^\theta_i = 0 \\
\hat{W}(y) & \text{if } m^\theta_i = a \\
\hat{W}(z) & \text{if } m^\theta_i = b \\
\hat{W}(\max\{y, z\}) & \text{if } m^\theta_i = h 
\end{cases} \quad (5.4.2)$$

$5.5$ Payoffs

The payoffs are received by the players at the end of the game according to the actions chosen. The payoff functions are as follows:

**Network A.** The profits made by Network A in markets $\varsigma, \theta$ are

$$\Upsilon_A(\mathbf{f}_\varsigma, \mathbf{m}_\varsigma) = \frac{1}{3} \mu.Q(x) \sum_{i \in \{1,2,3\}} A(\mathbf{f}_\varsigma, \mathbf{m}^\varsigma_i)$$

$$\Upsilon_A(\mathbf{f}_\theta, \mathbf{m}_\theta) = \frac{1}{3} \mu.Q(y) \sum_{i \in \{1,2,3\}} A(\mathbf{f}_\theta, \mathbf{m}^\theta_i)$$

where

$$Q(x) \equiv \frac{1}{2\pi} (\omega + x)$$

$$Q(y) \equiv \frac{1}{2\pi} (\omega + y)$$

and$^2$

$$A(\mathbf{f}_\varsigma, \mathbf{m}^\varsigma_i) = \mathbf{1}(m^\varsigma_i \in \{a, h\}) \quad (5.5.1)$$

$$A(\mathbf{f}_\theta, \mathbf{m}^\theta_i) \equiv \mathbf{1}(m^\theta_i = a) + \mathbf{1}(m^\theta_i = h).\mathbf{1}(z \leq y) \quad (5.5.2)$$

A network’s payoff corresponds to its total profit. Since Network A is active in both markets, their payoff is

$$\Pi_A(\mathbf{F}, \mathbf{M}) = \sum_{k \in \{\varsigma, \theta\}} \Upsilon_A(\mathbf{f}_k, \mathbf{m}_k),$$

$^1$Since $\mathbf{f}_\varsigma = (x, -\omega)$ and $x \geq -\omega$, Network A always has the higher seller-fee in market $\varsigma$. It follows that if a seller, multihomes, then they’re on Network A. That is, $W(\mathbf{f}_\varsigma, h) = \hat{W}(x)$.

Since $\mathbf{f}_\varsigma = (x, -\omega)$ and $\hat{W}(-\omega) = 0$, it follows that if a seller joins Network B in market $\theta$, then they receive no extra-surplus in market $\varsigma$. That is, $W(\mathbf{f}_\theta, b) = \hat{W}(-\omega) = 0$.

$^2$Since $\mathbf{f}_\varsigma = (x, -\omega)$ and $x \geq -\omega$, Network A always has the higher seller-fee. It follows that if a seller multihomes, $m^\varsigma_i = h$, then they’re on Network A in market $\varsigma$. Hence, the expression for $A(\mathbf{f}_\varsigma, \mathbf{m}^\varsigma_i)$ can be simplified appropriately.
where the fee matrix is
\[
F \equiv (f_k^T)_{k \in \{c,d\}} = \begin{pmatrix} x & y \\ -\omega & z \end{pmatrix}
\]
and the membership matrix is
\[
M \equiv (m_k^T)_{k \in \{c,d\}} = \begin{pmatrix} m_1^c & m_1^d \\ m_2^c & m_2^d \\ m_3^c & m_3^d \end{pmatrix}
\]

**Network B.** Network B is inactive in market \(c\) (because Platform C is switched off). Hence, Network B makes no profit in market \(c\). However, Network B is (potentially) active in market \(d\). Their profit in market \(d\) is
\[
\Upsilon_B(f_d, m_d) = \frac{1}{3} \mu Q(z) \sum_{i \in \{1,2,3\}} B(f_d, m_d^i),
\]
where
\[
Q(z) = \frac{1}{2\pi} (\omega + z)
\]
and
\[
B(f_d, m_d^i) = 1(m_d^i = b) + 1(m_d^i = h).1(z > y)
\]
Network B is only active in market \(d\). Hence, the payoff of Network B is
\[
\Pi_B(f_d, m_d) = \Upsilon_B(f_d, m_d)
\]

**Sellers.** Seller \(i\) competes against two sellers, whose prices in market \(k\) are denoted by \(p_{-i}^k\). The average price set by the seller’s competitors is
\[
\langle p_{-i}^k \rangle = \frac{\sum_{j \neq i} p_j^k}{2}
\]
The profit made by seller \(i \in \{1,2,3\}\) in market \(k \in \{c,d\}\) is
\[
\Gamma_i(f_k, m_i^k, p_k) \equiv (p_i^k - C(f_k, m_i^k)) \cdot D(p_i^k, \langle p_{-i}^k \rangle),
\]
where their partial-demand is
\[
D(p_i^k, \langle p_{-i}^k \rangle) \equiv \frac{\sigma + \langle p_{-i}^k \rangle - p_i^k}{3\sigma}
\]
and their marginal-cost is
\[
C(f_k, m_i^k) \equiv \delta + \tau - W(f_k, m_i^k)
\]
A seller’s payoff corresponds to its total profit. Since they are active in both markets, their payoff is
\[
\Pi_i(F, (m_i^k)_{k \in \{c,d\}}, P) \equiv \sum_{k \in \{c,d\}} \Gamma_i(f_k, m_i^k, p_k),
\]
where the price matrix is

\[
P \equiv (p^T_k)_{k \in \{c,d\}} = \begin{pmatrix} p^c_1 & p^d_1 \\ p^c_2 & p^d_2 \\ p^c_3 & p^d_3 \end{pmatrix}
\]

The Sub-Game Perfect Nash equilibrium (SPNE) of this game can be found using the technique of backward induction. Stage two has a range of multiple equilibria. The following decision rule is used to select particular types of SPNE at stage 2; these are referred to as Restricted SPNE (RSPNE).

### 5.6 Decision Rule

It is assumed that all things being equal, sellers prefer to join the maximum number of platforms because a small number of buyers may single-home. This can be represented by the following decision rule: The membership options, in a given market, can be arranged in the following order: \( o, b, a, h \). When indifferent between two options a seller chooses the one that is furthest to the right in the above list.

This can be operationalized as follows: Firstly, put the elements of \( M \) in one-to-one correspondence with those of \( \{0, 1, 2, 3\} \): \( o \equiv 0; b \equiv 1; a \equiv 2; h \equiv 3 \). Secondly, let \( \hat{M}(F) \) denote a seller’s set of optimal membership options given fees, \( F \). The seller’s choice becomes \( \max \hat{M}(F) \).
Stage 3: Sellers’ Prices
Chapter 6
Sellers Set Prices

Chapter 6 analyzes the prices set by sellers in the final stage of the game. Section (1) derives a first-order condition. Section (2) finds the subgame-perfect Nash equilibrium (SPNE). Section (3) finds an expression a seller’s payoff given the fees set by the platforms and the sellers’ membership decisions during stage 2.

6.1 Payoffs and Best Response

This section analyzes the prices set by sellers in a sub-game in which the payoff of seller $i$ is

$$\Pi_i = \sum_{k \in \{1, \ldots \}} \Gamma_i(f_k, m^k_i, p_k)$$

and where fees, $f_k$, and membership, $m_k$, are fixed.

A seller is able to choose different prices in each market, which implies that sellers maximize profits in each market separately. The profit of seller $i$ in market $k$ is defined as

$$\Gamma_i(f_k, m^k_i, p_k) \equiv (p^k_i - C(f_k, m^k_i)) \cdot D(p^k_i, \langle p^{k-1}_i \rangle),$$

where their partial-demand is

$$D(p^k_i, \langle p^{k-1}_i \rangle) = \frac{\sigma + \langle p^{k-1}_i \rangle - p^k_i}{3\sigma}$$

Fees, $f_k$, and membership, $m_k$, have been determined, which implies that a seller’s marginal cost, $C(f_k, m^k_i)$, is fixed.

It can be seen that:

$$\frac{\partial \Gamma_i}{\partial p^k_i} = D(p^k_i, \langle p^{k-1}_i \rangle) - \frac{1}{3\sigma} (p^k_i - C(f_k, m^k_i))$$

and

$$\frac{\partial^2 \Gamma_i}{\partial (p^k_i)^2} = -\frac{2}{3\sigma}$$

Since the profit function is concave in $p^k_i$, it follows that its optimum is characterized by the following FOC:

$$C(f_k, m^k_i) + 3\sigma D(p^k_i, \langle p^{k-1}_i \rangle) - p^k_i = 0$$
This can be re-expressed as

\[ C(f_k, m_i^k) + \sigma + \langle p_{-i}^k \rangle - 2p_i^k = 0 \]

By definition, the average price is

\[ \langle p_k \rangle = \frac{\sum_j p_j^k}{3} \] (6.1.1)

and the average price set by the rivals of seller \( i \) is

\[ \langle p_{-i}^k \rangle = \frac{\sum_{j \neq i} p_j^k}{2} \]

From these definitions, it can be seen that

\[ \langle p_{-i}^k \rangle = \frac{1}{2} \left( 3 \langle p_k \rangle - p_i^k \right) \]

Hence, the FOC becomes

\[ 2C(f_k, m_i^k) + 2\sigma + 3 \langle p_k \rangle - 5p_i^k = 0 \] (6.1.2)

This FOC gives the "best response" to the average price.

### 6.2 The SPNE

The fees chosen by the sellers must satisfy the following FOCs:

\[
\begin{align*}
2C(f_k, m_i^k) + 2\sigma + 3 \langle p_k \rangle - 5p_i^k &= 0 \\
2C(f_k, m_2^k) + 2\sigma + 3 \langle p_k \rangle - 5p_2^k &= 0 \\
2C(f_k, m_3^k) + 2\sigma + 3 \langle p_k \rangle - 5p_3^k &= 0
\end{align*}
\]

The set of SPNE’s in market \( k \) is denoted by \( N_k(f_k, m_k)(\subseteq \mathbb{R}^3) \). Hence, \( \tilde{p}_k \in N_k(f_k, m_k) \) iff \( \tilde{p}_k \) satisfies the three simultaneous equations. Therefore, the equilibrium condition is

\[ 2 \sum_i C(f_k, m_i^k) + 6\sigma + 9 \langle p_k \rangle - 5 \sum_i \tilde{p}_i^k = 0 \]

Since \( 3 \langle \tilde{p}_k \rangle = \sum_j \tilde{p}_j^k \), this becomes

\[ \sum_i C(f_k, m_i^k) + 3\sigma - 3 \langle \tilde{p}_k \rangle = 0 \]

Hence, if \( \tilde{p}_k \in N(f_k, m_k) \), then

\[ \langle \tilde{p}_k \rangle = \sigma + \frac{1}{3} \sum_j C(f_k, m_j^k) \] (6.2.1)

Substituting this result into a seller’s best-response function and re-arranging gives:

\[ 2C(f_k, m_i^k) + 5\sigma + \sum_j C(f_k, m_j^k) - 5\tilde{p}_i^k = 0 \]
It can be seen that the best-response depends on a seller’s own membership and the membership of rival sellers. Seller \( i \) has two competitors whose membership is denoted by \( m_{-i}^k \). Hence, it’s useful to introduce the following vectors:

\[
\begin{align*}
    m_{-1}^k &= (m_{k1}^i, m_{k2}^i) \\
    m_{-2}^k &= (m_{k1}^i, m_{k3}^i) \\
    m_{-3}^k &= (m_{k1}^i, m_{k2}^i)
\end{align*}
\]

It follows that:

**Proposition 6.1 (Price-Setting)** Given fees, \( f_k \), and membership, \( m_{-i}^k, m_i^k \), there is a unique SPNE in which prices in market \( k \) are as follows:

\[
\begin{align*}
    e_p^k i \in N_k(f_k, m_i^k),
    e_p^k = e_p^k(f_k, m_{-i}^k, m_i^k)
\end{align*}
\]

**6.3 Demands, Markups and Payoffs**

Since \( \langle \tilde{p}_{-i}^k \rangle \equiv \frac{3}{2} \langle \tilde{p}_i^k \rangle - \tilde{p}_i^k \), it follows that:

\[
\langle \tilde{p}_{-i}^k \rangle = \frac{1}{10} \left\{ 15 \langle \tilde{p}_i^k \rangle - 5\sigma - 2C(f_k, m_i^k) - \sum_j C(f_k, m_j^k) \right\}
\]

Substituting for \( \langle \tilde{p}_k \rangle \) using the equilibrium condition gives

\[
\langle \tilde{p}_{-i}^k \rangle = \frac{1}{5} \left\{ 5\sigma + 2 \sum_j C(f_k, m_j^k) - C(f_k, m_i^k) \right\}
\]

Hence, the price-differential is

\[
\langle \tilde{p}_{-i}^k \rangle - \tilde{p}_i^k = \frac{1}{5} \left\{ \sum_j C(f_k, m_j^k) - 3C(f_k, m_i^k) \right\}
\]

A seller’s partial-demand was defined as

\[
D(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle) = \frac{\sigma + \langle \tilde{p}_{-i}^k \rangle - \tilde{p}_i^k}{3\sigma}
\]

Therefore, in equilibrium, the partial-demand of seller \( i \) becomes

\[
D(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle) = \frac{1}{15\sigma} \left\{ 5\sigma + \sum_j C(f_k, m_j^k) - 3C(f_k, m_i^k) \right\}
\]

Finally, the markup of seller \( i \) is

\[
\tilde{p}_i^k - C(f_k, m_i^k) = \frac{1}{5} \left\{ 5\sigma + \sum_j C(f_k, m_j^k) - 3C(f_k, m_i^k) \right\}
\]

A seller’s profit is its markup multiplied by its partial demand. This implies that the profit of seller \( i \) becomes

\[
\Gamma_i(f_k, m_i^k, \tilde{p}_k) = \frac{1}{75\sigma} \left\{ 5\sigma + \sum_j C(f_k, m_j^k) - 3C(f_k, m_i^k) \right\}^2
\]

76
It can be seen that if a seller is more efficient than their rivals, then they have the higher market share and a higher markup. The marginal cost of seller $i$ was defined as

$$C(f_k, m^k_i) \equiv \delta + \tau - W(f_k, m^k_i),$$

where $W(f_k, m^k_i)$ is the extra-surplus received from platforms. Hence, a seller’s efficiency depends on the extra-surplus received from platforms. It can be seen that the most efficient seller is the one that receives the higher extra-surplus from platforms. The result can be summarized follows:

**Proposition 6.2 (Seller’s Profits)** Given fees, $f_k$, and membership, $m^k_{-i}, m^k_i$, the payoff received by seller $i$ becomes:

$$\Pi_i = \Pi_i(F, M) \equiv \sum_{k \in \{\varepsilon, d\}} \Gamma_i(f_k, m^k_{-i}, m^k_i),$$

where the profit of seller $i$ in market $k$ is

$$\Gamma_i(f_k, m^k_{-i}, m^k_i) \equiv \frac{1}{\bar{5}\sigma} \left\{5\sigma + 3W(f_k, m^k_i) - \sum_j W(f_k, m^k_j)\right\}^2$$

### 6.4 Properties of the Profit Function

A seller’s payoff depends on its extra-surplus, $W(f_k, m^k_i)$. It can be seen that

$$W(f_k, m^k_i) \leq \max \left\{\hat{W}(f) : f \in [-\omega, \rho]\right\} = \frac{\omega^2}{4\tau}$$

$$W(f_k, m^k_i) \geq \min \left\{\hat{W}(f) : f \in [-\omega, \rho]\right\} = \min\{0, \tau - \rho\}$$

Hence, the extra-surplus is bounded as follows:

$$\min\{0, \tau - \rho\} \leq W(f_k, m^k_i) \leq \frac{\omega^2}{4\tau}$$

With fees, $f_k$, and membership, $m^k_{-i}, m^k_i$, a seller’s payoff becomes

$$\Gamma_i(f_k, m^k_{-i}, m^k_i) \equiv \frac{1}{\bar{5}\sigma} \left\{5\sigma + 2W(f_k, m^k_i) - \sum_{j \neq i} W(f_k, m^k_j)\right\}^2,$$

where $\sum_{j \neq i} W(f_k, m^k_j)$ is the aggregate surplus received by a seller’s competitors. It can be seen that

$$2W(f_k, m^k_i) - \sum_{j \neq i} W(f_k, m^k_j) \geq \min\{0, 2(\tau - \rho)\} - \frac{\omega^2}{2\tau}$$

Furthermore, if $\rho \in (0, 2\tau)$, then

$$\min\{0, 2(\tau - \rho)\} - \frac{\omega^2}{2\tau} \geq -2\tau$$

(see proof). Finally, since $\sigma > 2\tau$, this implies that

$$5\sigma + 2W(f_k, m^k_i) - \sum_{j \neq i} W(f_k, m^k_j) > 0$$
Therefore, a seller’s payoff is a strictly increasing function of the extra-surplus received from platforms. This implies that sellers will select their membership so as to maximize the extra-surplus they (and their customers) receive.

Lemma 6.1 \textit{(Profit Increases as the Surplus Increases.)} If $W(f_k, \hat{m}) > W(f_k, \hat{m})$, then

$$\tilde{\Gamma}_i(f_k, m^{-i}_k, \hat{m}) > \tilde{\Gamma}_i(f_k, m^{-i}_k, \hat{m})$$

\textbf{Proof.} It can be seen that:

$$\min\{0, 2(\tau - \rho)\} - \frac{1}{2r} \omega^2 = \frac{1}{2r} \min\{-\omega^2, 2\tau \omega - 2\tau \rho - \omega^2\}$$

$$= -\frac{1}{2r} \max\{\omega^2, \omega^2 + 2\tau \rho - 2\tau \omega\} \text{ [definition of } \omega]\]$$

$$= -\frac{1}{2r} \left[\omega^2 + \max\{0, 2\tau(\rho - \omega)\}\right]$$

By substituting for $\omega$, we get

$$\omega^2 + \max\{0, 2\tau(\rho - \omega)\} = (2\tau - \rho)^2 + \max\{0, 4\tau(\rho - \tau)\},$$

where the RHS is a function of $\rho \in [0, 2\tau]$.

The RHS composed of two functions with the following properties: Firstly, $(2\tau - \rho)^2$ is a positive quadratic function of $\rho$ with a unique root (minimum) at $2\tau$. Secondly,

$$\max\{0, 4\tau(\rho - \tau)\} = \begin{cases} 4\tau(\rho - \tau) & \text{if } \rho \geq \tau \\ 0 & \text{if } \rho < \tau \end{cases}$$

which implies that

$$\max\left\{\max_{\rho}\{0, 4\tau(\rho - \tau)\} : \rho \in [0, 2\tau]\right\} = 4\tau^2$$

$$\min\left\{\max_{\rho}\{0, 4\tau(\rho - \tau)\} : \rho \in [0, 2\tau]\right\} = 0$$

Hence,

$$(2\tau - \rho)^2 + \max\{0, 4\tau(\rho - \tau)\}$$

has a minimum at $\rho = \tau$ and maxima at $\rho = 0$, $\rho = 2\tau$. Finally, both maxima are of equal magnitude, namely $4\tau^2$.

Therefore, this analysis shows that:

$$\min\{0, 2(\tau - \rho)\} - \frac{1}{2r} \omega^2 \geq -2\tau$$

\blacksquare
Stage 2: Sellers’ Membership
Chapter 7

Sellers’ Membership Decisions

Chapter 7 analyzes a seller’s optimal choice of membership in each market. Section (1) defines a seller’s optimization problem in a given market. Section (2) finds a seller’s best-response in market \( \varsigma \). Section (3) investigates a seller’s best-response in market \( \delta \).

7.1 Defining a Seller’s Best-Response

At the start of the second stage of the game seller-fees have been determined: \( f_\varsigma = (x, -\omega) \), \( f_\delta = (y, z) \). It has been shown that a seller’s payoff becomes:

\[
\tilde{\Pi}_i = \sum_{k \in \{\varsigma, \delta\}} \tilde{\Gamma}_i(f_k, m^k_{-i}, m^i_k),
\]

where the profit of seller \( i \) in market \( k \) is

\[
\tilde{\Gamma}_i(f_k, m^k_{-i}, m^i_k) \equiv \frac{1}{\sigma} \left\{5\sigma + 3W(f_k, m^k_i) - \sum_j W(f_k, m^j_k)\right\}^2
\]

Hence, their profit depends on the difference between the extra-surplus they receive and the extra-surplus received by their rival. Furthermore, their profit in market \( \varsigma \) depends on \( W(f_\varsigma, m^\varsigma_i) \) but not on \( W(f_\delta, m^\delta_i) \); and vice versa for market \( \delta \). This is a consequence of allowing sellers to choose different prices in each market.

In the second stage of the game sellers make their membership decisions. In market \( k \in \{\varsigma, \delta\} \) seller \( i \) selects \( m^k_i \in \mathbb{M} \), where \( \mathbb{M} = \{o, b, a, h\} \). (To operationalize the decision rule, values are attached to these options as follows: \( o = 0 \), \( b = 1 \), \( a = 2 \), \( h = 3 \).

Sellers make their membership decision so as to maximize their payoff. The extra-surplus that a seller receives from platforms depends only on \( m^k_i \). Hence, the profit in market \( \varsigma \) depends on \( m^\varsigma_i \) but not on \( m^\delta_i \). Similarly, the profit in market \( \delta \) depends on \( m^\delta_i \) but not on \( m^\varsigma_i \). Since the tie is not enforced, \( m^\varsigma_i \) and \( m^\delta_i \) can be chosen independently. This implies that the profits from each market can be maximized separately. Hence, a seller selects \( m^k_i \in \mathbb{M} \) so as to maximize \( \tilde{\Gamma}_i(f_k, m^k_{-i}, m^i_k) \). However, there may be more than one optimal choice, and in such situations they obey the decision-rule. Therefore, a seller’s
best-response is defined as:

$$
\hat{m} = \max \left\{ \hat{m} : \hat{m} \in \arg \max_{m \in \mathbb{M}} \Gamma_i(f_k, m_{-i}, m) \right\}
$$

Since $\Gamma_i(f_k, m_{-i}, m)$ is a strictly increasing function of $W(f_k, m_i)$, the problem in market $k$ can be reformulated as follows:

$$
\hat{m}_k(f_k) = \max \hat{M}(f_k),
$$

where the set of optimal membership options is

$$
\hat{M}(f_k) \equiv \arg \max_{m \in \mathbb{M}} W(f_k, m)
$$

Note that: (i) the solution doesn’t depend on the choice made by the other sellers; and (ii) because seller-fees have been determined the end-user benefit on each of the platforms is fixed.

Finally, it’s convenient to introduce the following concepts. Let $m'$ be referred to as "irrelevant" in market $k$ iff $\exists m \in \mathbb{M} : W(f_k, m) > W(f_k, m')$ or $W(f_k, m) \geq W(f_k, m')$, $m \geq m'$. An option is said to be "relevant" in market $k$ iff it’s not "irrelevant". When fees are $f_k$, the set of relevant membership options in market $k$ is denoted by $\hat{M}(f_k)$.

The set of equilibria is denoted by $\mathcal{N}(f_k)$. By definition, $\hat{m}_k \in \mathcal{N}(f_k)$ iff $\hat{m}_i(f_k) = \hat{m}_k(f_k)$, for all $i \in \{1, 2, 3\}$. A simplified game can be created by removing irrelevant options from the strategy set. Since these options aren’t ever chosen by sellers it follows that the simplified game has the same set of equilibria.

### 7.2 Sellers’ Best-Response in Market $\varsigma$

In market $\varsigma$ the fees are $f_\varsigma = (x, -\omega)$, where $x \in [-\omega, \rho]$, and the extra-surplus becomes

$$
W(f_\varsigma, m) = \begin{cases} 
0 & \text{if } m \in \{0, 1\} \\
\hat{W}(x) & \text{if } m \in \{a, b\}
\end{cases}
$$

where

$$
\hat{W}(x) = \frac{1}{4}(\omega^2 - x^2)
$$

(Note that Platform $C_B$ has been switched off.) It can be seen that: (1) if $|x| < \omega$, then $\hat{W}(x) > 0$; (2) if $|x| = \omega$, then $\hat{W}(x) = 0$; and (3) if $|x| > \omega$, then $\hat{W}(x) < 0$. It follows that the set of optimal membership options is

$$
\hat{M}(f_\varsigma) = \begin{cases} 
\{a, b\} & \text{if } |x| < \omega \\
\{0, b, a, b\} & \text{if } |x| = \omega \\
\{0, b\} & \text{if } |x| > \omega
\end{cases}
$$

In cases where a seller is indifferent between two (or more) options, they choose the one that occurs furthest to the right in the following list: $0, b, a, b$. A sellers’ best response becomes $\hat{m}_\varsigma(f_\varsigma) = \max \hat{M}(f_\varsigma)$. 81
Lemma 7.1 (Seller’s best response in market \( \mathcal{C} \).) A seller’s best-response in market \( \mathcal{C} \) becomes

\[
\tilde{m}_\mathcal{C}(f_\mathcal{C}) = \begin{cases} 
\mathfrak{b} & \text{if } |x| \leq \omega \\
\mathfrak{b} & \text{if } |x| > \omega 
\end{cases}
\]

This implies that

\[
A(f_\mathcal{C}, \tilde{m}_\mathcal{C}(f_\mathcal{C})) = \begin{cases} 
1 & \text{if } |x| \leq \omega \\
0 & \text{if } |x| > \omega 
\end{cases}
\]

Since all sellers make the same choice, it follows that there is a dominant strategy subgame equilibrium in which

\[
\tilde{m}_\mathcal{C}(f_\mathcal{C}) \in \mathcal{N}(f_\mathcal{C}) \iff \tilde{m}_\mathcal{C}(f_\mathcal{C}) = \begin{cases} 
\mathfrak{b} & \text{if } |x| \leq \omega \\
\mathfrak{b} & \text{if } |x| > \omega 
\end{cases}
\]

Hence, the profit from Platform \( \mathcal{C}_A \) becomes

\[
\Upsilon_A(f_\mathcal{C}, \tilde{m}_\mathcal{C}(f_\mathcal{C})) = \tilde{\Upsilon}_A(x) = \begin{cases} 
\mu \cdot Q(x) & \text{if } x \leq \omega \\
0 & \text{if } x > \omega 
\end{cases}
\]

7.3 Sellers’ Best-Response in Market \( \mathcal{D} \)

In market \( \mathcal{D} \) the fees are \( f_\mathcal{D} = (y, z) \), where \( y, z \in [-\rho, \rho] \), and the extra-surplus becomes

\[
W(f_\mathcal{D}, m) = \begin{cases} 
0 & \text{if } m = \mathfrak{o} \\
\hat{W}(z) & \text{if } m = \mathfrak{b} \\
\hat{W}(y) & \text{if } m = \mathfrak{a} \\
\hat{W}(\max\{y, z\}) & \text{if } m = \mathfrak{h}
\end{cases}
\]

To characterize a seller’s best-response it’s useful to categorize subgames according to the seller-fee chosen by Network \( \mathcal{B} \), namely, \( z \). There are three classes of subgame:

[1] Network \( \mathcal{B} \) subsidizes sellers: \( z \leq 0 \).

[2] Network \( \mathcal{B} \) charges sellers and offers a positive end-user benefit: \( 0 < z \leq \omega \).

[3] Network \( \mathcal{B} \) charges sellers and offers a negative end-user benefit: \( z > \omega \).

7.3.1 Subgame [1]

If \( z \leq 0 \), then the extra-surplus from multihoming is

\[
\hat{W}(\max\{y, z\}) = \begin{cases} 
\hat{W}(y) & \text{if } y \geq -|z| \\
\hat{W}(z) & \text{if } y < -|z|
\end{cases}
\]

and the maximum extra-surplus is

\[
\max_m W(f_\mathcal{D}, m) = \begin{cases} 
\hat{W}(y) & \text{if } |y| \leq |z| \\
\hat{W}(z) & \text{if } |y| > |z|
\end{cases}
\]
Furthermore, it can be seen that:

\[ W(f_d, b) = \max_m W(f_d, m) \text{ iff } |y| \geq |z| \]
\[ W(f_d, a) = \max_m W(f_d, m) \text{ iff } |y| \leq |z| \]
\[ W(f_d, h) = \max_m W(f_d, m) \text{ iff } y \leq |z| \]

This implies that the set of optimal membership options (excluding \( o \)) is

\[ \tilde{M}(f_d) \setminus \{o\} = \begin{cases} 
\{b\} & \text{if } y > |z| \\
\{b, a, h\} & \text{if } |y| = |z| \\
\{a, h\} & \text{if } |y| < |z| \\
\{b, h\} & \text{if } y < -|z| 
\end{cases} \]

(Note that \( o \) is irrelevant because \( \hat{W}(z) \geq 0 \).) A sellers’ best response becomes \( \hat{m}_d(f_d) = \max \tilde{M}(f_d) \setminus \{o\} \).

**Lemma 7.2 (Sellers’ best response in SG [2].)** If \( z \leq 0 \), then a sellers’ best-response in market \( d \) becomes

\[ \hat{m}_d(f_d) = \begin{cases} 
b & \text{if } y > |z| \\
h & \text{if } y \leq |z| 
\end{cases} \]

This implies that:

\[ A(f_d, \hat{m}_d(f_d)) = \begin{cases} 
1 & \text{if } -|z| \leq y \leq |z| \\
0 & \text{if } y < -|z| \text{ or } y > |z| 
\end{cases} \]
\[ B(f_d, \hat{m}_d(f_d)) = \begin{cases} 
1 & \text{if } y < -|z| \text{ or } y > |z| \\
0 & \text{if } -|z| \leq y \leq |z| 
\end{cases} \]
Since all sellers make the same choice, it follows that if \( z \leq 0 \), then there is a dominant strategy subgame equilibrium in which

\[
\bar{m}_d(f_d) \in N(f_d) \text{ iff } \bar{m}_d(f_d) = \begin{cases} b & \text{if } y > |z| \\ h & \text{if } y \leq |z| \end{cases}
\]

Finally, the networks’ profits in market \( d \) become

\[
\tilde{\gamma}_A(f_d, \bar{m}_d(f_d)) = \tilde{\gamma}_A(y, z) = \begin{cases} \mu Q(y) & \text{if } -|z| \leq y \leq |z| \\ 0 & \text{if } y < -|z| \text{ or } y > |z| \end{cases}
\]

\[
\tilde{\gamma}_B(f_d, \bar{m}_d(f_d)) = \tilde{\gamma}_B(y, z) = \begin{cases} \mu Q(z) & \text{if } y < -|z| \text{ or } y > |z| \\ 0 & \text{if } -|z| \leq y \leq |z| \end{cases}
\]

7.3.2 Subgame [2]

If \( 0 < z \leq \omega \), then the extra-surplus from multihoming is

\[
\hat{W}(\max\{y, z\}) = \begin{cases} \hat{W}(y) & \text{if } y \geq |z| \\ \hat{W}(z) & \text{if } y < |z| \end{cases}
\]

and the maximum extra-surplus is

\[
\max_m W(f_d, m) = \begin{cases} \hat{W}(y) & \text{if } |y| \leq |z| \\ \hat{W}(z) & \text{if } |y| > |z| \end{cases}
\]

Furthermore, it can be seen that:

\[
W(f_d, b) = \max_m W(f_d, m) \text{ iff } |y| \geq |z|
\]

\[
W(f_d, a) = \max_m W(f_d, m) \text{ iff } |y| \leq |z|
\]

\[
W(f_d, h) = \max_m W(f_d, m) \text{ iff } y < -|z| \text{ or } |y| = |z|
\]
This implies that the set of optimal membership options (excluding o) is
\[ \mathcal{M}(f_d) \setminus \{o\} = \begin{cases} \{b\} & \text{if } y > |z| \\ \{b, a, h\} & \text{if } |y| = |z| \\ \{a\} & \text{if } |y| < |z| \\ \{b, h\} & \text{if } y < -|z| \end{cases} \]
(Note that o is irrelevant because \( \hat{W}(z) \geq 0 \).) A sellers’ best response becomes
\[ \hat{m}_d(f_d) = \max \mathcal{M}(f_d) \setminus \{o\}. \]

Lemma 7.3 (Sellers’ best-response in SG [2].) If \( 0 < z \leq \omega \), then a seller’s best-response in market \( d \) becomes
\[ \hat{m}_d(f_d) = \begin{cases} b & \text{if } y > |z| \\ a & \text{if } |y| < |z| \\ h & \text{if } |y| = |z| \text{ or } y < -|z| \end{cases} \]

This implies that:
\[ A(f_d, \hat{m}_d(f_d)) = \begin{cases} 1 & \text{if } -|z| < y \leq |z| \\ 0 & \text{if } y \leq -|z| \text{ or } y > |z| \end{cases} \]
\[ B(f_d, \hat{m}_d(f_d)) = \begin{cases} 1 & \text{if } y \leq -|z| \text{ or } y > |z| \\ 0 & \text{if } -|z| < y \leq |z| \end{cases} \]
Since all sellers make the same choice, it follows that if \( 0 < z \leq \omega \), then there is a dominant strategy subgame equilibrium in which
\[ \tilde{m}_d(f_d) \in N(f_d) \text{ iff } \tilde{m}_d(f_d) = \begin{cases} b & \text{if } y > |z| \\ a & \text{if } |y| < |z| \\ h & \text{if } |y| = |z| \text{ or } y < -|z| \end{cases} \]

Finally, the networks’ profits in market \( d \) become
\[ \Upsilon_A(f_d, \tilde{m}_d(f_d)) = \tilde{\Upsilon}_A(y, z) = \begin{cases} \mu.Q(y) & \text{if } -|z| < y \leq |z| \\ 0 & \text{if } y \leq -|z| \text{ or } y > |z| \end{cases} \]
\[ \Upsilon_B(f_d, \tilde{m}_d(f_d)) = \tilde{\Upsilon}_B(y, z) = \begin{cases} \mu.Q(z) & \text{if } y \leq -|z| \text{ or } y > |z| \\ 0 & \text{if } -|z| < y \leq |z| \end{cases} \]

7.3.3 Subgame [3]
If \( z > -\omega \), then the extra-surplus from multihoming is
\[ \hat{W}(\max\{y, z\}) = \begin{cases} \hat{W}(y) & \text{if } y \geq z \\ \hat{W}(z) & \text{if } y < z \end{cases} \]
and the maximum extra-surplus is
\[ \max_m W(f_d, m) = \begin{cases} \hat{W}(y) & \text{if } y \leq \omega \\ 0 & \text{if } y > \omega \end{cases} \]
Furthermore, it can be seen that:

\[ W(f_q, o) = \max_m W(f_q, m) \] iff \( y \geq \omega \)
\[ W(f_q, a) = \max_m W(f_q, m) \] iff \( y \leq \omega \)

\[ W(f_q, h) < \max_m W(f_q, m) \]

This implies that the set of optimal membership options (excluding \( b \)) is

\[ \hat{M}(f_q) \setminus \{b\} = \begin{cases} \{a\} & \text{if } y < \omega \\ \{o, a\} & \text{if } y = \omega \\ \{o\} & \text{if } y > \omega \end{cases} \]

(Note that \( b \) is irrelevant because \( \hat{W}(z) < 0 \).)

**Lemma 7.4 (Sellers’ best response in SG [3].)** If \( z > \omega \), then a sellers’ best-response becomes

\[ \hat{m}_q(f_q) = \begin{cases} a & \text{if } y \leq \omega \\ o & \text{if } y > \omega \end{cases} \]

This implies that:

\[ A(f_q, \hat{m}_q(f_q)) = \begin{cases} 1 & \text{if } y \leq \omega \\ 0 & \text{if } y > \omega \end{cases} \]

\[ B(f_q, \hat{m}_q(f_q)) = 0 \]

Since all sellers make the same choice, it follows that if \( z > \omega \), then there is a dominant strategy subgame equilibrium in which

\[ \hat{m}_q(f_q) \in \mathcal{N}(f_q) \] iff \( \hat{m}_q(f_q) = \begin{cases} a & \text{if } y \leq \omega \\ o & \text{if } y > \omega \end{cases} \]
Finally, the networks’ profits in market $q$ become

$$
\gamma_A(f_q, \bar{m}_q(f_q)) = \tilde{\gamma}_A(y, z) = \begin{cases} 
\mu Q(y) & \text{if } y \leq \omega \\
0 & \text{if } y > \omega
\end{cases}
$$

$$
\gamma_B(f_q, \bar{m}_q(f_q)) = \tilde{\gamma}_B(y, z) = 0
$$
Stage 1: Networks’ Prices
Chapter 8

Network A’s Best-Response

Chapter 8 characterizes Network A’s best response. Section (1) finds the optimal seller-fee on a monopoly platform. Section (2) finds Network A’s best-response, in market $d$, to the seller-fee chosen by Network B.

Imagine that Network A knows the fee set by Network B in market $d$, namely, $z$. (Note that the networks actually set fees simultaneously.) What is Network A’s best-response to $z$? It can be seen that $z$ does not affect the profit made by Network A in market $c$. That is, Network A has a monopoly in market $c$. Whereas, $z$ does affect Network A’s profit in market $d$. Therefore, the two networks compete for sellers in market $d$ but Network A sets the monopoly fee in market $c$.

8.1 The Monopoly Fee in Market $c$

It has been shown that the profit from Platform $C_A$ becomes

$$\hat{\gamma}_A(x) = \begin{cases} 
\mu Q(x) & \text{if } x \leq \omega \\
0 & \text{if } x > \omega 
\end{cases}$$

Let $\hat{x}$ denote the monopoly fee. Since $Q(x)$ is positive and strictly increasing, the problem becomes:

$$\hat{x} = \arg \max_{x \in [-\omega, \rho]} \{ Q(x) : x \leq \omega \}$$

It can be seen that one of the constraints must be binding.

Lemma 8.1 (Monopoly fee.) Network A’s best-response in market $c$ becomes

$$\hat{x} = \min\{\rho, \omega\}$$

Since $\omega = 2\tau - \rho$ it follows that $\rho > \omega$ iff $\rho > \tau$, which implies that the seller fee becomes

$$\hat{x} = \begin{cases} 
\omega & \text{if } \rho > \tau \\
\rho & \text{if } \rho \leq \tau 
\end{cases}$$

It follows that the buyer fee is zero iff $\rho \leq \tau$. 

89
8.2 Best-Response in Market $\mathfrak{d}$

Subgames were classified according to the value of $z$. There are three classes of subgame: [1] $z \leq 0$; [2] $0 < z \leq \omega$; [3] $z > \omega$. In general, the profit functions differ across these subgames:

$$
\begin{align*}
z \leq 0 \Rightarrow \mathcal{Y}_A(f_{\mathfrak{d}}, \tilde{m}_\mathfrak{d}(f_{\mathfrak{d}})) &= \begin{cases} 
\mu \cdot Q(y) & \text{if } -|z| \leq y \leq |z| \\
0 & \text{if } y < -|z| \text{ or } y > |z|
\end{cases} \\
0 < z \leq \omega \Rightarrow \mathcal{Y}_A(f_{\mathfrak{d}}, \tilde{m}_\mathfrak{d}(f_{\mathfrak{d}})) &= \begin{cases} 
\mu \cdot Q(y) & \text{if } -|z| < y \leq |z| \\
0 & \text{if } y < -|z| \text{ or } y > |z|
\end{cases} \\
z > \omega \Rightarrow \mathcal{Y}_A(f_{\mathfrak{d}}, \tilde{m}_\mathfrak{d}(f_{\mathfrak{d}})) &= \begin{cases} 
\mu \cdot Q(y) & \text{if } y \leq \omega \\
0 & \text{if } y > \omega
\end{cases}
\end{align*}
$$

Let $\hat{y}$ denote the best-response. It can be seen that in all subgames, if $y < 0$, then $y$ is strictly dominated. Hence, Network $A$ sets a positive seller-fee: $\hat{y} \geq 0$. If $z \leq \omega$ and $y \geq 0$, then

$$
\tilde{\mathcal{Y}}_A(y, z) = \begin{cases} 
\mu \cdot Q(y) & \text{if } y \leq |z| \\
0 & \text{if } y > |z|
\end{cases}
$$

Whereas, if $z > \omega$ and $y \geq 0$, then

$$
\tilde{\mathcal{Y}}_A(y, z) = \begin{cases} 
\mu \cdot Q(y) & \text{if } y \leq \omega \\
0 & \text{if } y > \omega
\end{cases}
$$

Hence, if $y \geq 0$, then the profit function becomes

$$
\tilde{\mathcal{Y}}_A(y, z) = \begin{cases} 
\mu \cdot Q(y) & \text{if } y \leq \min\{\omega, |z|\} \\
0 & \text{if } y > \min\{\omega, |z|\}
\end{cases}
$$

Since $Q(x)$ is positive and strictly increasing, the problem becomes

$$
\hat{y}(z) = \arg \max_{y \in [0, \omega]} \{Q(y) : y \leq \min\{\omega, |z|\}\}
$$

It can be seen that one of the constraints must be binding. Hence, the best-response becomes

$$
\hat{y}(z) = \begin{cases} 
\min\{\rho, |z|\} & \text{if } |z| \leq \omega \\
\min\{\rho, \omega\} & \text{if } |z| > \omega
\end{cases}
$$

Since $z \in [-\omega, \rho]$, $\omega = 2\tau - \rho$ and $\rho \in (0, 2\tau)$, it follows that:

$$
|z| \leq \max\{\omega, \rho\} = \rho
$$

Hence,

$$
\hat{y}(z) = \begin{cases} 
|z| & \text{if } z \leq \omega \\
\omega & \text{if } z > \omega
\end{cases}
$$

**Lemma 8.2 (Competitive fee.)** Network $A$’s best-response in market $\mathfrak{d}$ becomes

$$
\hat{y}(z) = \min\{|z|, \omega\}
$$

It’s useful to consider Network $A$’s best-response in the special case where Network $B$ makes its platform free to sellers; that is, its best-response when
$z = 0$. Since $\omega > 0$, it follows that:

$$\hat{y}(0) = 0$$

Hence, when Network $B$ sets makes its platform free to sellers, Network $A$ follows suite. Finally, if $z < 0$, then $\hat{y}(z) = |z| > 0$. Similarly, if $z > 0$, then $\hat{y}(z) > 0$. Hence, $\hat{y}(z) = 0$ iff $z = 0$. 
Chapter 9

Equilibrium Outcomes

Chapter 9 determines the seller-fees that will be set by networks and equilibrium outcomes. Section (1) finds the equilibrium seller-fees. Section (2) finds the end-user benefit on Network A’s platforms. Section (3) finds the prices set by sellers. Section (4) combines the results to get an expression for the consumer surplus.

9.1 Equilibrium Fees

It has been shown that Network A’s optimal fees are:

\[ \hat{x} = \min\{\rho, \omega\} \]
\[ \hat{y}(z) = \min\{|z|, \omega\} \]

Since market \( d \) is contested, the networks play a game in which the payoffs are \( \tilde{\tau}_A(y, z) \), \( \tilde{\tau}_B(y, z) \) and the actions are \( y, z \in [-\omega, \rho] \).

Lemma 9.1 (Existence.) There exists an equilibrium in which the seller-fees in market \( d \) are: \( \tilde{y} = \tilde{z} = 0 \).

Proof. Suppose the fees are: \( \tilde{y} = \tilde{z} = 0 \). The profits made by the networks in market \( d \) become

\[ \tilde{\tau}_A(0) = \mu.Q(0) > 0 \]
\[ \tilde{\tau}_B(0) = 0 \]

(The form of these functions depend on the size of \( z \).) Does either network have an incentive to deviate?

Network A. Deviation by Network A can analyzed as follows. The the fee set by Network A matches its best-response when \( z = \rho \). Hence, Network A can’t profit through unilateral deviation.

Network B. Deviation by Network B can be analyzed as follows. The subgames played by sellers at Stage 2 were classified according to the size of \( z \). These subgames can be analyzed in turn.
[1] If $z < 0$, then

$$\tilde{\mathcal{T}}_B(f_d) = \left\{ \begin{array}{ll}
\mu Q(z) & \text{if } y < -|z| \text{ or } y > |z| \\
0 & \text{if } -|z| \leq y \leq |z|
\end{array} \right.$$ 

Since $z \neq 0$, it follows that $0 \leq |z|$. Hence, $\tilde{\mathcal{T}}_B(0, z) = 0$.

[2] If $0 < z \leq \omega$, then

$$\tilde{\mathcal{T}}_B(f_d) = \left\{ \begin{array}{ll}
\mu Q(z) & \text{if } y \leq -|z| \text{ or } y > |z| \\
0 & \text{if } -|z| < y \leq |z|
\end{array} \right.$$ 

Since $z \neq 0$, it follows that $0 \leq |z|$. Hence, $\tilde{\mathcal{T}}_B(0, z) = 0$.

[3] If $z > \omega$, then $\tilde{\mathcal{T}}_B(f_d) = 0$.

It has been shown that when $y = 0$, there is no way that Network B can make a positive profit. Therefore, Network B has no incentive to deviate.

Lemma 9.2 (Uniqueness.) If $\tilde{z} \neq 0$, then $y = \tilde{y}(\tilde{z})$, $z = \tilde{z}$ is not an equilibrium; because Network B has an incentive to deviate.

Proof. Suppose that the fee chosen by Network B is $\tilde{z} \neq 0$. For there to be an equilibrium Network A must choose its best-response, namely, $\tilde{y}(\tilde{z})$. (Note that for all $z \in [-\omega, \rho]$, $\tilde{g}(z) \geq 0$.) Deviation by Network B can be analyzed as follows. It has been shown that if $z \leq 0$, then

$$\tilde{\mathcal{T}}_B(\tilde{y}(\tilde{z}), z) = \left\{ \begin{array}{ll}
\mu Q(z) & \text{if } \tilde{y}(\tilde{z}) < -|z| \text{ or } \tilde{y}(\tilde{z}) > |z| \\
0 & \text{if } -|z| \leq \tilde{y}(\tilde{z}) \leq |z|
\end{array} \right.$$ 

Hence, if $z = 0$, then

$$\tilde{\mathcal{T}}_B(\tilde{y}(\tilde{z}), 0) = \left\{ \begin{array}{ll}
\mu Q(0) & \text{if } \tilde{y}(\tilde{z}) > 0 \\
0 & \text{if } \tilde{y}(\tilde{z}) = 0
\end{array} \right.$$ 

Since $\tilde{z} \neq 0$, it follows that $\tilde{y}(\tilde{z}) > 0$; which implies that

$$\tilde{\mathcal{T}}_B(\tilde{y}(\tilde{z}), 0) = \mu Q(0) = \frac{1}{\tilde{z}}\mu > 0$$

Therefore, Network B can profit through unilateral deviation. This completes the proof that the equilibrium is unique.

The two lemmas can be combined to give:

Proposition 9.1 There is a unique equilibrium in which the seller-fees in market $d$ are $y = \tilde{z} = 0$. It follows that buyer-fees are $\rho$ on both networks.

9.2 End-User Benefit

It has been shown that Network B is excluded. Hence, the extra-surplus received by sellers and their customers depends on the end-user benefit generated
by Network $A$:
\[
\begin{align*}
\hat{W}(\hat{x}) &= \frac{1}{4\tau}(\omega^2 - \hat{x}^2) \\
\hat{W}(\hat{y}) &= \frac{1}{4\tau}(\omega^2 - \hat{y}^2)
\end{align*}
\]

**Market $\mathcal{C}$**. Network $A$ sets the monopoly fee: $\hat{x} = \min\{\rho, \omega\}$. Hence, the end-user benefit in market $\mathcal{C}$ is
\[
\begin{align*}
\hat{W}(\hat{x}) &= \hat{W}(\min\{\rho, \omega\}) \\
&= \frac{1}{4\tau}(\omega^2 - \min\{\rho, \omega\}^2) \\
&= \max\{0, \frac{1}{4\tau}(\omega^2 - \rho^2)\} \\
&= \max\{0, \tau - \rho\}
\end{align*}
\]

**Market $\mathcal{D}$**. Network $A$ sets the competitive fee in market $\mathcal{D}$: $\hat{y} = 0$. Hence, the end-user benefit in market $\mathcal{D}$ is
\[
\begin{align*}
\hat{W}(\hat{y}) &= \hat{W}(0) \\
&= \frac{1}{4\tau}(\omega^2 - \hat{y}^2) \\
&= \frac{1}{4\tau}(2\tau - \rho)^2
\end{align*}
\]

### 9.3 Membership and Prices

Having determined the fees set by Network $A$ it’s possible to find the membership of sellers and the prices they set.

**Market $\mathcal{C}$**. It has been shown that a seller’s optimal membership choice in market $\mathcal{C}$ is
\[
\tilde{m}_i^\mathcal{C} = \tilde{m}_i^\mathcal{C}(\hat{x}, -\omega) = \begin{cases} 
\text{b} & \text{if } \hat{x} \leq \omega \\
\text{h} & \text{if } \hat{x} > \omega
\end{cases}
\]

Since $\hat{x} = \min\{\rho, \omega\} \leq \omega$, it follows that the sellers multihome:
\[
\tilde{m}_i^\mathcal{C} = \text{h}
\]

However, since Platform $C_B$ is switched off, the sellers are on Platform $C_A$: $A(\min\{\rho, \omega\}, -\omega; \text{h}) = 1$.

It has been shown that a seller’s optimal price is
\[
\tilde{p}_i^\mathcal{C}(\tilde{f}_i^\mathcal{C}, \tilde{m}_i^\mathcal{C}, \tilde{m}_i^\mathcal{C}) = \frac{1}{5}\left\{5\sigma + 2C(\tilde{f}_i^\mathcal{C}, \tilde{m}_i^\mathcal{C}) + \sum_j C(\tilde{f}_i^\mathcal{C}, \tilde{m}_j^\mathcal{C})\right\},
\]

where the marginal cost is
\[
C(\tilde{f}_i^\mathcal{C}, \tilde{m}_i^\mathcal{C}) = \delta + \tau - \hat{W}(\tilde{f}_i^\mathcal{C}, \tilde{m}_i^\mathcal{C})
= \delta + \tau - \hat{W}(\min\{\rho, \omega\}, -\omega; \text{h})
= \delta + \tau - \hat{W}(\min\{\rho, \omega\})
\]

94
Since all sellers have the same marginal cost, the price becomes

\[
\tilde{p}_i^d(\tilde{f}_d^i, \tilde{m}_d^i, \tilde{m}_i^d) = \sigma + C(\tilde{f}_d^i, \tilde{m}_i^d)
\]

\[= \sigma + \delta + \tau - \tilde{W}(\min\{\rho, \omega\})
\]

**Market d.** It has been shown that if \(z \leq 0\), then a seller’s optimal membership choice in market \(d\) is

\[
\tilde{m}_d(y, z) = \begin{cases} 
 b & \text{if } y > |z| \\
 h & \text{if } y \leq |z|
\end{cases}
\]

Since \(\tilde{y} = 0\), \(\tilde{z} = 0\), it follows that the sellers multihome:

\[
\tilde{m}_i^d = h
\]

However, due to buyers’ inertia, only Network A is active: \(A(0, 0; h) = 1\).

It has been shown that a seller’s optimal price is

\[
\tilde{p}_i^d(\tilde{f}_d^i, \tilde{m}_d^i, \tilde{m}_i^d) = \frac{1}{5} \left\{ 5\sigma + 2C(\tilde{f}_d^i, \tilde{m}_i^d) + \sum_j C(\tilde{f}_d^i, \tilde{m}_j^d) \right\},
\]

where the marginal cost is

\[
C(\tilde{f}_d^i, \tilde{m}_i^d) = \delta + \tau - \tilde{W}(\tilde{f}_d^i, \tilde{m}_i^d)
\]

\[= \delta + \tau - \tilde{W}(0, 0; h)
\]

\[= \delta + \tau - \tilde{W}(0)
\]

Since all sellers have the same marginal cost, the price becomes

\[
\tilde{p}_i^d(\tilde{f}_d^i, \tilde{m}_d^i, \tilde{m}_i^d) = \sigma + C(\tilde{f}_d^i, \tilde{m}_i^d)
\]

\[= \sigma + \delta + \tau - \tilde{W}(0)
\]

### 9.4 Consumer-Surplus

The consumer-surplus is

\[
\Phi(P) = \sum_{k \in \{c, d\}} \sum_{i \in \{1, 2, 3\}} \Theta(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle)
\]

where

\[
\Theta(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle) = \left\{ \nu - \tilde{p}_i^k - \frac{1}{2} \sigma D(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle) \right\} . D(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle)
\]

Since the sellers set the same fee in a given market, this becomes

\[
\Phi(P) = 3 \sum_{k \in \{c, d\}} \Theta(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle),
\]

where

\[
\Theta(\tilde{p}_i^k, \langle \tilde{p}_{-i}^k \rangle) = \frac{1}{3} \left( \nu - \tilde{p}_i^k - \frac{1}{6} \sigma \right)
\]
Hence,
\[ \tilde{\Phi} = 2\nu - \tilde{p}_i^2 - \tilde{p}_i^j - \frac{1}{3}\sigma, \]
which implies that
\[ \tilde{\Phi} = 2(\nu - \sigma - \delta - \tau) + \hat{W}(\min\{\rho, \omega\}) + \hat{W}(0) - \frac{1}{3}\sigma. \]

If there were no platforms the then the marginal cost would be \( \delta + \tau \) and equilibrium prices would be \( \sigma + \delta + \tau \). Hence, the consumer-surplus without platforms would become \( 2(\nu - \sigma - \delta - \tau) - \frac{1}{3}\sigma \). It follows that:

**Theorem 9.1 (Consumer-Surplus when Tying is prohibited.)** In equilibrium, the extra consumer-surplus generated by the platforms becomes:

\[ \Delta\tilde{\Phi}_I = \hat{W}(\min\{\rho, \omega\}) + \hat{W}(0) \]

It’s useful to understand the relationship between \( \Delta\tilde{\Phi}_I \) and \( \rho \). Firstly, \( \Delta\tilde{\Phi}_I \) can be re-expressed as follows

\[ \Delta\tilde{\Phi}_I = \begin{cases} 
\frac{1}{\bar{\tau}} [2\omega(\rho)^2 - \rho^2] & \text{if } \rho < \tau \\
\frac{1}{\bar{\tau}} \omega(\rho)^2 & \text{if } \rho \geq \tau 
\end{cases} \]

From this it can be shown that its continuous at \( \rho = \tau \) but not differentiable. Secondly, since \( \omega(\rho) \) decreases as \( \rho \) increases, it follows that both functions decrease as \( \rho \) increases. Hence, the extra-consumer surplus declines as the fee rises. Finally, if \( \rho = 2\tau \), then the extra-consumer surplus goes to zero. Figure 4 illustrates the relationship between \( \rho \) and the extra-consumer surplus.

\[ Figure 4. \] Extra-consumer surplus without tying.
Part III

Membership of a Network’s Platforms is Tied
Outline of Part III

The Game

Chapter 10 sets up the game played by sellers and networks when tying is permitted.

Stage 2: Sellers’ Membership Decisions

Chapter 11 analyzes the options available to a seller under tying.

Chapter 12 shows that there are eight possible membership configurations.

Chapter 13 classifies subgames and finds possible configurations for each type of subgame.

Chapter 14 investigates the necessary conditions for all the sellers to be on a single network.

Chapter 15 investigates sellers’ membership when $z \leq 0$ and $y \geq -|z|$.

Chapter 16 investigates sellers’ membership when $z \leq 0$ and $y < -|z|$.

Chapter 17 combines the results from the last two chapters in order to find sellers’ membership when $z \leq 0$.

Stage 1: Networks’ Fees

Chapter 18 shows that Network A can (and will) always exclude Network B.

Chapter 19 characterizes Network A’s best-response to the fee chosen by Network B.

Chapter 20 finds the equilibrium outcome.
The Game
Chapter 10

*Game II: Tying is Enforced*

Chapter 10 sets up the game played by sellers and networks when tying is permitted. Section (1) introduces the players. Section (2) describes the parameters. Section (3) explains the timing and the actions available to the players. Section (4) summarizes the concept of end-user benefit. Section (5) defines the players payoffs. Section (6) explains the sellers’ decision rule, which applies in situations where they are indifferent between two or more membership options.

Suppose that the membership of platforms is tied. That is, sellers can only join a particular network if they agree to be members of the network in both markets: $m_i^s = m_i^d$. Hence, we can drop the superscript.

### 10.1 Players

This part of the study investigates what happens when membership platforms is tied. The framework can be used to construct a model, *Game II*, in which sellers are restricted in their choice of membership. The players in the game are:

- Network $A$ ($n = A$)
- Network $B$ ($n = B$)
- Seller $1$ ($i = 1$)
- Seller $2$ ($i = 2$)
- Seller $3$ ($i = 3$)

### 10.2 Parameters

The parameters of the model can be summarized as follows:

- The gross benefit (money value) from consuming one unit of the good is $\nu$. This is assumed to be at least an order of magnitude larger than the other parameters.
- The wholesale price of the good is $\delta$.
- The travel cost parameter is $\sigma$. 

100
• Payment related transaction costs are uniformly distributed over \([0, 2\tau]\), where the average transaction cost is \(\tau\), where \(\tau < \frac{1}{10}\sigma\).

• The marginal cost of processing a transaction on a platform (payment-card) is \(\gamma\).

• The price-level on a network’s platforms is \(\rho \in (0, 2\tau)\). Its value is exogenously determined by the degree of competition between affiliated banks.

From these parameters the following quantities can be derived:

• A platform’s markup is \(\mu = \rho - \gamma\).

• The maximum surplus-per-transaction (MSPT) received by end-users (buyers and sellers) is \(\omega = 2\tau - \rho\).

### 10.3 Sequence of Actions

The sequence of actions in *Game II* is the same as that for *Game I* except for the second stage. In the second stage of *Game II*, a seller decides which network (if any) to join. That is, sellers join networks rather than platforms.

**Stage 1.** Networks set seller-fees on their platforms: Networks A is active in both markets, namely, \(\varsigma, \vartheta\). Whereas, Network B is only active in market \(\vartheta\) (because Platform \(C_B\) is switched off). The seller-fees set by Network A in markets \(\varsigma, \vartheta\) are \(x, y \in [-\omega, \rho]\), respectively. The seller-fee set by Network B in market \(\vartheta\) is \(z \in [-\omega, \rho]\). The fee vectors in markets \(\varsigma, \vartheta\) are \(f_\varsigma = (x, -\omega)\), \(f_\vartheta = (y, z)\), respectively.

**Stage 2.** The set of membership options is \(M \equiv \{o, b, a, h\}\), where the elements are as follows: \(o\) denotes outside-options (cash); \(b\) denotes membership of Network B; \(a\) denotes membership of Network A; and \(h\) denotes multihoming (multiple memberships). Seller \(i \in \{1, 2, 3\}\) decides which networks (if any) to join: \(m_i \in M\). The membership vectors is \(m = (m_1, m_2, m_3)\).

**Stage 3.** Seller \(i \in \{1, 2, 3\}\) chooses effective prices in each market: \(p_1^i \in \mathbb{R}\), \(p_3^i \in \mathbb{R}\). The price vectors in markets \(\varsigma, \vartheta\) are \(p_\varsigma = (p_1^\varsigma, p_2^\varsigma, p_3^\varsigma)\), \(p_\vartheta = (p_1^\vartheta, p_2^\vartheta, p_3^\vartheta)\), respectively.

It is assumed that everything that has occurred in earlier stages is common knowledge.

### 10.4 Extra-Surplus

The end-user benefit generated by a platform with seller-fee \(f\) is

\[
\hat{W}(f) = \frac{1}{4\tau}(\omega^2 - f^2)
\]

This is the increased surplus available to buyers and sellers due to a decrease in transaction costs.
The extra-surplus is the change in the surplus available to a seller and their customers due to reduced transaction costs. This depends on seller-fees and which platforms a seller joined. The extra-surplus available to seller \( i \) and their customers in market \( c \) is

\[
W(f_c, m_i) = \begin{cases} 
0 & \text{if } m_i \in \{a, b\} \\
\bar{W}(x) & \text{if } m_i \in \{a, h\}
\end{cases}
\]

The extra-surplus available to seller \( i \) and their customers in market \( d \) is

\[
W(f_d, m_i) = \begin{cases} 
0 & \text{if } m_i = a \\
\bar{W}(y) & \text{if } m_i = a \\
\bar{W}(z) & \text{if } m_i = b \\
\bar{W}(\max\{y, z\}) & \text{if } m_i = h
\end{cases}
\]

10.5 Payoffs

The payoffs are received by the players at the end of the game according to the actions chosen. The payoff functions are as follows:

**Network A.** The profits made by Network A in markets \( c, d \) are

\[
\Upsilon_A(f_c, m) = \frac{1}{3} \mu Q(x) \sum_{i \in \{1, 2, 3\}} A(f_c, m_i)
\]

\[
\Upsilon_A(f_d, m) = \frac{1}{3} \mu Q(y) \sum_{i \in \{1, 2, 3\}} A(f_d, m_i)
\]

where

\[
Q(x) = \frac{1}{2\pi} (\omega + x)
\]

\[
Q(y) = \frac{1}{2\pi} (\omega + y)
\]

and

\[
A(f_c, m_i) = 1(m_i \in \{a, h\})
\]

\[
A(f_d, m_i) = 1(m_i = a) + 1(m_i = h).1(z \leq y)
\]

A network’s payoff corresponds to its total profit. Since Network A is active in both markets, their payoff is

\[
\Pi_A(F, m) = \sum_{k \in \{c, d\}} \Upsilon_A(f_k, m),
\]

where the fee matrix is

\[
F = (f_k^T)_{k \in \{c, d\}} = \begin{pmatrix} x & y \\ -\omega & z \end{pmatrix}
\]

**Network B.** Network B is inactive in market \( c \) (because Platform \( C_B \) is switched off). Hence, Network B makes no profit in market \( c \). However,
Network $B$ is (potentially) active in market $q$. Their profit in market $q$ is

$$
\Upsilon_B(f_q, m) \equiv \frac{1}{3} \mu Q(z) \sum_{i \in \{1,2,3\}} B(f_q, m_i),
$$

where

$$
Q(z) \equiv \frac{1}{2\sigma} (\omega + z)
$$

and

$$
B(f_q, m_i) \equiv 1(m_i = b) + 1(m_i = b).1(z > y)
$$

Network $B$ is only active in market $q$. Hence, the payoff of Network $B$ is

$$
\Pi_B(f_q, m) \equiv \Upsilon_B(f_q, m)
$$

**Sellers.** Seller $i \in \{1,2,3\}$ competes against two sellers, whose prices in market $k \in \{c,d\}$ are denoted by $p_{k,i}$. The average price set by the seller’s competitors is

$$
\langle p_{k,i} \rangle \equiv \frac{\sum_{j \neq i} p_{j}^k}{2}
$$

The profit made by seller $i \in \{1,2,3\}$ in market $k \in \{c,d\}$ is

$$
\Gamma_i(f_k, m_i, p_k) \equiv (p_i^k - C(f_k, m_i)) \cdot D(p_i^k, \langle p_{-i} \rangle),
$$

where their partial-demand is

$$
D(p_i^k, \langle p_{-i} \rangle) \equiv \frac{\sigma + \langle p_{-i} \rangle - p_i^k}{3\sigma}
$$

and their marginal-cost is

$$
C(f_k, m_i) \equiv \delta + \tau - W(f_k, m_i)
$$

A seller’s payoff corresponds to its total profit. Since they are active in both markets, their payoff is

$$
\Pi_i(F, m_i, P) \equiv \sum_{k \in \{c,d\}} \Gamma_i(f_k, m_i, p_k),
$$

where the price matrix is

$$
P \equiv \begin{pmatrix} p_{c,i}^1 & p_{c,i}^2 & p_{c,i}^3 \\ p_{d,i}^1 & p_{d,i}^2 & p_{d,i}^3 \end{pmatrix}
$$

The Sub-Game Perfect Nash equilibrium (SPNE) of this game can be found using the technique of backward induction. Stage two has a range of multiple equilibria. The following decision rule is used to select particular types of SPNE at stage 2; these are referred to as Restricted SPNE (RSPNE).
10.6 Sellers’ Decision Rule

A seller may be indifferent between two or more options. In this situation they obey the following decision rule. First, arrange options in the following order: 0, b, a, h. (That is, in increasing order of the number of platforms that a seller joins.) Second, strike out all options except those that are optimal. Finally, choose the option that is furthest to the right.

The decision rule can be operationalized as follows: Firstly, put the elements of $M$ in one-to-one correspondence with those of $\{0, 1, 2, 3\}$: 0 $\equiv 0$, b $\equiv 1$, a $\equiv 2$, h $\equiv 3$. Secondly, let $\tilde{M}(F, m_{-i})$ denote a seller’s set of optimal membership options given fees, $F$, and the membership of the other two sellers, $m_{-i} \in M^2$. Hence, when a seller obeys the decision rule, their choice becomes:

$$\tilde{m}_i(F, m_{-i}) = \max \tilde{M}(F, m_{-i})$$

10.7 Sellers’ Price-Setting Decision

Seller $i \in \{1, 2, 3\}$ competes against two sellers and, in general, their behavior is influenced by the membership decisions of their rivals. The membership of sellers who compete with seller $i$ is denoted by $m_{-i} \in M^2$. There are three sellers and for each seller there’s a corresponding $(1 \times 2)$ vector. These vectors are as follows:

$$m_{-1} \equiv (m_2, m_3)
\quad m_{-2} \equiv (m_1, m_3)
\quad m_{-3} \equiv (m_1, m_2)$$

In the final stage of the game, sellers set prices. Membership, $m_{-i}, m_i$, and fees, $F$, are already determined and can be treated as constants. Since tying does not directly effect the final stage, the analysis of a seller’s price setting is the same as in Game I. Hence, prices and payoffs in a SPNE are the same as those we found before. These are summarized below:

**Prices.** Given fees, $f_k$, and membership, $m_{-i}, m_i$, the price set by seller $i$, in market $k \in \{\varsigma, \varphi\}$, becomes:

$$\tilde{p}_i^k = \tilde{p}_i^k(f_k, m_{-i}, m_i) \equiv \frac{1}{5} \left\{ 5\sigma + 2C(f_k, m_i) + \sum_j C(f_k, m_j) \right\}$$

**Sellers’ Payoffs.** Given fees, $f_k$, and membership, $m_{-i}, m_i$, the payoff of seller $i$, in market $k \in \{\varsigma, \varphi\}$, becomes:

$$\tilde{\Pi}_i = \tilde{\Pi}_i(F, m_{-i}, m_i) \equiv \sum_{k \in \{\varsigma, \varphi\}} \tilde{\Gamma}_i(f_k, m_{-i}, m_i),$$

where their profit in market $k$ is

$$\tilde{\Gamma}_i(f_k, m_{-i}, m_i) \equiv \frac{1}{5\sigma} \left\{ 5\sigma + 3W(f_k, m_i) - \sum_j W(f_k, m_j) \right\}^2$$

Furthermore, the profit made by seller $i$ in market $k$, $\tilde{\Gamma}_i(f_k, m_{-i}, m_i)$, is a
strictly increasing function of the extra-surplus received by seller $i$ in market $k$, $W(f_k, m_i)$. 
Stage 2: Sellers’ Membership
Chapter 11
The Sellers’ Dichotomy

Chapter 11 analyzes the options available to a seller under tying. Section (1) defines a seller’s best response. Section (2) investigates when one membership option dominates another membership option. Section (3) defines a Restricted SPNE and explains its relation to a conventional SPNE. Section (4) defines the set of relevant membership options and explains that irrelevant options can be deleted without altering the equilibrium. Section (5) shows that no subgame can have more than two relevant membership options. Section (6) claims that a Restricted SPNE always exists and that it is unique up to an interchange of sellers.

Once fees have been set, the sellers make their membership decisions. This chapter provides some preliminary analysis of their membership choices when platforms are tied.

11.1 Defining a Seller’s Best Response

In the second stage of the game, fees have been determined and sellers make their membership decision: \( m_i \in \mathbb{M} \equiv \{a, b, a, b\} \). Seller \( i \) maximizes their payoff, \( \sum_k \Gamma_i(f_k, m_{-i}, m_i) \), given the fees, \( F \), and the membership of the two other sellers, \( m_{-i} \). The set of optimal membership options is

\[
\hat{\mathbb{M}}(F, m_{-i}) \equiv \arg\max_{m_i \in \mathbb{M}} \sum_k \Gamma_i(f_k, m_{-i}, m_i)
\]

Finally, in situations where the best-response isn’t unique, they select an element from the optimal set, \( \hat{\mathbb{M}}(F, m_{-i}) \), according to the decision-rule. Therefore, the best-response of seller \( i \) is

\[
\hat{m}_i(F, m_{-i}) = \max \hat{\mathbb{M}}(F, m_{-i})
\]

11.2 Dominance, Equivalence and Feasibility

Before investigating the second stage of the game it’s useful to provide some preliminary analysis of the relationship between a seller’s payoff and their membership decision. As in the case of Game I, it can be shown that a seller’s profit in market \( k \) is an increasing function of the extra-surplus it receives.
That is, for all \( m_{-i} \in M \),
\[
\bar{\Gamma}_i(f_k, m_{-i}, \hat{m}) \leq \bar{\Gamma}_i(f_k, m_{-i}, \hat{m}) \iff W(f_k, \hat{m}) \leq W(f_k, \hat{m})
\]
This gives the following results regarding the dominance, feasibility and equivalence of membership options.

**Dominance.** One option, \( \hat{m} \), strictly dominates another, \( \check{m} \), if it leads to a higher end-user benefit in one market and at least matches the end-user benefit in the other market. That is, \( \check{m} > \hat{m} \) iff \( W(f_\check{c}, \check{m}) > W(f_\check{c}, \hat{m}) \), \( W(f_\check{d}, \check{m}) \geq W(f_\check{d}, \hat{m}) \) or \( W(f_\check{d}, \check{m}) \geq W(f_\check{d}, \hat{m}) \), \( W(f_\check{c}, \check{m}) > W(f_\check{c}, \hat{m}) \). Note that the dominance of one option relative to another depends only on the fees, \( F \).

**Equivalence.** Two options are "equivalent" iff they offer the same extra-surplus in both markets: \( m \sim m' \) iff \( W(f_k, m) = W(f_k, m') \), for all \( k \in \{c, d\} \). It can be seen that an option is trivially equivalent to itself. Hence, let two options be referred to as "distinct " iff they are not equivalent. Finally, note that if \( m \sim m' \), then \( \hat{M}(F, m) = \hat{M}(F, m') \) and so fees, \( F \), are such that a seller’s best-response to \( m \) is identical to their best-response to \( m' \).

**Weak Dominance.** One option, \( m \), weakly dominates another, \( m' \), if they are equivalent or \( m \) dominates \( m' \). That is, \( m \succeq m' \) iff \( m \sim m' \) or \( m' \succ m \).

**Feasibility.** The set of feasible options is the set of options that are not strictly dominated. Hence, \( m \) is "feasible" iff \( \nexists m' \in M : m' \succ m \).

### 11.3 Criteria for a Restricted SPNE

My criterion for a Restricted SPNE (RSPNE) is that unilateral deviation isn’t profitable and all sellers obey the decision rule in cases where they are indifferent. Essentially, the seller’s decision-rule acts as a mechanism for selecting among multiple equilibria. Let \( \mathcal{N}(F) (\subseteq M^3) \) denote the set of Restricted SPNE’s when fees are \( F \). Let subgame perfect Nash equilibria which also satisfy the decision rule be referred to as Restricted SPNE. My criterion for a RSPNE is based on fixed-points of the best-response functions. That is, \( \hat{m} \in \mathcal{N}(F) \) iff the following conditions are satisfied:

\[
\begin{align*}
\hat{m}_1 & = \max \hat{M}(F, \hat{m}_2, \hat{m}_3) \\
\hat{m}_2 & = \max \hat{M}(F, \hat{m}_1, \hat{m}_3) \\
\hat{m}_3 & = \max \hat{M}(F, \hat{m}_1, \hat{m}_2)
\end{align*}
\]

It’s useful to introduce the following terminology: a membership vector, \( m = (m_1, m_2, m_3) \), is said to be "stable" iff neither seller can profit through unilateral deviation. It can be seen that stability is a necessary condition for a RSPNE: if \( \hat{m} \in \mathcal{N}(F) \), then:

\[
\begin{align*}
\hat{m}_1 & \in \hat{M}(F; \hat{m}_2, \hat{m}_3) \\
\hat{m}_2 & \in \hat{M}(F; \hat{m}_1, \hat{m}_3) \\
\hat{m}_3 & \in \hat{M}(F; \hat{m}_1, \hat{m}_2)
\end{align*}
\]
This necessary condition for a RSPNE can be re-expressed more usefully as follows:

\[
\sum_k \tilde{\Gamma}_1(f_k; \tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \geq \max_{m \in M} \sum_k \tilde{\Gamma}_1(f_k; m, \tilde{m}_2, \tilde{m}_3) \\
\sum_k \tilde{\Gamma}_2(f_k; \tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \geq \max_{m \in \tilde{M}} \sum_k \tilde{\Gamma}_2(f_k; m_1, m, \tilde{m}_3) \\
\sum_k \tilde{\Gamma}_3(f_k; \tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \geq \max_{m \in \tilde{M}} \sum_k \tilde{\Gamma}_3(f_k; m_1, \tilde{m}_2, m)
\]

It often happens that, for a given combination of fees, \( F \), there is more than one stable membership vector. However, the decision rule makes it possible to select a unique outcome.

**Lemma 11.1 (Comparing Stable Vectors.)** Suppose that, given fees, \( F \), there are two stable membership vectors, \( m \) and \( \tilde{m} \). The decision rule can be used to select a unique outcome:

1. Suppose that \( m \) and \( \tilde{m} \) are identical except for the \( i^{th} \) element. Let the \( i^{th} \) element be higher in the second vector. That is, \( m_i < \tilde{m}_i \) and if \( i \neq j \), then \( m_j = \tilde{m}_j \). In this situation, the first vector, \( m \), is not a valid RSPNE: \( m \notin \mathcal{N}(F) \).

2. Suppose that for all \( j \in \{1, 2, 3\} \), \( m_j \sim \tilde{m}_j \) and \( m_j \leq \tilde{m}_j \). That is, either elements are identical or they are equivalent in which case \( \tilde{m} \) has the higher element. In this situation, the first vector, \( m \), is not a valid RSPNE: \( m \notin \mathcal{N}(F) \).

**Proof.** Consider the following examples:

Suppose that \( \tilde{m} = (\overline{m}, \tilde{m}_2, \tilde{m}_3) \) and \( m = (m, \tilde{m}_2, \tilde{m}_3) \) (where \( \overline{m} > m \)) are both stable, given a particular combination of fees, \( F \). By construction, these vectors differ only with respect to the membership of Seller 1. By assumption, both vectors are stable, which implies that \( \overline{m} \in \tilde{M}(F, \tilde{m}_2, \tilde{m}_3) \) and \( m \in \tilde{M}(F, m_2, m_3) \). Finally, since \( \overline{m} > m \), it follows that \( m \neq \max \tilde{M}(F, \tilde{m}_2, \tilde{m}_3) \). Therefore, \( m \notin \mathcal{N}(F) \).

Suppose that for all \( j \in \{1, 2, 3\} \), \( m_j \sim \tilde{m}_j \) and \( m_j \leq \tilde{m}_j \). Since \( m_j \sim \tilde{m}_j \), it follows that \( m_1 \in \tilde{M}(F, m_2, m_3) \) if \( m_1 \in \tilde{M}(F, \tilde{m}_2, \tilde{m}_3) \). (The same holds true if the sellers are rotated.) Hence, \( \max \tilde{M}(F, m_2, m_3) \neq m_1 \). Therefore, \( m \notin \mathcal{N}(F) \). \( \blacksquare \)

### 11.4 Independence of "Irrelevant" Alternatives

If \( m \succ m' \), then \( m' \) doesn’t belong to the set of feasible membership options. Hence, deleting \( m' \) from the set of membership options can’t affect a seller’s decision. Nor can it affect the set of RSPNE’s.

Now consider the equilibrium of a subgame in which fees, \( F \), are such that one option weakly dominates the other: \( m > m' \) and \( m \succeq m' \). Let \( \mathcal{N}(F) \)
denote the set of RSPNE for this subgame. Now consider the effect of removing \( m' \) from the seller’s set of options. That is, suppose that \( M \) becomes \( M \setminus \{m'\} \). Could this affect the set of RSPNE’s? There are two possibilities: (i) \( m > m' \); and (ii) \( m \sim m' \). These are analyzed in turn.

(i) If \( m > m' \), then \( m' \notin \hat{M}(F, m_i) \). This implies that a seller’s choice is not influenced by the presence of \( m' \). Hence, removing the option can have no effect on \( N(F) \).

(ii) If \( m \sim m' \), then \( m \in \hat{M}(F, m_i) \) iff \( m' \in \hat{M}(F, m_i) \). Since \( m' < m \), it follows that if \( m \sim m' \), then \( \max \hat{M}(F, m_i) \neq m' \). (For example, if \( b \geq a \), then \( \max \hat{M}(F, m_i) \neq a \) and so the option could be removed without affecting their choice.) Again, removing \( m' \) can’t affect \( N(F) \). Therefore, if \( m > m' \) or \( m \geq m' \), then

\[
\mathcal{N}(F) = \mathcal{N}(F) \setminus \{m'\} \times M^2
\]

That is, removing \( m' \) from a seller’s set of membership options has no effect on their choice or the RSPNE.

Let \( m' \) be referred to as "irrelevant" iff \( \exists m \in M: m > m' \) or \( m \geq m' \), \( m > m' \). An option is said to be "relevant" iff it’s not "irrelevant". When fees are \( F \), the set of relevant membership options is denoted by \( \hat{M}(F) \). It follows that

\[
\hat{M}(F) \equiv \left\{ m' \in M : \forall m \in M \setminus \{m'\}, \begin{cases} m \leq m' \\ m < m' \text{ or } m < m \end{cases} \right\}
\]

The concept of relevance is useful because it allows "irrelevant" strategies to be removed from the set of membership options without affecting the outcome. Let "simplification" refer to the process of removing irrelevant membership options in order to create "simplified" game. The original game and the simplified game have an identical set of RSPNE’s.

### 11.5 No More than Two Relevant Options

#### 11.5.1 Outside Options versus Joining Network \( B \)

The extra-surplus received by a seller when \( m_i = b \) is \( W(f, b) = \hat{W}(-\omega) = 0 \), \( W(f, b) = \hat{W}(y) \); where the end-user benefit generated by a platform with buyer-fee \( f \) is \( \hat{W}(f) \equiv \frac{1}{2\gamma}(\omega^2 - f^2) \). Previous analysis showed that: \( \hat{W}(f) < 0 \) iff \( f > \omega \). Hence, Network \( B \) offers zero end-user benefit in market \( \zeta \) (but a potentially positive end-user benefit in market \( \eta \)). Outside-options, \( m_i = o \), offer zero end-user benefit in both markets, which implies that:

- \( b < o \) iff \( z > \omega \)
- \( b \sim o \) iff \( z \leq \omega \)

That is, either one option dominates the other or they are equivalent. Furthermore, since \( o < b \), it follows that there are no fees, \( F \), such that \( \{o, b\} \subseteq \hat{M}(F) \). In other words, either \( o \) is irrelevant or \( b \) is irrelevant.
11.5.2 Multihoming versus Joining Network $A$

The extra-surplus received by a seller when $m_i = a$ is $W(f^*_i, a) = \hat{W}(x)$, $W(f^*_j, a) = \hat{W}(y)$. And the extra-surplus received by a seller when $m_i = h$ is $W(f^*_i, h) = \hat{W}(x)$, $W(f^*_j, h) = \hat{W}(\max\{y, z\})$. Hence, both options offer the same end-user benefit in market $\zeta$, which implies that

$$
\begin{align*}
  h &< a \iff \hat{W}(\max\{y, z\}) < \hat{W}(y) \\
  h &\geq a \iff \hat{W}(\max\{y, z\}) \geq \hat{W}(y)
\end{align*}
$$

Multihoming gives a strictly lower extra-surplus in market $\eta$ than that offered by Platform $D_A$ iff $\hat{W}(\max\{y, z\}) < \hat{W}(y)$. That is, either the options are equivalent or one option dominates the other. Finally, since $a < h$, it follows that there are no fees, $F$, such that $\{a, h\} \subseteq M(F)$. In other words, either $a$ is irrelevant or $h$ is irrelevant.\(^{1}\)

11.5.3 The Relevant Set

It has been shown that $\{o, b\} \not\subseteq M(F)$, $\{a, h\} \not\subseteq M(F)$. It follows that:

**Lemma 11.2 (Sellers’ Dichotomy.)** The fees in markets $\zeta, \eta$ are $f^*_\zeta = (x, -\omega)$, $f^*_\eta = (y, z)$ where $z, y, z \in [-\omega, \rho]$. Let $M(F)(\subseteq M)$ denote the set of "relevant" membership options given a particular matrix of fees, $F = (f^*_T, f^*_T)$. It follows that:

1. Sellers never have more than two "relevant" membership options: $|M(F)| \leq 2$, $\forall F \in F$.

2. The elements in $M(F)$ depend on $x, y, z$. There are four possible situations:

   $$
   \begin{align*}
   &\hat{W}(z) < 0, \hat{W}(\max\{y, z\}) < \hat{W}(y) \Rightarrow M(F) \subseteq \{o, a\} \\
   &\hat{W}(z) < 0, \hat{W}(\max\{y, z\}) \geq \hat{W}(y) \Rightarrow M(F) \subseteq \{o, h\} \\
   &\hat{W}(z) \geq 0, \hat{W}(\max\{y, z\}) < \hat{W}(y) \Rightarrow M(F) \subseteq \{b, a\} \\
   &\hat{W}(z) \geq 0, \hat{W}(\max\{y, z\}) \geq \hat{W}(y) \Rightarrow M(F) \subseteq \{b, h\}
   \end{align*}
$$

11.6 Existence and Uniqueness

It has been shown that there are never more than two relevant membership options. It can be shown that this implies that there always exists a RSPNE. Furthermore, the equilibrium is unique up to an interchange of sellers. (See Appendix A.)

\(^{1}\)If $h \sim a$, then $h \in \tilde{M}(F, m_{-i})$ iff $a \in \tilde{M}(F, m_{-i})$. Furthermore, since $a = 2 < 3 = h$, it follows that if $h \sim a$, then $\max\tilde{M}(F, m_{-i}) \neq a$. Therefore, if $h \geq a$, then $\max\tilde{M}(F, m_{-i}) \neq a$ and so the option could be removed without affecting their choice.
Lemma 11.3 (Existence and Uniqueness of RSPNE’s.) If there are no more than two relevant membership options, then a RSPNE exists and is unique up to a relabelling of the sellers. That is, if $|\mathbb{M}(F)| \leq 2$, then the set of RSPNE’s, $\mathcal{N}(F)$, has the following properties:

1. The set of equilibria isn’t empty: $\mathcal{N}(F) \neq \emptyset$, $\forall F \in \mathbb{F}$.

2. If $\tilde{m} \in \mathcal{N}(F)$, then for all $m' \in \mathbb{M}^3$, if there exists $m \in \mathbb{M}$ such that $\sum_j 1(\tilde{m}_j = m) \neq \sum_j 1(m'_j = m)$, then $m' \notin \mathcal{N}(F)$. 
Chapter 12

Eight Membership Configurations

Chapter 12 shows that there are eight possible membership configurations. The chapter contains the following sections. Section (1) shows that a seller’s extra-surplus depends on which network they’re on. Section (2) classifies configurations by introducing a three digit-code based on the number of sellers on each platform. Section (3) states that the number of sellers on a platform can’t exceed the total number of sellers in the game. Section (4) states that it’s impossible to have an equilibrium in which two sellers receive the same extra-surplus in market $\mathcal{C}$ but an unequal extra-surplus in market $\mathcal{D}$. Section (5) shows that the number of sellers on platforms $D_A$ and $D_B$. Section (6) combines all the proceeding results to show that there are only eight possible configurations.

12.1 Being "on" a Platform

Before analyzing the RSPNE it is useful to discuss the criteria for a seller to be "on" a particular platform. Sellers can join both networks; that is, they can "multihome". However, since it’s customers who decide which (if any) platform to use, it’s possible for sellers to join a platform without being "on" it. Because of this it’s useful to introduce some extra terminology: Firstly, a seller is "on" a platform if it’s a member and at least some of its customers use it:

- A seller is "on" Platform $C_A$ iff $A(f_\mathcal{C}, m_i) = 1$.
- A seller is "on" Platform $D_A$ iff $A(f_\mathcal{D}, m_i) = 1$.
- A seller is "on" Platform $D_B$ iff $B(f_\mathcal{D}, m_i) = 1$.

Secondly, a seller is "on" Network $A$ if it’s on both Platform $C_A$ and Platform $D_A$.

To be "on" a platform, a seller has to be a member, but there is an additional complication in market $\mathcal{D}$: if it’s a member of both Platform $D_A$ and Platform $D_B$, then they are "on" whichever has the lower buyer-fee because, given the choice, buyers never use the more expensive platform. The necessary and sufficient conditions for being on the platforms are as follows:

- A seller is "on" Platform $C_A$ iff $m_i \in \{a, b\}$.
• A seller is "on" Platform D_A iff one of the following is true: either $m_i = a$ or $m_i = h$ and $y \geq z$.

• A seller is "on" Platform D_B iff one of the following is true: either $m_i = b$ or $m_i = h$ and $y < z$.

Finally, the extra-surplus received by a seller depends on which platforms they are on. The extra-surplus received by a seller in market $c$ is

$$W(f_c, m_i) = \begin{cases} \hat{W}(x) & \text{if } A(f_c, m_i) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and the extra-surplus received by a seller in market $d$ is

$$W(f_d, m_i) = \begin{cases} \hat{W}(y) & \text{if } A(f_d, m_i) = 1 \\ \hat{W}(z) & \text{if } B(f_d, m_i) = 1 \\ 0 & \text{otherwise} \end{cases}$$

### 12.2 Classifying Equilibria

Suppose that in equilibrium the sellers’ membership is $\tilde{m} \in \mathcal{N}(F)$. The number of sellers "on" Network A in markets $c$ and $d$ becomes

$$\tilde{\Sigma}_A(f_c, \tilde{m}) = \sum_i A(f_c, \tilde{m}_i) = \sum_i 1(m_i \in \{a, h\})$$

and

$$\tilde{\Sigma}_A(f_d, \tilde{m}) = \sum_i A(f_d, \tilde{m}_i) = \sum_i 1(m_i = a) + \sum_i 1(m_i = h).1(z \leq y)$$

The number of sellers on Network B in market $d$ is

$$\tilde{\Sigma}_B(f_d, \tilde{m}) = \sum_i B(f_d, \tilde{m}_i) = \sum_i 1(m_i = b) + \sum_i 1(m_i = h).1(z > y)$$

It is useful to classify the equilibria within $\mathcal{N}(F)$ according to the number of sellers "on" each platform. Let $\mathcal{N}_{IJK}(F)(\subseteq \mathcal{N}(F))$ denote the set of equilibria in which the number of sellers "on" platforms C_A, D_A, D_B is I, J, K, respectively. From these definitions it can be seen that: $\tilde{m} \in \mathcal{N}_{IJK}(F)$ iff $\tilde{m} \in \mathcal{N}(F)$ and $\tilde{\Sigma}_A(f_c, \tilde{m}) = I$, $\tilde{\Sigma}_A(f_d, \tilde{m}) = J$, $\tilde{\Sigma}_B(f_d, \tilde{m}) = K$.

The sets are mutually exclusive and exhaust the possibilities. However, some of these sets could be empty. Indeed, a number of these configurations can be ruled out as impossible. It’s useful to identify such configurations because it reduces the number of situations that need to be analyzed.
12.3 Adding Up Rules

A seller is "on" Platform $D_A$ iff $A(f_q, m_i) = 1$, which requires that either $m_i = a$ or $m_i = h$, $z \leq y$. Similarly, a seller is "on" Platform $D_B$ iff $B(f_q, m_i) = 1$, which requires that either $m_i = b$ or $m_i = h$, $z > y$. Therefore, it's impossible for a seller to be "on" more than one platform in market $q$. Since there are three sellers, this implies that: $J + K \leq 3$.

Firstly, a seller is "on" Platform $D_A$ iff $A(f_q, m_i) = 1$, a necessary condition for which is $m_i \in \{a, h\}$. Secondly, $A(f_q', a) = A(f_q', h) = 1$, where $f_q = (x, -\omega)$. Therefore, if $A(f_q, m_i) = 1$, then $A(f_q', m_i) = 1$. That is, it's impossible for a seller to be "on" Network $A$ in market $q$ without also being "on" Network $A$ in market $c$. This implies that: $J \leq I$.

12.4 Consistency of Choices

Suppose that sellers’ membership, $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$, is such that all sellers receive the same extra-surplus in market $c$, e.g. $W(f_c', \tilde{m}_1) = W(f_c', \tilde{m}_2)$, but receive different levels of extra-surplus in market $q$, e.g. $W(f_q', \tilde{m}_1) \neq W(f_q', \tilde{m}_2)$. It can be shown that $\tilde{m}$ can’t be a RSPNE.

**Lemma 12.1** (Parity in $c$ implies Parity in $q$.) If $W(f_c', \tilde{m}_1) = W(f_c', \tilde{m}_2)$ and $W(f_q', \tilde{m}_1) \neq W(f_q', \tilde{m}_2)$, then $(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \notin N(F)$. Finally, the result holds for any permutation of the sellers.

**Proof.** Firstly, any RSPNE must be stable in the sense that neither seller can have an incentive to unilaterally deviate. Secondly, $m_1$ and $m_2$ deliver the same extra-surplus in market $c$ but differing levels of extra-surplus in market $q$. Hence, either $m_1 < m_2$ or $m_1 > m_2$; that is, one option must strictly dominate the other. Therefore, one of the sellers has an incentive to unilaterally deviate. It follows that there is no RSPNE in which two sellers receive the same extra-surplus in market $c$ but receive different levels of extra-surplus in market $q$.

For any RSPNE, if two sellers are on Platform $C_A$, then they made the same membership choice.

**Lemma 12.2** (All Sellers "on" $C_A$, Chose the Same Option.)

If $A(f_c, \tilde{m}_1) = A(f_c, \tilde{m}_2)$ and $\tilde{m} \in N(F)$, then $\tilde{m}_1 = \tilde{m}_2$. Finally, the result holds for any pair of sellers.

**Proof.** Suppose $\tilde{m} \in N(F)$ and $A(f_c, \tilde{m}_1) = A(f_c, \tilde{m}_2)$. That is, there is a RSPNE in which two sellers are [or are not] on Platform $C_A$. This implies that they receive the same extra-surplus in market $c$. Hence, a necessary condition for this to be a RSPNE is that these sellers receive the same extra-surplus in market $q$. Hence, the previous lemma implies that if $\tilde{m}$ is a RSPNE, then the sellers must have chosen equivalent membership options, $\tilde{m}_1 \sim \tilde{m}_2$. Finally, $\tilde{m} \in N(F)$ iff it is a fixed-point of the sellers’ best-response functions:
\[ m_i = \max \tilde{M}(F, \tilde{m}_{-i}) \] for \( i = 1, 2, 3 \). Furthermore, since \( m_1 \sim m_2 \), it follows that \( \tilde{M}(F, \tilde{m}_{-1}) = \tilde{M}(F, \tilde{m}_{-2}) \). This implies that:
\[ \tilde{m}_1 = \max \tilde{M}(F, \tilde{m}_2, \tilde{m}_3) = \max \tilde{M}(F, \tilde{m}_1, \tilde{m}_3) = \tilde{m}_2 \]

Therefore, the sellers must have made identical membership choices, \( \tilde{m}_1 = \tilde{m}_2 \).

Suppose that sellers’ membership, \( \tilde{m} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \), is such that all sellers receive the same extra-surplus in market \( q \), \( W(f_q, \tilde{m}_1) = W(f_q, \tilde{m}_2) \), but receive different levels of extra-surplus in market \( c \), \( W(f_c, \tilde{m}_1) \neq W(f_c, \tilde{m}_2) \). It can be shown that \( \tilde{m} \) can’t be a RSPNE.

**Lemma 12.3 (Parity in \( q \) implies Parity in \( c \).)** If \( W(f_q, \tilde{m}_1) = W(f_q, \tilde{m}_2) \) and \( W(f_c, \tilde{m}_1) \neq W(f_c, \tilde{m}_2) \), then \( (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \notin \mathcal{N}(F) \). Finally, this holds for any pair of sellers.

It follows that:

**Lemma 12.4 (All Sellers "on" \( D_B \), Chose the Same Option.)** If \( B(f_q, \tilde{m}_1) = 1 \), \( B(f_q, \tilde{m}_2) = 1 \) and \( \tilde{m} \in \mathcal{N}(F) \), then \( \tilde{m}_1 = \tilde{m}_2 \). Finally, this holds for any pair of sellers.

**Proof.** Suppose \( \tilde{m} \in \mathcal{N}(F) \) and \( B(f_q, \tilde{m}_1) = 1 \), \( B(f_q, \tilde{m}_2) = 1 \). That is, there is a RSPNE in which two sellers are on Platform \( D_B \). This implies that they receive the same extra-surplus in market \( q \). Hence, a necessary condition for this to be a RSPNE is that these sellers receive the same extra-surplus in market \( c \). Hence, the previous lemma implies that if \( \tilde{m} \) is a RSPNE, then the sellers must have chosen equivalent membership options, \( \tilde{m}_1 \sim \tilde{m}_2 \). Finally, \( \tilde{m} \in \mathcal{N}(F) \) iff it’s a fixed-point of the sellers’ best-response functions:
\[ \tilde{m}_i = \max \tilde{M}(F, \tilde{m}_{-i}) \] for \( i = 1, 2, 3 \). Furthermore, since \( \tilde{m}_1 \sim \tilde{m}_2 \), it follows that \( \tilde{M}(F, \tilde{m}_{-1}) = \tilde{M}(F, \tilde{m}_{-2}) \). This implies that:
\[ \tilde{m}_1 = \max \tilde{M}(F, \tilde{m}_2, \tilde{m}_3) = \max \tilde{M}(F, \tilde{m}_1, \tilde{m}_3) = \tilde{m}_2 \]

Therefore, sellers must have made identical membership choices, \( \tilde{m}_1 = \tilde{m}_2 \).

This implies that if \( K = 3 \), then \( I \in \{0, 3\} \). That is, if all sellers are "on" Platform \( D_B \), then either they’re are all "on" Platform \( C_A \) or no sellers are "on" Platform \( C_A \).

### 12.5 Constraints on \( J \) and \( K \), given \( I \)

Suppose that \( I \) sellers are "on" Platform \( C_A \). In any RSPNE, sellers divide into two sets. \( \text{Set}(1) \subseteq \{1, 2, 3\} \) denotes the set of sellers "on" Platform \( C_A \); and \( \text{Set}(2) \subseteq \{1, 2, 3\} \) denotes the set of sellers not "on" Platform \( C_A \). Note that, by definition, \( |\text{Set}(1)| = I \) and \( |\text{Set}(2)| = 3 - I \).
Lemma 12.5 (Constraints on J.) If \( \Sigma_A(f, \bar{m}) = I \), \( \Sigma_A(f, \bar{m}) = J \) and \( \bar{m} \in N(F) \), then \( J \in \{0, I\} \).

Proof. The first set of sellers are either all "on" Platform D_A or none of them are "on" Platform D_A. However, it’s not possible for the second set of sellers to be on Platform D_A (given that they must have chosen o or b). It follows that \( J = I.A(f, \bar{m}) \), where \( \bar{m} \in \{a, b\} \).

Lemma 12.6 (Constraints on K.) If \( \Sigma_A(f, \bar{m}) = I \), \( \Sigma_B(f, \bar{m}) = K \) and \( \bar{m} \in N(F) \), then \( K \in \{0, I, 3 - I, 3\} \).

Proof. The first set of sellers are either all "on" Platform D_B or none of them are "on" Platform D_B. (The first outcome only occurs if they multihome and Network B has the lower buyer-fee.) Similarly, the second set of sellers are either all "on" Platform D_B or none of them are "on" Platform D_B. It follows that
\[
K = I.B(f, \bar{m}) + (3 - I).B(f, \bar{m}),
\]
where \( \bar{m} \in \{a, b\} \) and \( \bar{m} \in \{a, b\} \). This suggests that \( K \in \{0, I, 3 - I, 3\} \).

It can be shown that in any RSPNE in which Seller 1 is "on" Platform C_A and Seller 2 isn’t "on" Platform C_A, only Seller 1 is "on" Platform D_A (whereas, Seller 2 is either "on" Platform D_B or uses outside-options). Hence, if sellers are asymmetric in market \( \mathcal{G} \), then they are also asymmetric in market \( \mathcal{D} \).

Lemma 12.7 (Further Constraints on J.) If \( \Sigma_A(f, \bar{m}) = I \), \( \Sigma_A(f, \bar{m}) = J \) and \( \bar{m} \in N(F) \), then \( I \notin \{0, 3\} \Rightarrow J = I \).

Proof. This proof has two parts:

It can be shown that if \( I \notin \{0, 3\} \), then \( J \neq 0 \). This uses a proof by contradiction:

Suppose that \( I \notin \{0, 3\} \) and \( J = 0 \). Firstly, this implies that there are two sets of sellers: Set (1) denotes the set of sellers on Platform C_A; and Set (2) denotes the set of sellers not on Platform C_A. It’s known that all the sellers in a given set made the same membership decision. The membership of sellers in Set (1) is \( \bar{m} \in \{a, b\} \). And the membership of sellers in Set (2) is \( \bar{m} \in \{a, b\} \).

Secondly, by assumption, no sellers are on Platform D_A (that is, \( J = 0 \)). It follows that \( \bar{m} = b \) and \( z > y \). That is, sellers in Set (1) multihome and Network B has the lower buyer-fee. Hence, the sellers in Set
(1) are on Platform $D_B$. In other words, $B(f_d, \hat{m}) = 1$. This implies that $K = I + (3 - I)B(f_d, \hat{m})$, where $\hat{m} \in \{a, b\}$. Therefore, $K \in \{I, 3\}$.

Thirdly, it can be seen that $K \neq 3$. Otherwise, all sellers would receive the same extra-surplus in market $d$, which is known to be incompatible with $I \notin \{0, 3\}$. Hence, $K = I$.

Finally, consider the sellers in Set (2). It can be seen that $B(f_d, \hat{m}) = 0$, otherwise, the number of sellers on Platform $D_B$ would exceed $I$. Hence, the sellers in Set (2) must use outside-options: $\hat{m} = o$. This requires $b < a$, which implies that $z > \omega$.

To summarize: $z > y$ and $z > \omega$. From a sketch of $W(f)$, it can be seen that Platform $D_A$ offers a strictly higher end-user benefit than Platform $D_B$. Since $W(y) > W(z)$, it follows that $a > h$, which is incompatible with $\hat{m} = h$. (That is, sellers in Set (1) can profitably deviate.) Therefore, the original supposition must be rejected. This implies that if $I \notin \{0, 3\}$, then $J \neq 0$.

It has been shown that $J \in \{0, I\}$. Since $I \notin \{0, 3\} \Rightarrow J \neq 0$, this implies that $I \notin \{0, 3\} \Rightarrow J = I$. 

If $I \notin \{0, 3\}$, then there are two groups of sellers: Set (1) denotes those on Platform $C_A$; and Set (2) denotes the set of sellers not on Platform $C_A$. Sellers in the first set chose $\hat{m} \in \{a, h\}$ and those in the second set chose $\hat{m} \in \{a, b\}$. In this situation, only sellers in Set (2) must use outside-options, $m_i = o$, then Network $B$ is inactive: $K = 0$.

**Lemma 12.8** (*Further Constraints on K.*) If $\hat{\Sigma}_A(f', \hat{m}) = I$, $\hat{\Sigma}_A(f_d, \hat{m}) = J$ and $\hat{m} \in \mathcal{N}(F)$, then $I \notin \{0, 3\} \Rightarrow K \in \{0, 3 - I\}$.

**Proof.** By assumption $I \notin \{0, 3\}$. This implies that there are two groups of sellers: Set (1) denotes those on Platform $C_A$ who chose $\hat{m} \in \{a, h\}$; and Set (2) denotes the set of sellers not on Platform $C_A$ who chose $\hat{m} \in \{a, b\}$. The sellers in Set (1) are either all "on" Platform $D_B$ or none of them are "on" Platform $D_B$. (The first outcome only occurs if they multihome and Network B has the lower buyer-fee.) Similarly, sellers in Set (2) are either all "on" Platform $D_B$ or none of them are "on" Platform $D_B$. It follows that

$$K = I.B(f_d, \hat{m}) + (3 - I).B(f_d, \hat{m}),$$

where $\hat{m} \in \{a, h\}$ and $\hat{m} \in \{a, b\}$. This suggests that $K \in \{0, I, 3 - I, 3\}$. The following analysis show that $I \notin \{0, 3\} \Rightarrow K \in \{0, 3 - I\}$. The argument has two parts:

It’s already been shown that $I \notin \{0, 3\} \Rightarrow J = I$. Since $J + K \leq 3$, it follows that $K \leq 3 - I$. Therefore, $I \notin \{0, 3\} \Rightarrow K \neq 3$.

It can be shown that there is no RSPNE in which $K = I$ (where $I \neq 3 - I$). This can be seen using proof by contradiction:

Suppose that there was a RSPNE in which $K = I$ (and $I \neq 3 - I$). Firstly, this outcome requires: (i) $B(f_d, \hat{m}) = 1$, where $\hat{m} \in \{a, h\}$; and (ii) $B(f_d, \hat{m}) = 0$, where $\hat{m} \in \{a, b\}$.
Secondly, it can be seen that \( B(f_d, \hat{m}) = 1 \) iff \( \hat{m} = h, \ z > y \). That is, the first set of sellers multihome and are on Platform \( D_B \) because it has the lower buyer-fee. A necessary condition for this to occur in a RSPNE is \( h \succeq a \).

Thirdly, it can be seen that \( B(f_d, \hat{m}) = 0 \) iff \( \hat{m} = o \). That is, the second set of sellers use outside-options and so aren’t on Platform \( D_B \). A necessary condition for this to occur in a RSPNE is \( b \). This is consistent with the earlier requirement that \( h \succeq a \). Therefore, the original supposition must be rejected. It follows that \( K \neq I \).

12.6 Possible Membership Configurations

There are three possibilities: (1) \( I = 0 \); (2) \( I \notin \{0, 3\} \); and (3) \( I = 3 \). These can be analyzed in turn to determine the possible values of \( J \):

Firstly, if \( I = 0 \), then all sellers receive the same extra-surplus in market \( c \) (that is, none). This implies that all sellers made the same membership decision: \( \hat{m} \in \{a, b\} \). Since no sellers are "on" Network \( A \), it follows that \( J = 0 \).

Secondly, if \( I \notin \{0, 3\} \), then \( J = I \).

Finally, if \( I = 3 \), then all sellers receive the same extra-surplus in market \( c \); which implies that all sellers made the same choice: \( \hat{m} \in \{a, h\} \). This implies that either all sellers are "on" Network \( D_A \) or no sellers are "on" Network \( D_A \).

The findings can be summarized as follows:

1. If \( I = 0 \), then \( J = 0 \).
2. If \( I \notin \{0, 3\} \), then \( J = I \).
3. If \( I = 3 \), then \( J \in \{0, 3\} \).

The next stage is to analyze the possible values of \( K \) in each case.

Firstly, if \( I = 0 \), then all sellers receive the same extra-surplus in market \( c \); which implies that all sellers made the same choice, \( \hat{m} \in \{a, b\} \). This implies that \( K \in \{0, 3\} \).

Secondly, if \( I \notin \{0, 3\} \), then \( K \in \{0, 3 - I\} \).

Finally, if \( I = 3 \), then all sellers receive the same extra-surplus in market \( c \); which implies that all sellers made the same choice, \( \hat{m} \in \{a, h\} \). This is only possible if \( K \in \{0, 3\} \), \( J \in \{0, 3\} \). Furthermore, the sellers must be on one of the platforms, which implies that \( J = 3 \) or \( K = 3 \). Since \( J + K \leq 3 \), it follows that there are two possibilities: (a) \( J = 3, K = 0 \); (b) \( J = 0, K = 3 \).

Proposition 12.1 (Eight Membership Configurations.) Suppose that \( \Sigma_A(f_c, \hat{m}) = I, \Sigma_A(f_d, \hat{m}) = J \) and \( \Sigma_B(f_d, \hat{m}) = K \), where \( \hat{m} \in N(F) \). The membership configurations which can occur in a RSPNE are summarized as follows:

\[ \text{It can be seen that } J = 0 \text{ would require: } \hat{m} = h \text{ and } z > y. \text{ Otherwise, } J = 3. \]
1. If $I = 0$, then $J = 0$ and $K \in \{0, 3\}$

2. If $I \notin \{0, 3\}$, then $J = I$ and $K \in \{0, 3 - I\}$

3. If $I = 3$, then one of the following occurs:

   (a) $J = 3$, $K = 0$;
   
   (b) $J = 0$, $K = 3$.

It can be seen that there are eight possible configurations. The three-digit codes ($IJK$) for these configurations are as follows:

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<th>$\Sigma_A(f_d, \tilde{m})$</th>
<th>$\Sigma_B(f_d, \tilde{m})$</th>
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<tr>
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</tr>
</tbody>
</table>

*Table 1. Eight Configurations*
Chapter 13

Three Classes of Subgame

Chapter 13 classifies subgames and finds possible configurations for each type of subgame. The sections are as follows. Section (1) identifies conditions under which the tie is "slack". Section (2) classifies subgames according to the size of $z$. Section (3) identifies possible configurations when $z \leq 0$. Section (4) identifies possible configurations when $0 < z \leq \omega$. Section (5) identifies possible configurations when $z > \omega$. Section (6) summarizes the results.

13.1 The Tie is either "Slack" or "Binding"

By tying its platforms Network $A$ threatens to bar sellers from Platform $C_A$ (its monopoly platform) unless they agree to join Platform $D_A$. However, sellers always have the option of multihoming. Furthermore, if Network $B$ has a higher seller-fee, then any seller that multihomes is on Platform $D_B$ (rather than Platform $D_A$). That is, if Network $B$ has the higher seller-fee, then sellers can be on Platform $C_A$ and Platform $D_B$. In this situation, the tie doesn’t alter their membership decision. It’s useful to introduce the following terminology:

The tie is "slack" iff both of the following conditions are satisfied:

- Network $B$ has the (strictly) higher seller-fee: $y < z$.
- Network $B$ offers the higher end-user benefit in market $d$, $\hat{W}(y) \leq \hat{W}(z)$.
  This requires: $|y| \geq |z|$.

Whereas, the tie is "binding" iff one of the following conditions is satisfied:

- Network $A$ has the higher seller-fee: $y \geq z$.
- Network $A$ offers the higher end-user benefit in market $d$, $\hat{W}(y) \geq \hat{W}(z)$.
  This requires: $|y| \leq |z|$.
Figure 5. Slackness when $z \leq 0$

Figure 6. Slackness when $z > 0$
13.2 Classifying Subgames

Once fees have been determined sellers make their membership decision. It has been shown that there are eight possible membership configurations. By making further assumptions about the fees it’s possible to identify the configurations that can occur in a Restricted SPNE (RSPNE). The subgames (SG) can be classified according to the nature of the seller-fees set by the networks in market $q$:

[1] Suppose that $W(z) \geq 0$ and $z \leq 0$. This requires: $-\omega \leq z \leq 0$. There are two sub-cases:

- The tie is binding: $y \geq -|z| = z$.
- The tie is slack: $y < -|z| = z$.

[2] Suppose that $W(z) \geq 0$ and $z > 0$. This requires: $0 < z \leq \omega$. There are two sub-cases:

- The tie is binding: $y > -z$.
- The tie is slack: $y \leq -z$.

[3] Suppose that $W(z) < 0$. This requires: $\omega < z \leq \rho$.

It can be seen that these types of subgame are mutually exclusive and exhaust the possibilities.

![Figure 7. Classifying Subgames](image-url)
13.3 Configurations in Subgame [1]

In this section suppose that Network B subsidizes sellers: \( z \leq 0 \). It’s useful to consider two sub-cases: (a) \( y \geq -|z| \); and (b) \( y < -|z| \).

Lemma 13.1 (Possible Configurations in SG [1a].) Suppose that \( z \leq 0 \) and \( y \geq -|z| \). It follows that: \( \mathcal{N}_{220}(F) = \emptyset \), \( \mathcal{N}_{110}(F) = \emptyset \), \( \mathcal{N}_{000}(F) = \emptyset \), \( \mathcal{N}_{303}(F) = \emptyset \).

Proof. If \( z \leq 0 \), then \( \tilde{W}(z) \geq 0 \), which implies that \( b \succeq a \). Since \( b > a \), it follows that \( a \notin \mathcal{M}(F) \). Hence, all sellers join one of the networks. That is, if \( \tilde{m} \in \mathcal{N}(F) \subseteq \mathcal{M}(F)^3 \), then:

\[
\tilde{\Sigma}_A(f_{\tilde{q}}, \tilde{m}) + \tilde{\Sigma}_B(f_{\tilde{q}}, \tilde{m}) = 3
\]

Therefore, \( \mathcal{N}_{220}(F) = \emptyset \), \( \mathcal{N}_{110}(F) = \emptyset \), \( \mathcal{N}_{000}(F) = \emptyset \).

If \( \mathcal{N}(F) \subseteq \mathcal{N}_{303}(F) \), then \( \tilde{\Sigma}_A(f_{\tilde{q}}, \tilde{m}) = 3 \), \( \tilde{\Sigma}_B(f_{\tilde{q}}, \tilde{m}) = 3 \). It follows that all the sellers multihome, \( \mathcal{N}(F) = \{e\}^3 \), and Network B has the lower seller-fee, \( y < z \). However, by assumption, \( y \geq z \). Therefore, \( \mathcal{N}_{303}(F) = \emptyset \). \( \blacksquare \)

Lemma 13.2 (Possible Configurations in SG [1b].) Suppose that \( z \leq 0 \) and \( y < -|z| \). It follows that:

\( \mathcal{N}(F) \subseteq \mathcal{N}_{303}(F) \cup \mathcal{N}_{003}(F) \)

Proof. It can be shown that if \( z \leq 0 \) and \( y < z \), then \( \mathcal{M}(F) \subseteq \{b, h\} \). The argument is as follows. Firstly, it’s been shown that if \( z \leq 0 \), then \( a \notin \mathcal{M}(F) \).

Secondly, if \( z \leq 0 \) and \( y < z \), then \( \tilde{W}(\max\{y, z\}) > \tilde{W}(y) \), which implies that \( h > a \). (Make a sketch of \( \tilde{W}(y) \) and \( \tilde{W}(\max\{y, z\}) \).) It follows that \( a \notin \mathcal{M}(F) \). Therefore, \( \mathcal{M}(F) \subseteq \{b, h\} \).

Furthermore, since \( y < z \), it follows that \( \tilde{W}(\max\{y, z\}) = \tilde{W}(z) \), which implies that either \( h \succeq b \) or \( h < b \). (The first case occurs when \( \tilde{W}(x) \geq 0 \). The second case occurs when \( \tilde{W}(x) < 0 \).) Since \( h > b \), it follows that either \( \mathcal{M}(F)^3 = \{h\}^3 \) (where \( y < z \)) or \( \mathcal{M}(F)^3 = \{b\}^3 \). Therefore,

\( \mathcal{N}(F) \subseteq \mathcal{N}_{303}(F) \cup \mathcal{N}_{003}(F) \)

\( \blacksquare \)

13.4 Configurations in Subgame [2]

In this section suppose that Network B offers a positive end-user benefit and sets a positive seller-fee: \( 0 < z \leq \omega \). It’s useful to consider two sub-cases: (a) \( y > -z \); and (b) \( y \leq -z \).

Lemma 13.3 (Possible Configurations in SG [2a].) Suppose that \( 0 < z \leq \omega \) and \( y > -z \). It follows that: \( \mathcal{N}_{220}(F) = \emptyset \), \( \mathcal{N}_{110}(F) = \emptyset \), \( \mathcal{N}_{000}(F) = \emptyset \), \( \mathcal{N}_{303}(F) = \emptyset \).
**Proof.** By assumption \( \hat{W}(z) \geq 0 \), which implies that \( b \geq a \). Since \( b > a \), it follows that \( a \notin \mathbb{M}(F) \). Hence, all sellers join one of the networks. That is, if \( \hat{m} \in \mathcal{N}(F) \subseteq \mathbb{M}(F)^3 \), then:

\[
\sum_A(f_d, \hat{m}) + \sum_B(f_d, \hat{m}) = 3
\]

Therefore, \( \mathcal{N}_{220}(F) = \emptyset \), \( \mathcal{N}_{110}(F) = \emptyset \), \( \mathcal{N}_{000}(F) = \emptyset \). If \( \mathcal{N}(F) \subseteq \mathcal{N}_{303}(F) \), then \( \sum_A(f_d, \hat{m}) = 3 \), \( \sum_B(f_d, \hat{m}) = 3 \). It follows that:

- All the sellers multihome, \( \mathcal{N}(F) = \{b\}^3 \subseteq \mathbb{M}(F)^3 \).
- Network B has the lower seller-fee, \( y < z \).

However, if \( 0 < z \leq \omega \) and \( y > -z \), then these conditions are never jointly satisfied. The argument is as follows. Firstly, if \( 0 < z \leq \omega \) and \( z > y > -z \), then \( W(\max\{y, z\}) < W(y) \), which implies that \( \hat{h} < a \). (Make a sketch of \( W(\max\{y, z\}) \) and \( W(y) \).) It follows that \( \hat{h} \notin \mathbb{M}(F) \). Hence, the first condition isn’t satisfied. Secondly, if \( y \geq z \), then clearly the second condition isn’t satisfied. Therefore, if \( 0 < z \leq \omega \) and \( y > -z \), then \( \mathcal{N}_{303}(F) = \emptyset \).

**Lemma 13.4 (Possible Configurations in SG [2b].)** Suppose that \( 0 < z \leq \omega \) and \( y \leq -z \). It follows that:

\[
\mathcal{N}(F) \subseteq \mathcal{N}_{303}(F) \cup \mathcal{N}_{003}(F)
\]

**Proof.** It can be shown that if \( z > 0 \) and \( y \leq -z \), then \( \mathbb{M}(F) \subseteq \{b, \hat{h}\} \). The argument is as follows. Firstly, it’s been shown that \( a \notin \mathbb{M}(F) \). Secondly, if \( 0 < z \leq \omega \) and \( y \leq -z \), then \( \hat{W}(\max\{y, z\}) \geq \hat{W}(y) \), which implies that \( \hat{h} \geq a \). (Make a sketch of \( \hat{W}(y) \) and \( \hat{W}(\max\{y, z\}) \).) Since \( \hat{h} > a \), it follows that \( a \notin \mathbb{M}(F) \). Therefore, \( \mathbb{M}(F) \subseteq \{b, \hat{h}\} \).

Furthermore, since \( y \leq -z < z \), it follows that \( \hat{W}(\max\{y, z\}) = \hat{W}(z) \), which implies that either \( \hat{h} \geq b \) or \( \hat{h} < b \). (The first case occurs when \( \hat{W}(x) \geq 0 \). The second case occurs when \( \hat{W}(x) < 0 \).) Since \( \hat{h} > b \), it follows that either \( \mathbb{M}(F)^3 = \{b\}^3 \) (where \( y < z \)) or \( \mathbb{M}(F)^3 = \{b\}^3 \). Therefore,

\[
\mathcal{N}(F) \subseteq \mathcal{N}_{303}(F) \cup \mathcal{N}_{003}(F)
\]

**13.5 Configurations in Subgame [3]**

In this section suppose that Network B offers a negative end-user benefit: \( z > \omega \).

**Lemma 13.5 (Possible Configurations in SG [3].)** Suppose that \( z > \omega \). It follows that: \( \mathcal{N}_{221}(F) = \emptyset \), \( \mathcal{N}_{112}(F) = \emptyset \), \( \mathcal{N}_{003}(F) = \emptyset \), \( \mathcal{N}_{303}(F) = \emptyset \).

**Proof.** Suppose that \( \hat{W}(z) < 0 \), which requires \( z > \omega \). It can be shown that if \( \hat{m} \in \mathcal{N}(F) \), then \( \sum_B(f_d, \hat{m}) = 0 \). The argument is as follows. By assumption, \( \hat{W}(z) < 0 \), which implies that \( b < a \). It follows that \( b \notin \mathbb{M}(F) \). Therefore, if \( \hat{m} \in \mathcal{N}(F) \subseteq \mathbb{M}(F)^3 \) and \( \sum_B(f_d, \hat{m}) > 0 \), then:
• One of the sellers multihomes: \( \tilde{m} \in \{ \tilde{h} \} \times \overline{M}(F)^2 \subseteq \overline{M}(F)^3 \).

• Network \( B \) has the higher seller-fee: \( y < z \).

However, if \( z > \omega \), then these conditions are never jointly satisfied. The argument is as follows. Firstly, if \( z > \omega \) and \( y < z \), then \( W(\max\{y, z\}) < W(y) \), which implies that \( \tilde{h} \succ \tilde{a} \). (Make a sketch of \( W(\max\{y, z\}) \) and \( W(y) \).) It follows that \( \tilde{h} \notin \overline{M}(F) \). Hence, the first condition isn’t satisfied. Secondly, if \( y \geq z \), then clearly the second condition isn’t satisfied. Therefore, if \( z > \omega \) and \( \tilde{m} \in \mathcal{N}(F) \), then \( \tilde{\Sigma}_B(\tilde{f}, \tilde{m}) = 0 \). It follows that \( \mathcal{N}_{221}(F) = \emptyset \), \( \mathcal{N}_{112}(F) = \emptyset \), \( \mathcal{N}_{003}(F) = \emptyset \), \( \mathcal{N}_{303}(F) = \emptyset \). ■

### 13.6 Summary

The table below illustrates the pattern of exclusions. It can be seen that the same pattern of exclusions occurs more than once. That is, there are three groups of subgames; and within each group the pattern of exclusions is the same. These groups are as follows:

- In subgames [1a], [2a] the only possible configurations are: \((003); (112); (221); (330)\).

- In subgames [1b], [2b] the only possible configurations are: \((003); (303)\).

- In Subgame [3] the only possible configurations are: \((000); (110); (220); (330)\).

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*Table 2. Impossible Configurations within a Subgame*
The five subgames are very similar. Indeed, subgames [1a], [2a] generate identical outcomes with regard to sellers’ membership. And the same is true of subgames [1b], [2b]. The only reason for separating out the cases is that it makes the analysis more straightforward.

At first sight Subgame [3] appears quite different from the others because Network $B$ is necessarily inactive. However, the "structure" of the configurations it generates is very similar to that found in subgames [1a], [2a] (where outside options play the role of Network $B$.) Furthermore, Subgame [3] is actually slightly less complex than the other subgames because there is no need to distinguish between cases where the tie is binding and cases where the tie is slack.

Since the subgames are so similar, I have focussed on Subgame [1]. The approach used to analyze Subgame [1] is a template for the other subgames.
Chapter 14

Elliptic Curves

Chapter 14 investigates the necessary conditions for all the sellers to be on a single network. Section (1) characterizes the "inner oval", within which all sellers are on Network A. Section (2) characterizes the "outer oval", outside of which all sellers are on Network B.\footnote{Configuration (330) occurs when there are three sellers on Network A and none on Network B. Configuration (003) occurs when there are three sellers on Network B and none on Network A.}

When making their membership decision a seller considers the fees set by the networks and the membership of other sellers. It’s useful to imagine that the networks’ fees belong to an action space, where different locations in this space generate different membership outcomes.

It has been shown that there is a limited number of possible membership configurations. Hence, the action-space is divided into a number of regions, where each region is paired with a particular configuration. If the networks’ fees belong to the centre of such a region, then unilateral deviation by any seller from the corresponding configuration leads to them receiving a strictly lower payoff. However, between two adjacent regions, there’s a boundary and for fees on this boundary one (or more) of the sellers is indifferent between two (or more) configurations. Therefore, the boundaries can be characterized by finding fees that make one of the sellers indifferent about their choice of membership.

It turns out that Configuration (330) and Configuration (003) are particularly significant. This chapter identifies some necessary conditions for these outcomes.

14.1 The Action-Space of Network A

Network A sets fees on platforms $C_A$ and $D_A$. Consider a two-dimensional space in which the x-axis correspond to the fee on Platform $C_A$ and the y-axis correspond to the fee on Platform $D_A$. This corresponds to a square with corners at: $(\rho, \rho)$, $(\rho, -\omega)$, $(-\omega, \rho)$, $(-\omega, -\omega)$. This is the action-space of Network A. It’s useful to understand how coordinates in this space are related to the end-user benefit on platforms $C_A$ and $D_A$.\footnote{Configuration (330) occurs when there are three sellers on Network A and none on Network B. Configuration (003) occurs when there are three sellers on Network B and none on Network A.}
The aggregate end-user benefit (AEUB) offered by Network A is

\[ \hat{W}(x) + \hat{W}(y) = \frac{1}{4\tau}(2\omega^2 - x^2 - y^2) \]

Hence, the AEUB of Network A is a function of \( x^2 + y^2 \), which implies that the iso-AEUB curves correspond to a family of circles, centred on the origin. As the radius decreases, the aggregate end-user benefit increases. Therefore, Network A offers its highest possible AEUB at the origin.

The aggregate end-user benefit offered by Network B is

\[ \hat{W}(-\omega) + \hat{W}(z) = \frac{1}{4\tau}(\omega^2 - z^2) \]

Hence, the condition for Network A to offer a higher AEUB than Network B is \( x^2 + y^2 < \omega^2 + z^2 \). In Network A’s action-space, this corresponds to the interior of a circle with radius \( \sqrt{\omega^2 + z^2} \).

### 14.2 Preconditions for the Stability of (303)

It has been shown that a seller’s payoff depends on which network(s) it joins but it also depends on the membership their competitors. Given fees, \( f_k \), and membership, \( m_{-i}, m_i \), the payoff of seller \( i \), in market \( k \in \{\varsigma, \delta\} \), becomes:

\[ \tilde{\Pi}_i = \tilde{\Pi}_i(F, m_{-i}, m_i) \equiv \sum_{k \in \{\varsigma, \delta\}} \tilde{\Gamma}_i(f_k, m_{-i}, m_i), \]

where their profit in market \( k \) is

\[ \tilde{\Gamma}_i(f_k, m_{-i}, m_i) \equiv \frac{1}{75\sigma} \left\{ 5\sigma + 3\hat{W}(f_k, m_i) - \sum_j W(f_k, m_j) \right\}^2 \]

If a seller is on Network A, then their extra-surplus is \( W(f_\varsigma, m_i) = \hat{W}(x) \), \( W(f_\delta, m_i) = \hat{W}(y) \). Whereas, if a seller is on Network B, then their extra-surplus is \( W(f_\varsigma, m_i) = 0 \), \( W(f_\delta, m_i) = \hat{W}(z) \).

If all sellers are on Network A, then they receive the same extra-surplus. Hence, their payoff is \( \tilde{\Pi}_i = \frac{2}{3}\sigma \). If one of the sellers were to unilaterally join Network B, instead, then their payoff would become

\[ \tilde{\Pi}_i = \frac{1}{75\sigma} \left\{ 5\sigma + 2\hat{W}(-\omega) - 2\hat{W}(x) \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + 2\hat{W}(z) - 2\hat{W}(y) \right\}^2 \]

Hence, a precondition for an outcome in which all sellers are on Network A is

\[ \left\{ 5\sigma + 2\hat{W}(-\omega) - 2\hat{W}(x) \right\}^2 + \left\{ 5\sigma + 2\hat{W}(z) - 2\hat{W}(y) \right\}^2 \leq 50\sigma^2, \]

where \( \hat{W}(f) = \frac{1}{4\tau}(\omega^2 - f^2) \). This can be re-expressed as:

\[ G(x, y, z) \leq 200\tau^2\sigma^2, \quad \text{(inner oval)} \]

where

\[ G(x, y, z) \equiv \left\{ 10\sigma\tau + x^2 - \omega^2 \right\}^2 + \left\{ 10\sigma\tau + y^2 - z^2 \right\}^2 \]
The parameter assumptions imply that $10\sigma > x^2 - \omega^2$ and $10\sigma > y^2 - z^2$. Hence,

$$\frac{\partial}{\partial x} G(x, y, z) \begin{cases} > 0 & \text{if } x > 0 \\ \leq 0 & \text{if } x \leq 0 \end{cases}$$

and

$$\frac{\partial}{\partial y} G(x, y, z) \begin{cases} > 0 & \text{if } y > 0 \\ \leq 0 & \text{if } y \leq 0 \end{cases}$$

whereas,

$$\frac{\partial}{\partial z} G(x, y, z) \begin{cases} < 0 & \text{if } z > 0 \\ \geq 0 & \text{if } z \leq 0 \end{cases}.$$

For reasons that will become clear later, this boundary is referred to as the inner oval. Since the inner oval occurs frequently throughout the analysis it’s useful to summarize its properties (see Appendix C). It can be shown that:

1. $G(x, y, z)$ corresponds to an oval shaped curve with mirror symmetry in both axes.

2. The radius of the curve is always positive and is enclosed by a circle of radius $\sqrt{\omega^2 + z^2}$.

3. The radius decreases to a minimum (strictly positive) value where the curve crosses an axis.

4. The maximum radius is $\sqrt{\omega^2 + z^2}$. This maximum occurs at $x = \pm \omega$, $y = \pm z$.

### 14.3 Preconditions for the Stability of (003)

If all sellers are on Network $B$, then they receive the same extra-surplus. Hence, their payoff is $\overline{\Pi}_i = \frac{2}{3}\sigma$. If one of the sellers were to unilaterally join Network
A, instead, then their payoff would become

$$\tilde{\Pi}_i = \frac{1}{75\sigma} \left\{ 5\sigma + 2\hat{W}(x) - 2\hat{W}(-\omega) \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + 2\hat{W}(y) - 2\hat{W}(z) \right\}^2$$

Hence, a precondition for an outcome in which all sellers are on Network B is

$$\left\{ 5\sigma + 2\hat{W}(x) - 2\hat{W}(-\omega) \right\}^2 + \left\{ 5\sigma + 2\hat{W}(y) - 2\hat{W}(z) \right\}^2 \leq 50\sigma^2,$$

where $\hat{W}(f) = \frac{1}{x\tau} (\omega^2 - f^2)$. This can be re-expressed as:

$$H(x, y, z) \leq 200\tau^2\sigma^2,$$

(outer oval)

where

$$H(x, y, z) \equiv \left\{ 10\sigma\tau + \omega^2 - x^2 \right\}^2 + \left\{ 10\sigma\tau + z^2 - y^2 \right\}^2.$$

Figure 9. Outer Oval

The parameter assumptions imply that $10\sigma\tau > \omega^2 - x^2$ and $10\sigma\tau > z^2 - y^2$. Hence,

$$\frac{\partial}{\partial x} H(x, y, z) \begin{cases} < 0 & \text{if } x > 0 \\ \geq 0 & \text{if } x \leq 0 \end{cases}$$

and

$$\frac{\partial}{\partial y} H(x, y, z) \begin{cases} < 0 & \text{if } y > 0 \\ \geq 0 & \text{if } y \leq 0 \end{cases} ;$$

whereas,

$$\frac{\partial}{\partial z} H(x, y, z) \begin{cases} > 0 & \text{if } z > 0 \\ \leq 0 & \text{if } z \leq 0 \end{cases} .$$

For reasons that will become clear later, this boundary is referred to as the outer oval. Since the outer oval occurs frequently throughout the analysis it’s useful to summarize its properties (see Appendix D). It can be shown that:
1. $H(x, y, z)$ corresponds to an oval shaped curve with mirror symmetry in both axes.

2. The radius of the curve is always positive and encloses a circle of radius $\sqrt{\omega^2 + z^2}$.

3. The maximum radius occurs where the curve crosses an axis.

4. The minimum radius is $\sqrt{\omega^2 + z^2}$. This minimum occurs at $x = \pm \omega$, $y = \pm z$. 
Chapter 15

Membership in Subgame [1a]

Chapter 15 investigates sellers’ membership when \( z \leq 0 \) and \( y \geq -|z| \). That is, the chapter investigates membership in Subgame [1a]. The chapter contains the following sections. Section (1) identifies necessary conditions each possible configuration. Section (2) identifies the part of the action-space where a given configuration is stable. That is, section (2) finds four regions of stability, which intersect at their boundaries and cover the action space. Section (3) constructs a series of mutually exclusive areas by allocating each boundary to one specific area. The areas are labelled as follows: Core, \( 1^{st} \) Shell, \( 2^{nd} \) Shell, and Exterior. Section (4) identifies the relevant options (\( \mathcal{M}(\mathbf{F}) \subseteq \{b,h\} \)) and summarizes the correspondence between membership vectors and configurations. Section (5) finds that within the Core all sellers multihome. Section (6) finds that within the \( 1^{st} \) Shell two sellers multihome and the third seller joins Network B. Section (7) finds that within the \( 2^{nd} \) Shell two sellers join Network B and the third seller multihomes. Section (8) finds that within the Exterior all the sellers join Network B. Section (9) summarizes the results and shows that there is a one-to-one correspondence between the four areas and the four possible configurations.

15.1 Four Possible Configurations

It’s been shown that in Subgame [1a] there are only four possible configurations: (330), (221), (112), (003). The following sections identify necessary and sufficient conditions for the stability of each configuration.

15.1.1 The Stability of (330)

If \( \Sigma_A(f_\uparrow, m) = 3 \), \( \Sigma_A(f_\downarrow, m) = 3 \), \( \Sigma_B(f_\downarrow, m) = 0 \), then unilateral deviation isn’t profitable for any of the sellers iff the following conditions hold:

\[
\left( 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right)^2 + \left( 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right)^2 \leq 50\sigma^2 \tag{1}
\]

\[
\left( 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right)^2 + \left( 5\sigma - \frac{z^2 - y^2}{2\tau} \right)^2 \leq 50\sigma^2 \tag{2}
\]

\[ y \geq z \text{ or } y^2 \leq z^2 \tag{3}\]
These conditions can be simplified as follows: Firstly, in Subgame [1a], \( z \leq \omega \). Therefore, any fees that satisfy (2), necessarily satisfy (1). Hence, (1) is redundant. Secondly, in Subgame [1a], \( y \geq -|z| = z \). Hence, (3) is necessarily satisfied. It follows that, the necessary and sufficient condition for stability is

\[
\left(5\sigma - \frac{\omega^2 - x^2}{2\tau}\right)^2 + \left(5\sigma - \frac{z^2 - y^2}{2\tau}\right)^2 \leq 50\sigma^2
\]

This can be re-expressed as \( G(x, y, z) \leq 200\tau^2\sigma^2 \), where

\[
G(x, y, z) \equiv \left\{10\sigma\tau + x^2 - \omega^2\right\}^2 + \left\{10\sigma\tau + y^2 - z^2\right\}^2,
\]

which implies that unilaterally switching to Network B isn’t profitable. It follows that:

**Lemma 15.1 (Stability of (330) in SG [1a].)** Let \( f_c = (x, -\omega) \), \( f_d = (y, z) \), where \( x, y, z \in [-\omega, \rho] \), and suppose that \( z \leq 0 \) and \( y \geq -|z| \). It follows that if \( m \in M^3 \) and \( \Sigma_A(f_c, m) = 3 \), \( \Sigma_A(f_d, m) = 3 \), \( \Sigma_B(f_d, m) = 0 \), then \( m \) is stable under \( F = (f_c^T, f_d^T) \) iff \( G(x, y, z) \leq 200\tau^2\sigma^2 \).

### 15.1.2 The Stability of (221)

This section identifies the necessary and sufficient conditions that \( F = (f_c^T, f_d^T) \) must satisfy to ensure that \( m \in M^3 \) is stable whenever

\[
\Sigma_A(f_c, m) = 2 \ , \ \Sigma_A(f_d, m) = 2 \ , \ \Sigma_B(f_d, m) = 1
\]

Suppose that the sellers are distributed as follows: Seller 1 and Seller 2 are on Network A; and Seller 3 is on Network B. The necessary and sufficient conditions for the sellers not to deviate can be found as follows (see Appendix E):

**Seller 1.** It can be shown that unilateral deviation isn’t profitable for Seller 1 iff the following conditions are satisfied:

\[
0 \leq \left(5\sigma + \frac{\omega^2 - x^2}{4\tau}\right)^2 + \left(5\sigma + \frac{z^2 - y^2}{4\tau}\right)^2
- \left(5\sigma - \frac{\omega^2 - x^2}{4\tau}\right)^2 - \left(5\sigma - \frac{2\omega^2 - z^2 - y^2}{4\tau}\right)^2
\]

\[
0 \leq \left(5\sigma + \frac{\omega^2 - x^2}{4\tau}\right)^2 + \left(5\sigma + \frac{z^2 - y^2}{4\tau}\right)^2
- \left(5\sigma - \frac{\omega^2 - x^2}{4\tau}\right)^2 - \left(5\sigma - \frac{z^2 - y^2}{4\tau}\right)^2
\]

\[
y \geq z \text{ or } y^2 \leq z^2
\]

These conditions can be simplified as follows: Firstly, in Subgame [1a], \( z \leq \omega \). Hence, any fees that satisfy (2) also satisfy (1). Therefore, (1) is redundant.
Secondly, using the "difference of two squares", (2) can be re-expressed as 
\[ x^2 + y^2 \leq \omega^2 + z^2. \] 
Finally, in Subgame [1a], \( y \geq -|z| = z \). Hence, (3) is necessarily satisfied. Therefore, Seller 1 will not unilaterally deviate iff the following conditions is satisfied:

\[ x^2 + y^2 \leq \omega^2 + z^2 \]

This condition implies that Network A offers the higher aggregate end-user benefit: \( \bar{W}(x) + \bar{W}(y) \geq \bar{W}(z) \).

**Seller 3.** It can be shown that unilateral deviation isn’t profitable for Seller 3 iff the following conditions are satisfied:

\[ z^2 \leq \omega^2 \]  
\[ 50\sigma^2 \leq \left( 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right)^2 + \left( 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right)^2 \]  
\[ 25\sigma^2 \leq \left( 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right)^2 + \left( 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right)^2 - \left( 5\sigma - \frac{\max\{y, z\}^2 - y^2}{2\tau} \right)^2 \]  

These conditions can be simplified as follows: Firstly, in Subgame [1a], \( z \leq \omega \). Hence, (4) is necessarily satisfied. Secondly, it can be seen that any fees that satisfy (5) also satisfy (6). It follows that (6) is redundant. Therefore, if Seller 3 will not unilaterally deviate iff the following condition is satisfied:

\[ 50\sigma^2 \leq \left( 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right)^2 + \left( 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right)^2 \]

This condition can be re-expressed as \( G(x, y, z) \geq 200\tau^2\sigma^2 \), where

\[ G(x, y, z) \equiv \left\{ 10\sigma \tau + x^2 - \omega^2 \right\}^2 + \left\{ 10\sigma \tau + y^2 - z^2 \right\}^2 , \]

which implies that if all sellers were on Network A, then unilaterally switching to Network B would be profitable.

**Lemma 15.2 (Stability of (221) in SG [1a].)** Let \( f_c = (x, -\omega), f_d = (y, z), \) where \( x, y, z \in [-\omega, \rho] \), and suppose that \( z \leq 0 \) and \( y \geq -|z| \). It follows that if \( m \in \mathbb{M}_3 \) and \( \Sigma_A(f_c, m) = 2, \Sigma_A(f_d, m) = 2, \Sigma_B(f_d, m) = 1 \), then \( m \) is stable under \( F = (f_c^T, f_d^T) \) iff the following conditions are satisfied:

1. Network A offers a higher aggregate end-user benefit: \( x^2 + y^2 \leq \omega^2 + z^2 \);
2. If all sellers were on Network A, then unilaterally switching to Network B would be profitable: \( G(x, y, z) \geq 200\tau^2\sigma^2 \).
15.1.3 The Stability of $(112)$

This section identifies the necessary and sufficient conditions that \( F = (f_{\varsigma}^T, f_q^T) \) must satisfy to ensure that \( m \in M^3 \) is stable whenever

\[
\Sigma_A(f_{\varsigma}, m) = 1, \quad \Sigma_A(f_q, m) = 1, \quad \Sigma_B(f_q, m) = 2
\]

Suppose that the sellers are distributed as follows: Seller 1 and Seller 2 are on Network B; and Seller 3 is on Network A. The necessary and sufficient conditions for the sellers not to deviate can be found as follows:

**Seller 1.** Unilateral deviation isn’t profitable for Seller 1 iff the following conditions are satisfied:

\[
z^2 \leq \omega^2 \quad (1)
\]

\[
0 \leq \left(5\sigma - \frac{\omega^2 - x^2}{4\tau}\right)^2 + \left(5\sigma - \frac{z^2 - y^2}{4\tau}\right)^2 - \left(5\sigma + \frac{\omega^2 - x^2}{4\tau}\right)^2 - \left(5\sigma + \frac{z^2 - y^2}{4\tau}\right)^2 \quad (2)
\]

\[
0 \leq \left(5\sigma - \frac{x^2 - \omega^2}{4\tau}\right)^2 + \left(5\sigma - \frac{z^2 - y^2}{4\tau}\right)^2 - \left(5\sigma + \frac{x^2 - \omega^2}{4\tau}\right)^2 - \left(5\sigma + \frac{z^2 - \max\{y, z\}}{4\tau}\right)^2 \quad (3)
\]

(See Appendix E.) These conditions can be simplified as follows: Firstly, by assumption, \( z \leq \omega \). Hence, (1) is necessarily satisfied. Secondly, it can be seen that any fees that satisfy (2) also satisfy (3). It follows that (3) is redundant. Finally, using the "difference of two squares", (2) becomes \( x^2 + y^2 \geq \omega^2 + z^2 \). Therefore, Seller 1 doesn’t deviate iff the following condition is satisfied: \( x^2 + y^2 \geq \omega^2 + z^2 \). This implies that the aggregate end-user benefit on Network B exceeds that on Network A.

**Seller 3.** Unilateral deviation isn’t profitable for Seller 3 iff the following conditions are satisfied:

\[
25\sigma^2 \leq \left(5\sigma + \frac{\omega^2 - x^2}{2\tau}\right)^2 + \left(5\sigma + \frac{z^2 - y^2}{2\tau}\right)^2 - \left(5\sigma - \frac{\omega^2 - z^2}{2\tau}\right)^2 \quad (4)
\]

\[
50\sigma^2 \leq \left(5\sigma + \frac{\omega^2 - x^2}{2\tau}\right)^2 + \left(5\sigma + \frac{z^2 - y^2}{2\tau}\right)^2 \quad y \geq z \text{ or } y^2 \leq z^2 \quad (5)
\]

(See Appendix E.) These conditions can be simplified as follows: Firstly, by assumption, \( z \leq \omega \). Hence, any fees that satisfy (5) also satisfy (4). It follows that (4) is redundant. Secondly, in Subgame [1a.], \( y \geq -|z| = z \). Hence, (6) is necessarily satisfied. Therefore, Seller 3 doesn’t deviate iff the following
condition is satisfied:

\[
\left(5\sigma + \frac{\omega^2 - x^2}{2\tau}\right)^2 + \left(5\sigma + \frac{z^2 - y^2}{2\tau}\right)^2 \geq 50\sigma^2
\]

This condition can be re-expressed as \(H(x, y, z) \geq 200\tau^2\sigma^2\), where

\[
H(x, y, z) \equiv \left(10\sigma\tau + \omega^2 - x^2\right)^2 + \left(10\sigma\tau + z^2 - y^2\right)^2
\]

This implies that if all sellers are on Network B, then unilaterally switching to Network A is profitable.

**Lemma 15.3** *(Stability of (112) in SG [1a].)* Let \(f_c = (x, -\omega)\), \(f_d = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0\) \& \(y \leq -|z|\). It follows that if \(m \in \mathbb{M}^3\) and \(\Sigma_A(f_c, m) = 1\), \(\Sigma_A(f_d, m) = 1\), \(\Sigma_B(f_d, m) = 2\), then \(m\) is stable under \(F = (f_c^T, f_d^T)\) iff the following conditions are satisfied:

1. Network B offers the higher aggregate end-user benefit: \(x^2 + y^2 \geq \omega^2 + z^2\)

2. If all sellers were on Network B, then unilaterally switching to Network A would be profitable: \(H(x, y, z) \geq 200\tau^2\sigma^2\).

**15.1.4 The Stability of (003)**

This section identifies the necessary and sufficient conditions that \(F = (f_c^T, f_d^T)\) must satisfy to ensure that \(m \in \mathbb{M}^3\) is stable whenever

\[
\Sigma_A(f_c, m) = 0, \quad \Sigma_A(f_d, m) = 0, \quad \Sigma_B(f_d, m) = 3
\]

It can be shown that when \(y \geq -|z|\), unilateral deviation from Configuration (003) isn’t profitable iff the following conditions are satisfied:

\[
z^2 \leq \omega^2
\]

(1)

\[
\left\{5\sigma + \frac{\omega^2 - x^2}{2\tau}\right\}^2 + \left\{5\sigma + \frac{z^2 - y^2}{2\tau}\right\}^2 \leq 50\sigma^2
\]

(2)

(See Appendix.) These conditions can be simplified as follows: By assumption, \(z \leq \omega\). Hence, (1) is always satisfied. Therefore, if \(y \geq z\), then Configuration (003) is stable iff the following conditions is satisfied:

\[
\left\{5\sigma + \frac{\omega^2 - x^2}{2\tau}\right\}^2 + \left\{5\sigma + \frac{z^2 - y^2}{2\tau}\right\}^2 \leq 50\sigma^2
\]

This can be re-expressed as \(H(x, y, z) \leq 200\tau^2\sigma^2\), where

\[
H(x, y, z) \equiv \left\{10\sigma\tau + \omega^2 - x^2\right\}^2 + \left\{10\sigma\tau + z^2 - y^2\right\}^2
\]

It follows that:

**Lemma 15.4** *(Stability of (003) in SG [1a].)* Let \(f_c = (x, -\omega)\), \(f_d = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0\) \& \(y \leq -|z|\). It follows that
if \( m \in M^3 \) and \( \Sigma_A(f, m) = 0, \Sigma_A(f, f) = 0, \Sigma_B(f, m) = 3 \), then \( m \) is stable under \( F = (f^T, f^T) \) iff \( H(x, y, z) \leq 200\gamma^2\sigma^2 \).

### 15.2 Regions of Stability

A configuration is stable iff none of the sellers can profit through unilateral deviation. Whether or not a particular configuration is stable depends on the value of \( z \) and the location of \((x, y)\) within the action space. The action-space can be divided into a number of regions, where each region is paired with a particular configuration. Hence, it’s useful to introduce the following notation: configuration \((IJK)\) is stable under \( F \in \mathcal{F} \) iff \((x, y) \in U_{IJK}(z) \subseteq [-\omega, \rho]^2\). The set \( U_{IJK}(z) \) will be referred to as the region of stability of Configuration \((IJK)\) when the seller-fee on Network \( B \) is \( z \). If a configuration is never stable, then the corresponding set is empty. (For example, \( U_{010}(z) = \emptyset \).)

Regions of stability have been identified for the four configurations that occur in Subgame [1a]. These regions are as follows:

- **Stability of \((330)\)**: If \( z \leq 0, y \leq -|z| \), then \((x, y) \in U_{330}(z)\) iff the following condition is satisfied:
  \[
  G(x, y, z) \leq 200\gamma^2\sigma^2
  \]

- **Stability of \((221)\)**: If \( z \leq 0, y \leq -|z| \), then \((x, y) \in U_{221}(z)\) iff the following conditions are satisfied:
  \[
  x^2 + y^2 \leq \omega^2 + z^2
  \quad G(x, y, z) \geq 200\gamma^2\sigma^2
  \]

- **Stability of \((112)\)**: If \( z \leq 0, y \leq -|z| \), then \((x, y) \in U_{112}(z)\) iff the following conditions are satisfied:
  \[
  x^2 + y^2 \geq \omega^2 + z^2
  \quad H(x, y, z) \geq 200\gamma^2\sigma^2
  \]

- **Stability of \((003)\)**: If \( z \leq 0, y \leq -|z| \), then \((x, y) \in U_{003}(z)\) iff the following condition is satisfied:
  \[
  H(x, y, z) \leq 200\gamma^2\sigma^2
  \]

It can be seen that each region is circumscribed by a series of boundaries. The boundaries divide the action space into four regions, each corresponding to a different configuration. Within its corresponding region the necessary and sufficient conditions for the stability of a particular configuration are satisfied. It can be seen that the regions are mutually exclusive except at the boundaries, where they intersect. Furthermore, they cover the entire action space, \([-\omega, \rho]^2\).
15.3 Mutually Exclusive Areas

Since the regions only intersect at the boundary, it follows that within the interior of a region the necessary conditions for just one type of outcome are satisfied. Since stability is a necessary condition for a RSPNE (and there must exist an equilibrium configuration), it follows that if \((x, y) \in \text{int} \ U_{IJK}(z)\), then \(\mathcal{N}(\mathbf{F}) \subseteq \mathcal{N}_{IJK}(\mathbf{F})\). However, because adjacent regions intersect at the boundaries, there appears to be more than one possible outcome at these points. (Although, there can only be one configuration which satisfies the sufficient conditions for a RSPNE.) To help resolve the ambiguity, it’s useful to construct a series of mutually exclusive areas:

<table>
<thead>
<tr>
<th>Area</th>
<th>Boundaries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core:</td>
<td>(U_{330}(z)) (G(x, y, z) \leq 200r^2\sigma^2)</td>
</tr>
</tbody>
</table>
| 1st Shell: | \(U_{221}(z) \setminus U_{330}(z)\) \(
\begin{align*}
x^2 + y^2 &\leq \omega^2 + z^2 \\
G(x, y, z) &> 200r^2\sigma^2
\end{align*}\) |
| 2nd Shell: | \(U_{112}(z) \setminus U_{221}(z)\) \(
\begin{align*}
x^2 + y^2 &> \omega^2 + z^2 \\
H(x, y, z) &\geq 200r^2\sigma^2
\end{align*}\) |
| Exterior: | \(U_{003}(z) \setminus U_{112}(z)\) \(H(x, y, z) < 200r^2\sigma^2\) |

Table 3. Areas of the Action-Space in Subgame [1a]

These sets are defined in such a way that a given boundary only belongs to one set. The boundaries along which adjacent regions intersect are as follows:
• **Core** has two closed, adjacent boundaries:
  \[ x \neq \pm \omega, \ G(x, y, z) = 200\tau^2\sigma^2 \text{ (curve)}; \]
  \[ x = \pm \omega, \ y = |z| = -z \text{ (points)} \]

• **1st Shell** has one closed, adjacent boundary:
  \[ x \neq \pm \omega, \ x^2 + y^2 = \omega^2 + z^2 \text{ (curve)}. \]

• **2nd Shell** has one closed, adjacent boundary:
  \[ x \neq \pm \omega, \ H(x, y, z) = 200\tau^2\sigma^2 \text{ (curve)}. \]

• **Exterior** has no closed, adjacent boundary.

### 15.4 Relevant Membership Options

A Restricted SPNE (RSPNE) is a fixed point of the sellers best-response functions. That is, \( \widetilde{m} \in \mathcal{N}(F) \) iff \( \widetilde{m}_i = \max \widetilde{M}(F, \widetilde{m}_{-i}) \), for \( i = 1, 2, 3 \). Let \( \mathcal{N}(F) \subseteq \mathbb{M}^3 \) denote the set of membership vectors which satisfy the criteria for a RSPNE when fees are \( F \in \mathbb{F} \). Also, let \( \mathcal{N}_{IJK}(F) \) denote the subset of equilibria in which Configuration \( (IJK) \) occurs; where

\[
\Sigma_A(f_d, m) = I, \ \Sigma_A(f_v, m) = J, \ \Sigma_B(f_d, m) = K
\]

An option, \( m \in \mathbb{M} \), is said to be "irrelevant" if there is another option, \( m' \in \mathbb{M} \), such that \( m' > m \) or \( m' \sim m \), \( m' > m \). An option is said to be relevant if it’s not irrelevant. Let \( \overline{M}(F) \subseteq \mathbb{M} \) denote the set of relevant membership options. The set of relevant options when \( z \leq 0 \) can be found as follows. Firstly, by assumption, \( \overline{W}(z) \geq 0 \), which implies that \( b \geq a \). Since \( b > a \), it follows that \( a \notin \overline{M}(F) \). Secondly, if \( -\omega \leq z \leq 0 \), then \( \overline{W}(\max\{y, z\}) \geq \overline{W}(y) \), which implies that \( h \geq a \). Since \( h > a \), it follows that \( a \notin \overline{M}(F) \). This analysis implies that: \( \overline{M}(F) \subseteq \{b, h\} \). Furthermore, it’s been shown that if \( \overline{M}(F) \subseteq \{b, h\} \), then:

\[
\widetilde{m} \in \mathcal{N}_{330}(F) \text{ iff } \widetilde{m} \in \{h\}^3 \subseteq \mathcal{N}(F)
\]
\[
\widetilde{m} \in \mathcal{N}_{221}(F) \text{ iff } \widetilde{m} \in \{h\}^2 \times \{b\} \subseteq \mathcal{N}(F)
\]
\[
\widetilde{m} \in \mathcal{N}_{112}(F) \text{ iff } \widetilde{m} \in \{h\} \times \{b\}^2 \subseteq \mathcal{N}(F)
\]
\[
\widetilde{m} \in \mathcal{N}_{003}(F) \text{ iff } \widetilde{m} \in \{b\}^3 \subseteq \mathcal{N}(F)
\]

(See Configurations and Relevant Options, Appendix B.)

Five mutually exclusive areas have been defined. (A given boundary has been allocated to just one area.) It will be shown that each area corresponds to a particular membership class of vector.

---

1. A seller often has more than one best response. Best response functions were created using a lexicographic decision rule. A RSPNE is a fixed point of the sellers best-response functions.
15.5 Membership in the Core

By definition, \((x, y)\) belongs to the Core in Subgame [1a] iff \(G(x, y, z) \leq 200r^2\sigma^2\). It can be shown that if \((x, y)\) belongs to the Core, then all sellers multihome. Furthermore, in Subgame [1a], \(y \geq -|z|\), which implies that multihoming is equivalent to joining Network \(A\).

**Lemma 15.5 (In the Core all sellers Multihome.)** Let \(f = (x, -\omega)\), \(f_g = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0\) and \(y \geq -|z|\). It follows that if \(G(x, y, z) \leq 200r^2\sigma^2\) and \(\vec{m} \in \mathcal{N}(F)\), then \(\vec{m} \in \{b\}^3\).

**Proof.** Suppose that \((x, y) \in U_{330}(z)\). Either \((x, y)\) is in the interior of the area or it’s on the boundary. These are analyzed in turn:

**Interior.** In Subgame [1a] there were four possible configurations: \((330)\), \((221)\), \((112)\), \((003)\). If \((x, y) \in \text{int} U_{330}(z)\), then \((x, y) \notin U_{221}(z)\), \((x, y) \notin U_{112}(z)\), \((x, y) \notin U_{003}(z)\). This implies that \(\mathcal{N}_{221}(F) = \emptyset\), \(\mathcal{N}_{112}(F) = \emptyset\), \(\mathcal{N}_{003}(F) = \emptyset\). It follows that \(\mathcal{N}(F) \subseteq \mathcal{N}_{330}(F)\). Since \(\mathcal{M}(F) \subseteq \{b, h\}\), it follows that \(\vec{m} \in \mathcal{N}_{330}(F)\) iff \(\vec{m} \in \{b\}^3\). (See Configurations and Relevant Options, Appendix.) Therefore, all sellers multihome: \(\mathcal{N}(F) = \{b\}^3\).

**Boundary.** If \((x, y)\) is on the boundary of \(U_{330}(z)\), then there are two possibilities: (a) \(x \neq \pm \omega, y > -|z| = z, G(x, y, z) = 200r^2\sigma^2\); and (b) \(x = \pm \omega, y = |z| = -z\). In Case (a) the Core intersects the 1st Shell; whereas, in Case (b) the Core intersects the other three areas. These are analyzed in turn:

Case (a): If \(x \neq \pm \omega, y > -|z| = z, G(x, y, z) = 200r^2\sigma^2\), then \((x, y) \notin U_{112}(z), (x, y) \notin U_{003}(z)\). This implies that \(\mathcal{N}_{112}(F) = \emptyset\), \(\mathcal{N}_{003}(F) = \emptyset\). Since there were only four potentially stable configurations to start with, this implies that \(\mathcal{N}(F) \subseteq \mathcal{N}_{330}(F) \cup \mathcal{N}_{221}(F)\).

Furthermore, because \(\mathcal{M}(F) \subseteq \{b, h\}\), it follows that: \(\vec{m} \in \mathcal{N}_{330}(F)\) iff \(\vec{m} = \{b\}^3\); and \(\vec{m} \in \mathcal{N}_{221}(F)\) iff \(\vec{m} \in \{b\}^2 \times \{b\}\). (See Configurations and Relevant Options, Appendix B.) It can be seen that there appear to be two possible configurations. However, in equilibrium, it’s not possible to have more than one type of configuration, because equilibria are unique up to a relabelling of the sellers. A unique outcome can be found as follows:

Firstly, if \(m = \{b\}^3\) and \(y > -|z| = z\), then \(\Sigma_A(f, m) = 3, \Sigma_A(f_d, m) = 3, \Sigma_B(f, m) = 0\) and the fact that \(y \geq z\), then \(A(f, h) = 1, B(f_d, h) = 0\).

Secondly, by assumption, \((x, y) \in U_{330}(z)\). Furthermore, if \((x, y) \in U_{330}(z)\), then \(m\) is stable whenever it’s the case that \(\Sigma_A(f, m) = 3, \Sigma_A(f_d, m) = 3, \Sigma_B(f, m) = 0\). This observation comes straight from the definition of \(U_{330}(z)\).

Hence, if \(m \in \{b\}^3\), then \(m\) is stable, which implies that: \(h \in \mathcal{M}(F, h, h)\). That is, the fees on the boundary are such that if all sellers multihome, then none of them has an incentive to unilaterally deviate. This information can be used to show that there is no RSPNE in which one of the sellers joins Network.
B. The argument is as follows:

\[ h \in \hat{\mathcal{M}}(F, h, h) \Rightarrow \max \hat{\mathcal{M}}(F, h, h) \neq b \quad \text{[using } h > b \text{]} \]
\[ \Rightarrow \{h\}^2 \times \{b\} \notin \mathcal{N}(F) \]
\[ \Rightarrow \mathcal{N}_{221}(F) = \emptyset \]

Therefore, \( \mathcal{N}(F) = \{h\}^3 \).

Case (b): Suppose that \( x = \pm \omega, y = |z| = -z \). If \( x = \pm \omega, y = |z| = -z \), then \( b \sim h \). (This is because \( \hat{W}(|\omega|) = 0, \hat{W}(y) = \hat{W}(-z) \).) Since \( \hat{\mathcal{M}}(F) \subseteq \{b, h\} \) and \( h > b \), it follows that \( \hat{\mathcal{M}}(F) = \{h\} \). Therefore, \( \mathcal{N}(F) = \{h\}^3 \). ■

15.6 Membership in the 1st Shell

By definition, \((x, y)\) belongs to the 1st Shell iff the following conditions are satisfied: \( y \geq -|z|, x^2 + y^2 \leq \omega^2 + z^2, G(x, y, z) > 200r^2\sigma \). It can be shown that if \((x, y)\) is within the 1st Shell, then two sellers multihome while the other seller joins Network \(B\). Since \( y \geq -|z| > z \), the tie is binding; which implies that if a seller multihomes, then they’re on Network \(A\) in market \(d\). Hence, the sellers divide into two groups. It follows that, in equilibrium, Network \(A\) has two-thirds of the market and Network \(B\) has one-third of the market.

Lemma 15.6 (In the 1st Shell two Multihome and one joins Network B.) Let \( f_x = (x, -\omega), f_y = (y, z) \), where \( x, y, z \in [-\omega, \rho] \), and suppose that \( z \leq 0 \) and \( y \geq -|z| \). It follows that if \( G(x, y, z) > 200r^2\sigma^2, x^2 + y^2 \leq \omega^2 + z^2 \) and \( \hat{m} \in \mathcal{N}(F) \), then \( \hat{m} \in \{h\}^2 \times \{b\} \).

Proof. In this subsection suppose that \((x, y) \in \mathbb{U}_{221}(z) \setminus \mathbb{U}_{330}(z)\). Either \((x, y)\) is in the interior of the area or it’s on the boundary. These are analyzed in turn:

Interior. In Subgame [1a] there are four possible configurations: \((330), (221), (112), (003)\). If \((x, y) \in \text{int } \mathbb{U}_{221}(z) \setminus \mathbb{U}_{330}(z)\), then \((x, y) \notin \mathbb{U}_{330}(z), (x, y) \notin \mathbb{U}_{112}(z), (x, y) \notin \mathbb{U}_{003}(z)\). This implies that \( \mathcal{N}_{330}(F) = \emptyset, \mathcal{N}_{112}(F) = \emptyset, \mathcal{N}_{003}(F) = \emptyset \). Furthermore, there were only four potentially stable configurations to start with. Hence, \( \mathcal{N}(F) \subseteq \mathcal{N}_{221}(F) \). Since \( \hat{\mathcal{M}}(F) \subseteq \{b, h\} \), it follows that \( \hat{\mathcal{M}} \in \mathcal{N}_{221}(F) \) iff \( \hat{\mathcal{M}} \in \{h\}^2 \times \{b\} \). Therefore,

\[ \mathcal{N}(F) \subseteq \mathcal{N}_{221}(F) = \{h\}^2 \times \{b\} \]

Boundary. It can be seen that \( \mathbb{U}_{221}(z) \setminus \mathbb{U}_{330}(z) \) has only one closed boundary, along which \( x \neq \pm \omega, y > -|z| \) and \( x^2 + y^2 = \omega^2 + z^2 \). Furthermore, if \((x, y)\) is on the boundary, then \((x, y) \notin \mathbb{U}_{330}(z), (x, y) \notin \mathbb{U}_{003}(z)\). This implies that \( \mathcal{N}_{330}(F) = \emptyset, \mathcal{N}_{003}(F) = \emptyset \). Since there were only four potentially stable configurations, it follows that:

\[ \mathcal{N}(F) \subseteq \mathcal{N}_{221}(F) \cup \mathcal{N}_{112}(F) \]
Since $\mathbb{M}(F) \subseteq \{b, h\}$, it follows that: $\widehat{m} \in \mathcal{N}_{221}(F)$ iff $\widehat{m} \in \{b\}^2 \times \{b\}$; and $\widehat{m} \in \mathcal{N}_{112}(F)$ iff $\widehat{m} \in \{h\} \times \{b\}^2$. (See Configurations and Relevant Options, Appendix B.) It can be seen that there appear to be two possible configurations. However, in equilibrium, it’s not possible to have more than one type of configuration, because equilibria are unique up to a relabelling of the sellers.

A unique outcome can be found as follows:

Firstly, if $m \in \{b\} \times \{b\}^2$ and $y > -|z|$, then $\Sigma_A(f_d, m) = 2$, $\Sigma_A(f_d, m) = 2$, $\Sigma_B(f_d, m) = 0$. This observation comes from the definition of $\Sigma_A(f_k, m)$, $\Sigma_B(f_d, m)$ and the fact that if $y > z$, then: $A(f_k, b) = 1$, $B(f_d, b) = 0$; whereas, $A(f_k, b) = 0$, $B(f_d, b) = 1$.

Secondly, by assumption, $(x, y) \in \mathbb{U}_{221}(z)$. Furthermore, if $(x, y) \in \mathbb{U}_{221}(z)$, then $m$ is stable whenever the case that $\Sigma_A(f_{c'}, m) = 2$, $\Sigma_A(f_d, m) = 2$, $\Sigma_B(f_d, m) = 0$. This comes straight from the definition of $\mathbb{U}_{221}(z)$.

Hence, if $m \in \{b\} \times \{b\}^2$, then $m$ is stable, which implies that: $b \in \mathbb{M}(F, b, h)$, $h \in \mathbb{M}(F, b, h)$, $h \in \mathbb{M}(F, b, b)$. That is, fees are such that if two sellers are on Network $B$ and the other seller multihomes, then none of the sellers have an incentive to deviate. It follows that:

$$
\begin{align*}
\mathbb{M}(F, b, h) & \quad \Rightarrow \quad \max\mathbb{M}(F, b, h) \neq b \quad [\text{using } b < h] \\
& \quad \Rightarrow \quad \{b\} \times \{b\}^2 \notin \mathbb{N}(F) \\
& \quad \Rightarrow \quad \mathbb{N}_{112}(F) = \emptyset
\end{align*}
$$

Therefore,

$$
\mathbb{N}(F) \subseteq \mathbb{N}_{221}(F) = \{b\}^2 \times \{b\}
$$

15.7 Membership in the 2$^\text{nd}$ Shell

By definition, $(x, y)$ belongs to the 2$^\text{nd}$ Shell iff the following conditions are satisfied: $y > -|z|$, $x^2 + y^2 > \omega^2 + z^2$, $H(x, y, z) \geq 200r^2\sigma^2$. It can be shown that if $(x, y)$ is within the 2$^\text{nd}$ Shell, then two sellers join Network $B$ while the other seller multihomes. Since $y \geq -|z|$, the tie is binding; which implies that if a seller multihomes, then they’re on Network $A$ in market $g$. Hence, the sellers divide into two groups. Network $B$ has two-thirds of the market and Network $A$ has one-third of the market.

**Lemma 15.7 (In the 2$^\text{nd}$ Shell one Multihomes and two join Network $B$.)** Let $f_{c'} = (x, -\omega)$, $f_d = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $z \leq 0$ and $y \geq -|z|$. It follows that if $x^2 + y^2 > \omega^2 + z^2$, $H(x, y, z) \geq 200r^2\sigma^2$ and $\overline{m} \in \mathbb{N}(F)$, then $\overline{m} \in \{b\}^2 \times \{b\}$.

**Proof.** Suppose that $(x, y) \in \mathbb{U}_{112}(z) \setminus \mathbb{U}_{221}(z)$. Either $(x, y)$ is in the interior of the area or it’s on the boundary. These are analyzed in turn:

**Interior.** In Subgame [1a] there are four possible configurations: $(330)$, $(221)$, $(112)$, $(003)$. If $(x, y)$ is in the interior of $\mathbb{U}_{112}(z) \setminus \mathbb{U}_{221}(z)$, then $(x, y) \notin \mathbb{U}_{330}(z)$, $(x, y) \notin \mathbb{U}_{221}(z)$, $(x, y) \notin \mathbb{U}_{003}(z)$. This implies that $\mathbb{N}_{330}(F) = \emptyset$, $\mathbb{N}_{221}(F) = \emptyset$, $\mathbb{N}_{003}(F) = \emptyset$. 

143
Lemma 15.8 Membership in the Exterior

It can be seen that \( U_{112}(z) \setminus U_{221}(z) \) has only one closed boundary, along which \( x \neq \pm \omega, \ y > -|z| \) and \( H(x, y, z) = 200\tau^2\sigma^2 \). If \((x, y)\) is on the boundary, then \((x, y) \notin U_{330}(z), (x, y) \notin U_{221}(z)\). This implies that \( \mathcal{N}_{330}(F) = \emptyset, \mathcal{N}_{221}(F) = \emptyset \). Hence,

\[
\mathcal{N}(F) \subseteq \mathcal{N}_{112}(F) \cup \mathcal{N}_{003}(F)
\]

Since \( \mathcal{M}(F) \subseteq \{b, h\} \), it follows that: \( \widetilde{m} \in \mathcal{N}_{112}(F) \) iff \( \widetilde{m} \in \{b\}^2 \times \{h\} \); and \( \widetilde{m} \in \mathcal{N}_{003}(F) \) iff \( \widetilde{m} \in \{b\}^3 \). (See Configurations and Relevant Options, Appendix B.) It can be seen that there appear to be two possible configurations. However, in equilibrium, it’s not possible to have more than one type of configuration, because equilibria are unique up to a relabelling of the sellers. A unique outcome can be found as follows:

Firstly, if \( m \in \{b\}^2 \times \{h\} \) and \( y > -|z| = z \), then \( \Sigma_A(f, m) = 1, \Sigma_A(f, m) = 1, \Sigma_B(f, m) = 2 \). This observation comes straight from the definitions of \( \Sigma_A(f, m), \Sigma_B(f, m) \) and the fact that \( y > z \), then: \( A(f, h) = 1, B(f, h) = 0 \); whereas, \( A(f, b) = 0, B(f, b) = 1 \).

Secondly, by definition, \((x, y) \in U_{112}(z)\). Furthermore, if \((x, y) \in U_{112}(z)\), then \( m \) is stable whenever it’s the case that \( \Sigma_A(f, m) = 1, \Sigma_A(f, m) = 1, \Sigma_B(f, m) = 2 \). This comes straight from the definition of \( U_{112}(z) \).

Hence, if \( m \in \{b\}^2 \times \{h\} \), then \( m \) is stable, which implies that: \( h \in \mathcal{M}(F, b, b), b \in \mathcal{M}(F, b, h), b \in \mathcal{M}(F, h, b) \). That is, fees are such that, if two sellers join Network \( B \) and the other seller multihomes, then none of the sellers has an incentive to deviate. It follows that:

\[
\begin{align*}
\mathcal{M}(F, b, b) & \Rightarrow \max \mathcal{M}(F, b, b) \neq b \quad \text{[using } b < h \text{]} \\
\Rightarrow \{b\}^3 & \not\subseteq \mathcal{N}(F) \\
\Rightarrow \mathcal{N}_{003}(F) & = \emptyset
\end{align*}
\]

Therefore,

\[
\mathcal{N}(F) \subseteq \mathcal{N}_{112}(F) = \{b\}^2 \times \{h\}
\]

\[
\blacksquare
\]

15.8 Membership in the Exterior

By definition, \((x, y)\) belongs to the Exterior iff the following conditions are satisfied: \( y \geq -|z|, H(x, y, z) < 200\tau^2\sigma^2 \). It can be shown that if \((x, y)\) is within the Exterior, then all sellers are join Network \( B \). This implies that Network \( A \) is inactive and all the sellers are on Network \( A \).

Lemma 15.8 (In the Exterior all sellers join Network \( B \)). Let \( f \_\omega = (x, -\omega), f \_\rho = (y, z), \) where \( x, y, z \in [-\omega, \rho], \) and suppose that \( z \leq 0 \) and \( y \geq -|z| \). It follows that if \( H(x, y, z) < 200\tau^2\sigma^2 \) and \( \widetilde{m} \in \mathcal{N}(F) \), then \( \widetilde{m} \in \{b\}^3 \).
Proof. Suppose that \((x, y) \in \mathbb{U}_{003}(z) \setminus \mathbb{U}_{112}(z)\). Either \((x, y)\) is in the interior of the area or it’s on the boundary. These are analyzed in turn:

Interior. In Subgame [1a] there are four possible configurations: \((330), (221), (112), (003)\). If \((x, y)\) is in the interior of \(\mathbb{U}_{003}(z) \setminus \mathbb{U}_{112}(z)\), then \((x, y) \notin \mathbb{U}_{330}(z), (x, y) \notin \mathbb{U}_{221}(z), (x, y) \notin \mathbb{U}_{112}(z)\). This implies that \(\mathbb{N}_{330}(F) = \emptyset, \mathbb{N}_{221}(F) = \emptyset, \mathbb{N}_{112}(F) = \emptyset\). Hence, \(\mathbb{N}(F) \subseteq \mathbb{N}_{003}(F)\). Since \(\mathbb{M}(F) \subseteq \{b, h\}\), it follows that \(\overline{m} \in \mathbb{N}_{003}(F)\) iff \(\overline{m} \in \{b\}^3\). Therefore,

\[ \mathbb{N}(F) \subseteq \mathbb{N}_{003}(F) = \{b\}^3 \]

Boundary. It can be seen that \(\mathbb{U}_{003}(z) \setminus \mathbb{U}_{112}(z)\) has no boundaries along which it intersects an adjacent region. ■

15.9 Equilibrium Outcomes in Subgame [1a]

Since the areas are mutually exclusive and cover the entire action-space it follows that the conditions are necessary and sufficient. For example, if \((x, y)\) is within the Core, then all the sellers multihome. Furthermore, it’s been shown that in the other areas some (or all) of the sellers don’t multihome. Therefore, all the sellers multihome iff \((x, y)\) is within the Core. The same argument can be applied to the other sorts of outcome. It follows that:

**Proposition 15.1 (Membership in SG [1a].)** Let \(f_c = (x, -\omega), f_d = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0 \) and \(y \geq -|z|\). It follows that if \(\overline{m} \in \mathbb{N}(F)\), then:

\[ \overline{m} \in \{b\}^3 \iff G(x, y, z) \leq 200\tau^2\sigma^2 \]

\[ \overline{m} \in \{b\}^2 \times \{b\} \iff \begin{cases} G(x, y, z) > 200\tau^2\sigma^2, \\ x^2 + y^2 \leq \omega^2 + z^2 \end{cases} \]

\[ \overline{m} \in \{b\} \times \{b\}^2 \iff \begin{cases} x^2 + y^2 > \omega^2 + z^2, \\ H(x, y, z) \geq 200\tau^2\sigma^2 \end{cases} \]

\[ \overline{m} \in \{b\}^3 \iff H(x, y, z) < 200\tau^2\sigma^2 \]

**Proposition 15.2 (Equilibrium Configurations in SG [1a].)** Let \(f_c = (x, -\omega), f_d = (y, z)\) and suppose that \(z \leq 0, y \geq -|z|\). If \(\overline{m} \in \mathbb{N}(F)\), then:

\[ G(x, y, z) \leq 200\tau^2\sigma^2 \iff \begin{cases} \hat{\Sigma}_A(f_c, \overline{m}) = 3, \\ \hat{\Sigma}_A(f_d, \overline{m}) = 3, \\ \hat{\Sigma}_B(f_d, \overline{m}) = 0 \end{cases} \]

\[ G(x, y, z) > 200\tau^2\sigma^2, \\ x^2 + y^2 \leq \omega^2 + z^2 \iff \begin{cases} \hat{\Sigma}_A(f_c, \overline{m}) = 2, \\ \hat{\Sigma}_A(f_d, \overline{m}) = 2, \\ \hat{\Sigma}_B(f_d, \overline{m}) = 1 \end{cases} \]
Proof. Each part of the proposition can be considered in turn:

Part 1. If \( y \geq z \), \( (x, y) \in \mathbb{U}_{330}(z) \) and \( \bar{m} \in \mathcal{N}(F) \), then \( \bar{m} = \{h\}^3 \). Furthermore, if \( y \geq z \), then
\[
A(f_c, h) = 1, \quad A(f_d, h) = 1, \quad B(f_d, h) = 0
\]
This is because if \( y \geq z \), then \( A(f_k, h) = 1 \), \( B(f_d, h) = 0 \). Therefore, we have the result.

If \( \bar{m} \) is stable under \( F = (f_c^T, f_d^T) \). (Stability is a necessary condition for a RSPNE.) Hence, \( (x, y) \in \mathbb{U}_{330}(z) \).

Part 2. If \( y \geq z \), \( (x, y) \in \mathbb{U}_{221}(z) \setminus \mathbb{U}_{330}(z) \) and \( \bar{m} \in \mathcal{N}(F) \), then \( \bar{m} = \{h\} \times \{b\} \).

If \( y \geq z \), then
\[
A(f_c, h) = 1, \quad A(f_d, h) = 1, \quad B(f_d, h) = 0
\]
and
\[
A(f_c, b) = 0, \quad A(f_d, b) = 0, \quad B(f_d, b) = 1
\]
Since two sellers choose \( h \) and the other chose \( b \), we have the result.

If \( \bar{m} \) is stable under \( F = (f_c^T, f_d^T) \). (Stability is a necessary condition for a RSPNE.) Hence, \( (x, y) \in \mathbb{U}_{221}(z) \). If \( (x, y) \in \text{int} \ \mathbb{U}_{221}(z) \), then \( (221) \) is the only stable configuration. However, if \( (x, y) \in \mathbb{U}_{221}(z) \cap \mathbb{U}_{330}(z) \), then all sellers multihome, \( \bar{m} \in \{h\}^3 \). Since \( y > z \), multihoming generates Configuration \((330)\). Therefore, if Configuration \((221)\) occurs, then
\[
(x, y) \in \mathbb{U}_{221}(z) \setminus \mathbb{U}_{330}(z)
\]

Parts 3 - 4 are proved in an exactly parallel manner. ■
Chapter 16

Membership in Subgame [1b]

Chapter 16 investigates sellers’ membership when \( z \leq 0 \) and \( y < -|z| \). The chapter has the following sections. Section (1) identifies necessary conditions for each possible configuration. Section (2) identifies the part of the action-space where a given configuration is stable. That is, section (2) finds two regions of stability, which intersect at their boundaries and cover the action space. Section (3) constructs mutually exclusive areas by allocating each boundary to one specific area. The areas are labelled as follows: LHS, and RHS. Section (4) identifies the relevant options (\( \mathbb{M}(F) \subseteq \{b, h\} \)) and summarizes the correspondence between membership vectors and configurations. Section (5) finds that within the RHS all sellers multihome. Section (6) finds that within the LHS two sellers multihome and the third seller joins Network B. Section (7) summarizes the results and shows that there is a one-to-one correspondence between the four areas and the four possible configurations.

16.1 Two Possible Configurations

In Subgame [1b] there are two possible Configurations: (003) and (303). The following sections find necessary and sufficient conditions for the stability of these configurations.

16.1.1 The Stability of (003)

This section identifies the necessary and sufficient conditions that \( F = (f_\xi^T, f_\zeta^T) \) must satisfy to ensure that \( m \in \mathbb{M}^3 \) is stable whenever

\[
\Sigma_A(f_\xi, m) = 0, \quad \Sigma_A(f_\zeta, m) = 0, \quad \Sigma_B(f_\zeta, m) = 3
\]

It can be shown that unilateral deviation from Configuration (003) isn’t profitable iff the following conditions are satisfied:

\[
z^2 \leq \omega^2 \tag{3}
\]

\[
\left\{ \frac{5\sigma + \frac{\omega^2 - x^2}{2\tau}}{\omega^2} \right\}^2 + \left\{ \frac{5\sigma + \frac{z^2 - y^2}{2\tau}}{\omega^2} \right\}^2 \leq 50\sigma^2 \tag{4}
\]

\[
x^2 \geq \omega^2 \tag{5}
\]
These conditions can be simplified as follows: Firstly, in Subgame [1b], $z \leq \omega$. Hence, (3) is always satisfied. Secondly, in Subgame [1b], $y < -|z| < 0$, which implies that $y^2 > z^2$. Hence, (4) holds whenever (5) holds. It follows that (4) is redundant. Therefore, if $y < -|z|$, then Configuration (003) is stable iff

$$|x| \geq \omega,$$

which implies that Network A offers a (weakly) negative end-user benefit in market $\varsigma$. It follows that:

**Lemma 16.1 (Stability of (003) in SG [1b].)** Let $f_\varsigma = (x, -\omega)$, $f_q = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $z \leq 0$ and $y < -|z| = z$. It follows that if $m \in \mathbb{M}^3$ and $\Sigma_A(f_\varsigma, m) = 0$, $\Sigma_A(f_q, m) = 0$, $\Sigma_B(f_q, m) = 3$, then $m$ is stable under $F = (f_\varsigma^T, f_q^T)$ iff Network A offers a (weakly) negative end-user benefit in market $\varsigma$, $|x| \geq \omega$.

### 16.1.2 The Stability of (303)

This section identifies the necessary and sufficient conditions that $F = (f_\varsigma^T, f_q^T)$ must satisfy to ensure that $m \in \mathbb{M}^3$ is stable whenever

$$\Sigma_A(f_\varsigma, m) = 3, \quad \Sigma_A(f_q, m) = 0, \quad \Sigma_B(f_q, m) = 3$$

It can be shown that unilateral deviation from Configuration (303) isn’t profitable iff the following conditions are satisfied:

$$\left\{5\sigma - \frac{\omega^2 - x^2}{2\tau}\right\}^2 + \left\{5\sigma - \frac{\omega^2 - z^2}{2\tau}\right\} \leq 50\sigma^2 \quad (1)$$

$$y^2 \geq z^2 \quad (2)$$

$$x^2 \leq \omega^2 \quad (3)$$

(See Appendix E.) These conditions can be simplified as follows: Firstly, in Subgame [1b], $y < -|z|$, which implies that $y^2 \geq z^2$. Hence, (2) is necessarily satisfied. Secondly, in Subgame [1b], $z \leq \omega$. It follows that if (3) is satisfied, then (1) is satisfied. Hence, (1) is redundant. It follows that:

**Lemma 16.2 (Stability of (303) in SG [1b].)** Let $f_\varsigma = (x, -\omega)$, $f_q = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $z \leq 0$ and $y < -|z| = z$. It follows that if $m \in \mathbb{M}^3$ and $\Sigma_A(f_\varsigma, m) = 3$, $\Sigma_A(f_q, m) = 0$, $\Sigma_B(f_q, m) = 3$, then $m$ is stable under $F = (f_\varsigma^T, f_q^T)$ iff Network A offers a positive end-user benefit in market $\varsigma$, $|x| \leq \omega$.

### 16.2 Regions of Stability

A configuration is stable iff none of the sellers can profit through unilateral deviation. Whether or not a particular configuration is stable depends on the value of $z$ and the location of $(x, y)$ within the action space. The action-space
can be divided into a number of regions, where each region is paired with a particular configuration. Hence, it’s useful to introduce the following notation: configuration \((IJK)\) is stable under \(F \in \mathcal{F}\) iff \(x, y) \in \mathcal{V}_{IJK} \subseteq [-\omega, \rho]^2\). The set \(\mathcal{V}_{IJK}\) will be referred to as the region of stability of Configuration \((IJK)\).

Regions of stability have been identified for the two configurations that occur in Subgame \([1b]\). These regions are as follows:

- **Stability of \((003)\):** If \(z \leq 0\) and \(y < -|z|\), then \((x, y) \in \mathcal{V}_{003}\) iff \(|x| \geq \omega\).
- **Stability of \((303)\):** If \(z \leq 0\) and \(y < -|z|\), then \((x, y) \in \mathcal{V}_{303}\) iff \(|x| \leq \omega\).

It can be seen that each region is circumscribed by a series of boundaries. The boundaries divide the action space into two regions, each corresponding to a different configuration. Within its corresponding region the necessary and sufficient conditions for the stability of a particular configuration are satisfied. It can be seen that the regions are mutually exclusive except at the boundaries, where they intersect. Furthermore, they cover the entire action space, \([-\omega, \rho]^2\).

![Figure 11. Regions of Stability in Subgame [1b]](image)

### 16.3 Mutually Exclusive Areas

Since the regions only intersect at the boundary, it follows that within the interior of a region the necessary conditions for just one type of outcome are satisfied. Since stability is a necessary condition for a RSPNE (and there must exist an equilibrium configuration), it follows that if \((x, y) \in \text{int} \mathcal{V}_{IJK}\), then \(\mathcal{N}(F) \subseteq \mathcal{N}_{IJK}(F)\). However, because adjacent regions intersect at the boundaries, there appears to be more than one possible outcome at these points. (Although, there can only be one configuration which satisfies the sufficient conditions for a RSPNE.) To help resolve the ambiguity, it’s useful to construct a series of mutually exclusive areas:
### Table 4. Areas of the Action-Space in Subgame [1b]

<table>
<thead>
<tr>
<th>Area</th>
<th>Boundaries</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHS</td>
<td>$V_{303}$</td>
</tr>
<tr>
<td>RHS</td>
<td>$V_{003} \setminus V_{303}$</td>
</tr>
</tbody>
</table>

These sets are defined in such a way that a given boundary only belongs to one set. The boundaries along which adjacent regions intersect are as follows:

- **LHS** has one closed, adjacent boundary:
  
  $x = \omega$ (vertical line).

- **RHS** has no closed, adjacent boundary.

#### 16.4 Relevant Membership Options

A Restricted SPNE (RSPNE) is a fixed point of the seller's best-response functions. That is, $\bar{m} \in \mathcal{N}(F)$ iff $\bar{m}_i = \max \bar{M}(F, \bar{m}_{-i})$, for $i = 1, 2, 3$. Let $\mathcal{N}(F) \subseteq \{b, h\}^3$ denote the set of membership vectors which satisfy the criteria for a RSPNE when fees are $F \in \mathbb{F}$. Also, let $\mathcal{N}_{ijk}(F)$ denote the subset of equilibria in which Configuration $(IJK)$ occurs; where

$$\Sigma_A(f, m) = I, \quad \Sigma_A(f, c, m) = J, \quad \Sigma_B(f, q, m) = K$$

An option, $m \in M$, is said to be "irrelevant" if there is another option, $m' \in M$, such that $m' \geq m$ or $m' \sim m$, $m > m$. An option is said to be relevant if it's not irrelevant. Let $\bar{M}(F)(\subseteq M)$ denote the set of relevant membership options. The set of relevant options when $z \leq 0$ can be found as follows. Firstly, by assumption, $\bar{W}(z) \geq 0$, which implies that $b \succeq a$. Since $b > a$, it follows that $a \notin M(F)$. Secondly, if $-\omega \leq z \leq 0$, then $\bar{W}(\max\{y, z\}) \geq \bar{W}(y)$, which implies that $h \succeq a$. Since $h > a$, it follows that $a \notin M(F)$. This analysis implies that: $M(F) \subseteq \{b, h\}$. Furthermore, it’s been shown that if $M(F) \subseteq \{b, h\}$, then:

$$\bar{m} \in \mathcal{N}_{303}(F) \iff \bar{m} \in \{h\}^3 \subseteq \mathcal{N}(F)$$

$$\bar{m} \in \mathcal{N}_{003}(F) \iff \bar{m} \in \{b\}^3 \subseteq \mathcal{N}(F)$$

(See Appendix B.)

---

1A seller often has more than one best response. Best response functions were created using a lexicographic decision rule. A RSPNE is a fixed point of the sellers best-response functions.
16.5 Membership on the RHS

By definition, \((x, y)\) is on the RHS iff \(x > \omega\). It can be shown that if \((x, y)\) is on the RHS, then all sellers join Network \(B\). This implies that Network \(A\) is inactive and all the sellers are on Network \(B\).

**Lemma 16.3** (On the RHS all sellers join Network \(B\)). Let \(f_0 = (x, -\omega)\), \(f_d = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0\) and \(y < -|z|\). It follows that if \(x > \omega\) and \(\overline{m} \in \mathcal{N}(F)\), then \(\overline{m} \in \{b\}^3\).

**Proof.** Suppose that \((x, y) \in V_{003} \setminus V_{303}\). Either \((x, y)\) is in the interior of the area or it’s on the boundary. These are analyzed in turn:

**Interior.** In Subgame [1b] there are two possible configurations: (003) and (303). If \((x, y)\) is in the interior of \(V_{003} \setminus V_{303}\), then \((x, y) \notin V_{303}\). This implies that \(N_{303}(F) = \emptyset\). Hence, \(\mathcal{N}(F) \subseteq N_{003}(F)\). Since \(\mathcal{M}(F) \subseteq \{b, \overline{h}\}\), it follows that \(\overline{m} \in N_{003}(F)\) iff \(\overline{m} \in \{b\}^3\). Therefore,

\[
\mathcal{N}(F) \subseteq N_{003}(F) = \{b\}^3
\]

**Boundary.** It can be seen that \(V_{003} \setminus V_{303}\) has no boundary along which it intersects an adjacent region.

16.6 Membership on the LHS

By definition, \((x, y)\) is on the LHS iff \(y < -|z|\) and \(x \leq \omega\). It can be shown that in this area all sellers multihome. If \(y < -|z|\), then the tie is slack, which means that Network \(B\) has a higher seller-fee and offers the higher end-user benefit in market \(d\). It follows that the sellers are on platforms \(C_A\) and \(D_B\).

**Lemma 16.4** (On the LHS all sellers Multihome). Let \(f_0 = (x, -\omega)\), \(f_d = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0\) and \(y < -|z|\). It follows that if \(x \leq \omega\) and \(\overline{m} \in \mathcal{N}(F)\), then \(\overline{m} \in \{b\}^3\).

**Proof.** Suppose that \((x, y) \in V_{303}\). Either \((x, y)\) is in the interior of the area or it’s on the boundary. These are analyzed in turn:

**Interior.** In Subgame [1b] there are two possible configurations: (303) and (003). If \((x, y)\) is in the interior of \(V_{303}\), then \((x, y) \notin V_{003}\). This implies that \(N_{003}(F) = \emptyset\). It follows that \(\mathcal{N}(F) \subseteq N_{303}(F)\). Since \(\mathcal{M}(F) \subseteq \{b, \overline{h}\}\), it follows that \(\overline{m} \in N_{303}(F)\) iff \(\overline{m} \in \{b\}^3\). In either case, all sellers multihome: \(\mathcal{N}(F) = \{b\}^3\).

**Boundary.** If \((x, y)\) is on the boundary of \(V_{303}\), then \(x = \omega, -\omega \leq y < -|z|\). If \(x = \omega\) and \(y < -|z| = z\), then \(b \sim \overline{h}\). Since \(\mathcal{M}(F) \subseteq \{b, \overline{h}\}\) and \(\overline{h} > b\), it follows that \(\mathcal{M}(F) = \{\overline{h}\}\). Therefore, \(\mathcal{N}(F) = \{\overline{h}\}^3\). ■
16.7 Equilibrium Outcomes in Subgame [1b]

Since the areas are mutually exclusive and cover the entire action-space it follows that the conditions are necessary and sufficient. For example, Lemma # says that if \((x, y)\) belongs to the LHS, then all the sellers multihome. Furthermore, it’s been shown that in the other area the sellers don’t multihome. Therefore, all the sellers multihome iff \((x, y)\) is within the LHS. The same argument can be applied to the other outcome. It follows that:

**Proposition 16.1 (Membership in SG [1b].)** Let \(f_\varsigma = (x, -\omega), f_\eta = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0\) and \(y < -|z|\). It follows that if \(\bar{m} \in N(F)\), then:

\[
\begin{align*}
\bar{m} & \in \{b\}^3 \text{ iff } x \leq \omega \\
\bar{m} & \in \{b\}^3 \text{ iff } x > \omega
\end{align*}
\]

**Proposition 16.2 (Equilibrium Configurations in SG [1b].)** Let \(f_\varsigma = (x, -\omega), f_\eta = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(z \leq 0\) and \(y < -|z|\). If \(\bar{m} \in N(F)\), then:

\[
\begin{align*}
x \leq \omega \text{ iff } & \quad \Sigma_A(f_\varsigma, \bar{m}) = 3, \\
& \quad \Sigma_A(f_\eta, \bar{m}) = 0, \\
& \quad \Sigma_B(f_\eta, \bar{m}) = 3 \\
x > \omega \text{ iff } & \quad \Sigma_A(f_\varsigma, \bar{m}) = 0, \\
& \quad \Sigma_A(f_\eta, \bar{m}) = 0, \\
& \quad \Sigma_B(f_\eta, \bar{m}) = 3
\end{align*}
\]
Chapter 17

Membership in Subgame [1]

Chapter 17 combines the results from the last two chapters in order to find sellers’ membership when $z \leq 0$. The chapter has the following sections. Section (1) uses previous results to characterize sellers’ demand for each platform. Section (2) summarizes the results regarding sellers membership and provides an intuitive explanation for the pattern that was found. Section (3) finds the networks’ payoffs using the results for sellers’ demand. Section (4) notes that membership in the other subgames is very similar.

17.1 Sellers’ Demand for Platforms

In Subgame [1] Network $B$ offers a positive end-user benefit, $\hat{W}(z) \geq 0$, and subsidizes sellers, $z \leq 0$. This chapter summarizes the equilibrium outcomes for this class of subgames.

It has been shown that in Subgame [1] the possible configurations are: $(330), (221), (112), (003), (303)$. The necessary and sufficient conditions for the existence of a RSPNE in which a particular configuration occurs are as follows:
Table 5. Configurations in Subgame [1]

Using the three-digit codes it’s possible to find the values of \( \bar{\Sigma}_A(f_{\xi}, \bar{m}) \), \( \bar{\Sigma}_A(f_{\eta}, \bar{m}) \), \( \bar{\Sigma}_B(f_{\eta}, \bar{m}) \). The results are summarized in the following propositions.

**Proposition 17.1 (Sellers’ demand for \( C_A \) in SG [1].)** Let \( f_{\xi} = (x, -\omega) \), \( f_{\eta} = (y, z) \), where \( x, y, z \in [-\omega, \rho] \), and suppose that \( z \leq 0 \). If \( \bar{m} \in \mathcal{N}(\mathbf{F}) \), then

\[
\bar{\Sigma}_A(f_{\xi}, \bar{m}) = \bar{\Sigma}_A(x, y, z),
\]

where \( \bar{\Sigma}_A(,\) is defined as follows:

\[
\bar{\Sigma}_A(x, y, z) = 3 \text{ iff } \begin{cases} y < -|z|, & \text{or } \begin{cases} y \geq -|z|, \\ x \leq \omega \end{cases} \\ G(x, y, z) \leq 200\tau^2\sigma^2 \end{cases}
\]

\[
\bar{\Sigma}_A(x, y, z) = 2 \text{ iff } \begin{cases} y \geq -|z|, \\ G(x, y, z) > 200\tau^2\sigma^2, \\ x^2 + y^2 \leq \omega^2 + z^2 \end{cases}
\]

\[
\bar{\Sigma}_A(x, y, z) = 1 \text{ iff } \begin{cases} y \geq -|z|, \\ x^2 + y^2 > \omega^2 + z^2, \\ H(x, y, z) \geq 200\tau^2\sigma^2 \end{cases}
\]

\[
\bar{\Sigma}_A(x, y, z) = 0 \text{ iff } \begin{cases} y < -|z|, \\ x > \omega \end{cases} \text{ or } \begin{cases} y \geq -|z|, \\ H(x, y, z) < 200\tau^2\sigma^2 \end{cases}
\]

The four scenarios are mutually exclusive and exhaustive.
Proposition 17.2 (Sellers’ demand for $D_A$ in SG [1].) Let $f_{\xi} = (x, -\omega)$, $f_y = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $z \leq 0$. If $\bar{m} \in N(F)$, then

$$\Sigma_A(f_y, \bar{m}) = \Sigma_A^d(x, y, z),$$

where $\Sigma_A^d(\cdot)$ are defined as follows:

$$\Sigma_A^d(x, y, z) = 3 \text{ iff } \begin{cases} y \geq -|z|, \\ G(x, y, z) \leq 200\tau^2\sigma^2 \end{cases}$$

$$\Sigma_A^d(x, y, z) = 2 \text{ iff } \begin{cases} y \geq -|z|, \\ G(x, y, z) > 200\tau^2\sigma^2, \\ x^2 + y^2 \leq \omega^2 + z^2 \end{cases}$$

$$\Sigma_A^d(x, y, z) = 1 \text{ iff } \begin{cases} y \geq -|z|, \\ x^2 + y^2 > \omega^2 + z^2, \\ H(x, y, z) \geq 200\tau^2\sigma^2 \end{cases}$$

$$\Sigma_A^d(x, y, z) = 0 \text{ iff } y < -|z| \text{ or } \begin{cases} y \geq -|z|, \\ H(x, y, z) < 200\tau^2\sigma^2 \end{cases}$$

The four scenarios are mutually exclusive and exhaustive.
Proposition 17.3 (Sellers’ demand for $D_B$ in SG [1].) Let $f_x = (x, -\omega)$, $f_y = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $z \leq 0$. If $\bar{m} \in \mathcal{N}(F)$, then

\[ \bar{\Sigma}_B(f_y, \bar{m}) = \bar{\Sigma}_B(x, y, z), \]

where $\bar{\Sigma}_B(.)$ is defined as follows:

\[ \bar{\Sigma}_B(x, y, z) = 3 \text{ iff } y < -|z| \text{ or } \begin{cases} y \geq -|z|, \\ H(x, y, z) < 200\tau^2\sigma^2 \end{cases} \]

\[ \bar{\Sigma}_B(x, y, z) = 2 \text{ iff } \begin{cases} y \geq -|z|, \\ x^2 + y^2 > \omega^2 + z^2, \\ H(x, y, z) \geq 200\tau^2\sigma^2 \end{cases} \]

\[ \bar{\Sigma}_B(x, y, z) = 1 \text{ iff } \begin{cases} y \geq -|z|, \\ G(x, y, z) > 200\tau^2\sigma^2, \\ x^2 + y^2 \leq \omega^2 + z^2 \end{cases} \]

\[ \bar{\Sigma}_B(x, y, z) = 0 \text{ iff } \begin{cases} y \geq -|z|, \\ G(x, y, z) \leq 200\tau^2\sigma^2 \end{cases} \]

The four scenarios are mutually exclusive and exhaustive.

It has been shown that if $W(z) \geq 0$, then the nature of the outcome depends on whether or not the tie is "binding". When $z \leq 0$, the tie is "slack" iff $y < -|z|$ and the tie is "binding" iff $y \geq -|z|$.
17.2 The "Shell Model"

17.2.1 Sellers’ Membership when the Tie is "Slack"

If \( y < -|z| \), then the tie is slack. Hence, sellers have a free choice over which combination of platforms they’re on. If \( W(x) \geq 0 \), then sellers multihome so that they’re on Platform \( C_A \) and Platform \( D_B \). Whereas, if \( W(x) < 0 \), then sellers join Network \( B \) so that they’re only on Platform \( D_B \). Therefore, if the tie is slack, \( y < -|z| \), then Network \( A \) is always inactive in market \( \varnothing \). However, they may attract sellers in market \( \varsigma \).

17.2.2 Sellers’ Membership when the Tie is "Binding"

The part of the action-space where the tie is binding has the following structure. The origin of the coordinate system is in the centre of the Core. Here, Network \( A \) offers the maximum level of end-user benefit on both its platforms; which is sufficient to ensure ensure that all the sellers join. The inner oval is surrounded by a series of concentric shells (or lobes). As \( x^2, y^2 \) increase, we move out of the Core and pass through a series of shells. The 1st Shell contains two sellers; and 2nd Shell contains one seller. If \( x^2, y^2 \) increase further, then we move out of the shells into the final region, Exterior, where there are no sellers on Network \( A \). This structure can be explained as follows:

If \( y \geq -|z| \), then the tie is binding and so sellers have to choose between Network \( A \) and Network \( B \). Therefore, the sellers divide across the networks. Since Network \( A \) is active in both markets, it can always offer a higher aggregate end-user benefit. Furthermore, if its aggregate end-user benefit is sufficiently large, then Network \( A \) can attract all the sellers. However, Network \( A \) is unable to exclude Network \( B \), just by matching the end-user benefit on its platform because a seller, generally, prefers an asymmetric outcome. That is, when making their membership decision a seller considers two things: firstly, the aggregate end-user benefit on each network; and, secondly, the number of other sellers on each network. This can be explained as follows:

Consider the following situation: (a) networks offer the same aggregate end-user benefit, \( \bar{W}(x) + \bar{W}(y) = \bar{W}(z) \); (b) networks don’t set the same fees, \( x \neq \pm \omega \) or \( y \neq \pm z \); and (c) the tie is binding, \( y \geq z \). It follows that the networks are differentiated: Network \( A \) offers a higher end-user benefit in market \( \varsigma \) and Network \( B \) offers a higher end-user benefit in market \( \varnothing \). With regard to sellers’ membership, there are two possible outcomes: (1) all sellers are on the same network; and (2) the sellers join different networks.

- If all sellers are on one network, then they receive the same extra-surplus and are equally competitive. In this case, competition forces sellers to pass the extra-surplus on to their customers.

---

1The boundary between the shells is a circle, \( x^2 + y^2 = \omega^2 + z^2 \), and at points on this circle the networks offer the same aggregate end-user benefit. The reason for this type of boundary is as follows. Suppose the two seller have made their membership choice and the final seller is considering his options. If there’s already one seller on each network, then there will necessarily be an asymmetric outcome. It follows that the third seller joins the network that offers the higher aggregate end-user benefit. At points on the circle they are indifferent between the networks, which implies that the circle must be the boundary.
• If the sellers join different networks, then the seller(s) on Network A are more competitive in market $\varsigma$; whereas, the seller(s) on Network B are more competitive in market $\vartheta$. This allows sellers to retain part of the surplus created by having access to platforms. In other words, the sellers are able to put further "distance" between them and their rivals by joining different networks, which gives them an additional source of rents from increased specialization.

Therefore, all other things being equal, sellers prefer an asymmetric outcome in which sellers divide themselves across the networks, which creates a form of repulsion between the sellers. To overcome this repulsion, Network A needs to more than match the end-user benefit on Network B if it’s to attract all the sellers. This explains why there’s an intermediate region in Network A’s action space where fees are such that both networks are active.

17.2.3 Atomic Analogy

There is a useful analogy between the way the sellers arrange themselves in the presence of platforms and the way electrons arrange themselves in the presence of atomic nuclei. In both cases a shell structure is created by the interplay of attractive and repulsive forces coupled with the discreet nature of the entities. Sellers have a tendency not to agglomerate; whereas, electrons are repelled by other electrons (like charges repel one another). In equilibrium, the repulsion between sellers is balanced by their wish to transact with buyers on the platform; whereas, electrons are attracted by the positively charged nucleus.

17.3 Networks’ Payoffs

This section summarizes the payoffs received by the networks in Subgame [1].

**Network A.** In Subgame 1 the profits made by Network A in markets $\varsigma$, $\vartheta$ are

$$\gamma_A(f_\varsigma, \overline{m}) = \frac{1}{2}\mu Q(x) \tilde{\Sigma}_A^\varsigma(x, y, z),$$

$$\gamma_A(f_\vartheta, \overline{m}) = \frac{1}{2}\mu Q(y) \tilde{\Sigma}_A^\vartheta(x, y, z),$$

where $\overline{m} \in N(F)$ and $Q(x) \equiv \frac{1}{2\tau}(\omega + x)$, $Q(y) \equiv \frac{1}{2\tau}(\omega + y)$. It follows that:

**Proposition 17.4 (Network A’s Payoff in SG [1]).** Let $f_\varsigma = (x, -\omega), f_\vartheta = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $z \leq 0$. If $\overline{m} \in N(F)$, then

$$\Pi_A(F, \overline{m}) = \Pi_A(x, y, z),$$
where $\Pi_A(.)$ is defined as follows:

$$
\Pi_A(x, y, z) = \begin{cases} 
\frac{\mu}{2\tau} (2\omega + x + y) & \text{iff } \begin{cases} y \geq -|z|, \\
G(x, y, z) \leq 200\tau^2\sigma^2 
\end{cases} \\
\frac{\mu}{3\tau} (2\omega + x + y) & \text{iff } \begin{cases} y \geq -|z|, \\
G(x, y, z) > 200\tau^2\sigma^2, \\
x^2 + y^2 \leq \omega^2 + z^2 
\end{cases} \\
\frac{\mu}{6\tau} (2\omega + x + y) & \text{iff } \begin{cases} y \geq -|z|, \\
x^2 + y^2 > \omega^2 + z^2 
\end{cases} \\
\frac{\mu}{9\tau} (\omega + x) & \text{iff } y < -|z|, \\
x \leq \omega \\
0 & \text{iff } \begin{cases} y < -|z|, \\
x > \omega \\
H(x, y, z) < 200\tau^2\sigma^2 
\end{cases} \\
0 & \text{iff } \begin{cases} y < -|z|, \\
G(x, y, z) > 200\tau^2\sigma^2 
\end{cases} 
\end{cases}
$$

The five scenarios are mutually exclusive and exhaustive.

**Network B.** Since Platform $C_B$ is switched off, Network $B$ makes no profit in market $c$. However, Network $B$ is (potentially) active in market $d$. Their profit in market $d$ is

$$
\Upsilon_B(f_d, \tilde{m}) = \frac{1}{3} \mu Q(z) \tilde{\Sigma}_B(x, y, z),
$$

where $\tilde{m} \in N(F)$ and $Q(z) \equiv \frac{1}{2\tau} (\omega + z)$. It follows that:

**Proposition 17.5 (Network B’s Payoff in SG [1].)** Let $f_c = (x, -\omega)$, $f_d = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $z \leq 0$. If $\tilde{m} \in N(F)$, then

$$
\Pi_B(F, \tilde{m}) = \Pi_B(x, y, z),
$$

where $\Pi_B(.)$ is defined as follows:

$$
\Pi_B(x, y, z) = \begin{cases} 
\frac{\mu}{2\tau} (\omega + z) & \text{iff } y < -|z| \text{ or } \begin{cases} y \geq -|z|, \\
H(x, y, z) < 200\tau^2\sigma^2 
\end{cases} \\
\frac{\mu}{3\tau} (\omega + z) & \text{iff } \begin{cases} y \geq -|z|, \\
H(x, y, z) \geq 200\tau^2\sigma^2, \\
x^2 + y^2 > \omega^2 + z^2 
\end{cases} \\
\frac{\mu}{6\tau} (\omega + z) & \text{iff } \begin{cases} y \geq -|z|, \\
x^2 + y^2 \leq \omega^2 + z^2 
\end{cases} \\
\frac{\mu}{9\tau} (\omega + z) & \text{iff } G(x, y, z) > 200\tau^2\sigma^2 \\
0 & \text{iff } \begin{cases} y \geq -|z|, \\
G(x, y, z) \leq 200\tau^2\sigma^2 
\end{cases} 
\end{cases}
$$

These four scenarios are mutually exclusive and exhaustive.
17.4 Membership in Other Subgames

Subgame [2] and [3] can be analyzed in the same way that Subgame [1] was analyzed in the proceeding chapters. Furthermore, the outcomes for subgames [2] and [3] are very similar to that found for Subgame [1]. Therefore, rather than repeat the analysis, the results for the remaining subgames are given in the Appendix F and Appendix G.
Stage 1: Networks’ Fees
Chapter 18

Exclusion of Network B

Chapter 18 shows that Network A can (and will) always exclude Network B. Section (1) claims that Network A will always choose positive seller fees. Section (2) restates Network A’s optimization problem. In particular, it characterizes the objective function. Section (3) shows that by setting its "default fees" Network A can always exclude Network B. Section (4) compares the payoff from the default-fees with the maximum payoff that could come from accommodating Network B. Section (4) summarizes the results and shows that it is never in the interest of Network A to share the market with Network B.

18.1 Positive Seller-Fees

The proceeding chapters analyzed the sellers’ membership decisions given a particular value of $x, y, z$. From this it was possible to find the number of sellers on each platform and the networks payoffs. The networks choose their fees so as to maximize their payoffs, which often comes down to attracting the maximum number of sellers. This chapter begins to investigate the fees set by Network A.

Imagine that $z$ is known and Network A is called upon to set its fees: $x, y \in [-\omega, \rho]$. (Note that fees are actually set simultaneously.) It can be seen that $(x, y)$ can be anywhere within a square that has its vertices in each quadrant of $\mathbb{R}^2$. Hence, the seller fees on one (or more) of Network A’s platforms could be negative. However, it can be shown that for any such fees there is an alternative choice where the fees are positive and produce a higher profit. (See Appendix II.)

Lemma 18.1 (Positive Seller-Fees.) If $(\tilde{x}, \tilde{y}) \in \arg \max_{x,y} \tilde{\Pi}_A(x, y, z)$, then $	ilde{x} \geq 0, \tilde{y} \geq 0$. That is, Network A sets positive seller-fees.

18.2 Restating Network A’s Problem

Network A selects $x, y$ so as to maximize $\tilde{\Pi}_A(x, y, z)$, given $z$. It has been shown that if $(x, y)$ is outside the 1st Quadrant (that is, $x > 0$ or $y > 0$), then $(x, y)$ is dominated. Therefore, if $(\tilde{x}, \tilde{y}) \in \arg \max_{x,y} \tilde{\Pi}_A(x, y, z)$, then $\tilde{x} \geq 0$, $\tilde{y} \geq 0$. Since any point outside the 1st Quadrant is dominated, the other three quadrants aren’t relevant. This information can be used to redefine Network A’s maximization problem.
The form of Network A’s payoff function depends on the value of $z$. It’s been shown that the form of the payoff function differs across the subgames. However, once we restrict attention to the 1st Quadrant, the three subgames generate a very similar objective function. There are two cases to consider: $z \leq \omega$ and $z > \omega$. The first of these correspond to subgames [1], [2] and the second corresponds to Subgame [3]. The payoff function for Subgame [1] has been derived in the preceding chapters; and the payoff functions for the other two subgames are stated in the Appendix.

It can be shown that if $z \leq \omega$ and $x \geq 0, y \geq 0$, then Network A’s payoff becomes

$$
\tilde{\Pi}_A(x, y, z) = \frac{\mu}{2\tau} (2\omega + x + y) \quad \text{if} \quad G(x, y, z) \leq 200\tau^2\sigma^2
$$

$$
\tilde{\Pi}_A(x, y, z) = \frac{\mu}{\sqrt{\tau}} (2\omega + x + y) \quad \text{if} \quad \begin{cases} 
G(x, y, z) > 200\tau^2\sigma^2 \\
x^2 + y^2 \leq \omega^2 + z^2
\end{cases}
$$

$$
\tilde{\Pi}_A(x, y, z) = \frac{\mu}{6\tau} (2\omega + x + y) \quad \text{if} \quad \begin{cases} 
x^2 + y^2 > \omega^2 + z^2 \\
H(x, y, z) \geq 200\tau^2\sigma^2
\end{cases}
$$

$$
\tilde{\Pi}_A(x, y, z) = 0 \quad \text{if} \quad H(x, y, z) < 200\tau^2\sigma^2
$$

Whereas, if $z > \omega$ and $x \geq 0, y \geq 0$, then Network A’s payoff becomes

$$
\tilde{\Pi}_A(x, y, z) = \frac{\mu}{2\tau} (2\omega + x + y) \quad \text{if} \quad G(x, y, \omega) \leq 200\tau^2\sigma^2
$$

$$
\tilde{\Pi}_A(x, y, z) = \frac{\mu}{\sqrt{\tau}} (2\omega + x + y) \quad \text{if} \quad \begin{cases} 
G(x, y, \omega) > 200\tau^2\sigma^2 \\
x^2 + y^2 \leq 2\omega^2
\end{cases}
$$

$$
\tilde{\Pi}_A(x, y, z) = \frac{\mu}{6\tau} (2\omega + x + y) \quad \text{if} \quad \begin{cases} 
x^2 + y^2 > 2\omega^2 \\
H(x, y, \omega) \geq 200\tau^2\sigma^2
\end{cases}
$$

$$
\tilde{\Pi}_A(x, y, z) = 0 \quad \text{if} \quad H(x, y, \omega) < 200\tau^2\sigma^2
$$

It’s useful to compare these payoff functions. In both cases, the 1st Quadrant is divided into four regions and the form of the function varies with the region. However, the functions differ with respect to how the boundaries of these regions are defined. If $z \leq \omega$, then $z$ enters into the equations that define the boundaries. Whereas, if $z > \omega$, then $z$ is replaced by $\omega$ in the boundary equations. It follows that Network A selects $x, y \in [0, \rho]$, so as to maximize $R(x, y, z)$, where $R(.)$ is defined as follows:

$$
R(x, y, z) = \frac{\mu}{2\tau} (2\omega + x + y) \quad \text{iff} \quad G(x, y, \min\{\omega, z\}) \leq 200\tau^2\sigma^2
$$

$$
R(x, y, z) = \frac{\mu}{\sqrt{\tau}} (2\omega + x + y) \quad \text{iff} \quad \begin{cases} 
G(x, y, \min\{\omega, z\}) > 200\tau^2\sigma^2 \\
x^2 + y^2 \leq \omega^2 + (\min\{\omega, z\})^2
\end{cases}
$$

$$
R(x, y, z) = \frac{\mu}{6\tau} (2\omega + x + y) \quad \text{iff} \quad \begin{cases} 
x^2 + y^2 > \omega^2 + (\min\{\omega, z\})^2 \\
H(x, y, \min\{\omega, z\}) \geq 200\tau^2\sigma^2
\end{cases}
$$

$$
R(x, y, z) = 0 \quad \text{iff} \quad H(x, y, \min\{\omega, z\}) < 200\tau^2\sigma^2
$$
18.3 Default-Fees

It has been shown that Network A’s optimal choice of $(x,y)$ belongs to the 1st Quadrant. The 1st Quadrant is divided into four regions and Network A must select a location within one of these regions. The number of sellers on each network (and the form of the payoff function) depends on which region is chosen. These regions are: Core, 1st Shell, 2nd Shell, Exterior.

If a point in the Core is selected, then all three sellers are on Network A and Network B is inactive. Whereas, if a point outside of the Core is selected, then Network B is able to attract one (or more) of the sellers. Let the first case be referred to as "Exclusion" and the second case be referred to "Accommodation".

It can be shown that Network A is always able to exclude Network B. The argument is as follows. If $x = \min\{\omega, \rho\}$, $y = \min\{\omega, |z|\}$, then

$$G(x, y, \min\{\omega, z\}) = \begin{cases} 200\sigma^2\tau^2 & \text{if } \rho \geq \omega \\ (10\sigma\tau + \rho^2 - \omega^2)^2 + 100\sigma^2\tau^2 & \text{if } \rho < \omega \end{cases}$$

It follows that:

**Lemma 18.2 (Default Fees.)** For all $z \in [-\omega, \rho]$, if $x = \min\{\omega, \rho\}$, $y = \min\{\omega, |z|\}$, then

$$G(\min\{\omega, \rho\}, \min\{\omega, |z|\}, \min\{\omega, z\}) \leq 200\sigma^2\tau^2,$$

which implies that $(x, y)$ is within the Core and Network B is inactive.

Let $x = \min\{\omega, \rho\}$, $y = \min\{\omega, |z|\}$ be referred to as the "default-fees", which are identical to the fees chosen by Network A when tying is prohibited. These fees generate the following "default payoff":

$$R(\min\{\omega, \rho\}, \min\{\omega, |z|\}, z) = \frac{\mu}{2\tau}(2\omega + \min\{\omega, \rho\} + \min\{\omega, |z|\})$$

It can be seen that the default-payoff depends on the relative size of $\omega$ and $\rho$. It can be seen that Network A can (and will) veto any outcome that generates a lower payoff than the default-payoff.

18.4 Accommodation vs Exclusion

Buyers’ demand increases as the seller-fee increases (and the buyer-fee decreases). The maximum possible seller-fee is $\rho$. Hence, $(x,y)$ is located with a square with vertices: $(0,0)$, $(0, \rho)$, $(\rho, 0)$, $(\rho, \rho)$. It can be seen that the 1st
Quadrant is made up of four regions, which are defined as follows:

<table>
<thead>
<tr>
<th>Region</th>
<th>Boundaries</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core:</td>
<td>( G(x, y, \min{z, \omega}) \leq 200\tau^2\sigma^2 )</td>
<td>Exclusion</td>
</tr>
</tbody>
</table>
| 1\textsuperscript{st} Shell: | \[
\begin{align*}
\left\{ \begin{array}{l}
x^2 + y^2 & \leq \omega^2 + (\min\{z, \omega\})^2 \\
G(x, y, \min\{z, \omega\}) & > 200\tau^2\sigma^2 
\end{array} \right.
\] & Accommodation \|
| 2\textsuperscript{nd} Shell: | \[
\begin{align*}
\left\{ \begin{array}{l}
x^2 + y^2 > \omega^2 + (\min\{z, \omega\})^2 \\
H(x, y, \min\{z, \omega\}) & \geq 200\tau^2\sigma^2 
\end{array} \right.
\] & Accommodation \|
| Exterior:  | \( H(x, y, \min\{z, \omega\}) < 200\tau^2\sigma^2 \) & Strictly Dominated |

Table 6. Areas in the 1\textsuperscript{st} Quadrant

There are four cases to consider: (1) the point \((\rho, \rho)\) is in the Core; (2) the point \((\rho, \rho)\) is within the 1\textsuperscript{st} Shell; (3) the point \((\rho, \rho)\) is within the 2\textsuperscript{nd} Shell; (4) the point \((\rho, \rho)\) is in the Exterior.

18.4.1 When only the Core is Accessible

If the point \((\rho, \rho)\) is in the Core, then Network A necessarily excludes Network B because accommodation isn’t possible. This is because the square defined by \((0, 0), (0, \rho), (\rho, 0), (\rho, \rho)\) doesn’t intersect either shell.

Lemma 18.3 (Case 1.) If \(G(\rho, \rho, \min\{z, \omega\}) \leq 200\tau^2\sigma^2\), then accommodation isn’t possible.

18.4.2 When 1\textsuperscript{st} Shell is Accessible

If the point \((\rho, \rho)\) is in the 1\textsuperscript{st} Shell, then the default-payoff exceeds the maximum payoff that could be received by accommodating Network B. The argument is as follows:

Suppose that \((\rho, \rho)\) is within the 1\textsuperscript{st} Shell. Accommodation occurs iff Network A selects a point in the 1\textsuperscript{st} Shell. (This is because the only other accessible region is the Core; and within this region Network B is excluded.)

Within a region, the maximum payoff is found by maximizing \(x + y\) subject to the constraint that \(x \leq \rho, y \leq \rho\) and \((x, y)\) remains within the region. If \((\rho, \rho)\) is within the 1\textsuperscript{st} Shell, then the maximum payoff occurs at \(x = \rho, y = \rho\). It can be seen that
\[
\frac{\mu}{3\tau} (2\omega + 2\rho) = \frac{4}{3}\mu
\]
Therefore, the maximum payoff from accommodation is \(\frac{4}{3}\mu\). This can be compared with the default -payoff.

The default-payoff depends on the relative size of \(\rho\) and \(\omega\). If \((\rho, \rho)\) is within the 1\textsuperscript{st} Shell, then \(\rho \leq \omega\) and \(\rho \leq \tau\). The argument is as follows. Since \((\rho, \rho)\) is within the 1\textsuperscript{st} Shell, it follows that:
\[
2\rho^2 \leq \omega^2 + (\min\{z, \omega\})^2 \leq 2\omega^2
\]
Hence, \( \rho \leq \omega \). Finally, since \( \omega = 2\tau - \rho \), this implies that \( \rho \leq \tau \). Using this information a lower bound for the default-payoff can be found as follows:

\[
R(\min\{\rho, \omega\}, \min\{|z|, \omega\}, z) = R(\rho, \min\{|z|, \omega\}, z) [\rho \leq \omega] = \frac{\mu}{2\tau} (2\omega + \rho + \min\{|z|, \omega\}) \\
\geq \frac{\mu}{2\tau} (2\omega + \rho) = \frac{\mu}{2\tau} (4\tau - \rho) [\omega = 2\tau - \rho] \\
\geq \mu \left(2 - \frac{\mu}{2\tau}\right) \\
\geq \frac{3}{2\mu} [\rho \leq \tau]
\]

Therefore, the default-payoff exceeds \( \frac{3}{2}\mu \).

The results can be summarized as follows. It has been shown that the default-payoff exceeds \( \frac{3}{2}\mu \); whereas, the maximum payoff from accommodation is \( \frac{4}{3}\mu \). Therefore, the default-payoff exceeds the maximum payoff from accommodating Network \( B \).

**Lemma 18.4 (Case 2.)** If \( 2\rho^2 \leq \omega^2 + (\min\{z, \omega\})^2 \), \( G(\rho, \rho, \min\{z, \omega\}) > 200\tau^2 \sigma^2 \), then for all \( z \in [-\omega, \rho] \), it can be shown that: (i) the maximum payoff from accommodation is \( \frac{4}{3}\mu \); whereas, (ii) the default-payoff exceeds \( \frac{3}{2}\mu \).

### 18.4.3 When 2nd Shell is Accessible

If the point \((\rho, \rho)\) is in the 2nd Shell, then the default payoff exceeds the maximum payoff that could be received by accommodating Network \( B \). The argument is as follows:

Suppose that \((\rho, \rho)\) is in the 2nd Shell. The maximum payoff from accommodation occurs in the 2nd Shell or the 1st Shell. These cases are analyzed as follows.

Within a region, the maximum payoff is found by maximizing \( x + y \) subject to the constraint that \( x \leq \rho \), \( y \leq \rho \) and \( (x, y) \) remains within the region. If \((\rho, \rho)\) is within the 2nd Shell, then the maximum payoff occurs at \( x = \rho \), \( y = \rho \). It can be seen that

\[
\frac{\mu}{6\tau} (2\omega + 2\rho) \equiv \frac{2}{3}\mu
\]

(This uses \( \omega = 2\tau - \rho \).) Therefore, the maximum payoff within the 2nd Shell is \( \frac{2}{3}\mu \).

It can be shown that the maximum payoff from a point within the 1st Shell is strictly less than \( \frac{4}{3}\mu \). The argument is as follows. In the 1st Shell, the maximum payoff occurs somewhere on the outer boundary. That is, if

\[
(\hat{x}_1(z), \hat{y}_1(z)) = \arg \max_{x, y \in [0, \rho]} \{x + y : \begin{cases} x^2 + y^2 \leq \omega^2 + (\min\{z, \omega\})^2 \\ G(x, y, \min\{z, \omega\}) > 200\tau^2 \sigma^2 \end{cases} \},
\]

then

\[
(\hat{x}_1(z), \hat{y}_1(z)) = \arg \max_{x, y \in [0, \rho]} \{x + y : x^2 + y^2 = \omega^2 + (\min\{z, \omega\})^2 \},
\]

166
Since \((\rho, \rho)\) is in the 2nd Shell, it follows that \(\hat{x}_1(z) \leq \rho\), \(\hat{y}_1(z) \leq \rho\) (and it's not the case that \(\hat{x}_1(z) = \rho\), \(\hat{y}_1(z) = \rho\)). This implies that for all \(z \in [-\omega, \rho]\), \(\hat{x}_1(z) + \hat{y}_1(z) \leq 2\rho\). Hence,

\[
\frac{\mu}{3\tau} (2\omega + \hat{x}_1(z) + \hat{y}_1(z)) \leq \frac{\mu}{3\tau} (2\omega + 2\rho) \equiv \frac{4}{3}\mu
\]

Therefore, the maximum payoff within the 1st Shell is less than \(\frac{4}{3}\mu\).

The maximum payoff from accommodation occurs at either \((\rho, \rho)\) or \((\hat{x}_1(z), \hat{y}_1(z))\). However, either way this maximum is less than \(\frac{4}{3}\mu\). This upper bound can be compared with the default-payoff.

The default-payoff depends on the relative size of \(\rho\) and \(\omega\). It can be shown that if \((\rho, \rho)\) is within the 2nd Shell, then \(\rho \leq \omega\) and \(\rho \leq \tau\). The argument is as follows. By assumption, \(H(\rho, \rho, \min\{z, \omega\}) \geq 200\tau^2\sigma^2\). Since \((\min\{z, \omega\})^2 \leq \omega^2\), it follows that:

\[
200\tau^2\sigma^2 \leq H(\rho, \rho, \min\{z, \omega\}) \leq 2 \cdot (10\tau\sigma + \omega^2 - \rho^2)^2
\]

Hence, \(\omega \geq \rho\). Finally, since \(\omega = 2\tau - \rho\), this implies that \(\tau \geq \rho\). This information can be used to show that the default-payoff exceeds \(\frac{4}{3}\mu\) (see Case 2).

The results can be summarized as follows: It has been shown that the default-payoff exceeds \(\frac{4}{3}\mu\); whereas, the maximum payoff from accommodation is less than \(\frac{4}{3}\mu\). Therefore, the default-payoff exceeds the maximum payoff from accommodating Network \(B\).

**Lemma 18.5 (Case 3.)** If \(2\rho^2 > \omega^2 + (\min\{z, \omega\})^2\), \(H(\rho, \rho, \min\{z, \omega\}) \geq 200\tau^2\sigma^2\), then for all \(z \in [-\omega, \rho]\), it can be shown that: (i) the maximum payoff from accommodation is less than \(\frac{4}{3}\mu\); whereas, (ii) the default-payoff exceeds \(\frac{3}{2}\mu\).

### 18.4.4 When all Regions are Accessible

If the point \((\rho, \rho)\) is in the Exterior, then the default payoff exceeds the maximum payoff that could be received by accommodating Network \(B\). The argument is as follows:

Suppose that \((\rho, \rho)\) is in the Exterior. If Network \(A\) were to chose a point in the Exterior, then it makes this is because no sellers join the its network. Hence, the maximum payoff from accommodation occurs in the 2nd Shell or the 1st Shell. These cases are analyzed as follows.

Within a region, the maximum payoff is found by maximizing \(x + y\). Since \((\rho, \rho)\) belongs to the Exterior, the maximum payoff within the 2nd Shell will occur on the outer boundary. That is, if

\[
(\hat{x}_2(z), \hat{y}_2(z)) \equiv \arg \max_{x,y \in [0,\rho]} \{x + y : \begin{cases} x^2 + y^2 > \omega^2 + (\min\{z, \omega\})^2, \\ H(x, y, \min\{z, \omega\}) \geq 200\tau^2\sigma^2 \end{cases} \}.
\]

then

\[
(\hat{x}_2(z), \hat{y}_2(z)) = \arg \max_{x,y \in [0,\rho]} \{x + y : H(x, y, \min\{z, \omega\}) = 200\tau^2\sigma^2 \}
\]

It can be shown that the outer boundary expands as \(|z|\) increases. This expansion continues up to the point where \(z = \omega\); after which any further increases
in $z$ have no effect on the boundary. Hence, the constraint imposed by the boundary becomes less binding as $|z|$ increases. That is, for all $z \in [-\omega, \rho],
\hat{x}_2(\omega) + \hat{y}_2(\omega) \geq \hat{x}_2(z) + \hat{y}_2(z).

Furthermore, it can be shown that:

$$\arg \max_{x, y \in [0, \rho]} \{x + y : H(x, y, \omega) \geq 200r^2 \sigma^2\} = \begin{cases} (\omega, \omega) & \text{if } \omega \leq \rho \\ (\rho, \rho) & \text{if } \omega > \rho \end{cases}$$

(Note that just because $(\rho, \rho)$ is in the Exterior for a particular value of $z$, there is no guarantee that it’s in the exterior when $z = \omega$.) Therefore, for all $z \in [-\omega, \rho],
\hat{x}_2(z) + \hat{y}_2(z) \leq 2 \min\{\omega, \rho\}.

It can be seen that

$$\frac{\mu}{2r} (2\omega + \hat{x}_2(z) + \hat{y}_2(z)) \leq \frac{\mu}{6\tau} (2\omega + 2 \min\{\omega, \rho\}) \equiv \frac{\mu}{6\tau} (\omega + \min\{\omega, \rho\})$$

This implies that the maximum payoff within the 2nd Shell is less than $\frac{\mu}{6\tau} (\omega + \min\{\omega, \rho\})$.

Since $(\rho, \rho)$ belongs to the Exterior, the maximum payoff within the 1st Shell will occur on the outer boundary. A very similar argument to that given above shows that maximum payoff within the 1st Shell is less than $\frac{2\mu}{3\tau} (\omega + \min\{\omega, \rho\})$. Therefore, an upper bound for the maximum payoff from accommodation is less than $\frac{2\mu}{3\tau} (\omega + \min\{\omega, \rho\})$.

A lower bound for the default-payoff can be found as follows:

$$\frac{\mu}{2r} (2\omega + \min\{\rho, \omega\} + \min\{|z|, \omega\}) \geq \frac{\mu}{2r} (2\omega + \min\{\rho, \omega\})$$

Hence, the default-payoff exceeds $\frac{\mu}{2r} (2\omega + \min\{\rho, \omega\})$.

The results so far can be summarized as follows: The maximum payoff from accommodation is less than $\frac{2\mu}{3\tau} (\omega + \min\{\omega, \rho\})$; whereas, the default-payoff exceeds $\frac{\mu}{2r} (2\omega + \min\{\rho, \omega\})$. It can be seen that:

$$\frac{1}{2} (2\omega + \min\{\rho, \omega\}) - \frac{2\mu}{3\tau} (\omega + \min\{\omega, \rho\}) \equiv \frac{1}{3} \omega - \frac{1}{6} \min\{\omega, \rho\} \geq \frac{1}{6} \omega$$

Hence,

$$\frac{\mu}{2r} (2\omega + \min\{\rho, \omega\}) - \frac{2\mu}{3\tau} (\omega + \min\{\omega, \rho\}) \geq \frac{\mu \omega}{6\tau}$$

Therefore, the default-payoff exceeds the maximum payoff from accommodation.

**Lemma 18.6 (Case 4.)** If $H(\rho, \rho, \min\{z, \omega\}) < 200r^2 \sigma^2$, then for all $z \in [-\omega, \rho], the maximum payoff from accommodation is strictly less than the default-payoff (and the gap exceeds $\frac{\mu \omega}{6\tau}$).

### 18.5 Exclusion

In the preceding sections, the default-payoff has been compared against the maximum payoff that could be received by accommodating Network B. It was shown that in each case, the default-payoff strictly exceeds that from accommodating Network B. It follows that:
Proposition 18.1 *(Network A excludes Network B.)* There are two cases:

1. If \( G(\rho, \rho, \min\{z, \omega\}) \leq 200\tau^2\sigma^2 \), then the top corner of the action-space, \((\rho, \rho)\), is in the Core. Hence, it’s not possible for Network A to accommodate Network B.

2. If \( G(\rho, \rho, \min\{z, \omega\}) > 200\tau^2\sigma^2 \), then the top corner of the action-space, \((\rho, \rho)\), is outside the Core, which implies that accommodation is possible. However, the default-fees generate a strictly higher payoff than the maximum payoff from fees located outside the Core. That is, if 

\[
(\tilde{x}, \tilde{y}) \in \arg \max_{x, y \in [0, \rho]} R(x, y, z),
\]

then 

\[
G(\tilde{x}, \tilde{y}, \min\{z, \omega\}) \leq 200\tau^2\sigma^2.
\]

Therefore, Network A excludes Network B.
Chapter 19

Network A’s Best-Response

Chapter 19 characterizes Network A’s best-response to the fee chosen by Network B. Section (1) restates Network A’s problem: maximize \( x + y \) subject to excluding Network B. Section (2) shows that there is a point where the inner oval is tangential to an iso-payoff line. Section (3) finds the best-response when the top-right corner of the action-space, \((\rho, \rho)\), is within the Core. Section (4) finds the best-response \((\rho, \rho)\) is outside the Core. The best-response is qualitatively different depending on the size of \( W(x) \). Hence, there are a series of thresholds. Section (5) investigates the properties of the threshold functions. Section (6) uses the previous results to derive a best-response function. Section (7) consider the special case where \( z = 0 \).

19.1 Restating Network A’s Problem (again)

Let \( \hat{x}(z) \), \( \hat{y}(z) \) denote Network A’s best-response. The best-response can be analysed as follows. Firstly, Network A sets positive seller-fees. That is, \( \hat{x}(z) \geq 0 \), \( \hat{y}(z) \geq 0 \). Secondly, Network A excludes Network B. That is, \((\hat{x}(z), \hat{y}(z))\) is within the Core, which requires

\[
G(\hat{x}(z), \hat{y}(z), \min\{z, \omega\}) \leq 200r^2\sigma^2
\]

Finally, since all sellers are "on" Network A, its payoff function is \( \hat{\Pi}_A = \frac{\mu}{2}\sigma (2\omega + x + y) \). The iso-payoff lines are a family of straight lines, \( x + y = constant \), with gradients of \(-1\). Therefore, Network A maximizes \( x + y \) within the Core. Network A’s best-response can be characterized as follows:

\[
(\hat{x}(z), \hat{y}(z)) = \arg \max_{x, y \in [0, \rho]} \{ x + y : G(x, y, \min\{z, \omega\}) \leq 200r^2\sigma^2 \}
\]

It can be seen that there are two types of constraint: firstly, unilateral deviation can’t be profitable for a seller; and, secondly, seller-fees can’t exceed the price-level. It can be shown that these constraints are binding. That is, one (or more) of the following holds:

\[
G(\hat{x}(z), \hat{y}(z), \min\{z, \omega\}) = 200r^2\sigma^2
\]

\[
\hat{x}(z) = \rho
\]

\[
\hat{y}(z) = \rho
\]
The argument is as follows:

Network A will maximize \( x + y \) subject to the constraints. Hence, the maximum possible payoff occurs at \( x = y = \rho \). If this \((\rho, \rho)\) is within the Core, then \( \hat{x}(z) = \hat{y}(z) = \rho \). Therefore, the last two constraints are binding.

However, if \((\rho, \rho)\) is outside the Core, then one or more of the sellers isn’t on Network A when \( x = y = \rho \). Furthermore, it’s been shown that Network A always excludes Network B. Hence, we must decrease \( x, y \) until we reach the boundary of the Core. Consider a point in the interior of the Core. If \((x, y)\) is in the interior, then it’s possible to increase \( x \) and \( y \) while still satisfying the constraints. Hence, the point can’t be optimal. Therefore, the optimum must be on the boundary. Therefore, the first constraint is binding. (One of the other constraints could also be binding.)

The analysis shows that it is sufficient to consider two cases: (1) \((\rho, \rho)\) is within the Core, which requires \( G(\rho, \rho, \min\{z, \omega\}) \leq 200 \tau^2 \sigma^2 \); and (2) \((\rho, \rho)\) is outside the Core, which requires \( G(\rho, \rho, \min\{z, \omega\}) > 200 \tau^2 \sigma^2 \).

Before analyzing each case it’s useful to consider the point where the boundary of the Core is tangential to an iso-payoff line. This is a potential optimum in Case (2).

### 19.2 The Point of Tangency

The iso-payoff lines (within the Core) are weakly convex and the slope of these lines is \(-1\). Whereas, within the 1\textsuperscript{st} Quadrant, the constraint-curve is strictly concave. The slope of the constraint-curve is zero at the point where it crosses the y-axis and becomes \(-\infty\) at the point where it crosses the x-axis. Since the curve is smooth and concave, there must exist a point on the boundary where the slope is \(-1\). Hence, there exists a unique point of tangency between the constraint-curve and an iso-payoff line. The point of tangency is denoted by \((x^*, y^*) \in \mathbb{R}_+^2\). Furthermore, it can be seen that

\[
(x^*, y^*) = \arg \max_{x, y \in \mathbb{R}_+} \{x + y : G(x, y, \min\{z, \omega\}) = 200 \tau^2 \sigma^2\}
\]

It is useful to characterise the point of tangency as the solution of two simultaneous equations. This can be done as follows. Using the definition of \(G(.)\), the constraint-curve can be written as

\[
(10 \tau \sigma + x^2 - \omega^2)^2 + (10 \tau \sigma + y^2 - (\min\{z, \omega\}))^2 = 200 \tau^2 \sigma^2,
\]

where the RHS is a constant. Total differentiation with respect to \(x\) and some rearrangement gives:

\[
\frac{dy}{dx} = -\left(\frac{x}{y}\right) \cdot \left(\frac{10 \tau \sigma + x^2 - \omega^2}{10 \tau \sigma + y^2 - (\min\{z, \omega\})^2}\right)
\]

This is the slope of the constraint curve. (Note the negative sign.) Since the iso-payoff lines have a slope of \(-1\), the tangency condition becomes:

\[
\left(\frac{x}{y}\right) \cdot \left(\frac{10 \tau \sigma + x^2 - \omega^2}{10 \tau \sigma + y^2 - (\min\{z, \omega\})^2}\right) = 1
\]
Hence, $x^*, y^*$ is the value of $x, y$ that solves the following simultaneous equations:

$$\left[10\tau\sigma + x^2 - \omega^2\right]^2 + \left[10\tau\sigma + y^2 - (\min\{z, \omega\})^2\right]^2 = 200\tau^2\sigma^2$$

$$x. \left[10\tau\sigma + x^2 - \omega^2\right] = y. \left[10\tau\sigma + y^2 - (\min\{z, \omega\})^2\right]$$

Finally, it’s been shown that if $x, y, z \in [-\omega, \rho], \text{ then } x^2 < 2\tau\sigma, y^2 < 2\tau\sigma, z^2 < 2\tau\sigma$. (See Properties of the Inner Oval, Appendix C.) It follows that $x^2 - \omega^2 > 10\tau\sigma$ and $(\min\{z, \omega\})^2 > 10\tau\sigma$; which implies that the quantities in () brackets can never be negative.

The pair of simultaneous equations can be replaced by a single implicit equation for $x^*$. This is done as follows. Firstly, from the constraint-curve, we get:

$$\left[10\tau\sigma + y^2 - (\min\{z, \omega\})^2\right]^2 = \Lambda^2 (x^2 - \omega^2)$$

where

$$\Lambda^2 (x^2 - \omega^2) \equiv 200\tau^2\sigma^2 - \left[10\tau\sigma + x^2 - \omega^2\right]^2$$

and

$$y^2 = \Lambda(x^2 - \omega^2) + (\min\{z, \omega\})^2 - 10\tau\sigma$$

Secondly, square both side of the tangency condition and substitute for the terms containing $y^2$. This gives

$$x^2. \left[10\tau\sigma + x^2 - \omega^2\right]^2 = \left[\Lambda(x^2 - \omega^2) + (\min\{z, \omega\})^2 - 10\tau\sigma\right] \times \Lambda^2 (x^2 - \omega^2)$$

which is an implicit equation for $x^*$ in terms of $(\min\{z, \omega\})^2$ and $\omega^2$.

It’s useful to carry out some comparative statics. Consider each side of the equation as an expression that depends on $x$; and imagine that LHS and RHS are plotted against $x$ on the same graph. Firstly, it can be seen that LHS is increasing in $x^2$; whereas, the RHS is decreasing in $x^2$. Hence, there is a unique crossing point. Since the point of tangency corresponds to the point where LHS = RHS, there is a unique point of tangency.

Secondly, it can be seen that the RHS increases as $(\min\{z, \omega\})^2$ increases. It follows that $x^*$ increases as $(\min\{z, \omega\})^2$ increases. (Make a sketch of LHS and RHS and find the point of intersection. Now, shift the RHS upwards and observe how the point of intersection varies.) Therefore,

$$\frac{\partial x^*}{\partial z}(z) \left\{ \begin{array}{ll} < 0 & \text{if } z < 0 \\ > 0 & \text{if } 0 \leq z \leq \omega \\ = 0 & \text{if } z > \omega \end{array} \right.$$
where

\[ \Lambda^2(y^2 - (\min\{z, \omega\})^2) \equiv 200\tau^2\sigma^2 - \left[10\tau\sigma + y^2 - (\min\{z, \omega\})^2\right]^2 \]

and

\[ x^2 = \Lambda(y^2 - (\min\{z, \omega\})^2) + \omega^2 - 10\tau\sigma \]

where

\[ \Lambda(y^2 - (\min\{z, \omega\})^2) \equiv \sqrt{200\tau^2\sigma^2 - \left[10\tau\sigma + y^2 - (\min\{z, \omega\})^2\right]^2} \]

Secondly, square both side of the tangency condition and substitute for the terms containing \( x^2 \). This gives

\[ y^2 \cdot \left[10\tau\sigma + y^2 - (\min\{z, \omega\})^2\right] = \left[\Lambda(y^2 - (\min\{z, \omega\})^2) + \omega^2 - 10\tau\sigma\right] \times \Lambda^2(y^2 - (\min\{z, \omega\})^2) \]

which is an implicit equation for \( y^* \) in terms of \( (\min\{z, \omega\})^2 \) and \( \omega^2 \).

It’s useful to carry out some comparative statics. Consider each side of the equation as an expression that depends on \( y \); and imagine that LHS and RHS are plotted against \( y \) on the same graph. Firstly, it can be seen that LHS is increasing in \( y^2 \); whereas, the RHS is decreasing in \( y^2 \). Hence, there is a unique crossing point. Since the point of tangency corresponds to the point where LHS = RHS, there is a unique point of tangency.

Secondly, it can be seen that the RHS increases as \( (\min\{z, \omega\})^2 \) increases; whereas, the LHS decreases. It follows that \( y^* \) increases as \( (\min\{z, \omega\})^2 \) increases. (Make a sketch of LHS and RHS and find the point of intersection. Now, shift the RHS upwards and the LHS downwards. Finally, observe how the point of intersection varies.) Therefore,

\[ \frac{\partial y^*}{\partial z}(z) = \begin{cases} < 0 & \text{if } z < 0 \\ > 0 & \text{if } 0 \leq z \leq \omega \\ = 0 & \text{if } z > \omega \end{cases} \]

In general, it’s very difficult to find a closed form expression for \( x^*, y^* \). The only exception is the spacial case where \( (\min\{z, \omega\})^2 = \omega^2 \). In this situation the equation becomes

\[ x^2 \cdot \left[10\tau\sigma + x^2 - \omega^2\right] = \left\{ \sqrt{200\tau^2\sigma^2 - \left[10\tau\sigma + x^2 - \omega^2\right]^2} + \omega^2 - 10\tau\sigma \right\} \times \left\{ 200\tau^2\sigma^2 - \left[10\tau\sigma + x^2 - \omega^2\right]^2 \right\} \]

It can be seen that if \( x^2 = \omega^2 \), then: the LHS becomes \( 100\tau^2\sigma^2\omega^2 \); and (after some manipulation) the RHS becomes \( 100\tau^2\sigma^2\omega^2 \). Hence, if \( (\min\{z, \omega\})^2 = \omega^2 \), then \( x^* = \omega^2 \). Substituting this result into the constraint equation, gives \( y^* = \omega^2 \).

How does \( \frac{y^*}{x^*} \) vary with \( (\min\{z, \omega\})^2 \)? It has been shown that the point of tangency occurs at the place where the slope of the constraint-curve is \(-1\). Consider the point where the constraint-curve intersects the 45-degree line. At

173
this point the slope is

\[
\frac{dy}{dx} = -\left( \frac{10\tau\sigma + x^2 - \omega^2}{10\tau\sigma + x^2 - (\min\{z, \omega\})^2} \right)
\]

Since \((\min\{z, \omega\})^2 \leq \omega^2\), it follows that if \(x = y\), then \(\frac{dy}{dx} \leq 1\). Furthermore, as we move up the constraint curve (increasing \(y\)), the magnitude of the slope decreases. Therefore, the point of tangency can’t occur above the 45-degree line. It follows that:

\[
x^*(z) \begin{cases} 
> y^*(z) & \text{if } |z| < \omega \\
= y^*(z) & \text{if } |z| \geq \omega
\end{cases}
\]

### 19.3 Best-Response when \((\rho, \rho)\) is within the Core

The maximum possible payoff is \(\tilde{\Pi}_A = 2\mu\) and occurs when sellers pay the entire transaction fee: \(x = \rho\), \(y = \rho\). Therefore, if \((\rho, \rho)\) is within the Core, then \(\hat{x}(z) = \rho\), \(\hat{y}(z) = \rho\). Since the price level is fixed, this implies that buyers receive the service for free.

The only reason Network A wouldn’t choose these fees is that the high seller fees may make it impossible to attract all sellers. That is, \((\rho, \rho)\) may occur above the constraint curve. This implies that:

\[
\hat{x}(z) = \hat{y}(z) = \rho \iff G(\rho, \rho, \min\{z, \omega\}) \leq 200\tau^2\sigma^2
\]

Hence, Network A makes its platforms free to buyers iff \(G(\rho, \rho, \min\{z, \omega\})\) is "small" relative to \(\tau\) and \(\sigma\).

It has been shown that \(G(\rho, \rho, \min\{z, \omega\})\) decreases as \(\min\{z, \omega\}\) increases; which implies that there exists a unique value of \(\min\{z, \omega\}\) such that

\[
G(\rho, \rho, \min\{z, \omega\}) = 200\tau^2\sigma^2
\]

Let \(\theta(\sigma, \rho)\) be implicitly defined by the equation:

\[
G(\rho, \rho, \theta(\sigma, \rho)) = 200\tau^2\sigma^2
\]

(Note that the dimension of \(\theta(\sigma, \rho)\) is \(L^2\).) From this we get:

\[
\theta(\sigma, \rho) \equiv 10\tau\sigma + \rho^2 - \Lambda(\rho^2 - \omega(\rho)^2),
\]

where

\[
\Lambda(\rho^2 - \omega(\rho)^2) \equiv \sqrt{200\tau^2\sigma^2 - (10\tau\sigma + \rho^2 - \omega(\rho)^2)^2}
\]

(Remember that \(\omega(\rho)\) is itself a function of the underlying parameters.) It follows that:

**Lemma 19.1 (When is \((\rho, \rho)\) within the Core?)** The location of \((\rho, \rho)\) within the action-space depends on how \(\min\{z, \omega(\rho)\}\) compares to \(\theta(\sigma, \rho)\) (relative sizes). It can be shown that:

(i) If \((\min\{z, \omega(\rho)\})^2 \geq \theta(\sigma, \rho)\), then \((\rho, \rho)\) is within the Core; that is,

\[
G(\rho, \rho, \theta(\sigma, \rho)) \leq 200\tau^2\sigma^2
\]

174
(ii) If \((\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)\), then \((\rho, \rho)\) is outside the Core; that is,
\[
G(\rho, \rho, \theta(\sigma, \rho)) > 200\tau^2\sigma^2.
\]

This lemma has the following implications:

- If \((\min\{z, \omega(\rho)\})^2 \geq \theta(\sigma, \rho)\), then \(\hat{x}(z) = \rho, \hat{y}(z) = \rho\).
- If \((\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)\), then \(\hat{x}(z) \neq \rho\) or \(\hat{y}(z) \neq \rho\).

It’s useful to know how \(\theta(\cdot)\) varies as the price-level changes. It can be seen that
\[
\frac{\partial \theta}{\partial \rho}(\sigma, \rho) = 2\rho + \frac{4\tau \cdot (10\tau \sigma + \rho^2 - \omega(\rho)^2)}{\sqrt{200\tau^2\sigma^2 - (10\tau \sigma + \rho^2 - \omega(\rho)^2)^2}},
\]
which implies that \(\frac{\partial \theta}{\partial \rho}(\sigma, \rho) > 0\). That is, \(\theta(\sigma, \rho)\) rises as the price-level, \(\rho\), increases. It follows that:

**Lemma 19.2** (Solve \(\theta(\sigma, \rho) = 0\), given \(\sigma\)) There exists a threshold, \(\lambda(\sigma) \in (0, \tau)\), such that if \(\rho \leq \lambda(\sigma)\), then \(\theta(\sigma, \rho) \leq 0\); whereas, if \(\rho > \lambda(\sigma)\), then \(\theta(\sigma, \rho) > 0\).

**Proof.** Since \(\theta(\sigma, \rho)\) is an increasing function of \(\rho\), it follows that there exists a threshold, \(\lambda(\sigma) \in \mathbb{R}_+\), such that: if \(\rho \leq \lambda(\sigma)\), then \(\theta(\sigma, \rho) \leq 0\); whereas, if \(\rho > \lambda(\sigma)\), then \(\theta(\sigma, \rho) > 0\).

It can be seen that \(\lambda(\sigma)\) is implicitly defined by \(\theta(\sigma, \lambda(\sigma)) = 0\). Upper and lower bounds for the solution can be found as follows. Firstly, consider the case where \(\rho \to 0\) so that \(\omega(\rho) \to 4\tau^2\). It follows that:
\[
\theta(\sigma, 0) = 10\tau \sigma - \Lambda(-4\tau^2)
\]
\[
= 10\tau \sigma - \sqrt{200\tau^2\sigma^2 - (10\tau \sigma - 4\tau^2)^2}
\]
\[
< 10\tau \sigma - \sqrt{200\tau^2\sigma^2 - (10\tau \sigma)^2}
\]
\[
= 0
\]

Hence, \(\theta(\sigma, 0) < 0\). Secondly, consider the case where \(\rho = \tau\) so that \(\omega(\rho) = \tau\). It can be seen that:
\[
\theta(\sigma, \tau) = 10\tau \sigma + \tau^2 - \Lambda(0)
\]
\[
= \tau^2
\]
\[
> 0
\]

Hence, \(\theta(\sigma, \tau) > 0\). Therefore, there exists a value of \(\rho\) between 0 and \(\tau\) such that \(\theta(\sigma, \rho) = 0\). From the definition of \(\lambda(\sigma)\), it follows that \(0 < \lambda(\sigma) < \tau\). 

This lemma is useful when analyzing the best-response where Network B offers the maximum possible level of end-user benefit. That is, the lemma helps to characterize \(\hat{x}(0), \hat{y}(0)\).
19.4 Best-Response when \((\rho, \rho)\) is Outside the Core

Suppose that \((\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)\) so that \((\rho, \rho)\) is above the constraint curve. It follows that the best-response is on the constraint curve. It can be shown that either the optimum is at the point of tangency or there’s a corner solution. (The corner solution is at the point where the horizontal line \(y = \rho\) intersects the constraint curve.) The argument is as follows. Firstly, it can be seen that Network A’s problem becomes

\[
\max x + y \text{ s.t. } \begin{cases} 
G(x, y, \min\{z, \omega\}) = 200\tau^2\sigma^2 \\
x \leq \rho, \ y \leq \rho
\end{cases},
\]

which implies that

\[
(\hat{x}(z), \hat{y}(z)) = \arg \max_{x,y \in [0,\rho]} \{x + y : G(x, y, \min\{z, \omega\}) = 200\tau^2\sigma^2\}
\]

Secondly, by definition,

\[
(x^*(z), y^*(z)) = \arg \max_{x,y \in \mathbb{R}_+} \{x + y : G(x, y, \min\{z, \omega\}) = 200\tau^2\sigma^2\}
\]

It follows that: if \(x^*(z) \leq \rho\), then \(\hat{x}(z) = x^*(z), \ \hat{y}(z) = y^*(z)\); whereas, if \(x^*(z) > \rho\), then there’s a corner solution in which \(\hat{x}(z) = \rho, \ \hat{y}(z) = y^{**}(z)\), where

\[
y^{**}(z) = \max \{y : G(\rho, y, \min\{z, \omega\}) = 200\tau^2\sigma^2\}
\]

(Remember that \(x^*(z) \geq y^*(z)\). Hence, if \(x^*(z) \leq \rho\), then \(y^*(z) \leq \rho\).) These results can be summarized as follows: if \((\min\{z, \omega\})^2 \leq \theta(\sigma, \rho)\), then

\[
\hat{x}(z) = x^*(z), \ \hat{y}(z) = y^*(z) \quad \text{iff} \quad x^*(z) \leq \rho
\]

and

\[
\hat{x}(z) = \rho, \ \hat{y}(z) = y^{**}(z) \quad \text{iff} \quad x^*(z) > \rho
\]

It has been shown that \(x^*(z)\) is implicitly defined by the following equation:

\[
x^*(z)^2, [10\tau\sigma + x^*(z)^2 - \omega(\rho)]^2 = \left[\Lambda \left(x^*(z)^2 - \omega(\rho)^2\right) + (\min\{z, \omega(\rho)\})^2 - 10\tau\sigma\right] \\
x \cdot \Lambda^2 \left(x^*(z)^2 - \omega(\rho)^2\right)
\]

where

\[
\Lambda(x^2 - \omega(\rho)^2) = \sqrt{200\tau^2\sigma^2 - [10\tau\sigma + x^2 - \omega(\rho)^2]^2}
\]

\[
\Lambda^2(x^2 - \omega(\rho)^2) = 200\tau^2\sigma^2 - [10\tau\sigma + x^2 - \omega(\rho)^2]^2
\]

Furthermore, it’s known that \(x^*(z)\) increases with \((\min\{z, \omega(\rho)\})^2\); which implies that there exists a unique value of \((\min\{z, \omega(\rho)\})^2\) such that \(x^*(z) = \rho\). Let \(\phi(\sigma, \rho)\) be implicitly defined by the following equation:

\[
\rho^2, [10\tau\sigma + \rho^2 - \omega(\rho)]^2 = \left[\Lambda \left(\rho^2 - \omega(\rho)^2\right) + \phi(\sigma, \rho) - 10\tau\sigma\right] \\
x \cdot \Lambda^2 \left(\rho^2 - \omega(\rho)^2\right)
\]

(Note that the dimension of \(\phi(\sigma, \rho)\) is \(L^2\).)
Lemma 19.3 (When is \(x^*(z) \leq \rho\)?) The location of \((x^*(z), y^*(z))\) within the action-space depends on how \(\min\{z, \omega(\rho)\}\) compares to \(\phi(\sigma, \rho)\). It can be shown that:

(i) If \((\min\{z, \omega(\rho)\})^2 \leq \phi(\sigma, \rho)\), then \(x^*(z) \leq \rho\).

(ii) If \((\min\{z, \omega(\rho)\})^2 > \phi(\sigma, \rho)\), then \(x^*(z) > \rho\).

This lemma has the following implications:

- If \(\phi(\sigma, \rho) < (\min\{z, \omega\})^2 < \theta(\sigma, \rho)\), then there’s a corner solution in which 
  \[\hat{x}(z) = \rho, \quad \hat{y}(z) = y^{**}(z)\]

- If \((\min\{z, \omega\})^2 < \theta(\sigma, \rho)\) and \((\min\{z, \omega\})^2 \leq \phi(\sigma, \rho)\), then 
  \[\hat{x}(z) = x^*(z), \quad \hat{y}(z) = y^*(z)\]

Hence, the nature of Network A’s best-response depends on the order of \((\min\{z, \omega\})^2, \theta(\sigma, \rho)\) and \(\phi(\sigma, \rho)\). (That is, the relative sizes of the three quantities.) It will be shown that the order of these quantities depends (in part) on how \(\rho\) compares to \(\tau\).

It’s useful to know how \(\phi(.)\) varies as the price-level changes. Consider each side of the equation as an expression that depends on \(\phi\); and imagine that LHS and RHS are plotted against \(\phi\) on the same graph. It can be seen that: the LHS is a constant that depends on the parameter values; whereas, the RHS increases as \(\phi\) increases. Hence, given particular values of the parameters, there’s a unique value of \(\phi\) for which LHS = RHS; and this value is what defines \(\phi(\sigma, \rho)\).

How does \(\phi(\sigma, \rho)\) vary as we increase \(\rho\)? It can be seen that: the LHS increases as \(\rho\) increases; whereas, the RHS decreases as \(\rho\) increases. (Remember that \(\omega(\rho) = 2\tau - \rho\).) It can be seen that the point where LHS = RHS shifts to the right. Hence,

\[\frac{\partial \phi}{\partial \rho}(\sigma, \rho) > 0\]

It’s useful to find the point where \(\omega(\rho)\) and \(\phi(\sigma, \rho)\) intersect. Since \(\omega(\rho)\) is decreasing in \(\rho\); whereas, \(\phi(\sigma, \rho)\) is increasing in \(\rho\) there is a unique value of \(\rho\) at which \(\phi(\sigma, \rho) = \omega(\rho)^2\). Furthermore, if \(\rho = \tau\), then \(\omega(\rho)^2 = \rho^2\); which implies that \(\phi(\sigma, \rho) = \rho^2 = \omega(\rho)^2\). Therefore, \(\phi(\sigma, \rho) = \omega(\rho)^2\) iff \(\rho = \tau\). Finally, since \(\phi(\sigma, \rho)\) is increasing in \(\rho\) there exists a unique value of \(\rho\) for which \(\phi(\sigma, \rho) = 0\).

Lemma 19.4 (Solve \(\phi(\sigma, \rho) = 0\), given \(\sigma\).) There exists a threshold, \(\eta(\sigma) \in \mathbb{R}_+\), such that if \(\rho < \eta(\sigma)\), then \(\phi(\sigma, \rho) < 0\); whereas, if \(\rho \geq \eta(\sigma)\), then \(\phi(\sigma, \rho) \geq 0\).

This lemma is useful when analyzing the best-response when Network B offers the maximum level of end-user benefit. That is, the lemma helps to characterize \(\hat{x}(0), \hat{y}(0)\).
19.5 Threshold Functions

19.5.1 Ordering $\theta(\sigma, \rho)$ and $\omega(\rho)^2$.

This section investigates how $\rho$ determines the ordering of $\theta(\sigma, \rho)$ and $\omega(\rho)^2$. It has been shown that $\frac{\partial \theta}{\partial \rho}(\sigma, \rho) > 0$; whereas, $\frac{\partial \omega}{\partial \rho} = -1 < 0$. This implies that there is unique value of $\rho$ for which $\theta(\sigma, \rho) = \omega(\rho)$. (Make a sketch.) Finally, it can be seen that

$$\theta(\sigma, \tau) = \tau^2 = \omega(\tau)^2$$

Hence, $\rho^2 = \omega(\rho)^2$ iff $\rho = \tau$. This analysis implies that:

**Lemma 19.5** (Ordering of $\omega(\rho)$ and $\theta(\sigma, \rho)$.) The order of $\omega(\rho)$ and $\theta(\sigma, \rho)$ (relative sizes) depends on how $\rho$ compares to $\tau$. It can be shown that:

(i) If $\rho < \tau$, then $\omega(\rho)^2 > \theta(\sigma, \rho)$.

(ii) If $\rho = \tau$, then $\omega(\rho)^2 = \theta(\sigma, \rho)$.

(iii) If $\rho > \tau$, then $\omega(\rho)^2 < \theta(\sigma, \rho)$.

It was shown that if $(\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)$, then $(\rho, \rho)$ is outside the Core. Since $(\min\{z, \omega(\rho)\})^2 \leq \omega(\rho)^2$, it follows that if $\omega(\rho)^2 < \theta(\sigma, \rho)$, then

$$(\min\{z, \omega(\rho)\})^2 \leq \omega(\rho)^2 < \theta(\sigma, \rho);$$

which implies that $(\rho, \rho)$ is outside the Core. In other words, a sufficient condition for $(\rho, \rho)$ to be outside the Core is that $\omega(\rho)^2 < \theta(\sigma, \rho)$.

\[\text{Figure 14. } \theta(\rho) \text{ and } \omega(\rho) \text{ versus } \rho\]
19.5.2 Ordering $\theta(\sigma, \rho)$ and $\phi(\sigma, \rho)$.

It can be shown that $\phi(\sigma, \rho) = \theta(\sigma, \rho)$ iff $\rho = \tau$. The argument is as follows. Rearranging the expression for $\theta(\sigma, \rho)$ gives

$$10\tau \sigma + \rho^2 - \theta(\sigma, \rho) = \sqrt{200\tau^2 \sigma^2 - (10\tau \sigma + \rho^2 - \omega(\rho)^2)^2},$$

Hence, if $\phi(\sigma, \rho) = \theta(\sigma, \rho)$, then

$$\rho^2 = \sqrt{200\tau^2 \sigma^2 - [10\tau \sigma + \rho^2 - \omega(\rho)^2]^2 + \phi(\sigma, \rho) - 10\tau \sigma}$$

and

$$200\tau^2 \sigma^2 - [10\tau \sigma + \rho^2 - \phi(\sigma, \rho)]^2 = [10\tau \sigma + \rho^2 - \omega(\rho)^2]^2$$

The implicit expression for $\phi(\sigma, \rho)$ is

$$\rho^2, [10\tau \sigma + \rho^2 - \omega(\rho)^2]^2 = \left[\sqrt{200\tau^2 \sigma^2 - [10\tau \sigma + \rho^2 - \omega(\rho)^2]^2 + \phi(\sigma, \rho) - 10\tau \sigma}\right]$$

$$\times \left[200\tau^2 \sigma^2 - [10\tau \sigma + \rho^2 - \omega(\rho)^2]^2\right]$$

Substituting the two results given above into this expression gives

$$\rho^2, [200\tau^2 \sigma^2 - [10\tau \sigma + \rho^2 - \phi(\sigma, \rho)]^2] = \rho^2, [200\tau^2 \sigma^2 - [10\tau \sigma + \rho^2 - \omega(\rho)^2]^2]$$

Therefore, if $\theta(\sigma, \rho) = \phi(\sigma, \rho)$, then $\phi(\sigma, \rho) = \omega(\rho)^2$, which implies that $\rho = \tau$.

It remains to prove that the direction of the implication can be reversed. If $\rho = \tau$, then $\rho^2 = \omega(\rho)^2$. Firstly, this implies that $\theta(\sigma, \rho) = \rho^2$. Secondly, it follows that $\phi(\sigma, \rho) = \rho^2$. Therefore, if $\rho = \tau$, then $\phi(\sigma, \rho) = \theta(\sigma, \rho)$.

Let

$$\theta'(\sigma, \tau) \equiv \frac{\partial}{\partial \rho} \theta(\sigma, \rho) \bigg|_{\rho=\tau}$$

$$\phi'(\sigma, \tau) \equiv \frac{\partial}{\partial \rho} \phi(\sigma, \rho) \bigg|_{\rho=\tau}$$

It can be shown that $\theta'(\sigma, \tau) > \phi'(\sigma, \tau)$. The argument is as follows. Firstly, it has been shown that

$$\frac{\partial \theta}{\partial \rho}(\sigma, \rho) = 2\rho + \frac{4\tau (10\tau \sigma + \rho^2 - \omega(\rho)^2)}{\sqrt{200\tau^2 \sigma^2 - (10\tau \sigma + \rho^2 - \omega(\rho)^2)^2}}$$

Since $\omega(\tau) = \tau$, it follows that

$$\theta'(\sigma, \tau) = 6\tau$$

Secondly, $\phi'(\sigma, \tau)$ can be found as follows. Let

$$L(\sigma; \rho) = \rho^2 [10\tau \sigma + \rho^2 - \omega(\rho)^2]^2$$
\[ R(\sigma; \rho; \phi(\sigma, \rho)) = \left\{ \sqrt{200\tau^2\sigma^2 - \left[10\tau\sigma + \rho^2 - \omega(\rho)^2\right]^2} + \phi(\sigma, \rho) - 10\tau\sigma \right\} \times \left\{ 200\tau^2\sigma^2 - \left[10\tau\sigma + \rho^2 - \omega(\rho)^2\right]^2 \right\} \]

These expressions correspond to the LHS and RHS of the implicit equation for \( \phi(\sigma, \rho) \). Differentiating each expression with respect to \( \rho \) and evaluating the results at \( \rho = \tau \) gives the following results:

\[
\frac{dL}{d\rho}(\sigma; \tau) = 200\tau^3\sigma^2 + 80\tau^4\sigma
\]

\[
\frac{dR}{d\rho}(\sigma; \tau; \phi(\sigma, \tau)) = 100\tau^2\sigma^2, \phi'(\sigma, \tau) - 400\tau^3\sigma^2 - 80\tau^4\sigma
\]

Equating the derivatives and collecting like terms gives:

\[
600\tau^3\sigma^2 + 160\tau^4\sigma = 100\tau^2\sigma^2, \phi'(\sigma, \tau)
\]

After simplifying and rearranging we get

\[
\phi'(\sigma, \tau) = \frac{60\tau\sigma + 16\tau^2}{10\sigma}
\]

Finally, the slopes can be compared as follows:

\[
\theta'(\sigma, \tau) - \phi'(\sigma, \tau) = 6\tau - \frac{60\tau\sigma + 16\tau^2}{10\sigma} = \frac{16\tau^2}{10\sigma} < 0
\]

Therefore, at the point where the curves cross \( \phi(.) \) has a higher slope than \( \theta(.) \). That is,

\[ \theta'(\sigma, \tau) < \phi'(\sigma, \tau) \]

The results of this section can be summarized as follows: Firstly, \( \theta(\sigma, \rho) \) and \( \phi(\sigma, \rho) \) are both increasing functions of \( \rho \). Secondly, they intersect once (and only once) at \( \rho = \tau \). Finally, \( \theta'(\sigma, \tau) < \phi'(\sigma, \tau) \). From these results it follows that:

**Lemma 19.6 (Ordering of \( \theta(\sigma, \rho) \) and \( \phi(\sigma, \rho) \).)** The order of \( \theta(\sigma, \rho) \) and \( \phi(\sigma, \rho) \) (relative sizes) depends on how \( \rho \) compares to \( \tau \). It can be shown that:

(i) If \( \rho < \tau \), then \( \theta(\sigma, \rho) > \phi(\sigma, \rho) \).

(ii) If \( \rho = \tau \), then \( \theta(\sigma, \rho) = \phi(\sigma, \rho) \).

\[ \frac{d}{d\rho} \sqrt{200\tau^2\sigma^2 - \left[10\tau\sigma + \rho^2 - \omega(\rho)^2\right]^2} \big|_{\rho=\tau} = -4\tau \]

Also, note that \( \phi(\sigma, \tau) = \theta(\sigma, \tau) = \tau^2 \).
(iii) If $\rho > \tau$, then $\theta(\sigma, \rho) < \phi(\sigma, \rho)$.

Figure 15. $\theta(\rho)$ and $\phi(\rho)$ versus $\rho$

19.6 The Best-Response Function

It has been shown that, in principle, there are three types of response open to Network A. These are as follows: Firstly, Network A can make both its platforms free to buyers, which means sellers pay the entire transaction fee. That is, $x = \rho$, $y = \rho$. Secondly, Network A can make Platform $D_A$ free to buyers but charge both sides on Platform $C_A$ (corner solution). That is, $x = \rho$, $y = y^*(z)$. Finally, Network A can charge both sides on both platforms (point of tangency): $x = x^*(z)$, $y = y^*(z)$. Network A’s best response depends on how $(\min\{z, \omega(\rho)\})^2$ compares to the $\theta(\sigma, \rho)$, $\phi(\sigma, \rho)$. It has been shown that:

- If $(\min\{z, \omega(\rho)\})^2 \geq \theta(\sigma, \rho)$, then
  $$\hat{x}(z) = \rho, \quad \hat{y}(z) = \rho$$ (free for buyers)

- If $\phi(\sigma, \rho) \leq (\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)$, then
  $$\hat{x}(z) = \rho, \quad \hat{y}(z) = y^*(z)$$ (corner solution)

- If $(\min\{z, \omega(\rho)\})^2 < \phi(\sigma, \rho)$ and $(\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)$, then
  $$\hat{x}(z) = x^*(z), \quad \hat{y}(z) = y^*(z)$$ (point of tangency).
It has been shown that the best-response depends on the ordering of \((\min\{z, \omega(\rho)\})^2\) and \(\theta(\sigma, \rho), \phi(\sigma, \rho)\). Since \((\min\{z, \omega(\rho)\})^2 \leq \omega(\rho)^2\), this ordering is partly determined by the relative size of \(\rho\) and \(\tau\). By combining results from the last two sections, it can be shown that:

- If \(\rho < \tau\), then \(\phi(\sigma, \rho) < \theta(\sigma, \rho) < \omega(\rho)^2\)
- If \(\rho = \tau\), then \(\phi(\sigma, \rho) = \theta(\sigma, \rho) = \omega(\rho)^2\)
- If \(\rho > \tau\), then \(\phi(\sigma, \rho) > \theta(\sigma, \rho) > \omega(\rho)^2\)

\[\text{Figure 16. } \theta(\rho), \phi(\rho) \text{ and } \omega(\rho) \text{ versus } \rho\]

**Case (1).** Suppose that \(\rho < \tau\). There are three sub-cases. Firstly, if \((\min\{z, \omega(\rho)\})^2 < \phi(\sigma, \rho)\), then
\[
(\min\{z, \omega(\rho)\})^2 < \phi(\sigma, \rho) < \theta(\sigma, \rho)\;,
\]
which implies that \(\hat{x}(z) = x^*(z), \; \hat{y}(z) = y^*(z)\). Secondly, if
\[
\phi(\sigma, \rho) \leq (\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)\;,
\]
then \(\hat{x}(z) = \rho, \; \hat{y}(z) = y^{**}(z)\). Finally, if \((\min\{z, \omega(\rho)\})^2 \geq \theta(\sigma, \rho)\), then \(\hat{x}(z) = \rho, \; \hat{y}(z) = \rho\). These cases are mutually exclusive and exhaustive.

**Case (2).** Suppose that \(\rho = \tau\). There are two cases. If \((\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho)\), then \(\hat{x}(z) = \rho, \; \hat{y}(z) = y^{**}(z)\). Whereas, if \((\min\{z, \omega(\rho)\})^2 \geq \theta(\sigma, \rho)\), then \(\hat{x}(z) = \rho, \; \hat{y}(z) = \rho\). These cases are mutually exclusive and exhaustive.
Case (3). Suppose that $\rho > \tau$. It follows that
\[ \theta(\sigma, \rho) > \phi(\sigma, \rho) > \omega(\rho)^2 \geq (\min\{z, \omega(\rho)\})^2 , \]
which implies that $\hat{x}(z) = x^*(z)$, $\hat{y}(z) = y^*(z)$.

**Proposition 19.1 (Network A’s Best-Response.)** The best-response of Network A is
\[ (\hat{x}(z), \hat{y}(z)) = \begin{cases} (x^*(z), y^*(z)) \text{ if } \rho > \tau \text{ or } \rho \leq \tau, \quad (\min\{z, \omega(\rho)\})^2 < \phi(\sigma, \rho) \\ (\rho, y^{**}(z)) \text{ if } \phi(\sigma, \rho) \leq (\min\{z, \omega(\rho)\})^2 < \theta(\sigma, \rho) \\ (\rho, \rho) \text{ if } (\min\{z, \omega(\rho)\})^2 \geq \theta(\sigma, \rho) \end{cases} \]

19.7 Special Case where $z = 0$

Sellers are attracted to networks whose platforms offer a high end-user benefit. Furthermore, $\hat{W}(f)$ increases as $|f|$ decreases.

It has been shown that Network A can always exclude Network B. However, exclusion becomes more difficult as Network B makes itself more attractive to sellers by setting a low seller-fee. That is, as $|z|$ decreases, Network A has to lower its seller fees in order to attract sellers.

It’s useful to conclude this chapter by discussing the fees set by Network A when $z = 0$. In this situation, Network A’s best-response is as follows:
\[ (\hat{x}(0), \hat{y}(0)) = \begin{cases} (x^*(0), y^*(0)) \text{ if } \rho > \tau \text{ or } \rho \leq \tau, \quad \phi(\sigma, \rho) > 0 \\ (\rho, y^{**}(0)) \text{ if } \phi(\sigma, \rho) \leq 0 < \theta(\sigma, \rho) \\ (\rho, \rho) \text{ if } \theta(\sigma, \rho) \leq 0 \end{cases} \]

However, if $\rho > \tau$, then $\phi(\sigma, \rho) > \tau > 0$; whereas, if $\phi(\sigma, \rho) > 0$, then there is no guarantee that $\rho > \tau$. It follows that the best-response can be re-expressed as
\[ (\hat{x}(0), \hat{y}(0)) = \begin{cases} (x^*(0), y^*(0)) \text{ if } \phi(\sigma, \rho) > 0 \\ (\rho, y^{**}(0)) \text{ if } \phi(\sigma, \rho) \leq 0 < \theta(\sigma, \rho) \\ (\rho, \rho) \text{ if } \theta(\sigma, \rho) \leq 0 \end{cases} \]

Finally, there exist two thresholds, namely, $\lambda(\sigma)$ and $\eta(\sigma)$, which are defined as follows:

- If $\rho \leq \lambda(\sigma)$, then $\theta(\sigma, \rho) \leq 0$; whereas, if $\rho > \lambda(\sigma)$, then $\theta(\sigma, \rho) > 0$.
- If $\rho \leq \eta(\sigma)$, then $\phi(\sigma, \rho) \leq 0$; whereas, if $\rho > \eta(\sigma)$, then $\phi(\sigma, \rho) > 0$.

Therefore,
\[ (\hat{x}(0), \hat{y}(0)) = \begin{cases} (x^*(0), y^*(0)) \text{ if } \rho > \eta(\sigma) \\ (\rho, y^{**}(0)) \text{ if } \eta(\sigma) \geq \rho > \lambda(\sigma) \\ (\rho, \rho) \text{ if } \rho \leq \lambda(\sigma) \end{cases} \]

This implies that if the price-level, $\rho$, is low enough, then Network A is able to attract all the sellers even when it charges the highest possible seller-fees and Network B is free for sellers.
Chapter 20

Equilibrium Outcomes

Chapter 20 finds the equilibrium outcome. Section (1) finds the equilibrium fees. Section (2) finds expressions for the end-user benefit on network A’s platforms. Section (3) finds sellers’ membership and the prices they set. Section (4) uses the results to characterize the consumer surplus.

20.1 Equilibrium Fees

At the first stage of the game both networks set fees simultaneously. Network A chooses $x, y \in [-\omega, \rho]$ and Network B chooses $z \in [-\omega, \rho]$. Let $N(\sigma, \rho)$ denote the equilibrium of a game in which the parameters are $\sigma, \rho$.

There are two cases to consider; these are: (i) $\rho \leq \lambda(\sigma)$; and (ii) $\rho > \lambda(\sigma)$. In the first case, the price level $\rho$ is so low relative to the average transaction cost, $\tau$, that Network A is able to exclude Network A even when its platforms are made free to buyers (and sellers pay the entire transaction fee). Whereas, in the second case, the price level $\rho$ is sufficiently high that sellers prefer to join Network B if forced to pay the entire fee by Network A.

If $\rho \leq \lambda(\sigma)$, then Network A can exclude Network B regardless of what fee it sets. Intuitively, this is because a very low price-level (relative to the average transaction cost) makes the end-user benefit on Platform $C_A$ is so large that sellers want to be on Network A, even though Platform $D_A$ is expensive relative to Platform $D_B$. Since Network A is able to attract all sellers whatever fees it chooses, it makes its platforms free to buyers because this maximizes the volume of transactions (and given that its markup is fixed this maximizes its payoff). Finally, this outcome occurs regardless of the fee set by Network B. Hence, equilibria are indexed by Network B’s choice of buyer-fee.

Lemma 20.1 (Equilibria when $\rho$ is small.) If $\rho \leq \lambda(\sigma)$, then there’s a continuum of multiple equilibria in which the fees are $\tilde{x} = \rho$, $\tilde{y} = \rho$, $\tilde{z} \in [-\omega, \rho]$.

If $\rho > \lambda(\sigma)$, then Network A can’t necessarily attract all sellers unless buyers are made to pay part of the transaction fee. In particular, if Network B makes its platform free to sellers, then Network A can’t attract all sellers unless it chooses a price structure in which buyers pay part of the transaction fee.
Lemma 20.2 (Existence.) If \( \rho > \lambda(\sigma) \), then there exists an equilibrium in which \( \tilde{x} = \tilde{x}(0) \), \( \tilde{y} = \tilde{y}(0) \), \( \tilde{z} = 0 \).

Proof. \( \tilde{x} = \tilde{x}(0) \), \( \tilde{y} = \tilde{y}(0) \), \( \tilde{z} = 0 \) is a valid equilibrium iff neither network has an incentive to unilaterally deviate.

First, consider whether it’s possible for Network A to profit by choosing other fees. Since \( \tilde{x}, \tilde{y} \) equals Network A’s best-response to \( z = 0 \), it’s not possible Network A to do better through unilateral deviation.

Second, consider whether it’s possible for Network B to do better by choosing an alternative fee. Since \( \tilde{x}, \tilde{y} \) corresponds to Network A’s best-response and Network A’s best response always satisfies the constraints, it follows that: \( G(\tilde{x}, \tilde{y}, 0) \leq 200\tau^2\sigma^2 \) and \( \tilde{x} \leq \rho, \tilde{y} \leq \rho \). Hence, \( (\tilde{x}, \tilde{y}) \) is within the Core; which implies that all the sellers are on Network A; and Network B is inactive. Furthermore, since \( G(x, y, z) \) is decreasing in \( |z| \) (for \( |z| < \omega \)) it follows that if \( z \neq 0 \), then
\[
G(\tilde{x}, \tilde{y}, z) < G(\tilde{x}, \tilde{y}, 0) \leq 200\tau^2\sigma^2
\]
Hence, for all \( z \in [-\omega, \rho] \), \( G(\tilde{x}, \tilde{y}, z) \leq 200\tau^2\sigma^2 \); which implies that there is no way that Network B can attract any of the sellers. Therefore, Network B can’t do better through unilateral deviation.

It has been shown that neither network can profit by unilaterally deviating from \( \tilde{x} = \tilde{x}(0), \tilde{y} = \tilde{y}(0), \tilde{z} = 0 \) and so these are equilibrium fees. □

Lemma 20.3 (Uniqueness.) If \( \rho > \lambda(\sigma) \) and \( \tilde{z} \neq 0 \), then \( x = \tilde{x}(\tilde{z}), y = \tilde{y}(\tilde{z}), z = \tilde{z} \) is not an equilibrium because Network B can profit through unilateral deviation.

Proof. Suppose that the fee set by Network B is \( \tilde{z} \neq 0 \). For there to be an equilibrium Network A must choose the best-response to \( \tilde{z} \), namely, \( \tilde{x}(\tilde{z}), \tilde{y}(\tilde{z}) \).

There are two cases to consider:

1. \( \min\{\tilde{z}, \omega\}\)^2 \( \geq \theta(\sigma, \rho) \);
2. \( \min\{\tilde{z}, \omega\}\)^2 \( < \theta(\sigma, \rho) \).

The argument for uniqueness uses proof by contradiction: assume that an alternative equilibrium exists and then show that one of the networks has an incentive to deviate. Each case is considered in turn:

Case (1). Suppose that there exists an equilibrium in which \( z = \tilde{z} \neq 0 \) and \( \min\{\tilde{z}, \omega\}\)^2 \( \geq \theta(\sigma, \rho) \). Network A’s best-response would be \( \tilde{x}(\tilde{z}) = \rho, \tilde{y}(\tilde{z}) = \rho \); which ensures that all sellers are "on" Network A and Network B is inactive. In this situation, Network B makes no profit. Does Network B have an incentive to unilaterally deviate?

Suppose that Network B chose \( z = 0 \), instead. Consider sellers’ membership in the subgame where \( x = \rho, y = \rho, z = 0 \). Firstly, this is a particular example of SG [1]. Secondly, since \( x > 0, y > 0 \), it follows that \( (x, y) \) is within the 1st Quadrant of the action-space; and the 1st Quadrant divides into four regions: Core; 1st Shell; 2nd Shell; and Exterior. Finally, since \( \rho > \lambda(\sigma) \), it follows that \( \theta(\sigma, \rho) > 0 \); which implies that \( G(\rho, \rho, 0) > 200\tau^2\sigma^2 \). Hence, \( (\rho, \rho) \) is outside the Core. Therefore, \( (\rho, \rho) \) is in one of the other regions (1st...
Shell, 2nd Shell, Exterior). In all these regions, one (or more) of the sellers is on Network B; which implies that Network B makes a positive profit.

Since Network B can profit through unilateral deviation, there is no equilibrium in which \( z = \hat{z} \neq 0 \) and \( (\min\{\hat{z}, \omega\})^2 \geq \theta(\sigma, \rho) \).

**Case (2).** Suppose that there exist an equilibrium in which \( z = \hat{z} = 0 \) and \( (\min\{\hat{z}, \omega\})^2 < \theta(\sigma, \rho) \). The best-response of Network A becomes

\[
\hat{x}(\hat{z}) = \begin{cases} \rho & \text{if } (\min\{\hat{z}, \omega\})^2 \geq \phi(\sigma, \rho) \\ x^*(\hat{z}) & \text{if } (\min\{\hat{z}, \omega\})^2 < \phi(\sigma, \rho) \end{cases}
\]

\[
\hat{y}(\hat{z}) = \begin{cases} y^{**}(\hat{z}) & \text{if } (\min\{\hat{z}, \omega\})^2 \geq \phi(\sigma, \rho) \\ y^*(\hat{z}) & \text{if } (\min\{\hat{z}, \omega\})^2 < \phi(\sigma, \rho) \end{cases}
\]

Furthermore, Network A’s best-response ensures that all sellers are "on" Network A. (That is, the point of tangency and the corner solution are on the constraint curve.) Hence, Network B is inactive and receives zero payoff. Does Network B have an incentive to unilaterally deviate?

Suppose that Network B chose \( z = 0 \), instead. Consider sellers’ membership in the subgame where \( x = \hat{x}(\hat{z}), y = \hat{y}(\hat{z}), z = 0 \). Firstly, this is a particular example of SG [1]. Secondly, since \( x > 0, y > 0 \), it follows that \((x, y)\) is within the 1st Quadrant of the action-space; and the 1st Quadrant divides into four regions: Core, 1st Shell; 2nd Shell; and Exterior. Finally, since \( 0 < (\min\{\hat{z}, \omega\})^2 \) and \( G(\hat{x}(\hat{z}), \hat{y}(\hat{z}), \hat{z}) \) is decreasing in \( (\min\{z, \omega\})^2 \), it follows that

\[
G(\hat{x}(\hat{z}), \hat{y}(\hat{z}), 0) > G(\hat{x}(\hat{z}), \hat{y}(\hat{z}), \hat{z}) = 200\tau^2 \sigma^2,
\]

This implies that if \( z = 0 \), then \((\hat{x}(\hat{z}), \hat{y}(\hat{z}))\) is outside the Core. Therefore, \((\hat{x}(\hat{z}), \hat{y}(\hat{z}))\) is in one of the other regions (1st Shell, 2nd Shell, Exterior). In all these regions, one (or more) of the sellers is on Network B; which implies that Network B makes a positive profit.

Since Network B can profit through unilateral deviation, there is no equilibrium in which \( z = \hat{z} \neq 0 \) and \( (\min\{\hat{z}, \omega\})^2 < \theta(\sigma, \rho) \).

It follows that:

**Proposition 20.1 (Equilibrium Fees.)** The seller-fees set by the networks in equilibrium are as follows:

\[
\tilde{x} = \begin{cases} \rho & \text{if } \rho \leq \eta(\sigma) \\ x^*(0) & \text{if } \rho > \eta(\sigma) \end{cases}
\]

\[
\tilde{y} = \begin{cases} \rho & \text{if } \rho \leq \lambda(\sigma) \\ y^{**}(0) & \text{if } \lambda(\sigma) < \rho \leq \eta(\sigma) \\ y^*(0) & \text{if } \rho > \eta(\sigma) \end{cases}
\]

\[
\tilde{z} \in \begin{cases} [-\omega, \rho] & \text{if } \rho \leq \lambda(\sigma) \\ \{0\} & \text{if } \rho > \lambda(\sigma) \end{cases}
\]

### 20.2 End-User Benefit

The equilibrium fees determine the end-user benefit on Network A’s platforms. Furthermore, since all sellers are on Network A in every equilibrium, the fees
on Network A’s platforms determine the extra-surplus received by sellers and their customers. It has been shown that the nature of the equilibrium depends on the size of $\rho$ relative to $\lambda(\sigma)$ and $\eta(\sigma)$ (where $\eta(\sigma) > \lambda(\sigma)$). There are three scenarios to consider: (i) $\rho \leq \lambda(\sigma)$; (ii) $\lambda(\sigma) < \rho \leq \eta(\sigma)$; and (iii) $\rho > \eta(\sigma)$. (Note that which scenario actually occurs is purely determined by the parameter values.)

**Scenario (i).** If $\rho \leq \lambda(\sigma)$, then Network A charges sellers the entire transaction fee and makes its platforms free to buyers. That is, sellers pay $\rho$ on each sale made using the platform and buyers pay nothing. Hence, the aggregate end-user benefit offered by Network A becomes

$$\hat{W}(\tilde{x}) + \hat{W}(\tilde{y}) = 2\hat{W}(\rho),$$

where

$$\hat{W}(\rho) = \tau - \rho.$$

**Scenario (ii).** If $\lambda(\sigma) < \rho \leq \eta(\sigma)$, then Platform $C_A$ is made free to buyers but both sides of the market are charged when using Platform $D_A$. In market $c$ the seller-fee is $\rho$ and buyers pay nothing; whereas, in market $d$ the seller-fee is $y^{**}(0)$ and buyers pay $\rho - y^{**}(0)$. Hence, the aggregate end-user benefit offered by Network A becomes

$$\hat{W}(\tilde{x}) + \hat{W}(\tilde{y}) = \hat{W}(\rho) + \hat{W}(y^{**}(0))$$

where

$$\hat{W}(\rho) < \hat{W}(y^{**}(0)) < \hat{W}(0).$$

**Scenario (iii).** If $\rho > \eta(\sigma)$, then both sides of the market pay part of the transaction fee. In market $c$ sellers pay $x^*(0)$ and buyers pay $\rho - x^*(0)$; whereas, in market $d$ sellers pay $y^*(0)$ and buyers pay $\rho - x^*(0)$. Hence, the aggregate end-user benefit offered by Network A becomes

$$\hat{W}(\tilde{x}) + \hat{W}(\tilde{y}) = \hat{W}(x^*(0)) + \hat{W}(y^*(0)),$$

where

$$\hat{W}(x^*(0)) = \frac{1}{4\tau}(\omega^2 - x^*(0)^2)$$

and

$$\hat{W}(y^*(0)) = \frac{1}{4\tau}(\omega^2 - y^*(0)^2).$$

### 20.3 Membership and Prices

Having determined the fees set by Network A it’s possible to find the membership of sellers and the prices they set.

**Scenario (i).** If $\rho \leq \lambda(\sigma)$, then $\tilde{x} = \rho$, $\tilde{y} = \rho$, $\tilde{z} \in [-\omega, \rho]$. Firstly, $(\tilde{x}, \tilde{y})$ is always within the Core. Hence, $G(\rho, \rho, \tilde{z}) < 200\tau^2\sigma^2$. It follows that

$$\mathcal{N}(\bar{F}) \subseteq \mathcal{N}_{330}(\bar{F}).$$
Secondly, \( \max\{\rho, \hat{z}\} = \rho \), which implies that \( \max\{\bar{y}, \hat{z}\} = \bar{y} \). It’s been shown that: if \( \max\{\bar{y}, \hat{z}\} = \bar{y} \) and \( \hat{W}(\hat{z}) < 0 \), then
\[
\mathcal{M}(\tilde{F}) \subseteq \{a, b\};
\]
whereas, if \( \max\{\bar{y}, \hat{z}\} = \bar{y} \) and \( \hat{W}(\hat{z}) \geq 0 \), then
\[
\mathcal{M}(\tilde{F}) \subseteq \{b, \bar{y}\}.
\]
(See Sellers’ Dichotomy.) Finally, it’s been shown that: if \( \max\{\bar{y}, \hat{z}\} = \bar{y} \) and \( \hat{W}(\hat{z}) < 0 \), then
\[
\mathcal{M}(\tilde{F}) \subseteq \{a, b\};
\]
whereas, if \( \max\{\bar{y}, \hat{z}\} = \bar{y} \) and \( \hat{W}(\hat{z}) \geq 0 \), then
\[
\mathcal{M}(\tilde{F}) \subseteq \{b, \bar{y}\}.
\]
(See Configurations and Relevant Options.) Therefore, the sellers multihome:
\[\tilde{m}_i = b\]
However, since \( \max\{\bar{y}, \hat{z}\} = \bar{y} \), only Network A is active: \( A(\rho, \hat{z}; \bar{y}) = 1 \) and \( A(\rho, -\omega; \bar{y}) = 1 \).

It has been shown that a seller’s optimal price in market \( k \) becomes
\[
\tilde{p}_k^i(\tilde{f}_k, \tilde{m}_{-i}, \tilde{m}_i) = \frac{1}{5} \left\{5\sigma + 2C(\tilde{f}_k, \tilde{m}_i) + \sum_j C(\tilde{f}_k, \tilde{m}_j) \right\},
\]
where the marginal cost in market \( \zeta \) is
\[
C(\tilde{f}_\zeta, \tilde{m}_i) = \delta + \tau - \hat{W}(\tilde{f}_\zeta, \tilde{m}_i)
\]
\[
= \delta + \tau - \hat{W}(\rho, -\omega; \bar{y})
\]
\[
= \delta + \tau - \hat{W}(\rho)
\]
and the marginal cost in market \( \eta \) is
\[
C(\tilde{f}_\eta, \tilde{m}_i) = \delta + \tau - \hat{W}(\tilde{f}_\eta, \tilde{m}_i)
\]
\[
= \delta + \tau - \hat{W}(\rho, \hat{z}; \bar{y})
\]
\[
= \delta + \tau - \hat{W}(\rho)
\]
Since all the sellers have the same marginal costs, the prices become:
\[
\tilde{p}_k^i(\tilde{f}_\zeta, \tilde{m}_{-i}, \tilde{m}_i) = \sigma + C(\tilde{f}_\zeta, \tilde{m}_i)
\]
\[
= \sigma + \delta + \tau - \hat{W}(\rho)
\]
\[
\tilde{p}_k^j(\tilde{f}_\eta, \tilde{m}_{-i}, \tilde{m}_i) = \sigma + C(\tilde{f}_\eta, \tilde{m}_i)
\]
\[
= \sigma + \delta + \tau - \hat{W}(\rho)
\]

**Scenario (ii).** If \( \lambda(\sigma) < \rho \leq \eta(\sigma) \), then \( \tilde{x} = \rho, \ \tilde{y} = y^{**}(0), \ \tilde{z} = 0 \). Firstly, \((\tilde{x}, \tilde{y})\) is always within the Core. Hence, \( G(\rho, y^{**}(0), 0) = 200\tau^2\sigma^2 \). It follows
that
\[ \mathcal{N}(\tilde{\mathcal{F}}) \subseteq \mathcal{N}_{330}(\tilde{\mathcal{F}}) \]
Secondly, \( \max\{\tilde{y}, \tilde{z}\} = \tilde{y} \) and \( \tilde{W}(\tilde{z}) = \tilde{W}(0) > 0 \); which implies that:
\[ \mathcal{M}(\tilde{\mathcal{F}}) \subseteq \{b, h\} \]
(See Sellers' Dichotomy.) Finally, it’s been shown that if \( \mathcal{M}(\mathcal{F}) \subseteq \{b, h\} \), then:
\[ \tilde{m} \in \mathcal{N}_{330}(\mathcal{F}) \iff \tilde{m} \in \{b\}^3 \subseteq \mathcal{N}(\mathcal{F}) \]
(See Configurations and Relevant Options.) Therefore, the sellers multihome:
\[ \tilde{m}_i = b \]
It has been shown that a seller’s optimal price in market \( k \) becomes
\[ \tilde{p}_i^k(\tilde{f}_k, \tilde{m}_{-i}, \tilde{m}_i) = \frac{1}{5} \left\{ 5\sigma + 2C(\tilde{f}_k, \tilde{m}_i) + \sum_j C(\tilde{f}_k, \tilde{m}_j) \right\}, \]
where the marginal cost in market \( \varsigma \) is
\[ C(\tilde{f}_\varsigma, \tilde{m}_i) = \delta + \tau - \tilde{W}(\tilde{f}_\varsigma, \tilde{m}_i) \]
\[ = \delta + \tau - W(\rho, -\omega; b) \]
\[ = \delta + \tau - \tilde{W}(\rho) \]
and the marginal cost in market \( q \) is
\[ C(\tilde{f}_q, \tilde{m}_i) = \delta + \tau - \tilde{W}(\tilde{f}_q, \tilde{m}_i) \]
\[ = \delta + \tau - W(\tilde{y}^*(0), 0; b) \]
\[ = \delta + \tau - \tilde{W}(\tilde{y}^*(0)) \]
Since all the sellers have the same marginal costs, the prices become:
\[ \tilde{p}_i^\varsigma(\tilde{f}_\varsigma, \tilde{m}_{-i}, \tilde{m}_i) = \sigma + C(\tilde{f}_\varsigma, \tilde{m}_i) \]
\[ = \sigma + \delta + \tau - \tilde{W}(\rho) \]
\[ \tilde{p}_i^q(\tilde{f}_q, \tilde{m}_{-i}, \tilde{m}_i) = \sigma + C(\tilde{f}_q, \tilde{m}_i) \]
\[ = \sigma + \delta + \tau - \tilde{W}(\tilde{y}^*(0)) \]
**Scenario (iii).** If \( \rho > \eta(\sigma) \), then \( \tilde{x} = x^*(0), \tilde{y} = y^*(0), \tilde{z} = 0 \). Repeating the analysis in Scenario (ii) shows that the sellers multihome:
\[ \tilde{m}_i = b \]
It has been shown that a seller’s optimal price in market \( k \) becomes
\[ \tilde{p}_i^k(\tilde{f}_k, \tilde{m}_{-i}, \tilde{m}_i) = \frac{1}{5} \left\{ 5\sigma + 2C(\tilde{f}_k, \tilde{m}_i) + \sum_j C(\tilde{f}_k, \tilde{m}_j) \right\}, \]
where the marginal cost in market $\zeta$ is

\[
C(\tilde{f}_\zeta, \tilde{m}_i) = \delta + \tau - W(\tilde{f}_\zeta, \tilde{m}_i) \\
= \delta + \tau - W(x^*(0), -\omega; \tilde{h}) \\
= \delta + \tau - \tilde{W}(x^*(0))
\]

and the marginal cost in market $\delta$ is

\[
C(\tilde{f}_\delta, \tilde{m}_i) = \delta + \tau - W(\tilde{f}_\delta, \tilde{m}_i) \\
= \delta + \tau - W(y^*(0), 0; \tilde{h}) \\
= \delta + \tau - \tilde{W}(y^*(0))
\]

Since all the sellers have the same marginal costs, the prices become:

\[
\tilde{p}^\zeta_i(\tilde{f}_\zeta, \tilde{m}_{-i}, \tilde{m}_i) = \sigma + C(\tilde{f}_\zeta, \tilde{m}_i) \\
= \sigma + \delta + \tau - \tilde{W}(x^*(0))
\]

\[
\tilde{p}^\delta_i(\tilde{f}_\delta, \tilde{m}_{-i}, \tilde{m}_i) = \sigma + C(\tilde{f}_\delta, \tilde{m}_i) \\
= \sigma + \delta + \tau - \tilde{W}(y^*(0))
\]

### 20.4 Consumer-Surplus

The consumer-surplus is

\[
\Phi(\tilde{P}) = \sum_{k \in \{\zeta, \delta\}} \sum_{i \in \{1, 2, 3\}} \Theta(\tilde{p}^k_i, \langle \tilde{p}^k_{-i} \rangle)
\]

where

\[
\Theta(\tilde{p}^k_i, \langle \tilde{p}^k_{-i} \rangle) = \left\{ \nu - \tilde{p}^k_i - \frac{1}{2} \sigma D(\tilde{p}^k_i, \langle \tilde{p}^k_{-i} \rangle) \right\} . D(\tilde{p}^k_i, \langle \tilde{p}^k_{-i} \rangle)
\]

In all three scenarios, the sellers make the same membership decision and set the same prices. Hence, the consumer-surplus becomes:

\[
\Phi(\tilde{P}) = 3 \sum_{k \in \{\zeta, \delta\}} \Theta(\tilde{p}^k_i, \langle \tilde{p}^k_{-i} \rangle),
\]

where

\[
\Theta(\tilde{p}^k_i, \langle \tilde{p}^k_{-i} \rangle) = \frac{1}{3} \left( \nu - \tilde{p}^k_i - \frac{1}{6} \sigma \right)
\]

Hence,

\[
\tilde{\Phi} = 2\nu - \tilde{p}^\zeta_i - \tilde{p}^\delta_i - \frac{1}{3} \sigma
\]

Each of the three scenarios can be considered separately:

**Scenario (i).** If $\rho \leq \lambda(\sigma)$, then

\[
\tilde{p}^\zeta_i = \sigma + \delta + \tau - \tilde{W}(\rho) \quad \text{and} \quad \tilde{p}^\delta_i = \sigma + \delta + \tau - \tilde{W}(\rho),
\]

190
which implies that
\[ \tilde{\Phi} = 2(\nu - \sigma - \delta - \tau) + 2\hat{W}(\rho) - \frac{1}{3}\sigma \]

**Scenario (ii).** If \( \lambda(\sigma) < \rho \leq \eta(\sigma) \), then
\[ \tilde{p}_i^\sigma = \sigma + \delta + \tau - \hat{W}(\rho) \quad \text{and} \quad \tilde{p}_i^\jmath = \sigma + \delta + \tau - \hat{W}(y^{**}(0)) ; \]
which implies that
\[ \tilde{\Phi} = 2(\nu - \sigma - \delta - \tau) + \hat{W}(\rho) + \hat{W}(y^{**}(0)) - \frac{1}{3}\sigma \]

**Scenario (iii).** If \( \rho > \eta(\sigma) \), then
\[ \tilde{p}_i^\sigma = \sigma + \delta + \tau - \hat{W}(x^*(0)) \quad \text{and} \quad \tilde{p}_i^\jmath = \sigma + \delta + \tau - \hat{W}(y^*(0)) ; \]
which implies that
\[ \tilde{\Phi} = 2(\nu - \sigma - \delta - \tau) + \hat{W}(x^*(0)) + \hat{W}(y^*(0)) - \frac{1}{3}\sigma \]

If there were no platforms, then the marginal cost would be \( \delta + \tau \) and equilibrium prices would be \( \sigma + \delta + \tau \). Hence, the consumer-surplus without platforms would become \( 2(\nu - \sigma - \delta - \tau) - \frac{1}{3}\sigma \). It follows that:

**Theorem 20.1 (Consumer-Surplus when Membership is Tied.)** In equilibrium, the extra consumer-surplus generated by the platforms becomes:

\[ \Delta \tilde{\Phi}_{II} = \begin{cases} 
2\hat{W}(\rho) & \text{if } \rho \leq \lambda(\sigma) \\
\hat{W}(\rho) + \hat{W}(y^{**}(0)) & \text{if } \lambda(\sigma) < \rho \leq \eta(\sigma) \\
\hat{W}(x^*(0)) + \hat{W}(y^*(0)) & \text{if } \rho > \eta(\sigma)
\end{cases} \]

It’s useful to consider how \( \Delta \tilde{\Phi}_{II} \) varies with \( \rho \). Firstly, \( \Delta \tilde{\Phi}_{II} \) is continuous but may not be differentiable at the thresholds. Secondly, \( \hat{W}(\rho) \) and \( \hat{W}(y^{**}(0)) \) decrease as \( \rho \) increases but it’s unclear whether \( \hat{W}(x^*(0)) \) and \( \hat{W}(y^*(0)) \) are monotonically decreasing. However, it can be seen that
\[ 2\hat{W}(0) \geq \hat{W}(x^*(0)) + \hat{W}(y^*(0)) \geq \hat{W}(0), \]
where \( \hat{W}(0) \) decreases as \( \rho \) increases. Hence, the envelope containing the sum of these functions decreases. Finally, it can be shown that if \( \rho = 2\tau \), then \( x^*(0) = y^*(0) = 0 \).\(^1\) This implies that if \( \rho = 2\tau \), then the extra consumer-surplus goes to zero. Figure 17 illustrates the relationship between \( \rho \) and the extra-consumer surplus generated by platforms when tying is permitted.

\(^1\)It can be seen that \( x = 0 \) and \( y = 0 \) satisfy the implicit equations for \( x^*(0) \) and \( y^*(0) \) when \( z = 0 \) and \( \rho = 2\tau \).
Figure 17. Extra consumer surplus with tying.
Part IV

Comparing Outcomes, with and without Tying
Chapter 21

The Pass on Test

In recent years, there has been a move in the European Community and the US to judge tying cases under what has become known as the "rule of reason". For example, the Treaty of the European Communities (TEC), under Article 81(3), permits practices that might otherwise be seen as harmful (such as, abuse of a dominant position) when they are indispensable to creating economic benefit through improved market efficiency. However, consumers must receive a fair share of the resulting benefit. That is, some of the extra-surplus must be passed on to consumers; this requirement is referred to as the "pass on test".

21.1 Consumer-Surplus Generated by Platforms

It has been shown that if tying is prohibited (Game I), then the consumer-surplus generated by the platforms becomes:

\[ \Delta \Phi_I = \hat{W}(\min\{\rho, \omega\}) + \hat{W}(0) \]

Whereas, if tying is enforced (Game II), then the consumer-surplus generated by the platforms becomes:

\[ \Delta \Phi_{II} = \begin{cases} 
2\hat{W}(\rho) & \text{if } \rho \leq \lambda(\sigma) \\
\hat{W}(\rho) + \hat{W}(y^*(0)) & \text{if } \lambda(\sigma) < \rho \leq \eta(\sigma) \\
\hat{W}(x^*(0)) + \hat{W}(y^*(0)) & \text{if } \rho > \eta(\sigma) 
\end{cases} \]

It follows that tying satisfies the pass on test iff

\[ \Delta \Phi_{II} > \Delta \Phi_I \]

21.2 Levels of Inter-Network Competition

It can be seen that there are three scenarios: (i) \( \rho \leq \lambda(\sigma) \); (ii) \( \lambda(\sigma) < \rho \leq \eta(\sigma) \); and (iii) \( \rho > \eta(\sigma) \). These scenarios depend on the parameter values: total transaction-fee, \( \rho \); degree of horizontal product differentiation, \( \sigma \); and average transaction-cost, \( \tau \). In principle, these parameters could be estimated as follows: Firstly, the total transaction-fee, \( \rho \), is directly observable. Secondly, the level of product differentiation, \( \sigma \), can be estimated from the total profit made
by retailers:

\[ \sigma = \sum_i \tilde{\Gamma}_i \]

Finally, the average transaction-cost incurred by buyers, \( \tau \), can be inferred from the total transaction-fee, \( \rho \), and volume of transactions in market \( q \) when tying is prohibited, \( Q(0) \). These two pieces of information can be combined to get:\(^1\)

\[ \tau = \frac{\rho}{2[1 - Q(0)]} \]

The fundamental assumption, made in my model, was that a network’s price-level can be treated as a fixed parameter:

\[ \text{buyer-fee} + \text{seller-fee} = \rho, \]

where \( \rho \in (0, 2\tau) \). It was argued that \( \rho \) is exogenously determined by the level of competition between the banks in the payment-card association. That is, \( \rho \) decreases as the competition within the association increases. If we make the further assumption that the cost of processing transactions is negligible (that is, \( \gamma \to 0 \)), then the markup, \( \mu \), accounts for most of the transaction-fee, \( \rho \). It follows that:

- Scenario (i) occurs when there is "strong" intra-network competition.
- Scenario (ii) occurs when there is "moderate" intra-network competition.
- Scenario (iii) occurs when there is "weak" intra-network competition.

The value of the thresholds, \( \lambda(\sigma), \eta(\sigma) \), depends on the parameters in the model. Hence, if we had values of the other parameters in the model, then we could find value for these thresholds.\(^2\) The observed value of \( \rho \) could then be compared with these thresholds to identify the relevant scenario.

### 21.3 "Strong" Intra-Network Competition

In scenario (i), \( \rho \leq \lambda(\sigma) \); which implies that \( \rho < \tau \). Since \( \omega = 2\tau - \rho \), it follows that \( \rho < \omega \). Hence, when tying is prohibited, the extra consumer-surplus generated by platforms is

\[ \Delta \tilde{\Phi}_I = \hat{W}(\rho) + \hat{W}(0) \]

Whereas, when tying is enforced, the extra consumer-surplus generated by platforms becomes

\[ \Delta \tilde{\Phi}_{II} = 2\hat{W}(\rho) \]

Since \( \hat{W}(f) \) increases as \( |f| \) decreases, it follows that:

---

\(^1\)Note that if \( Q(0) < \frac{1}{2} \), then \( \rho > \tau \). That is, if buyers pay the entire transaction-fee, then the transaction-fee exceeds the average transaction-cost if and only if less than 50% of retail sales are made by card.

\(^2\)This would involve numerically solving the implicit equations that define \( \lambda(\sigma), \eta(\sigma) \).
Proposition 21.1 ("strong" intra-network competition.) If \( \rho \leq \lambda(\sigma) \), then the effect of tying on the consumer-surplus becomes:

\[
\Delta \Phi_{II} - \Delta \Phi_I = \hat{W}(\rho) - \hat{W}(0) < 0
\]

Hence, if there’s "strong" intra-network competition, then tying fails the pass on test. Indeed, consumers are actually harmed by tying.

21.4 "Moderate" Intra-Network Competition

In Scenario (ii), \( \lambda(\sigma) < \rho \leq \eta(\sigma) \), which implies that \( \rho \leq \tau \). Since \( \omega = 2\tau - \rho \), it follows that \( \rho \leq \omega \). Hence, when tying is prohibited, the extra consumer-surplus generated by platforms is

\[
\Delta \Phi_I = \hat{W}(\rho) + \hat{W}(0)
\]

Whereas, when tying is enforced, the extra consumer-surplus generated by platforms is

\[
\Delta \Phi_{II} = \hat{W}(\rho) + \hat{W}(y^*(0)),
\]

where

\[
0 < y^*(0) < \rho
\]

Since \( \hat{W}(f) \) increases as \(|f|\) decreases, it follows that:

Proposition 21.2 ("Moderate" intra-network competition.) If \( \lambda(\sigma) < \rho \leq \eta(\sigma) \), then the effect of tying on the consumer-surplus becomes:

\[
\Delta \Phi_{II} - \Delta \Phi_I = \hat{W}(y^*(0)) - \hat{W}(0) < 0
\]

Hence, if there’s "moderate" intra-network competition, then tying fails the pass on test. Indeed, consumers are actually harmed by tying.

21.5 "Weak" Intra-Network Competition

In Scenario (iii), \( \rho > \eta(\sigma) \). Since \( \eta(\sigma) < \tau \), there are two sub-cases to consider: \( \eta(\sigma) < \rho \leq \tau \); and \( \tau < \rho < 2\tau \). These cases are considered in turn:

Firstly, if \( \rho \leq \tau \), then \( \rho \leq \omega \). Hence, when tying is prohibited, the extra consumer-surplus generated by platforms is

\[
\Delta \Phi_I = \hat{W}(\rho) + \hat{W}(0)
\]

Whereas, when tying is enforced, the extra consumer-surplus generated by platforms is

\[
\Delta \Phi_{II} = \hat{W}(x^*(0)) + \hat{W}(y^*(0))
\]

Secondly, if \( \rho > \tau \), then \( \rho > \omega \). Hence, when tying is prohibited, the extra consumer-surplus generated by platforms is

\[
\Delta \Phi_I = \hat{W}(\omega) + \hat{W}(0),
\]
where
\[ \hat{W}(\omega) = 0 \]

Whereas, when tying is enforced, the extra consumer-suplus generated by platforms is
\[ \Delta \hat{\Phi}_{II} = \hat{W}(x^*(0)) + \hat{W}(y^*(0)) \]

**Lemma 21.1 (Lower-bound for the AEUB offered by Network A.)** If \( z = 0 \), then at the point of tangency, \((x, y) = (x^*(0), y^*(0))\), the aggregate end-user benefit (AEUB) offered by Network A exceeds the maximum end-user benefit that a single platform can generate. That is,
\[ \hat{W}(x^*(0)) + \hat{W}(y^*(0)) > \hat{W}(0) \]

**Proof.** The point of tangency is defined by
\[ (x^*(0), y^*(0)) = \arg \max_{x,y} \{x + y : G(x, y, 0) = 200\tau^2\sigma^2\} \]

Since \((x^*(0), y^*(0))\) is on the constraint-curve, it follows that:
\[ (10\tau\sigma - \omega^2 + x^*(0)^2)^2 + (10\tau\sigma + y^*(0)^2)^2 = 200\tau^2\sigma^2 \]

Furthermore, by substituting \(x = \omega, y = 0\) into the implicit equations for \(x^*(0)\) and \(y^*(0)\) it can been shown that \((x^*(0), y^*(0)) \neq (\omega, 0)\) (providing \(\omega > 0\)).

The equation for the curve can be re-expressed as
\[ 20\tau\sigma.(x^2 + y^2 - \omega^2) + (\omega^2 - x^2)^2 + y^4 = 0, \]
where
\[ x > 0 \]
\[ y > 0 \]

When expressed in polar coordinates the curve becomes
\[ 20\tau\sigma.(a^2 - \omega^2) + \left(a^2 \cos^2 \hat{A} - \omega^2\right)^2 + a^4 \sin^4 \hat{A} = 0 \]

where
\[ a = \sqrt{x^2 + y^2} \]
\[ \tan \hat{A} = \frac{y}{x} \]

In polar coordinates the curve becomes
\[ 20\tau\sigma.(\omega^2 - a^2) = \left(a^2 \cos^2 \hat{A} - \omega^2\right)^2 + a^4 \sin^4 \hat{A} \]

Firstly, it can be seen that \(a > 0\) (because \(\tau\sigma > 0\)). Secondly, since the RHS can’t be negative, it follows that \(\omega^2 - a^2 \geq 0\); which implies that \(a \leq \omega\). Hence, the curve is enclosed by a circle of radius \(\omega\).
Moreover, it can be shown that \( a = \omega \) iff \( \hat{A} = 0 \). The argument is as follows. Firstly, because \( 20\tau\sigma > \omega^2 - a^2 \), it follows that if \( \hat{A} = 0 \), then \( \omega^2 = a^2 \). Secondly, if \( \omega^2 = a^2 \), then

\[
0 = \left( a^2 \cos^2 \hat{A} - \omega^2 \right)^2 + a^4 \sin^4 \hat{A}
\]

\[
= a^4 \left[ \left( \cos^2 \hat{A} - 1 \right)^2 + \sin^4 \hat{A} \right]
\]

\[
= 2a^4 \sin^4 \hat{A}
\]

Since \( a > 0 \), it follows that \( \sin \hat{A} = 0 \). Since \( x, y \) are positive, this requires that \( \hat{A} = 0 \). This proves that the curve only intersects the circle at \( x = \omega, y = 0 \).

The results of this analysis can be summarized as follows. Firstly, if \( x > y \) and \( G(x, y, 0) = 200\tau^2\sigma^2 \), then \( x^2 + y^2 < \omega^2 \). Secondly, \( (x^*(0), y^*(0)) \neq (\omega, 0) \) and \( G(x^*(0), y^*(0), 0) = 200\tau^2\sigma^2 \). Therefore,

\[
x^*(0)^2 + y^*(0)^2 < \omega^2
\]

Since \( \tilde{W}(f) = \frac{1}{4\tau} (\omega(\rho)^2 - f^2) \), it follows that

\[
\tilde{W}(x^*(0)) + \tilde{W}(y^*(0)) = \frac{1}{4\tau} (2\omega(\rho)^2 - x^*(0)^2 - y^*(0)^2)
\]

\[
> \frac{1}{4\tau} \omega(\rho)^2
\]

\[
= \tilde{W}(0)
\]

Lemma 21.2 (A Unique Threshold.) There exists a threshold, \( \kappa(\sigma) \in (\eta(\sigma), \tau) \), such that

\[
\tilde{W}(x^*(0)) + \tilde{W}(y^*(0)) > \tilde{W}(\rho) + \tilde{W}(0) \iff \rho > \kappa(\sigma)
\]

Proof. It’s useful to begin by analyzing the relationship between \( x^*(0), y^*(0) \) and \( \rho \). These results will then be used to analyze the relationship between \( \rho \) and the consumer surplus.

Part 1. It can be shown that \( x^*(0) \) decreases as \( \rho \) increases; and the argument is as follows. Firstly, \( x^*(0) \) corresponds to the value of \( x \) that satisfies the following implicit equation:

\[
x^2 \left[ 10\tau\sigma + x^2 - \omega(\rho)^2 \right]^2 = \left[ \Lambda \left( x^2 - \omega(\rho)^2 \right) - 10\tau\sigma \right] \cdot \Lambda^2 (x^2 - \omega(\rho)^2)
\]

where

\[
\Lambda(x^2 - \omega(\rho)^2) \equiv \sqrt{200\tau^2\sigma^2 - [10\tau\sigma + x^2 - \omega(\rho)^2]^2}
\]
Secondly, we can imagine a sketch of the LHS and RHS as functions of $x$. It can be seen that the LHS is increasing in $x$, whereas, the RHS is decreasing in $x$. Hence, the LHS corresponds to an upward sloping curve and the RHS corresponds to a downwards sloping curve. This implies that $x^*(0)$ corresponds to the point where the curves intersect. Now consider the effect of an increase in $\rho$ on $x^*(0)$. Since $\omega(\rho)$ decreases as $\rho$ increases, it follows that the LHS increases, whereas, the RHS decreases. It can be seen that the point of intersection moves to the left. Hence, $x^*(0)$ decreases as $\rho$ increases.

Similarly, it can be shown that $y^*(0)$ decreases as $\rho$ increases; and the argument is as follows. Firstly, $y^*(0)$ corresponds to the value of $x$ that satisfies the following implicit equation:

$$y^2 \cdot [10\pi + y^2]^2 = [\Lambda(y^2) + \omega(\rho)^2 - 10\pi \cdot \Lambda^2(y^2)]$$

Secondly, we can imagine a sketch of the LHS and the RHS as functions of $y$. It can be seen that the LHS is increasing in $y$ while the RHS is decreasing in $y$. Hence, the LHS corresponds to an upward sloping curve and the RHS corresponds to a downwards sloping curve. This implies that $y^*(0)$ corresponds to the point where the curves intersect. Now consider the effect of an increase in $\rho$ on $y^*(0)$. Since $\omega(\rho)$ decreases as $\rho$ increases, it follows that the RHS decreases, while, the LHS is invariant. It can be seen that the point of intersection moves to the left. Hence, $y^*(0)$ decreases as $\rho$ increases.

These results can be summarized as: $\frac{\partial}{\partial \rho} x^* < 0, \frac{\partial}{\partial \rho} y^* < 0$.

Part 2. It can be seen that

$$\tilde{W}(x) + \tilde{W}(y) > \tilde{W}(\rho) + \tilde{W}(0) \iff \rho^2 > x^2 + y^2$$

Firstly, if $\rho = \eta$, then without tying $x = \rho, y = 0$, whereas, with tying $x = \rho, y = y^{**}(0) > 0$; which implies that tying lowers the consumer surplus. Secondly, if $\rho = \tau$, then without tying $x = \rho = \omega(\rho), y = 0$, whereas, with tying $x = x^*(0), y = y^*(0)$; which implies that tying raises the consumer surplus because $x^*(0)^2 + y^*(0)^2 < \omega(\rho)^2$. Hence, $\Delta \tilde{\Phi}_{II} < \Delta \tilde{\Phi}_{I}$ at the start of the interval $(\rho = \eta)$ but $\Delta \tilde{\Phi}_{II} > \Delta \tilde{\Phi}_{I}$ at the end of the interval $(\rho = \tau)$. Hence, there must exist at least one point between $\eta$ and $\tau$ at which $\rho^2 = x^*(0)^2 + y^*(0)^2$. Since $x^*(0)^2, y^*(0)^2$ decrease as $\rho$ increases, it follows that there is a unique point where $\rho^2 = x^*(0)^2 + y^*(0)^2$. Therefore, there exists a unique value of $\rho$, somewhere between $\eta$ and $\tau$, such that $\rho^2 = x^*(0)^2 + y^*(0)^2$. $\blacksquare$

It follows that:

**Proposition 21.3 ("Weak" intra-network competition.)** If $\rho > \eta(\sigma)$, then there are two cases to consider. These are: $\eta(\sigma) < \rho \leq \kappa(\sigma)$ and $\kappa(\sigma) < \rho < 2\tau$.

1. If $\rho \leq \kappa$, then

$$\Delta \tilde{\Phi}_{II} - \Delta \tilde{\Phi}_{I} \leq 0$$

Hence, if $\rho \leq \kappa$, then tying fails the pass on test.

2. If $\rho > \kappa$, then

$$\Delta \tilde{\Phi}_{II} - \Delta \tilde{\Phi}_{I} > 0$$

Hence, if $\rho > \tau$, then tying satisfies the pass on test.
21.6 The Main Theorem

It has been shown that the outcome differs depending on the size of $\rho$. Three cases were considered: (i) $\rho \leq \lambda(\sigma)$; (ii) $\lambda(\sigma) < \rho \leq \eta(\sigma)$; and (iii) $\rho > \eta(\sigma)$.

It has been shown that in the first two cases trying always lowers the extra-consumer surplus generated by tying. However, in the final case, if $\rho$ exceeds a certain threshold, namely, $\kappa(\sigma)$, then tying increases the extra-consumer surplus generated by platforms. It follows that:

\textbf{Theorem 21.1 (Main Theorem.)} There exists a unique threshold, $\kappa(\sigma) \in (\eta(\sigma), \tau)$, such that: (i) if $\rho < \kappa(\sigma)$, then $\Delta \Phi_{II} < \Delta \Phi_{I}$; (ii) if $\rho = \kappa(\sigma)$, then $\Delta \Phi_{II} = \Delta \Phi_{I}$; and (iii) if $\rho > \kappa(\sigma)$, then $\Delta \Phi_{II} > \Delta \Phi_{I}$. That is, tying satisfies the pass on test iff $\rho > \kappa(\sigma)$.

To visualize the effect of tying on the consumer surplus it’s useful to construct a sketch of $\Delta \Phi_{II}$ and $\Delta \Phi_{I}$ against $\rho$. It can be seen that the two curves cross at $\kappa$. (Note that by assumption $\rho < 2\tau$ because otherwise platforms are too expensive to generate any benefit for their users.)

\begin{center}
\includegraphics[width=0.8\textwidth]{figure18.png}
\end{center}

\textit{Figure 18. Unique point of intersection.}

It can be seen that the effect of tying depends on the value of $\rho$. Furthermore, there is a unique threshold, $\kappa$, above which tying raises the extra consumer surplus generated by platforms. The relationship between $\rho$ and the effect of tying on the extra-consumer surplus is summarized in the sketch below.
Figure 19. Effect of tying on consumer surplus.
Chapter 22

Conclusion

22.1 Summary

This thesis analyses the effect of tying sellers’ membership of a monopoly platform to membership of another platform, which operates in an otherwise competitive market. Visa’s contentious use of the honour-all-cards rule to tie their debit and credit cards is an example of such a tie-in.

There has been a move in the European Community and the US to judge tying cases under what has become known as the "rule of reason". For example, the Treaty of the European Communities (TEC), under Article 81(3), permits practices that might otherwise be seen as harmful when they are indispensable to creating economic benefit through improved market efficiency. However, a fair proportion of the extra-surplus must be passed on to consumers; this requirement is referred to as the "pass on test".

Rochet and Tirole (2008) found that allowing Visa to tie its credit and debit cards raised "social welfare", which was defined as the sum of Visa’s profit and the surplus received by merchants and cardholders. However, it can reasonably be argued that their study doesn’t fully address the concerns of regulators because it doesn’t show that consumers benefit from the imposition of a tie. Hence, my thesis investigates whether tying platforms that operate in separate markets satisfies the "pass on test".

Framework. In Rochet and Tirole (2008) sellers operate in two independent markets (ç and q) and each platform serves one particular market. A multi-platform network (Network A) runs platforms in both markets; and a rival network (Network B) only operates in market q. The price-level (buyer-fee plus seller-fee) on a network’s platforms is exogenously determined (by inter-bank competition) but they can choose the price-structure. My study extended this framework by explicitly modelling competition between sellers offering differentiated goods.

No Tying. Networks want low buyer-fees to increase the volume of transactions, whereas, sellers want higher buyer-fees to deter the excessive use of payment-cards. Hence, platform competition leads to a price-structure that maximizes the net-benefit received by buyers and sellers. In contrast, a monopoly platform extracts most of the surplus by encouraging excessive use of payment-cards. Therefore, if tying is prohibited, then competition for sellers in market q leads to an optimal price-structure. However, Network
A extracts most of the surplus created by its monopoly platform in market $\zeta$. Finally, if the average transaction-cost, $\tau$, exceeds the price-level, $\rho$, then the net-benefit generated by a monopoly platform remains strictly positive; otherwise, it delivers no net-benefit.

**Tying.** If a tie-in is enforced, then sellers have to choose between networks rather than platforms and so sellers have to consider the total net-benefit on a network’s platforms. Hence, Network $A$ (multi-platform network) can always exclude Network $B$ (single-platform network). However, Network $A$ is unable to exclude Network $B$ just by matching the maximum net-benefit that a single-platform network can generate; rather, it must "compensate" sellers for the extra competition they face from being on the same network. Therefore, if tying is permitted, then the total net-benefit on Network $A$ exceeds the maximum benefit that can be generated by a single platform.

**Pass-On Test.** It was found that if transaction-fees, $\rho$, are high relative to transaction-costs, $\tau$, then tying always increases the consumer surplus. However, if transaction-fees, $\rho$, are low relative to transaction-costs, $\tau$, then tying generally doesn’t benefit consumers and will reduce the consumer-surplus if their transaction-costs are sufficiently high. The intuition for this result is as follows.

**Cheap Platforms.** Consider the case where platforms are relatively cheap and tying is prohibited. Platforms become more profitable as seller-fees rise and buyer-fees fall. However, competition for sellers’ membership forces both networks to offer the maximum end-user benefit in market $\delta$ and to make their platforms free for sellers ($f_A^d = f_B^d = 0$). But Network $A$ has a monopoly in market $\zeta$, which makes it possible to attract all the sellers in this market, providing it offers a non-negative end-user benefit. Moreover, if $\rho$ is low relative to $\tau$, then sellers will join Platform $C_A$ even when they are required to pay the entire fee ($f_A^\delta = \rho$) because it still offers a positive end-user benefit. This implies that the extra-surplus generated by platforms becomes $\Delta \Phi_I = W(0) + W(\rho)$.

Now consider the case where tying is enforced. Network $B$ will still try to resist exclusion by making its platform free to sellers ($f_B^d = 0$). However, Network $A$ now has much greater flexibility over its choice of fees but the combined benefit on its platforms necessarily exceeds $2W(\rho)$. Moreover, if fees are sufficiently low, then $2W(\rho)$ can become almost double the size of $W(0)$. That is, if platforms are relatively cheap, then access to a platform in market $\zeta$ is important to sellers because it significantly affects their competitiveness even when sellers pay entire fee. Furthermore, as $\rho$ decreases, any loss suffered from being on Platform $D_A$ rather than Platform $D_B$ becomes ever more minimal. It follows that if platforms are relatively cheap, then tying gives Network $A$ substantial leverage over sellers. Moreover, this leverage can be strong enough to exclude Network $B$ despite setting the maximum possible seller fees on both their platforms. Hence, for sufficiently low values of $\rho$, the combined surplus generated by platforms when tying is enforced becomes $\Delta \Phi_{II} = 2W(\rho)$.

It can be seen that if platforms are cheap, then $\Delta \Phi_{II} - \Delta \Phi_I = W(\rho) - W(0)$. Finally, since $W(0) > W(\rho)$, it follows that tying reduces the surplus and fails the pass-on-test.

**Expensive Platforms.** Consider the case where platforms are relatively expensive and tying is prohibited. In market $\delta$ competition for sellers forces...
both networks to offer the maximum possible end-user benefit on their platforms \((f_A^d = f_B^d = 0)\). However, Network A has a monopoly in market \(c\) and so can attract all the sellers in this market providing that it offers a non-negative end-user benefit on Platform \(C_A\). If \(\rho > \tau\), then Network A is able to increase the seller-fee until it meets merchants’ resistance \((f_A^c = \omega)\). Hence, the combined surplus generated by platforms when tying is prohibited becomes \(W(0) + W(\omega)\). Finally, since \(W(\omega) = 0\), it follows that \(\Delta \Phi_I = W(0)\).

Now consider the case where platforms are relatively expensive and tying is enforced. Network B will still try to resist exclusion by making its platform free to sellers \((f_B^d = 0)\). However, by tying its platforms Network A can exclude Network B despite setting a positive seller-fee in market \(d\). Since \(f_B^d = 0\) and \(f_A^d > 0\), it follows that neither network has the best platform in both markets, which gives sellers the option to specialize by joining different networks (asymmetric membership). Furthermore, such specialization would enable sellers to retain part of the surplus created by platforms, which gives sellers an incentive to specialize and creates a kind of repulsion between them. In order to attract all the sellers, and exclude Network B, Network A must overcome this repulsion. Hence, Network A must offer a strictly higher total benefit than that offered by Network B. That is, \(W(f_A^c) + W(f_B^d) > W(0)\). Therefore, the combined surplus generated by platforms when tying is enforced becomes \(\Delta \Phi_{II} > W(0)\).

It can be seen that if platforms are expensive, then \(\Delta \Phi_{II} - \Delta \Phi_I > 0\), which implies that tying increases the surplus and satisfies the pass-on-test.

### 22.2 Further work

Rochet and Tirole carried out robustness testing by introducing a number of alternative assumptions. They claim that in each case the result is either unchanged or strengthened; that is, "social welfare" is increased by tying. However, they do not explicitly discuss the effect on consumers. Hence, it would be useful to carry out similar robustness tests of my results. I have five pieces of further work in mind:

- Markets \(c\) and \(d\) could be of different sizes. This would involve weighting demands and profits by \(N_c, N_d\), where \(N_c + N_d = 1\).
- Sellers’ could have transaction costs of their own. This would mean that a platform’s end-user benefit is maximized by setting a positive seller fee: \(\text{arg max } W(f) > 0\).
- The markets may not be fully independent (debit-card substitution). This would involve allowing debit-card users to use a credit-card instead. However, since substitution is imperfect they incur a small penalty.
- The entrant (Network \(B\)) could have control over his price level. There could be an additional stage to the game in which the entrant decides the level of his price, \(\rho_B \in (0, 2\tau)\).
- It may be possible to calibrate the model using information on fees and the volume of transactions.
Bibliography


Part V

Appendix
Appendix A

Existence and Uniqueness

The analysis of sellers’ membership decisions when platforms are tied is complicated. Hence, it’s useful to begin by considering the simpler case where there are only two relevant membership options. Moreover, since there are never more than two relevant membership options, this is sufficient to cover all eventualities.

A.1 Set Up: Two Membership Options

Suppose that fees, \( F \), are such that there are no more than two feasible membership options. These options are denoted \( m_i = 0 \) and \( m_i = 1 \). (Note that 0 and 1 need not refer to \( a \) and \( b \).) Hence, if \( m_i \in M(F) \), then \( m_i \in \{0, 1\} \). Since sellers will never choose options outside of \( M(F) \), we can analyze a simplified game in which 0 and 1 are the only options available. This game an the original game will have the same set of RSPNE’s.

Let the extra-surplus received as a consequence of choosing these options be measured in units of \( \sigma \). The extra-surplus received in market \( k \) from choosing \( m_i = 0 \) is \( \sigma v_0^k \), where \( v_0^k \in (-1, 1) \). Similarly, the extra-surplus received in market \( k \) from choosing \( m_i = 0 \) is \( \sigma v_1^k \), where \( v_1^k \in (-1, 1) \). Finally, in market \( k \), the advantage or disadvantage from choosing \( m_i = 1 \) (over choosing \( m_i = 0 \)) is

\[
\Delta v_k = v_1^k - v_0^k
\]

Finally, it’s assumed that the options aren’t equivalent: \( \Delta v_0 \neq 0 \) and/or \( \Delta v_1 \neq 0 \). (If \( m_i = 0 \) and \( m_i = 1 \) were equivalent, then sellers choose \( m_i = 1 \).)

The number of sellers that choose \( m_j = 1 \) is \( N(m) = \sum_j m_j \) and the number of sellers that choose \( m_j = 0 \) is \( 3 - N(m) \). Hence, the aggregate extra-surplus is

\[
\sum_j W(f, m_j) = \sigma \left[ (3 - N).v_0^k + N(m).v_1^k \right]
\]

\[\text{1} \text{Furthermore, it’s been shown that if } f \in [-\omega, \rho], \text{ then } -\tau < \tilde{W}(f) < \tau. \text{ Since } \tau < \sigma, \text{ it follows that if } f \in [-\omega, \rho], \text{ then }
\]

\[
-1 < \frac{\tilde{W}(f)}{\sigma} < 1
\]
This can be re-expressed as:

$$\sum_j W(f_k, m_j) = \sigma \left[ 3v_k^j + \Delta v_k.N(m) \right]$$

It can be seen that there are two groups of sellers: seller $i$ is in *Set 0* if $m_i = 0$; and seller $i$ is in *Set 1* if $m_i = 1$. Let $\mathcal{N}(F) (\subseteq \{0, 1\}^3)$ denote the set of RSPNE’s when fees are $F$. Since there are only two options, the criterion for a RSPNE can be simplified as follows: Firstly, if $\sum_j \tilde{m}_j = 0$, then:

$$\tilde{m} \in \mathcal{N}(F) \iff 1 \notin \tilde{M}(F, \tilde{m}_{-i} \langle \tilde{m}_{-i} \rangle = 0)$$

Secondly, if $\sum_j \tilde{m}_j = N(m) \notin \{0, 3\}$, then:

$$\tilde{m} \in \mathcal{N}(F) \iff \begin{cases} 1 \in \tilde{M}(F, \tilde{m}_{-i} \langle \tilde{m}_{-i} \rangle = \frac{N(m)-1}{2}, \\ 1 \notin \tilde{M}(F, \tilde{m}_{-i} \langle \tilde{m}_{-i} \rangle = \frac{N(m)}{2}). \end{cases}$$

Finally, if $\sum_j \tilde{m}_j = 3$, then:

$$\tilde{m} \in \mathcal{N}(F) \iff 1 \in \tilde{M}(F, \tilde{m}_{-i} \langle \tilde{m}_{-i} \rangle = 1).$$

### A.2 Payoffs

The profit of seller $i$ in market $k$ is

$$\Gamma_i(f_k, m_{-i}, m_i) = \frac{1}{t^5 \sigma} \left\{ 5\sigma + 3W(f_k, m_i) - \sum_j W(f_k, m_j) \right\}^2$$

The profit received by a seller depends on which set they belong to.

(1) **Sellers in Set 0.** The profit of a seller in *Set 0* ($N(m) < 3$) becomes

$$\tilde{\Gamma}_i(f_k, 0, m_{-i} \langle m_{-i} \rangle = \frac{N(m)}{2}) = \frac{\sigma}{t^5} \left[ 5 - \Delta v_k.N(m) \right]^2$$

Hence, the payoff of a seller in *Set 0* is

$$\tilde{\Pi}_i = \frac{\sigma}{t^5} \sum_k \left[ 5 - \Delta v_k.N(m) \right]^2, \text{ for } i > N(m)$$

(2) **Sellers in Set 1.** The profit of a seller in *Set 1* ($N(m) \geq 1$) becomes

$$\tilde{\Gamma}_i(f_k, 1, m_{-i} \langle m_{-i} \rangle = \frac{N(m)-1}{2}) = \frac{\sigma}{t^5} \left[ 5 + \Delta v_k.(3-N(m)) \right]^2$$

Hence, the payoff of a seller in *Set 1* is

$$\tilde{\Pi}_i = \frac{\sigma}{t^5} \sum_k \left[ 5 + \Delta v_k.(3-N(m)) \right]^2, \text{ for } i \leq N(m)$$
A.3 Unilateral Deviation

Suppose that initially there are \(N(m)\) sellers in Set 1. It follows that if seller \(i\) is in Set 0, then \(m_{-i} = \frac{N(m)}{2}\). Whereas, if seller \(i\) is in Set 1, then \(m_{-i} = \frac{N(m)-1}{2}\). Now, consider the effect of unilateral deviation on a seller’s payoff:

(1) From Set 0 to Set 1. If a seller in Set 0 \((N(m) < 3)\) were to unilaterally deviate, then their profit would become

\[
\tilde{\Gamma}_i(f_k, 1, m_{-i} | m_{-i}) = \frac{\sigma}{75} [5 + \Delta v_k (2 - N(m))]^2
\]

Hence, if a seller was "initially" in Set 0 and they unilaterally switched to Set 1, then their payoff would become:

\[
\tilde{\Pi}_i = \frac{\sigma}{75} \sum_k [5 + \Delta v_k (2 - N(m))]^2, \text{ for } i > N(m)
\]

Therefore, unilateral deviation isn’t profitable for sellers in Set 0 iff

\[
\sum_k [5 - \Delta v_k N(m)]^2 \geq \sum_k [5 + \Delta v_k (2 - N(m))]^2
\]

It can be seen that

\[
[5 + \Delta v_k (2 - N(m))]^2 - [5 - \Delta v_k N(m)]^2 = 4 \Delta v_k (5 + \Delta v_k - \Delta v_k N(m))
\]

Hence, the condition becomes

\[
\sum_k \{5 \Delta v_k + (\Delta v_k)^2 - (\Delta v_k)^2 \cdot N(m)\} \leq 0,
\]

which is equivalent to

\[
\frac{5 \sum_k \Delta v_k}{\sum_k (\Delta v_k)^2} \leq N(m) - 1
\]

(2) From Set 1 to Set 0. If a seller in Set 1 \((N(m) \geq 1)\) were to unilaterally deviate, then their profit would become

\[
\tilde{\Gamma}_i(f_k, 0, m_{-i} | m_{-i}) = \frac{\sigma N(m)-1}{75} [5 - \Delta v_k (N(m) - 1)]^2
\]

Hence, if a seller was "initially" in Set 1 and they unilaterally switched to Set 0, then their payoff would become:

\[
\tilde{\Pi}_i = \frac{\sigma}{75} \sum_k [5 - \Delta v_k (N(m) - 1)]^2, \text{ for } i \leq N(m)
\]

Therefore, unilateral deviation isn’t profitable for sellers in Set 1 iff

\[
\sum_k [5 + \Delta v_k (3 - N(m))]^2 \geq \sum_k [5 - \Delta v_k (N(m) - 1)]^2
\]
It can be seen that
\[ [5 + \Delta v_k.(3 - N(m))]^2 - [5 - \Delta v_k.(N(m) - 1)]^2 = 4\Delta v_k. (5 + 2.\Delta v_k - \Delta v_k.N(m)) \]
Hence, the condition becomes
\[ \sum_k \{5.\Delta v_k + 2.(\Delta v_k)^2 - (\Delta v_k)^2.N(m)\} \geq 0 \]
which is equivalent to
\[ \frac{5 \sum_k \Delta v_k}{\sum_k (\Delta v_k)^2} \geq N(m) - 2 \]

A.4 Criteria for RSPNE’s

There are four possible types of RSPNE, the criteria for which are as follows:

1. If \( \sum_j \tilde{m}_j = 0 \), then \( \widetilde{m} \in \mathcal{N}(F) \) iff \( 1 \notin \widehat{M}(F, \tilde{m}_{-i}| \langle \tilde{m}_{-i} \rangle = 0) \), which requires
\[ \frac{5 \sum_k \Delta v_k}{\sum_k (\Delta v_k)^2} < -1 \]

2. If \( \sum_j \tilde{m}_j = 1 \), then \( \widetilde{m} \in \mathcal{N}(F) \) iff
\[ 1 \in \widehat{M}(F, \tilde{m}_{-i}| \langle \tilde{m}_{-i} \rangle = 0) \]
and
\[ 1 \notin \widehat{M}(F, \tilde{m}_{-i}| \langle \tilde{m}_{-i} \rangle = \frac{1}{2}) \]
which requires
\[ -1 \leq \frac{5 \sum_k \Delta v_k}{\sum_k (\Delta v_k)^2} < 0 \]

3. If \( \sum_j \tilde{m}_j = 2 \), then \( \widetilde{m} \in \mathcal{N}(F) \) iff
\[ 1 \in \widehat{M}(F, \tilde{m}_{-i}| \langle \tilde{m}_{-i} \rangle = \frac{1}{2}) \]
and
\[ 1 \notin \widehat{M}(F, \tilde{m}_{-i}| \langle \tilde{m}_{-i} \rangle = 1) \]
which requires
\[ 0 \leq \frac{5 \sum_k \Delta v_k}{\sum_k (\Delta v_k)^2} < 1 \]

4. If \( \sum_j \tilde{m}_j = 3 \), then \( \widetilde{m} \in \mathcal{N}(F) \) iff \( 1 \in \widehat{M}(F, \tilde{m}_{-i}| \langle \tilde{m}_{-i} \rangle = 1) \), which requires
\[ 1 \leq \frac{5 \sum_k \Delta v_k}{\sum_k (\Delta v_k)^2} \]
A.5 Existence and "Uniqueness" of RSPNE's

It can be seen that the outcome depends on

$$\frac{5 \sum_k \Delta v_k}{\sum_k (\Delta v_k)^2}$$

Furthermore, the criteria are mutually exclusive and exhaustive of the possibilities (one of them must occur). This implies that if there are only two options available, then a RSPNE exists and is unique up to a relabelling of the sellers. Moreover, the same must be true of all cases where there are no more than two relevant membership options, because availability of the other options can’t alter the set of RSPNE’s.

**Lemma A.1 (Existence and Uniqueness of RSPNE’s.)** If there are no more than two relevant membership options, then a RSPNE exists and is unique up to a relabelling of the sellers. That is, if $|\mathbb{M}(F)| \leq 2$, then the set of RSPNE’s, $\mathcal{N}(F)$, has the following properties:

1. The set of equilibria isn’t empty: $\mathcal{N}(F) \neq \emptyset$, $\forall F \in \mathcal{F}$.
2. If $\tilde{m} \in \mathcal{N}(F)$, then for all $m' \in \mathbb{M}^3$, if there exists $m \in \mathbb{M}$ such that $\sum_j 1(\tilde{m}_j = m) \neq \sum_j 1(m'_j = m)$, then $m' \notin \mathcal{N}(F)$. 
Appendix B

Configurations and Relevant Options

An option, $m$, is said to be "irrelevant" if there is another option, $m'$, such that $m' > m$ or $m' \sim m$ and $m' > m$. An option is said to be relevant if it is not irrelevant. Let $\bar{M}(F) \subseteq M$ denote the set of relevant membership options. In order to reduce the number of configurations we need to analyze it's useful to identify (and remove) "irrelevant" membership options. Furthermore, it's been shown that sellers never have more than two relevant membership options: $|\bar{M}(F)| \leq 2$. (This is the sellers' dichotomy.) Previous analysis showed that the possibilities are as follows:

\[
\begin{align*}
\hat{W}(z) < 0, \hat{W}(\max\{y,z\}) < \hat{W}(y) & \Rightarrow \bar{M}(F) \subseteq \{o,a\} \\
\hat{W}(z) < 0, \hat{W}(\max\{y,z\}) \geq \hat{W}(y) & \Rightarrow \bar{M}(F) \subseteq \{o,b\} \\
\hat{W}(z) \geq 0, \hat{W}(\max\{y,z\}) < \hat{W}(y) & \Rightarrow \bar{M}(F) \subseteq \{b,a\} \\
\hat{W}(z) \geq 0, \hat{W}(\max\{y,z\}) \geq \hat{W}(y) & \Rightarrow \bar{M}(F) \subseteq \{b,h\}
\end{align*}
\]

B.1 Membership when $\bar{M}(F) \subseteq \{o,a\}$

Suppose that $z > \omega$ and $\hat{W}(\max\{y,z\}) < \hat{W}(y)$, which requires $y < z$ (draw a sketch). Since there are only two feasible options, $\bar{M}(F) \subseteq \{o,a\}$, each outcome corresponds to a specific set of membership vectors:

\[
\begin{align*}
\tilde{m} & \in N_{330}(F) \iff \tilde{m} \in \{a\}^3 \subseteq N(F) \\
\tilde{m} & \in N_{220}(F) \iff \tilde{m} \in \{a\}^2 \times \{o\} \subseteq N(F) \\
\tilde{m} & \in N_{110}(F) \iff \tilde{m} \in \{a\} \times \{o\}^2 \subseteq N(F) \\
\tilde{m} & \in N_{000}(F) \iff \tilde{m} \in \{o\}^3 \subseteq N(F)
\end{align*}
\]

**Proof.** Each part is considered in turn:
Part (1). If \( \vec{m} = \{a\}^3 \in \mathcal{N}(F) \), then:

\[
\widetilde{\Sigma}_A(f, \vec{m}) = 3, \widetilde{\Sigma}_A(f, \vec{m}) = 3, \widetilde{\Sigma}_B(f, \vec{m}) = 0
\]

(This follows from the definitions of \( \widetilde{\Sigma}_A(f, m) \), \( \widetilde{\Sigma}_B(f, m) \).) This implies that \( \vec{m} \in \mathcal{N}_{330}(F) \). The corresponding set of membership vectors can be found as follows. Firstly, all the elements in \( \vec{m} \) must be "relevant". \( \vec{m} \in \mathcal{M}(F)^3 \). By assumption, \( \mathcal{M}(F) \subseteq \{o, a\} \). Hence, \( \vec{m} \in \{o, a\}^3 \). Secondly, if \( \vec{m} \in \mathcal{N}_{330}(F) \), then \( \widetilde{\Sigma}_A(f, \vec{m}) = 3 \), which requires \( \vec{m} \in \{a, b\}^3 \). (This follows from the definition of \( \widetilde{\Sigma}_A(f, \vec{m}) \).) Therefore,

\[
\vec{m} \in \{a, b\}^3 \cap \{o, a\}^3 = \{a\}^3
\]

Part (2). If \( \vec{m} \in \{a\}^2 \times \{o\} \subseteq \mathcal{N}(F) \), then:

\[
\widetilde{\Sigma}_A(f, \vec{m}) = 2, \widetilde{\Sigma}_A(f, \vec{m}) = 2, \widetilde{\Sigma}_B(f, \vec{m}) = 0
\]

Therefore, \( \vec{m} \in \mathcal{N}_{330}(F) \). The corresponding set of membership vectors can be found as follows. Firstly, all the elements in \( \vec{m} \) must be "relevant". By assumption, \( \mathcal{M}(F) \subseteq \{o, a\} \). Hence, \( \vec{m} \in \{o, a\}^3 \). Secondly, if \( \vec{m} \in \mathcal{N}_{220}(F) \), then \( \widetilde{\Sigma}_A(f, \vec{m}) = 2 \), which requires \( \vec{m} \in \{a, b\}^2 \times \{o, b\} \). (This follows from the definition of \( \widetilde{\Sigma}_A(f, \vec{m}) \).) Therefore,

\[
\vec{m} \in \{a, b\}^2 \times \{o, b\} \cap \{o, a\}^3 = \{a\}^2 \times \{o\}
\]

Part (3) and Part (4) are very similar. \( \blacksquare \)

Since the sellers are initially identical, I don’t distinguish between outcomes that are identical up to an interchange of sellers. (For example, \( \{a, a, b\} \) and \( \{a, b, a\} \) are basically the same sort of outcome.) Hence, there are four distinct types of membership vector.

### B.2 Membership when \( \mathcal{M}(F) \subseteq \{o, h\} \)

Suppose that \( z > \omega \) and \( W(\max\{y, z\}) \geq W(y) \), which requires \( y \geq z \) (draw a sketch). Since there are only two feasible options, \( \mathcal{M}(F) \subseteq \{o, h\} \), each outcome corresponds to a specific set of membership vectors:

**Lemma B.2 (Membership when \( \mathcal{M}(F) \subseteq \{o, h\} \).)** Suppose that \( \mathcal{M}(F) \subseteq \{o, h\} \). It follows that:

- \( \vec{m} \in \mathcal{N}_{330}(F) \) iff \( \vec{m} \in \{h\}^3 \subseteq \mathcal{N}(F) \).
- \( \vec{m} \in \mathcal{N}_{220}(F) \) iff \( \vec{m} \in \{h\}^2 \times \{o\} \subseteq \mathcal{N}(F) \).
- \( \vec{m} \in \mathcal{N}_{110}(F) \) iff \( \vec{m} \in \{h\} \times \{o\}^2 \subseteq \mathcal{N}(F) \).
- \( \vec{m} \in \mathcal{N}_{000}(F) \) iff \( \vec{m} \in \{o\}^3 \subseteq \mathcal{N}(F) \).

**Proof.** Each part is considered in turn:
Proof. Each part is considered in turn:

Part (1). If \( \widetilde{m} = \{b\}^3 \in \mathcal{N}(F) \), then:

\[
\Sigma_A(f_c, \widetilde{m}) = 3, \quad \Sigma_A(f_d, \widetilde{m}) = 3.1(y \geq z), \quad \Sigma_B(f_d, \widetilde{m}) = 3.1(y < z)
\]

(This follows from the definitions of \( \Sigma_A(f_k, m) \), \( \Sigma_A(f_d, m) \).) It can be shown that if \( z > \omega \), \( \hat{W}(\max\{y, z\}) \geq \hat{W}(y) \), then \( y \geq z \); which implies that \( I = 3, J = 3, K = 0 \). Therefore, \( \widetilde{m} \in \mathcal{N}\{330\}(F) \). The corresponding membership vector can be found as follows. Firstly, all the elements in \( \widetilde{m} \) must be "relevant". \( \widetilde{m} \in \mathcal{M}(F)\}^3 \). By assumption, \( \mathcal{M}(F) \subseteq \{a, b\} \). Hence, \( \widetilde{m} \in \{a, b\}^3 \). Secondly, if \( \widetilde{m} \in \mathcal{N}\{330\}(F) \), then \( \Sigma_A(f_c, \widetilde{m}) = 3 \), which requires \( \widetilde{m} \in \{a, b\}^3 \). (This follows from the definition of \( \Sigma_A(f_c, \widetilde{m}) \).) Therefore,

\[
\widetilde{m} \in \{a, b\}^3 \cap \{a, b\}^3 = \{b\}^3
\]

Part (2). If \( \widetilde{m} \in \{b, \} \times \{a\} \subseteq \mathcal{N}(F) \), then:

\[
\Sigma_A(f_c, \widetilde{m}) = 2, \quad \Sigma_A(f_d, \widetilde{m}) = 2.1(y \geq z), \quad \Sigma_B(f_d, \widetilde{m}) = 0
\]

Hence, if \( y \geq z \), then \( I = 2, J = 2, K = 0 \) (otherwise, \( I = 2, J = 0, K = 2 \)). This suggests that \( \widetilde{m} \in \mathcal{N}\{220\}(F) \cup \mathcal{N}\{202\}(F) \). However, it’s been shown that there’s no RSPNE in which \( I = 2, J = 0, K = 2 \). That is, \( \mathcal{N}\{202\}(F) = \emptyset \). Therefore, \( \widetilde{m} \in \mathcal{N}\{220\}(F) \). The corresponding set of membership vectors can be found as follows. Firstly, all the elements in \( \widetilde{m} \) must be "relevant". By assumption, \( \mathcal{M}(F) \subseteq \{a, b\} \). Hence, \( \widetilde{m} \in \{a, b\}^3 \). Secondly, if \( \widetilde{m} \in \mathcal{N}\{220\}(F) \), then \( \Sigma_A(f_c, \widetilde{m}) = 2 \), which requires \( \widetilde{m} \in \{a, b\}^2 \times \{a, b\} \). (This follows from the definition of \( \Sigma_A(f_c, \widetilde{m}) \).) Therefore,

\[
\widetilde{m} \in \{a, b\}^2 \times \{a, b\} \cap \{a, b\}^3 = \{b\}^2 \times \{a\}
\]

Part (3) and Part (4) are very similar. ■

B.3 Membership when \( \mathcal{M}(F) \subseteq \{b, a\} \)

If \( z \leq \omega \) and \( \hat{W}(\max\{y, z\}) < \hat{W}(y) \), then \( \mathcal{M}(F) \subseteq \{b, a\} \). Since there are only two feasible options, \( \mathcal{M}(F) \subseteq \{b, a\} \), each outcome corresponds to a specific set of membership vectors:

Lemma B.3 (Membership when \( \mathcal{M}(F) \subseteq \{b, a\} \).) Suppose that \( \mathcal{M}(F) \subseteq \{b, a\} \).

It follows that:

\[
\begin{align*}
\widetilde{m} & \in \mathcal{N}\{330\}(F) \text{ iff } \widetilde{m} \in \{a\}^3 \subseteq \mathcal{N}(F) \\
\widetilde{m} & \in \mathcal{N}\{221\}(F) \text{ iff } \widetilde{m} \in \{a\}^2 \times \{b\} \subseteq \mathcal{N}(F) \\
\widetilde{m} & \in \mathcal{N}\{112\}(F) \text{ iff } \widetilde{m} \in \{a\} \times \{b\}^2 \subseteq \mathcal{N}(F) \\
\widetilde{m} & \in \mathcal{N}\{003\}(F) \text{ iff } \widetilde{m} \in \{b\}^3 \subseteq \mathcal{N}(F)
\end{align*}
\]

Proof. Each part is considered in turn:
Part (1). If $\vec{m} \in \{a\}^3 \subseteq \mathcal{N}(F)$, then:

$$\Sigma_A(f_c, \vec{m}) = 3, \quad \Sigma_A(f_d, \vec{m}) = 3, \quad \Sigma_B(f_d, \vec{m}) = 0$$

(This follows from the definitions of $\Sigma_A(f_k, m)$, $\Sigma_B(f_d, m)$.) This implies that either $\vec{m} \in \mathcal{N}_{330}(F)$. The corresponding set of membership vectors can be found as follows. Firstly, all the elements in $\vec{m}$ must be "relevant": $\vec{m} \in \mathcal{M}(F)^3$. By assumption, $\mathcal{M}(F) \subseteq \{b, a\}$. Hence, $\vec{m} \in \{b, a\}^3$. Secondly, if $\vec{m} \in \mathcal{N}_{330}(F)$, then $\Sigma_A(f_c, \vec{m}) = 3$, which requires $\vec{m} \in \{a, b\}^3$. Therefore,

$$\vec{m} \in \{a, b\}^3 \cap \{b, a\}^3 = \{a\}^3$$

Part (2). If $\vec{m} \in \{a\}^2 \times \{b\} \subseteq \mathcal{N}(F)$, then:

$$\Sigma_A(f_c, \vec{m}) = 2, \quad \Sigma_A(f_d, \vec{m}) = 2, \quad \Sigma_B(f_d, \vec{m}) = 1$$

Therefore, $\vec{m} \in \mathcal{N}_{221}(F)$. The corresponding set of membership vectors can be found as follows. Firstly, all the elements in $\vec{m}$ must be "relevant". By assumption, $\mathcal{M}(F) \subseteq \{b, a\}$. Hence, $\vec{m} \in \{b, a\}^3$. Secondly, if $\vec{m} \in \mathcal{N}_{221}(F)$, then $\Sigma_A(f_c, \vec{m}) = 2$, which this requires $\vec{m} \in \{a, b\}^2 \times \{a, b\}$. Therefore,

$$\vec{m} \in \{a, b\}^2 \times \{a, b\} \cap \{b, a\}^3 = \{a\}^2 \times \{b\}$$

Part (3) and Part (4) are very similar. ■

B.4 Membership when $\mathcal{M}(F) \subseteq \{b, h\}$

Suppose that $z \leq \omega$ and $\hat{W}(\max\{y, z\}) \geq \hat{W}(y)$. Since there are only two feasible options, $\mathcal{M}(F) \subseteq \{b, h\}$, each outcome corresponds to a specific set of membership vectors:

Lemma B.4 (Membership when $\mathcal{M}(F) \subseteq \{b, h\}$.) Suppose that $\mathcal{M}(F) \subseteq \{b, h\}$.

It follows that:

$$\vec{m} \in \mathcal{N}_{330}(F) \cup \mathcal{N}_{303}(F) \quad \text{iff} \quad \vec{m} \in \{h\}^3 \subseteq \mathcal{N}(F)$$

$$\vec{m} \in \mathcal{N}_{221}(F) \quad \text{iff} \quad \vec{m} \in \{h\}^2 \times \{b\} \subseteq \mathcal{N}(F)$$

$$\vec{m} \in \mathcal{N}_{112}(F) \quad \text{iff} \quad \vec{m} \in \{h\} \times \{b\}^2 \subseteq \mathcal{N}(F)$$

$$\vec{m} \in \mathcal{N}_{003}(F) \quad \text{iff} \quad \vec{m} \in \{b\}^3 \subseteq \mathcal{N}(F)$$

Proof. Each part is considered in turn:

Part (1). If $\vec{m} \in \{h\}^3 \subseteq \mathcal{N}(F)$, then:

$$\Sigma_A(f_c, \vec{m}) = 3, \quad \Sigma_A(f_d, \vec{m}) = 3.1(y \geq z), \quad \Sigma_B(f_d, \vec{m}) = 3.1(y < z)$$

(This follows from the definitions of $\Sigma_A(f_k, m)$ and $\Sigma_B(f_d, m)$.) Hence, if $y \geq z$, then $I = 3$, $J = 3$ and $K = 0$; otherwise, $I = 3$, $J = 0$, and $K = 3$. This implies that either $\vec{m} \in \mathcal{N}_{330}(F)$ or $\vec{m} \in \mathcal{N}_{303}(F)$. The corresponding
set of membership vectors can be found as follows. Firstly, all the elements in \( \tilde{m} \) must be "relevant": \( \tilde{m} \in \mathbb{M}(F)^3 \). By assumption, \( \mathbb{M}(F) \subseteq \{b, h\} \). Hence, \( \tilde{m} \in \{b, h\}^3 \). Secondly, if \( \tilde{m} \in N_{303}(F) \cup N_{303}(F) \), then \( \Sigma_A(f_{\xi}, \tilde{m}) = 3 \), which requires \( \tilde{m} \in \{a, h\}^3 \). (This follows from the definition of \( \Sigma_A(f_{\xi}, \tilde{m}) \).) Therefore,

\[
\tilde{m} \in \{a, h\}^3 \cap \{b, h\}^3 = \{h\}^3
\]

**Part (2).** If \( \tilde{m} \in \{b\}^2 \times \{b\} \subseteq \mathcal{N}(F) \), then:

\[
\tilde{\Sigma}_A(f_{\xi}, \tilde{m}) = 2, \quad \tilde{\Sigma}_A(f_{\eta}, \tilde{m}) = 2.1(y \geq z), \quad \tilde{\Sigma}_B(f_{\eta}, \tilde{m}) = 1 + 2.1(y < z)
\]

Hence, if \( y \geq z \), then \( I = 2, J = 2, K = 1 \); otherwise, \( I = 2, J = 0, K = 3 \). This suggests that \( \tilde{m} \in \mathcal{N}_{203}(F) \cup \mathcal{N}_{221}(F) \). However, it’s been shown that there’s no RSPNE in which \( I = 2, J = 0, K = 3 \). That is, \( \mathcal{N}_{203}(F) = \emptyset \). Therefore, \( \tilde{m} \in \mathcal{N}_{221}(F) \). The corresponding set of membership vectors can be found as follows. Firstly, all the elements in \( \tilde{m} \) must be "relevant". By assumption, \( \mathbb{M}(F) \subseteq \{b, h\} \). Hence, \( \tilde{m} \in \{b, h\}^3 \). Secondly, if \( \tilde{m} \in \mathcal{N}_{221}(F) \), then \( \Sigma_A(f_{\xi}, \tilde{m}) = 2 \), which requires \( \tilde{m} \in \{a, b\}^2 \times \{a, b\} \). (This follows from the definition of \( \Sigma_A(f_{\xi}, \tilde{m}) \).) Therefore,

\[
\tilde{m} \in \{a, b\}^2 \times \{a, b\} \cap \{b, h\}^3 = \{h\}^2 \times \{b\}
\]

**Part (3) and Part (4) are very similar.**
Appendix C

Properties of the Inner Oval

The first boundary curve is

$$(10\tau\sigma - \omega^2 + x^2)^2 + (10\tau\sigma - z^2 + y^2)^2 = 200\tau^2\sigma^2,$$

where

$$\omega = \text{constant} = 2\tau - \rho,$$
$$z = \text{constant} \in [-\omega, \omega]$$

and

$$(x, y) \in [-\rho, \omega]^2$$

This section investigates the properties of this boundary curve.

C.1 An Upper Bound for $f^2$

These boundary curves depend on $x^2, y^2$ and $\omega^2, z^2$. Hence, it’s useful to find upper-bounds for these quantities.

Lemma C.1 (Upper Bound for $\omega^2$.) It can be shown that $\omega^2 < 2\tau\sigma$.

Proof. The original parameter assumptions require, $\omega = 2\tau - \rho < 2\tau$ and $2\tau > \rho > 0$. Also, by assumption, the the level of product differentiation, $\sigma$, exceeds the maximum possible transaction cost: $\sigma > 2\tau$. This implies that $\omega^2 < 4\tau^2 < 2\tau\sigma$. ■

Lemma C.2 (Upper Bounds for $x^2, y^2$ and $z^2$.) It can be shown that $x^2 < 2\tau\sigma$. The same holds for $y^2$ and $z^2$.

Proof. Since $x \in [-\omega, \rho]$, it follows that $x^2 \leq \max\{\rho^2, \omega^2\}$, where $\omega = 2\tau - \rho$ and $2\tau > \rho > 0$. It follows that

$$x^2 \leq \max\{\rho^2, \omega^2\} = \begin{cases} \rho^2 & \text{if } \rho \geq \tau \\ \omega^2 & \text{if } \rho < \tau \end{cases}$$

which implies that $x^2 \leq 4\tau^2$. Furthermore, the level of product differentiation, $\sigma$, exceeds the maximum possible transaction cost: $\sigma > 2\tau$. This implies that $x^2 < 4\tau^2 < 2\tau\sigma$. The inequality also holds for $y^2$ and $z^2$. ■
The Slope: \( \frac{dy}{dx} \).

Total differentiation with respect to \( x \) gives:

\[
x(10\tau\sigma - \omega^2 + x^2) + y(10\tau\sigma - z^2 + y^2) \frac{dy}{dx} = 0
\]

The gradient of the curve is

\[
\frac{dy}{dx} = -\frac{x}{y} \left( \frac{10\tau\sigma - \omega^2 + x^2}{10\tau\sigma - z^2 + y^2} \right)
\]

It has been shown that

\[
\omega^2 < 2\tau\sigma \\
z^2 < 2\tau\sigma
\]

and

\[
x^2 < 2\tau\sigma \\
y^2 < 2\tau\sigma
\]

Hence, it follows that:

\[
10\tau\sigma - z^2 + y^2 \geq 8\tau\sigma
\]

and

\[
12\tau\sigma - \omega^2 + x^2 \geq 8\tau\sigma
\]

Therefore, the second factor in the expression for \( \frac{dy}{dx} \) is positive, which implies that the sign is determined by the ratio of \( x \) and \( y \). In particular, it can be seen that the sign of the slope depends on whether \( x \) and \( y \) have the same sign or opposite signs.

- The slope is negative if \( x \) and \( y \) have the same sign (first and third quadrant).
- The slope is positive if \( x \) and \( y \) have opposite signs (second and last quadrants).
- The curve is horizontal at \( x = 0 \) (and \( y \neq 0 \)).
- The curve is vertical at \( y = 0 \) (and \( x \neq 0 \)).

The sign of \( \frac{dy}{dx} \) can be summarized as follows:

<table>
<thead>
<tr>
<th>( y &lt; 0 )</th>
<th>( x &lt; 0 )</th>
<th>( x &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>

It can be seen from this result that the equation corresponds to a closed curve.
C.3 Convexity and Concavity

Differentiating a second time gives

\[ 0 = (10.\tau \sigma - w^2 + x^2) + 2x^2 \]
\[ + (10.\tau \sigma - z^2 + y^2) \cdot \left( \frac{dy}{dx} \right)^2 + 2y^2 \cdot \left( \frac{dy}{dx} \right)^2 \]
\[ + y \cdot (10.\tau \sigma - z^2 + y^2) \cdot \frac{d^2y}{dx^2} \]

which can be re-expressed as

\[ 0 = 10.\tau \sigma - w^2 + 3x^2 \]
\[ + (10.\tau \sigma - z^2 + 3y^2) \cdot \left( \frac{dy}{dx} \right)^2 \]
\[ + y \cdot (10.\tau \sigma - z^2 + y^2) \cdot \frac{d^2y}{dx^2} \]

It has already been shown that

\[ 10.\tau \sigma - z^2 + 3y^2 \geq 8.\tau \sigma \]

and

\[ 12.\tau \sigma - \omega^2 + 3x^2 \geq 8.\tau \sigma \]

This implies that the first and second term are positive. Therefore, the sign of \( \frac{d^2y}{dx^2} \) is determined by the sign of \( y \).

- If \( y > 0 \), then the curve is concave.
- If \( y < 0 \), then the curve is convex.

The sign of \( \frac{d^2y}{dx^2} \) can be summarized as follows:

<table>
<thead>
<tr>
<th></th>
<th>( x &lt; 0 )</th>
<th>( x &gt; 0 )</th>
<th>( y &lt; 0 )</th>
<th>( y &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

C.4 Polar Coordinates

The first boundary curve can be re-expressed as

\[ 20\tau \sigma \cdot (x^2 + y^2 - \omega^2 - z^2) + (\omega^2 - x^2)^2 + (z^2 - y^2)^2 = 0 \]

When expressed in polar coordinates the curve becomes

\[ 20\tau \sigma \cdot (a^2 - b^2) + \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right)^2 + \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right)^2 = 0 \]
where

\[ a = \sqrt{x^2 + y^2} \]
\[ \tan \hat{A} = \frac{y}{x} \]

and

\[ b = \sqrt{w^2 + z^2} \]
\[ \tan \hat{B} = \frac{z}{w} \]

### C.5 Upper Bound on the Radius, \( a \)

In polar coordinates the curve becomes

\[
20\tau \sigma \cdot (b^2 - a^2) = \left(a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B}\right)^2 + \left(a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B}\right)^2
\]

Hence, \( b^2 - a^2 \geq 0 \), which implies that

\[ a \leq b = \sqrt{w^2 + z^2} \]

Hence, the curve has a maximum possible radius. That is, the curve is enclosed by a circle of radius \( b \).

### C.6 Lower Bound on the Radius, \( a \)

If \( a \to 0 \), then the boundary equation requires

\[
20\tau \sigma = b^2 \cdot \left(\cos^4 \hat{B} + \sin^4 \hat{B}\right)
\]

However, it has been shown that

\[ b^2 = \omega^2 + z^2 < 4\tau \sigma \]

This implies that

\[ b^2 \cdot \left(\cos^4 \hat{B} + \sin^4 \hat{B}\right) \leq 2b^2 < 8\tau \sigma \]

Hence, the condition can’t be satisfied. Therefore, the curve never passes through the origin:

\[ a > 0 \]
C.7 Stationary Points

On the curve, the radius, $a$, is an implicit function of the angle, $\hat{A}$. Total differentiation with respect to $\hat{A}$ gives

$$
-40\tau \sigma \cdot a \cdot \frac{da}{d\hat{A}} = 2 \left( 2a \cos^2 \hat{A} \cdot \frac{da}{d\hat{A}} - 2a^2 \sin \hat{A} \cos \hat{A} \right) \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) \\
+ 2 \left( 2a \sin^2 \hat{A} \cdot \frac{da}{d\hat{A}} + 2a^2 \sin \hat{A} \cos \hat{A} \right) \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right)
$$

First-Order Conditions. At the stationary points this becomes

$$
0 = 2 \sin \hat{A} \cos \hat{A} a^2 \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right) \\
- 2 \sin \hat{A} \cos \hat{A} a^2 \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right)
$$

Rearrangement gives

$$
2 \sin \hat{A} \cos \hat{A} a^2 \left[ b^2 \left( \cos^2 \hat{B} - \sin^2 \hat{B} \right) - a^2 \left( \cos^2 \hat{A} - \sin^2 \hat{A} \right) \right] = 0
$$

Using the double-angle formulae we get

$$
a^2 \sin 2\hat{A} \left[ b^2 \cos^2 \hat{B} - a^2 \cos^2 \hat{A} + a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right] = 0
$$

Finally, it’s convenient to re-express the last factor in terms of cartesian coordinates:

$$
a^2 \sin 2\hat{A} \left[ w^2 - x^2 + y^2 - z^2 \right] = 0
$$

Hence, there are two possibilities (because $a > 0$):

$$
y^2 + w^2 - x^2 - z^2 = 0 \\
\sin 2\hat{A} = 0
$$

At least one of these must be satisfied at a stationary point.

Case 1. Consider the first possibility:

$$
w^2 - x^2 = z^2 - y^2
$$

This implies that the boundary equation becomes

$$
\left( 10\tau \sigma + x^2 - w^2 \right)^2 = 100\tau^2 \sigma^2
$$

Hence, $x^2 = w^2$, which gives $x = \pm w$ (and $y = \pm z$). That is, there are stationary points at the following coordinates:

$$(x, y) = (w, z)$$
$$(x, y) = (-w, z)$$
$$(x, y) = (w, -z)$$
$$(x, y) = (-w, -z)$$
Case 2. Consider the second possibility:

\[ \sin 2\hat{A} = 0 \]

It has been shown that \( \hat{A} \) takes four possible values: \( 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi \). Furthermore, they divide into two sets: \( \{0, \pi\} \) and \( \{\frac{1}{2}\pi, \frac{3}{2}\pi\} \). These are now analysed in turn:

If \( \hat{A} = 0 \) or \( \hat{A} = \pi \), then

\[ 20\tau\sigma. (b^2 - a^2) = \left( a^2 - b^2 \cos^2 \hat{B} \right)^2 + \left( b^2 \sin^2 \hat{B} \right)^2 \]

which can be re-expressed as

\[ 20\tau\sigma. (\omega^2 + z^2 - a^2) = (a^2 - \omega^2)^2 + z^4 \]

Further re-arrangement gives

\[ (a^2 - \omega^2)^2 + 20\tau\sigma. (a^2 - \omega^2) - (20\tau\sigma - z^2).z^2 = 0 \]

Hence,

\[ a^2 - \omega^2 = -10\tau\sigma \pm \sqrt{(10\tau\sigma)^2 + (20\tau\sigma - z^2).z^2} \]

Since \( \omega^2 < 4\tau^2 < 2\tau\sigma \), it follows that \( a^2 - \omega^2 > -2\tau\sigma \). However, the negative root, implies that \( a^2 - \omega^2 < -10\tau\sigma \). Therefore, only the positive root is feasible:

\[ a^2 = \omega^2 + \sqrt{(10\tau\sigma)^2 + (20\tau\sigma - z^2).z^2 - 10\tau\sigma} \]

Note that this implies \( a^2 > \omega^2 \).

If \( \hat{A} = \frac{1}{2}\pi \) or \( \hat{A} = \frac{3}{2}\pi \), then

\[ 20\tau\sigma. (b^2 - a^2) = \left( b^2 \cos^2 \hat{B} \right)^2 + \left( a^2 - b^2 \sin^2 \hat{B} \right)^2 \]

which can be re-expressed as

\[ 20\tau\sigma. (\omega^2 + z^2 - a^2) = \omega^4 + (a^2 - z^2)^2 \]

Further re-arrangement gives

\[ (a^2 - z^2)^2 + 20\tau\sigma. (a^2 - z^2) - (20\tau\sigma - \omega^2).\omega^2 = 0 \]

Using similar arguments to those above, we get

\[ a^2 = z^2 + \sqrt{(10\tau\sigma)^2 + (20\tau\sigma - \omega^2).\omega^2 - 10\tau\sigma} \]
C.8 Maximum and Minimum Radius

In order to determine the nature of these stationary points it is necessary to find the sign of \( \frac{d^2a}{dA^2} \). Differentiating with respect to \( A \) a second time gives

\[
-10\tau \sigma a \frac{d^2a}{dA^2} = \left( \cos^2 \hat{A} \left( \frac{da}{dA} \right)^2 - 2a \sin 2\hat{A} \frac{da}{dA} + a \cos^2 \hat{A} \frac{d^2a}{dA^2} - a^2 \cos 2\hat{A} \right) \times \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) + \frac{1}{2} a^2 \left( 2 \cos^2 \hat{A} \frac{da}{dA} - a \sin 2\hat{A} \right)^2 + \left( \sin^2 \hat{A} \left( \frac{da}{dA} \right)^2 + 2a \sin 2\hat{A} \frac{da}{dA} + a \sin^2 \hat{A} \frac{d^2a}{dA^2} + a^2 \cos 2\hat{A} \right) \times \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right) + \frac{1}{2} a^2 \left( 2 \sin^2 \hat{A} \frac{da}{dA} + a \sin 2\hat{A} \right)^2
\]

At the stationary points this becomes

\[
-10\tau \sigma \frac{d^2a}{dA^2} = \left( \sin^2 \hat{A} \frac{d^2a}{dA^2} - a \cos(2\hat{A}) \right) \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) + \left( \sin^2 \hat{A} \frac{d^2a}{dA^2} + a \cos(2\hat{A}) \right) \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right) + a^3 \sin^2(2\hat{A})
\]

**Four Local Maxima.** It has been shown that there a stationary points at \( x = \pm w \) and \( y = \pm z \). At these coordinates:

\[
a = b
\]

and

\[
\cos^2 \hat{A} = \cos^2 \hat{B} \\
\sin^2 \hat{A} = \sin^2 \hat{B}
\]

Substituting these into the implicit expression for \( \frac{d^2a}{dA^2} \) gives

\[
-10\tau \sigma \frac{d^2a}{dA^2} = a^3 \sin^2(2\hat{B}),
\]

which implies that

\[
\frac{d^2a}{dA^2} < 0
\]

Therefore, local maxima must occur at

\[
(x, y) = (w, z) \\
(x, y) = (-w, z) \\
(x, y) = (w, -z) \\
(x, y) = (-w, -z)
\]
Four Local Minima. It has been shown that there are stationary points at:

\[ \hat{A} = 0 \]
\[ \hat{A} = \frac{1}{2}\pi \]
\[ \hat{A} = \pi \]
\[ \hat{A} = \frac{3}{2}\pi \]

At these points \( \sin 2\hat{A} = 0 \), which implies the second order expression becomes

\[
-10\tau\sigma \frac{d^2a}{d\hat{A}^2} = \left( \sin^2 \hat{A} \frac{d^2a}{d\hat{A}^2} - a \cos(2\hat{A}) \right) \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) + \left( \sin^2 \hat{A} \frac{d^2a}{d\hat{A}^2} + a \cos(2\hat{A}) \right) \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right)
\]

This can be re-expressed as

\[
-10\tau\sigma \frac{d^2a}{d\hat{A}^2} = \left( a^2 \cos^2 \hat{A} + a^2 \sin^2 \hat{A} - b^2 \cos^2 \hat{B} - b^2 \sin^2 \hat{B} \right) \sin^2 \hat{A} \frac{d^2a}{d\hat{A}^2} + a \cos(2\hat{A}). \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} + b^2 \cos^2 \hat{B} - a^2 \cos^2 \hat{A} \right)
\]

Using the trigonometric identities this becomes

\[
-10\tau\sigma \frac{d^2a}{d\hat{A}^2} = \left( a^2 - b^2 \right) \sin^2 \hat{A} \frac{d^2a}{d\hat{A}^2} + a \cos(2\hat{A}). \left( b^2 \cos(2\hat{B}) - a^2 \cos(2\hat{A}) \right)
\]

It has been shown that \( \hat{A} \) takes four possible values: 0, \( \frac{1}{2}\pi \), \( \pi \), \( \frac{3}{2}\pi \). Furthermore, they divide into two sets: \( \{0, \pi\} \) and \( \{\frac{1}{2}\pi, \frac{3}{2}\pi\} \). These are now analysed in turn:

If \( \hat{A} = 0 \) or \( \hat{A} = \pi \), then the implicit expression for \( \frac{d^2a}{d\hat{A}^2} \) becomes

\[
-10\tau\sigma \frac{d^2a}{d\hat{A}^2} = a. \left( b^2 \cos(2\hat{B}) - a^2 \right)
\]

which can be re-expressed as

\[
-10\tau\sigma \frac{d^2a}{d\hat{A}^2} = a. \left( \omega^2 - z^2 - a^2 \right)
\]

(This uses \( b^2 \cos(2\hat{B}) = b^2 \cos^2 \hat{B} - b^2 \sin^2 \hat{B} = \omega^2 - z^2 \).) Hence,

\[
10\tau\sigma \frac{d^2a}{d\hat{A}^2} = a. \left( z^2 + \omega^2 - a^2 \right)
\]

Furthermore, it has been shown that if \( \hat{A} = 0 \) or \( \hat{A} = \pi \), then \( a^2 > \omega^2 \); which implies that

\[
\frac{d^2a}{d\hat{A}^2} > 0
\]
Therefore, local minima occur at

\[ \dot{A} = 0 \]
\[ A = \pi \]

If \( \dot{A} = \frac{1}{2}\pi \) or \( \dot{A} = \frac{3}{2}\pi \), then the implicit expression for \( \frac{d^2a}{dA^2} \) becomes

\[-10\sigma \cdot \frac{d^2a}{dA^2} = (a^2 - b^2) \cdot \frac{d^2a}{dA^2} - a \cdot (b^2 \cos(2\hat{B}) + a^2) \]

which can be re-expressed as

\[-10\sigma \cdot \frac{d^2a}{dA^2} = (a^2 - b^2) \cdot \frac{d^2a}{dA^2} - a \cdot (\omega^2 - z^2 + a^2) \]

(This uses \( b^2 \cos(2\hat{B}) = b^2 \cos^2 \hat{B} - b^2 \sin^2 \hat{B} = \omega^2 - z^2 \).) Hence,

\[(10\sigma + a^2 - b^2) \cdot \frac{d^2a}{dA^2} = a \cdot (\omega^2 - z^2 + a^2) \]

Firstly, by assumption \( z^2 < \omega^2 \). Secondly, it has been shown that \( b^2 < 4\pi\sigma \). It follows that

\[ \frac{d^2a}{dA^2} > 0 \]

Therefore, there are local minima at

\[ \dot{A} = \frac{1}{2}\pi \]
\[ A = \frac{3}{2}\pi \]

C.9 Summary (Inner Oval)

The properties of the inner boundary curve can be summarized as follows:

- Oval shaped curve centred on the origin with mirror symmetry in both axes.
- The radius of the curve is always positive and is enclosed by a circle of radius \( \sqrt{\omega^2 + z^2} \).
- The radius decreases to a minimum (positive) value where the curve crosses an axis.
- The radius increases to a maximum of \( \sqrt{\omega^2 + z^2} \) at points where \( x = \pm \omega, y = \pm z \).
Appendix D

Properties of the Outer Oval

The second boundary curve is

\[(10\tau\sigma + \omega^2 - x^2)^2 + (10\tau\sigma + z^2 - y^2)^2 = 200\tau^2\sigma^2\]

where

\[
\omega = \text{constant} = 2\tau - \rho \\
z = \text{constant} \in [-\omega, \omega]
\]

and

\[(x, y) \in [-\rho, \omega]^2\]

This section investigates the properties of the outer boundary curve.

D.1 An Upper Bound for \(f^2\)

These boundary curves depend on \(x^2, y^2\) and \(\omega^2, z^2\). Hence, it’s useful to find upper-bounds for these quantities.

Lemma D.1 (Upper Bound for \(\omega^2\).) It can be shown that \(\omega^2 < 2\tau\sigma\).

Proof. The original parameter assumptions require, \(\omega = 2\tau - \rho < 2\tau\) and \(2\tau > \rho > 0\). Also, by assumption, the the level of product differentiation, \(\sigma\), exceeds the maximum possible transaction cost: \(\sigma > 2\tau\). This implies that \(\omega^2 < 4\tau^2 < 2\tau\sigma\).  

Lemma D.2 (Upper Bounds for \(x^2, y^2\) and \(z^2\).) It can be shown that \(x^2 < 2\tau\sigma\). The same holds for \(y^2\) and \(z^2\).

Proof. Since \(x \in [-\omega, \rho]\), it follows that \(x^2 \leq \max\{\rho^2, \omega^2\}\), where \(\omega = 2\tau - \rho\) and \(2\tau > \rho > 0\). It follows that

\[
x^2 \leq \max\{\rho^2, \omega^2\} = \begin{cases} 
\rho^2 & \text{if } \rho \geq \tau \\
\omega^2 & \text{if } \rho < \tau 
\end{cases}
\]

which implies that \(x^2 \leq 4\tau^2\). Furthermore, the level of product differentiation, \(\sigma\), exceeds the maximum possible transaction cost: \(\sigma > 2\tau\). This implies that \(x^2 < 4\tau^2 < 2\tau\sigma\). The inequality also holds for \(y^2\) and \(z^2\).  

228
D.2  The Slope: $\frac{dy}{dx}$.

Total differentiation with respect to $x$ gives:

$$x \cdot (10\tau \sigma + \omega^2 - x^2) + y \cdot (10\tau \sigma + z^2 - y^2) \frac{dy}{dx} = 0$$

The gradient of the curve is

$$\frac{dy}{dx} = -\frac{x}{y} \left(10\tau \sigma + \omega^2 - x^2\right) \left(10\tau \sigma + z^2 - y^2\right)$$

It can be shown that $\omega^2 < 2\tau \sigma$, $x^2 < 2\tau \sigma$ and $y^2 < 2\tau \sigma$. Hence, the quantity in brackets is positive:

$$10\tau \sigma + \omega^2 - x^2 > 8\tau \sigma$$
$$10\tau \sigma + z^2 - y^2 > 8\tau \sigma$$

This implies that the sign of the slope depends on whether $x$ and $y$ have the same sign or opposite signs.

- The slope is negative if $x$ and $y$ have the same sign (first and third quadrant).
- The slope is positive if $x$ and $y$ have opposite signs (second and last quadrants).
- The curve is horizontal at $x = 0$ (and $y \neq 0$).
- The curve is vertical at $y = 0$ (and $x \neq 0$).

The sign of $\frac{dy}{dx}$ can be summarized as follows:

<table>
<thead>
<tr>
<th></th>
<th>$x &lt; 0$</th>
<th>$x &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y &lt; 0$</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$y &gt; 0$</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

D.3  Convexity and Concavity

Differentiating a second time gives

$$0 = (10\tau \sigma + \omega^2 - x^2) - 2x^2$$
$$+ (10\tau \sigma + z^2 - y^2) \cdot \left(\frac{dy}{dx}\right)^2 - 2y^2 \cdot \left(\frac{dy}{dx}\right)^2$$
$$+ y \cdot (10\tau \sigma + z^2 - y^2) \cdot \frac{d^2y}{dx^2}$$
which can be re-expressed as

\[ 0 = 10\tau\sigma + w^2 - 3x^2 + (10\tau\sigma + z^2 - 3y^2) \cdot \left( \frac{dy}{dx} \right)^2 + y(10\tau\sigma + z^2 - y^2) \cdot \frac{d^2y}{dx^2} \]

It can be shown that \( \omega^2 < 2\tau\sigma, \ x^2 < 2\tau\sigma \) and \( y^2 < 2\tau\sigma \). Hence, it can be shown that

\[ 10\tau\sigma + \omega^2 - 3x^2 \geq 4\tau\sigma \]
\[ 10\tau\sigma + z^2 - 3y^2 \geq 4\tau\sigma \]

and

\[ 10\tau\sigma + z^2 - y^2 \geq 8\tau\sigma \]

This implies that the first and second term are positive. Therefore, the sign of \( \frac{d^2y}{dx^2} \) depends on the sign of \( y \):

- If \( y > 0 \), then the curve is concave.
- If \( y < 0 \), then the curve is convex.

The sign of \( \frac{d^2y}{dx^2} \) can be summarized as follows:

<table>
<thead>
<tr>
<th></th>
<th>( y &lt; 0 )</th>
<th>( x &lt; 0 )</th>
<th>( x &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y &gt; 0 )</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

### D.4 Polar Coordinates

The second boundary curve can be re-expressed as

\[ 20\tau\sigma.(\omega^2 + z^2 - x^2 - y^2) + (\omega^2 - x^2)^2 + (z^2 - y^2)^2 = 0 \]

When expressed in polar coordinates the curve becomes

\[ 20\tau\sigma.(b^2 - a^2) + \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right)^2 + \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right)^2 = 0 \]

where

\[ a = \sqrt{x^2 + y^2} \]
\[ \tan \hat{A} = \frac{y}{x} \]

and

\[ b = \sqrt{w^2 + z^2} \]
\[ \tan \hat{B} = \frac{z}{w} \]
D.5 Lower Bound on the Radius, \( a \)

In polar coordinates the curve becomes

\[
20\tau \sigma. (a^2 - b^2) = \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right)^2 + \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right)^2
\]

Hence, \( a^2 - b^2 \geq 0 \), which implies that

\[
a \geq b = \sqrt{w^2 + z^2}
\]

Hence, the curve has a minimum possible radius. That is, the curve encloses a circle of radius \( b \).

D.6 Stationary Points

On the curve, the radius, \( a \), is an implicit function of the angle, \( \hat{A} \). Total differentiation with respect to \( \hat{A} \) gives

\[
40\tau \sigma. a \frac{da}{d\hat{A}} = 2 \left( 2a \cos^2 \hat{A} \frac{da}{d\hat{A}} - 2a^2 \sin \hat{A} \cos \hat{A} \right) \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) + 2 \left( 2a \sin^2 \hat{A} \frac{da}{d\hat{A}} + 2a^2 \sin \hat{A} \cos \hat{A} \right) \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right)
\]

First-Order Conditions. At the stationary points this becomes

\[
0 = 2 \sin \hat{A} \cos \hat{A} a^2 \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right) - 2 \sin \hat{A} \cos \hat{A} a^2 \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right)
\]

Note that this expression is identical to expression that characterizes the stationary points of the first curve. Therefore, stationary points satisfy

\[
a^2 \sin 2\hat{A}. [w^2 - x^2 + y^2 - z^2] = 0
\]

which implies there are two possibilities (because \( a > 0 \)):

\[
y^2 + w^2 - x^2 - z^2 = 0
\]

\[
\sin 2\hat{A} = 0
\]

At least one of these must be satisfied at a stationary point.

Case 1. Consider the first possibility:

\[
w^2 - x^2 = z^2 - y^2
\]
It follows that there are stationary points at the following coordinates:

\[
\begin{align*}
(x, y) &= (w, z) \\
(x, y) &= (-w, z) \\
(x, y) &= (w, -z) \\
(x, y) &= (-w, -z)
\end{align*}
\]

**Case 2.** Consider the second possibility:

\[\sin 2\hat{A} = 0\]

It has been shown that \(\hat{A}\) takes four possible values: 0, \(\frac{1}{2}\pi\), \(\pi\), and \(\frac{3}{2}\pi\). Furthermore, they divide into two sets: \(\{0, \pi\}\) and \(\{\frac{1}{2}\pi, \frac{3}{2}\pi\}\). These are now analysed in turn:

If \(\hat{A} = 0\) or \(\hat{A} = \pi\), then

\[
20\tau\sigma(\omega^2 - b^2) = \left(b^2 - b^2 \cos^2 \hat{B}\right)^2 + \left(b^2 \sin^2 \hat{B}\right)^2
\]

which can be re-expressed as

\[
20\tau\sigma(\omega^2 - \omega^2 - z^2) = (\omega^2 - \omega^2)^2 + z^4
\]

Further re-arrangement gives

\[
(a^2 - \omega^2)^2 - 20\tau\sigma(a^2 - \omega^2) + (20\tau\sigma + z^2)z^2 = 0
\]

Hence,

\[
a^2 - \omega^2 = 10\tau\sigma \pm \sqrt{(10\tau\sigma)^2 - (20\tau\sigma + z^2).z^2}
\]

The positive root gives \(a^2 > \omega^2 + 10\tau\sigma\). Since \(a^2 < 4\tau\sigma\), it follows that only the negative root is feasible:

\[
a^2 = \omega^2 + 10\tau\sigma - \sqrt{(10\tau\sigma)^2 - (20\tau\sigma + z^2).z^2}
\]

Note that this implies \(a^2 > \omega^2\).

If \(\hat{A} = \frac{1}{2}\pi\) or \(\hat{A} = \frac{3}{2}\pi\), then

\[
20\tau\sigma(\omega^2 - b^2) = \left(b^2 \cos^2 \hat{B}\right)^2 + \left(b^2 - b^2 \sin^2 \hat{B}\right)^2
\]

which can be re-expressed as

\[
20\tau\sigma(\omega^2 - \omega^2 - z^2) = \omega^4 + (\omega^2 - z^2)^2
\]

Further re-arrangement gives

\[
(a^2 - \omega^2)^2 - 20\tau\sigma(a^2 - \omega^2) + (20\tau\sigma + \omega^2)\omega^2 = 0
\]

Using similar arguments to those above, we get

\[
a^2 = z^2 + 10\tau\sigma \pm \sqrt{(10\tau\sigma)^2 - (20\tau\sigma + \omega^2)\omega^2}
\]

\[\text{232}\]
D.7 Maximum and Minimum Radius

In order to determine the nature of these stationary points it is necessary to find the sign of \( \frac{d^2a}{dA^2} \). Differentiating with respect to \( A \) a second time gives

\[
10\tau \sigma . a \frac{d^2a}{dA^2} = \left( \cos^2 \hat{A} \left( \frac{da}{d\hat{A}} \right)^2 - 2a \sin 2\hat{A} \frac{da}{d\hat{A}} + a \cos^2 \hat{A} \frac{d^2a}{d\hat{A}^2} - a^2 \cos 2\hat{A} \right) \times \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) + \frac{1}{2} a^2 \left( 2 \cos^2 \hat{A} \frac{da}{d\hat{A}} - a \sin 2\hat{A} \right)^2 + \left( \sin^2 \hat{A} \left( \frac{da}{d\hat{A}} \right)^2 + 2a \sin 2\hat{A} \frac{da}{d\hat{A}} + a \sin^2 \hat{A} \frac{d^2a}{d\hat{A}^2} + a^2 \cos 2\hat{A} \right) \times \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right) + \frac{1}{2} a^2 \left( 2 \sin^2 \hat{A} \frac{da}{d\hat{A}} + a \sin 2\hat{A} \right)^2
\]

At the stationary points this becomes

\[
10\tau \sigma . a \frac{d^2a}{dA^2} = \left( \sin^2 \hat{A} \frac{d^2a}{d\hat{A}^2} - a \cos(2\hat{A}) \right) \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) + \left( \sin^2 \hat{A} \frac{d^2a}{d\hat{A}^2} + a \cos(2\hat{A}) \right) \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right) + a^3 \sin^2(2\hat{A})
\]

Four Local Minima. It has been shown that there are stationary points at \( x = \pm w \) and \( y = \pm z \). At these coordinates:

\[
a = b
\]

and

\[
\cos^2 \hat{A} = \cos^2 \hat{B} \\
\sin^2 \hat{A} = \sin^2 \hat{B}
\]

Substituting these into the implicit expression for \( \frac{d^2a}{dA^2} \) gives

\[
10\tau \sigma . a \frac{d^2a}{dA^2} = a^3 \sin^2(2\hat{B}),
\]

which implies that

\[
\frac{d^2a}{dA^2} > 0
\]

Therefore, local minima must occur at

\[
(x, y) = (w, z) \\
(x, y) = (-w, z) \\
(x, y) = (w, -z) \\
(x, y) = (-w, -z)
\]
Four Local Maxima. It has been shown that there are stationary points at:

\[
\begin{align*}
\hat{A} &= 0 \\
\hat{A} &= \frac{1}{2} \pi \\
\hat{A} &= \pi \\
\hat{A} &= \frac{3}{2} \pi
\end{align*}
\]

At these points \( \sin 2\alpha = 0 \), which implies the second order expression becomes

\[
10\tau\sigma \frac{d^2 a}{d\hat{A}^2} = \left( \sin^2 \hat{A} \frac{d^2 a}{d\hat{A}^2} - a \cos(2\hat{A}) \right) \left( a^2 \cos^2 \hat{A} - b^2 \cos^2 \hat{B} \right) + \left( \sin^2 \hat{A} \frac{d^2 a}{d\hat{A}^2} + a \cos(2\hat{A}) \right) \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} \right)
\]

This can be re-expressed as

\[
10\tau\sigma \frac{d^2 a}{d\hat{A}^2} = \left( a^2 \cos^2 \hat{A} + a^2 \sin^2 \hat{A} - b^2 \cos^2 \hat{B} - b^2 \sin^2 \hat{B} \right) \cdot \sin^2 \hat{A} \frac{d^2 a}{d\hat{A}^2} + a \cos(2\hat{A}) \left( a^2 \sin^2 \hat{A} - b^2 \sin^2 \hat{B} + b^2 \cos^2 \hat{B} - a^2 \cos^2 \hat{A} \right)
\]

Using the trigonometric identities this becomes

\[
10\tau\sigma \frac{d^2 a}{d\hat{A}^2} = \left( a^2 - b^3 \right) \sin^2 \hat{A} \frac{d^2 a}{d\hat{A}^2} + a \cos(2\hat{A}) \left( b^2 \cos(2\beta) - a^2 \cos(2\hat{A}) \right)
\]

It has been shown that \( \hat{A} \) takes four possible values: \( 0, \frac{1}{2} \pi, \pi, \frac{3}{2} \pi \). Furthermore, they divide into two sets: \( \{0, \pi\} \) and \( \{\frac{1}{2} \pi, \frac{3}{2} \pi\} \). These are now analysed in turn:

If \( \hat{A} = 0 \) or \( \hat{A} = \pi \), then the implicit expression for \( \frac{d^2 a}{d\hat{A}^2} \) becomes

\[
10\tau\sigma \frac{d^2 a}{d\hat{A}^2} = a \left( b^2 \cos(2\hat{B}) - a^2 \right)
\]

which can be re-expressed as

\[
10\tau\sigma \frac{d^2 a}{d\hat{A}^2} = a \left( \omega^2 - z^2 - a^2 \right)
\]

(This uses \( b^2 \cos(2\hat{B}) = b^2 \cos^2 \hat{B} - b^2 \sin^2 \hat{B} = \omega^2 - z^2 \).) Hence,

\[
10\tau\sigma \frac{d^2 a}{d\hat{A}^2} = -a \left( z^2 + a^2 - \omega^2 \right)
\]

Furthermore, it has been shown that \( a^2 > b^2 = \omega^2 + z^2 \). This implies that \( a^2 > \omega^2 \) and so

\[
\frac{d^2 a}{d\hat{A}^2} < 0
\]
Therefore, local maxima occur at
\[
\begin{align*}
\hat{A} &= 0 \\
\bar{A} &= \pi
\end{align*}
\]

If \( \hat{A} = \frac{1}{2}\pi \) or \( \bar{A} = \frac{3}{2}\pi \), then the implicit expression for \( \frac{d^2 a}{dA^2} \) becomes
\[
10\pi \sigma \cdot \frac{d^2 a}{dA^2} = (a^2 - b^2) \cdot \frac{d^2 a}{dA^2} - a \cdot \left(b^2 \cos(2\hat{B}) + a^2\right)
\]
which can be re-expressed as
\[
10\pi \sigma \cdot \frac{d^2 a}{dA^2} = (a^2 - b^2) \cdot \frac{d^2 a}{dA^2} - a \cdot (\omega^2 - z^2 + a^2)
\]
(This uses \( b^2 \cos(2\hat{B}) = b^2 \cos^2 \hat{B} - b^2 \sin^2 \hat{B} = \omega^2 - z^2 \).) Hence,
\[
(10\pi \sigma + b^2 - a^2) \cdot \frac{d^2 a}{dA^2} = -a \cdot (\omega^2 + a^2 - z^2)
\]
Firstly, by assumption, \( z^2 < \omega^2 \). Secondly, it has been shown that \( b^2 < 4\pi \sigma \).
It follows that
\[
\frac{d^2 a}{dA^2} < 0
\]
Therefore, there are local maxima at
\[
\begin{align*}
\hat{A} &= \frac{1}{2}\pi \\
\bar{A} &= \frac{3}{2}\pi
\end{align*}
\]

### D.8 Summary (Outer Oval)

The properties of the second boundary curve can be summarized as follows:

- Oval shaped curve centred on the origin with mirror symmetry in both axes.
- The radius of the curve is always positive and encloses a circle of radius \( \sqrt{\omega^2 + z^2} \).
- The radius increases to a maximum value at points where the curve crosses an axis.
- The radius decreases to a maximum of \( \sqrt{\omega^2 + z^2} \) at points where \( x = \pm \omega, y = \pm z \).
Appendix E

Stability Conditions in Subgame \([1]\)

In Subgames \([1]\) there are five possible configurations: \((330)\), \((221)\), \((112)\), \((003)\), \((303)\). This chapter finds necessary and sufficient conditions for these configurations to be stable when \(W(z) \geq 0\).

E.1 Necessary and Sufficient Conditions for Stability

A configuration is stable iff none of the sellers can profit through unilateral deviation. Since stability is a prerequisite for a configuration to occur in equilibrium, it’s useful to identify the necessary and sufficient conditions under which each configurations is stable. By identifying and excluding unstable configurations, it’s possible to home in on the equilibrium outcome. Before starting the analysis it’s useful to provide an overview of the process.

Suppose that the fee on Network \(B\) has been set and imagine varying the fees on Network \(A\). That is, \(z\) is determined but \(x, y \in [-\omega, \rho]\) are variable. The action-space of Network \(A\) can be divided into a number of regions, where each region corresponds to the set of necessary and sufficient conditions that \((x, y)\) must satisfy in order for a specific configuration to be stable. The procedure for identifying these regions can be summarized as follows:

1. Allocate sellers to platforms and find expressions for their payoff in the proposed configuration. If sellers are on different platforms, then there’s more than one group of sellers to consider.

2. Find expressions for a seller’s payoff if they were to unilaterally deviate. In general, the sellers have three alternative membership options. Hence, there are three alternative payoffs to consider.

3. Unilateral deviation can’t be profitable if the proposed outcome is a stable configuration. Hence, none of the alternative payoffs can offer a higher payoff than the original payoff. From this we can construct three inequalities.

4. It’s often possible to show that one (or more) of the conditions is redundant (because it’s necessarily implied by one of the other conditions). Hence, the set of conditions can be simplified.
E.2 Five Possible Outcomes

In this chapter it is assumed that Network B offers positive end-user benefit: \( W(z) \geq 0 \). In general, there are eight possible configurations but three of them can be ruled out because if \( W(z) \geq 0 \), then \( o \notin \mathcal{M}(F) \). The five remaining configurations are: \((330), (221), (112), (003), (303)\). Necessary conditions for each type of outcome can be found by considering the conditions for profitable unilateral deviation by one of the sellers.

E.3 Configuration \((330)\)

This section identifies the necessary and sufficient conditions that \( F = (f^T_\xi, f^T_d) \) must satisfy to ensure that \( m \in \mathcal{M}^3 \) is stable whenever \( \Sigma_A(f_\xi, m) = 3, \Sigma_A(f_d, m) = 3, \Sigma_B(f_d, m) = 0 \).

Payoff. Since all sellers are on Network A they receive the same extra-surplus: \( W(f_\xi, m_i) = \hat{W}(x), W(f_d, m_i) = \hat{W}(y) \). Therefore, their payoffs are:

\[
\hat{\Pi}_i = \frac{2}{3} \sigma
\]

Unilateral Deviation. Suppose that Seller 1 were to unilaterally deviate from a situation in which \( \Sigma_A(f_\xi, m) = 3, \Sigma_A(f_d, m) = 3, \Sigma_B(f_d, m) = 0 \). This requires that: \( A(f_\xi, m_1) = 0 \) or \( A(f_d, m_1) = 0 \) or \( B(f_d, m_1) = 1 \). Hence, one of the following occurs:

- Seller 1 uses outside options, \( m_1 = o \), and their payoff becomes

\[
\hat{\Pi}_1 = \frac{1}{75} \sigma \left\{ 5 \sigma + 2 \hat{W}(-\omega) - 2 \hat{W}(x) \right\}^2 + \frac{1}{75} \sigma \left\{ 5 \sigma + 2 \hat{W}(\omega) - 2 \hat{W}(y) \right\}^2,
\]

which can be re-expressed as

\[
\hat{\Pi}_1 = \frac{1}{75} \sigma \left\{ 5 \sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{75} \sigma \left\{ 5 \sigma - \frac{\omega^2 - y^2}{2\tau} \right\}^2
\]

- Seller 1 joins Network B, \( m_1 = b \), and their payoff becomes

\[
\hat{\Pi}_1 = \frac{1}{75} \sigma \left\{ 5 \sigma + 2 \hat{W}(-\omega) - 2 \hat{W}(x) \right\}^2 + \frac{1}{75} \sigma \left\{ 5 \sigma + 2 \hat{W}(z) - 2 \hat{W}(y) \right\}^2,
\]

which can be re-expressed as

\[
\hat{\Pi}_1 = \frac{1}{75} \sigma \left\{ 5 \sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{75} \sigma \left\{ 5 \sigma - \frac{z^2 - y^2}{2\tau} \right\}^2
\]

- Seller 1 multihomes, \( m_1 = h \), and Network B has the higher seller-fee in market \( d \): \( z > y \). Hence, their payoff becomes

\[
\hat{\Pi}_1 = \frac{1}{3} \sigma + \frac{1}{75} \sigma \left\{ 5 \sigma + 2 \hat{W}(z) - 2 \hat{W}(y) \right\}^2,
\]
which can be re-expressed as

\[ \Pi_1 = \frac{1}{3} \sigma + \frac{1}{75\sigma} \left\{ 5\sigma - \frac{z^2 - y^2}{2\tau} \right\}^2 \]

where \( z > y \).

**Necessary Conditions.** Unilateral deviation isn’t profitable iff the following conditions are satisfied:

\[
\begin{align*}
\left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \left\{ 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right\}^2 &\leq 50\sigma^2 \\
\left\{ 5\sigma - \frac{\omega^2 - x^2 y^2}{2\tau} \right\}^2 + \left\{ 5\sigma - \frac{z^2 - y^2}{2\tau} \right\}^2 &\leq 50\sigma^2 \\
\neg \left( z > y \land \left\{ 5\sigma - \frac{z^2 - y^2}{2\tau} \right\}^2 > 25\sigma^2 \right) 
\end{align*}
\]

These can be re-expressed as

\[
\begin{align*}
\left\{ 10\sigma + x^2 - \omega^2 \right\}^2 + \left\{ 10\sigma + y^2 - \omega^2 \right\}^2 &\leq 200\tau^2\sigma^2 \\
\left\{ 10\sigma + x^2 - \omega^2 \right\}^2 + \left\{ 10\sigma + y^2 - z^2 \right\}^2 &\leq 200\tau^2\sigma^2 \\
y \geq z \text{ or } y^2 \leq z^2
\end{align*}
\]

Note that if \( -\omega \leq z \leq 0 \), then \( y \geq z \) or \( |y| \leq |z| \) iff \( y \geq -|z| = z \). Whereas, if \( 0 < z \leq \omega \), then \( y \geq z \) or \( |y| \leq |z| \) iff \( y \geq -z \).

**E.4 Configuration (221)**

This section identifies the necessary and sufficient conditions that \( \mathbf{F} = (\mathbf{f}_q^T, \mathbf{f}_q^T) \) must satisfy to ensure that \( \mathbf{m} \in \mathbb{M}^3 \) is stable whenever \( \Sigma_A(\mathbf{f}_q, \mathbf{m}) = 2 \), \( \Sigma_A(\mathbf{f}_q, \mathbf{m}) = 2 \), \( \Sigma_B(\mathbf{f}_q, \mathbf{m}) = 1 \).

**Payoff.** Let Seller 1 and Seller 2 be on Network A and Seller 3 be on Network B:

\[
\begin{align*}
A(\mathbf{f}_q, m_i) &= 1 \text{ for } i = 1, 2 \\
A(\mathbf{f}_q, m_i) &= 1 \text{ for } i = 1, 2 \\
and
B(\mathbf{f}_q, m_3) &= 1 \\
B(\mathbf{f}_q, m_3) &= 1
\end{align*}
\]

The extra-surplus received by the sellers becomes:

\[
\begin{align*}
W(\mathbf{f}_q, m_i) &= \hat{W}(x) \text{ for } i = 1, 2 \\
W(\mathbf{f}_q, m_i) &= \hat{W}(y) \text{ for } i = 1, 2
\end{align*}
\]

238
and
\[ W(f_c, m_3) = \tilde{W}(-\omega) \]
\[ W(f_d, m_3) = \tilde{W}(y) \]

Hence, their payoffs are
\[ \tilde{\Pi}_i = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \tilde{W}(x) - \tilde{W}(-\omega) \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \tilde{W}(y) - \tilde{W}(z) \right\}^2 \quad \text{for } i = 1, 2 \]
\[ \tilde{\Pi}_3 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + 2\tilde{W}(-\omega) - 2\tilde{W}(x) \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + 2\tilde{W}(z) - 2\tilde{W}(y) \right\}^2 \]

which can be re-expressed as
\[ \tilde{\Pi}_i = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \frac{z^2 - y^2}{4\tau} \right\}^2 \quad \text{for } i = 1, 2 \]
\[ \tilde{\Pi}_3 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma - \frac{z^2 - y^2}{2\tau} \right\}^2 \]

**Unilateral Deviation by Seller 1.** Suppose that Seller 1 (who was originally on Network A) were to unilaterally deviate from a situation in which \( \Sigma_A(f_c, m) = 2, \Sigma_A(f_d, m) = 2, \Sigma_B(f_d, m) = 1 \). This requires that: \( A(f_c, m_1) = 0 \) or \( A(f_d, m_1) = 0 \) or \( B(f_d, m_1) = 1 \). There are three possibilities:

- **Seller 1** uses outside options, \( m_1 = o \), and their payoff becomes
  \[ \tilde{\Pi}_1 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \tilde{W}(-\omega) - \tilde{W}(x) \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + 2\tilde{W}(-\omega) - \tilde{W}(z) - \tilde{W}(y) \right\}^2, \]
  which can be re-expressed as
  \[ \tilde{\Pi}_1 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma - \frac{2\omega^2 - y^2 - z^2}{4\tau} \right\}^2 \]

- **Seller 1** joins Network B, \( m_1 = b \), and their payoff becomes
  \[ \tilde{\Pi}_1 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \tilde{W}(-\omega) - \tilde{W}(x) \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \tilde{W}(z) - \tilde{W}(y) \right\}^2, \]
  which can be re-expressed as
  \[ \tilde{\Pi}_1 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma - \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma - \frac{z^2 - y^2}{4\tau} \right\}^2 \]

- **Seller 1** multihomes, \( m_1 = h \), and Network B has the higher seller-fee in market \( f \): \( z > y \). Their payoff becomes
  \[ \tilde{\Pi}_1 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \tilde{W}(x) - \tilde{W}(-\omega) \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma + \tilde{W}(z) - \tilde{W}(y) \right\}^2, \]
  which can be re-expressed as
  \[ \tilde{\Pi}_1 = \frac{1}{\tau_5 \sigma} \left\{ 5\sigma - \frac{x^2 - \omega^2}{4\tau} \right\}^2 + \frac{1}{\tau_5 \sigma} \left\{ 5\sigma - \frac{z^2 - y^2}{4\tau} \right\}^2 \]
Unilateral Deviation by Seller 3. Suppose that Seller 3 (who was originally on Network B) unilaterally deviates from a situation in which $\Sigma_A(f, m) = 2$, $\Sigma_A(f, m) = 2$, $\Sigma_B(f, m) = 1$. This requires: $A(f, m_3) = 1$ or $A(f, m_3) = 1$ or $B(f, m_3) = 0$. There are three possibilities:

- Seller 3 uses outside options, $m_3 = \phi$, and their payoff becomes

$$\tilde{\Pi}_3 = \frac{1}{15\sigma} \left\{ 5\sigma + 2W(-\omega) - 2W(x) \right\}^2 + \frac{1}{15\sigma} \left\{ 5\sigma + 2W(-\omega) - 2W(y) \right\}^2,$$

which can be re-expressed as

$$\tilde{\Pi}_3 = \frac{1}{15\sigma} \left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{15\sigma} \left\{ 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right\}^2.$$

- Seller 3 joins Network A, $m_3 = \alpha$, and their payoff becomes

$$\tilde{\Pi}_3 = \frac{2}{3}\sigma$$

- Seller 3 multihomes, $m_3 = \phi$, and their payoff becomes

$$\tilde{\Pi}_3 = \frac{1}{3}\sigma + \frac{1}{15\sigma} \left\{ 5\sigma + 2W(\max\{y, z\}) - 2W(y) \right\}^2,$$

which can be re-expressed as

$$\tilde{\Pi}_3 = \frac{1}{3}\sigma + \frac{1}{15\sigma} \left\{ 5\sigma - \frac{(\max\{y, z\})^2 - y^2}{2\tau} \right\}^2.$$

**Necessary Conditions.** The necessary conditions for neither seller to deviate can be found as follows:

**Seller 1.** Unilateral deviation isn’t profitable for Seller 1 iff the following conditions are satisfied:

$$\left\{ 5\sigma - \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \left\{ 5\sigma - \frac{\omega^2 - z^2 - y^2}{4\tau} \right\}^2 \leq \left\{ 5\sigma + \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \left\{ 5\sigma + \frac{z^2 - y^2}{4\tau} \right\}^2,$$

$$\left\{ 5\sigma - \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \left\{ 5\sigma - \frac{z^2 - y^2}{4\tau} \right\}^2 \leq \left\{ 5\sigma + \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \left\{ 5\sigma + \frac{z^2 - y^2}{4\tau} \right\}^2.$$

Hence,

$$\left\{ 20\sigma \tau - (\omega^2 - x^2) \right\}^2 + \left\{ 20\sigma \tau - (\omega^2 - z^2 - y^2) \right\}^2 \leq \left\{ 20\sigma \tau + (\omega^2 - x^2) \right\}^2 + \left\{ 20\sigma \tau + (z^2 - y^2) \right\}^2,$$

$$\left\{ 20\sigma \tau - (\omega^2 - x^2) \right\}^2 + \left\{ 20\sigma \tau - (z^2 - y^2) \right\}^2 \leq \left\{ 20\sigma \tau + (\omega^2 - x^2) \right\}^2 + \left\{ 20\sigma \tau + (z^2 - y^2) \right\}^2.$$

Note that if $-\omega \leq z \leq 0$, then $y \geq z$ or $|y| \leq |z|$ iff $y \geq -|z| = z$. Whereas, if $0 < z \leq \omega$, then $y \geq z$ or $|y| \leq |z|$ iff $y > -z$.  

240
Seller 3. Unilateral deviation isn’t profitable for Seller 3 providing that:

\[
\left\{ 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right\}^2 \leq \left\{ 5\sigma - \frac{z^2 - y^2}{2\tau} \right\}^2
\]

\[
50\sigma^2 \leq \left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \left\{ 5\sigma - \frac{z^2 - y^2}{2\tau} \right\}^2
\]

\[
25\sigma^2 + \left\{ 5\sigma - \frac{(\max\{y, z\})^2 - y^2}{2\tau} \right\}^2 \leq \left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \left\{ 5\sigma - \frac{z^2 - y^2}{2\tau} \right\}^2
\]

Hence,

\[
z^2 \leq \omega^2
\]

\[
\{10\sigma + x^2 - \omega^2\}^2 + \{10\sigma + y^2 - z^2\}^2 \geq 200\sigma^2\tau^2
\]

\[
\{10\sigma + x^2 - \omega^2\}^2 + \{10\sigma + y^2 - z^2\}^2 \geq 100\sigma^2\tau^2 + \{10\sigma + y^2 - (\max\{y, z\})^2\}^2
\]

### E.5 Configuration (112)

This section identifies the necessary and sufficient conditions that \( F = (f_{\xi}, f_{\eta}) \) must satisfy to ensure that \( \mathbf{m} \in \mathbb{M}^3 \) is stable whenever \( \Sigma_A(f_{\xi}, \mathbf{m}) = 1 \), \( \Sigma_A(f_{\eta}, \mathbf{m}) = 1 \), \( \Sigma_B(f_{\eta}, \mathbf{m}) = 2 \).

**Payoff.** Let Seller 1 and Seller 2 be on Network B and Seller 3 be on Network A:

\[
B(f_{\xi}, m_i) = 1 \text{ for } i = 1, 2
\]

\[
B(f_{\eta}, m_i) = 1 \text{ for } i = 1, 2
\]

and

\[
A(f_{\xi}, m_3) = 1
\]

\[
A(f_{\eta}, m_3) = 1
\]

The extra-surplus received by the sellers becomes:

\[
W(f_{\xi}, m_i) = \hat{W}(-\omega) \text{ for } i = 1, 2
\]

\[
W(f_{\eta}, m_i) = \hat{W}(z) \text{ for } i = 1, 2
\]

and

\[
W(f_{\xi}, m_3) = \hat{W}(x)
\]

\[
W(f_{\eta}, m_3) = \hat{W}(y)
\]

Hence, their payoffs are

\[
\hat{\Pi}_i = \frac{1}{\tau_5} \left\{ 5\sigma + \hat{W}(-\omega) - \hat{W}(x) \right\}^2 + \frac{1}{\tau_5} \left\{ 5\sigma + \hat{W}(z) - \hat{W}(y) \right\}^2 \text{ for } i = 1, 2
\]

\[
\hat{\Pi}_3 = \frac{1}{\tau_5} \left\{ 5\sigma + 2\hat{W}(x) - 2\hat{W}(-\omega) \right\}^2 + \frac{1}{\tau_5} \left\{ 5\sigma + 2\hat{W}(y) - 2\hat{W}(z) \right\}^2
\]
which can be re-expressed as

\[
\begin{align*}
\Pi_i &= \frac{1}{75\sigma} \left\{ 5\sigma - \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma - \frac{z^2 - y^2}{4\tau} \right\}^2 \\
\Pi_3 &= \frac{1}{75\sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + \frac{z^2 - y^2}{2\tau} \right\}^2
\end{align*}
\]

for \( i = 1, 2 \)

**Unilateral Deviation by Seller 1.** Suppose that Seller 1 (who was originally on Network B) were to unilaterally deviate from a situation in which \( \Sigma_A(f_c, m) = 1, \Sigma_A(f_d, m) = 1, \Sigma_B(f_d, m) = 2. \) This requires: \( A(f_c, m_1) = 1 \) or \( A(f_d, m_1) = 1 \) or \( B(f_d, m_1) = 0 \). There are three possibilities:

- **Seller 1 uses outside options,** \( m_1 = o \), and their payoff becomes

\[
\Pi_1 = \frac{1}{75\sigma} \left\{ 5\sigma + \tilde{W}(-\omega) - \tilde{W}(x) \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + 2\tilde{W}(-\omega) - \tilde{W}(z) - \tilde{W}(y) \right\}^2,
\]

which can be re-expressed as

\[
\Pi_1 = \frac{1}{75\sigma} \left\{ 5\sigma - \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma - \frac{2\omega^2 - y^2 - z^2}{4\tau} \right\}^2
\]

- **Seller 1 joins Network A,** \( m_1 = a \), and their payoff becomes

\[
\Pi_1 = \frac{1}{75\sigma} \left\{ 5\sigma + \tilde{W}(x) - \tilde{W}(-\omega) \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + \tilde{W}(y) - \tilde{W}(z) \right\}^2,
\]

which can be re-expressed as

\[
\Pi_1 = \frac{1}{75\sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + \frac{z^2 - y^2}{4\tau} \right\}^2
\]

- **Seller 1 multihomes,** \( m_1 = h \), and their payoff becomes

\[
\Pi_1 = \frac{1}{75\sigma} \left\{ 5\sigma + \tilde{W}(-\omega) - \tilde{W}(x) \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + \tilde{W}(\max\{y, z\}) - \tilde{W}(z) \right\}^2,
\]

which can be re-expressed as

\[
\Pi_1 = \frac{1}{75\sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{4\tau} \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma + \frac{z^2 - \max\{y, z\}}{4\tau} \right\}^2
\]

**Unilateral Deviation by Seller 3.** Suppose that Seller 3 (who was originally on Network A) unilaterally deviates from a situation in which \( \Sigma_A(f_c, m) = 1, \Sigma_A(f_d, m) = 1, \Sigma_B(f_d, m) = 2. \) This requires: \( A(f_c, m_3) = 0 \) or \( A(f_d, m_3) = 0 \) or \( B(f_d, m_3) = 1 \). There are three possibilities:

- **Seller 3 uses outside options,** \( m_3 = o \), and their payoff becomes

\[
\Pi_3 = \frac{1}{3\sigma} + \frac{1}{75\sigma} \left\{ 5\sigma + 2\tilde{W}(-\omega) - 2\tilde{W}(z) \right\}^2,
\]

242
which can be re-expressed as

\[ \tilde{\Pi}_3 = \frac{1}{3} \sigma + \frac{1}{75 \sigma} \left\{ 5 \sigma - \frac{w^2 - z^2}{2 \tau} \right\}^2 \]

- Seller 3 joins Network B, \( m_3 = b \), and their payoff becomes

\[ \tilde{\Pi}_3 = \frac{2}{3} \sigma \]

- Seller 3 multihomes, \( m_3 = h \), and Network B has the higher seller-fee: \( z > y \). Their payoff becomes

\[ \tilde{\Pi}_3 = \frac{1}{75 \sigma} \left\{ 5 \sigma + 2 \bar{W}(x) - 2 \bar{W}(-\omega) \right\}^2 + \frac{1}{3} \sigma, \]

which can be re-expressed as

\[ \tilde{\Pi}_3 = \frac{1}{75 \sigma} \left\{ 5 \sigma + \frac{\omega^2 - x^2}{2 \tau} \right\}^2 + \frac{1}{3} \sigma, \]

**Necessary Conditions.** The necessary conditions for neither seller to deviate can be found as follows:

**Seller 1.** Unilateral deviation isn’t profitable for Seller 1 iff the following conditions are satisfied:

\[
\left\{ 5 \sigma - \frac{2 \omega^2 - z^2 - y^2}{4 \tau} \right\} \leq \left\{ 5 \sigma - \frac{z^2 - y^2}{4 \tau} \right\}^2
\]
\[
\left\{ 5 \sigma + \frac{\omega^2 - x^2}{4 \tau} \right\}^2 + \left\{ 5 \sigma + \frac{z^2 - y^2}{4 \tau} \right\}^2 \leq \left\{ 5 \sigma - \frac{\omega^2 - x^2}{4 \tau} \right\} + \left\{ 5 \sigma - \frac{z^2 - y^2}{4 \tau} \right\}^2
\]
\[
\left\{ 5 \sigma + \frac{x^2 - \omega^2}{4 \tau} \right\}^2 + \left\{ 5 \sigma + \frac{z^2 - \max\{y, z\}}{4 \tau} \right\}^2 \leq \left\{ 5 \sigma - \frac{\omega^2 - x^2}{4 \tau} \right\}^2 + \left\{ 5 \sigma - \frac{z^2 - y^2}{4 \tau} \right\}^2
\]

Hence,

\[
z^2 \leq \omega^2
\]
\[
\{20 \sigma \tau + \omega^2 - x^2\}^2 + \{20 \sigma \tau + z^2 - y^2\}^2 \leq \{20 \sigma \tau - (\omega^2 - x^2)\}^2 + \{20 \sigma \tau - (z^2 - y^2)\}^2
\]
\[
\{20 \sigma \tau + x^2 - \omega^2\}^2 + \{20 \sigma \tau + z^2 - \max\{y, z\}\}^2 \leq \{20 \sigma \tau - (\omega^2 - x^2)\}^2 + \{20 \sigma \tau - (z^2 - y^2)\}^2
\]

**Seller 3.** Unilateral deviation isn’t profitable for Seller 3 iff the following conditions are satisfied:

\[
25 \sigma^2 + \left\{ 5 \sigma - \frac{\omega^2 - z^2}{2 \tau} \right\}^2 \leq \left\{ 5 \sigma + \frac{\omega^2 - x^2}{2 \tau} \right\}^2 + \left\{ 5 \sigma + \frac{z^2 - y^2}{2 \tau} \right\}^2
\]
\[
50 \sigma^2 \leq \left\{ 5 \sigma + \frac{\omega^2 - x^2}{2 \tau} \right\}^2 + \left\{ 5 \sigma + \frac{z^2 - y^2}{2 \tau} \right\}^2
\]
\[
- \left( z > y \land 25 \sigma^2 \geq \left\{ 5 \sigma + \frac{z^2 - y^2}{2 \tau} \right\}^2 \right)
\]
Hence,
\[ 100\sigma^2 \tau^2 + \left( 10\sigma \tau + z^2 - \omega^2 \right)^2 \leq \left( 10\sigma \tau + \omega^2 - x^2 \right)^2 + \left( 10\sigma \tau + z^2 - y^2 \right)^2 \]
\[ 200\sigma^2 \tau^2 \leq \left( 10\sigma \tau + \omega^2 - x^2 \right)^2 + \left( 10\sigma \tau + z^2 - y^2 \right)^2 \]
\[ y \geq z \text{ or } y^2 \leq z^2 \]

Note that if \(-\omega \leq z \leq 0\), then \(y \geq z \) or \(|y| \leq |z|\) iff \(y \geq -|z| = z\). Whereas, if \(0 < z \leq \omega\), then \(y \geq z \) or \(|y| \leq |z|\) iff \(y > -z\).

E.6 Configuration (003)

This section identifies the necessary and sufficient conditions that \(F = (f_{d}^T, f_{q}^T)\) must satisfy to ensure that \(m \in \mathbb{M}^3\) is stable whenever \(\Sigma_A(f_{c}, m) = 0\), \(\Sigma_A(f_{q}, m) = 0\), \(\Sigma_B(f_{q}, m) = 3\). It’s useful to consider two cases: (1) \(y \geq -|z|\); and (2) \(y < -|z|\).

E.6.1 Case (1)

This subsection considers the necessary and sufficient conditions that \(F\) (where \(y \geq -|z|\)) must satisfy for Configuration (003) to be stable. Throughout this subsection it’s assumed that \(y \geq -|z|\).

Payoff. Since all sellers are on Network \(B\) they receive the same extra-surplus:
\[ W(f_{c}, m_i) = \hat{W}(-\omega), W(f_{q}, m_i) = \hat{W}(z) \]. Therefore, their payoffs are:
\[ \hat{\Pi}_i = \frac{2}{3} \sigma \]

Unilateral Deviation. Suppose that Seller 1 were to unilaterally deviate from a situation in which \(\Sigma_A(f_{c}, m) = 0\), \(\Sigma_A(f_{q}, m) = 0\), \(\Sigma_B(f_{q}, m) = 3\). This requires: \(A(f_{c}, m_1) = 1\) or \(A(f_{q}, m_1) = 1\) or \(A(f_{q}, m_1) = 0\). Hence, one of the following occurs:

- Seller 1 uses outside options, \(m_1 = a\), and their payoff becomes
  \[ \hat{\Pi}_1 = \frac{1}{3} \sigma + \frac{1}{7 \sigma} \left\{ 5\sigma + 2\hat{W}(-\omega) - 2\hat{W}(z) \right\}^2 \]
  which can be re-expressed as
  \[ \hat{\Pi}_1 = \frac{1}{3} \sigma + \frac{1}{7 \sigma} \left\{ 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right\}^2 \]

- Seller 1 joins Network \(A\) or multihomes, \(m_1 \in \{a, b\}\), and their payoff becomes
  \[ \hat{\Pi}_1 = \frac{1}{7 \sigma} \left\{ 5\sigma + 2\hat{W}(x) - 2\hat{W}(-\omega) \right\}^2 + \frac{1}{7 \sigma} \left\{ 5\sigma + 2\hat{W}(y) - 2\hat{W}(z) \right\}^2 \]
which can be re-expressed as
\[ \Pi_1 = \frac{1}{\tau \sigma} \left( 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right)^2 + \frac{1}{\tau \sigma} \left( 5\sigma + \frac{z^2 - y^2}{2\tau} \right)^2 \]

**Necessary Conditions.** Unilateral deviation isn’t profitable iff the following conditions are satisfied:
\[ \left\{ 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right\}^2 \leq 25\sigma^2 \]
\[ \left\{ 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \left\{ 5\sigma + \frac{z^2 - y^2}{2\tau} \right\}^2 \leq 50\sigma^2 \]

Hence,
\[ z^2 \leq \omega^2 \]
\[ \{ 10\sigma + \omega^2 - x^2 \}^2 + \{ 10\sigma + z^2 - y^2 \}^2 \leq 200\tau^2 \sigma^2 \]

**E.6.2 Case (2)**

This sub-section considers the necessary and sufficient conditions that \( F \) (where \( y < -|z| \)) must satisfy for Configuration (003) to be stable. Throughout this subsection it’s assumed that \( y < -|z| \).

**Payoff.** Since all sellers are on Network \( B \) they receive the same extra-surplus: \( W(f_c, m_i) = \hat{W}(-\omega), W(f_d, m_i) = \hat{W}(z) \). Therefore, their payoffs are:
\[ \Pi_i = \frac{2}{3} \sigma \]

**Unilateral Deviation.** Suppose that Seller 1 were to unilaterally deviate from a situation in which \( \Sigma_A(f_c, m) = 0, \Sigma_A(f_d, m) = 0, \Sigma_B(f_d, m) = 3 \). This requires: \( A(f_c, m_1) = 1 \) or \( A(f_d, m_1) = 1 \) or \( A(f_d, m_1) = 0 \). Hence, one of the following occurs:

- **Seller 1 uses outside options, \( m_1 = o \), and their payoff becomes**
  \[ \Pi_1 = \frac{1}{3} \sigma + \frac{1}{\tau \sigma} \left\{ 5\sigma + 2\hat{W}(-\omega) - 2\hat{W}(z) \right\}^2, \]
  which can be re-expressed as
  \[ \Pi_1 = \frac{1}{3} \sigma + \frac{1}{\tau \sigma} \left\{ 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right\}^2 \]

- **Seller 1 joins Network \( A \), \( m_1 = a \), and their payoff becomes**
  \[ \Pi_1 = \frac{1}{\tau \sigma} \left\{ 5\sigma + 2\hat{W}(x) - 2\hat{W}(-\omega) \right\}^2 + \frac{1}{\tau \sigma} \left\{ 5\sigma + 2\hat{W}(y) - 2\hat{W}(z) \right\}^2, \]
  which can be re-expressed as
  \[ \Pi_1 = \frac{1}{\tau \sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{\tau \sigma} \left\{ 5\sigma + \frac{z^2 - y^2}{2\tau} \right\}^2 \]
Seller 1 multihomes, $m_1 = h$, and their payoff becomes

$$\tilde{\Pi}_1 = \frac{1}{16\sigma} \left\{ 5\sigma + 2\hat{W}(x) - 2\hat{W}(-\omega) \right\}^2 + \frac{1}{3}\sigma$$

which can be re-expressed as

$$\tilde{\Pi}_1 = \frac{1}{16\sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{3}\sigma$$

**Necessary Conditions.** Unilateral deviation isn’t profitable iff the following conditions are satisfied:

$$\left\{ 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right\}^2 \leq 25\sigma^2$$

$$\left\{ 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \left\{ 5\sigma + \frac{z^2 - y^2}{2\tau} \right\}^2 \leq 50\sigma^2$$

$$\left\{ 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right\}^2 \leq 25\sigma^2$$

Hence,

$$z^2 \leq \omega^2$$

$$\{ 10\sigma \tau + \omega^2 - x^2 \}^2 + \{ 10\sigma \tau + z^2 - y^2 \}^2 \leq 200\tau^2\sigma^2$$

$$x^2 \geq \omega^2$$

**E.7 Configuration (303)**

This section identifies the necessary and sufficient conditions that $F = (f_4^T, f_4^T)$ must satisfy to ensure that $m \in M^3$ is stable whenever $\Sigma_A(f_4, m) = 3$, $\Sigma_A(f_4, m) = 0$, $\Sigma_B(f_4, m) = 3$.

**Prerequisites.** If a seller is on $C_A$ and $D_B$, then they must multihome and Network $B$ has the higher seller-fee. Hence, preconditions for (303) are $y < z$ and $h \in M(F)$. Furthermore, if $h \in M(F)$, then $h \succeq a$, which requires $\hat{W}(\max\{y, z\}) \geq \hat{W}(y)$. Since $y < z$, it follows that $\hat{W}(z) \geq \hat{W}(y)$. Therefore, prerequisites for (303) are:

$$y < z$$

$$|z| \leq |y|$$

There are two cases to consider: $z \leq 0$; and $z > 0$. If $z \leq 0$, then $y < z$, $|z| \leq |y|$ iff $y < -|z| = z$. Whereas, if $z > 0$, then $y < z$, $|z| \leq |y|$ iff $y \leq -z$. Therefore, a prerequisite for Configuration (303) is

$$y \leq -|z|$$

**Payoff.** Since all sellers are on the same combination of platforms, they receive the same extra-surplus: $W(f_4, m_i) = \hat{W}(x)$, $W(f_4, m_i) = \hat{W}(z)$. Therefore,
their payoffs are:

$$\tilde{\Pi}_i = \frac{2}{3}\sigma$$

**Unilateral Deviation.** Assume that $y < z$, $|z| \leq |y|$ so that a seller who multihomes, $m_i = h$, is on Platform $D_B$ in market $d$. This is a prerequisite for the type of outcome under consideration. Suppose that Seller 1 were to unilaterally deviate from a situation in which $\Sigma_A(f_i, m) = 3$, $\Sigma_A(f_{-i}, m) = 0$, $\Sigma_B(f_{-i}, m) = 3$. This requires: $A(f_i, m_1) = 0$ or $A(f_{-i}, m_1) = 1$ or $B(f_{-i}, m_1) = 0$. Hence, one of the following occurs:

- Seller 1 uses outside options, $m_1 = o$, and their payoff becomes

$$\tilde{\Pi}_1 = \frac{1}{75\sigma} \left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right\}^2$$

which can be re-expressed as

$$\tilde{\Pi}_1 = \frac{1}{75\sigma} \left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{75\sigma} \left\{ 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right\}^2$$

- Seller 1 joins Network $A$, $m_1 = a$, and their payoff becomes

$$\tilde{\Pi}_1 = \frac{1}{3}\sigma + \frac{1}{75\sigma} \left\{ 5\sigma + 2\tilde{W}(y) - 2\tilde{W}(z) \right\}^2$$

which can be re-expressed as

$$\tilde{\Pi}_1 = \frac{1}{75\sigma} \left\{ 5\sigma + \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \frac{1}{3}\sigma$$

- Seller 1 joins Network $B$, $m_1 = b$, and their payoff becomes

$$\tilde{\Pi}_1 = \frac{1}{75\sigma} \left\{ 5\sigma + 2\tilde{W}(x) - 2\tilde{W}(z) \right\}^2 + \frac{1}{3}\sigma$$

which can be re-expressed as

$$\tilde{\Pi}_1 = \frac{1}{75\sigma} \left\{ 5\sigma - \frac{\omega^2 - y^2}{2\tau} \right\}^2 + \frac{1}{3}\sigma$$

**Necessary Conditions.** Unilateral deviation isn’t profitable iff the following conditions are satisfied:

$$\left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 + \left\{ 5\sigma - \frac{\omega^2 - z^2}{2\tau} \right\}^2 \leq 50\sigma^2$$

$$\left\{ 5\sigma + \frac{\omega^2 - y^2}{2\tau} \right\}^2 \leq 25\sigma^2$$

$$\left\{ 5\sigma - \frac{\omega^2 - x^2}{2\tau} \right\}^2 \leq 25\sigma^2$$

Hence,

$$\left\{ 10\sigma + x^2 - \omega^2 \right\}^2 + \left\{ 10\sigma + z^2 - \omega^2 \right\}^2 \leq 200\tau^2\sigma^2$$
Finally, a prerequisite for (303) is

\[ y^2 \geq z^2 \]
\[ x^2 \leq \omega^2 \]

\[ y < z \]

**E.8 Stable Configurations in Subgame [2]**

Subgames [1] and [2] have the same group of potentially configurations. The results in this appendix are immediately applicable to Subgame [2].
Appendix F

Subgame [2]

F.1 Equilibrium Outcomes

In Subgame [2] Network B offers a positive end-user benefit and sets a positive seller-fee. This requires: $0 < z \leq \omega$. It has been shown that in Subgame [2] the possible configurations are: $(330), (221), (112), (003), (303)$. The necessary and sufficient conditions for the existence of a RSPNE in which a particular configuration occurs are as follows:

<table>
<thead>
<tr>
<th></th>
<th>[2a]</th>
<th>[2b]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0 &lt; z \leq \omega,$</td>
<td>$0 &lt; z \leq \omega,$</td>
</tr>
<tr>
<td></td>
<td>$y &gt; -z$</td>
<td>$y \leq -z$</td>
</tr>
</tbody>
</table>

*Table 7. Configurations in Subgame [2]*

It can be seen that the outcome is almost identical to that in Subgame [1]. There are only two differences: firstly, $z$ is replaced by $-z$; and secondly, $y \geq -|z|$ is replaced by $y > -z$. That is, if $z > 0$, then for the tie to be binding the inequality has to be strict.

It can be seen that there are five possible configurations and each of these outcomes is paired with a unique region of the action-space. The value taken by
\(\Sigma_A(f_k, \tilde{m}), \Sigma_B(f_d, \tilde{m})\) in each equilibrium can be found from the three-digit codes, \((IJK)\), where

\[
\Sigma_A(f_c, \tilde{m}) = I, \Sigma_A(f_d, \tilde{m}) = J, \Sigma_B(f_d, \tilde{m}) = K
\]

From this it’s possible to find expressions for the networks’ payoffs.

**F.2 Network A’s Payoff**

In Subgame [2] the payoff function of Network A is as follows:

**Proposition F.1** Let \(f_c = (x, -\omega), f_d = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(0 < z \leq \omega\). If \(\tilde{m} \in \mathcal{N}(F)\), then

\[
\Pi_A(F, \tilde{m}) = \Pi_A(x, y, z),
\]

where \(\Pi_A(.)\) is defined as follows:

\[
\Pi_A(x, y, z) = \frac{\alpha}{2\pi} (2\omega + x + y) \quad \text{iff} \quad \begin{cases} 
 y > -z, \\
 G(x, y, z) \leq 200\tau^2\sigma^2
\end{cases}
\]

\[
\Pi_A(x, y, z) = \frac{\alpha}{3\pi} (2\omega + x + y) \quad \text{iff} \quad \begin{cases} 
 y > -z, \\
 G(x, y, z) > 200\tau^2\sigma^2, \\
x^2 + y^2 \leq \omega^2 + z^2
\end{cases}
\]

\[
\Pi_A(x, y, z) = \frac{\alpha}{6\pi} (2\omega + x + y) \quad \text{iff} \quad \begin{cases} 
 y > -z, \\
 x^2 + y^2 > \omega^2 + z^2, \\
 H(x, y, z) \geq 200\tau^2\sigma^2
\end{cases}
\]

\[
\Pi_A(x, y, z) = \frac{\alpha}{2\pi} (\omega + x) \quad \text{iff} \quad \begin{cases} 
 y \leq -z, \\
x \leq \omega
\end{cases}
\]

\[
\Pi_A(x, y, z) = 0 \quad \text{iff} \quad \begin{cases} 
 y > -z, \\
 H(x, y, z) < 200\tau^2\sigma^2 \quad \text{or} \quad y \leq -z, \\
x > \omega
\end{cases}
\]

The five scenarios are mutually exclusive and exhaustive.

It can be shown that if \((x, y)\) is outside of the 1st Quadrant (that is, \(x < 0\) or \(y < 0\)), then \((x, y)\) is strictly dominated. If \(x \geq 0, y \geq 0\), then Network A’s payoff function becomes:

\[
\Pi_A(x, y, z) = \frac{\alpha}{2\pi} (2\omega + x + y) \quad \text{iff} \quad G(x, y, z) \leq 200\tau^2\sigma^2
\]

\[
\Pi_A(x, y, z) = \frac{\alpha}{3\pi} (2\omega + x + y) \quad \text{iff} \quad \begin{cases} 
 G(x, y, z) > 200\tau^2\sigma^2, \\
x^2 + y^2 \leq \omega^2 + z^2
\end{cases}
\]

\[
\Pi_A(x, y, z) = \frac{\alpha}{6\pi} (2\omega + x + y) \quad \text{iff} \quad \begin{cases} 
 x^2 + y^2 > \omega^2 + z^2, \\
 H(x, y, z) \geq 200\tau^2\sigma^2
\end{cases}
\]

\[
\Pi_A(x, y, z) = 0 \quad \text{iff} \quad H(x, y, z) < 200\tau^2\sigma^2
\]
Network $A$ selects $x \geq 0, y \geq 0$, so as to maximize this payoff function.

### F.3 Network $B$’s Payoff

In Subgame [2] the payoff function of Network $B$ is as follows:

**Proposition F.2** Let $f_\ell = (x, -\omega), f_d = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $0 < z \leq \omega$. If $\tilde{m} \in N(F)$, then

$$\Pi_B(F, \tilde{m}) = \Pi_B(x, y, z),$$

where $\Pi_B(.)$ is defined as follows:

- $\Pi_B(x, y, z) = \frac{\mu}{2\tau} (\omega + z)$ if $y \leq -z$ or $\left\{ \begin{array}{l} y > -z, \\ H(x, y, z) < 200\tau^2\sigma^2 \end{array} \right.$

- $\Pi_B(x, y, z) = \frac{\mu}{2\tau} (\omega + z)$ if $y > -z$ and $\left\{ \begin{array}{l} H(x, y, z) \geq 200\tau^2\sigma^2, \\ x^2 + y^2 > \omega^2 + z^2, \end{array} \right.$

- $\Pi_B(x, y, z) = \frac{\mu}{2\tau} (\omega + z)$ if $y > -z$ and $\left\{ \begin{array}{l} x^2 + y^2 \leq \omega^2 + z^2, \\ G(x, y, z) > 200\tau^2\sigma^2 \end{array} \right.$

- $\Pi_B(x, y, z) = 0$ if $y > -z$ and $\left\{ \begin{array}{l} G(x, y, z) \leq 200\tau^2\sigma^2 \end{array} \right.$

*The four scenarios are mutually exclusive and exhaustive.*
Appendix G

Subgame [3]

G.1 Equilibrium Outcomes

In Subgame [3] Network B offers a negative end-user benefit. This requires: \( \omega < z \leq \rho \). It has been shown that in Subgame [3] the possible configurations are: (330), (220), (110), (000). The necessary and sufficient conditions for the existence of a RSPNE in which a particular configuration occurs are as follows:

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(330)</td>
<td>( G(x, y, \omega) \leq 200\tau^2\sigma^2 )</td>
</tr>
<tr>
<td>(220)</td>
<td>( x^2 + y^2 \leq 2\omega^2 )</td>
</tr>
<tr>
<td>(110)</td>
<td>( x^2 + y^2 &gt; 2\omega^2 ) ( H(x, y, \omega) \geq 200\tau^2\sigma^2 )</td>
</tr>
<tr>
<td>(000)</td>
<td>( H(x, y, \omega) &lt; 200\tau^2\sigma^2 )</td>
</tr>
</tbody>
</table>

Table 8. Configurations in Subgame [3]

It can be seen that the boundaries are very similar to those that occur in Subgame [1a] (and Subgame [2a]). The only difference is that \( z \) has been replaced by \( \omega \). This can be explained as follows: Firstly, if \( z > \omega \), then \( b < a \) and \( h \leq a \). Secondly, outside options, \( a \), play the same type of role that joining Network B, \( b \), did in subgames [1], [2]. That is, Network A "competes" against outside options.

It can be seen that there are four possible configurations and each of these outcomes is paired with a unique region of the action-space. The value taken by \( \sum_A(f_k, \tilde{m}) \), \( \sum_B(f_q, \tilde{m}) \) in each equilibrium can be found from the three-digit
codes, \((IJK)\), where
\[
\sum_A(f, \bar{m}) = I, \sum_A(f, \bar{m}) = J, \sum_B(f, \bar{m}) = K = 0
\]
From this it’s possible to find expressions for the networks’ payoffs.

### G.2 Network A’s Payoff

In Subgame [3] the payoff function of Network A is as follows:

**Proposition G.1** Let \(f = (x, -\omega), f = (y, z)\), where \(x, y, z \in [-\omega, \rho]\), and suppose that \(\omega < z\). If \(\bar{m} \in N(F)\), then
\[
\Pi_A(F, \bar{m}) = \bar{\Pi}_A(x, y, z)
\]
where \(\bar{\Pi}_A(.)\) is defined as follows:
\[
\bar{\Pi}_A(x, y, z) = \frac{\mu}{2\sigma} (2\omega + x + y) \text{ iff } G(x, y, \omega) \leq 200\tau^2\sigma^2
\]
\[
\bar{\Pi}_A(x, y, z) = \frac{\mu}{3\sigma} (2\omega + x + y) \text{ iff } \begin{cases} 
G(x, y, \omega) > 200\tau^2\sigma^2, \\
x^2 + y^2 \leq 2\omega^2
\end{cases}
\]
\[
\bar{\Pi}_A(x, y, z) = \frac{\mu}{6\sigma} (2\omega + x + y) \text{ iff } \begin{cases} 
x^2 + y^2 > 2\omega^2, \\
H(x, y, \omega) \geq 200\tau^2\sigma^2
\end{cases}
\]
\[
\bar{\Pi}_A(x, y, z) = 0 \text{ iff } H(x, y, \omega) < 200\tau^2\sigma^2
\]
The five scenarios are mutually exclusive and exhaustive.

It can be shown that if \((x, y)\) is outside of the 1st Quadrant (that is, \(x < 0\) or \(y < 0\)), then \((x, y)\) is strictly dominated. If \(x \geq 0, y \geq 0\), then Network A’s payoff function becomes:
\[
\bar{\Pi}_A(x, y, z) = \frac{\mu}{2\sigma} (2\omega + x + y) \text{ iff } G(x, y, \omega) \leq 200\tau^2\sigma^2
\]
\[
\bar{\Pi}_A(x, y, z) = \frac{\mu}{3\sigma} (2\omega + x + y) \text{ iff } \begin{cases} 
G(x, y, \omega) > 200\tau^2\sigma^2, \\
x^2 + y^2 \leq 2\omega^2
\end{cases}
\]
\[
\bar{\Pi}_A(x, y, z) = \frac{\mu}{6\sigma} (2\omega + x + y) \text{ iff } \begin{cases} 
x^2 + y^2 > 2\omega^2, \\
H(x, y, \omega) \geq 200\tau^2\sigma^2
\end{cases}
\]
\[
\bar{\Pi}_A(x, y, z) = 0 \text{ iff } H(x, y, \omega) < 200\tau^2\sigma^2
\]
Network A selects \(x \geq 0, y \geq 0\), so as to maximize this payoff function.

### G.3 Network B’s Payoff

In Subgame [3] the payoff function of Network B is as follows:
Proposition G.2 Let $f_\varepsilon = (x, -\omega), f_\delta = (y, z)$, where $x, y, z \in [-\omega, \rho]$, and suppose that $0 < z \leq \omega$. If $\overline{m} \in \mathcal{N}(F)$, then

$$\Pi_B(F, \overline{m}) = 0$$
Appendix H

Network A Sets Positive Seller-Fees

The proceeding chapters analyzed the sellers’ membership decisions given a particular value of $x, y, z$. From this it was possible to find the number of sellers on each platform and the networks payoffs. The networks choose their fees so as to maximize their payoffs, which often comes down to attracting the maximum number of sellers. This chapter begins to investigate the fees set by Network A.

Imagine that $z$ is known and Network A is called upon to set its fees: $x, y \in [-\omega, \rho]$. (Note that fees are set simultaneously.) It can be seen that $(x, y)$ can be anywhere within a square that has its vertices in each quadrant of $\mathbb{R}^2$. Hence, the seller fees on one (or more) of Network A’s platforms could be negative. However, this chapter shows that any point outside of the 1$^{st}$ Quadrant is strictly dominated, where $(x, y)$ is within the 1$^{st}$ Quadrant if $x \geq 0, y \geq 0$.

Subgames were classified according to the size of $z$. Three classes of subgame were defined: [1] $z \leq 0$; [2] $0 < z \leq \omega$; and [3] $\omega < z$. In principle, we should consider the nature of Network A’s best-response in each case separately. However, the membership outcomes (and, hence, payoff functions) are very similar in each case. That is, the action-space is divided up as follows. There is a central oval within which all sellers are on Network A. This oval is enclosed by a series of concentric shells (lobes). As we move away from the inner oval, the number of sellers in each shell decreases.

Since the three cases are qualitatively similar from Network A’s prespective, I will only analyze the first case: $z \leq 0$. However, I claim that the result can be replicated for the other two cases.

The analysis will proceed by taking each quadrant in turn. It will be shown that for any point located within the quadrant under consideration, there exists another point on the boundary of the 1$^{st}$ Quadrant that offers a strictly higher payoff.

H.1 2$^{nd}$ Quadrant

In this section suppose that $x < 0, y \geq 0$. Firstly, in the 2$^{nd}$ Quadrant,

$$\tilde{\Sigma}_A^x(x, y, z) = \tilde{\Sigma}_A^d(x, y, z),$$
which implies that we only need to consider changes in $\tilde{\Sigma}_A^e(x, y, z)$. Secondly, in the 2nd Quadrant, there are four regions: Exterior; 2nd Shell, 1st Shell, Core.

It’s useful to consider two cases. Firstly, if $H(x, y, z) < 200\tau^2\sigma^2$, then $(x, y)$ is in the Exterior. Secondly, if $H(x, y, z) \geq 200\tau^2\sigma^2$, then $(x, y)$ belongs to one of the other regions: 2nd Shell, 1st Shell, or Core. These cases are analyzed in turn:

*Case (a).* Suppose that $H(x, y, z) < 200\tau^2\sigma^2$. If $(x, y)$ is in the Exterior, then no sellers are on Network $A$ and so it makes no profit. Whereas, if $x = y = 0$, then $(x, y)$ is in the Core and all the sellers are on Network $A$. Its payoff becomes

$$\tilde{\Pi}_A(0, 0, z) = \frac{\mu \omega}{\tau} > 0$$

It can be seen that any point in the Exterior is strictly dominated by the origin.

*Case (b).* Suppose that $H(x, y, z) \geq 200\tau^2\sigma^2$. If $(x, y)$ isn’t in the Exterior, then some sellers are on Network $A$. To analyze this situation it’s useful to introduce the following sets:

$$S_1 = \{(x, y) : \tilde{\Sigma}_A^e(x, y, z) \geq 1\} = \{(x, y) : H(x, y, z) \geq 200\tau^2\sigma^2\}$$

$$S_2 = \{(x, y) : \tilde{\Sigma}_A^e(x, y, z) \geq 2\} = \{(x, y) : x^2 + y^2 \leq \omega^2 + z^2\}$$

$$S_3 = \{(x, y) : \tilde{\Sigma}_A^e(x, y, z) = 3\} = \{(x, y) : G(x, y, z) \leq 200\tau^2\sigma^2\}$$

Imagine starting at a point in the 2nd Quadrant, $(x, y)$, and increasing $x$ (decreasing $|x|$) until you reach the $y$-axis, $(0, y)$. Consider how the value of $\tilde{\Sigma}_A^e(x, y, z)$ changes as we switch from $(x, y)$ to $(0, y)$. There are three sub-cases to consider: (i) $(x, y) \in S_1 \setminus S_2$; (ii) $(x, y) \in S_2 \setminus S_3$; and (iii) $(x, y) \in S_3$. These are analyzed as follows:

(i) Suppose that $(x, y) \in S_1 \setminus S_2$. Since $x < 0$ it follows that $\frac{\partial}{\partial x}H(x, y, z) > 0$; which implies that if $(x, y) \in S_1$, then $(0, y) \in S_1$. It follows that: $\tilde{\Sigma}_A^e(x, y, z) = 1$; whereas, $\tilde{\Sigma}_A^e(0, y, z) \geq 1$. Therefore,

$$\tilde{\Sigma}_A^e(0, y, z) \geq \tilde{\Sigma}_A^e(x, y, z)$$

(ii) Suppose that $(x, y) \in S_2 \setminus S_3$. Since $x < 0$, it follows that $\frac{\partial}{\partial x}(x^2) < 0$; which implies that if $(x, y) \in S_2$, then $(0, y) \in S_2$. It follows that: $\tilde{\Sigma}_A^e(x, y, z) = 2$; whereas, $\tilde{\Sigma}_A^e(0, y, z) \geq 2$. Therefore,

$$\tilde{\Sigma}_A^e(0, y, z) \geq \tilde{\Sigma}_A^e(x, y, z)$$

256
(iii) Suppose that \( (x, y) \in S_3 \). Since \( x < 0 \), it follows that \( \frac{\partial}{\partial x} G(x, y, z) < 0 \); which implies that if \( (x, y) \in S_3 \), then \( (0, y) \in S_3 \). It follows that:

\[
\tilde{\Sigma}_A(x, y, z) = \tilde{\Sigma}_A(0, y, z) = 3
\]

This implies that by switching from \( (x, y) \) to \( (0, y) \), Network \( A \) can increase \( Q(x) \) without decreasing \( \tilde{\Sigma}_A(0, y, z) \). Therefore, for any point in the 2nd Quadrant, \( (x, y) \), there exists a point on the y-axis, \( (0, y) \), which offers a strictly higher payoff. That is, if \( (x, y) \) is within the 2nd Quadrant, then \( (x, y) < (0, y) \).

**Lemma H.1 (2nd Quadrant.)** Suppose that \( x < 0 \), \( y \geq 0 \) and \( z \leq 0 \). It follows that: if \( H(x, y, z) < 200r^2\sigma^2 \), then \( \tilde{\Pi}_A(x, y, z) < \tilde{\Pi}_A(0, 0, z) \); whereas, if \( H(x, y, z) \geq 200r^2\sigma^2 \), then \( \tilde{\Pi}_A(x, y, z) < \tilde{\Pi}_A(0, y, z) \).

**H.2 3rd Quadrant**

In this section suppose that \( x < 0 \), \( y < 0 \). In the 3rd Quadrant, there are two regions: LHS; and Core. If \( y < -|z| \), then \( (x, y) \) is within the LHS. Whereas, if \( y \geq -|z| \), then \( (x, y) \) is within the Core. These are analyzed in turn:

Case (1). Suppose that \( y < -|z| \). If \( (x, y) \) is within the LHS, then \( \tilde{\Sigma}_A(x, y, z) = 3 \), \( \tilde{\Sigma}_A(0, 0, z) = 0 \). Whereas, \( \tilde{\Sigma}_A(0, 0, z) = 3 \), \( \tilde{\Sigma}_A(0, 0, z) = 3 \). (This is because the origin is at the centre of the Core.) Hence, by switching from \( (x, y) \) to \( (0, 0) \), Network \( A \) is able to increase the number of sellers on its platforms. Furthermore, \( Q(0) > Q(x) \) and \( Q(0) > Q(y) \). That is, switching from \( (x, y) \) to \( (0, 0) \), increases buyers’ demand. Therefore, the origin strictly dominates any point within the LHS.

Case (2). Suppose that \( y \geq -|z| \). If \( (x, y) \) is within the Core, then \( \tilde{\Sigma}_A(x, y, z) = 3 \), \( \tilde{\Sigma}_A(x, y, z) = 3 \). Since the origin is at the centre of the Core, switching from \( (x, y) \) to \( (0, 0) \) can’t change the number of sellers on Network \( A \). However, \( Q(0) > Q(x) \) and \( Q(0) > Q(y) \). Hence, switching from \( (x, y) \) to \( (0, 0) \), increases buyers’ demand. Therefore, the origin strictly dominates \( (x, y) \).

**Lemma H.2 (3rd Quadrant.)** Suppose that \( x < 0 \), \( y < 0 \) and \( z \leq 0 \). It follows that: \( \tilde{\Pi}_A(x, y, z) < \tilde{\Pi}_A(0, 0, z) \)

**H.3 4th Quadrant**

In this section suppose that \( x \geq 0 \), \( y < 0 \). Consider two cases: (1) \( y \geq -|z| \); and (2) \( y < -|z| \). These are analyzed in turn:

Case (1). Suppose that \( y \geq -|z| \). There are four regions: Exterior; 2nd Shell, 1st Shell, Core. There are two cases to consider. Firstly, if \( H(x, y, z) < 200r^2\sigma^2 \), then \( (x, y) \) is in the Exterior. Secondly, if \( H(x, y, z) \geq 200r^2\sigma^2 \), then \( (x, y) \) belongs to one of the other regions: 2nd Shell, 1st Shell, or Core.

(a) Suppose that \( H(x, y, z) < 200r^2\sigma^2 \). If \((x, y)\) is in the Exterior, then \((x, y)\) is strictly dominated by the origin.
(b) Suppose that \( H(x, y, z) \geq 200\tau^2\sigma^2 \). If \((x, y)\) is in one of the other regions, then there must exist a point on the \( x \)-axis, \((x, 0)\), that offers a higher payoff. The proof is very similar to that already given for points in the 2\(^{nd}\) Quadrant.

Case (2). Suppose that \( y < -|z| \). There are two regions: \( \text{RHS} \) and \( \text{LHS} \). If \( x > \omega \), then \((x, y)\) is on the \( \text{RHS} \); whereas, if \( x \leq \omega \), then \((x, y)\) is on the \( \text{LHS} \). These cases are analyzed as follows:

(a) Suppose that \( x > \omega \). If \((x, y)\) is on the \( \text{RHS} \), then no sellers are on Network \( A \) and it makes no profit. Hence, any such point is strictly dominated by the origin.

(b) Suppose that \( x \leq \omega \). If \( x \leq \omega \), then \((x, 0)\) is within the \( \text{Core} \) (see diagram). If \((x, y)\) is on the \( \text{LHS} \), then \( \bar{\Sigma}_A^x(x, y, z) = 3 \), \( \bar{\Sigma}_A^y(x, y, z) = 0 \). Whereas, \( \bar{\Sigma}_A^x(x, 0, z) = 3 \), \( \bar{\Sigma}_A^y(x, 0, z) = 3 \). Hence, by switching from \((x, y)\) to \((x, 0)\), Network \( A \) can increase the number of sellers on their platforms, while increasing buyers’ demand at the same time. Therefore, \((x, y)\) is strictly dominated by \((x, 0)\).

Lemma H.3 (4\(^{th}\) Quadrant.) Suppose that \( z \leq 0 \) and \( x \geq 0 \), \( y < 0 \). It follows that: if \( y \geq -z \), \( H(x, y, z) < 200\tau^2\sigma^2 \) or \( y < -z \), \( x > \omega \), then

\[
\bar{\Pi}_A(x, y, z) < \bar{\Pi}_A(0, 0, z)
\]

whereas, if \( y \geq -z \), \( H(x, y, z) > 200\tau^2\sigma^2 \) or \( y < -z \), \( x \leq \omega \), then

\[
\bar{\Pi}_A(x, y, z) < \bar{\Pi}_A(x, 0, z)
\]