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THE RESTRICTED ERMOLAEV ALGEBRA AND $F_4$

THOMAS PURSLOW

Abstract. We investigate the simple Lie algebra of type $F_4$ over an algebraically closed field of characteristic three. In this article, we show that the first Ermolaev algebra makes an appearance as a maximal subalgebra of $F_4$, and prove this using old results of Kuznetsov, Kostrikin and Ostrik about graded depth-one simple Lie algebras over fields of characteristic three.

Introduction

Let $k$ be an algebraically closed field of characteristic three, and $G$ be an algebraic group of type $F_4$ with $\mathfrak{g} = \text{Lie}(G)$. Over the complex numbers the classification of maximal subalgebras of the simple Lie algebras was achieved by Dynkin [Dynkin, 1952], and the process of extending this classification to the modular case is underway with [Herpel and Stewart, 2016] and [Premet, 2015] for $p \geq 5$.

We focus on [Herpel and Stewart, 2016, Remark 1.4], which included an example of a 26-dimensional simple subalgebra $L$ in $\mathfrak{g}$, that is neither classical nor $W(1; 1)$. This shows the requirement that $p$ is good in [Herpel and Stewart, 2016, Theorem 1.3] is necessary. We confirm that $L$ is isomorphic to the restricted Ermolaev algebra and provide our own argument of maximality as the main result in this article.

Theorem 1. Let $e := e_{1000} + e_{0100} + e_{0001} + e_{0120}$ be a representative for the nilpotent orbit denoted $F_4(a_1)$. For $f := f_{1232}$, we have that $L := \langle e, f \rangle \cong \text{Er}(1; 1)^{(1)}$ is a maximal subalgebra of $\mathfrak{g}$.

A consequence of working with algebraically closed fields of characteristic $p < 5$ is we lack the Premet-Strade classification [Premet and Strade, 2008], which presents issues when it comes to identifying simple Lie algebras. We are extremely lucky that graded simple Lie algebras of depth-one have been well studied with [Kostrikin and Ostrik, 1995] producing a
recognition theorem, and considering simple Lie algebras exhibiting depth-one gradings with non-semisimple zero component.

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1. Preliminaries

1.1. Notation. Throughout, let $G$ be an algebraic group of type $F_4$ with $\mathfrak{g} = \text{Lie}(G)$. We define as usual $\Pi = \{\alpha_i\}$ to be a basis of simple roots for the root system $\Phi$ of $G$. We express any root as a linear combination of such simple roots using the ordering found in [Bourbaki, 2002]. To shorten the notation further, we specify the roots by giving the list of coefficients; for example, the highest root in $F_4$ will be expressed as $\{2342\}$.

This well-known construction allows us to define the simple Lie algebra $\mathfrak{g}$ over an algebraically closed field $k$ of positive characteristic using a Chevalley basis [Chevalley, 1956], taking basis elements indexed by the roots with basis $\{e_\alpha : \alpha \in \Phi\}$ and $\{h_\alpha : \alpha \in \Pi\}$. We write $e_{\alpha_1+\alpha_2}$ as $e_{1100}$, and for the negative roots use $f_\alpha$ to denote $e_{-\alpha}$.

1.2. Nilpotent orbits. We familiarise ourselves with nilpotent orbits and the classification of such orbits. For every nilpotent element $e \in \mathfrak{g}$ we can form the orbit of $e$ under the adjoint action of $G$ on $\mathfrak{g}$.

Definition 1.2.1. A nilpotent element $e \in \mathfrak{g}$ is said to be distinguished if $C_G(e)^0$ is a unipotent group.

For every nilpotent element $e \in \mathfrak{g}$, there is a Levi subgroup $L$ such that $e$ is distinguished in the Lie algebra of the derived subgroup $L'$ of $L$. For example, take a maximal torus $T$ of $C_G(e)$ and consider the Levi subgroup $L = C_G(T)$. It follows from the structure of the restricted Lie algebra $\text{Lie}(L)$ that $e \in \text{Lie}(L')$. 
In fields of characteristic zero or \( p \gg 0 \) the Jacobson–Morozov theorem associates an \( \mathfrak{sl}(2) \) triple to every nilpotent element. This observation is used to prove the Bala–Carter classification of nilpotent orbits for \( p \) sufficiently large [Bala and Carter, 1976a], [Bala and Carter, 1976b]. To obtain the same classification in good characteristic we use cocharacters to replace \( \mathfrak{sl}(2) \) triples, [Premet, 2003]. For us, cocharacters give rise to useful gradings on \( \mathfrak{g} \).

**Definition 1.2.2.** Let \( e \) be a nilpotent element in \( \mathfrak{g} \) such that \( e \) is distinguished in some Levi subgroup \( L \). A cocharacter \( \tau : k^* \to G \) is associated to \( e \) if both

\[
e \in \mathfrak{g}(\tau, 2) \quad \text{and} \quad \text{im}(\tau) \subseteq [L, L].
\]

Any cocharacter produces a \( \mathbb{Z} \)-grading \( \mathfrak{g} = \bigoplus_i \mathfrak{g}(\tau, i) \) such that \( [\mathfrak{g}(\tau, i), \mathfrak{g}(\tau, j)] \subseteq \mathfrak{g}(\tau, i+j) \). The classification of all nilpotent orbits in the exceptional Lie algebras is well known at this stage. For us, the main reference [Lawther and Testerman, 2011] contains an incredible amount of information including tables containing orbit representatives, cocharacters and much more.

It must be noted that for bad primes many results on nilpotent orbits do not hold. The main concern for this article is the possible lack of associated cocharacters. It is possible we may only have \( e \in \bigoplus_{i \geq 2} \mathfrak{g}(\tau, i) \) for nilpotent elements in bad characteristic. We are fortunate for the work of [VIGRE Algebra Group, 2005], which provides a list of nilpotent orbit representatives in bad characteristic with some errors corrected by [Stewart, 2016, Section 1.1], and for [Holt and Spaltenstein, 1985] producing a result regarding the number of new nilpotent orbits.

We study the nilpotent orbit with representative \( e := e_{1000} + e_{0100} + e_{0001} + e_{0120} \). It is easy to check using GAP that \( \dim(\mathfrak{g}_e) = 6 \), and so the orbit \( \mathcal{O}(e) \) is subregular since there is only one nilpotent orbit of codimension 6 in \( F_4 \) [Holt and Spaltenstein, 1985, Theorem 4]. Hence we label the nilpotent orbit \( F_4(a_1) \) as in the characteristic zero case and continue to use the associated cocharacter from [Lawther and Testerman, 2007, pg81], making use of [Clarke and Premet, 2013].

**1.3. The restricted Ermolaev algebra.** The Ermolaev algebras, first constructed in [Ermolaev, 1982], are a class of simple Lie algebras, different to the Cartan or classical type, and only appearing in algebraically closed fields of characteristic \( p = 3 \). We construct the restricted case using the description in [Strade, 2004, Section 4.4], which gives a nice grading on such a class of simple Lie algebras. In fact, for general notation on modular Lie algebras, and specifically for the Jacobson–Witt algebras we refer to [Strade, 2004].
We have a map

$$\text{div} : W(2; 1) \to \mathcal{O}(2; 1),$$

such that $\text{div}(\sum f_i \partial_i) = \sum \partial_i (f_i)$, and, for any $\alpha \in \mathbb{k}$, there is a $W(2; 1)$-module denoted by $\mathcal{O}(2; 1)(\alpha \text{div})$ obtained by taking $\mathcal{O}(2; 1)$ under the action

$$D \cdot f := D(f) + \alpha \text{div}(D)f,$$

for all $D \in W(2; 1)$ and $f \in \mathcal{O}(2; 1)$.

The restricted Ermolaev algebra as a vector space is $W(2; 1) \oplus \mathcal{O}(2; 1)(\text{div})$, denoted by $\text{Er}(1; 1)$, which admits an automorphism of order two with 1-eigenspace $W(2; 1)$ and $(-1)$-eigenspace $\mathcal{O}(2; 1)(\text{div})$. The Lie bracket is given by

$$[f, g] := (f \partial_2(g) - g \partial_2(f))\partial_1 + (g \partial_1(f) - f \partial_1(g))\partial_2,$$

for all $f, g \in \mathcal{O}(2; 1)$ with all other products defined canonically.

To obtain a simple Lie algebra from this, note that if $\alpha = 1$ then $\mathcal{O}(2; 1)(\alpha \text{div})$ has a submodule of codimension 1 [Strade, 2004, Proposition 4.3.2, (1)], denoted by $\mathcal{O}'(2; 1)(\text{div})$, and the derived subalgebra of $\text{Er}(1; 1)$ is equal to

$$W(2; 1) \oplus \mathcal{O}'(2; 1)(\text{div}).$$

This is a simple Lie algebra of dimension 26 called the restricted Ermolaev algebra.

$\text{Er}(1; 1)$ inherits a $\mathbb{Z}$-grading from the usual grading of $W(2; 1)$ with

$$\text{Er}(1, 1)_i := W(2; 1)_i \oplus \mathcal{O}(2; 1)_{i+1},$$

for all $i \geq -1$. From the grading, we immediately see that

$$\text{Er}(1, 1)_{-1} = \mathbb{k}\partial_1 + \mathbb{k}\partial_2 + \mathbb{k}1,$$

and the zero component is a 6-dimensional subalgebra containing a 3-dimensional radical. The radical is both nilpotent and non-central, in the sense it is not contained in the centre. Taking the quotient of the zero component by its radical gives a Lie algebra isomorphic to $\mathfrak{sl}(2)$.

Remark 1.4. By [Strade, 2004, Proposition 4.3.2 (1)] we know that $\mathcal{O}'(2; 1)(\text{div})$ is a proper submodule that does not contain $x_1^2 x_2^2$ and as a consequence the top component of $\text{Er}(1; 1)^{(1)}$ is two-dimensional.
It should be noted that this construction can be generalised to all vectors \( \underline{n} \in \mathbb{N}^2 \) where we can consider
\[
\text{Er}(n_1, n_2) := W(2; \underline{n}) \oplus O(2; \underline{n})_{(\text{div})},
\]
to produce simple Lie algebras of dimension \( 3^{n_1+n_2+1} - 1 \). This is the so called Ermolaev series. We could attempt this for all \( p > 0 \), however for \( p \neq 3 \) the Jacobi identity fails to hold.

To see this consider
\[
J(x_1 \partial_1, x_1, x_2) = [x_1 \partial_1, [x_1, x_2]] + [x_2, [x_1 \partial_1, x_1]] + [x_1, [x_2, x_1 \partial_1]] = 3(x_1 \partial_1 + x_2 \partial_2).
\]

1.5. **GAP calculations.** We frequently use [GAP, 2016] in this article, so we list the type of commands we use. We obtain the exceptional simple Lie algebras including \( F_4 \) along with a Chevalley basis with the following commands:

```gap
gap> g:=SimpleLieAlgebra( "F", 4, GF(3));;
gap> b:=Basis( g );;
```

GAP does not produce \( F_4 \) in the same order as the usual Bourbaki ordering; it differs by some permutation on the roots. We find a full list of all the basis elements in [de Graaf, 2008] for all exceptional Lie algebras corresponding to their usual ordering. As an example, \( e \) as above produces \( e_{0100} + e_{1000} + e_{0120} + e_{0001} \). We ask GAP to compute subalgebras on occasion; for example to obtain \( \langle e, f \rangle \) and its dimension we enter:

```gap
gap> h:=Subalgebra( g, [ e, f ] );;
gap> Dimension( h );
26
```

We use the well-known MeatAxe package based on [Holt and Rees, 1994] to check simplicity of our Lie algebra. To do this, we input the adjoint module as a set of matrices \( \{ e_i \} \), and use the MeatAxe to check the irreducibility of such a module.

The MeatAxe was initially intended to be used in the case of groups, but we can still use this to check that the adjoint module of our Lie algebra is irreducible. For the set \( \{ e_i \} \) one can find \( \lambda_i \in k \) such that \( \lambda_i I + e_i \) is an invertible matrix. Thus \( \{ \lambda_i I + e_i \} \) generates a subgroup of \( GL(V) \), and there is a subspace stable under the \( \lambda_i I + e_i \) if and only if there is one under the \( e_i \).
gap> bh:=Basis(h);;
gap> Mats:=List(bh,x->AdjointMatrix(bh,x));;
gap> gm:=GModuleByMats(Mats,GF(3));;
gap> MTX.IsAbsolutelyIrreducible(gm); true

This checks if our algebra is absolutely simple. We do this because we have defined the Lie algebras over finite fields, and being simple over a finite field does not necessarily imply the algebra is simple over an algebraically closed field. At times we will consider the normaliser of a subalgebra and generate vector spaces using:

gap> N:=LieNormaliser(g,h);;
gap> V:=VectorSpace(GF(3),[b[1],b[7]]);;
gap> Dimension(V);
2

2. Er(1; 1)\((1)\) is a subalgebra of \(F_4\)

We prove the restricted Ermolaev algebra is a subalgebra of \(\mathfrak{g}\). Using the commands considered in the previous section we are able to obtain the following information using GAP.

1. The subalgebra defined in Theorem \([1]\) \(L := \langle e, f \rangle\), is simple and of dimension 26.
2. The normaliser of \(L\) in \(\mathfrak{g}\) is isomorphic to \(L\).
3. For \(f' := f_{1222} - f_{1242} \in L\), the subalgebra \(W := \langle e, f' \rangle\) is simple and of dimension 18.

It turns out that \(W\) is isomorphic to \(W(2; 1)\). To prove \(L\) is isomorphic to the Ermolaev algebra we need to locate a complementary subspace to \(W\) in \(L\). The subalgebra \(L\) obtains a grading using the cocharacter \(\tau\) associated to \(e\) from \([\text{Lawther and Testerman, 2011, pg81}]\). We then regrade \(L\) to conclude \(L \cong \text{Er}(1; 1)\)(1), using \([\text{Kuznetsov, 1989, Theorem 1}]\).

**Proposition 2.1.** The subalgebra \(L\) is isomorphic to \(\text{Er}(1; 1)\)(1).

**Proof.** We apply the cocharacter \(\tau\) to the basis elements of \(L\) to establish the degree of each homogenous element of \(L\), and thus obtain a grading. This gives the following table:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(-14)</th>
<th>(-12)</th>
<th>(-10)</th>
<th>(-8)</th>
<th>(-6)</th>
<th>(-4)</th>
<th>(-2)</th>
<th>(0)</th>
<th>(2)</th>
<th>(4)</th>
<th>(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\dim(L(\tau,i)))</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
To find a suitable complementary module to $W$ in $L$, we attempt to find the element $1 \in O'(2;1)_{(\text{div})}$ as an element of $L$, and then try to construct an 8-dimensional vector space $V$ with the action of $W$ on such an element. For this, consider $\ker(\text{ad} e) \cap L(4)$, which is one-dimensional with basis element $v := e_{0111} - e_{1110}$. Computing the $k$-span of all brackets $[x,v]$ for $x \in W$ produces the required module $V$ such that $L = W \oplus V$.

From section 1.3, it is clear that $W(2;1)$ is generated as a Lie algebra by all brackets of the form $[u,v]$ for $u,v \in O(2;1)_{(\text{div})}$. To show this holds for our $W$, we consider $[V,V] := \text{span}_k\{[u,v] : u,v \in V\}$, and generate a subalgebra of $L$ by these elements. This produces a simple 18-dimensional Lie algebra, containing both $e$ and $f'$, and hence is isomorphic to $W$.

Using our cocharacter $\tau$, we see that $V$ is graded in degrees $-10 \leq i \leq 4$ with each even graded component one-dimensional. In the table below we give a basis for each homogenous component together with an integer $d(i)$ which will replace its degree in the new grading on $L$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>Basis element of $V(\tau, i)$</th>
<th>$d(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$e_{0111} - e_{1110}$</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>$e_{0011} - e_{0110}$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$e_{0010}$</td>
<td>1</td>
</tr>
<tr>
<td>-2</td>
<td>$f_{0011} + f_{0110}$</td>
<td>0</td>
</tr>
<tr>
<td>-4</td>
<td>$f_{0111} + f_{1110}$</td>
<td>1</td>
</tr>
<tr>
<td>-6</td>
<td>$f_{1111}$</td>
<td>2</td>
</tr>
<tr>
<td>-8</td>
<td>$f_{1231}$</td>
<td>1</td>
</tr>
<tr>
<td>-10</td>
<td>$f_{1232}$</td>
<td>2</td>
</tr>
</tbody>
</table>

From now on, we write $v(i)$ to denote the basis elements in the table; for example, we let $v(4) := e_{0111} - e_{1110}$. These new degrees allow us to regrade the simple Lie algebra $L$ with a 3-dimensional $L_{-1}$ component with basis elements,

$$\{e_{0111} - e_{1110}, e_{1121} + e_{0122} - e_{1220}, e_{0001} + e_{1000} + e_{0100}\},$$

obtained by computing the non-zero elements $v(4), [v(4), v(2)]$ and $[v(4), v(-2)]$. 
To obtain $L_0$, we take the obvious Lie brackets of elements along with the elements of degree 0 from $V$ to produce

$$L_0 = \langle v(2), v(-2), [v(4), v(0)], [v(4), v(-4)], [v(4), v(-8)], [v(2), v(-2)] \rangle.$$

This is a 6-dimensional Lie algebra consisting of an $\mathfrak{sl}(2)$ triple

$$\{e_1, f_1, h_1\},$$

with $e_1 := e_{0121} + e_{1120}$, $f_1 := f_{0121} + f_{1120}$ and $h_1 := h_{\alpha_1} + h_{\alpha_4}$ along with a 3-dimensional radical with basis $\{v(2), v(-2), [v(-2), v(2)]\}$.

This gives a depth-one graded simple Lie algebra whose $L_0$ component contains a non-central nilpotent radical. Hence we can use [Kuznetsov, 1989, Theorem 1] along with the grading to see $L \cong \text{Er}(1; 1)^{(1)}$.

**Corollary 2.2.** The subalgebra $W := \langle e, f' \rangle$ is isomorphic to $W(2; 1)$.

**Proof.** Every element of $W$ is obtained via the $k$-span of Lie brackets $[u, v]$ from $V$ and so we obtain a grading on $W$ as follows.

For $W_{-1}$ we have a two-dimensional vector space with basis elements $e_{1121} + e_{0122} - e_{1220}$ and $e_{0001} + e_{1000} + e_{0100}$. These are obtained taking the brackets $[v(4), v(2)]$ and $[v(4), v(-2)]$ respectively. We calculate the degree 0 component in the same way to obtain

$$W_0 = \langle [v(4), v(0)], [v(4), v(-4)], [v(4), v(-8)], [v(2), v(-2)] \rangle,$$

which consists of an $\mathfrak{sl}(2)$ triple

$$\{e_1, f_1, h_1\},$$

with $e_1 := e_{0121} + e_{1120}$, $f_1 := f_{0121} + f_{1120}$, $h_1 := h_{\alpha_1} + h_{\alpha_4}$ and central element $h_{\alpha_2} + h_{\alpha_4}$.

This gives a depth-one graded simple Lie algebra with classical simple $W_0$ (modulo its centre). Using [Kostrikin and Ostrik, 1995, Theorem 1] in combination with dim($W$) = 18 says that $W \cong W(2; 1)$. $\square$

3. **The Ermolaev algebra is maximal**

In the final section we complete the proof of Theorem 1 by proving that $L$ is a maximal subalgebra of $F_4$. We will make use of the fact that for $p \geq 3$, simple Lie algebras of type $F_4$ admit a non-degenerate symmetric form.

**Lemma 3.1.** The adjoint module of the Lie algebra $L := \text{Er}(1; 1)^{(1)}$ is not self-dual.
Proof. By construction we have \( \dim(L_{-1}) = 3 \) and \( \dim(L_3) = 2 \) from Remark \ref{remark1.4} hence \( L_{-1} \not\cong (L_3)^* \) as \( L_0 \)-modules. It follows by \cite[Lemma 4]{Premet1985} that \( L \) is not self-dual. \( \square \)

Using this together with the fact that \( L = N_{\mathfrak{g}}(L) \) will allow us to conclude \( L \) is a maximal subalgebra in \( F_4 \).

**Proposition 3.2.** The subalgebra \( L \) is maximal in the exceptional Lie algebra of type \( F_4 \).

**Proof.** For \( p = 3 \) the adjoint module of \( F_4 \) is self-dual and admits a non-degenerate invariant symmetric form denoted by \( \kappa \), the so-called normalised Killing form defined explicitly \cite[pg 661]{Clarke2013}. Since \( \dim(L) = \frac{1}{2} \dim(\mathfrak{g}) = 26 \) and the adjoint module of \( L \) is not self-dual, we have that \( L \) is a maximal totally isotropic subspace of \( F_4 \) with respect to \( \kappa \).

Suppose it were not maximal. Then there exists \( M \) such that \( L \subsetneq M \subsetneq \mathfrak{g} \). The restriction of \( \kappa \) to \( M \) is non-zero with non-zero radical \( R \), since \( L \) is totally isotropic and \( \dim(L) > \frac{1}{2} \dim(M) \). If \( L \cap R = 0 \), then \( L \oplus R \) is a maximal totally isotropic subalgebra containing \( L \) — which is a contradiction. Since \( L \) is simple and \( L \cap R \neq 0 \), it must be the case \( R \cap L = L \). In particular we conclude \( R = L \), and so \( M \subsetneq N_{\mathfrak{g}}(L) = L \) proving \( L \) is maximal. This completes the proof of Theorem \ref{thm:1} \( \square \)

**References**


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